# Flow Matching

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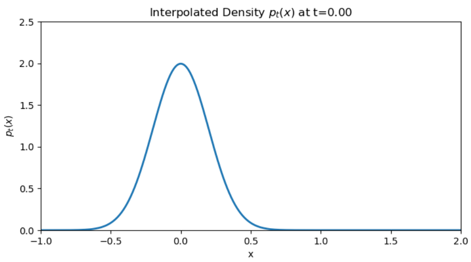
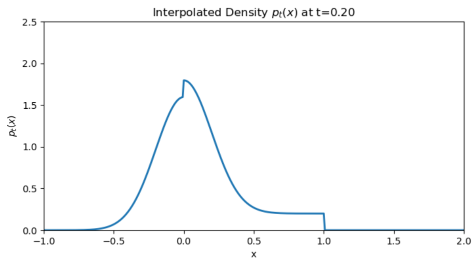
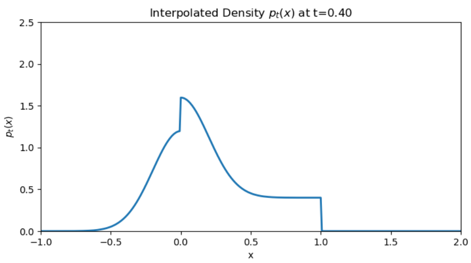
We start by defining a probability density path $p : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ meaning that for each time $t$, $p_t$ is density function i.e. $\int p_t(x)dx = 1$.
A simple example of such a path is a path $p$ interpolating two density $p_0$ and $p_1$ with $p_t = tp_1 + (1-t)p_0$



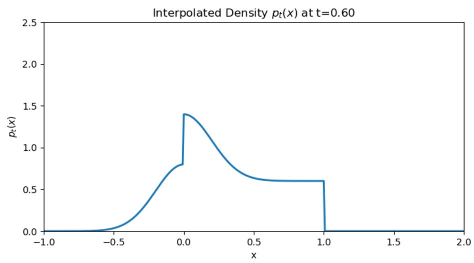

(a)



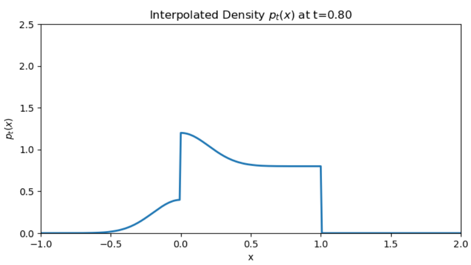

(b)



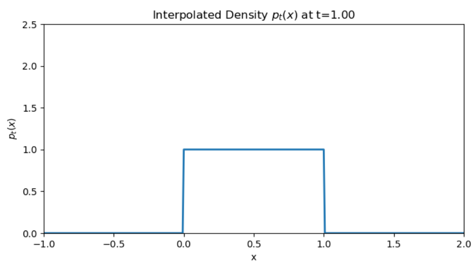

(c)




(d)




(e)




(f)

Figure 1: A probability path interpolating $\mathcal{N}(0, 0.2)$ and $\mathcal{U}([0,1])$

Next we introduce a core object, a time dependant vector field $v : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ which can be used to construct a map $\phi : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$, called a flow, by the following ODE

$$\begin{aligned} \frac{d}{dt}\phi_t(x) &= v_t(\phi_t(x)) \\ \phi_0(x) &= x \end{aligned} \tag{1}$$

The link between the flow and the probability path is given by the change of variables formula

$$p_t(x) = q(\phi_t^{-1}(x)) \det \left[ \frac{\partial \phi_t^{-1}}{\partial x}(x) \right] \tag{2}$$