Machine Learning 1 - HW2 – 2025 – Paper 1

1 Naive Bayes Modeling of Personality Types (10 points)

Naive Bayes (NB) is a particular form of classification that makes strong independence assumptions regarding the features of the data, conditional on the classes (see Bishop section 4.2.3). Specifically, NB assumes each feature is independent given the class label. In contrast, when we looked at probabilistic generative models for classification in the lecture, we used a full-covariance Gaussian to model data from each class, which incorporates correlation between all the input features (i.e. they are not conditionally independent).

If correlated features are treated independently, the evidence for a class will be overcounted. However, Naive Bayes is very simple to construct, because by ignoring correlations the class-conditional likelihood, $p(\mathbf{x}|\mathcal{C}_k)$, is a product of D univariate distributions, each of which is simple to learn:

$$p(\mathbf{x}|\mathcal{C}_k) = \prod_{d=1}^{D} p(x_d|\mathcal{C}_k)$$

Consider the task to classify people into different personality types. For example, the "Big Five" are a well-established scheme for assigning people their personality type depending on how extroverted, how conscientious, how agreeable, how neurotic, and how open to experiences they are. We assume there are K different personality types $\mathcal{C}_k$. The last class, $\mathcal{C}_K$, means "ambiguous", and in this case, a human will be asked for help.

In your experiment, you want to find out if the personality type can be predicted based on the consumer-behaviour of people. Thus, you come up with D different life-areas in which people can spend money, e.g. hobbies, clothes, food, gadgets, and their home. Thus, for each of the N people in your dataset, you obtain one vector $\mathbf{x}_n$ of dimension D, and x_{nd} is the amount of money that person n spends per month in life area d. Thus, $x_{nd} \in \mathbb{R} \geq 0$. Overall, your training set consists of an $N \times D$ matrix. The target matrix $\mathbf{T}$, who's rows consist of the row vectors $\mathbf{t}_n^T = (t_{n1}, \ldots, t_{nK})$, one-hot-encoded such that $\mathbf{t}_n^T = (0, \ldots, 1, \ldots, 0)$ with the scalar 1 at position k if $n \in \mathcal{C}_k$.

Assume we know $p(\mathcal{C}_k) = \pi_k$ (with the constraint $\sum_{k=1}^{K} \pi_k = 1$). We can model the amount of money spent in a life area with different distributions. For this question, we will use an exponential model. Thus, the amount of money spent in a life area is modeled with one parameter λ_{dk}, when conditioned on class $\mathcal{C}_k$:

$$p(\mathbf{x}|\mathcal{C}_k, \lambda_{1k}, \ldots, \lambda_{Dk}) = \prod_{d=1}^{D} \lambda_{dk} e^{-\lambda_{dk} \cdot x_d}.$$

With this information answer the following questions:

(a) Write down the data likelihood,

$$p(\mathbf{T}, \mathbf{X}|\mathbf{\Lambda})$$

without the Naive Bayes conditional independence assumption at first. Then, derive the data likelihood for the general K classes naive Bayes classifier, stating where you make use of the product rule and the naive Bayes assumption.

You should write the likelihood in terms of $p(x_{nd} \mid \mathcal{C}_k)$, meaning you should not assume the explicit exponential distribution. [1 point]

(b) How does the number of parameters change if you make use of the naive Bayes assumption? Why is the assumption called naive? Can you think of an example in which this assumption does not hold? [1 point]

(c) Write down the data log-likelihood $\ln p(\mathbf{T}, \mathbf{X} \mid \mathbf{\Lambda})$ for the given likelihood model. [1 point]

(d) Solve for the MLE estimator of λ_{dk}. Express in your own words how the result can be interpreted. [2 points]

(e) Specify $p(\mathcal{C}_1|\mathbf{x})$ for the general k classes naive Bayes classifier in terms of the prior and class conditional distributions. [1 point]

Specify your final answer in terms of π and λ.

(f) Write $p(\mathcal{C}_1|\mathbf{x})$ for the exponential model explicitly in terms of the modeling choices (exponential distribution, and π_k). [1 point]

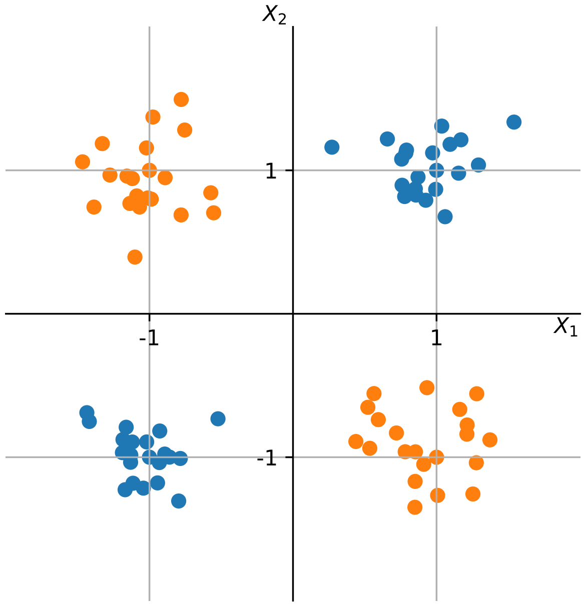
(g) Write down as an inequality the region for which $\mathbf{x}$ is predicted to be in class C_1 compared to all other classes in terms of the posterior from the previous question 1f. Subsequently, specify the written inequality as a linear inequality of the form: $\mathbf{x}^T\mathbf{a} > c$. [2 points]

(h) Is the region where $\mathbf{x}$ is predicted to be in $\mathcal{C}_1$ convex? Why?

Hint: Recall that a region is convex if for any points x_0 and x_1 that belong to C_1, all the points in the straight line defined as $x_0 * (1-\lambda) + \lambda x_1$, with $\lambda \in (0,1)$ will belong to C_1 as well. [1 point]

2 Binary classification (9 points)

Now, suppose you are given the following datasets shown in the figure below



(a) For each of these following methods, explain why they can/cannot classify of the dataset. [3 points]

(i) Logistic regression with linear features, $y = \sigma(\mathbf{w}^T\mathbf{x})$.

(ii) Logistic regression with non linear basis functions, $y = \sigma(\mathbf{w}^T\boldsymbol{\phi})$

(iii) Multilayer Perceptron with 1 hidden layer

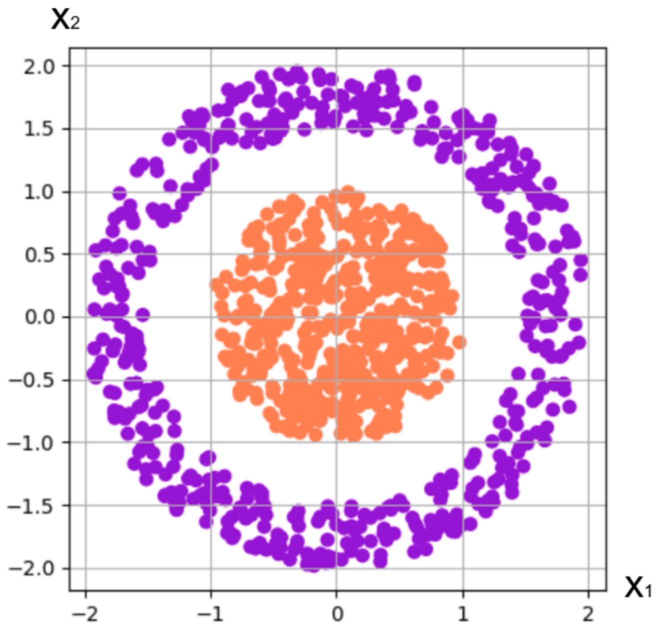
(b) Can the above dataset be classified using Naive Bayes? Remember that we are free to set the marginal densities $p(x_i|C_j)$ as we please. If the answer is yes, provide the marginal densities. If the answer is no, prove that it.

Hint: Consider the subset $\{(-1, 1), (1, 1), (1, -1), (-1, -1)\}$ *of the dataset.*
[3 points]

(*c*) Now consider the following dataset. Can it be classified using Naive Bayes (NB)? How is your reasoning different from the one you provided in question 2*b*?

Hint: Can you think of a distribution that fits this data? (you can assume x_1 *and* x_2 *to be continuous).*

[2 points]



(*d*) What is the main difference between logistic regression with non linear basis functions $\boldsymbol{\phi}$ and multilayer perceptrons, in terms of $\boldsymbol{\phi}$? [1 point]

3 Regularized Logistic Regression (11 points)

Consider logistic regression for K classes with N training vectors $\{\mathbf{x}_n\}_{n=1}^N$, each of which is mapped to a different feature vector

$$\boldsymbol{\phi}_n = \boldsymbol{\phi}(\mathbf{x_n}) = (\phi_0(\mathbf{x}_n), \phi_1(\mathbf{x}_n), \ldots, \phi_{M-1}(\mathbf{x}_n))^T$$

using basis functions $\phi_j(\mathbf{x}_n)$ with $j = 1, \ldots, M-1$, and $\phi_0(\mathbf{x}_n) = 1$. Each vector $\mathbf{x}_n$ has a corresponding target vector $\mathbf{t}_n$ of size K: $\mathbf{t}_n = (t_{n1}, t_{n2}, \ldots, t_{nK})^T$, where $t_{nk} = 1$ if $\mathbf{x}_n \in \mathcal{C}_k$, and $t_{nj} = 0$ for all $j \neq k$. The input data can be collected in a matrix $\mathbf{X}$ such that the n-th row is given by $\mathbf{x}_n^T$ and the targets can be collected in a target matrix $\mathbf{T}$, such that the n-th row is given by $\mathbf{t}_n^T$.

$$\mathbf{X} = \begin{pmatrix} —\mathbf{x}_1^T— \\ \vdots \\ —\mathbf{x}_N^T— \end{pmatrix}, \; \mathbf{T} = \begin{pmatrix} —\mathbf{t}_1^T— \\ \vdots \\ —\mathbf{t}_N^T— \end{pmatrix}$$

The feature vectors can also be collected in a matrix $\mathbf{\Phi}$ such that the n-th row of $\mathbf{\Phi}$ contains ϕ_n^T:

$$\mathbf{\Phi} = \begin{pmatrix} \phi_0(\mathbf{x}_1) & \phi_1(\mathbf{x}_1) & \cdots & \phi_{M-1}(\mathbf{x}_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(\mathbf{x}_N) & \phi_1(\mathbf{x}_N) & \cdots & \phi_{M-1}(\mathbf{x}_N) \end{pmatrix}$$

Assuming i.i.d. data, the posterior class probabilities are modeled by

$$p(\mathcal{C}_k|\boldsymbol{\phi}(\mathbf{x}), \boldsymbol{w}_1, \ldots, \boldsymbol{w}_K) = y_k(\boldsymbol{\phi}) = \frac{\exp(a_k)}{\sum_{j=1}^K \exp(a_j)}$$

where the "activations" a_k are given by $a_k = \mathbf{w}_k^T \boldsymbol{\phi}$. Assume a Gaussian prior on the parameter vectors $\mathbf{w}_1, \ldots, \mathbf{w}_K$:

$$p(\mathbf{w}_1, \ldots, \mathbf{w}_K|\alpha) = \prod_{k=1}^K \mathcal{N}(\mathbf{w}_k|\mathbf{0}, \alpha^{-1}\mathbf{I})$$

(*a*) Write down the likelihood $p(\mathbf{T}|\mathbf{\Phi}, \mathbf{w}_1, \ldots, \mathbf{w}_K)$ as a product over N and K. Use the entries of $\mathbf{T}$ as selectors of the correct class. Then write down the log-likelihood $\log p(\mathbf{T}|\mathbf{\Phi}, \mathbf{w}_1, \ldots, \mathbf{w}_K)$. [1 point]

(b) Write down the explicit form of the prior $p(\mathbf{w}_1, \ldots, \mathbf{w}_K|\alpha)$. Compute the logarithm of the prior $\log p(\mathbf{w}_1, \ldots, \mathbf{w}_K|\alpha)$. How does computing the logarithm help us during computations? [1.5 points]

(c) Write down an expression for the posterior

$$p(\mathbf{w}_1, \ldots, \mathbf{w}_K|\mathbf{\Phi}, \mathbf{T}, \alpha)$$

over $\mathbf{w}_1, \ldots, \mathbf{w}_K$ by applying Bayes rule. [1 point]

(d) Show that obtaining a Maximum A Posteriori (MAP) estimate for $\mathbf{w}_1, \ldots, \mathbf{w}_K$ is equivalent to performing regularized logistic regression for K classes, where we minimize the function

$$-\sum_{n=1}^{N}\sum_{k=1}^{K} t_{nk} \log y_k(\boldsymbol{\phi}_n) + \frac{\alpha}{2}\sum_{k=1}^{K}\sum_{m=0}^{M-1} |w_{km}|^2$$

with respect to $\mathbf{w}_1, \ldots, \mathbf{w}_K$. [2 points]

(e) Say you're quite confident that the weights should lie around 0. What variable would change in the equation from question (d) and why? Why does this actually make the weights be closer to 0? Use the equation from question (d) to answer. [1 point]

(f) Now assume you are very unsure of the values the weights should take. You still model your prior with a Gaussian. Now, prove that in this case, the more uncertain you are, the closer the MAP solution approaches the MLE solution. Hint: this can be solved by using the equation from question (d). [1.5 points]

(g) Derive the gradient of the log-likelihood $\nabla_{\mathbf{w}_j} \log p(\mathbf{T}|\mathbf{\Phi}, \mathbf{w}_1, \ldots, \mathbf{w}_K)$. Your final answer should be a single sum over n. You can make use of the derivative of the softmax function $\frac{\partial y_k}{\partial a_j} = y_k(\mathbb{I}_{kj} - y_j)$ (you computed this for the first assignment). [2 points]

(h) Derive the gradient of the log-prior $\nabla_{\mathbf{w}_j} \log p(\mathbf{w}_1, \ldots, \mathbf{w}_K|\alpha)$. Derive the gradient of the log-posterior $\nabla_{\mathbf{w}_j} \log p(\mathbf{w}_1, \ldots, \mathbf{w}_K|\mathbf{T}, \mathbf{\Phi}, \alpha)$. [1 point]