STS - Sport Tournament Scheduling

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1 Introduction

The problem addressed in this project is the the Sports Tournament Scheduling (STS). We approach it using Constraint Programming (CP), SAT solvers, Satisfiability Modulo Theory (SMT), and Mixed Integer Linear Programming (MILP). All the developed models share a common formalization, described in this section.

Input Parameters.

- n: Number of teams (an even integer)
- $w = n - 1$: Number of weeks
- $p = \frac{n}{2}$: Number of periods
- Teams are indexed by $t \in \{1, \ldots, n\} = T$
- Weeks are indexed by $w \in \{1, \ldots, n-1\} = W$
- Periods are indexed by $p \in \{1, \ldots, n/2\} = P$

1.1 Objective variable.

The objective function is the same across all proposed approaches. Specifically, we aim to minimize the maximum absolute difference between the number of home and away matches played by any team:

$$M = \max_{t \in T} |H_t - A_t|$$

where H_t denotes the number of home games of team t, A_t denotes the number of away games of team t. The objective variable is bounded between a lower limit of 1 and an upper limit of $n - 1$.

Moreover, initially, we considered an alternative formulation:

$$M = \sum_{t \in T} |H_t - A_t|,$$

but empirical evaluation showed that minimizing the maximum imbalance (first formulation) consistently produced significantly better results across all the techniques used.

1.2 Constraints.

All models adhere to the following fundamental constraints:

- every team plays with every other team only once;
- every team plays once a week;
- exactly one match is scheduled in each period of every week;
- every team plays at most twice in the same period over the tournament.

1.3 Pre-solving with the Circle Method.

In each of the four approaches addressed in this report we used a common pre-processing step that helped improving the quality of the found solutions. This is because we avoid additional decision variables by precomputing which teams (i, j) play in each week w using the fast and classical *circle method* for round-robin tournaments [1].

The circle method, is a constructive procedure for generating round-robin tournament schedules in which each team must play every other team exactly once. The method begins by arranging all teams in a circular formation, fixing one team, the pivot, in a constant position, while the remaining teams are placed around it. In each week, the teams are paired according to their positions in the circle, where each team plays against the team directly opposite to it. After every week, all teams except the fixed one are rotated clockwise by one position, producing a new set of pairings for the following week. This process continues until all the weeks have been completed. By construction two core constraints of the problem are satisfied:

- every team plays against every other team exactly once
- every team plays once a week.

Example for $n = 6$:

$$\text{Week 1: } (6,1),(2,5),(3,4) \quad \text{Week 2: } (6,2),(3,1),(4,5),\dots$$

1.4 Work division

The work has been roughly split in the following way: Cotič did the CP part, Centanni worked on SAT, Uskov did the SMT part and Lodi completed the MILP part.

2 CP Model

2.1 Decision Variables

The CP model relies on the following decision variables:

- For each unordered pair of teams (i, j) we define $\mathbf{period}_{i,j} \in P$ as the slot in which team i plays against team j.
- Similarly, for each unordered pair of teams (i, j) we define the variables $\mathbf{home}_{i,j} \in \{0, 1\}$ in such a way that $\text{home}_{i,j} = 1$ when i plays against j at home.

Moreover, using the previously described *circle method*, all week assignments are precomputed and passed to the model in the variables $\mathbf{week}_{i,j} \in W$, where $\text{week}_{i,j} = w$ means that i plays against j in week w.

2.2 Objective Function

As already described above, the objective function is defined as

$$\max_{t \in T} |H_t - A_t|,$$

where H_t is the number of home games of team t, A_t is the number of away games of team t, and T is the set of teams.

In the CP model, the number of games each team plays at home and away are stored in two arrays:

$$\forall t \in T \quad \texttt{home\_count}[\texttt{t}] \in \{1, \dots, n-1\}, \qquad \texttt{away\_count}[\texttt{t}] \in \{1, \dots, n-1\}.$$

The imbalance for each team is stored in

$$\texttt{imbalance}[\texttt{t}] = |\texttt{home\_count}[\texttt{t}] - \texttt{away\_count}[\texttt{t}]|$$

and the optimization variable is taken as the max over the array.

A final consideration regarding the objective function is that the CP models performed significantly better when switching from minimizing the sum of home–away imbalances to minimizing the maximum imbalance.

2.3 Constraints

Thanks to the previously described *circle method* only two constraints had to be implemented to solve the problem in a consistent way:

Each team plays at most twice in the same period:

$$\forall\, t \in T,\ \forall\, p \in P: \quad \sum_{j \in T \setminus \{t\}} \chi_{\{\mathrm{period}_{t,j} = p\}} \ \leq\ 2$$

We implemented this constraint using the global constraint `count_geq`, which yielded better results than both the intuitive formulation and the alternative global constraint `global_cardinality`.

Each period in each week contains exactly one match:

$$\forall\, w \in W,\ \forall\, p \in P: \quad \sum_{(i,j) \in M} \chi_{\{\,\mathrm{week}_{i,j} = w \,\wedge\, \mathrm{period}_{i,j} = p\,\}} = 1,$$

where M is the set of all the possible matches:

$$M \ = \ \big\{\, (i,j) \ :\ i,j \in T,\ i < j \,\big\}.$$

We tried to tackle this constraint using the global constraint `alldifferent` but it weakened the performance on basically all instances, which is why we didn't implement this constraint using the available global one.

Additional constraints were required due to the way we defined the decision variables. Specifically, we allowed the slot variable to take the value 0, even though it does not correspond to a valid slot in which teams can play. Furthermore, the slot matrix must be symmetric, and the home team assignment also exhibits a particular structure by design:

- $\forall\, t \in T: \quad \text{period}_{t,t} = 0$
- $\forall\, i, j \in T,\, i \neq j: \begin{array}{l} \text{period}_{i,j} \neq 0, \\ \text{period}_{i,j} = \text{period}_{j,i}, \\ \text{home}_{j,i} = 1 - \text{home}_{i,j} \end{array}$

Symmetry Breaking Constraints

We identified some symmetries in the problem that, if left unaddressed, enlarged the search space explored by the solver. To reduce redundant exploration, we introduced symmetry-breaking constraints:

SB1: Fixing the slots of the first team:

$$\forall\, t \in T: \quad \text{period}_{1,t} \leq \text{period}_{1,t+1}$$

This constraint ensures that the slots assigned to the first team are strictly increasing, effectively removing symmetries that arise from permuting slot orders. We implemented it using the MiniZinc global constraint `increasing`.

SB2: Team 1's home/away pattern is fixed:

$$\forall\, w \in W: \quad \text{home}_{1,\,w+1} = \begin{cases} 1, & 1 \leq w \leq \lfloor \frac{\text{num\_teams}}{2} \rfloor, \\ 0, & \lfloor \frac{\text{num\_teams}}{2} \rfloor + 1 \leq w \leq \text{num\_teams} - 1 \end{cases}$$

This constraint eliminates symmetries that come from flipping home-away status forcing the first team to play the first half of the season at home and the second away.

SB3: Imbalance ordering:

$$\forall\, t \in T: \quad \text{imbalance}_t \geq \text{imbalance}_{t+1}$$

This last symmetry breaking constraint binds the imbalance array to be decreasing and was of course tested only on the optimization version of the problem.

2.4 Validation

The models were implemented in MiniZinc and coupled with a Python script that takes the input parameters, determines week assignments, and subsequently executes the corresponding MiniZinc model.

2.4.1 Experimental design

We experimented our models using **Gecode** and **Chuffed** with different search strategies. All experiments were conducted respecting the given timeout of 300 s, with the solvers in their sequential version and using 42 as `random_seed` in order to obtain a deterministic behavior.

2.4.2 Experimental Results (Decision version)

In Figure 1, we present the results for the decision version of the problem. The best configuration combines Chuffed with the random-order variable selection heuristic and Luby restarts, using a restart scale of 50. Another clear observation is that Chuffed consistently outperforms Gecode. Overall, the results are very similar to those of the optimization version, which is why the discussion of the chosen approaches is deferred to the following section.

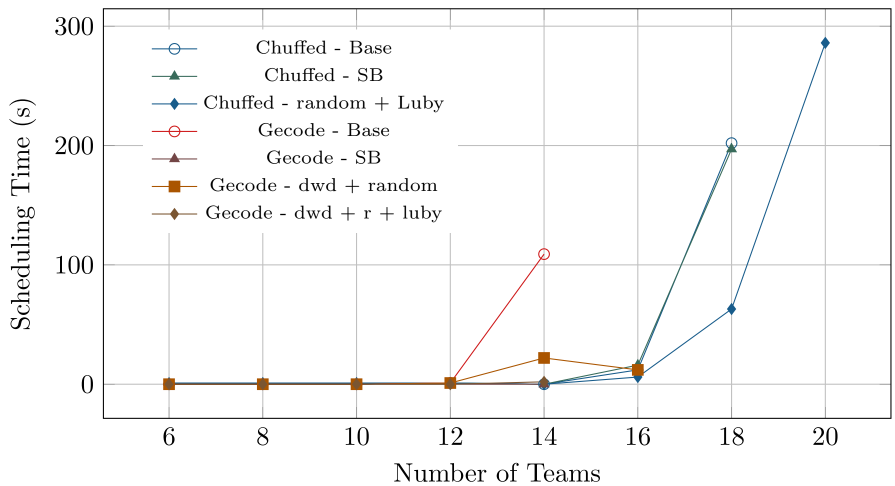


Figure 1: Decision version: runtime in seconds for different solving configurations.

2.4.3 Experimental Results (Optimization version)

Our first objective was to evaluate the impact of the proposed symmetry-breaking constraints. As shown in Table 1, the inclusion of symmetry breaking yields clear benefits for both Gecode and Chuffed. In particular, Chuffed was able to solve the case $n = 16$ to optimality, which was not possible without symmetry breaking and it found a better solution for $n = 18$. Similarly, Gecode, when combined with symmetry breaking, successfully solved the case $n = 14$ to optimality. Among the three symmetry-breaking strategies described before, the best performance was obtained using the first two, while the third did not lead to any improvement, which is why we did not use it.

Having established the advantages of symmetry breaking, we next investigated alternative search strategies to further enhance performance of both solvers. All subsequent experiments were conducted with symmetry breaking enabled.

For the optimization version with Chuffed, we were not able to identify search strategies that improved the model's solutions, highlighting the effectiveness of Lazy Clause Generation. Among the tested configurations, the best was `random_order` for variable selection combined with a `Luby`(50) restart strategy. We also experimented with `first_fail`, but regardless of the chosen value-assignment heuristic, no improvement was observed, although `indomain_split` performed slightly better than the others.

We additionally tested enabling free search, allowing the solver to switch between the specified search annotations and its default search. However, this significantly degraded performance: in fact, the models were unable to solve even the case $n = 16$ to optimality.

For Gecode, we initially experimented with the `first_fail` variable selection heuristic, but it did not yield any performance improvement. Switching to the domain over weighted degree heuristic, combined with random value assignment, we obtained a significant improvement. Using this approach, Gecode was able to solve instances up to $n = 16$ optimally, although, the time required to solve $n = 14$ increased noticeably.

We also attempted to combine this strategy with restarts, specifically `Luby`, but the results weakened. This trend was consistent across several restart scales arbitrarily chosen within $[20, 150]$.

Additionally, we evaluated the large neighborhood search (LNS) strategy; however, it provided no benefit regardless of the retainment percentage. In fact the results were the same as without it. As to be expected, this approach was only tested for the optimization version of the problem.

Overall, Chuffed consistently outperformed Gecode, producing better results across nearly all non-trivial instances, with the exception of some faster solutions that Gecode was able to find for $n = 16$.

n	**Chuffed**			**Gecode**			
	Base	SB	random+luby(50)	Base	SB	dwd+random	dwd+r+luby(50)
10	0‖**1**	0‖**1**	0‖**1**	0‖**1**	0‖**1**	0‖**1**	0‖**1**
12	1‖**1**	1‖**1**	2‖**1**	1‖**1**	0‖**1**	1‖**1**	1‖**1**
14	1‖**1**	6‖**1**	6‖**1**	300‖7	0‖**1**	36‖**1**	12‖**1**
16	300‖3	22‖**1**	87‖**1**	N/A	N/A	14‖**1**	300‖9
18	300‖11	300‖5	300‖7	N/A	N/A	N/A	N/A

Table 1: Optimization version: runtime in seconds and found objective value for different combinations of models, solvers and search strategies. Smaller instances $(6, 8)$ are left out because all configurations solved them instantly.

3 SAT Model

3.1 Decision Variables

To encode the Sports Tournament Scheduling (STS) problem, we define the following decision variables:

1. **match_period_vars**$[i, j, p] \in \{\text{True}, \text{False}\}$, for all $i, j \in T$ with $i < j$, and $p \in P$.

This variable is true if and only if team i plays against team j during period p. The week for each match-up (i, j) is precomputed (see Section 4.1.1) and fixed.

2. **home_vars**$[i, j] \in \{\text{True}, \text{False}\}$, for all $i, j \in T$ with $i < j$.
   This variable is true if and only if team i plays at home against team j.

3.2 Objective Function

The goal of the STS problem is to minimize the maximum home-away imbalance among all teams:

$$\min \max_{t \in T} |H_t - A_t|,$$

where H_t is the number of home games for team t, and A_t is the number of away games for team t.

Since SAT solvers handle only boolean variables, we solve this optimization problem via a *binary search* on the maximum allowed imbalance k.

For each team $t \in T$, the number of home games H_t is defined as

$$H_t = \sum_{\substack{(i,j) \in \text{pair\_to\_week} \\ p \in P}} \begin{cases} 1 & \text{if } t = i \land \text{match\_period\_vars}_{i,j,p} \land \text{home\_vars}_{i,j} \\ 1 & \text{if } t = j \land \text{match\_period\_vars}_{i,j,p} \land \neg\text{home\_vars}_{i,j} \\ 0 & \text{otherwise} \end{cases}$$

To enforce that the home-away imbalance for each team t does not exceed k, we use two pseudo-boolean constraints. Let NUM_GAMES $= n - 1$. For a given k, the constraints are:

1. $H_t \geq \text{lower\_bound}$, where $\text{lower\_bound} = \left\lceil \frac{\text{NUM\_GAMES} - k}{2} \right\rceil$,

2. $H_t \leq \text{upper\_bound}$, where $\text{upper\_bound} = \left\lfloor \frac{\text{NUM\_GAMES} + k}{2} \right\rfloor$.

These bounds ensure that the number of home games for each team respects the maximum imbalance k. Since the maximum number of matches a team can play is $|W|$, which is odd, then the optimal value we want to achieve is exactly $k = 1$.

For other details, see section 3.4.

3.3 Constraints

Thanks to the *circle method*, some constraints are inherently satisfied:

1. Each team plays every other team exactly once.

2. Each team plays exactly once every week.

This reduces the constraints needed, leaving the solver to decide only the specific period for each match (i, j). In particular, we implemented the following constraints:

Each match is assigned to exactly one period:

$$\forall (i,j) \in \text{pair\_to\_week} : \quad \sum_{p \in P} \text{match\_period\_vars}_{i,j,p} = 1$$

Each period in each week contains exactly one match:

$$\forall w \in W, \forall p \in P : \quad \sum_{(i,j) \in \mathcal{M}_w} \text{match\_period\_vars}_{i,j,p} = 1$$

Each team plays at most twice in the same period:

$$\forall t \in T, \forall p \in P : \quad \sum_{\substack{(i,j) \in \text{pair\_to\_week} \\ t \in \{i,j\}}} \text{match\_period\_vars}_{i,j,p} \leq 2$$

Symmetry Breaking Constraints

To reduce redundant search caused by symmetric solutions, we experimented with several symmetry breaking (SB) constraints.

SB1: Match between teams 0 and $n-1$ is in the first period:

$$\text{match\_period\_vars}_{0,n-1,0} = 1$$

This fixes the match between the pivot team $n-1$ and team 0 in period 0, breaking rotational symmetry.

We also investigated two further constraints, which are commented in the code and reported here only for completeness, as they did not yield improvements in practice. Since they introduced additional complexity that outweighed any potential benefits, they were not considered in the final results.

SB2: Team 0's home/away pattern is fixed:

$$\forall (i,j) \in \text{pair\_to\_week with } 0 \in \{i,j\} : \begin{cases} \text{home\_vars}_{i,j} & \text{if } w(i,j) \text{ is even and } i = 0 \\ \neg\text{home\_vars}_{i,j} & \text{if } w(i,j) \text{ is even and } j = 0 \\ \neg\text{home\_vars}_{i,j} & \text{if } w(i,j) \text{ is odd and } i = 0 \\ \text{home\_vars}_{i,j} & \text{if } w(i,j) \text{ is odd and } j = 0 \end{cases}$$

This enforces a fixed home/away assignment for team 0, removing the global symmetry of flipping all home/away statuses.

SB3: Lexicographical ordering of matches in week 0:

$$\forall a \in \{0, \dots, |\mathcal{M}_0| - 2\} : \quad \text{lex\_less}(V_a, V_{a+1})$$

This constraint enforces a lexicographical order on the period assignments of matches in week 0, preventing symmetric permutations.

Encoding Methods

We implemented all the encoding methods covered in class:

1. For the *exactly-one* constraint: Pairwise, Bitwise, Sequential, and Heule encodings.
2. For the *at-most-k* constraint: Pairwise, Sequential, and Totalizer encodings.

Additionally, we employed the **Totalizer encoding** [2] to efficiently model cardinality constraints. This encoding builds a balanced binary tree of adders and is known for scalability and efficiency in SAT formulations. It introduces $O(n \log n)$ auxiliary variables and up to $O(n^2)$ clauses in the worst case.

3.4 Validation

The model was implemented in Python using Z3's API. All experiments were conducted respecting the given timeout of 300 s and with an increasing number of instances (i.e., number of teams, n).

In the following, we present tables and plots to compare the Z3 model using different encoding techniques (used for both optimization and decision versions), both with and without symmetry breaking constraints:

1. Pairwise + Pairwise encodings
2. Heule + Sequential encodings
3. Heule + Totalizer encodings

3.4.1 Experimental Results (Decision version)

As shown in Table 2, even a simple symmetry breaking (SB) constraint can effectively guide the solver, leading to noticeably faster search times.

Table 2: Runtime in seconds for the SAT (decision) version using the Z3 solver, with and without symmetry breaking (SB).

N	np-np + SB	np-np w/out SB	heule-seq + SB	heule-seq w/out SB	heule-tot + SB	heule-tot w/out SB
12	1.0	1.0	0.0	0.0	0.0	0.0
14	3.0	2.0	0.0	0.0	1.0	0.0
16	4.0	26.0	7.0	3.0	5.0	10.0
18	17.0	80.0	8.0	11.0	7.0	15.0
20	N/A	N/A	79.0	N/A	150.0	151.0

3.4.2 Experimental Results (Optimization version)

The optimization version of the Sports Tournament Scheduling (STS) problem was solved using a binary search approach on the objective function value.

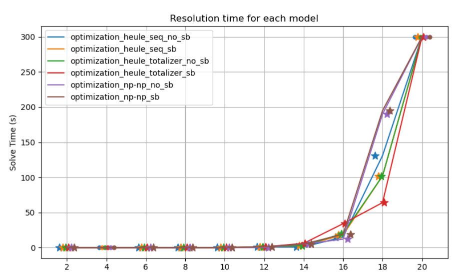
As shown in Table 3, the results confirm the expected performance of the encodings. The Heule + Totalizer combination generally proved to be very efficient, particularly for larger instances. Conversely, the Pairwise + Pairwise approach, known for its larger number of clauses, was the slowest.

Generally, except for some rare cases, the SB constraint proved effective at guiding the solver to an optimal solution faster, significantly reducing the search space.

The table also shows that all tested configurations successfully found an **optimal solution** for $N \leq 18$.

Table 3: SAT results. Results are of the type "time|**optimal value**" or "time|suboptimal value".

N	np-np + SB	np-np w/out SB	heule-seq + SB	heule-seq w/out SB	heule-tot + SB	heule-tot w/out SB
10	1|**1**	1|**1**	0|**1**	0|**1**	0|**1**	0|**1**
12	2|**1**	3|**1**	0|**1**	1|**1**	1|**1**	1|**1**
14	7|**1**	7|**1**	3|**1**	3|**1**	2|**1**	5|**1**
16	46|**1**	35|**1**	37|**1**	25|**1**	36|**1**	34|**1**
18	207|**1**	292|**1**	143|**1**	160|**1**	150|**1**	98|**1**
20	N/A	N/A	300|5	N/A	300|10	N/A



(a) Resolution time

Figure 2: Runtimes for the optimization version of the Sports Tournament Scheduling (STS) problem. Data points are marked to indicate the solution status: a star ($\star$) indicates that a solution (optimal or sub-optimal) was found, while a circle ($\bullet$) indicates that no solution was found within the time limit.

4 SMT Model

4.1 Decision Variables

Let n be the number of teams, and let $W = \{1, \dots, n-1\}$ denote the set of tournament weeks. Each week is divided into $n/2$ periods. Let $\mathcal{M}_w$ be the set of matches scheduled in week $w \in W$.

We define an integer decision variable:

$$p_{w,i,j} \in \{1, \dots, n/2\}$$

for each match between any pair of teams (i, j), where $(i, j) \in \mathcal{M}_w$, indicating the period during which this match takes place in week w.

The model is encoded using the theory of Linear Integer Arithmetic (LIA), extended with pseudo-Boolean constraints for cardinality.

4.1.1 Pre-solving with the Circle Method

We avoid additional decision variables by precomputing which teams (i, j) play in each week w using the fast and classical *circle method* for round-robin tournaments [1], which was already explained above. In practice, this greatly reduces the number of SMT variables and constraints. Thanks to pre-solving, we can solve instances with up to $n = 20$ teams—and usually even $n = 22$ sometimes — within the 5-minute limit. This is 2 to 4 teams more than our first SMT models that generated all possible matches as decision variables.

4.2 Objective Function

For the sake of solving the home-away balancing task, there is a simple mathematical algorithm. It does not need optimization with SMT. If (i,j) is a game between any two teams i and j then the modulo arithmetic's algorithm is this:

$$i, j \in \{1, \dots, n\}, \quad d \equiv j - i \pmod{n},\ 1 \le d \le n-1, \qquad \text{home}(i, j) = \begin{cases} i, & d < \frac{n}{2}, \\ j, & d \ge \frac{n}{2}, \end{cases}$$

By construction, it achieves an absolute disbalance of home/away games equal to 1 per each team. In total, the sum of the absolute disbalance values will be n. Theoretically, it is minimal for even n.

We use this mathematical approach in the decisional version, so the json results for the decisional version are also home/away optimal.

But to study the solver overhead of enforcing this balance via optimization, we performed a dedicated experiment. We define for team t - absolute difference between home and away games as:

$$\mathit{abs\_diff}_t = \left|(n-1) - 2 \cdot \mathit{away\_count}_t\right|$$

We introduced the global imbalance metric

$$\texttt{sumDif} \ = \ \sum_{t=1}^{n} abs\_diff_t$$

as our first attempt to construct an objective function. Also, we added the constraint

$$\texttt{sumDif} \ \geq \ n$$

We added following additional symmetry-breaking constraint (w.r.t to team numbering) on the per-team imbalances:

$$abs\_diff_{t_1} \ \geq \ abs\_diff_{t_2} \ \geq \ \cdots \geq \ abs\_diff_{t_n},$$

and then instructed Z3 to minimize

$$\max_t abs\_diff_t = abs\_diff_0$$

In our experiments, it appeared that instead of minimizing `sumDif`, it is better to minimize $\max_t abs\_diff_t$ objective, as the optimization becomes almost an order of magnitude faster.

4.3 Constraints

Domain Constraints. Each decision variable must be assigned a valid period:

$$\forall(w, i, j): \quad 1 \leq p_{w,i,j} \leq n/2$$

Unique Match per Period per Week. For each week $w \in W$ and period $k \in \{1, \ldots, n/2\}$, exactly one match must be assigned to that period:

$$\sum_{(i,j)\in\mathcal{M}_w} [p_{w,i,j} = k] = 1$$

Period Load per Team. Each team must appear at most twice in each period across the tournament:

$$\forall t \in [n], \forall k \in \{1, \ldots, n/2\}: \qquad \sum_{(w,i,j):t\in\{i,j\}} [p_{w,i,j} = k] \leq 2$$

Symmetry Breaking Constraints

Fixed Periods in First Week. To reduce the search space, we fix the period assignment of all matches in the first week:

$$\forall k \in \{1, \ldots, n/2\}, \quad p_{1,i_k,j_k} = k$$

where (i_k, j_k) is the k-th match in a lexicographically sorted list of matches in week 1. This removes the permutation symmetry for the period labels.

In our SMT tests more complicated constraints - like lexicographical constraints on period assignment vectors for chosen pairs of teams - did not bring any noticeable additional improvements, as well as adding redundant constraints.

4.4 Validation

Solver. The model was implemented in Python using the Z3 SMT solver. A custom tactic pipeline was used to enable pseudo-Boolean cardinality constraints translation into bit-vector constraints for speed.

Experimental Results (Decision version, SMT in Z3)

We tested the SMT model (without home-away optimization) using Z3 on increasing values of n, with and without symmetry breaking. All runs were limited to 300 seconds. The results in Table 4 show that enabling symmetry breaking significantly improves performance, especially for larger instances.

Table 4: Runtime in seconds for SMT (decision version), Z3 solver, with and without symmetry breaking (SB).

Number of Teams	SB Enabled	SB Disabled
6	0	0
8	0	0
10	0	0
12	0	1
14	0	0
16	0	2
18	8	17
20	5	42

For our SMT model, we observed that the simple version (first week periods were fixed) of symmetry breaking consistently reduced runtime for bigger n, especially beyond $n = 16$. Without SB, solving $n = 20$ takes 8 times longer.

Though we need to mention that for a fixed n there is still some variation between runs - due to the solver's heuristic choices adding some randomness. In some cases $n = 22$ gets solved on our hardware, while often it times out, so it was not included in the final table.

Experimental Results (Optimization version, SMT in Z3)

We extended the SMT model to solve the optimization version of the problem, where the objective is to minimize the maximal (w.r.t to all teams) absolute discrepancy in home/away balance. Theoretically, our objective - maximal absolute discrepancy - being equal to 1 is equivalent to reaching the optimum, and indeed this optimum value was reached in all tested instances.

Table 5 reports the total solving time (in seconds) for various team sizes, both with and without symmetry breaking (SB). All runs were performed with Z3 under a 5-minute timeout. In all cases, the solver found and proved optimal solutions within the time limit.

Table 5: Runtime in seconds for SMT (optimization version), Z3 solver, with and without symmetry breaking (SB). The objective function, $\max_t abs_d iff_t$, in every case was optimal, and the value is shown in bold.

Number of Teams	SB Enabled	SB Disabled
6	0.00|**1**	0.00|**1**
8	0.00|**1**	0.00|**1**
10	0.00|**1**	0.00|**1**
12	3.00|**1**	6.00|**1**
14	4.00|**1**	11.00|**1**
16	127.00|**1**	44.00|**1**
18	189.00|**1**	257.00|**1**
20	149.00|**1**	154.00|**1**

We observe that for the SMT optimization version, symmetry breaking has no such dramatic impact for bigger n, though it helped in every case except for $n = 16$. For $n = 12$ and $n = 14$, SB yields a 2 times and 3 times improvement.

Though there is a noticeable variance due to solver heuristic choices, the observations made above are true for the majority of the re-runs, if made.

5 MIP Model

The MIP formulation was developed with Pyomo, a Python library for writing solver-independent models. Two models were developed: a base version composed of a 4d array with symmetry breaking and implied constraints, and a better one, optimizing the number of variables featuring circle matching.

5.1 Variables

Other than the costants defined in 1, the MIP formulation needs two more. Let $wp = (w, p)$ be the week/period slot w, p; let $m_{i,j} = (i, j)$ be the match played by team i, j with $i < j$:

$$WP = (\{0, \dots, n-2\}, \{0, \dots, \frac{n}{2} - 1\}), \quad M = (\{0, \dots, n-1\}, \{0, \dots, n-1\})$$

Decision Variables

To encode the STS problem we define:

1. $\mathbf{Y}[wp, m] \in \{0, 1\} \quad \forall wp \in WP, \forall m = (i, j) \in M$ with $i < j$.
   $\mathbf{Y}[wp, m] = 1$ if and only if match m is scheduled in slot wp.

2. $\mathbf{H}[i, j] \in \{0, 1\} \quad \forall (i, j) \in M, \ i < j$.
   $\mathbf{H}[i, j] = 1$ if team i plays at home against j, 0 otherwise.

Other Variables

For the efficiency constraints, we also track whether a team plays in a given period:

$$\mathbf{Q}[i,p] \in \{0,1\}, \quad \forall i \in T,\ p \in P,$$

where $\mathbf{Q}[i,p] = 1$ if team i has at least one match in period p.

5.2 Objective Function

Variables In order to minimize the maximum imbalance, we need 3 extra variables:

1. $\mathbf{Home}[i] \in [0, n-1]$: the number of home matches played by team i
2. $\mathbf{Away}[i] \in [0, n-1]$: the number of away matches played by team i
3. $\mathbf{Z} \in [0, n-1]$: the maximum imbalance across teams

Constraints

1. **Home games**:

$$\forall i \in T: \quad \mathbf{Home}[i] = \sum_{j \in T}^{j} \mathbf{H}[(i,j)] + \sum_{j \in T}^{j} (1 - \mathbf{H}[(j,i)]) \quad \text{with } i < j$$

2. **Away games**:

$$\forall i \in T: \quad \mathbf{Away}[i] = \sum_{j \in T}^{j} (1 - \mathbf{H}[(i,j)]) + \sum_{j \in T}^{j} \mathbf{H}[(j,i)] \quad \text{with } i < j$$

3. **Maximum imbalance**:

$$\forall i \in T: \quad |\text{Home}[i] - \text{Away}[i]| \leq Z.$$

5.3 Constraints

With circle matching we get a matrix:

$$\text{presolved[w,i,j]} = \begin{cases} 1 & \text{if the match (i,j) is scheduled for week w} \\ 0 & \text{o.w.} \end{cases}$$

To take advantage of the precomputed matching schedule, we enforce:

$$\forall w \in W, \forall p \in P, \forall m \in M, m = (i,j) : \mathbf{Y}[(w,p), m] \leq \text{presolved}[w,i,j]$$

Circle matching already takes care of some of the problem constraints, but we have to enforce:

1. **One match per w/p slot**:

$$\forall wp \in WP: \quad \sum_{m \in M}^{m} \mathbf{Y}[wp, m] = 1$$

2. **One w/p slot per match**:

$$\forall m \in M: \quad \sum_{wp \in WP}^{wp} \mathbf{Y}[wp, m] = 1$$

3. **At most 2 matches in the same period**:

$$\forall p \in P, \forall k \in T: \sum_{w \in W}^{w} \sum_{j \in \{k+1 \ldots n\}}^{j} \mathbf{Y}[(w,p),(k,j)] + \sum_{w \in W}^{w} \sum_{i \in \{0 \ldots k\}}^{i} \mathbf{Y}[(w,p),(i,k)] <= 2$$

Constraints for efficiency

Especially useful in an optimization environment, this constraint aims to spread the matches of a team in different periods over the scheduling, to get a more balanced result. It also help with symmetry breaking.

$$\forall p \in P, \forall i \in T: \quad 2\mathbf{Q}[i,p] \geq \sum_{w \in W}^{w} \sum_{j \in \{i+1 \ldots n\}}^{j} \mathbf{Y}[(w,p),(i,j)] + \sum_{w \in W}^{w} \sum_{j \in \{0 \ldots i\}}^{j} \mathbf{Y}[(w,p),(j,i)]$$

$$\forall i \in T: \quad \sum p \in P^{p} \mathbf{Q}[i,p] \geq ceil((n-1)/2)$$

Symmetry breaking constraints

In this particular model the addition of symmetry breaking constraints did not improve the results, as circle matching already breaks most of the symmetries, and the addition of other constraints only increased the model weight.

5.4 4d array model

The simplest possible implementation of the STS problem, developed only for comparison as it is not nearly as efficient as the previous one. The only decision variable is X, a 4d array ($n-1 \times n/2 \times n \times n$) of binary values, where each cell represents a match, described as a combination of week, period, team1 and team2. The first team is the one playing home. The constraints, apart from the necessary ones, include two symmetry-breaking constraints and two implied constraints. The optimization is performed in the same way as in the previous model.

5.5 Validation

The model was implemented in Python using Pyomo 6.9.2. The results were obtained by running the models for 3 different solvers:

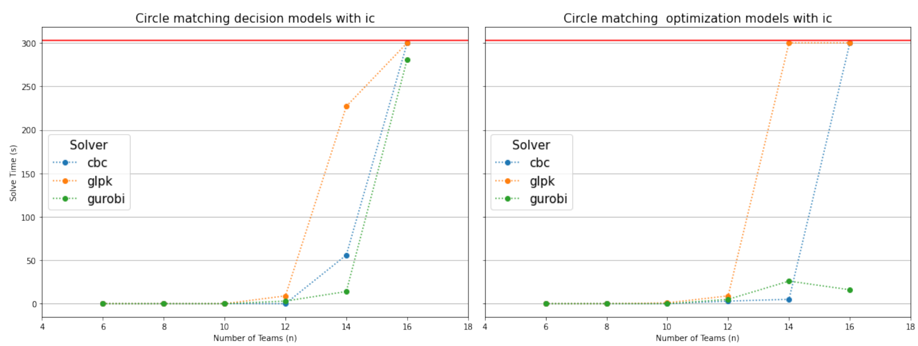
1. **Glpk**: open-source GNU MIP and LP solver, works well for small instances, but does not scale well.

2. **Cbc**: open-source solver based on CLP, thanks to its advanced heuristics and cutting planes performs well even on larger problems.

3. **Gurobi**: a commercial solver, under an academic license, which is required to reproduce its results.

5.5.1 Experimental Results

We tested the MIP formulations on different configurations, using all solvers and both model variants (4D array and Circle Matching, CM). For reproducibility, a more complete set of experiments is available in the `/res` folder of the project work. Since circle matching inherently reduces symmetries, additional symmetry-breaking constraints were not applied.



(a) Decision vs Optimization

Decision version Somewhat counterintuitively, the decision version required on average *more time* to solve (see Figure **??**). This behavior can be explained by the branch-and-bound exploration of MIP solvers: once the objective is removed, the search tends to focus on constraint satisfaction, which often leads to a larger feasible space and slower convergence. Moreover, efficiency-oriented constraints, designed to balance the schedule, turned out to be detrimental in this setting.

n	Model	Solver	Time	n	Model	Solver	Time
8	4D	Glpk	1s	12	CM	Gurobi	2s
10	4D	Glpk	300s	14	4D	Gurobi	24s
10	4D	Cbc	5s	14	CM	Glpk	300s
12	4D	Cbc	300s	14	CM	Cbc	27s
12	4D	Gurobi	23s	14	CM	Gurobi	36s
12	CM	Glpk	14s	16	CM	Cbc	300s
12	CM	Cbc	11s	16	CM	Gurobi	72s

Table 6: Results for the decision problem
All with efficiency constraints off

Optimization version As expected, the CM model consistently outperformed the 4D array formulation in both solution quality and runtime. For example, with $n = 12$, the CM model solved optimally in a few seconds with all solvers, while the 4D array version either timed out or required several minutes without guaranteeing optimality. This highlights the benefit of a stricter model, with a smaller associated search space.

Table 7 shows how solver performance followed the usual hierarchy: **Gurobi** was the most efficient, followed closely by **Cbc**, while **Glpk** struggled with scalability and often reached the time limit for larger instances. Interestingly, though, for some smaller instances Glpk performs better than Cbc.

n	M.	Solver	Time	Opt	n	M.	Solver	Time	Opt
10	4D	Cbc	298s	no	12	CM	Gurobi	5s	yes
10	4D	Glpk	44s	yes	14	CM	Cbc	5s	yes
10	4D	Gurobi	5s	yes	14	CM	Glpk	300s	no
12	4D	Cbc/Glpk	300s	no	14	CM	Gurobi	26s	yes
12	CM	Cbc	3s	yes	16	CM	Cbc	300s	no
12	CM	Glpk	9s	yes	16	CM	Gurobi	16s	yes

Table 7: Results for the optimization problem
All with efficiency constraints on

References

[1] Dominique de Werra. Scheduling Round-Robin Tournaments: An Overview. *Discrete Applied Mathematics*, 91(1–3):241–277, 1999.

[2] Olivier Bailleux and Yacine Boufkhad. Efficient CNF Encoding of Boolean Cardinality Constraints. In: *Principles and Practice of Constraint Programming (CP 2003)*, Lecture Notes in Computer Science, 2003.