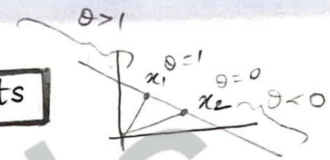
CONVEX SETS

Lines and Line segments

* $y = x_2 + \theta(x_1 - x_2)$



Affine sets

* A set $C \subseteq \mathbb{R}^n$ is affine, if line through any two distinct points in C lies in C.

Affine sets * $y = \theta x_1 + (1-\theta)x_2$ $\theta \in \mathbb{R}$ and $x_1, x_2 \in C$

Affine combination * $\theta_1 x_1 + \theta_2 x_2 + \dots \theta_k x_k \in C$; $\theta_1 + \theta_2 + \dots \theta_k = 1$

Subspace * $V = C - x_0 = \{x - x_0 \mid x \in C\}$; $x_0 \in C$ dim. of C = dim. of V.

$\{x \mid Ax = b\} \to$ sol^n set of sys. of LE is affine, subspace = nullspace of A

Affine hull * The set of all affine combination of points in C. 'aff C'. If S is any affine set with $C \subseteq S$, aff $C \subseteq S$.

Affine dim $\to$ dimension of affine hull

CONVEX SETS

* A set C is convex if the line segment b/w any two points in C lies in C, i.e. if for any $x_1, x_2 \in C$ and any θ with $0 \le \theta \le 1$, we have $\theta x_1 + (1-\theta)x_2 \in C$

Convex combination * $\theta_1 x_1 + \theta_2 x_2 + \dots \theta_k x_k$ $\theta_1 + \theta_2 + \dots \theta_k = 1$; $\theta_i \ge 0$

extended to probability

Convex hull 'conv C' – the smallest convex set that contains C. the set of all convex combination of points in C.

CONES

* A set C is called cone / non-negative homogeneous, if for every $x \in C$ and $\theta \ge 0$ we have $\theta x \in C$.

Convex cone – $\theta_1 x_1 + \theta_2 x_2 \in C$ $x_1, x_2 \in C$ & $\theta_1, \theta_2 \ge 0$.

set of PSD is convex cone

Conic combination * $\theta_1 x_1 + \theta_2 x_2 + \dots \theta_k x_k$ $\theta_1, \theta_2, \dots \theta_k \ge 0$.

Conic hull the set of all conic combinations of points in C. i.e. smallest convex cone that contains C.

DUAL CONES

* all vectors that have non-negative inner product with K.

* dual cone of subspace is its orthogonal complement

K-cone

$K^* = \{y \mid x^T y \ge 0 \; \forall x \in K\}$

K^* always convex and is a cone though K is not

SEPARATING & SUPPORTING HP

* if C is a convex set, then every boundary points of it contains a HP that clearly separates the convex set into 2 HS, such HP called supporting HP.

$x_0 \in \text{bd}\, C = \text{cl}\, C \setminus \text{int}\, C$

* Separating hyperplane theorem: $C \cap D = \phi$; $a^T x \le b$, $a^T x \ge b$

unique minimum S_1, x_1; S_2, x_2 not minimal unique

GENERALIZED INEQUALITIES

ordering in $\mathbb{R}^n$ space

K – convex, closed, solid – non-empty interior, pointed

$K \subseteq \mathbb{R}^n$ **Proper cone**

$x \preceq_K y \iff y - x \in K$

– as cone is directional we use it for comparison

OPERATIONS THAT PRESERVE CONVEXITY

1) Intersection

Polyhedron – HS + HP

PSD – ∞HS $\{X \in S^n \mid z^T X z \ge 0\}$

2) Affine function $f: \mathbb{R}^n \to \mathbb{R}^m$ (sum of a linear func. & constant)

$f(x) = Ax + b$. ; if $S \subseteq \mathbb{R}^n$ is convex then image of f under S ($f(S)$) is also convex

* scaling, translation.

3) Perspective func^n. $P: \mathbb{R}^{n+1} \to \mathbb{R}$ last component is one

$P(z, t) = z/t$

4) Linear fractional fun^n $f = P \circ a$

Projective f^n $\to$ prob. simplex

SIMPLEX

k-dimensional simplex in $\mathbb{R}^n$

$v_0, v_1, \dots v_k \in \mathbb{R}^n$ points affinely independent $\to v_1 - v_0, v_2 - v_0, \dots v_k - v_0$ is linearly independent

$C = \text{conv}\{v_0, v_1, \dots v_k\}$

The affine dimension of this (k+1) affinely indep points is k.

* By introducing simplex, we were able to [illegible] a dimension to convex set.

prob. simplex: (n-1) dim. simplex by unit vectors $e_1, e_2 \dots e_n$

each vectors correspond to a prob. distribution with n elements. $\to$ set of vectors $x \ge 0$, $1^T x = 1$.

unit simplex – n dimensional simplex determined by unit vectors $e_1, e_2 \dots e_n$ and 0.

1 dim simplex – linesegment

2 – triangle

3 – tetrahedron

NORM CONE

$C = \{(x, t) \mid \|x\| \le t\} \subseteq \mathbb{R}^{n+1}$

* when $\|\;\|_2 \Rightarrow$ 2nd order cone $\Rightarrow$ quadratic cone $\Rightarrow$ Lorentz cone $\Rightarrow$ ice-cream cone

POLYHEDRA

* solution set of finite linear inequalities & equalities

$P = \{x \mid a_j^T x \le b_j,\; C_j^T x = b_j\}$ j = 1 to m, j = 1 to p

intersection of finite no. of halfspaces and hyperplanes

* polyhedron – convex set

* bounded " – polytope

non-negative orthant is a polyhedral cone

HYPERPLANES AND HALFSPACES

$\{x \mid a^T x = b\}$, $a \in \mathbb{R}^n$, $a \ne 0$ and $b \in \mathbb{R}$

normal vector; constant offset from origin

* hyperplane contains offset x_0, plus all vectors orthogonal to the normal vector a.

1 HS, 1 HP, 1 HS

* closed HS $\{x \mid a^T x \le b\}$

x_0 plus any vector that makes an obtuse angle with normal vector a.

$> 90°$

* open HS $\{x \mid a^T x < b\}$