Politecnico di Milano
School of Industrial and Information Engineering
Master of Science in Mathematical Engineering

TITLE: very interesting subject, ain't it?

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Academic year 2017-2018

The secret to happiness is freedom.
And the secret to freedom is courage.

Thucydides

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Abstract

Including Bitcoin in an investment portfolio increases portfolio diversification.

Acknowledgements

add acknowledgements

Thank you.

Chapter 1

Introduction

1.1 Thesis structure

Chapter 2

Correlation Analysis

In order to get an initial insight on how Bitcoin is correlated with other assets, we will perform a correlation analysis based on the empirical time series of our data. We will focus our attention on the logarithmic returns it is the standard practice. We will often refer to logarithmic returns simply as returns, only specifying their nature when it is necessary to avoid confusion.

2.1 Empirical Correlation of Returns

We first start by performing some statistical analysis on the data in order to estimate the distribution from which they are sampled. For this part, we will consider our data as successive samples of a N-dimensional vector in $\mathbb{R}^N$, where N is the number of assets:

$$\mathbf{x}_j = \begin{pmatrix} x_{1,j} \\ x_{2,j} \\ \vdots \\ x_{N,j} \end{pmatrix}, j = 1 \dots N_{sample}$$

Each element i of the vector $\mathbf{x}_j$ represents the j^{th} realization of the returns for asset i.

Following basic statistics, we can now compute the *sample mean* of our vectors of returns as:

$$\bar{\mathbf{x}} = \frac{1}{N_{sample}} \sum_{j=1}^{N_{sample}} \mathbf{x}_j = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_N \end{pmatrix}$$

where $\bar{x}_i = \frac{1}{N_{sample}} \sum_{j=1}^{N_{sample}} x_{i,j}$ is the sample mean of component i.

Now we compute the *sample covariance matrix* through the following formula:

$$\bar{\Sigma} = \frac{1}{N_{sample} - 1} \sum_{j=1}^{N_{sample}} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})^T$$

where $\bar{\mathbf{x}}$ represent the sample mean of the returns just introduced.

All the information needed to obtain the *correlation matrix* C are already included in $\bar{\Sigma}$, we only need to perform some further calculations:

$$C_{i,j} = \frac{\bar{\Sigma}_{i,j}}{\sqrt{\bar{\Sigma}_{i,i}\bar{\Sigma}_{j,j}}} \tag{2.1}$$

We have thus obtained an empirical estimate of the correlation between our assets returns. The formula in (2.1) is often referred to as *Pearson correlation coefficient*, from the name of the English mathematician Karl Pearson who first formulated it.

Results are reported in the following tables.

******** ADD RESULT TABLES ********

We are mainly interested in the correlation between Bitcoin and other assets returns, so we will now focus on the first row (or equivalently column, by symmetry) of the correlation matrix.

All values are fairly close to zero, never exceeding 10% towards the positive or the negative side. One may thus wonder whether these correlations are *statistically significantly* different from zero. To answer this question, we will introduce two statistical tests to check the correlation significance.

2.2 Correlation Significance

The very core of Inferential Statistics, the branch of statistics that allows to draw conclusions from the information contained in a set of data, is hypothesis testing.

In our case, we are specifically interested in testing if the sample correlation coefficients are significantly different from zero or not. Both of the following tests are presented in the most general form for a sample of two variables, their distribution correlation ρ and their sample correlation $\hat{\rho}$.

Following standard testing procedure, we specify the *null hypothesis* and the *alternative hypothesis*:

$$\mathbf{H_0}: \quad \rho = 0 \quad vs. \quad \mathbf{H_1}: \quad \rho \neq 0$$

These will be common to both presented tests.

2.2.1 Pearson's *t*-test

Our first test is based on Student's t-distribution and the following t-statistic:

$$t = \hat{\rho}\sqrt{\frac{n-2}{1-\hat{\rho}^2}} \tag{2.2}$$

which under the null hypothesis is distributed as a Student's t with $n-2$ degrees of freedom, where n stands for the cardinality of the sample. We can thus proceed by computing the relative p-value and compare it to a given level of confidence α (usually $\alpha = 95\%$). The result of the test will be deduced as follows:

- $p - value < 1 - \alpha$: we have statistical evidence to state that the correlation is *significantly* different from zero;

- $p - value \geq 1 - \alpha$: there is *no statistical evidence* to state that the correlation is different from zero.

2.2.2 Permutation test

The permutation test is based on building an empirical distribution of values for the correlation by sampling different pairs of X and Y variable and then computing Pearson's correlation. If this is done a large enough number of times, we obtain an empirical distribution of possible values. From this distribution we can then obtain the p-value of the test and thus the final result in the same way as in the previous case.

2.2.3 Significance results

The values that we obtained for the correlation of Bitcoin with the other assets are reported in Table 2.1, including the resulting p-values for both of the tests that were introduced in the paragraph above.

Looking at the first line alone, we can see that the asset-Bitcoin correlation never surpasses 5% in absolute value. This is exactly what we would expect given that Bitcoin price seems to move on its and not really care about what is happening on the market (at least to some degree).

Moreover, if we also study the significance of the correlation level through Pearson's or the permutation test, we can see that the only asset that has a correlation that is *significantly different from zero*[1] is the Standard&Poor's

[1]Considering a confidence level of 5%.

	bric	sp500	eurostoxx	nasdaq	bond_europe	bond_us	bond_eur
Correlation	1,41%	4,37%	4,12%	3,59%	1,41%	-1,84%	1,92%
Pearson	52,70%	4,10%	5,40%	9,35%	50,35%	39,90%	37,95%
Permutation	51,21%	4,24%	5,55%	9,48%	51,29%	39,28%	37,20%

	eur	gbp	chf	jpy	gold	wti	grain	metal
Correlation	2,29%	0,73%	2,46%	-1,09%	-0,24%	0,77%	3,50%	2,69%
Pearson	29,00%	72,75%	24,75%	61,45%	91,00%	71,55%	10,10%	20,90%
Permutation	28,64%	73,33%	25,34%	61,32%	91,08%	72,00%	10,34%	21,13%

Table 2.1: Values of the correlation between Bitcoin and the other assets and their p-value using both Pearson's and the permutation test.

500 Index. Still, the correlation that we experience between S&P500 and Bitcoin returns is of 4,37 %, which is considerably low.

Thus, our results show that Bitcoin is fundamentally incorrelated to any of the asset that we are taking into consideration in our analysis.

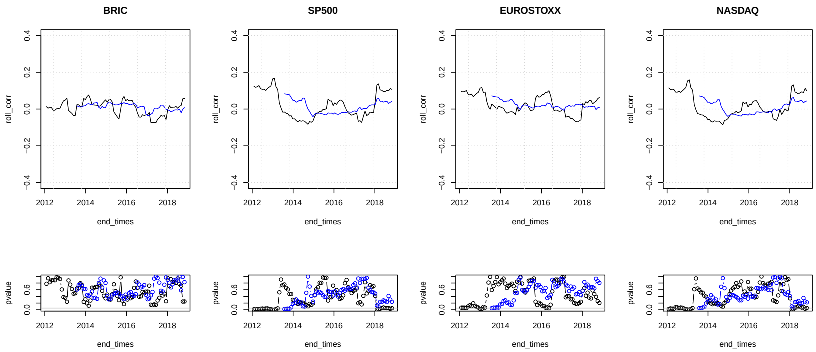
This result is what induced us to consider the possible diversification benefits of introducing Bitcoin in an investor's portfolio. We will see in Chapter 5 what great improvements in terms of increased return and decreased risk this addition brings to our reference portfolio.

2.3 Rolling Correlation

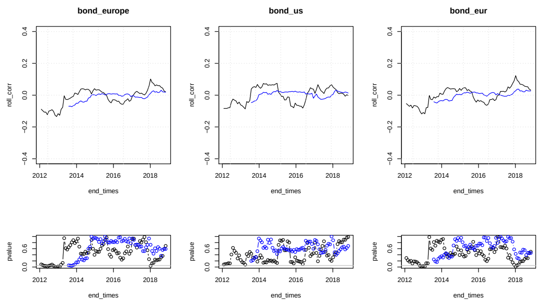
Our study so far has focused on the analysis of the dataset as a whole, with values spanning from July 2010 to December 2018. This is clearly important if we want to obtain a general overview of the period, but it is also interesting to see how the correlation between the assets has evolved through. Therefore, we present in Figure 2.1 the results obtained from calculating the correlation between Bitcoin and the other assets using rolling windows of 36 and 18 months, updated monthly.

There are two graphs for each asset: in the top plots levels of the rolling correlations are represented using two different colours, blue for the 3-year and black for the 18-month windows; in the bottom plots we included the significance of each rolling correlation through its p-value. The grey horizontal line represents the 5% level of significance[2].

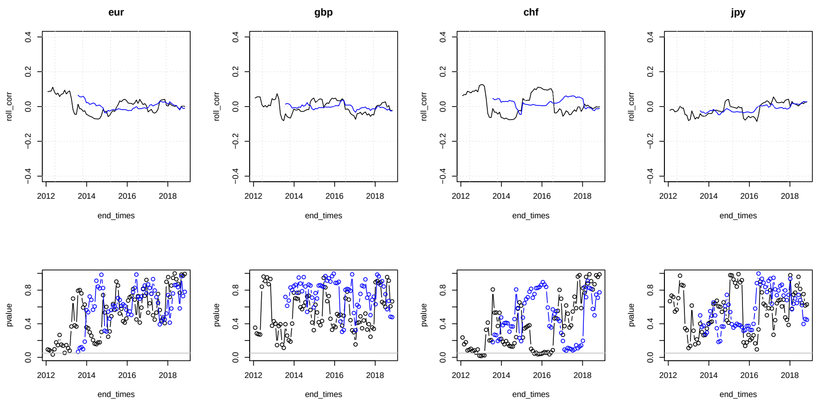
[2]As we explained in the previous section, to check that a sample correlation is *significantly* non zero, we compare the p-value of the test to a given level, here $1 - \alpha = 5\%$.



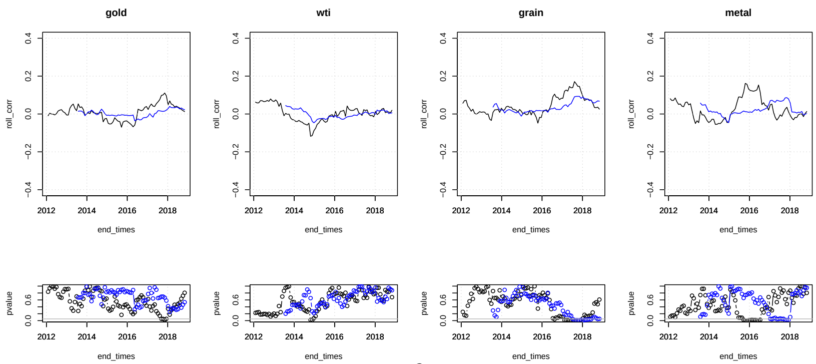
(a) Stocks



(b) Bonds



(c) Currency exchange



(d) Commodities

Figure 2.1: Plots of rolling correlation for the different asset classes (on top) and significance for each value (on bottom). Blue lines are the 3-year rolling correlations, while the black ones have a window of 18 months. Both computations are updated monthly.

The main conclusion we can draw from these images is that the correlation of any asset with Bitcoin is hardly ever significantly different from zero, and when it is, its absolute level is never more than greater than 20% for a small period of time.

To confirm the fact that Bitcoin is not correlated with any asset, we can also take a look at the path of the rolling correlations: there is no line that is always above zero, nor below. This indicates that there is no underlying trend, whether positive or negative, and the correlation one might find is only temporary.

**** maybe add more comments on the results ****

Graphically, whenever the dots are above the grey line in Figure 2.1, the corresponding correlation is *not* significantly different from zero.

Chapter 3

Presentation of the Models

In this chapter we will present the stochastic frameworks in which we developed our analysis. We first introduce a *jump diffusion* (JD) model presented in 1976 by R.C. Merton: he added log-normal jumps to the simple B&S dynamics of the asset price. Then we move to the *stochastic volatility* (SV) model of Heston 1993. Heston introduced a new stochastic process that accounts for the variance of the underlying which evolves as a B&S with a stochastic volatility term. The last model we will present was introduced by Bates in 1996 and it is the combination of the former two: an asset dynamics which include jumps and is driven by a stochastic volatility. All models are first introduced in the one dimensional case and then generalised to the n dimensional case which was then implemented in our code.

3.1 Preliminary Notions

In this section we will briefly present the equation of a geometric brownian motion, introduce the notion of Poisson process and present the CIR process. All of these building blocks will be required to fully understand the models to follow.

3.1.1 Geometric Brownian Motion

The simplest continuous dynamics to describe the price of an asset is that of a geometric brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{3.1}$$

where S_t represents the price of the asset at time t, μ is the (constant) drift and σ is the (constant) volatility. W_t is a Wiener process. This is the standard and most widespread stichastic differential equation to model asset dynamics, so we will only present those results that will be later used in our study.

Applying Ito's lemma to the previous equation, we can also explicitly express the dynamics of the log-returns $X_t = log(S_t)$, obtaining:

$$dX_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t \tag{3.2}$$

This stochastic differential has a simple solution which can be computed via stochastic integrals and allows us to describe the dynamics of the log-returns at each instant t starting from $t = 0$:

$$X_t = X_0 + (\mu - \frac{\sigma^2}{2})t + \sigma W_t \tag{3.3}$$

Thanks to (3.3) we can now express the price dynamics of the asset by inverting the relation with the returns: $S_t = e^{X_t}$. We thus obtain the solution to (3.1):

$$S_t = S_0 \, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \tag{3.4}$$

Given that $S_t = e^{X_t}$, that X_t by its equation is a generalized brownian motion and hence we have $X_t - X_0 \sim \mathcal{N}(\mu - \frac{\sigma^2}{2}, \sigma^2)$, the resulting distribution of prices at time t as units of the initial value is distibuted as a log-normal.

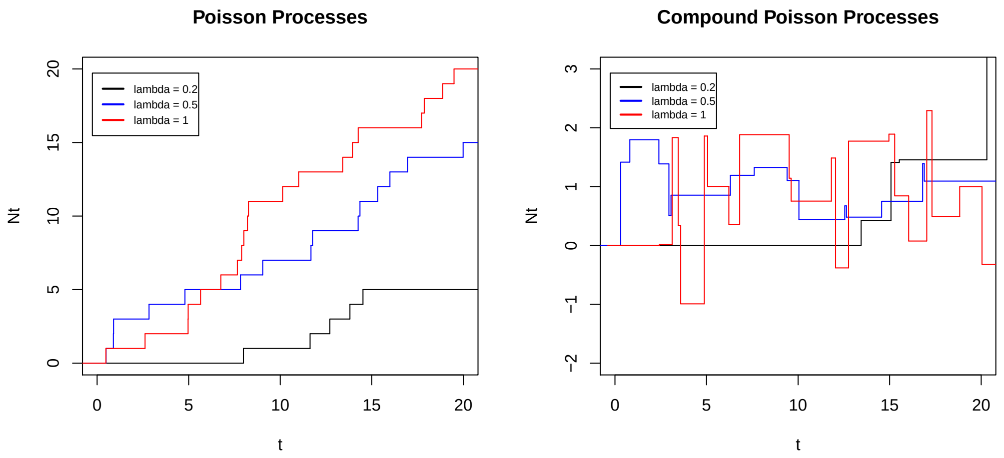
The great success of these framework comes from the simplicity of its dynamics. In particular, since the log-returns follow a Gaussian distribution, μ and σ are easy to calibrate from data and the formulas for pricing options are often explicit. As one can imagine, a simple model can only explain simple phenomena: that's why we have such a great deal of *new and improved* version of equation 3.1.

3.1.2 Poisson Process and Compound Poisson Process

Consider a sequence of *independent* exponential random variables $\{\tau_i\}_{i\geq 1}$ with parameter λ[1] and let $T_n = \sum_{i=1}^{n} \tau_i$. Then we can define the *Poisson process* N_t as

$$N_t = \sum_{n\geq 1} \mathbb{1}_{t\geq T_n} \tag{3.5}$$

[1] An exponential random variable τ of parameter λ has a cumulative distibution function of the form: $\mathbb{P}(\tau \geq y) = e^{-\lambda y}$



(a) Poisson processes.

(b) Compound with Gaussian jumps.

Figure 3.1: Poisson processes and compound Poisson processes with different λ.

where $\mathbb{1}_{condition}$ is 1 if the *condition* is true, 0 if it is false.

N_t is thus a piece-wise constant RCLL[2] process with jumps that happen at times T_n and are all of size 1, as we can see from Figure fig. 3.1a. An important property of Posson processes is that they have independent and stationary increments, meaning that the increment of $N_t - N_s$ (with $s \leq t$) is independent from the history of the process up to N_s and has the same law of N_{t-s}. At any time t, N_t is distributed as a Poisson of parameter λt, which means it is a discrete random variable on the integer set with

$$\mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \tag{3.6}$$

When working with jump diffusion process, it is often the case that there is no explicit formula for its density, thus usually we resort to characteristic functions. The characteristic function of N_t is given by

$$\phi_{N_t}(u) = e^{\lambda t (e^{iu} - 1)} \tag{3.7}$$

The computations to get 3.7 from 3.6 are carried out in Appendix A.

For financial applications, it is of little interest to have a process with a single possible jump size. The *compound* Poisson processes are a generalization of Poisson processes where the jump sizes can have an arbitrary

[2]RCLL is shorthand for right continuous with left limit.

distribution. More precisely, consider a Poisson process N_t with parameter λ and a sequence of i.i.d[3] variables $Y_{i i \geq 1}$ with law $f_Y(y)$. Then the process defined by

$$X_t = \sum_{n=1}^{N_t} Y_i \tag{3.8}$$

is a compound Poisson process. Examples of this kind of process are plotted in Figure fig. 3.1b.

As before, we developed the computations to obtain an expression for the characteristic function of X_t in Appendix A. The resulting expression depends on the distribution of Y, specifically from its characteristic function $\phi_Y(u)$:

$$\phi_{X_t}(u) = e^{\lambda t(\phi_Y(u)-1)} \tag{3.9}$$

Both in Merton's and in the Heston's models there will be a jump component driven by a compound Poisson process with Gaussian jump sizes, as we will see in the following paragraphs.

3.1.3 CIR Process

The CIR process was introduced in [4] in 1985 by Cox, Ingersoll and Ross (hence the name CIR) as a generalization of a Vasicek process to model the mean reverting dynamics of interest rates. Following their notation, the differential equation for the evolution of the rate is given by:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{3.10}$$

where we have three parameters that characterize it: θ is the *long-term value* of the rate, the asymptotic level which it tends to settle at in the long run;κ is the *mean-reversion rate*, the speed at which the rate is pulled back to the θ value; finally σ accounts for the *volatility* of the stochastic component. When $\kappa, \theta > 0$, equation (3.10) represents a first order mean-reverting autoregressive process. Moreover, thanks to *****add citation*******, we know that the process will not hit zero if the following condition is satisfied:

$$2\kappa\theta > \sigma^2 \tag{3.11}$$

This condition is usually referred to as *Feller* condition, from the author of the cited paper in which this result was first presented.

An example of a CIR process is shown in Figure (3.2), where the mean-reverting effect is clearly visible.

[3] i.i.d stands for independent and identically distributed.

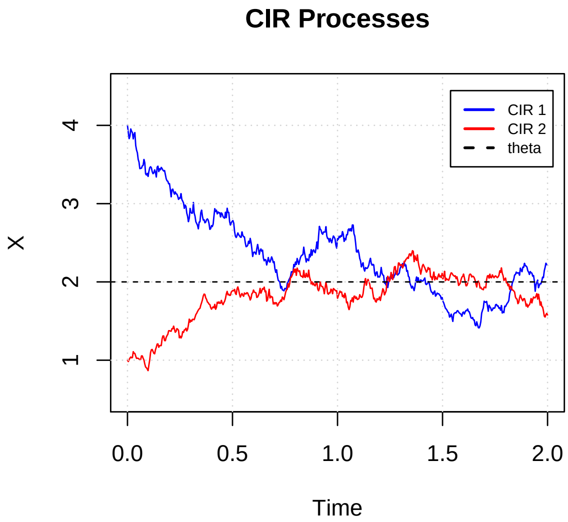


Figure 3.2: Two trajectories of the same CIR process: $dr_t = 3(2 - r_t)dt + 0.5\sqrt{r_t}dW_t$ with different starting point. We can see the mean-reverting effect that attracts both trajectories to the value $\theta = 2$.

When modelling interest rate in continuous time, having r_t hit zero is not an issue since when $r_t = 0$ equation (3.10) reduces to $dr_t = \kappa\theta dt$ which immediately brings the level back to positive values and hence the square root in the dynamics never loses meaning. Conversely, considering a *discretized* version of (3.10), as is the case in a simulation framework, one needs to pay attention on how he models the increments since a simple Euler discretization scheme may cause r_t to reach *negative* values[4] and invalidate the whole model representation.

The non negativity of the CIR process will become fundamental when we introduce Heston and Bates SV models, where the (stochastic) variance of the process driving the asset price will be model as a CIR process.

Since it will be useful later on in the paper, we also present the *stationary* distribution of r_t. Due to the mean reversion effect, r_t will approach a *gamma* distribution with density:

$$f_{r_\infty}(x) = \frac{\omega^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\omega x} \tag{3.12}$$

[4]This model was introduced having in mind the certainty that interest rates could never be negative, hence the introduction of a square root in the dynamics. Given recent years interest rates levels, this in no more the case.

where

$$\omega = \frac{2\kappa}{\sigma^2}\ ,\ \nu = \frac{2\kappa\theta}{\sigma^2}$$

Its *moment generating function*, which will be useful later as well, is defined as follows:

$$M_{r_\infty}(z) = \Big(\frac{\omega}{\omega - z}\Big)^{\nu} \tag{3.13}$$

3.2 Merton Model

******maybe add some words right here ********

3.2.1 Original Univariate Model

The first jump diffusion model was originally introduced in [11] in order to account for the leptokurtic distribution of real market returns and to model sudden falls (or rises) in prices due to the arrival of new information. The asset price dynamics S_t is modelled as a GBM to which a jump component driven by a compound Poisson process is added:

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t + (Y_t - 1)dN \tag{3.14}$$

where α and σ are respectively the drift and the diffusion of the continuous part, Y_t is a process modelling the intensity of the jumps and $N(t)$ is the Poisson process driving the arrival of the jumps and has parameter λ.

We can rewrite (3.14) in terms of the log-returns $X_t = log(S_t)$ and obtain, following the computations in [10] and using theory from [12]:

$$dX_t = (\alpha - \frac{\sigma^2}{2})dt + \sigma dW_t + \log(Y_t) \tag{3.15}$$

that has as solution:

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N(t)} \eta_k \tag{3.16}$$

where X_0 is the initial value of the log-returns, $\eta_k = \log(Y_k) = \log(Y_{t_k})$ and t_k is the time when the k^{th} Poisson shock from $N(t)$ happens. We use $\mu = \alpha - \frac{\sigma^2}{2}$ for ease of notation throughout the paper. Following [11], we take η_k *i.i.d.* (independent and identically distributed) and Gaussian, in particular $\eta \sim \mathcal{N}(\theta, \delta^2)$. Another choice for the distribution of η is given in [8].

It is often useful when dealing with market data that are by nature discrete, to consider a *discretized* version of (3.16) in which the values are sampled at intervals of Δt in $[0, T]$. We thus get that for $X_i = \log(\frac{S_{i+1}}{S_i})$:

$$X_i = \mu \Delta t + \sigma \sqrt{\Delta t}\, z + \sum_{k=1}^{N_{i+1}-N_i} Y_k \tag{3.17}$$

where we denote $X_i = X_{t_i}$, $N_i = N(t_i)$ and $t_i = i\Delta t$ with $i = 0 \dots N$, $t_N = N\Delta t = T$, z is distributed as a standard Gaussian $z \sim \mathcal{N}(0, 1)$.

The Poisson process $N(t)$ in (3.17) is computed at times t_{i+1} and t_i and these quantities are subtracted. Following basic stochastic analysis, one can prove that the resulting value $N_{i+1} - N_i$, is distributed as a Poisson random variable N of parameter $\lambda \Delta t$. This allows us to provide an explicit formulation for the transition density of the returns using the theorem of total probability:

$$f_{\Delta X}(x) = \sum_{k=0}^{\infty} \mathbb{P}(N = k) f_{\Delta X|N=k}(x) \tag{3.18}$$

This is an infinite mixture of Gaussian distributions, due to the infinite possible realization of the Poisson variable, and renders the estimation of the model through MLE technique intractable, see [7]. To solve this problem we introduce a first order approximation, as it's been proposed in [2]. Considering small Δt, so that also $\lambda \Delta t$ is small, we obtain that the only relevant terms in (3.18) are the one for $k = 0, 1$. The formula for the transition density becomes:

$$f_{\Delta X}(x) = \mathbb{P}(N = 0) f_{\Delta X|N=0}(x) + \mathbb{P}(N = 1) f_{\Delta X|N=1}(x)$$

expressing it explicitly:

$$f_{\Delta X}(x) = (1 - \lambda \Delta t)\ f_{\mathcal{N}}(x; \mu, \sigma^2) + (\lambda \Delta t)\ f_{\mathcal{N}}(x; \mu + \theta, \sigma^2 + \delta^2) \tag{3.19}$$

where $f_{\mathcal{N}}(x; \mu, \sigma^2)$ is the density of a Gaussian with parameters $\mathcal{N}(\mu, \sigma^2)$.

3.2.2 Multivariate Model

Starting from the univariate model introduced in [11], we developed a generalization to n assets including only idiosyncratic jumps:

$$\frac{dS_t^{(j)}}{S_t^{(j)}} = \alpha_j dt + \sigma_j dW_t^{(j)} + (Y_t^{(j)} - 1)dN_t^{(j)} \tag{3.20}$$

where $\mathbf{S}_t$ are the prices of the assets, $j = 1...n$ represents the asset, α_j are the drifts, σ_j are the diffusion coefficients, $W_t^{(j)}$ are the components of an n-dimensional Wiener process $\mathbf{W}_t$ with $dW^{(j)}dW^{(i)} = \rho_{j,i}$, η_j represent the intensities of the jumps and are distributed as Gaussian: $\eta_j \sim \mathcal{N}(\theta_j, \delta_j^2)$. Finally, $N^{(j)}(t)$ are Poisson processes with parameters λ_j, which are independent of $\mathbf{W}_t$ and of one another.

In order to calibrate the parameters to the value of the market log-returns, we used a Maximum Likelihood approach. We thus maximize:

$$\mathcal{L}(\psi|\Delta\mathbf{x}_{t_1}, \Delta\mathbf{x}_{t_2}, \dots, \Delta\mathbf{x}_{t_N}) = \sum_{i=1}^{N} f_{\Delta\mathbf{X}}(\Delta\mathbf{x}_{t_i}|\psi) \tag{3.21}$$

where $\psi = \{\{\mu_j\}, \{\sigma_j\}, \{\rho_{i,j}\}, \{\theta_j\}, \{\delta\}_j, \{\lambda_j\}\}$ are the model parameters, $f_{\Delta\mathbf{X}}$ is the transitional density of the log-returns which is computed approximately using the theorem of total probability. For a full insight on the model and the calibration procedure, please refer to the *APPENDIX LINK*

3.3 Heston Model

The Heston model was presented in 1993 in [6] as a new framework to model stochastic volatility in the asset price dynamics, which allows to better fit the skewness and kurtosis of the log-return distribution.

3.3.1 Univariate Heston Model

The Heston model belongs to the family of SD processes, generalizations of B&S model in which the volatility is no more constant but is itself stochastic. ***add examples?***

Starting with a GBM as in the B&S framework, we obtain the dynamics of the price process simply by allowing the volatility in (3.1) to evolve over time and specifying how this evolution takes place. In particular, the dynamics of the *instantaneous* variance $V_t = \sigma_t^2$ for the Heston model is described by a CIR process :

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^S \tag{3.22}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V \tag{3.23}$$

where μ represents the drift in the asset prices, $\kappa > 0$ and $\theta > 0$ are respectively the *mean-reversion* rate and the long-run level for the V_t process, $\sigma_V > 0$ is often referred to as the *volatility of volatility* parameter, in short vol-of-vol. The two brownian motions W_t^S and W_t^V are correlated with correlation coefficient equal to ρ. The variance process is always strictly positive if the Feller condition $2\kappa\theta > \sigma_V$ is satisfied.

Let us now consider the dynamics of the log-return $x_t = \log(S_t)$, as we did in the Merton case. Unfortunately, given the increased complexity of the model due to the SV part, an explicit formula for the density of the log-return is not available and it has to be computed from the Fourier inversion of the characteristic function:

$$f_{x_t}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_{x_t}(iu)e^{iux}du \tag{3.24}$$

In (3.24) we omitted the dependence of $f_{x_t}(x)$ and $\phi_{x_t}(u)$ on model parameters and on the initial values of the log-returns x_0 and of the variance process V_0.

To derive the expression of the characteristic function $\phi_{x_t}(u)$, one has to solve a couple of *Fokker-Planck* partial differential equation as is shown in the Appendix of the reference paper by Heston [6]. This procedure is beyond the scope of this thesis and thus we will only report the final result, which is a log-affine equation on the information at time $t = 0$:

$$\phi_{x_t}(u|x_0, V_0) = \exp\{A(t,u) + B(t,u)x_0 + C(t,u)V_0\}$$

$$A(t,u) = \mu uit + \frac{\kappa\theta}{\sigma_V^2}\left((\kappa - \rho\sigma_V ui + d)t - 2\log\left[\frac{1-ge^{dt}}{1-g}\right]\right) \tag{3.25a}$$

$$B(t,u) = iu \tag{3.25b}$$

$$C(t,u) = \frac{\kappa - \rho\sigma_V ui + d}{\sigma_V^2}\left[\frac{1-e^{dt}}{1+ge^{dt}}\right] \tag{3.25c}$$

where :

$$d = \sqrt{(\rho\sigma_V ui - \kappa)^2 + \sigma_V^2(ui + u^2)}$$

$$g = \frac{\kappa - \rho\sigma_V ui + d}{\kappa - \rho\sigma_V ui - d}$$

This formulation is the one proposed by Heston in his original paper [6] but it is shown in [1] that it has numerical issues when pricing Vanilla options using Fourier methods. We will not be pricing any instrument, however we may incur in the same errors when calibrating our model through the techniques explained in Chapter 4. For this reason, we will be using the alternative formulation presented in [1]:

$$\begin{aligned}
\phi^*_{x_t}(u|x_0, V_0) &= \exp\{A(t,u) + B(t,u)x_0 + C(t,u)V_0\} \\
A(t,u) &= \mu uit + \frac{\kappa\theta}{\sigma_V^2}\left((\kappa - \rho\sigma_V ui - d)t - 2\log\left[\frac{1-g^*e^{-dt}}{1-g^*}\right]\right) \\
B(t,u) &= iu \\
C(t,u) &= \frac{\kappa - \rho\sigma_V ui - d}{\sigma_V^2}\left[\frac{1-e^{-dt}}{1-g^*e^{-dt}}\right]
\end{aligned} \tag{3.26}$$

where :

$$\begin{aligned}
d &= \sqrt{(\rho\sigma_V ui - \kappa)^2 + \sigma_V^2(ui + u^2)} \\
g^* &= \frac{\kappa - \rho\sigma_V ui - d}{\kappa - \rho\sigma_V ui + d} = \frac{1}{g}
\end{aligned}$$

The only difference is that the signs of the d terms are all flipped: the origin of the two representations for the characteristic function lies in the fact that the complex root d has two possible values and the second value is exactly minus the first value. A uniform choice between the two possibilities cannot be made over all the complex plane to obtain a continuous function due to the presence of a branch cut in the graph of $\sqrt{z}$. Selecting the sign of d as in $\phi^*_{x_t}$ and g^* avoids the discontinuity affecting our future computations.

For this reason, from now on we will be using the second formulation while leaving out the star * from our notation.

Our characteristic function still depends on the initial values x_0 and V_0 and is thus often called *conditional* characteristic function on x_0 and V_0. We can easily remove the dependence on x_0 by considering the distribution of incremental returns, namely $\Delta x_t = \log(S_t/S_0) = x_t - x_0$

Applying the definition of characteristic function:

$$\begin{aligned}
\phi_{\Delta x_t}(u|V_0) &= \mathbb{E}\left[e^{iu\Delta x_t}|V_0\right] \\
&= \mathbb{E}\left[e^{iu(x_t - x_0)}|V_0\right] \\
&= \mathbb{E}\left[e^{iux_t}|V_0\right]e^{-iux_0} \\
&= \phi_{x_t}(u)e^{-iux_0} \\
&= \exp\{A(t,u) + (B(t,u) - iu)x_0 + C(t,u)V_0\}
\end{aligned}$$

and since $B(t,u) = iu$, the second term in the exponential is equal to zero and we are left with:

$$\phi_{\Delta x_t}(u|V_0) = \exp\{A(t,u) + C(t,u)V_0\} \tag{3.27}$$

This equation is however still dependent on the initial value of the variance process. In a simulation framework, this would not be an issue, since we can define the level of V_0 ourselves and then generate all the different scenarios. However, if we need to calibrate the model parameters from asset prices, the market data for the variance process are not available. We will address more in depth different ways to solve this problem later on in Chapter 4. As for now, we show that we can obtain an *unconditional* expression for the characteristic function of a Heston process by approximating the distribution of V_t with its stationary distribution.

$$\begin{aligned}
\phi_{\Delta x_t}(u) &= \mathbb{E}\big[e^{iu\Delta x_t}\big] \\
&= \mathbb{E}\big[\mathbb{E}\big[e^{iu\Delta x_t}|V_0\big]\big] \\
&= \int_0^\infty \mathbb{E}\big[e^{iu(x_t - x_0)}|V_0 = v\big] f_{V_0}(v)dv \\
&= \int_0^\infty \exp\{A(t,u) + C(t,u)v\} f_{V_0}(v)dv \\
&= \exp\{A(t,u)\} \int_0^\infty \exp\{C(t,u)v\} f_{V_0}(v)dv \\
&= \exp\{A(t,u)\} M_{V_0}(C(t,u))
\end{aligned}$$

where we have used the Law of Total Expectation and $M_{V_0}(z)$ indicates the *moment generating function* of V_0 as considered stationary, so that of a Gamma distribution. In particular, it will have the same expression as (3.13).

3.3.2 Parsimonious Multiasset Heston Model

The problem with generalizing Heston framework to include more than one underlying is that the model becomes quickly very complex: we need 5 parameters for each asset ($\mu, \kappa, \theta, \sigma_V$ and ρ), and we can also model all types of correlation between the different stochastic drivers. In particular, we have 4 types of correlation:

- ρ^{S_i,V_i}: correlation that we already have in the unidimensional case, that models the way the price and the variance processes are linked ;

- ρ^{S_i,S_j}: correlation between the movements in prices of two different assets ;
- ρ^{V_i,V_j}: correlation between the variance processes of two different assets. One might expect that asset of similar nature have a higher correlation both in the price and in the variance processes;
- ρ^{S_i,V_j}: correlation between the price process of an asset and the variance process of a different one.

To overcome this increase in both mathematical and computation complexity, we will use the *parsimonious* model introduced by Szimayer, Dimitroff and Lorenz in [5]. Their framework has two main characteristics: each single-asset sub-model forms a traditional Heston model and the parameters are composed by n single-asset Heston parameter sets and $n(n-1)/2$ *asset-asset* correlations.

The n-dimensional model hence becomes:

$$\begin{pmatrix} dS_i(t)/S_i(t) \\ dV_i(t) \end{pmatrix} = \begin{pmatrix} \mu_i \\ \kappa_i(\theta_i - V_i(t)) \end{pmatrix} dt + \begin{pmatrix} \sqrt{V_i(t)} & 0 \\ 0 & \sigma_{V_i}\sqrt{V_i(t)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \rho_i & \sqrt{1-\rho_i^2} \end{pmatrix} \begin{pmatrix} dW^{S_i}(t) \\ dW^{V_i}(t) \end{pmatrix} \tag{3.28}$$

where $dW^{S_i}(t)$ and $dW^{V_i}(t)$ are independent processes and we have written explicitly the dependence between the price and the variance processes.

To fully characterize the model, we need to describe how the different W^{S_i} and W^{V_i} are correlated. In accordance with what was stated earlier, the correlation structure is the following:

$$\Sigma^{(S,V)} = cor(\boldsymbol{W}^S, \boldsymbol{W}^V) = \begin{pmatrix} \Sigma^S & 0 \\ 0 & I_n \end{pmatrix} \tag{3.29}$$

in which $\Sigma^S = cor(\boldsymbol{W}^S)$. Equation (3.29) mathematically represents what we defined as the correlation structure for our model: we only explicitly define the *asset-asset* correlations through matrix Σ^S, the dependence of every variance on other processes is carried over via Σ^S and the correlations ρ_i. More clearly, let $(\Sigma^S)_{i,j} = \rho_{i,j}$[5]:

- $dW^{S_i}(t)dW^{S_j}(t) = \rho_{i,j}dt$
- $dW^{S_i}(t)dW^{V_j}(t) = \rho_{i,j}\rho_j dt$

[5] Of course we will have $\rho_{i,j} = 1$ whenever $i = j$.

- $dW^{V_i}(t)dW^{V_j}(t) = \rho_i\rho_{i,j}\rho_j dt$ if $i \neq j$, dt otherwise

A detailed representation of a 2-asset *parsimonious* Heston model can be found in the reference paper [5].

3.4 Bates Model

Bates model is a way of combining both of the characteristics of Merton (price jumps) and Heston (stochastic volatility) in a single framework. It was introduced in 1996 by David Bates, an American professor at University of Iowa.

3.4.1 Univariate Model

In his paper [3], Bates proposes his model as a way of capturing the leptokurtosis in the distribution of log-differences of the USD/DeutscheMark exchange rate. He suggested to improve the versatility of Heston model by including *log-normal, Poisson driven* jumps in the prices process, borrowing this addition from Merton model.

Thus, we will have to deal with a total of 8 parameters: $\mu, \kappa, \theta, \sigma_V$ and ρ for the stochastic volatility part, and μ_J, σ_J and λ for the jump component.

****** add jump component ******

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^S \tag{3.30}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t^V \tag{3.31}$$

Of course, as these equations are directly obtained from (3.22) and (3.23), we have that the variance process is strictly positive whenever the Feller condition $2\kappa\theta \geq \sigma_V$ is satisfied. ***** add explanation of jump part*****

Unfortunately, for the same reason as in previous section, an explicit formula for the distribution of the log-returns $x_t = log(S_t)$ is not available and thus we have to make reference to their characteristic function. The silver lining is, though, that to obtain expression of $\phi_{x_t}(u)$, we only have to include an extra additive term to the exponential in (3.32) to account for the jumps:

******* controllare jump component***********

$$\begin{aligned}
\phi^*_{x_t}(u|x_0, V_0) &= \exp\{A(t,u) + B(t,u)x_0 + C(t,u)V_0 + D(t,u)\} \\
A(t,u) &= \mu u i t + \frac{\kappa\theta}{\sigma_V^2}\left((\kappa - \rho\sigma_V u i - d)t - 2\log\left[\frac{1 - g^* e^{-dt}}{1 - g^*}\right]\right) \\
B(t,u) &= iu \\
C(t,u) &= \frac{\kappa - \rho\sigma_V u i - d}{\sigma_V^2}\left[\frac{1 - e^{-dt}}{1 - g^* e^{-dt}}\right] \\
D(t,u) &= -\lambda\mu_J u i t + \lambda t\left[(1+\mu_J)^{ui} e^{\sigma_J^2 (ui/2)(ui-1)} - 1\right]
\end{aligned} \tag{3.32}$$

where :

$$\begin{aligned}
d &= \sqrt{(\rho\sigma_V u i - \kappa)^2 + \sigma_V^2(ui + u^2)} \\
g^* &= \frac{\kappa - \rho\sigma_V u i - d}{\kappa - \rho\sigma_V u i + d} = \frac{1}{g}
\end{aligned}$$

We will used the * formulation for the same desirable numerical properties as already stated, while leaving out the * symbol from our notation from now on.

Chapter 4

Calibration of the Models

In this chapter we will explain how the different models were calibrated and what difficulties were overcome. Empirical results are include for each section.

Chapter 5

Optimal Portfolio Allocation

In this chapter we will explore what the optimal allocation is for our portfolio of assets. We will study the *efficient frontier* using two different risk measures, volatility and expected shortfall. In all our analyses, we will be comparing the effects that including Bitcoin in our portfolio has on the optimal allocation.

5.1 Markowitz Mean-Variance Portfolio Optimization

Modern Portfolio Theory (MPT) is a mathematical framework for creating a portfolio of asset by maximizing the expected return for a given level of risk or by minizing the risk while maintaining the same expected gain. Before the article [9] by Harry Markowitz in 1952, the concept of *diversification* (the old warning *not to put all your eggs in one basket*) was only driven by the experience of how markets behave. Moreover, investors used to base their decisions on expected return alone and thus when given a choice between two assets with different expected returns, they would put all their money on the top performing one.

With his article, that would later grant him the Nobel Prize in Economics, Markowitz introduced a more rigorous and mathematically sound framework to assembly a portfolio of assets. His key insight is that an asset's return and risk risk should not be assessed by itself, but rather by how it affects the overall portfolio risk and return. To do so, the *variance* is used as a proxy for risk. Hence the name *mean-variance* analysis that is often used as a substitute for MPT.

Let's introduce the assumption underlying the MPT:

1. Investors are *risk averse*: they will always choose the less risky asset, when two assets offer the same return. At the same time, an investor wanting a higher return has to be willing to accept a higher risk. This equally holds for portfolios as a whole: given two portfolio with the different risk profiles, he will choose the less risky in case of same return and the most remunerating in case of same risk.

2. Portfolio return is the weighted sum of the single assets' returns: in general $\mathbb{E}[R_{ptf}] = \sum_{i=1}^{N} w_i \mathbb{E}[R_i]$.

3. Portfolio variance is a function of both the assets variances and their correlations: $V_{ptf} = \sum_{i=1}^{N} w_i \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i, j=1}^{N} w_i w_j \rho_{i,j} \sigma_i \sigma_j$

Items 2 and 3 above can be more compactly stated using matrix notation, which will come in handy later on in our analysis:

$$r_{ptf}(\mathbf{w}) = \mathbf{w}^T \mathbf{r} \tag{5.1}$$

$$\sigma^2_{ptf}(\mathbf{w}) = \mathbf{w}^T \Sigma \mathbf{w} \tag{5.2}$$

where we have the weights vector $\mathbf{w} = [w_1, w_2, ..., w_N]^T$, $\mathbf{r} = [r_1, r_2, ..., r_N]^T$, using the shorthand $r_i = \mathbb{E}[R_i]$ and finally Σ is the NxN covariance matrix of the assets.

We can now state the *optimization problem* involving the minimization of the portfolio risk for a specified expected portfolio return in terms of the variable we have just introduced.

$$\begin{aligned}
&\min_{\mathbf{w} \in \mathbb{R}^N} && \sigma^2_{ptf}(\mathbf{w}) && \text{(5.3a)}\\
&\text{subject to} && \mathbf{e}^T \mathbf{w} = 1, && \text{(5.3b)}\\
& && \mathbf{r}^T \mathbf{w} = r_{target}, && \text{(5.3c)}\\
& && w_i \geq 0, \text{for } i = 1 \ldots N. && \text{(5.3d)}
\end{aligned}$$

where $\mathbf{e}$ indicates a vector of ones and the first constraint makes sure that the sum of the weights always equals to one. This is to represent a portfolio in which all the money available is allocated in the assets we are taking into consideration. The second constraint ensures that the portfolio allocation $\mathbf{w}$ produces the target expected return r_{target}. Finally, the last constraint is in fact optional and is only used to exclude the possibility to go short on any asset.

The optimization problem in (5.3) has a quadratic objective function given by (5.2) and only has linear constraints[1]. Thanks to this property, the optimization can be carried out numerically by any of the quadratic/linear optimizers that are available for most programming languages.

As we are going to explain in the following sections, we will be mainly focusing on the case where there is no short selling, as indeed so far there are no instruments on the market that allow an investor to go short on Bitcoin and our analysis shows that the main diversification advantage comes from including Bitcoin in our portfolio. Allowing short-selling improves our diversification capability only so slightly.

5.2 Markowitz Efficient Frontier

It is interesting to study the set of optimal allocation as a whole, rather than simply focus on one target return and minimizing the portfolio risk. To do so, we can consider a set of target returns and compute for each of them the respective minimum variance. We thus get a set of pairs (σ^2, r) that represents the best allocation in terms of the minimum risk.

We can thus plot those pairs on an X-Y graph and obtain a curve, the *portfolio frontier*, that intrinsically represents our portfolio of N assets. As a usual practice in finance, we will be plotting on the X-axis the volatility σ instead of the variance σ^2.

In Figure 5.1 we can see what the portfolio frontier looks like for our portfolio of assets. It is interesting to notice how the curve divides the plane in two region: the area to the left of the line includes all those pairs (σ, r) that are not attainable with our assets, since they have a volatility that is too low for that level of expected return. On the other hand, the region to the right of the portfolio frontier is made of all the pairs that are possible to obtain with a specific allocation $\mathbf{w}$ but that will never be chosen by an investor: moving to the left on the same level of return we eventually reach a point on the frontier. The portfolio represented by this point will dominate the one we started from in terms of risk, so it will always be a better choice.

We can proceed with the same argument arguing in terms of best return for a given level of risk: we can thus introduce the *efficient* frontier. For every level of volatility that has two corresponding points on the portfolio frontier, only the one with the higher expected return will be chosen by an investor in our reference framework: hence only the top half of the curve (from the vertex and up) will form the *efficient portfolio frontier.*

[1] The last positivity constraint can be easily expressed in matrix form by writing $\mathbf{I}_N\mathbf{w} \geq \mathbf{0}_N$ where $\mathbf{I}_N$ is the identity matrix of order N and $\mathbf{0}_N$ is N-dimensional vector of zeros.

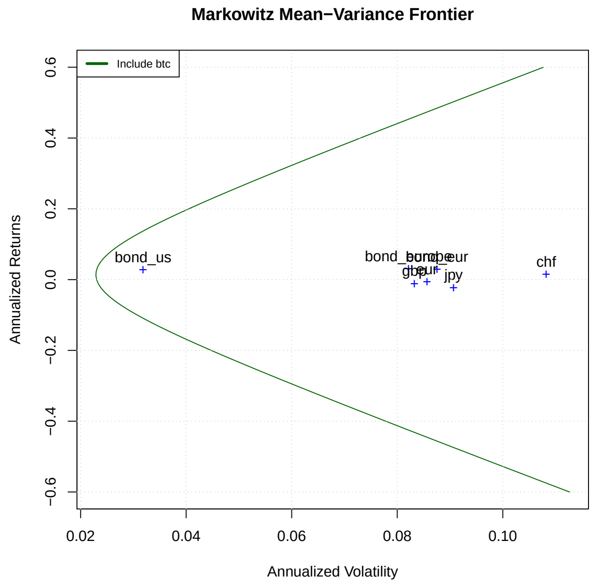


Figure 5.1: The Markowitz Mean-Variance frontier obtained from our portfolio of assets, that includes Bitcoin and allows short-selling.

As a summary, let us keep in mind that any point in the volatility-expected return plane *dominates* all the other portfolio allocations that are represented by points situated below and to the right of it. Conversely, that same point will be dominated by all other allocations above and to the left of it.

Another mathematical perk of considering only the efficient top section is that it renders the frontier a *bijective* function. Hence, for each (attainable) level of risk there is one and only one corresponding expected portfolio return, and vice-versa.

5.2.1 Efficient Frontier with and without Bitcoin

Let us now study how including Bitcoin in our portfolio of assets can help increase the diversification and obtain a higher (expected) return with a lower risk.

In Figure 5.2 we have plotted the efficient Markowitz frontier in all possible cases: including and excluding Bitcoin while allowing short-selling, and the same curves when short-selling is not allowed.

Comparing the two green lines, we can clearly see that the exclusion of short-selling does not penalize our efficient allocation by much. The same

Volatility Level	Return without Bitcoin	Return including Bitcoin
2,61%	3,00%	3,00%
2,75%	3,89%	7,70%
3,00%	4,59%	10,94%
3,25%	5,05%	13,37%
3,50%	5,43%	15,48%
3,75%	5,76%	17,41%
4,00%	6,06%	19,21%
4,25%	6,34%	20,94%
4,50%	6,61%	22,60%
4,75%	6,87%	24,21%
5,00%	7,12%	25,79%
5,25%	7,37%	27,34%
5,50%	7,61%	28,86%
5,75%	7,85%	30,37%
6,00%	8,08%	31,85%

Table 5.1: Expected return for different levels of volatility, both including and excluding Bitcoin.

can be stated for the red and orange curves, which represent our portfolio when excluding the digital asset. Thus, given our particular set of assets, allowing for short-selling does very little to improve the diversification of our portfolio.

Let us now take a look of what happens when we include Bitcoin in the reference portfolio: as we can see from Figure 5.2, we get a significant improvement in the expected return when considering each level of risk . Equivalently, for the same level of return we have a noticeable decrease in the volatility of our portfolio.

We can see some numerical proof of the diversification properties of adding Bitcoin to our portfolio in Table 5.1 and Table 5.2.

5.2.2 Portfolio Allocation

We have so far seen the implications of introducing the digital asset in our portfolio in terms of improvement in the expected return and of lowering the overall portfolio risk. Let us now take a look at how Markowitz MPT allocates the money in the different assets.

To do so, we can plot the values of $\mathbf{w}$ as resulting from (5.3) for different levels of volatilities (and hence returns).

Return Level	Volatiliy without Bitcoin	Volatility including Bitcoin
3,00%	2,61%	2,61%
3,50%	2,61%	2,67%
4,00%	2,61%	2,78%
4,50%	2,62%	2,96%
5,00%	2,63%	3,22%
5,50%	2,65%	3,55%
6,00%	2,67%	3,95%
6,50%	2,69%	4,40%
7,00%	2,71%	4,88%
7,50%	2,74%	5,39%
8,00%	2,77%	5,91%
8,50%	2,80%	6,45%
9,00%	2,84%	7,00%
9,50%	2,88%	7,56%
10,00%	2,92%	8,12%
10,50%	2,96%	8,69%
11,00%	3,01%	9,27%
11,50%	3,05%	9,85%
12,00%	3,10%	10,43%
12,50%	3,16%	11,01%
13,00%	3,21%	11,60%
13,50%	3,26%	12,19%
14,00%	3,32%	12,78%
14,50%	3,38%	13,37%
15,00%	3,44%	13,96%

Table 5.2: Volatility for different level of expected portfolio return, both including and excluding Bitcoin.

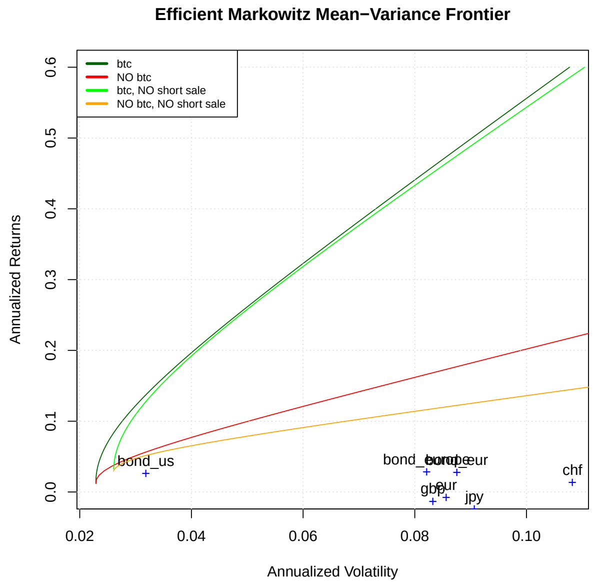


Figure 5.2: The *Efficient* Markowitz Mean-Variance frontier obtained from our portfolio of assets, both including and excluding Bitcoin and with short-selling or not.

5.3 Portfolio Optimization with CVaR as a Risk Measure

Chapter 6

Conclusions

Appendix A

Characteristic Function of a Compound Poisson Process

By definition, the characteristic function of a random variable X is given by:

$$\phi_X(u) = \mathbb{E}[e^{iuX}] \tag{A.1}$$

Let's first consider a simple Poisson process N_t of parameter λ. At each time $t > 0$, N_t has a discrete distribution that follows:

$$\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},\ n = 0, 1, 2, ... \tag{A.2}$$

Since N_t is a *discrete* random variable, the expectation in A.1 amounts to a sum over all the possible values of N_t:

$$\begin{aligned}
\phi_{N_t}(u) = \mathbb{E}[e^{iuN_t}] &= \sum_{n=0}^{\infty} e^{iun}\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} e^{iun} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t e^{iu})^n}{n!}
\end{aligned}$$

Using the definition of exponential $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ we then get the final result:

$$\begin{aligned}
\phi_{N_t}(u) &= e^{-\lambda t} e^{\lambda t e^{iu})} \\
&= e^{\lambda t(e^{iu}-1)}
\end{aligned}$$

Let's consider now a compound Poisson process defined by

$$X_t = \sum_{i=1}^{N_t} Y_i \tag{A.3}$$

where Y_i are i.i.d. and have density expressed by the function $f_Y(y)$.

To compute the characteristic function of X_t we can follow the same steps as in the simple Poisson case, but in addition we use the theorem of total expectation to first simplify the expression and then proceed exploiting the i.i.d property:

$$\begin{aligned}
\phi_{X_t}(u) &= \mathbb{E}[e^{iuX_t}] \\
&= \sum_{n=0}^{\infty} \mathbb{E}[e^{iuX_t} | N_t = n] \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}[\prod_{i=1}^{N_t} e^{iuY_i} | N_t = n] \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} \prod_{i=1}^{n} \mathbb{E}[e^{iuY_i} | N_t = n] \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} (\mathbb{E}[e^{iuY}])^n \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} (\phi_Y(u))^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t\, \phi_Y(u))^n}{n!} \\
&= e^{\lambda t(\phi_Y(u) - 1)}
\end{aligned}$$

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