Determining Conway Numbers Representing Various Structures in the Hackenbush Game to Quantify Player Advantages

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Abstract

In the combinatorial game Black and White Hackenbush first introduced by Conway in *On Numbers and Games*, two players, Left and Right, delete black and white edges, respectively, in grounded structures, until one player is unable to make a move. No general theorem is known to date for determining whether a player has a winning strategy in an arbitrary Hackenbush configuration, but certain simplifying principles, such as a recursive algorithm for branched trees, are known. In this paper, we use the theory of Conway numbers to analyze two Hackenbush structures: strings and loops. The Conway number representing these structures provides a quantitative measure of the advantages conferred to a player in a game of Black and White Hackenbush. Through induction and the properties of the Conway numbers, we derive an explicit formula for the value of a Hackenbush string, and through local analysis of the structure, determine the value of a Hackenbush loop in terms of its component strings. We thus extend the understanding of a game known for its computational complexity, and present techniques applicable to other combinatorial games and fundamental problems in computation theory.

1 Overview

Combinatorial game theory is an active field replete with open problems bearing applications in complexity theory, theoretical computer science, and artificial intelligence. The branch restricts itself to the analysis of two-player, sequential movement games with complete information and no elements of chance. The first major success in the field occurred in 1902, when Charles Bouton provided a complete solution for nim, an impartial combinatorial game, or one in which an identical set of moves is available to both players [3]. In the 1930s, Sprague and Grundy independently proved a theorem that demonstrated that *all* impartial games are equivalent to nim heaps [4, 5]. Their discovery sparked much interest in this branch of mathematics, as it suggested that major unifications of seemingly independent game theories were possible.

The other major class of combinatorial games, partisan games, involves different sets of moves for the two players, and was jointly introduced by Berlekamp, Conway, and Guy in the 1970s. They expanded the study of impartial games in their text *Winning Ways for Your Mathematical Plays*, which contained extensive analysis of various games of their own invention [2]. Conway's acclimated treatise, *On Numbers and Games*, further advanced the field by introducing a new class of numbers to analyze positions and player options in combinatorial games [1]. He used the endgame, represented as $\{ \mid \}$, in which neither player has an available move, as the inductive base for his number system, and assigned values to games based on the potential moves available to players "Left" and "Right."

In this paper, we investigate the two-player, partisan game Black and White Hackenbush introduced by Conway, Berlekamp, and Guy. In the game, two players, "Left" and "Right," delete black and white edges respectively from grounded structures consisting of nodes and rods. Deletion of an edge that destroys the path of another rod to the ground also destroys that edge, regardless of its color. We consider the standard variant of the game in which play ends when a player is unable to make a move and his or her opponent is declared the winner. We go beyond Berlekamp's recursive "Colon Principle," which applies to all Hackenbush structures but provides little specific information, by proving an explicit formula for the value of a Hackenbush string through induction. We then consider Hackenbush loops, proving a lemma regarding the best move options for players "Left" and "Right," and then using this intermediate fact to derive the Conway value of loops in general.

The ability to calculate the Conway numbers representing Hackenbush strings and loops constitutes a significant advance in the understanding of the game. This is because the values not only signify whether winning strategies exist for certain players, but provide an exact quantification of the advantage, in terms of number of moves, that a player possesses in a certain configuration. Knowing such precise information about the nature of a structure is highly powerful, especially in more complicated configurations involving the *sum* of several games. The approaches used in this paper rely broadly on the properties of Conway numbers, and are highly applicable to other combinatorial games.

2 Introduction to the Conway Numbers

In his 1976 book, *On Numbers and Games*, John Conway introduced a novel way to analyze combinatorial games, using a number system of his creation to describe the advantages of various positions in two-player, sequential movement games. In his system, a game X is represented as $\{X_L \mid X_R\}$, where X_L is the set of all possible game configurations after player "Left" has made a move in X, and X_R is the set of all configurations resulting from a move by player "Right" in X. Further, X_L and X_R are themselves games of the form $\{X_L \mid X_R\}$. A combinatorial game ends when a player can no longer make a move, in which case the set representing that player's options is empty, and the other player is declared the winner [1].

Conway assigned the number 0 to the game $\{ \mid \}$, in which neither player has a potential move. This game is stated to arise in the zeroth "generation," as it is constructed only from the empty set. In the first generation, the Conway games $\{0 \mid \}$, $\{ \mid 0\}$, and $\{0 \mid 0\}$ are constructed using the game 0 from the zeroth generation and the empty set. The game $\{0 \mid \}$ is a win for Left since Right has no moves regardless of who goes first, while Left can move to the game 0, or $\{ \mid \}$, and end the game.

The game $\{0 \mid \}$ is assigned the value 1, the motivation for which lies in Conway's comparison rule for games [1; p. 4]. In an analogous manner, the game $\{ \mid 0\}$, which is a win for Right, is assigned the value -1. In general, games of *positive* value are wins for Left, while games of *negative* value are wins for Right. Games such as $\{0 \mid 0\}$ are deemed *incomparable* to 0 and are first player wins, since regardless of whether it is Left or Right's move, each can move to 0 and end the game in their favor. Games equal to 0, such as $\{ \mid \}$, are second player wins, since the first player faces a game in which no moves are possible and thus automatically loses.

All games are constructed inductively from games defined in previous generations. For example, in the second generation, games such as $\{0 \mid 1\}$, $\{1 \mid \}$, and $\{ \mid -1\}$ arise. The motivation for the labels assigned to these games lies in Conway's definition of the *sum* of two games.

In a combinatorial game, a player faced by the sum of two games can make a move in the first game or in the second game, but not in both. The game ends when a player has no options in any game. Then, the sum of two games $G = \{G_L \mid G_R\}$ and $H = \{H_L \mid H_R\}$ is:

$$G + H = \{G_L + H, G + H_L \mid G_R + H, G + H_R\}$$

In this inductive definition for addition, the options for Left in $G + H$ consist of the Left options in G combined with the game H, along with the Left options in H combined with the game G, since Left must make a move in only one of G or H.

By this definition, $\{\,|\,\}$ plus the game $\{\,|\,\}$ is in fact $\{\,|\,\}$ itself, since if Left and Right have no options in the component games, Left and Right will have no options in the sum of the two. Consider the sum $G + 0$, where $G = \{G_L|\, G_R\}$ and $0 = \{\,|\,\}$. Using the definition of addition, we construct the left and right sets of $G + 0$, observing that since it is not possible to make a move in 0, the options $G + 0_L$ and $G + 0_R$ do not exist. Then, $G + 0 = \{G_L + 0 \mid G_R + 0\}$. If we assume as our inductive hypothesis that $G_L + 0$ is in fact G_L and $G_R + 0 = G_R$, then $G + 0 = G$. We thus establish that the game 0 satisfies the property of an additive identity.

Further, $\{0 \mid \} + \{0 \mid \} = \{1 \mid \}$, so 2 is an appropriate label for the game $\{1 \mid \}$ using the intuition of real number addition. More generally, the game $\{n - 1 \mid \ \}$ is assigned the number n from an inductive standpoint, as $\{n - 2 \mid \ \} + 1 = \{n - 1 \mid \ \}$, while the game $\{\ \mid -(n-1)\}$ can be assigned the number $-n$ from an analogous argument.

If we assume that a game $n - 1$ is a win for Left, then it is clear that n, or $\{n - 1 \mid \ \}$, is a win for Left regardless of which player goes first, since Left can move to the game $n - 1$ while Right has no moves. All games with positive integer values are thus wins for Left, while games with negative integer values are wins for Right, confirming that the outcome classes defined earlier are indeed appropriate.

Every Conway number also has an additive inverse, defined by:

$$-G = \{-G_R \mid -G_L\}$$

The game $-G$ is the negative of G, and has the property that all valid moves for Left in G are now valid moves for Right and vice versa. For example, if a game G involved tiling a board consisting of unit squares with 1x2 dominoes, with Left placing the dominoes horizontally and Right vertically, then the game $-G$ would involve the same board, but Left would now place dominoes vertically and Right horizontally. Thus, if the original game G was a Left win, $-G$ would be a Right win.

We can easily verify that $-0 = 0$ as expected. Furthermore, we can demonstrate that the sum $G + (-G)$ is equal to 0, which corroborates our definition for $-G$, by the following argument: any move made by the first player in G or $-G$ can be countered by the second player in the other game, so that the second player will always have a valid move. If the first player moves to a position X in game G, then the second player can move to –X in $-G$. Eventually, the first player will have no valid moves. Since $G + (-G)$ is always a second player win, its value is equal to 0 from the outcome classes described earlier.

The Conway numbers form a group under addition, as addition is closed, commutative, and associative, and both an additive identity and inverses exist. The Conway numbers further hold all properties of an ordered field, including a consistent definition for multiplication and the property of transitivity [6].

We now discuss the game $\{0 \mid 1\}$ that arose in the second generation. We show that this game should be labeled as $\frac{1}{2}$ by confirming that $\frac{1}{2} + \frac{1}{2} + (-1) = 0$. In other words, we show that in the game $\{0 \mid 1\} + \{0 \mid 1\} + \{\ \mid 0\}$, a *winning strategy* exists for the second player. Assume Left moves first. Then Left must move to 0 in either the first or second game (which are identical) leaving the configuration $\{\ \mid\ \} + \{0 \mid 1\} + \{\ \mid 0\}$, or simply $\{0 \mid 1\} + \{\ \mid 0\}$, for Right. If Right moves to 1 in $\{0 \mid 1\}$, the resulting configuration is $\{0 \mid\ \} + \{\ \mid 0\}$. Left must make its only possible move in the first game, allowing Right to move to 0 in the second game and win. It can similarly be shown that if Right moves first in $\{0 \mid 1\} + \{0 \mid 1\} + \{\ \mid 0\}$, Left can always win. Thus, a winning strategy exists for the second player.

In the third generation, it can similarly be established that $\{\, 0 \mid \frac{1}{2} \} = \frac{1}{4}$ and that $\{ \frac{1}{2} \mid 1 \} = \frac{3}{4}$. It is not hard to see that in the n^{th} generation, all dyadic rationals of the form $\frac{p}{2^{n-1}}$ exist, with p ∈ ℤ and -$2^{n-1} \leq p \leq 2^{n-1}$, in addition to all integers from $-n$ to n [1].

We now state *Conway's convention* for labeling a general game $G = \{G_L \mid G_R\}$. A game G is assigned a Conway number only if G_L and G_R are finite sets of numbers, and every possible game configuration in G_L is smaller in value than every game in G_R. If this is true, G takes on the "simplest" value lying between G_L and G_R. The "simplest" value is defined to equal $\frac{p}{2^k}$, where k is as small as possible, and $\frac{p}{2^k}$ has the least absolute value such that $G_L < \frac{p}{2^k} < G_R$. For example, the games $\left\{-\frac{23}{32} \mid \frac{3}{4}\right\}$, $\left\{-\frac{1}{2} \mid \frac{47}{8}\right\}$, and $\left\{-\frac{1}{64} \mid \frac{1}{4}\right\}$ are all equal to 0. Furthermore, if $G_L = \frac{p}{2^k}$ and $G_R = \frac{p+1}{2^k}$ for some p ∈ ℤ, $G = \frac{2p+1}{2^{k+1}}$. A justification of this convention is presented in Berlekamp's text *Winning Ways* [2; p. 22].

In games with multiple options in the left and right sets, certain options can be deleted without changing the value of the game. For example, $\left\{-1, 0, \frac{1}{2} \mid 1,\ 2\right\} = \left\{\frac{1}{2} \mid 1\right\} = \frac{3}{4}$. We now prove an important theorem of the Conway numbers used extensively later in the paper.

Theorem 2.1. Deleting all but the *highest value game in the left set* and the *lowest value game in the right set* produces a game of equivalent value [2].

Proof. Let $G = \{L_L, L_H, L_1, L_2, \ldots \mid R_L, R_H, R_1, R_2, \ldots\}$. We claim that if $L_L \leq L_H$ and $R_L \leq R_H$, L_L and R_H can be deleted from the options of G without changing the value of G. Let $H = \{L_H, L_1, L_2, \ldots \mid R_L, R_1, R_2, \ldots\}$. To show that $G = H$ is equivalent to demonstrating that $G + (-H) = 0$, or that $G + (-H)$ is a second player win.

From the definition of the inverse, $-H = \{-R_L, -R_1, -R_2, \ldots \mid -L_H, -L_1, -L_2, \ldots\}$. All moves X in G can be countered by the second player by a move to $-$X in $-H$ and vice versa, except a move by Left to L_L in G and a move by Right to R_H in G. But if Left, as the first player, moves to L_L in G, Right can move to $-L_H$ in H. The resulting game $L_L + (-L_H)$ is negative in value from our assumption above, so that the game is necessarily a win for Right, the second player. Similarly, if Right, as the first player, moves to R_H in G, Left can move to $-R_L$ in H. The resulting game $R_H + (-R_L)$ is positive in value and a win for the second player, Left. Thus, a winning strategy always exists for the second player, and G is indeed equal in value to H.

3 Black and White Hackenbush

This paper deals with a particularly difficult "partisan" game, Black and White Hackenbush, first introduced by Conway in *On Numbers and Games*. A partisan game is in one which the possible moves for Left and Right are not identical. Chess is one example, as a player can only move pieces of his or her color. In impartial games, however, all legal moves are open to both players, and the only possible advantage is order of play. In the 1930s, the Sprague–Grundy theorem demonstrated that all impartial games are equivalent to nim heaps, providing a complete theory for this class of combinatorial games [4, 5]. On the other hand, very few such general results for partisan games have been established.

In Black and White Hackenbush, Left is allowed to delete any black rod and Right is allowed to delete any white rod in grounded structures constructed of black and white edges. Any rod whose connection to the ground is severed by the deletion of a rod is deleted as well. The two players take turns deleting edges until one player can no longer maker a move, in which case the other player is declared the winner. The objective of the game is thus to maintain edges of one’s one color, so as to have open moves in later turns.

The following are illustrations of games of value 0, 1, and -1 respectively:

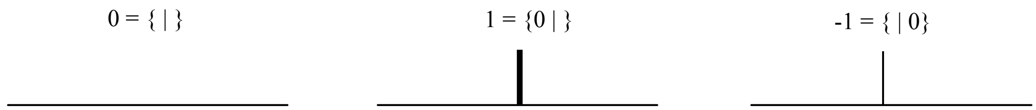


Fig. 3.1: Hackenbush Configurations of Various Values

The Conway number for a given structure, in addition to indicating whether the first, second, Left or Right player has a winning strategy, is a quantitative measure of the *advantage* a player has in a game. For example, the following game, which has a value of 2 (the sum of the components 3 and -1), confers an advantage of two moves to Left if played appropriately:

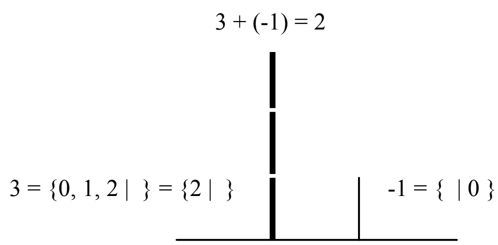


Fig. 3.2: Hackenbush Game of Value Two

Right’s only option is to delete the single white rod. Left, however, can make three moves in the preceding game, if it first deletes the top rod, then the middle, and finally the grounded one. If this game was combined with another arbitrary game, Left could count on advantage of *two moves* on Right, as corresponding to the game’s Conway value of two.

Thus, the Conway number for structures in Black and White Hackenbush is a quantification of the advantages conferred by various positions, and a powerful tool for determining winning strategies in games composed of multiple components.

4 Basic Theorems in Hackenbush

Theorem 4.1. Deletion of a black (white) rod always decreases (increases) the value of a game.

Proof. Let G be an arbitrary Hackenbush configuration, and G' the game resulting from the deletion of a black rod k, without loss of generality. Denote the component that is destroyed by deletion of k as G_k. We claim that $G + (-G') > 0$, or equivalently, that a winning strategy exists for Left in $G + (-G')$.

If Left moves first, it can delete the black rod k in G, leaving the game $G' + (-G')$ for Right to make a move in. This game has value 0, so regardless of what move Right makes, Left will have a winning strategy, as Left is the second player in this new game. If Right instead moves first, and deletes any rod in G_k, Left can delete k, rendering the same game $G' + (-G')$. If instead Right deletes any other rod in G or $-G'$, Left can delete the analogous black rod in the other game. Left will thus always have a reply to any move made by Right.

Corollary 4.2. Consider two edges of the same color E_1 and E_2 in a game G, such that deletion of E_1 destroys E_2, but deletion of E_2 does not destroy E_1. Then it is never the optimal move for a player to delete E_1.

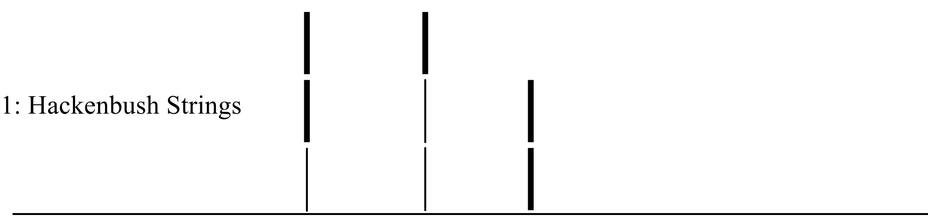
Proof. Without loss of generality, assume that E_1 and E_2 are black. From Theorem 3.1, we see that deletion of E_2 lowers the value of G by a positive value m, and that deletion of E_1 lowers the value of G by a positive value n. Direct deletion of E_1 is equivalent to deletion of E_2, a loss of m, followed by the deletion of E_1, a loss of some positive value k, because E_2 is dependent on E_1. Since $n = m + k > m$, deletion of E_1 results in a lower value game than deletion of E_2. From Theorem 2.1, we see that the higher value option is more favorable to Left, and thus it is never the optimal move to delete E_1. An analogous argument holds if E_1 and E_2 are white.

5 Hackenbush Strings

From the work of Karp and the results of Garey and Johnson, it can be shown that Black and White Hackenbush is NP-complete [7, 8]. Besides the Colon Principle, no general theorems exist to determine the Conway numbers of arbitrary configurations in Black and White Hackenbush [2]. Analysis of certain basic structures such as *strings* and *loops* that are components in many games thus proves invaluable. In this section, we present and prove by induction an explicit formula for the value of a Hackenbush string.

Definition. A string is a sequence of black or white rods, grounded at one end, with no branches.

Fig. 5.1: Hackenbush Strings



Theorem 5.1. Let X be a string represented by game $X = \{X_L \mid X_R\}$. Let $B(X)$ denote the game obtained by appending a black rod to the top of string *X*, and $W(X)$ denote the game obtained by appending a white rod to the top of *X*. Then $B(X) = \{X \mid X_R\}$ and $W(X) = \{X_L \mid X\}$.

Proof. Appending a black rod does not change the options for Right, so the right set of $B(X)$ is clearly still X_R. The best option for Left is to delete the black rod just appended, from Corollary 4.2, so the left set of $B(X)$ is the original game X. An analogous argument holds for $W(X)$.

Theorem 5.2. The game representing a string of K black rods is K. The game representing a string of K white rods is $-K$.

Proof. Consider the base case: one black rod. Deletion by Left results in the game $0 = \{\mid\}$.

Right has no moves. The value of one grounded black rod is thus $\{\, 0 \mid \} = 1$. Now, assume a string of R black rods is represented by game R for R < K. Consider the string with K black rods. Right has no moves. Left can delete the top rod, resulting in the game $K-1$, or a rod below the top rod, resulting in a game of lower value. Thus, a string with K black rods is represented by the game $\{K-1,\ K-2, \dots \mid\} = \{K-1 \mid \} = K$. An analogous argument holds for K white rods.

Theorem 5.3. The value (Conway number) representing any string is given by $\pm N \pm \frac{1}{2^1} \pm \frac{1}{2^2} \pm \cdots \pm \frac{1}{2^{\mathrm{k}}}$ where N is the number of consecutive rods of one color at the base, and $\pm\frac{1}{2^{\mathrm{i}}}$ is the value contributed by the i^{th} rod after the first color change, with a positive sign corresponding to a black rod and a negative sign corresponding to a white.

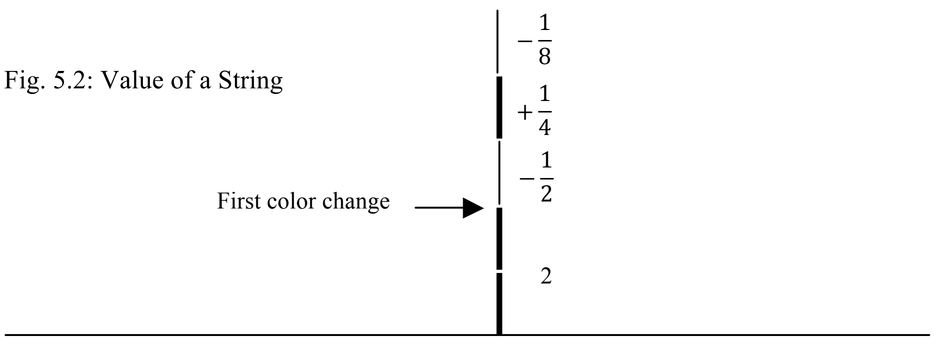


Fig. 5.2: Value of a String

Proof. We begin by establishing Conway's convention: for a game $G = \{\, G_L \mid G_R \,\}$, call G_L and G_R "close" if they are dyadic rationals of the form $\frac{p}{2^k}$ and $\frac{p+1}{2^k}$ for some $p, k \in \mathbb{Z}$. By Conway's convention, if G_L and G_R are "close", $G = \frac{1}{2}\,(G_L + G_R) = \frac{2p+1}{2^{k+1}}$.

We first analyze the base case. Without loss of generality, consider a string of N consecutive black rods, represented by the game N. Let N^{-1} be the game generated by deleting the top rod of N, and N^1 the game generated by appending a white rod to N. Then $N^1 = \{\, N^{-1} \mid N \,\}$ from Theorem 5.1. Note that $N^{-1} = N - 1$ and that N^1, applying Conway's convention, is the fraction $\frac{1}{2}$ $(N^{-1} + N)$. In addition, note that for N^1, $N_R - N_L = N - N^{-1} = \frac{1}{2^0}$ and $N^1 = \mathrm{N} - \frac{1}{2^1}$. This completes the base case.

Our inductive hypothesis stands as follows: $N^{k-1} = \{N_L \mid N_R\} = \mathrm{N} \pm \frac{1}{2^1} \pm \frac{1}{2^2} \pm \cdots \pm \frac{1}{2^{k-1}}$. Further, assume that N_L and N_R are dyadic rationals $\frac{p}{2^{k-2}}$ and $\frac{p+1}{2^{k-2}}$, so $N_R - N_L = \frac{1}{2^{k-2}}$.

Then N^k is either $\{N^{k-1} \mid N_R\}$ or $\{N_L \mid N^{k-1}\}$ depending on whether a black or white rod is appended to N^{k-1}. From Conway's convention, we see that $N^{k-1} = \frac{2p+1}{2^{k-1}}$, $N_L = \frac{p}{2^{k-2}} = \frac{2p}{2^{k-1}}$, and $N_R = \frac{p+1}{2^{k-2}} = \frac{2p+2}{2^{k-1}}$. Then, in either case, the left and right sets of N^k are "close", and further $N_R - N^{k-1} = \frac{1}{2^{k-1}}$ and $N^{k-1} - N_L = \frac{1}{2^{k-1}}$.

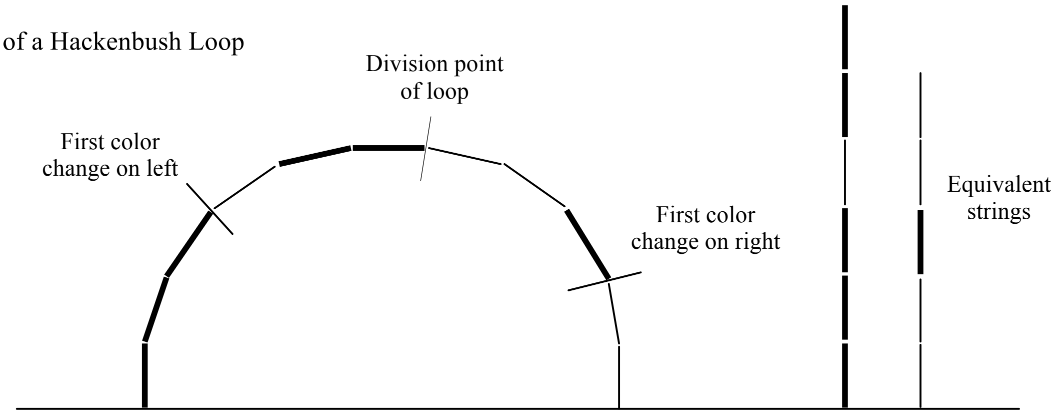
In the first case, if $N^k = \{N^{k-1} \mid N_R\}$ and a black rod was appended, $N^k = \frac{1}{2}(N^{k-1} + N_R) = N^{k-1} + \frac{1}{2^k}$. In the second case, if $N^k = \{N_L \mid N^{k-1}\}$ and a white rod was appended, $N^k = \frac{1}{2}(N_L + N^{k-1}) = N^{k-1} - \frac{1}{2^k}$. Thus N^k is in fact $\pm\mathrm{N} \pm \frac{1}{2^1} \pm \frac{1}{2^2} \pm \cdots \pm \frac{1}{2^k}$, with positive signs corresponding to black rods and negative signs corresponding to white rods.

6 Hackenbush Loops

Definition. A loop is a string that is grounded on both ends, so that deleting any black or white rod in the loop results in two strings.

Theorem 6.1. The value (Conway number) of any loop is found by separating the loop midway between the first color changes on either side, and summing the values of the two resulting strings. If the midway point is in fact a rod (i.e. there an odd number of rods between the color changes), then that rod remains on both strings.

Fig. 6.1: Value of a Hackenbush Loop



Lemma 6.1a. Let a rod R in a loop be denoted by (X, Y) where X is its distance from the first color change on the left and Y is its distance from the first color change on the right. Then, the best option for Left (Right) is to delete the black (white) rod with minimal difference $|X - Y|$.

Proof. Consider two consecutive black rods in the loop, such that white rods may lie between them but no black rods do so. Call the lower black rod in the illustration A, and the higher one B. Assume, without loss of generality, that $A_X \leq B_Y$ (A is closer the first color change on the left than B is to the first color change on the right) and that k white rods separate A and B.

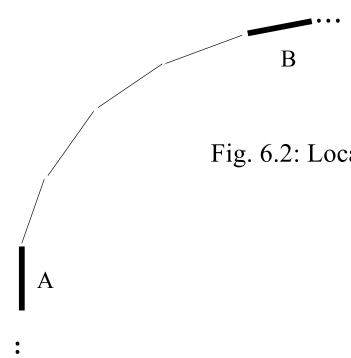


Fig. 6.2: Local Analysis of Consecutive Rods

Consider the value of the strings formed by deletion of either A or B. Let m denote the value of the string below A, and n the value of the string to the right of B. Then, deletion of A will sever the loop results in two strings—the string m below A and a string that includes n, B, and the k white rods—with values:

$$m \quad \text{and} \quad n + \frac{1}{2^{B_Y}} - \frac{1}{2^{B_Y+1}} - \frac{1}{2^{B_Y+2}} - \cdots - \frac{1}{2^{B_Y+k}} = n + \frac{1}{2^{B_Y+k}}$$

Deletion of B results in two strings with values:

$$n \quad \text{and} \quad m + \frac{1}{2^{A_X}} - \frac{1}{2^{A_X+1}} - \frac{1}{2^{A_X+2}} - \cdots - \frac{1}{2^{A_X+k}} = m + \frac{1}{2^{A_X+k}}$$

Since strings are independent games, their values can be added. Thus, the game upon deletion of A by Left has value $m + n + \frac{1}{2^{B_Y+k}}$ and the game upon deletion of B by Left has value $m + n + \frac{1}{2^{A_X+k}}$. Since we assumed $A_X \le B_Y$, $m + n + \frac{1}{2^{B_Y+k}} \le m + n + \frac{1}{2^{A_X+k}}$. Deletion of B is thus a better option for Left because it results in a higher value game. Note that $A_X \le B_Y$ implies $|B_Y - B_x| \le |A_Y - A_x|$ since B_x is in fact $A_x + k + 1$ and A_Y is $B_Y + k + 1$. Thus, it is always better for Left to delete the rod that is closer to the center of the loop.

If the preceding comparison process is extended to consecutive black rods along the length of the entire loop, it is seen that the *best* option for Left is to delete the rod with the minimal difference $|X - Y|$. An analogous argument holds for the best option for Right.

Theorem 6.1.

Proof. Case 1: An odd number of rods lie between the first color changes on either side.
Assume, without loss of generality, that the rod midway between the color changes is black, and that the coordinates of the rod are (k, k). Then, the best option for Left is to delete this rod. The best option for Right is to delete the white rod closest to the middle rod (Lemma), with coordinates $(k + a,\ k - a)$.

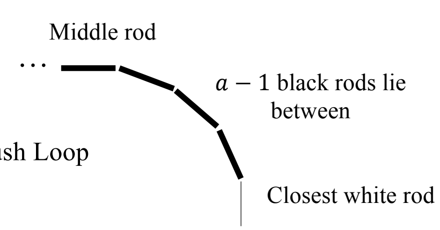


Fig. 6.3: Center of a Hackenbush Loop

We now calculate the values of the left and right options of the preceding game. Assume the string to the left of the middle rod has value m, and the string below the "closest white rod" has value n. Then, the values of the strings resulting from deletion of the middle rod by Left are:

$$m \quad \text{and} \quad n - \frac{1}{2^{k-a}} + \frac{1}{2^{k-a+1}} + \ldots + \frac{1}{2^{k-1}} = n - \frac{1}{2^{k-1}}$$

Deletion of the "closest white rod" by Right results in strings with values:

$$n \quad \text{and} \quad m + \frac{1}{2^{k}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+a-1}} = m + \frac{1}{2^{k-1}} - \frac{1}{2^{k+a-1}}$$

The sum $m + n$ is of the form $\frac{p}{2^{k-1}}$, where p $\in \mathbb{Z}$, because no rod in m or n is farther than $k-1$ from a color change. Then, the Left option for the loop is $\frac{p-1}{2^{k-1}}$ and the Right option is $\frac{p+1}{2^{k-1}} - \frac{1}{2^{k+a-1}}$. The "simplest" Conway number lying between these two numbers, by Conway's convention, is $\frac{p}{2^{k-1}} = m + n$, so this is the value of the loop. Splitting the loop midway between the color changes (and appending the middle rod to both strings) results in strings with values:

$$m + \frac{1}{2^{k}} \quad \text{and} \quad n - \frac{1}{2^{k-a}} + \frac{1}{2^{k-a+1}} + \ldots + \frac{1}{2^{k}} = n - \frac{1}{2^{k}}$$

The sum of the values of these two strings is $m + n$ as well, so Theorem 4 holds.

Case 2: An even number of rods lie between the first color changes on either side.

An analogous argument holds.

7 Conclusion and Next Steps

Our explicit formulas for the values of Hackenbush strings and loops facilitate analysis of complicated Hackenbush configurations, as deletion of rods in any game eventually results in independent string and loop structures. In addition, since all dyadic rationals can be represented by Hackenbush strings, a "disguised" game of an arbitrary value X can be constructed by combining a complex structure with known value Y with a string of value $X - Y$. When taken together with the Colon Principle stated below, our formulas for loops and strings significantly extend the set of structures whose Conway values can be efficiently calculated.

Colon Principle: At any node in a Hackenbush game, it is possible to replace a structure whose only connection to the grounded uses that node with a structure of equivalent value, without changing the value of the overall game [2].

Branched strings, or trees, for example, can now be replaced by equivalent strings whose value can be easily calculated by Theorem 5.3.

Configurations that involve loops *and* strings, such as the following, however, still remain unsolved:

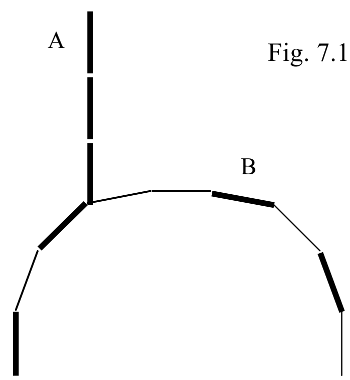


Fig. 7.1: String-Loop Conglomerate

$$\{\tfrac{1}{8} \mid \tfrac{9}{32}\} = \tfrac{1}{4}$$

In the preceding structure, intuition would suggest that deletion of A is a stronger option for Left then deletion of B, as A is five edges away from a color change on the left, while B is only three rods away from a color change on the right. In actuality, deletion of A and B result in structures of equivalent value. A theory to determine Conway numbers for such structures is thus yet to be established. Perhaps the solution lies in a similar inductive approach to the one used in this paper, with the base case involving a loop with a single appended rod.

References

[1] J. H. Conway. *On Numbers and Games*. London: Academic Press Inc., 1976.

[2] E. R. Berlekamp et al. *Winning Ways for Your Mathematical Plays: Volume 1*. Wellesley: A. K. Peters, Ltd., 2001.

[3] C. L. Bouton. Nim, A Game with a Complete Mathematical Theory. *Annals of Mathematics*, 3 (1/4): 35-39, 1902.

[4] R. P. Sprague. “Uber mathematische Kampfspiele.” *Tˆohoku Math. J.*, 41**:** 291-301, 1935.

[5] P. M. Grundy. Mathematics and Games. *Eureka*, 2: 6-8, 1939.

[6] G. Grimm. An Introduction to Surreal Numbers. Thesis, 2002.

[7] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* New York: W. H. Freeman, 1983.

[8] R. M. Karp. Reducibility Among Combinatorial Problems. *Complexity of Computer Computations*. New York: Plenum, 1972.