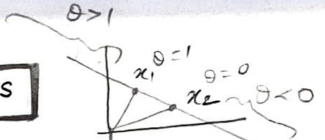
CONVEX SETS

Lines and Line segments



* $\boxed{y = x_2 + \theta(x_1 - x_2)}$

Affine sets

* A set $C \subseteq \mathbb{R}^n$ is affine, if line through any two distinct points in C lies in C.

Affine sets * $\boxed{y = \theta x_1 + (1-\theta) x_2}$ $\theta \in \mathbb{R}$ and $x_1, x_2 \in C$

Affine combinat^n * $\boxed{\theta_1 x_1 + \theta_2 x_2 + \dots \theta_k x_k}$ $\in C$; $\theta_1 + \theta_2 + \dots \theta_k = 1$

Subspace * $\boxed{V = C - x_0 = \{x - x_0 \mid x \in C\}}$; $x_0 \in C$
dim. of C = dim. of V.

$\{x \mid Ax = b\} \rightarrow$ sol^n set of sys. of LE is affine, subspace = nullspace of A.

Affine hull * The set of all affine combination of points in C.
'aff c'. If S is any affine set with $C \subseteq S$, aff $C \subseteq S$.

Affine dim → dimension of affine hull

CONVEX SETS

* A set C is convex if the linesegment b/w any two points in C lies in C, i.e. if for any $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have

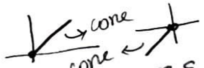
$\boxed{\theta x_1 + (1-\theta) x_2 \in C}$

Convex combination * $\boxed{\theta_1 x_1 + \theta_2 x_2 + \dots \theta_k x_k}$ $\theta_1 + \theta_2 + \dots \theta_k = 1$; $\theta_i \geq 0$.

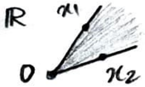
extended to probability

Convex hull 'conv C' – the smallest convex set that contains C.
the set of all convex combination of points in C.

CONES



* A set C is called cone /non-negative homogeneous, if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$.



Convex cone – $\boxed{\theta_1 x_1 + \theta_2 x_2 \in C}$ $x_1, x_2 \in C$ & $\theta_1, \theta_2 \geq 0$.
>set of PSD is convex cone

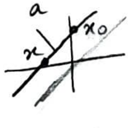
Conic combination * $\boxed{\theta_1 x_1 + \theta_2 x_2 + \dots \theta_k x_k}$ $\theta_1, \theta_2, \dots \theta_k \geq 0$.

Conic hull the set of all conic combinations of points in C.
i.e. smallest convex cone that contains C.

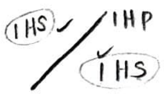
HYPERPLANES AND HALFSPACES

$\boxed{\{x \mid a^T x = b\}}$, $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$

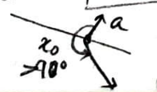
normal vector (a) – constant offset from origin (b)



* hyperplane contains offset x_0, plus all vectors orthogonal to the normal vector a.



* closed HS $\boxed{\{x \mid a^T x \leq b\}}$



x_0 plus any vector that makes an obtuse angle with normal vector a.

× open HS $\boxed{\{x \mid a^T x < b\}}$

NORM CONE

$\boxed{C = \{(x,t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}}$

* when $\|\,\|_2$ ⇒ 2nd order cone
⇒ quadratic cone ⇒ Lorentz cone
⇒ ice-cream cone

POLYHEDRA

* solution set of finite linear inequalities & equalities

$\boxed{P = \{x \mid \overset{\rightarrow j = 1 \text{ to } m}{a_j^T x \leq b_j}, \overset{\rightarrow j = 1 \text{ to } p}{c_j^T x = b_j}\}}$

intersection of finite no. of halfspaces and hyperplanes
*polyhedron – convex set
*bounded " – polytope

non-negative orthant is a polyhedral cone

SIMPLEX

k-dimensional simplex in $\mathbb{R}^n$

$\boxed{C = \text{conv}\{v_0, v_1, \dots v_k\}}$

$v_0, v_1, \dots v_k \in \mathbb{R}^n$. points affinely independent
⇒ $v_1 - v_0, v_2 - v_0, \dots v_k - v_0$ is linearly independent.

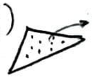
The affine dimension of this (k+1) affinely indep. points is k.

* By introducing simplex, we were able to [illegible] a dimension to convex set.

prob. simplex: (n-1) dim. simplex by unit vectors $e_1, e_2 \dots e_n$

→ unit simplex – n dimensional simplex determined by unit vectors $e_1, e_2 \dots e_n$ and 0.

→ 1 dim simplex – linesegment
2 – triangle
3 - tetrahedron



each vectors corresspond to a prob. distribution with n elements.
⇒set of vectors $\boxed{x \geq 0, 1^T x = 1.}$

OPERATIONS THAT PRESERVE CONVEXITY

1) **Intersection**
Polyhedron – HS + HP
PSD – ∞HS $\{X \in S^n \mid z^T X z \geq 0\}$

2) **Affine function** $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (sum of a linear func. & constant)
$f(x) = Ax + b$. ; if $S \subseteq \mathbb{R}^n$ is convex then image of f under S ($f(S)$) is also convex
* scaling, translation.

3) **Perspective func^n**. $P: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$
last component is one.
$P(z, t) = z/t$

4) **Linear fractional fun^n** / Projective f^n
$f = P \circ a$

GENERALIZED INEQUALITIES

ordering in $\mathbb{R}^n$ space

$x \preceq_K y \Leftrightarrow y - x \in K$

– as cone is directional we use it for comparison

K – convex, closed, solid - non-empty interior, pointed

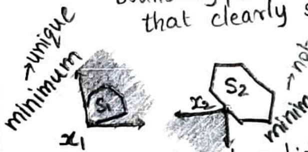
$\boxed{K \subseteq \mathbb{R}^n \text{ Proper cone}}$

SEPARATING & SUPPORTING HP

* Separating hyperplane theorem:
$C \cap D = \phi$
$a^T x \leq b$; $a^T x \geq b$

* if C is a convex set, then every boundary points of it contains a HP that clearly separates the convex set into 2 HS, such HP called supporting HP.

$x_0 \in \text{bd } C = \text{cl } C \setminus \text{int } C$



DUAL CONES

* all vectors that have non-negative inner product with K.

K-cone
$K^* = \{y \mid x^T y \geq 0 \ \forall x \in K\}$

K* always convex and is a cone though K is not

* dual cone of subspace is its orthogonal complement