COMB ASSOCIATIVE OPERADS

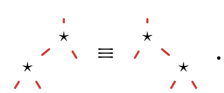
CYRILLE CHENAVIER, CHRISTOPHE CORDERO, AND SAMUELE GIRAUDO

Abstract. The associative operad is the quotient of the magmatic operad by the operad congruence identifying the two binary trees of degree 2. We introduce here a generalization of the associative operad depending on a nonnegative integer d, called d-comb associative operad, as the quotient of the magmatic operad by the operad congruence identifying the left and the right comb binary trees of degree d. We study the case $d = 3$ and provide an orientation of its space of relations by using rewrite systems on trees and the Buchberger algorithm for operads to obtain a convergent rewrite system.

Contents

Introduction

Associative algebras are spaces endowed with a binary product $\star$ satisfying among others the associativity law $(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)$. It is well-known that the associative algebras are representations of the associative (nonsymmetric) operad As. This operad can be seen as the quotient of the magmatic operad Mag (the free operad of binary trees on the binary generator $\star$) by the operad congruence $\equiv$ satisfying



(0.0.1)

These two binary trees are the syntax trees of the expressions appearing in the above associativity law.

In a more combinatorial context and regardless of the theory of operads, the Tamari order is a partial order on the set of the binary trees having a fixed number of internal

Date: May 30, 2018.

2010 *Mathematics Subject Classification.* TODO: [].

Key words and phrases. Operad; trees; Rewrite rule; Presentation; Enumeration.

nodes d. This order is generated by the covering relation consisting in rewriting a tree $\mathfrak{t}$ into a tree $\mathfrak{t}'$ by replacing a subtree of $\mathfrak{t}$ of the form of the left member of (0.0.1) into a tree of the form of the right member of (0.0.1). This transformation is known in a computer science context as the right rotation operation [Knu98] and intervenes in algorithms involving binary search trees [AVL62]. The partial order hence generated by the right rotation operation is known as the Tamari order [Tam62] and has a lot of combinatorial and algebraic properties (see for instance [HT72,Cha06]).

A first connection between the associative operad and the Tamari order is based upon the fact that the orientation of (0.0.1) from left to right provides a convergent orientation (a terminating and confluent rewrite relation) of the congruence $\equiv$. The normal forms of the rewrite relation induced by the rewrite rule obtained by orienting (0.0.1) from left to right are right comb binary trees and are hence in one-to-one correspondence with the elements of As.

This work is intended to be a first strike in the study of the eventual links between the Tamari order and some quotients of the operad Mag. In the long run, we would like to study quotients $\mathrm{Mag}/_{\equiv}$ of Mag where $\equiv$ is an operad congruence generated by equivalence classes of trees of a fixed degree. In particular, we would like to know if $\equiv$ is generated by equivalence classes of trees forming intervals of the Tamari order leads to algebraic properties for $\mathrm{Mag}/_{\equiv}$ (like the description of orientations of its space of relations, nice bases and Hilbert series).

We focus here on one of these quotients $\mathrm{CAs}^{(3)}$ which is the operad describing the category of the algebras equipped with a binary product $\star$ and subjected to the relation $((x_1 \star x_2) \star x_3) \star x_4 = x_1 \star (x_2 \star (x_3 \star x_4))$. This is a kind of associativity law in higher degree $d = 3$. This operad is generated by an equivalence class of trees which is not an interval for the Tamari order. As preliminary computer experiments show, $\mathrm{CAs}^{(3)}$ has oscillating first dimensions (see (3.0.13)), what is rather unusual among all known operads. In this paper, we provide an orientation of the space of relations of $\mathrm{CAs}^{(3)}$. For this, we use rewrite systems on trees [BN98] and the Buchberger algorithm for operads [DK10].

This text is presented as follows. Section 1 contains preliminaries about the magmatic operad and rewrite relations on trees. In Section 2, we define the operad $\mathrm{CAs}^{(3)}$ as a particular case of a more general construction of generalizations $\mathrm{CAs}^{(d)}$, $d \geqslant 1$, of As. Finally, Section 3 contains the orientation of the space of relations of $\mathrm{CAs}^{(3)}$ (Theorem 3.0.1). As consequences, we obtain for $\mathrm{CAs}^{(3)}$ the description of one of its Poincaré-Birkhoff-Witt bases (Proposition 3.0.2) and the description of its Hilbert series (Proposition 3.0.3).

1. The magmatic operad, quotients, and rewrite relations

We consider nonsymmetric set-theoretic operads. Let $\mathcal{O}$ be such an operad. We denote respectively by $\circ_i$ and $\circ$ the partial and complete compositions of $\mathcal{O}$. For any $n \geqslant 1$, $\mathcal{O}(n)$ is the set of the elements x of $\mathcal{O}$ of arity $|x| = n$. We denote by Mag the magmatic operad, that is the free operad over one binary generator $\star$, and we represent the elements of Mag by binary trees. The *arity* $|\mathfrak{t}|$ (resp. *degree* $\deg(\mathfrak{t})$) of a binary tree $\mathfrak{t}$ is its number of leaves (resp. internal nodes). Given a binary tree $\mathfrak{t}$, we denote by $\mathrm{p}(\mathfrak{t})$ the *prefix word* of $\mathfrak{t}$, that is the word on $\{0, 2\}$ obtained by a left to right depth-first traversal of $\mathfrak{t}$ and by writing

0 (resp. 2) when a leaf (resp. an internal node) is encountered. The set of all words on $\{0, 2\}$ is endowed with the lexicographic order $\leqslant$ induced by $0 < 2$.

If $\to$ is a rewrite rule on Mag such that $\mathfrak{s} \to \mathfrak{s}'$ implies $|\mathfrak{s}| = |\mathfrak{s}'|$, we denote by $\Rightarrow$ the *rewrite relation induced* by $\to$. Formally we have $\mathfrak{t} \circ_i (\mathfrak{s} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_n]) \Rightarrow \mathfrak{t} \circ_i (\mathfrak{s}' \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_n])$, if $\mathfrak{s} \to \mathfrak{s}'$ where $n = |\mathfrak{s}|$, and $\mathfrak{t}$, $\mathfrak{r}_1$, ..., $\mathfrak{r}_n$ are binary trees. In other words, one has $\mathfrak{t} \Rightarrow \mathfrak{t}'$ if it is possible to obtain $\mathfrak{t}'$ from $\mathfrak{t}$ by replacing a subtree $\mathfrak{s}$ of $\mathfrak{t}$ by $\mathfrak{s}'$ whenever $\mathfrak{s} \to \mathfrak{s}'$. We use here the standard terminology (*terminating*, *confluent*, *convergent*, *branching pair*, *joinable*, *normal form*, *etc.*) about rewrite relations and rewrite systems [BN98].

Given an operad $\mathcal{O} \simeq \mathrm{Mag}/_{\equiv}$ where $\equiv$ is an operad congruence of Mag, we say that $\to$ is an *orientation* of $\equiv$ if the reflexive, transitive, and symmetric closure of $\Rightarrow$ is $\equiv$. We say that $\to$ is a *convergent orientation* if $\Rightarrow$ is convergent. When $\to$ is a convergent orientation of $\equiv$, the set of all normal forms of $\Rightarrow$ is a *Poincaré-Birkhoff-Witt basis* of the operad $\mathcal{O}$ and its elements are exactly the binary trees avoiding, as subtrees, the trees appearing as left members in $\to$.

We shall use the following criterion to prove that a rewrite relation on Mag is terminating.

Lemma 1.0.1. *Let $\to$ be a rewrite rule on* Mag*. If for any $\mathfrak{t}, \mathfrak{t}' \in$ Mag such that $\mathfrak{t} \to \mathfrak{t}'$ one has $\mathrm{p}(\mathfrak{t}) > \mathrm{p}(\mathfrak{t}')$, then the rewrite relation induced by $\to$ is terminating.*

Moreover, we shall use the following result appearing in [Gir16] specialized on rewrite relation on Mag to prove that a terminating rewrite relation is convergent.

Lemma 1.0.2. *Let $\to$ be a rewrite rule on* Mag *wherein all trees $\mathfrak{t}$ and $\mathfrak{t}'$ such that $\mathfrak{t} \to \mathfrak{t}'$ have degrees at most ℓ. Then, if the rewrite relation $\Rightarrow$ induced by $\to$ is terminating and all its branching pairs of degrees at most $2\ell - 1$ are joinable, $\Rightarrow$ is convergent.*

2. Generalizations of the associative operad

It is known that the rewrite rule $\to$ orienting (0.0.1) from left to right is a convergent orientation of (0.0.1). Then, a Poincaré-Birkhoff-Witt basis of As is the set of all right comb binary trees.

Let us now define for any $d \geqslant 1$ the *γ-comb associative operad* $\mathrm{CAs}^{(\gamma)}$ as the quotient operad $\mathrm{Mag}/_{\equiv^{(\gamma)}}$ where $\equiv^{(\gamma)}$ is the smallest operad congruence of Mag satisfying

$$\underbrace{(\dots(\star \circ_1 \star) \circ_1 \dots) \circ_1 \star}_{\gamma \text{ operands}} \equiv^{(\gamma)} \underbrace{\star \circ_2 (\cdots \circ_2 (\star \circ_2 \star) \dots)}_{\gamma \text{ operands}}. \tag{2.0.1}$$

In words, (2.0.1) says that the left and the right comb binary trees of degree γ are equivalent for $\equiv^{(\gamma)}$. Notice that $\equiv^{(1)}$ is trivial so that $\mathrm{CAs}^{(1)} = \mathrm{Mag}$ and that $\equiv^{(2)}$ is the operad congruence defined by (0.0.1) so that $\mathrm{CAs}^{(2)} = \mathrm{As}$.

In the sequel we denote by $\mathrm{LC}^{(\gamma)}$ and $\mathrm{RC}^{(\gamma)}$ the left comb and the right comb of degree γ, respectively. Hence, $\equiv^{(\gamma)}$ is the congruence relation induced by $\mathrm{LC}^{(\gamma)} \equiv^{(\gamma)} \mathrm{RC}^{(\gamma)}$.

We denote by CAs the set of γ-comb associative operads where γ belongs the set of strictly positive integers:

$$\mathrm{CAs} = \left\{ \mathrm{CAs}^{(\gamma)} \mid \gamma \geqslant 1 \right\}.$$

Our purpose is to show that CAs admits a lattice structure. For that, we provide a description of operads morphisms between the elements of CAs.

Proposition 2.0.1. *Let γ and γ' be two strictly positive integers. There exists at most one operads morphism $\varphi : \mathrm{CAs}^{(\gamma')} \to \mathrm{CAs}^{(\gamma)}$. This morphism exists if and only if* $\mathrm{LC}(\gamma') \equiv^{(\gamma)} \mathrm{RC}(\gamma')$ *and it is surjective in this case. Moreover, φ is injective if and only if $\gamma = \gamma'$, that is if and only if φ is the identity morphism.*

Proof. The operad $\mathrm{CAs}^{(\gamma')}$ is generated by one binary generator $t^{(\gamma')}$, so that φ is entirely determined by $\varphi\left(t^{(\gamma')}\right)$. Moreover, $\varphi\left(t^{(\gamma')}\right)$ has to be of arity 2 in $\mathrm{CAs}^{(\gamma)}$, so that $\varphi\left(t^{(\gamma')}\right)$ has to be equal to $t^{(\gamma)}$. Hence, if φ exists, it is the unique operads morphism from $\mathrm{CAs}^{(\gamma')}$ and $\mathrm{CAs}^{(\gamma)}$, and in this case, $t^{(\gamma)}$ being in the image of φ, the latter is surjective. Moreover, φ is well defined if and only if $\varphi\left(\mathrm{LC}^{(\gamma')}\right)$ and $\varphi\left(\mathrm{RC}^{(\gamma')}\right)$ are equal in $\mathrm{CAs}^{(\gamma)}$, that is if and only if $\mathrm{LC}^{(\gamma')} \equiv^{(\gamma)} \mathrm{RC}^{(\gamma')}$. In particular, if γ' is strictly smaller than γ, then there does not exist any morphism $\varphi : \mathrm{CAs}^{(\gamma')} \to \mathrm{CAs}^{(\gamma)}$. Moreover, if γ' is strictly smaller than γ, the cardinality of $\mathrm{CAs}^{(\gamma')}(\gamma)$ is equal to the cardinality of $\mathrm{CAs}^{(\gamma)}(\gamma)$ plus one. Hence, if there exists an injective morphism $\varphi : \mathrm{CAs}^{(\gamma')} \to \mathrm{CAs}^{(\gamma)}$, we must have $\gamma = \gamma'$, and in this case, φ is the identity morphism. □

We define the binary relation $\preceq$ on CAs as follows: we have $\mathrm{CAs}^{(\gamma)} \preceq \mathrm{CAs}^{(\gamma')}$ if and only if there exists a morphism $\varphi : \mathrm{CAs}^{(\gamma')} \to \mathrm{CAs}^{(\gamma)}$.

Proposition 2.0.2. *The binary relation $\preceq$ is a partially order on* CAs*, that is* $(\mathrm{CAs}, \preceq)$ *is a poset.*

Proof. The binary relation $\preceq$ is reflexive since there exists the identity morphism of $\mathrm{CAs}^{(\gamma)}$ for every strictly positive integer γ. It is transitive since if there exist two morphisms $\varphi : \mathrm{CAs}^{(\gamma)} \to \mathrm{CAs}^{(\gamma')}$ and $\psi : \mathrm{CAs}^{(\gamma'')} \to \mathrm{CAs}^{(\gamma')}$, then $\psi \circ \varphi$ is a morphism from $\mathrm{CAs}^{(\gamma'')}$ to $\mathrm{CAs}^{(\gamma)}$. Now, let us assume that there exist morphisms $\varphi : \mathrm{CAs}^{(\gamma')} \to \mathrm{CAs}^{(\gamma)}$ and $\psi : \mathrm{CAs}^{(\gamma)} \to \mathrm{CAs}^{(\gamma')}$. In particular, $\psi \circ \varphi$ and $\varphi \circ \psi$ are endomorphisms of $\mathrm{CAs}^{(\gamma')}$ and $\mathrm{CAs}^{(\gamma)}$, respectively. From Proposition 2.0.1, these two morphisms are identity morphisms, so that φ and ψ are injective. From Proposition 2.0.1, γ and γ' are equal, which proves that $\preceq$ is anti-symmetric, hence a partial order. □

In order to show that the poset $(\mathrm{CAs}, \preceq)$ extends into a lattice, that is two elements of CAs admit lower-bounds and upper-bounds, we relate $(\mathrm{CAs}, \preceq)$ to the lattice of integers $(\mathbb{N}, |, \gcd, \mathrm{lcm})$, where $|$ denotes the division relation, gcd denotes the *greatest common divisor* and lcm the *least common multiple* operators, respectively. In order to obtain this relationship, we introduce the notion of *left rank* of an element $t \in \mathrm{Mag}$: this is the length $\mathrm{lr}(t)$ of the left branch beginning at the root of t. The left rank of t is represented as follows:

PICTURE

Lemma 2.0.3. *Let γ be a strictly positive integer and let t and t' be two elements of* Mag*. If $t \equiv^{(\gamma)} t'$, then $\mathrm{lr}(t)$ and $\mathrm{lr}(t')$ are equal modulo $\gamma - 1$: $\mathrm{lr}(t) \equiv \mathrm{lr}(t')\,[\gamma - 1]$.*

Proof. Let $\to$ be the rewrite relation on Mag induced by the rewrite rule $\mathrm{LC}^{(\gamma)} \to \mathrm{RC}^{(\gamma)}$. In the rest of the proof we fix two elements t and t' of Mag.

First, we show that $t \to t'$ implies that $\mathrm{lr}\,(t') - \mathrm{lr}\,(t)$ is divisible by $\gamma - 1$ **TODO**.

Assume that $t \xrightarrow{*} t'$, so that there exist elements $t_1, \cdots, t_n$ of Mag such that $t_1 = t$, $t_n = t'$ and for every $i \in \{1, \cdots, n-1\}$, we have $t_i \to t_{i+1}$. The integer $\mathrm{lr}\,(t_{i+1}) - \mathrm{lr}\,(t_i)$ is divisible by $\gamma - 1$ from the first part of the proof, so that $\mathrm{lr}\,(t') - \mathrm{lr}\,(t)$ is divisible by $\gamma - 1$.

Assume that $t \overset{*}{\leftrightarrow} t'$, so that there exist elements $t_1, \cdots, t_n$ of Mag such that $t_1 = t$, $t_n = t'$ and for every $i \in \{1, \cdots, n-1\}$, we have $t_i \xrightarrow{*} t_{i+1}$ or $t_{i+1} \xrightarrow{*} t_i$. From the second part of the proof, $\mathrm{lr}\,(t_{i+1}) - \mathrm{lr}\,(t_i)$ is divisible by $\gamma - 1$, so that $\mathrm{lr}\,(t') - \mathrm{lr}\,(t)$ is divisible by $\gamma - 1$.

The congruence relation $\equiv^{(\gamma)}$ is the equivalence relation induced by $\to$, that is we have $t \equiv^{(\gamma)} t'$ if and only if $t \overset{*}{\leftrightarrow} t'$. Hence, from the third part of the proof, $t \equiv^{(\gamma)} t'$ implies that $\mathrm{lr}\,(t') - \mathrm{lr}\,(t)$ is divisible by $\gamma - 1$. □

Proposition 2.0.4. *Let γ and γ' be two strictly positive integers. There exists a morphism $\varphi : \mathrm{CAs}^{(\gamma')} \to \mathrm{CAs}^{(\gamma)}$ if and only if $(\gamma - 1) \mid (\gamma' - 1)$.*

Proof. From Proposition 2.0.1, it is sufficient to show that $\mathrm{LC}^{(\gamma')} \equiv^{(\gamma)} \mathrm{RC}^{(\gamma')}$ if and only if $(\gamma - 1) \mid (\gamma' - 1)$. If $\mathrm{LC}^{(\gamma')} \equiv^{(d)} \mathrm{RC}^{(\gamma')}$, from Lemma 2.0.3, $\mathrm{lr}\left(\mathrm{LC}^{(\gamma')}\right) - \mathrm{lr}\left(\mathrm{RC}^{(\gamma')}\right) = \gamma' - 1$ is divisible by $\gamma - 1$, which shows the direct implication. Conversely, if $(\gamma - 1) \mid (\gamma' - 1)$, the rewrite rule $\mathrm{LC}^{(\gamma)} \to \mathrm{RC}^{(\gamma)}$ induces the following sequence of rewriting steps:

Rewriting sequence from left comb of degree d' to right comb of degree d'

Hence, we have $\mathrm{LC}^{(\gamma')} \equiv^{(\gamma)} \mathrm{RC}^{(\gamma')}$. □

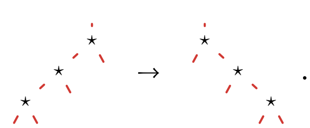
Proposition 2.0.4 implies the following:

Theorem 2.0.5. *The poset* $(\mathrm{CAs}, \preceq)$ *extends into a lattice* $(\mathrm{CAs}, \preceq, \wedge, \vee)$, *where the lower- bound $\wedge$ and the upper-bound $\vee$ are defined as follows:*

$$\mathrm{CAs}^{(\gamma)} \wedge \mathrm{CAs}^{(\gamma')} = \mathrm{CAs}^{(gcd(\gamma-1,\ \gamma'-1))} \text{ and } \mathrm{CAs}^{(\gamma)} \vee \mathrm{CAs}^{(\gamma')} = \mathrm{CAs}^{(lcm(\gamma-1,\ \gamma'-1))}.$$

3. The 3-comb associative operad

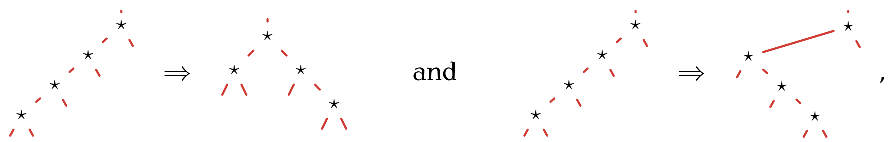
We now focus on the study of the operad $\mathrm{CAs}^{(3)}$. By definition, this operad is the quotient of Mag by the operad congruence spanned by the relation



(3.0.1)

This rewrite rule is compatible with the lexicographic order on prefix words presented at the beginning of Section 1 in the sense that the prefix word of the left member of (3.0.1) is lexicographically greater than the prefix word of the right one.

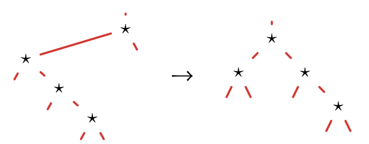
However, the rewrite relation ⇒ induced by → is not confluent. Indeed, we have



(3.0.2)

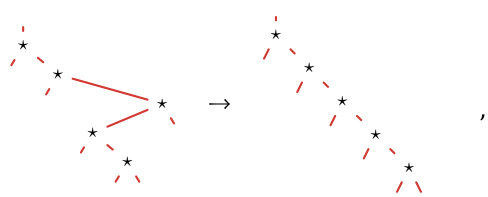
and the two right members of (3.0.2) form a branching pair which is not joinable.

In order to transform the rewrite relation induced by (3.0.1) into a convergent one, we apply the Buchberger algorithm for operads [DK10, Section 3.7] with respect to the lexicographic order on prefix words. Following this algorithm, we need to put the right members of (3.0.2) in relation by →. To respect the lexicographic property of the prefix words, this leads to the new relation

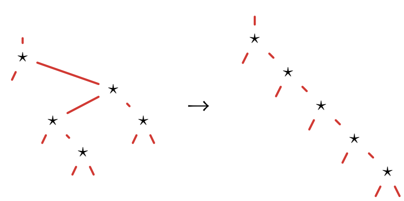


(3.0.3)

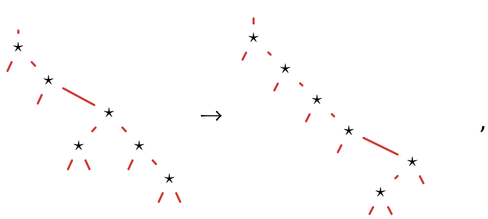
The Buchberger algorithm applied on binary trees of degrees 5, 6, and 7 provides the new relations



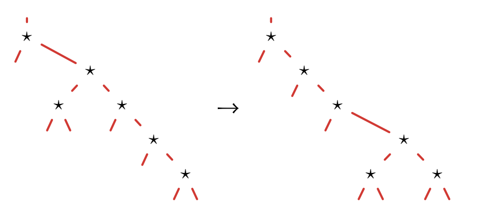
(3.0.4)



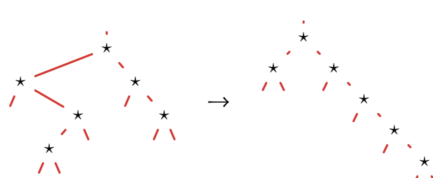
(3.0.5),



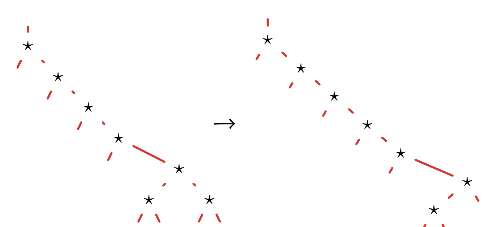
(3.0.6)



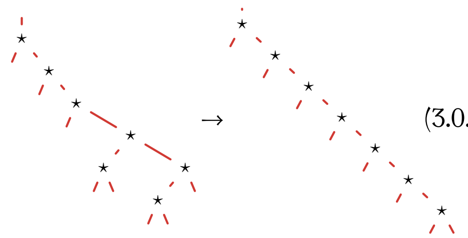
(3.0.7)



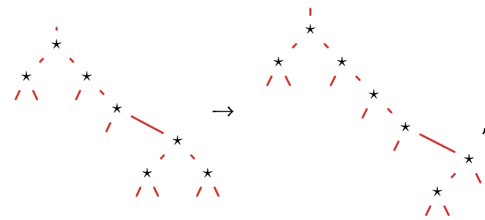
(3.0.8),



(3.0.9)



(3.0.10),



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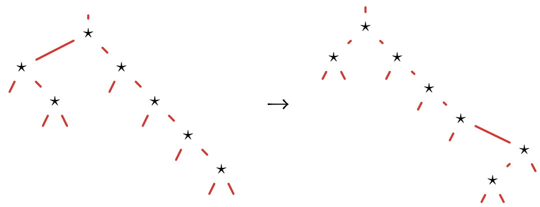
(3.0.11)



(3.0.12).

We claim that the rewrite relation $\Rightarrow$ induced by rewrite rule $\to$ satisfying (3.0.1), (3.0.3), (3.0.4)—(3.0.12) is convergent. First, for every relation $\mathrm{t} \to \mathrm{t}'$, we have $\mathrm{p}(\mathrm{t}) > \mathrm{p}(\mathrm{t}')$. Therefore, by Lemma 1.0.1, $\Rightarrow$ is terminating. Moreover, the greatest degree of a tree appearing in $\to$ is 7 so that, from Lemma 1.0.2, to show that $\Rightarrow$ is convergent, it is enough to prove that each tree of degree at most 13 admits exactly one normal form. Equivalently, this amounts to show that the number of normal forms of trees of arity n is equal to $\#\mathrm{CAs}^{(3)}(n)$. By computer exploration, we get the same sequence

$$1, 1, 2, 4, 8, 14, 20, 19, 16, 14, 14, 15, 16, 17 \tag{3.0.13}$$

for $\#\mathrm{CAs}^{(3)}(n)$ and for the numbers of normal forms of arity n, when $1 \leqslant n \leqslant 14$. Hence, we get our following main result.

Theorem 3.0.1. *The rewrite rule $\to$ satisfying (3.0.1), (3.0.3), (3.0.4)—(3.0.12) is a convergent orientation of the congruence $\equiv^{(3)}$ of $\mathrm{CAs}^{(3)}$.*

The rewrite rule $\to$ has, arity by arity, the cardinalities

$$0, 0, 0, 1, 1, 2, 3, 4, 0, \dots\ . \tag{3.0.14}$$

We obtain from Theorem 3.0.1 also the following consequences.

Proposition 3.0.2. *The set of the trees avoiding as subtrees the ones appearing as left members of $\to$ is a Poincaré-Birkhoff-Witt basis of $\mathrm{CAs}^{(3)}$.*

From Proposition 3.0.2, and by using a result of [Gir18] describing a system of equations for the generating series of syntax trees avoiding some sets of subtrees, we obtain the following result.

Proposition 3.0.3. *The Hilbert series of $\mathrm{CAs}^{(3)}$ is*

$$\mathcal{H}_{\mathrm{CAs}^{(3)}}(t) = \frac{t}{(1-t)^2}\left(1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11}\right). \tag{3.0.15}$$

For $n \leqslant 10$, the dimensions of $\mathrm{CAs}^{(3)}(n)$ are provided by Sequence (3.0.13) and for all $n \geqslant 11$, the Taylor expansion of (3.0.15) shows that $\#\mathrm{CAs}^{(3)}(n) = n + 3$.

Perspectives

Our first axis of perspectives consists in collecting properties about the operads $\mathsf{CAs}^{(d)}$. A natural question consists in finding all the morphisms between the operads $\mathsf{CAs}^{(d)}$. Some surjective morphisms are described by Proposition **??** and we can hope to a full description of these, as well as some possible injections. Moreover, we can try to obtain a convergent orientation of $\equiv^{(d)}$ and general expressions of the Hilbert series of $\mathsf{CAs}^{(d)}$ when $d \geqslant 4$. By computer exploration, we have the sequence

$$1, 1, 2, 5, 13, 35, 96, 264, 724, 1973, 5355, 14390 \tag{3.0.16}$$

for the first dimensions for $\mathsf{CAs}^{(4)}$. By applying the Buchberger algorithm on trees of degrees until 10, we obtain that a convergent orientation of $\equiv^{(4)}$ has, arity by arity, the sequence $0, 0, 0, 0, 1, 1, 0, 3, 4, 5, 18, 22$ for its first cardinalities. Moreover, for $\mathsf{CAs}^{(5)}$, we get the sequence

$$1, 1, 2, 5, 14, 41, 124, 384, 1210, 3861, 12440 \tag{3.0.17}$$

of dimensions and the first cardinalities $0, 0, 0, 0, 0, 1, 1, 0, 0, 4, 5$ for any convergent orientation of $\equiv^{(5)}$. Finally, for $\mathsf{CAs}^{(6)}$, we get the sequence

$$1, 1, 2, 5, 14, 42, 131, 420, 1375, 4576, 15431 \tag{3.0.18}$$

of dimensions and the first cardinalities $0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0$ for any convergent orientation of $\equiv^{(6)}$. We can notice that only $\mathsf{CAs}^{(3)}$ seems to have oscillating first dimensions.

A second axis concerns a complete understanding of $\mathsf{CAs}^{(3)}$. We can try to construct an explicit basis of this operad. Proposition 3.0.2 describes a basis in terms of trees avoiding some patterns but, we can hope to find a simpler description. This includes the description of a family of combinatorial objects forming a basis of $\mathsf{CAs}^{(3)}$ and an adequate definition of a partial composition map $\circ_i$ on these. Moreover, a natural question is to explore the suboperads $\mathsf{CAs}^{(3)}$ in the category of vector spaces.

In a last axis, we can consider further generalizations of As being quotients of Mag by congruences defined by identifying certain binary trees of a same fixed degree. A possible question is, as presented in the introduction, to investigate if combinatorial properties of the trees belonging to a same equivalence class imply algebraic properties on the obtained operads.

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