# Bitcoin Allocation

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November 6th

# Outline

# Original Merton Model (1976)

Original Merton 1976 [3] model:

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t + (Y_t - 1)dN \tag{1}$$

where

- $\alpha$ is the drift
- $\sigma$ is the diffusion coefficient
- $Y_t$ is a process modelling the intensity of the jumps
- $N(t)$ is the Poisson process driving the arrival of the jumps and has parameter $\lambda$

# Dynamics of the log-returns

We can rewrite (1) in terms of the log-returns $X_t = log(S_t)$ and obtain:

$$dX_t = (\alpha - \frac{\sigma^2}{2})dt + \sigma dW_t + \log(Y_t) \tag{2}$$

that has as solution:

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N(t)} \eta_k \tag{3}$$

where

- $X_0 = log(S_0)$
- $\eta_k = \log(Y_k) = \log(Y_{t_k})$ and $t_k$ is the time when the $k^{th}$ Poisson shock from $N(t)$ happens. $\eta_k$ are distributed as Gaussians: $\eta \sim \mathcal{N}(\theta, \delta^2)$
- $N^{(j)}(t)$ are Poisson processes with parameters $\lambda_j$, which are independent of $\mathbf{W}_t$ and of one another
- $\mu = \alpha - \frac{\sigma^2}{2}$ for ease of notation

# Discretized dynamics of log-returns

It is often useful when dealing with market data that are by nature discrete, to consider a *discretized* version of (3) in which the values are sampled at intervals of $\Delta t$ in $[0, T]$. We thus get that for $X_i = \log(\frac{S_{i+1}}{S_i})$:

$$X_i = \mu\Delta t + \sigma\sqrt{\Delta t}\, z + \sum_{k=1}^{N_{i+1}-N_i} Y_k \tag{4}$$

where we denote $X_i = X_{t_i}$, $N_i = N(t_i)$ and $t_i = i\Delta t$ with $i = 0 \dots N$, $t_N = N\Delta t = T$, $z$ is distributed as a standard Gaussian $z \sim \mathcal{N}(0, 1)$.

Following basic stochastic analysis, one can prove that the resulting value $N_{i+1} - N_i$, is distributed as a Poisson random variable $N$ of parameter $\lambda\Delta t$.

# Original Merton Model (1976)

Applying the Total probability theorem, we get the following expression for the transition density:

$$f_{\Delta X}(x) = \sum_{k=0}^{\infty} \mathbb{P}(N = k) f_{\Delta X|N=k}(x) \quad (5)$$

This infinite mixture of Gaussian random variable makes the MLE intractable(see Honore). To get a simpler definition of the log-return density, we limit our study to small values of $\lambda \Delta t$ so that the Poisson variable can only be in $0, 1$ with non-negligible probability.

$$\begin{aligned} f_{\Delta X}(x) &= \mathbb{P}(N = 0) f_{\Delta X|N=0}(x) + \mathbb{P}(N = 1) f_{\Delta X|N=1}(x) \\ &= (1 - \lambda \Delta t)\ f_{\mathcal{N}}(x; \mu, \sigma^2) + (\lambda \Delta t)\ f_{\mathcal{N}}(x; \mu + \theta, \sigma^2 + \delta^2) \end{aligned}$$

# Multivariate Merton

Multivariate generalization of Merton's 1976 *jump diffusion* for $n$ assets:

$$\frac{dS_t^{(j)}}{S_t^{(j)}} = \alpha_j dt + \sigma_j dW_t^{(j)} + (Y_t^{(j)} - 1) dN_t^{(j)} \tag{6}$$

where for $j = 1 \ldots n$ representing the assets:

- $\mathbf{S}_t$ is the price vector
- $\alpha_j$ are the drifts
- $\sigma_j$ are the diffusion coefficients
- $W_t^{(j)}$ are the components of an $n$-dimensional Wiener process $\mathbf{W}_t$ with $dW^{(j)} dW^{(i)} = \rho_{j,i}$
- $\eta_j$ represent the intensities of the jumps and are distributed as Gaussian: $\eta_j \sim \mathcal{N}(\theta_j, \delta_j^2)$
- $N^{(j)}(t)$ are Poisson processes with parameters $\lambda_j$, which are independent of $\mathbf{W}_t$ and of one another

# Multivariate transition density

We can proceed as in the univariate case in order to get an explicit formula for the transition density, conditioning on all the different Poisson processes. In the two asset case we have

$$
\begin{aligned}
f_{\Delta\mathbf{X}}(\mathbf{x}) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \mathbb{P}(N^{(1)}(t) = k_1, N^{(2)}(t) = k_2) \\
f_{\Delta\mathbf{X}}(\mathbf{x}|N^{(1)}(t) = k_1, N^{(2)}(t) = k_2) \\
= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \mathbb{P}(N^{(1)}(t) = k_1)\mathbb{P}(N^{(2)}(t) = k_2) \\
f_{\Delta\mathbf{X}}(\mathbf{x}|N^{(1)}(t) = k_1, N^{(2)}(t) = k_2)
\end{aligned}
$$

Considering small $\Delta t$:

$$
\begin{aligned}
f_{\Delta\mathbf{X}}(\mathbf{x}) = \sum_{k_1=0}^{1}\sum_{k_2=0}^{1} \mathbb{P}(N^{(1)}(t) = k_1)\mathbb{P}(N^{(2)}(t) = k_2) \\
f_{\Delta\mathbf{X}}(\mathbf{x}|N^{(1)}(t) = k_1, N^{(2)}(t) = k_2, N^{(Z)}(t) = k_Z)
\end{aligned}
$$

# MLE procedure

To recover the parameter of the model, we maximize the *Likelihood* function:

$$\mathcal{L}(\psi|\Delta\mathbf{x}_{t_1}, \Delta\mathbf{x}_{t_2}, \ldots, \Delta\mathbf{x}_{t_N}) = \sum_{i=1}^{N} f_{\Delta\mathbf{X}}(\Delta\mathbf{x}_{t_i}|\psi) \tag{7}$$

We first use a stochastic optimization in which we find where the global minimun should be and then proceed with an algorithm that computes the numerical gradient to reach the closest minimum. The estimation through *maximul likelihood* is a standard practice, see for instance [1].

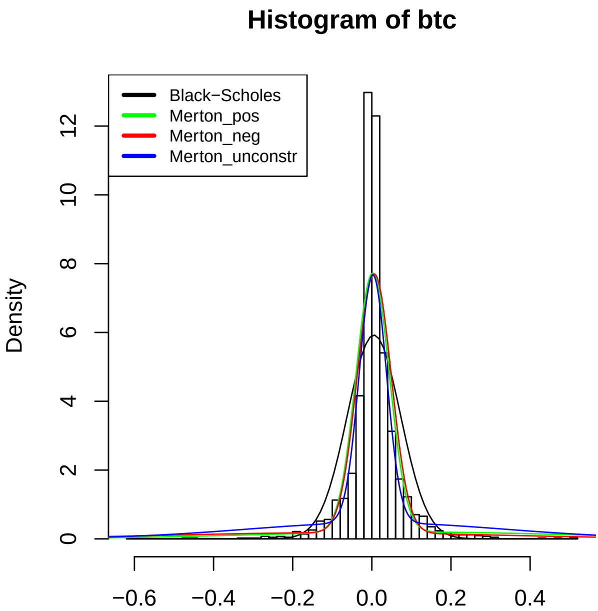
# Dataset

1. Stock indexes:
   - S&P500: American stock market index based on 500 large company with stock listed either on the NYSE or NASDAQ.
   - EURO STOXX 50: equity index of eurozone stocks, covering 50 stocks from 11 eurozone countries.
   - MSCI BRIC: market cap weighted index designed to measure the equity market performance across the emerging country indices of Brazil, Russia, India and China.
   - NASDAQ: market cap weighted index including all NASDAQ tiers: Global Select, Global Market and Capital Market.
2. Commodities:
   - Gold: price of gold measured in USD/Oz.
   - WTI: price of crude oil used as benchmark in oil pricing and as the underlying commodity in the NYMEX oil future contracts.
   - Grain: S&P GPSCI index that measures the performance of the grain commodity market.
   - Metals: S&P GSCI Industrial Metals index that measures the movements of industrial metal prices including aluminium, copper, zinc, nickel and lead.

# Dataset

3. Currencies:
   - EUR/USD
   - GBP/USD
   - CHF/USD
   - JPY/USD
4. Bond indexes:
   - BBG Pan European: Bloomberg Barclays Pan-European Aggregate Index that tracks fixed-rate, investment-grade securities issued in different European currencies.
   - BBG Pan US: Bloomberg Barclays US Aggregate Bond Index, a benchmark that measures investment grade, US dollar-denominated, fixed-rate taxable bond market.
   - BBG Pan EurAgg: similar to the Pan European but only considers securities issued in euros.
5. CBOE VIX: index that reflects the market estimate of future volatility, based on a weighted average of implied volatility on a wide range of strikes.

# Histogram of calibrated values



# Parameter Results

| | btc | bric | sp500 | eurostoxx | nasdaq | gold | wti |
|---|---|---|---|---|---|---|---|
| $\mu$ | 1.497 | 0.012 | 0.128 | 0.050 | 0.183 | 0.042 | -0.038 |
| $\theta$ | -0.100 | -0.100 | -0.100 | -0.100 | -0.100 | -0.100 | -0.100 |
| $\delta$ | 0.171 | 0.087 | 0.097 | 0.092 | 0.089 | 0.094 | 0.126 |
| $\lambda$ | 46.957 | 0.971 | 2.491 | 1.358 | 2.209 | 2.500 | 3.612 |

| | grain | metal | eur | gbp | chf | jpy |
|---|---|---|---|---|---|---|
| $\mu$ | -0.046 | -0.019 | -0.017 | -0.014 | 0.000 | -0.017 |
| $\theta$ | -0.100 | -0.100 | -0.100 | -0.100 | -0.100 | -0.100 |
| $\delta$ | 0.106 | 0.098 | 1.501 | 0.100 | 0.115 | 0.096 |
| $\lambda$ | 2.582 | 0.785 | 0.319 | 0.820 | 1.815 | 1.982 |

| | bond_europe | bond_us | bond_eur | vix |
|---|---|---|---|---|
| $\mu$ | 0.042 | 0.024 | 0.036 | -0.480 |
| $\theta$ | -0.100 | -0.100 | -0.100 | -0.100 |
| $\delta$ | 0.091 | 0.099 | 0.099 | 0.236 |
| $\lambda$ | 0.611 | 0.572 | 1.002 | 10.635 |

# Mean Return Comparison

| Asset | btc | bric | sp500 | eurostoxx | nasdaq | gold |
|---|---|---|---|---|---|---|
| Sample | 1.331 | -0.010 | 0.111 | 0.024 | 0.143 | 0.004 |
| Model | 1.497 | 0.012 | 0.128 | 0.050 | 0.183 | 0.042 |

| Asset | wti | grain | metal | eur | gbp | chf |
|---|---|---|---|---|---|---|
| Sample | -0.022 | -0.062 | -0.028 | -0.013 | -0.018 | 0.006 |
| Model | -0.038 | -0.046 | -0.019 | -0.017 | -0.014 | 0.000 |

| Asset | jpy | bond_europe | bond_us | bond_eur | vix |
|---|---|---|---|---|---|
| Sample | -0.030 | 0.036 | 0.024 | 0.035 | -0.034 |
| Model | -0.017 | 0.042 | 0.024 | 0.036 | -0.480 |

# Correlation Comparison

| | bric | sp500 | eurostoxx | nasdaq | gold |
|---|---|---|---|---|---|
| Sample | -0.0134 | -0.0123 | 0.0320 | -0.0119 | -0.0063 |
| Model | 0.0141 | 0.0437 | 0.0368 | 0.0359 | -0.0024 |

| | wti | grain | metal | eur | gbp | chf |
|---|---|---|---|---|---|---|
| Sample | 0.0232 | 0.0234 | 0.0072 | 0.0268 | 0.0182 | 0.0210 |
| Model | 0.0077 | 0.0350 | 0.0269 | 0.0229 | 0.0073 | 0.0246 |

| | jpy | bond_europe | bond_us | bond_eur | vix |
|---|---|---|---|---|---|
| Sample | -0.0123 | -0.0132 | -0.0216 | -0.0075 | 0.0182 |
| Model | -0.0109 | -0.0238 | -0.0184 | -0.0098 | -0.0509 |

# Correlation Comparison

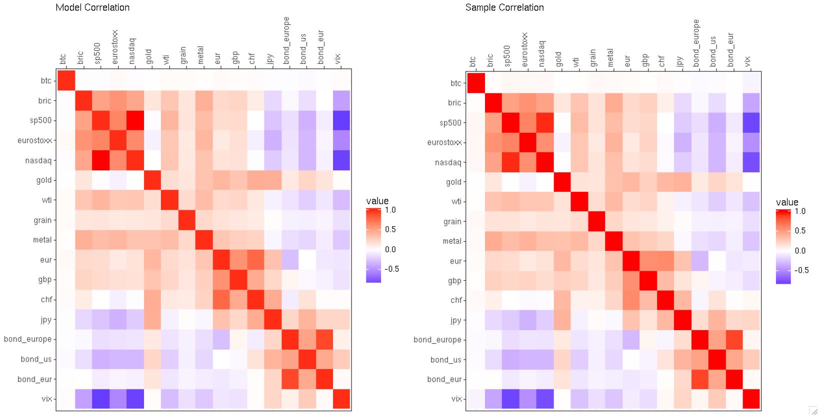



Figure 1: The calibrated model is able to reproduce the correlation structure in the dataset very closely.

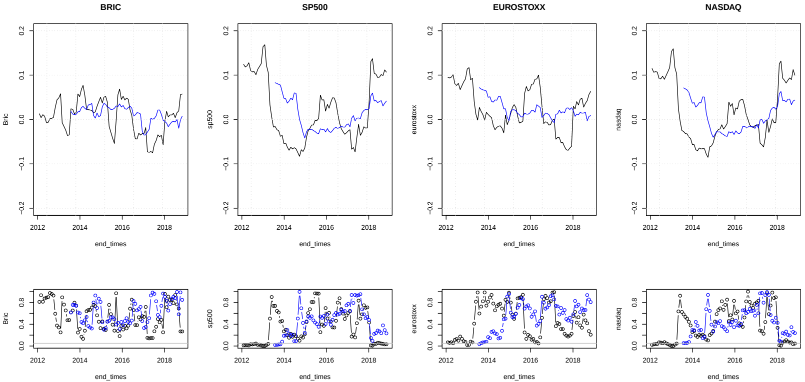
# Correlation Significance against Bitcoin

| | bric | sp500 | eurostoxx | nasdaq | gold | wti |
|---|---|---|---|---|---|---|
| Correlation | 0.0141 | 0.0437 | 0.0368 | 0.0359 | -0.0024 | 0.0077 |
| Pearson | 0.504 | 0.044 | 0.094 | 0.089 | 0.905 | 0.719 |
| Permutation | 0.512 | 0.042 | 0.087 | 0.095 | 0.911 | 0.720 |
| Spearman | 0.922 | 0.459 | 0.157 | 0.329 | 0.423 | 0.459 |

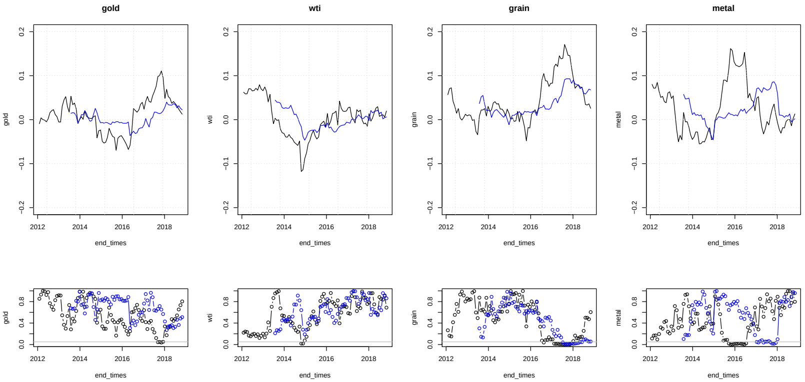
| | grain | metal | eur | gbp | chf | jpy |
|---|---|---|---|---|---|---|
| Correlation | 0.0350 | 0.0269 | 0.0229 | 0.0073 | 0.0246 | -0.0109 |
| Pearson | 0.108 | 0.195 | 0.291 | 0.747 | 0.239 | 0.608 |
| Permutation | 0.103 | 0.211 | 0.286 | 0.733 | 0.253 | 0.613 |
| Spearman | 0.170 | 0.798 | 0.453 | 0.697 | 0.905 | 0.474 |

| | bond_europe | bond_us | bond_eur | vix |
|---|---|---|---|---|
| Correlation | -0.0238 | -0.0184 | -0.0098 | -0.0509 |
| Pearson | 0.285 | 0.408 | 0.642 | 0.018 |
| Permutation | 0.268 | 0.393 | 0.649 | 0.018 |
| Spearman | 0.468 | 0.336 | 0.816 | 0.330 |

# Rolling Correlation



# Rolling Correlation



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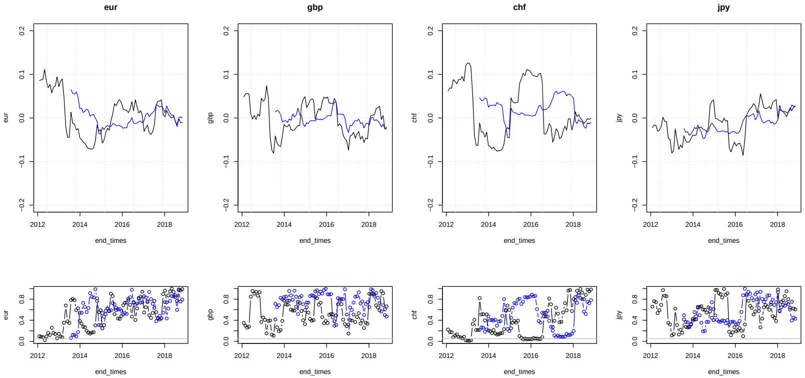
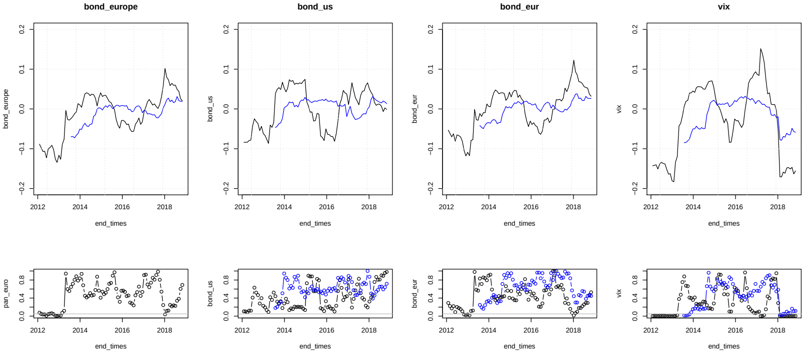


# Rolling Correlation



# Efficient Frontier and Optimal Allocation

The two approaches considered both involve the minimization of a *risk measure* given a level of target returns:

- *Markowitz Mean-Variance optimization*: minimize the portfolio variance (or equivalently the volatility) as originally proposed in [2]. This approach has explicit formulas for the solution in the unconstrained case.
- *CVaR optimization*: minimize the portfolio *conditional VaR* rather than the simple VaR due to the mathematical properties of the former (CVaR is a coherent risk measure). Numerical algorithms are required.

# Efficient Frontier and Optimal Allocation

$$\begin{aligned} \min_{\mathbf{w}\in\mathbb{R}^n} \quad & PtfRisk(\mathbf{w}) \\ \text{subject to} \quad & \mathbf{e}'\mathbf{w} = 1, \\ & \mathbf{r}'\mathbf{w} = r_{target}, \\ & w_i \geq 0, \text{for } i = 1 \dots n. \end{aligned}$$

where:

- $\mathbf{w}$ is the vector of weights,
- $\mathbf{r}$ is the vector of returns,
- $\mathbf{e}$ indicates a vector of ones,
- $r_{target}$ is the target portfolio return,
- the last constraint is included only when shortselling is not allowed.

In the Markowitz case the risk measure is the portfolio variance $\mathbf{w}'\Sigma\mathbf{w}$, in the CVaR case is the portfolio CVaR at level 95%.

# Markowitz Efficient Frontier

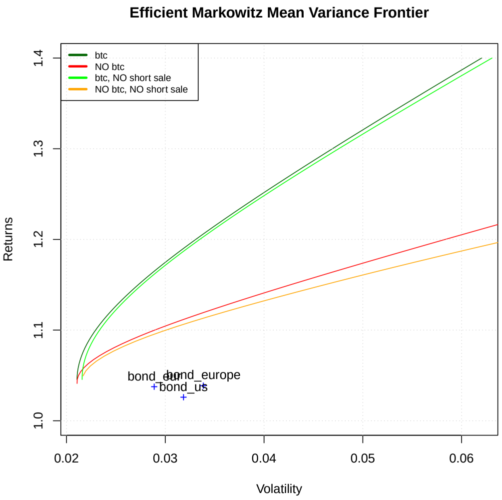



Figure 2: Efficient frontier for the sample percentage returns We can see how including Bitcoin effectively diversifies the portfolio.

# Markowitz allocation including Bitcoin

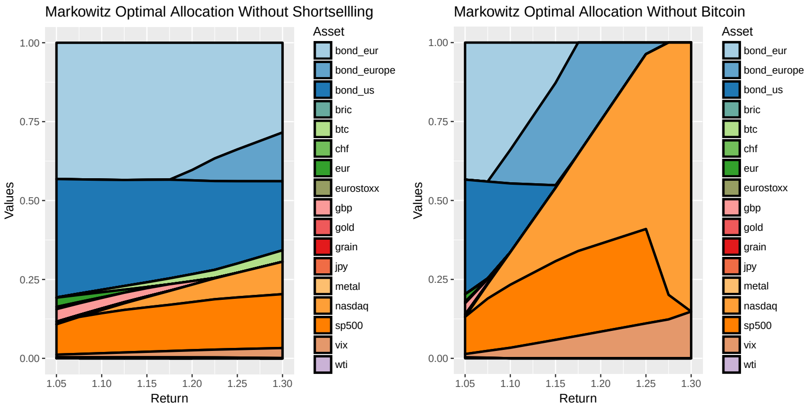


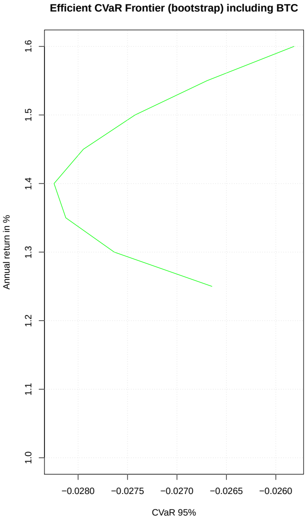

Figure 3: We can see in green that the allocation in Bitcoin is about 3% and increases with the target return.

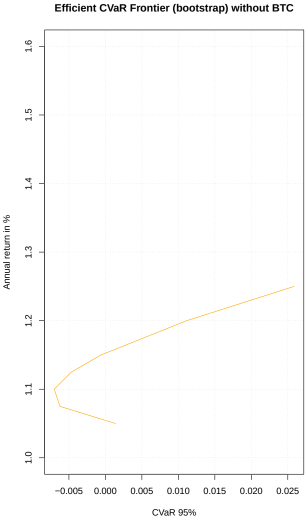
# CVaR Efficient Frontier

Since we only have the empirical distribution of the returns, we have to compute the CVaR empirically as well. We considered two approaches:

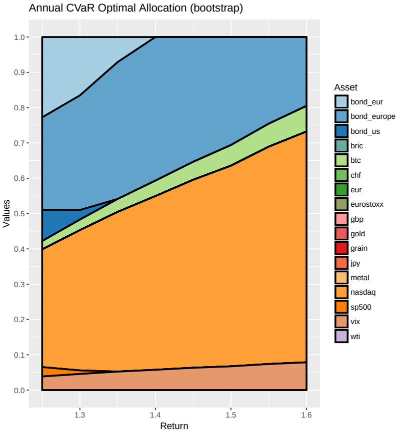
1. *Annual returns simulation*: given the set of all available daily returns, extract 255 of them to simulate one annual return scenario. Repeat to get a number of simulations (in our case 10,000). We will refer to this method as annual bootstrap.
2. *Daily return as scenarios*: consider the latest $N = 255 * 5 = 1275$ daily return and use them as the different scenario realizations.

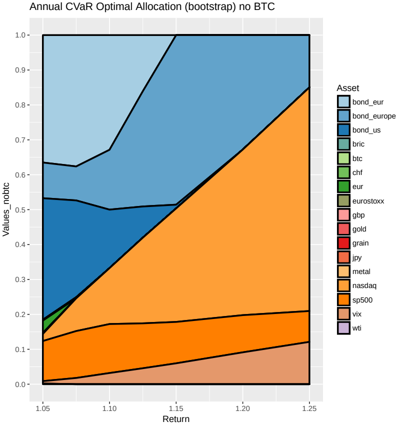
# CVaR Efficient Frontier (annual bootstrap)



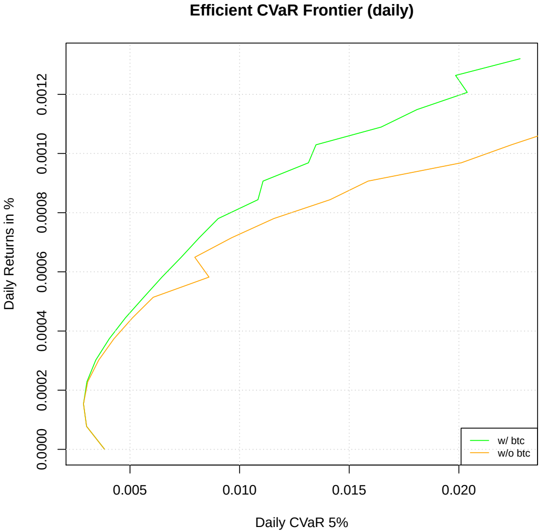



# Optimal Allocation (annual bootstrap)

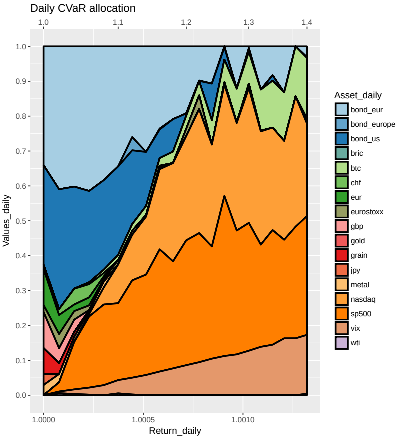


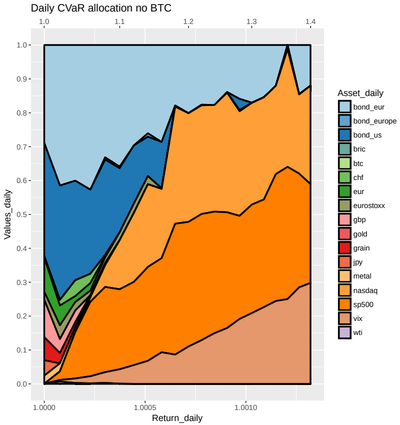



# CVaR Efficient Frontier (daily returns)



# Optimal Allocation (daily returns)



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