Politecnico di Milano
School of Industrial and Information Engineering
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TITLE: very interesting subject, ain't it?

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The secret to happiness is freedom.
And the secret to freedom is courage.

Thucydides

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Abstract

Including Bitcoin in an investment portfolio increases portfolio diversification.

Acknowledgements

add acknowledgements

Thank you.

Chapter 1

Introduction

1.1 Thesis structure

Chapter 2

Correlation Analysis

In order to get an initial insight on how Bitcoin is correlated with other assets, we will perform a correlation analysis based on the empirical time series of our data. We will focus our attention on the logarithmic returns it is the standard practice. We will often refer to logarithmic returns simply as returns, only specifying their nature when it is necessary to avoid confusion.

2.1 Empirical Correlation of Returns

We first start by performing some statistical analysis on the data in order to estimate the distribution from which they are sampled. For this part, we will consider our data as successive samples of a N-dimensional vector in $\mathbb{R}^N$, where N is the number of assets:

$$\mathbf{x}_j = \begin{pmatrix} x_{1,j} \\ x_{2,j} \\ \vdots \\ x_{N,j} \end{pmatrix}, j = 1 \dots N_{sample}$$

Each element i of the vector $\mathbf{x}_j$ represents the j^{th} realization of the returns for asset i.

Following basic statistics, we can now compute the *sample mean* of our vectors of returns as:

$$\bar{\mathbf{x}} = \frac{1}{N_{sample}} \sum_{j=1}^{N_{sample}} \mathbf{x}_j = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_N \end{pmatrix}$$

where $\bar{x}_i = \frac{1}{N_{sample}} \sum_{j=1}^{N_{sample}} x_{i,j}$ is the sample mean of component i.

Now we compute the *sample covariance matrix* through the following formula:

$$\bar{\Sigma} = \frac{1}{N_{sample} - 1} \sum_{j=1}^{N_{sample}} (\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})^T$$

where $\bar{\mathbf{x}}$ represent the sample mean of the returns just introduced.

All the information needed to obtain the *correlation matrix* C are already included in $\bar{\Sigma}$, we only need to perform some further calculations:

$$C_{i,j} = \frac{\bar{\Sigma}_{i,j}}{\sqrt{\bar{\Sigma}_{i,i}\bar{\Sigma}_{j,j}}} \tag{2.1}$$

We have thus obtained an empirical estimate of the correlation between our assets returns. The formula in (2.1) is often referred to as *Pearson correlation coefficient*, from the name of the English mathematician Karl Pearson who first formulated it.

Results are reported in the following tables.

******** ADD RESULT TABLES ********

We are mainly interested in the correlation between Bitcoin and other assets returns, so we will now focus on the first row (or equivalently column, by symmetry) of the correlation matrix.

All values are fairly close to zero, never exceeding 10% towards the positive or the negative side. One may thus wonder whether these correlations are *statistically significantly* different from zero. To answer this question, we will introduce two statistical tests to check the correlation significance.

2.2 Correlation Significance

The very core of Inferential Statistics, the branch of statistics that allows to draw conclusions from the information contained in a set of data, is hypothesis testing.

In our case, we are specifically interested in testing if the sample correlation coefficients are significantly different from zero or not. Both of the following tests are presented in the most general form for a sample of two variables, their distribution correlation ρ and their sample correlation $\hat{\rho}$.

Following standard testing procedure, we specify the *null hypothesis* and the *alternative hypothesis*:

$$\mathbf{H_0}: \quad \rho = 0 \quad vs. \quad \mathbf{H_1}: \quad \rho \neq 0$$

These will be common to both presented tests.

2.2.1 t-test

Our first test is based on Student's t-distribution and the following t-statistic:

$$t = \hat{\rho}\sqrt{\frac{n-2}{1-\hat{\rho}^2}} \tag{2.2}$$

which under the null hypothesis is distributed as a Student's t with $n-2$ degrees of freedom, where n stands for the cardinality of the sample. We can thus proceed by computing the relative p-value and compare it to a given level of confidence α (usually $\alpha = 95\%$). The result of the test will be deduced as follows:

- $p-value < 1-\alpha$: we have statistical evidence to state that the correlation is *significantly* different from zero;
- $p-value \geq 1-\alpha$: there is *no statistical evidence* to state that the correlation is different from zero.

2.2.2 Permutation test

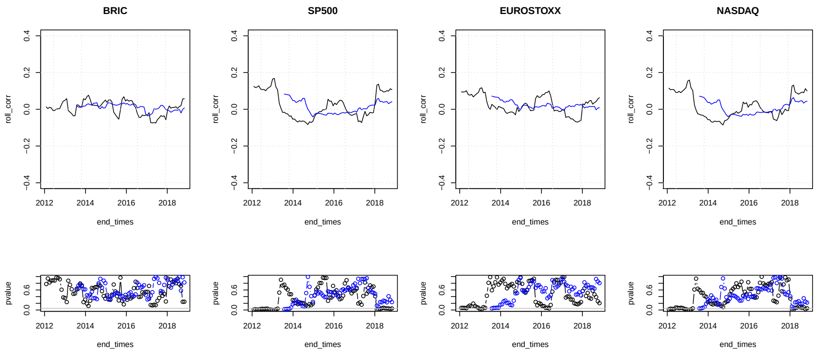
The permutation test is based on building an empirical distribution of values for the correlation by sampling different pairs of X and Y variable and then computing Pearson's correlation. If this is done a large enough number of times, we obtain an empirical distribution of possible values. From this distribution we can then obtain the p-value of the test and thus the final result in the same way as in the previous case.

2.2.3 Significance results

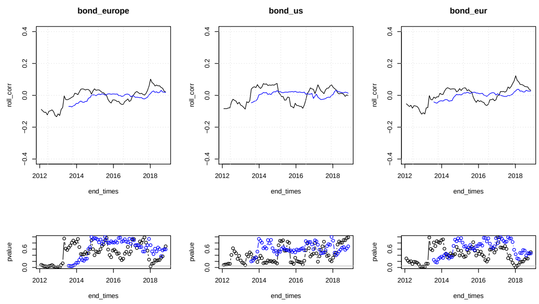
********Comments and numerical results***************

2.3 Rolling Correlation

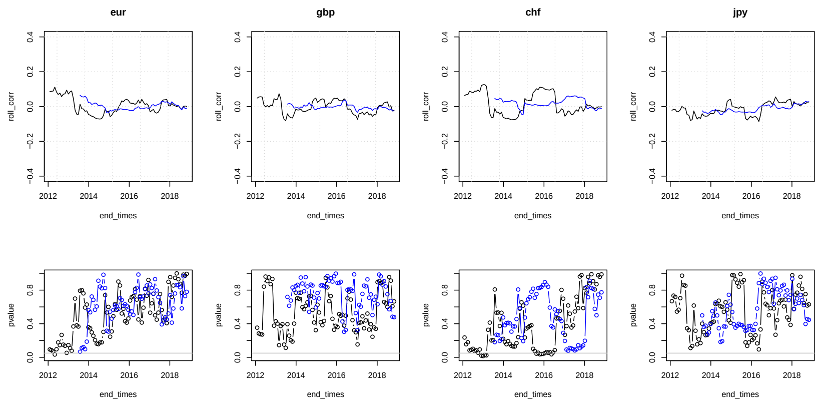
Our study so far has focused on the analysis of the dataset as a whole, with values spanning from 2010 to 2018. This is clearly important if we want to obtain a general overview of the period, but it is also interesting to see how the correlation between the assets has evolved through. Therefore, we present in Figure 2.1 the results obtained from calculating the correlation between Bitcoin and the other assets using rolling windows of 36 and 18 months, updated monthly.



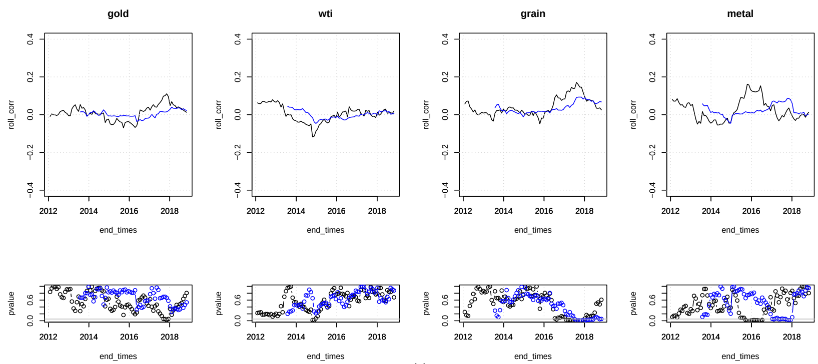
(a) Stocks



(b) Bonds



(c) Currency exchange



(d) Commodities

Figure 2.1: Plots of rolling correlation for the different asset classes (on top) and significance for each value (on bottom). Blue lines are the 3-year rolling correlations, while the black ones have a window of 18 months. Both computations are updated monthly.

There are two graphs for each asset: in the top plots levels of the rolling correlations are represented using two different colours, blue for the 3-year and black for the 18-month windows; in the bottom plots we included the significance of each rolling correlation through its p-value. The grey horizontal line represents the 5% level of significance[1].

The main conclusion we can draw from these images is that the correlation of any asset with Bitcoin is hardly ever significantly different from zero, and when it is, its absolute level is never more than greater than 20% for a small period of time.

To confirm the fact that Bitcoin is not correlated with any asset, we can also take a look at the path of the rolling correlations: there is no line that is always above zero, nor below. This indicates that there is no underlying trend, whether positive or negative, and the correlation one might find is only temporary.

**** maybe add more comments on the results ****

[1]As we explained in the previous section, to check that a sample correlation is *significantly* non zero, we compare the p-value of the test to a given level, here $1 - \alpha = 5\%$. Graphically, whenever the dots are above the grey line in Figure 2.1, the corresponding correlation is *not* significantly different from zero.

Chapter 3

Presentation of the Models

In this chapter we will present the stochastic frameworks in which we developed our analysis. We first introduce a *jump diffusion* model presented in 1976 by R.C. Merton: he added log-normal jumps to the simple B&S dynamics of the asset price. Then we move to the *stochastic volatility* model of Heston 1993. Heston introduced a new stochastic process that accounts for the variance of the underlying which evolves as a B&S with a stochastic volatility term. The last model we will present was introduced by Bates in 1996 and it is the combination of the former two: an asset dynamics which include jumps and is driven by a stochastic volatility. All models are first introduced in the one dimensional case and then generalised to the n dimensional case which was then implemented in our code.

3.1 Preliminary Notions

In this section we will briefly present the equation of a geometric brownian motion, introduce the notion of Poisson process and present the CIR process. All of these building blocks will be required to fully understand the models to follow.

3.1.1 Geometric Brownian Motion

The simplest continuous dynamics to describe the price of an asset is that of a geometric brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{3.1}$$

where S_t represents the price of the asset at time t, μ is the (constant) drift and σ is the (constant) volatility. W_t is a Wiener process. This is the standard and most widespread stichastic differential equation to model asset dynamics, so we will only present those results that will be later used in our study.

Applying Ito's lemma to the previous equation, we can also explicitly express the dynamics of the log-returns $X_t = log(S_t)$, obtaining:

$$dX_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t \tag{3.2}$$

This stochastic differential has a simple solution which can be computed via stochastic integrals and allows us to describe the dynamics of the log-returns at each instant t starting from $t = 0$:

$$X_t = X_0 + (\mu - \frac{\sigma^2}{2})t + \sigma W_t \tag{3.3}$$

Thanks to (3.3) we can now express the price dynamics of the asset by inverting the relation with the returns: $S_t = e^{X_t}$. We thus obtain the solution to (3.1):

$$S_t = S_0\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \tag{3.4}$$

Given that $S_t = e^{X_t}$, that X_t by its equation is a generalized brownian motion and hence we have $X_t - X_0 \sim \mathcal{N}(\mu - \frac{\sigma^2}{2}, \sigma^2)$, the resulting distribution of prices at time t as units of the initial value is distibuted as a log-normal.

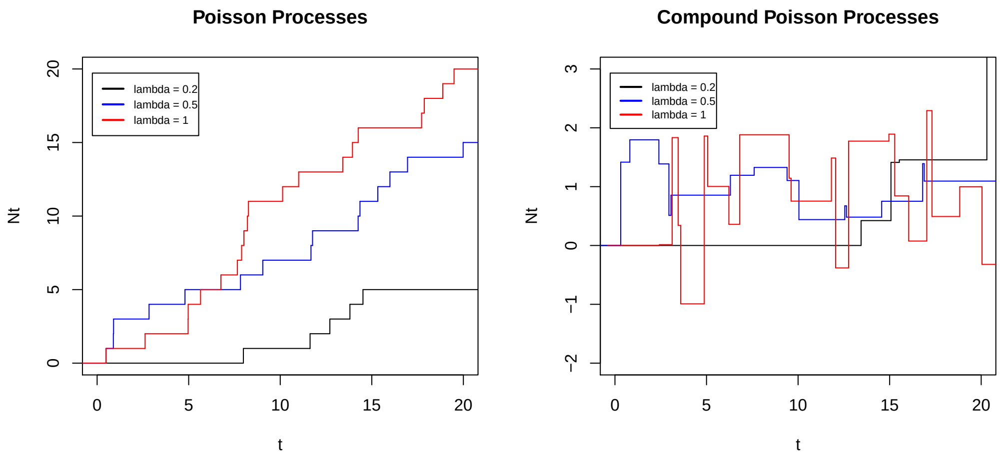
The great success of these framework comes from the simplicity of its dynamics. In particular, since the log-returns follow a Gaussian distribution, μ and σ are easy to calibrate from data and the formulas for pricing options are often explicit. As one can imagine, a simple model can only explain simple phenomena: that's why we have such a great deal of *new and improved* version of equation 3.1.

3.1.2 Poisson Process and Compound Poisson Process

Consider a sequence of *independent* exponential random variables $\{\tau_i\}_{i\geq 1}$ with parameter λ[1] and let $T_n = \sum_{i=1}^{n} \tau_i$. Then we can define the *Poisson process* N_t as

$$N_t = \sum_{n\geq 1} \mathbb{1}_{t\geq T_n} \tag{3.5}$$

[1]An exponential random variable τ of parameter λ has a cumulative distibution function of the form: $\mathbb{P}(\tau \geq y) = e^{-\lambda y}$



(a) Poisson processes. (b) Compound with Gaussian jumps.

Figure 3.1: Poisson processes and compound Poisson processes with different λ.

where $\mathbb{1}_{condition}$ is 1 if the *condition* is true, 0 if it is false.

N_t is thus a piece-wise constant RCLL[2] process with jumps that happen at times T_n and are all of size 1, as we can see from Figure fig. 3.1a. An important property of Posson processes is that they have independent and stationary increments, meaning that the increment of $N_t - N_s$ (with $s \leq t$) is independent from the history of the process up to N_s and has the same law of N_{t-s}. At any time t, N_t is distributed as a Poisson of parameter λt, which means it is a discrete random variable on the integer set with

$$\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \tag{3.6}$$

When working with jump diffusion process, it is often the case that there is no explicit formula for its density, thus usually we resort to characteristic functions. The characteristic function of N_t is given by

$$\phi_{N_t}(u) = e^{\lambda t(e^{iu}-1)} \tag{3.7}$$

The computations to get 3.7 from 3.6 are carried out in Appendix A.

For financial applications, it is of little interest to have a process with a single possible jump size. The *compound* Poisson processes are a generalization of Poisson processes where the jump sizes can have an arbitrary

[2]RCLL is shorthand for right continuous with left limit.

distribution. More precisely, consider a Poisson process N_t with parameter λ and a sequence of i.i.d[3] variables $Y_{i i\geq 1}$ with law $f_Y(y)$. Then the process defined by

$$X_t = \sum_{n=1}^{N_t} Y_i \tag{3.8}$$

is a compound Poisson process. Examples of this kind of process are plotted in Figure fig. 3.1b.

As before, we developed the computations to obtain an expression for the characteristic function of X_t in Appendix A. The resulting expression depends on the distribution of Y, specifically from its characteristic function:

$$\phi_{X_t}(u) = e^{\lambda t(\phi_Y(u)-1)} \tag{3.9}$$

3.1.3 CIR Process

3.2 Merton Model

******maybe add some words right here ********

3.2.1 Original Univariate Model

The first jump diffusion model was originally introduced in [5] in order to account for the leptokurtic distribution of real market returns and to model sudden fall (or rise) in prices due to the arrival of new information. The asset price dynamics is modelled as follows:

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t + (Y_t - 1)dN \tag{3.10}$$

where α and σ are respectively the drift and the diffusion of the continuous part, Y_t is a process modelling the intensity of the jumps and $N(t)$ is the Poisson process driving the arrival of the jumps and has parameter λ.

We can rewrite (3.10) in terms of the log-returns $X_t = log(S_t)$ and obtain, following the computations in [4] and using theory from [6]:

$$dX_t = (\alpha - \frac{\sigma^2}{2})dt + \sigma dW_t + \log(Y_t) \tag{3.11}$$

[3]i.i.d stands for independent and identically distributed.

that has as solution:

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N(t)} \eta_k \tag{3.12}$$

where X_0 is the initial value of the log-returns, $\eta_k = \log(Y_k) = \log(Y_{t_k})$ and t_k is the time when the k^{th} Poisson shock from $N(t)$ happens. We use $\mu = \alpha - \frac{\sigma^2}{2}$ for ease of notation throughout the paper. Following [5], we take η_k *i.i.d.* (independent and identically distributed) and Gaussian, in particular $\eta \sim \mathcal{N}(\theta, \delta^2)$. Another choice for the distribution of η is given in [3].

It is often useful when dealing with market data that are by nature discrete, to consider a *discretized* version of (3.12) in which the values are sampled at intervals of Δt in $[0, T]$. We thus get that for $X_i = \log(\frac{S_{i+1}}{S_i})$:

$$X_i = \mu \Delta t + \sigma \sqrt{\Delta t}\; z + \sum_{k=1}^{N_{i+1}-N_i} Y_k \tag{3.13}$$

where we denote $X_i = X_{t_i}$, $N_i = N(t_i)$ and $t_i = i\Delta t$ with $i = 0 \dots N$, $t_N = N\Delta t = T$, z is distributed as a standard Gaussian $z \sim \mathcal{N}(0, 1)$.

The Poisson process $N(t)$ in (3.13) is computed at times t_{i+1} and t_i and these quantities are subtracted. Following basic stochastic analysis, one can prove that the resulting value $N_{i+1} - N_i$, is distributed as a Poisson random variable N of parameter $\lambda \Delta t$. This allows us to provide an explicit formulation for the transition density of the returns using the theorem of total probability:

$$f_{\Delta X}(x) = \sum_{k=0}^{\infty} \mathbb{P}(N = k) f_{\Delta X|N=k}(x) \tag{3.14}$$

This is an infinite mixture of Gaussian distributions, due to the infinite possible realization of the Poisson variable, and renders the estimation of the model through MLE technique intractable, see [2]. To solve this problem we introduce a first order approximation, as it's been proposed in [1]. Considering small Δt, so that also $\lambda \Delta t$ is small, we obtain that the only relevant terms in (3.14) are the one for $k = 0, 1$. The formula for the transition density becomes:

$$f_{\Delta X}(x) = \mathbb{P}(N = 0) f_{\Delta X|N=0}(x) + \mathbb{P}(N = 1) f_{\Delta X|N=1}(x)$$

expressing it explicitly:

$$f_{\Delta X}(x) = (1 - \lambda \Delta t)\; f_{\mathcal{N}}(x; \mu, \sigma^2) + (\lambda \Delta t)\; f_{\mathcal{N}}(x; \mu + \theta, \sigma^2 + \delta^2) \tag{3.15}$$

where $f_{\mathcal{N}}(x; \mu, \sigma^2)$ is the density of a Gaussian with parameters $\mathcal{N}(\mu, \sigma^2)$.

3.2.2 Multivariate Model

Starting from the univariate model introduced in [5], we developed a generalization to n assets including only idiosyncratic jumps:

$$\frac{dS_t^{(j)}}{S_t^{(j)}} = \alpha_j dt + \sigma_j dW_t^{(j)} + (Y_t^{(j)} - 1)dN_t^{(j)} \tag{3.16}$$

where $\mathbf{S}_t$ are the prices of the assets, $j = 1...n$ represents the asset, α_j are the drifts, σ_j are the diffusion coefficients, $W_t^{(j)}$ are the components of an n-dimensional Wiener process $\mathbf{W}_t$ with $dW^{(j)}dW^{(i)} = \rho_{j,i}$, η_j represent the intensities of the jumps and are distributed as Gaussian: $\eta_j \sim \mathcal{N}(\theta_j, \delta_j^2)$. Finally, $N^{(j)}(t)$ are Poisson processes with parameters λ_j, which are independent of $\mathbf{W}_t$ and of one another.

In order to calibrate the parameters to the value of the market log-returns, we used a Maximum Likelihood approach. We thus maximize:

$$\mathcal{L}(\psi|\Delta\mathbf{x}_{t_1}, \Delta\mathbf{x}_{t_2}, \ldots, \Delta\mathbf{x}_{t_N}) = \sum_{i=1}^{N} f_{\Delta\mathbf{X}}(\Delta\mathbf{x}_{t_i}|\psi) \tag{3.17}$$

where $\psi = \{\{\mu_j\}, \{\sigma_j\}, \{\rho_{i,j}\}, \{\theta_j\}, \{\delta\}_j, \{\lambda_j\}\}$ are the model parameters, $f_{\Delta\mathbf{X}}$ is the transitional density of the log-returns which is computed approximately using the theorem of total probability. For a full insight on the model and the calibration procedure, please refer to the *APPENDIX LINK*

3.3 Heston Model

3.4 Bates Model

Chapter 4

Calibration of the Models

In this chapter we will explain how the different models were calibrated and what difficulties were overcome. Empirical results are include for each section.

Chapter 5

Markowitz Portfolio Optimization

Chapter 6

Conclusions

Appendix A

Characteristic Function of a Compound Poisson Process

By definition, the characteristic function of a random variable X is given by:

$$\phi_X(u) = \mathbb{E}[e^{iuX}] \tag{A.1}$$

Let's first consider a simple Poisson process N_t of parameter λ. At each time $t > 0$, N_t has a discrete distribution that follows:

$$\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},\ n = 0, 1, 2, ... \tag{A.2}$$

Since N_t is a *discrete* random variable, the expectation in A.1 amounts to a sum over all the possible values of N_t:

$$\begin{aligned}
\phi_{N_t}(u) = \mathbb{E}[e^{iuN_t}] &= \sum_{n=0}^{\infty} e^{iun}\mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} e^{iun} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t e^{iu})^n}{n!}
\end{aligned}$$

Using the definition of exponential $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ we then get the final result:

$$\begin{aligned}
\phi_{N_t}(u) &= e^{-\lambda t}e^{\lambda t e^{iu})} \\
&= e^{\lambda t(e^{iu}-1)}
\end{aligned}$$

Let's consider now a compound Poisson process defined by

$$X_t = \sum_{i=1}^{N_t} Y_i \tag{A.3}$$

where Y_i are i.i.d. and have density expressed by the function $f_Y(y)$.

To compute the characteristic function of X_t we can follow the same steps as in the simple Poisson case, but in addition we use the theorem of total expectation to first simplify the expression and then proceed exploiting the i.i.d property:

$$\begin{aligned}
\phi_{X_t}(u) &= \mathbb{E}[e^{iuX_t}] \\
&= \sum_{n=0}^{\infty} \mathbb{E}[e^{iuX_t} | N_t = n] \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} \mathbb{E}[\prod_{i=1}^{N_t} e^{iuY_i} | N_t = n] \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} \prod_{i=1}^{n} \mathbb{E}[e^{iuY_i} | N_t = n] \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} (\mathbb{E}[e^{iuY}])^n \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} (\phi_Y(u))^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t \, \phi_Y(u))^n}{n!} \\
&= e^{\lambda t (\phi_Y(u) - 1)}
\end{aligned}$$

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