Numerical Results

LU with Simple Gaussian Elimination

Stage 0.

$$\left[\begin{array}{rrrr|r} 4.000000 & -1.000000 & 0.000000 & 3.000000 & 1.000000 \\ 1.000000 & 15.500000 & 3.000000 & 8.000000 & 1.000000 \\ 0.000000 & -1.300000 & -4.000000 & 1.100000 & 1.000000 \\ 14.000000 & 5.000000 & -2.000000 & 30.000000 & 1.000000 \end{array}\right]$$

Stage 1.

$$\left[\begin{array}{rrrr|r} 4.000000 & -1.000000 & 0.000000 & 3.000000 & 1.000000 \\ 0.000000 & 15.750000 & 3.000000 & 7.250000 & 0.750000 \\ 0.000000 & -1.300000 & -4.000000 & 1.100000 & 1.000000 \\ 0.000000 & 8.500000 & -2.000000 & 19.500000 & -2.500000 \end{array}\right]$$

$$L = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.250000 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 3.500000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}, \quad U = \begin{bmatrix} 4.000000 & -1.000000 & 0.000000 & 3.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}$$

Stage 2.

$$\left[\begin{array}{rrrr|r} 4.000000 & -1.000000 & 0.000000 & 3.000000 & 1.000000 \\ 0.000000 & 15.750000 & 3.000000 & 7.250000 & 0.750000 \\ 0.000000 & 0.000000 & -3.752381 & 1.698413 & 1.061905 \\ 0.000000 & 0.000000 & -3.619048 & 15.587302 & -2.904762 \end{array}\right]$$

$$L = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.250000 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & -0.082540 & 1.000000 & 0.000000 \\ 3.500000 & 0.539683 & 0.000000 & 1.000000 \end{bmatrix}, \quad U = \begin{bmatrix} 4.000000 & -1.000000 & 0.000000 & 3.000000 \\ 0.000000 & 15.750000 & 3.000000 & 7.250000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}$$

Stage 3.

$$\left[\begin{array}{rrrr|r} 4.000000 & -1.000000 & 0.000000 & 3.000000 & 1.000000 \\ 0.000000 & 15.750000 & 3.000000 & 7.250000 & 0.750000 \\ 0.000000 & 0.000000 & -3.752381 & 1.698413 & 1.061905 \\ 0.000000 & 0.000000 & 0.000000 & 13.949239 & -3.928934 \end{array}\right]$$

$$L = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.250000 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & -0.082540 & 1.000000 & 0.000000 \\ 3.500000 & 0.539683 & 0.964467 & 1.000000 \end{bmatrix}, \quad U = \begin{bmatrix} 4.000000 & -1.000000 & 0.000000 & 3.000000 \\ 0.000000 & 15.750000 & 3.000000 & 7.250000 \\ 0.000000 & 0.000000 & -3.752381 & 1.698413 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}$$

Solution via forward/backward substitution.

$$x = \begin{bmatrix} 0.525109 \\ 0.255459 \\ -0.410480 \\ -0.281659 \end{bmatrix}$$

LU with Partial Pivoting

Stage 0.

$$\left[\begin{array}{rrrr|r} 4.000000 & -1.000000 & 0.000000 & 3.000000 & 1.000000 \\ 1.000000 & 15.500000 & 3.000000 & 8.000000 & 1.000000 \\ 0.000000 & -1.300000 & -4.000000 & 1.100000 & 1.000000 \\ 14.000000 & 5.000000 & -2.000000 & 30.000000 & 1.000000 \end{array}\right]$$

Stage 1.

$$\left[\begin{array}{rrrr|r} 14.000000 & 5.000000 & -2.000000 & 30.000000 & 1.000000 \\ 0.000000 & 15.142857 & 3.142857 & 5.857143 & 0.928571 \\ 0.000000 & -1.300000 & -4.000000 & 1.100000 & 1.000000 \\ 0.000000 & -2.428571 & 0.571429 & -5.571429 & 0.714286 \end{array}\right]$$

$$L = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.071429 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 0.285714 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}, \ U = \begin{bmatrix} 14.000000 & 5.000000 & -2.000000 & 30.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}, \ P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Stage 2.

$$\left[\begin{array}{rrrr|r} 14.000000 & 5.000000 & -2.000000 & 30.000000 & 1.000000 \\ 0.000000 & 15.142857 & 3.142857 & 5.857143 & 0.928571 \\ 0.000000 & 0.000000 & -3.730189 & 1.602830 & 1.079717 \\ 0.000000 & 0.000000 & 1.075472 & -4.632075 & 0.863208 \end{array}\right]$$

$$L = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.071429 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & -0.085849 & 1.000000 & 0.000000 \\ 0.285714 & -0.160377 & 0.000000 & 1.000000 \end{bmatrix}, \ U = \begin{bmatrix} 14.000000 & 5.000000 & -2.000000 & 30.000000 \\ 0.000000 & 15.142857 & 3.142857 & 5.857143 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}, \ P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Stage 3.

$$\left[\begin{array}{rrrr|r} 14.000000 & 5.000000 & -2.000000 & 30.000000 & 1.000000 \\ 0.000000 & 15.142857 & 3.142857 & 5.857143 & 0.928571 \\ 0.000000 & 0.000000 & -3.730189 & 1.602830 & 1.079717 \\ 0.000000 & 0.000000 & 0.000000 & -4.169954 & 1.174507 \end{array}\right]$$

$$L = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.071429 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & -0.085849 & 1.000000 & 0.000000 \\ 0.285714 & -0.160377 & -0.288316 & 1.000000 \end{bmatrix}, \ U = \begin{bmatrix} 14.000000 & 5.000000 & -2.000000 & 30.000000 \\ 0.000000 & 15.142857 & 3.142857 & 5.857143 \\ 0.000000 & 0.000000 & -3.730189 & 1.602830 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}, \ P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Solution.

$$x = \begin{bmatrix} 0.525109 \\ 0.255459 \\ -0.410480 \\ -0.281659 \end{bmatrix}$$

Crout

Stage 0 and Stage 1 (augmented matrix).

$$\left[\begin{array}{cccc|c} 4.000000 & -1.000000 & 0.000000 & 3.000000 & 1.000000 \\ 1.000000 & 15.500000 & 3.000000 & 8.000000 & 1.000000 \\ 0.000000 & -1.300000 & -4.000000 & 1.100000 & 1.000000 \\ 14.000000 & 5.000000 & -2.000000 & 30.000000 & 1.000000 \end{array}\right]$$

$$L = \begin{bmatrix} 4.000000 & 0.000000 & 0.000000 & 0.000000 \\ 1.000000 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 14.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}, \quad U = \begin{bmatrix} 1.000000 & -0.250000 & 0.000000 & 0.750000 \\ 0.000000 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}$$

Stage 2.

$$L = \begin{bmatrix} 4.000000 & 0.000000 & 0.000000 & 0.000000 \\ 1.000000 & 15.750000 & 0.000000 & 0.000000 \\ 0.000000 & -1.300000 & 1.000000 & 0.000000 \\ 14.000000 & 8.500000 & 0.000000 & 1.000000 \end{bmatrix}, \quad U = \begin{bmatrix} 1.000000 & -0.250000 & 0.000000 & 0.750000 \\ 0.000000 & 1.000000 & 0.190476 & 0.460317 \\ 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}$$

Stage 3.

$$L = \begin{bmatrix} 4.000000 & 0.000000 & 0.000000 & 0.000000 \\ 1.000000 & 15.750000 & 0.000000 & 0.000000 \\ 0.000000 & -1.300000 & -3.752381 & 0.000000 \\ 14.000000 & 8.500000 & -3.619048 & 1.000000 \end{bmatrix}, \quad U = \begin{bmatrix} 1.000000 & -0.250000 & 0.000000 & 0.750000 \\ 0.000000 & 1.000000 & 0.190476 & 0.460317 \\ 0.000000 & 0.000000 & 1.000000 & -0.452623 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}$$

Stage 4.

$$L = \begin{bmatrix} 4.000000 & 0.000000 & 0.000000 & 0.000000 \\ 1.000000 & 15.750000 & 0.000000 & 0.000000 \\ 0.000000 & -1.300000 & -3.752381 & 0.000000 \\ 14.000000 & 8.500000 & -3.619048 & 13.949239 \end{bmatrix}, \quad U = \begin{bmatrix} 1.000000 & -0.250000 & 0.000000 & 0.750000 \\ 0.000000 & 1.000000 & 0.190476 & 0.460317 \\ 0.000000 & 0.000000 & 1.000000 & -0.452623 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}$$

Solution.

$$x = \begin{bmatrix} 0.525109 \\ 0.255459 \\ -0.410480 \\ -0.281659 \end{bmatrix}$$

Doolittle

Stage 0–1.

$$\left[\begin{array}{cccc|c} 4.000000 & -1.000000 & 0.000000 & 3.000000 & 1.000000 \\ 1.000000 & 15.500000 & 3.000000 & 8.000000 & 1.000000 \\ 0.000000 & -1.300000 & -4.000000 & 1.100000 & 1.000000 \\ 14.000000 & 5.000000 & -2.000000 & 30.000000 & 1.000000 \end{array}\right], \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3.5 & 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Stage 2.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0 & -0.082540 & 1 & 0 \\ 3.5 & 0.539683 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & -1 & 0 & 3 \\ 0 & 15.75 & 3 & 7.25 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Stage 3.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0 & -0.082540 & 1 & 0 \\ 3.5 & 0.539683 & 0.964467 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & -1 & 0 & 3 \\ 0 & 15.75 & 3 & 7.25 \\ 0 & 0 & -3.752381 & 1.698413 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Stage 4.

$$U = \begin{bmatrix} 4 & -1 & 0 & 3 \\ 0 & 15.75 & 3 & 7.25 \\ 0 & 0 & -3.752381 & 1.698413 \\ 0 & 0 & 0 & 13.949239 \end{bmatrix}$$

$$x = \begin{bmatrix} 0.525109 \\ 0.255459 \\ -0.410480 \\ -0.281659 \end{bmatrix}$$

Cholesky

Error: Matrix is not positive definite (negative value -3.7523809523809524 at position $(2, 2)$).

Jacobi

$$T = \begin{bmatrix} 0.000000 & 0.250000 & 0.000000 & -0.750000 \\ -0.064516 & 0.000000 & -0.193548 & -0.516129 \\ 0.000000 & -0.325000 & 0.000000 & 0.275000 \\ -0.466667 & -0.166667 & 0.066667 & 0.000000 \end{bmatrix}, \quad C = \begin{bmatrix} 0.250000 \\ 0.064516 \\ -0.250000 \\ 0.033333 \end{bmatrix}$$

spectral radius $\rho(T) = 0.753517$

Table 1: Jacobi iterations ($x^{(k)}$).

#	$\lvert E\rvert$	x_1	x_2	x_3	x_4
0		0.250000	0.064516	-0.250000	0.033333
1	3.6e-1	0.250000	0.064516	-0.250000	-0.110753
2	1.5e-1	0.241129	0.079570	-0.261801	-0.110753
3	1.4e-1	0.352957	0.156793	-0.306317	-0.109909
4	7.5e-2	0.371630	0.157759	-0.331183	-0.177933

#	$\|E\|$	x_1	x_2	x_3	x_4
5	6.8e-2	0.422890	0.196476	-0.350203	-0.188466
6	4.0e-2	0.440469	0.202287	-0.365683	-0.220108
7	3.4e-2	0.465653	0.220480	-0.376273	-0.230312
8	2.2e-2	0.477854	0.226172	-0.384992	-0.245803
9	1.8e-2	0.490895	0.235067	-0.391102	-0.253027
10	1.3e-2	0.498537	0.239137	-0.395979	-0.261002
11	1.0e-2	0.505536	0.243705	-0.399495	-0.265572
12	7.3e-3	0.510105	0.246292	-0.402236	-0.269834
13	5.7e-3	0.513948	0.248727	-0.404249	-0.272580
14	4.2e-3	0.516617	0.250286	-0.405796	-0.274914
15	3.2e-3	0.518757	0.251618	-0.406944	-0.276522
16	2.4e-3	0.520296	0.252532	-0.407819	-0.277819
17	1.8e-3	0.521498	0.253272	-0.408473	-0.278748
18	1.4e-3	0.522379	0.253801	-0.408969	-0.279476
19	1.0e-3	0.523057	0.254215	-0.409341	-0.280008
20	7.7e-4	0.523560	0.254518	-0.409622	-0.280418
21	5.8e-4	0.523943	0.254752	-0.409834	-0.280723
22	4.4e-4	0.524230	0.254925	-0.409993	-0.280954
23	3.3e-4	0.524447	0.255057	-0.410113	-0.281128
24	2.5e-4	0.524610	0.255156	-0.410204	-0.281259
25	1.9e-4	0.524733	0.255231	-0.410272	-0.281358
26	1.4e-4	0.524826	0.255287	-0.410323	-0.281432
27	1.1e-4	0.524896	0.255329	-0.410362	-0.281488
28	8.0e-5	0.524948	0.255361	-0.410391	-0.281530
29	6.0e-5	0.524988	0.255385	-0.410413	-0.281562
30	4.6e-5	0.525018	0.255403	-0.410430	-0.281586
31	3.4e-5	0.525040	0.255417	-0.410442	-0.281604
32	2.6e-5	0.525057	0.255427	-0.410452	-0.281618
33	1.9e-5	0.525070	0.255435	-0.410459	-0.281628
34	1.5e-5	0.525080	0.255441	-0.410464	-0.281636
35	1.1e-5	0.525087	0.255445	-0.410468	-0.281642
36	8.3e-6	0.525092	0.255448	-0.410471	-0.281646
37	6.3e-6	0.525097	0.255451	-0.410473	-0.281649
38	4.7e-6	0.525100	0.255453	-0.410475	-0.281652
39	3.6e-6	0.525102	0.255454	-0.410476	-0.281654
40	2.7e-6	0.525104	0.255455	-0.410477	-0.281655
41	2.0e-6	0.525105	0.255456	-0.410478	-0.281656
42	1.5e-6	0.525106	0.255457	-0.410479	-0.281657
43	1.1e-6	0.525107	0.255457	-0.410479	-0.281658
44	8.7e-7	0.525107	0.255457	-0.410479	-0.281658
45	6.5e-7	0.525108	0.255458	-0.410480	-0.281658
46	4.9e-7	0.525108	0.255458	-0.410480	-0.281659
47	3.7e-7	0.525108	0.255458	-0.410480	-0.281659
48	2.8e-7	0.525109	0.255458	-0.410480	-0.281659
49	2.1e-7	0.525109	0.255458	-0.410480	-0.281659
50	1.6e-7	0.525109	0.255458	-0.410480	-0.281659

#	$\lvert E\rvert$	x_1	x_2	x_3	x_4
51	1.2e-7	0.525109	0.255458	-0.410480	-0.281659
52	9.0e-8	0.525109	0.255458	-0.410480	-0.281659

Gauss–Seidel

$$T = \begin{bmatrix} 0.000000 & 0.250000 & 0.000000 & -0.750000 \\ 0.000000 & -0.016129 & -0.193548 & -0.467742 \\ 0.000000 & 0.005242 & 0.062903 & 0.427016 \\ 0.000000 & -0.113629 & 0.036452 & 0.456425 \end{bmatrix}, \quad C = \begin{bmatrix} 0.250000 \\ 0.048387 \\ -0.265726 \\ -0.109113 \end{bmatrix}$$

$$\rho(T) = 0.599488$$

Table 2: Gauss–Seidel iterations ($x^{(k)}$).

#	$\lvert E\rvert$	x_1	x_2	x_3	x_4
0		0.250000	0.048387	-0.265726	-0.109113
1	3.8e-1	0.343931	0.150074	-0.328780	-0.174099
2	1.7e-1	0.418093	0.191035	-0.359964	-0.217613
3	1.0e-1	0.460969	0.216763	-0.380292	-0.243265
4	6.0e-2	0.486640	0.232281	-0.392389	-0.258638
5	3.6e-2	0.502049	0.241563	-0.399633	-0.267859
6	2.1e-2	0.511285	0.247128	-0.403978	-0.273386
7	1.3e-2	0.516822	0.250465	-0.406582	-0.276700
8	7.7e-3	0.520141	0.252465	-0.408143	-0.278686
9	4.6e-3	0.522131	0.253664	-0.409079	-0.279877
10	2.8e-3	0.523324	0.254383	-0.409640	-0.280591
11	1.7e-3	0.524039	0.254814	-0.409977	-0.281019
12	1.0e-3	0.524467	0.255072	-0.410179	-0.281275
13	6.0e-4	0.524724	0.255227	-0.410299	-0.281429
14	3.6e-4	0.524879	0.255320	-0.410372	-0.281521
15	2.1e-4	0.524971	0.255375	-0.410415	-0.281577
16	1.3e-4	0.525026	0.255409	-0.410441	-0.281610
17	7.7e-5	0.525059	0.255429	-0.410457	-0.281630
18	4.6e-5	0.525079	0.255441	-0.410466	-0.281642
19	2.8e-5	0.525091	0.255448	-0.410472	-0.281649
20	1.7e-5	0.525098	0.255452	-0.410475	-0.281653
21	1.0e-5	0.525103	0.255455	-0.410477	-0.281656
22	6.0e-6	0.525105	0.255456	-0.410479	-0.281657
23	3.6e-6	0.525107	0.255457	-0.410479	-0.281658
24	2.1e-6	0.525108	0.255458	-0.410480	-0.281659
25	1.3e-6	0.525108	0.255458	-0.410480	-0.281659
26	7.7e-7	0.525109	0.255458	-0.410480	-0.281659
27	4.6e-7	0.525109	0.255458	-0.410480	-0.281659
28	2.8e-7	0.525109	0.255458	-0.410480	-0.281659

#	$\lvert E\rvert$	x_1	x_2	x_3	x_4
29	1.7e-7	0.525109	0.255458	-0.410480	-0.281659
30	1.0e-7	0.525109	0.255458	-0.410480	-0.281659

SOR (relaxation)

$$T = \begin{bmatrix} -0.500000 & 0.375000 & 0.000000 & -1.125000 \\ 0.048387 & -0.536290 & -0.290323 & -0.665323 \\ -0.023589 & 0.261442 & -0.358468 & 0.736845 \\ 0.335544 & -0.102283 & 0.036734 & 0.527515 \end{bmatrix}, \quad C = \begin{bmatrix} 0.375000 \\ 0.060484 \\ -0.404486 \\ -0.268070 \end{bmatrix}$$

$$\rho(T) = 0.631208$$

Table 3: SOR iterations ($x^{(k)}$).

#	$\lvert E\rvert$	x_1	x_2	x_3	x_4
0		0.375000	0.060484	-0.404486	-0.268070
1	6.2e-1	0.511760	0.341976	-0.450049	-0.304696
2	3.2e-1	0.590144	0.235228	-0.390336	-0.308594
3	1.5e-1	0.515306	0.281526	-0.444371	-0.271236
4	1.1e-1	0.528060	0.243909	-0.383605	-0.283362
5	7.4e-2	0.521218	0.255125	-0.424458	-0.279399
6	4.3e-2	0.524387	0.258003	-0.403799	-0.282252
7	2.1e-2	0.527091	0.252514	-0.412630	-0.282229
8	1.1e-2	0.523655	0.258137	-0.410946	-0.281073
9	6.9e-3	0.526181	0.253697	-0.409146	-0.282129
10	5.5e-3	0.524441	0.256380	-0.411790	-0.281318
11	4.2e-3	0.525405	0.255085	-0.409503	-0.281846
12	2.8e-3	0.525031	0.255513	-0.411073	-0.281584
13	1.7e-3	0.525084	0.255547	-0.410197	-0.281673
14	8.8e-4	0.525171	0.255337	-0.410569	-0.281674
15	4.4e-4	0.525049	0.255562	-0.410493	-0.281637
16	2.7e-4	0.525153	0.255389	-0.410431	-0.281679
17	2.2e-4	0.525083	0.255497	-0.410532	-0.281646
18	1.7e-4	0.525122	0.255443	-0.410441	-0.281667
19	1.1e-4	0.525106	0.255461	-0.410504	-0.281656
20	6.8e-5	0.525108	0.255462	-0.410469	-0.281660
21	3.6e-5	0.525111	0.255454	-0.410484	-0.281660
22	1.8e-5	0.525107	0.255463	-0.410481	-0.281659
23	1.1e-5	0.525111	0.255456	-0.410479	-0.281660
24	8.4e-6	0.525108	0.255460	-0.410482	-0.281659
25	6.6e-6	0.525110	0.255458	-0.410479	-0.281660
26	4.5e-6	0.525109	0.255459	-0.410481	-0.281659
27	2.7e-6	0.525109	0.255459	-0.410480	-0.281659

#	$\lvert E\rvert$	x_1	x_2	x_3	x_4
28	1.5e-6	0.525109	0.255459	-0.410480	-0.281659
29	7.2e-7	0.525109	0.255458	-0.410481	-0.281659

Interpolation: Vandermonde

Vandermonde matrix.

$$V = \begin{bmatrix} -1.000000 & 1.000000 & -1.000000 & 1.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \\ 27.000000 & 9.000000 & 3.000000 & 1.000000 \\ 64.000000 & 16.000000 & 4.000000 & 1.000000 \end{bmatrix}$$

Polynomial coefficients.

$$\mathbf{a} = \begin{bmatrix} -1.141667 \\ 5.825000 \\ -5.533333 \\ 3.000000 \end{bmatrix}$$

$$p(x) = -1.141667\,x^3 + 5.825000\,x^2 - 5.533333\,x + 3.000000$$

Interpolation: Newton (divided differences)

Divided differences table (upper triangular format).

	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	15.500000	0.000000	0.000000	0.000000
x_1	3.000000	-12.500000	0.000000	0.000000
x_2	8.000000	1.666667	3.541667	0.000000
x_3	1.000000	-7.000000	-2.166667	-1.141667

$$\text{Newton coefficients } (a_0, a_1, a_2, a_3) = \begin{bmatrix} 15.500000 \\ -12.500000 \\ 3.541667 \\ -1.141667 \end{bmatrix}$$

$$p_N(x) = 15.500000 - 12.500000(x + 1.000000) + 3.541667(x + 1.000000)(x - 0.000000) - 1.141667(x + 1.000000)(x - 0.00$$

Interpolation: Lagrange

Basis polynomials $L_i(x)$.

$$\begin{aligned}
L_0(x) &= -0.050000x^3 + 0.350000x^2 - 0.600000x,\\
L_1(x) &= 0.083333x^3 - 0.500000x^2 + 0.416667x + 1.000000,\\
L_2(x) &= -0.083333x^3 + 0.250000x^2 + 0.333333x,\\
L_3(x) &= 0.050000x^3 - 0.100000x^2 - 0.150000x.
\end{aligned}$$

$$p_L(x) = 15.5\,L_0(x) + 3.0\,L_1(x) + 8.0\,L_2(x) + 1.0\,L_3(x)$$

Linear splines

Coefficients (m_i, b_i) per interval $[x_i, x_{i+1}]$.

Interval	m_i	b_i
$[-1, 0]$	-12.500000	3.000000
	1.666667	3.000000
	-7.000000	29.000000

$$\begin{aligned}
S_0(x) &= -12.500000\,x + 3.000000,\\
S_1(x) &= 1.666667\,x + 3.000000,\\
S_2(x) &= -7.000000\,x + 29.000000.
\end{aligned}$$

Quadratic splines

Coefficients (a_i, b_i, c_i) of $a_i x^2 + b_i x + c_i$.

Segment	a_i	b_i	c_i
S_0	0.000000	-12.500000	3.000000
S_1	4.722222	-12.500000	3.000000
S_2	-22.833333	152.833333	-245.000000

$$\begin{aligned}
S_0(x) &= 0.000000x^2 - 12.500000x + 3.000000,\\
S_1(x) &= 4.722222x^2 - 12.500000x + 3.000000,\\
S_2(x) &= -22.833333x^2 + 152.833333x - 245.000000.
\end{aligned}$$

Natural cubic spline

Segment form: $S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$.

Segment	Interval	a_i	b_i	c_i	d_i
$S_0(x)$	$[-1.00, 0.00]$	15.500 000	−15.033 333	0.000 000	2.533 333
$S_1(x)$	$[0.00, 3.00]$	3.000 000	−7.433 333	7.600 000	−1.522 222
$S_2(x)$	$[3.00, 4.00]$	8.000 000	−2.933 333	−6.100 000	2.033 333

$$S_0(x) = 15.5000 - 15.0333\,(x - (-1)) + 0.0000\,(x - (-1))^2 + 2.5333\,(x - (-1))^3, \quad x \in [-1, 0],$$
$$S_1(x) = 3.0000 - 7.4333\,(x - 0) + 7.6000\,(x - 0)^2 - 1.5222\,(x - 0)^3, \quad x \in [0, 3],$$
$$S_2(x) = 8.0000 - 2.9333\,(x - 3) - 6.1000\,(x - 3)^2 + 2.0333\,(x - 3)^3, \quad x \in [3, 4].$$