THE ASTROPHYSICAL JOURNAL, 591:L79–L82, 2003 July 10

PHASE CORRELATIONS IN NON-GAUSSIAN FIELDS

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Received 2003 May 7; accepted 2003 June 4; published 2003 June 23

ABSTRACT

We present the general relationship between phase correlations and the hierarchy of polyspectra in the Fourier space, and a new theoretical understanding of the phase information is provided. Phase correlations are related to the polyspectra only through the nonuniform distributions of the phase sum $\theta_{\boldsymbol{k}_1} + \cdots + \theta_{\boldsymbol{k}_N}$ with closed wavevectors, $\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_N = 0$. The exact relationship is given by the infinite series, which one can truncate in a consistent manner. The method to calculate the series to arbitrary order is explained, and the explicit expression of the first-order approximation is given. A numerical demonstration proves that the distribution of the phase sum is a robust estimator and provides an alternative statistic to search for the non-Gaussianity.

Subject headings: cosmology: theory — large-scale structure of universe — methods: statistical

1. INTRODUCTION

Quantifying the cosmic fields, such as the density fields, velocity fields, gravitational lensing fields, temperature fluctuations in the cosmic microwave background, etc., is undoubtedly crucial to study the origin and dynamics of the structure in the universe. The structure of these fields is believed to have emerged from primordial random Gaussian perturbations, as most of the inflationary models naturally predict nearly scale-invariant Gaussian fluctuations (Guth & Pi 1982; Starobinskii 1982; Hawking 1982; Bardeen, Steinhardt, & Turner 1983).

Even if the primordial perturbations are random Gaussian, the gravitationally nonlinear evolution produces non-Gaussianity in the cosmic fields. Quantifying the non-Gaussianity is not trivial since it depends on the full hierarchy of the higher order correlation functions in real space or of the polyspectra in Fourier space. Several members of such hierarchy can be observationally determined, with which only partial information on non-Gaussianity is quantified. Therefore, alternative statistics, such as the void probability function (White 1979), the genus statistic (Gott, Dickinson, & Melott 1986), the Minkowski functionals (Minkowski 1903; Mecke, Buchert, & Wagner 1994; Schmalzing & Buchert 1997), etc., that contain the information on the full hierarchy of higher order statistics should be useful.

Non-Gaussianity is frequently termed "phase correlations." This term reflects the fact that the Fourier phases of a random Gaussian field are randomly distributed without any correlation among different modes. Therefore, phase correlations, if any, obviously characterize the non-Gaussianity. However, what kind of phase correlations arise in a given non-Gaussian field have been far from obvious. Investigations along this line are quite limited in the literature despite its importance, apparently because of the lack of theoretical guidelines. Most of the earlier work (Ryden & Gramann 1991; Soda & Suto 1992; Jain & Bertschinger 1998) assesses only the nonlinear evolution of phases in individual Fourier modes without statistics. Phenomenological studies of N-body simulations have revealed that the one-point phase distribution remains uniform even in non-Gaussian fields (Suginohara & Suto 1991) and that the phase difference between neighboring Fourier modes is nonuniformly distributed (Scherrer, Melott, & Shandarin 1991; Coles & Chiang 2000; Chiang 2001; Chiang, Naselsky, & Coles 2002; Watts, Coles, & Melott 2003). However, the meaning of the discovered phase correlations is obscure in those works.

Since the hierarchy of the higher order statistics contains statistically all information on the distribution (Bertschinger 1992), there should be some connection between phase correlations and polyspectra, which is the key to theoretically understanding the phase correlations. Examining a toy model, Watts & Coles (2003) realized the importance of the phase sums with closed wavevectors in this connection, although they have never derived the exact relations. In this Letter, the connection in the general form is discovered for the first time. As a result, we will have much better theoretical understanding of the phase information than before.

2. PHASE CORRELATIONS AND POLYSPECTRA

Although the real part $\mathrm{Re}\, f_{\boldsymbol{k}}$ and the imaginary part $\mathrm{Im} f_{\boldsymbol{k}}$ of the Fourier transform $f_{\boldsymbol{k}}$ of a random field f are naturally the independent variables, one can also take their linear combinations $f_{\boldsymbol{k}} = \mathrm{Re}\, f_{\boldsymbol{k}} + i\, \mathrm{Im}\, f_{\boldsymbol{k}}$ and $f^*_{\boldsymbol{k}} = \mathrm{Re}\, f_{\boldsymbol{k}} - i\, \mathrm{Im}\, f_{\boldsymbol{k}}$ as another set of mutually independent variables. For calculational advantages, we use the latter choice. In this Letter, the reality of the random field f is assumed since most of the cosmic fields are real, although one can readily generalize the following analysis to the complex fields. Because of the reality condition, $f^*_{\boldsymbol{k}} = f_{-\boldsymbol{k}}$, the $f^*_{\boldsymbol{k}}$ are actually not independent variables, and the $f_{\boldsymbol{k}}$ of all modes $\boldsymbol{k}$ are taken as independent variables.

It is useful to define the normalized quantity $\alpha_{\boldsymbol{k}} \equiv f_{\boldsymbol{k}}/\sqrt{P(k)}$, where $P(k) = \langle |f_{\boldsymbol{k}}|^2 \rangle$ is the power spectrum of the random field. The key technique to derive the relation between phase correlations and polyspectra is given by previous work (Matsubara 1995, 2003): the joint probability function $P(\{\alpha_{\boldsymbol{k}}\})$ of having a particular set of $\alpha_{\boldsymbol{k}}$ is formally represented by

$$\mathcal{P}(\{\alpha_{\boldsymbol{k}}\}) = \exp\left[\sum_{N=3}^{\infty} \frac{(-)^N}{N!} \sum_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N} \left\langle \alpha_{\boldsymbol{k}_1} \cdots \alpha_{\boldsymbol{k}_N} \right\rangle_{\mathrm{c}} \frac{\partial^N}{\partial\alpha_{\boldsymbol{k}_1} \cdots \partial\alpha_{\boldsymbol{k}_N}}\right] \mathcal{P}_{\mathrm{G}}(\{\alpha_{\boldsymbol{k}}\}), \tag{1}$$

where $\langle \cdots \rangle_{\mathrm{c}}$ indicates the cumulants and $\mathcal{P}_{\mathrm{G}}(\{\alpha_{\boldsymbol{k}}\})$ is the multivariate Gaussian distribution function of variables $\{\alpha_{\boldsymbol{k}}\}$. In the present case, $\mathcal{P}_{\mathrm{G}}(\{\alpha_{\boldsymbol{k}}\}) \propto \exp\left(-\frac{1}{2} \sum_{\boldsymbol{k}} \alpha_{\boldsymbol{k}}\alpha_{-\boldsymbol{k}}\right)$, since the covariance matrix is $\langle \alpha_{\boldsymbol{k}} \alpha_{\boldsymbol{k}'} \rangle = \delta^{\rm K}_{\boldsymbol{k}+\boldsymbol{k}'}$, where the symbol $\delta^{\rm K}_{\boldsymbol{k}}$ is defined by $\delta^{\rm K}_{\boldsymbol{k}} = 1$ for $\boldsymbol{k} = 0$ and $\delta^{\rm K}_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \neq 0$. The periodic boundary condition with box size $V = L^3$ is assumed.

Since the polyspectra $P^{(N)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_N)$ are defined from the cumulants by

$$\langle f_{\boldsymbol{k}_1} \cdots f_{\boldsymbol{k}_N} \rangle_{\rm c} = V^{1-N/2} \delta^{\rm K}_{\boldsymbol{k}_1+\cdots+\boldsymbol{k}_N} P^{(N)}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{N-1}). \tag{2}$$

The above formula (1) provides the relation between polyspectra and joint distribution of the Fourier coefficients. Expanding the exponential in equation (1), each term in this expansion consists of the products of the polyspectra times derivatives of $\mathcal{P}_{\rm G}$. The derivatives of $\mathcal{P}_{\rm G}$ are given by a simple polynomial of $\alpha_{\boldsymbol{k}}$ times $\mathcal{P}_{\rm G}$. The general term in the expansion has the form

$$\sum_{\boldsymbol{k}'s} \delta^{\rm K}_{\boldsymbol{k}_1+\boldsymbol{k}_2} + \cdots \delta^{\rm K}_{\boldsymbol{k}'_1+\boldsymbol{k}'_2+\cdots} \cdots$$
$$\times\, p^{(N)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots) p^{(N)}(\boldsymbol{k}'_1, \boldsymbol{k}'_2, \ldots) \cdots H_{\boldsymbol{k}_1\boldsymbol{k}_2\cdots\boldsymbol{k}'_1\boldsymbol{k}'_2\cdots} \mathcal{P}_{\rm G}, \tag{3}$$

with appropriate coefficients, where

$$p^{(N)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_{N-1}) = \frac{P^{(N)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_{N-1})}{\sqrt{V^{N-2} P(k_1) P(k_2) \cdots P(k_{N-1}) P\left(|\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_{N-1}|\right)}} \tag{4}$$

are the dimensionless, normalized polyspectra of $\alpha_{\boldsymbol{k}}$, and

$$H_{\boldsymbol{k}_1\boldsymbol{k}_2\cdots} = \frac{1}{\mathcal{P}_{\rm G}} \left(-\frac{\partial}{\partial \alpha_{\boldsymbol{k}_1}}\right) \left(-\frac{\partial}{\partial \alpha_{\boldsymbol{k}_2}}\right) \cdots \mathcal{P}_{\rm G} \tag{5}$$

is a generalization of Hermite polynomials and is given by polynomials of $\alpha_{\boldsymbol{k}}$ and $\delta^{\rm K}_{\boldsymbol{k}}$. For example,

$$H_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} = \alpha_{-\boldsymbol{k}_1}\alpha_{-\boldsymbol{k}_2}\alpha_{-\boldsymbol{k}_3} - \delta^{\rm K}_{\boldsymbol{k}_1+\boldsymbol{k}_2}\alpha_{-\boldsymbol{k}_3} - \delta^{\rm K}_{\boldsymbol{k}_2+\boldsymbol{k}_3}\alpha_{-\boldsymbol{k}_1} - \delta^{\rm K}_{\boldsymbol{k}_3+\boldsymbol{k}_1}\alpha_{-\boldsymbol{k}_2}, \tag{6}$$

and so on. Thus, the joint probability function $\mathcal{P}(\{\alpha_{\boldsymbol{k}}\})$ is represented by $\mathcal{P}_{\rm G}$ times infinite sum of products by $\alpha_{\boldsymbol{k}}$, $\delta^{\rm K}_{\boldsymbol{k}}$, and normalized polyspectra.

Next step is to transform the complex variable $\alpha_{\boldsymbol{k}}$ into the modulus $|\alpha_{\boldsymbol{k}}|$ and the phase $\theta_{\boldsymbol{k}}$ by $\alpha_{\boldsymbol{k}} = |\alpha_{\boldsymbol{k}}| e^{i\theta_{\boldsymbol{k}}}$. This should be carefully done, since $\alpha_{\boldsymbol{k}}$ is considered dependent on $\alpha^*_{\boldsymbol{k}}$ so far. At this point, we restrict the wavenumber $\boldsymbol{k}$ in the upper half-sphere (uhs), $k_z \geq 0$, and the degrees of freedom in the lower half-sphere is relabeled by the reality relation, $\alpha_{\boldsymbol{k}} = \alpha^*_{-\boldsymbol{k}}$ for $k_z < 0$. The mode $\boldsymbol{k} = 0$ is excluded, which ensures zero mean of the original field f. The term (3) is accordingly relabeled, resulting in the sum of the products of $\alpha_{\boldsymbol{k}}$, $\alpha^*_{\boldsymbol{k}}$, $\delta^{\rm K}_{\boldsymbol{k}}$, where $\boldsymbol{k} \in$ uhs, and normalized polyspectra. With the above procedures, one can express the ratio $\mathcal{P}(\{\alpha_{\boldsymbol{k}}\})/\mathcal{P}_{\rm G}(\{\alpha_{\boldsymbol{k}}\})$ in terms of the normalized polyspectra, the modulus $|\alpha_{\boldsymbol{k}}|$, and the phase $\theta_{\boldsymbol{k}}$. The Jacobian of the transform from $\alpha_{\boldsymbol{k}}$ to $(|\alpha_{\boldsymbol{k}}|, \theta_{\boldsymbol{k}})$ is the same for the probability functions $\mathcal{P}(\{\alpha_{\boldsymbol{k}}\})$ and $\mathcal{P}_{\rm G}(\{\alpha_{\boldsymbol{k}}\})$. Therefore this meets our ends to relate the phase correlations and polyspectra, which is a completely new result.

Practically, one needs to truncate the infinite series by a consistent manner. Fortunately, the non-Gaussianity generated by gravitationally nonlinear evolution is known to approximately follow the hierarchical model of the higher order correlations, in which the polyspectra $P^{(N)}$ have the order $P^{(N)} \sim O[P(k)^{N-1}]$ (e.g., Bernardeau et al. 2002). This means $p^{(N)} \sim O(\epsilon^{N-2})$, where $\epsilon \sim \sqrt{P(k)/V}$. Therefore one can evaluate the phase correlations in perturbative manner as long as the expansion parameter ϵ is small. It is straightforward to perform the above procedure to express $\mathcal{P}(\{|\alpha_{\boldsymbol{k}}|, \theta_{\boldsymbol{k}}\})$ in terms of normalized polyspectra to arbitrary order in ϵ. In the lowest order approximation, only the normalized bispectrum $p^{(3)}$ gives the term of order $O(\epsilon^1)$. The result is

$$\mathcal{P}(\{|\alpha_{\boldsymbol{k}}|, \theta_{\boldsymbol{k}}\}) \prod_{\boldsymbol{k}\in{\rm uhs}} d|\alpha_{\boldsymbol{k}}| d\theta_{\boldsymbol{k}} = \Bigg[1 + \sum_{\boldsymbol{k}_1, \boldsymbol{k}_2 \in {\rm uhs}} |\alpha_{\boldsymbol{k}_1}||\alpha_{\boldsymbol{k}_2}||\alpha_{\boldsymbol{k}_1+\boldsymbol{k}_2}| \times \cos(\theta_{\boldsymbol{k}_1} + \theta_{\boldsymbol{k}_2} - \theta_{\boldsymbol{k}_1+\boldsymbol{k}_2}) p^{(3)}(\boldsymbol{k}_1, \boldsymbol{k}_2)\Bigg] \times \prod_{\boldsymbol{k}\in{\rm uhs}} 2|\alpha_{\boldsymbol{k}}| e^{-|\alpha_{\boldsymbol{k}}|^2} d|\alpha_{\boldsymbol{k}}| \frac{d\theta_{\boldsymbol{k}}}{2\pi}. \tag{7}$$

Higher order terms can be similarly calculated, although they are somehow tedious. For example, in the second-order approximation, $O(\epsilon^2)$, there appears the square of the first-order term, and terms like

$$|\alpha_{\boldsymbol{k}_1}||\alpha_{\boldsymbol{k}_2}||\alpha_{\boldsymbol{k}_3}||\alpha_{\boldsymbol{k}_1+\boldsymbol{k}_2\pm\boldsymbol{k}_3}| \times \cos(\theta_{\boldsymbol{k}_1} + \theta_{\boldsymbol{k}_2} \pm \theta_{\boldsymbol{k}_3} - \theta_{\boldsymbol{k}_1+\boldsymbol{k}_2\pm\boldsymbol{k}_3}) \tag{8}$$

with the appropriate normalized trispectrum or the product of normalized bispectra multiplied, and other terms which do not depend on phases.

The phases always contribute to the probability distribution by the combination of the form, $\cos(\theta_{\boldsymbol{k}_1} + \cdots + \theta_{\boldsymbol{k}_N})$, with closed wavevectors: $\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_N = 0$. This is generally true because the phase dependence in equation (3) is the exponential of the sum of phases, and the probability is the real number so that taking real parts gives the cosine function. The reason that phase correlations exist only among modes with closed wavevectors comes from the translational invariance. In equations (7) and (8), wavenumbers are restricted to the uhs so that the modes in the lower half-sphere are relabeled by $\theta\boldsymbol{k} = -\theta\_\boldsymbol{k}$.

The moduli $|\alpha_{\boldsymbol{k}}|$ are easily integrated in the first-order approximation of equation (7), resulting in

$$\mathcal{P}(\{\theta_{\boldsymbol{k}}\}) \propto 1 + \frac{\sqrt{\pi}}{2} \sum_{\boldsymbol{k}}^{\rm uhs} p^{(3)}(\boldsymbol{k}, \boldsymbol{k}) \cos(2\theta_{\boldsymbol{k}} - \theta_{2\boldsymbol{k}}) + \left(\frac{\sqrt{\pi}}{2}\right)^3 \sum_{\boldsymbol{k}\neq\boldsymbol{k}'}^{\rm uhs} p^{(3)}(\boldsymbol{k}, \boldsymbol{k}') \cos(\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{k}'} - \theta_{\boldsymbol{k}+\boldsymbol{k}'}). \tag{9}$$

The practically useful relations between phase correlations and the bispectrum are obtained by further integrating some phases in equation (9). One obtains

$$\mathcal{P}(\theta_{\boldsymbol{k}},\ \theta_{2\boldsymbol{k}}) \propto 1 + \frac{\sqrt{\pi}}{2} p^{(3)}(\boldsymbol{k},\ \boldsymbol{k}) \times \cos(2\theta_{\boldsymbol{k}} - \theta_{2\boldsymbol{k}}), \tag{10}$$

$$\mathcal{P}(\theta_{\boldsymbol{k}},\ \theta_{\boldsymbol{k}'},\ \theta_{\boldsymbol{k}+\boldsymbol{k}'}) \propto 1 + \frac{\pi^{3/2}}{4} p^{(3)}(\boldsymbol{k},\ \boldsymbol{k}') \times \cos(\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{k}'} - \theta_{\boldsymbol{k}+\boldsymbol{k}'}), \tag{11}$$

where $\boldsymbol{k} \neq \boldsymbol{k}'$. These are the explicit forms of the relation between phase correlations and the bispectrum in the first-order approximation. We find that the distribution of the "phase sum" $\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{k}'} - \theta_{\boldsymbol{k}+\boldsymbol{k}'}$ is determined only by the normalized bispectrum at the first-order level, although higher order normalized polyspectra can contribute in general. The higher order calculations show that the distribution of the phase sum $\theta_{\boldsymbol{k}_1} + \cdots + \theta_{\boldsymbol{k}_N}$ for the modes with closed wavevectors $\boldsymbol{k}_1 + \cdots + \boldsymbol{k}_N = 0$ is determined by normalized polyspectra of order 3 to N in the lowest order approximation, where the identification $\theta_{\boldsymbol{k}} = -\theta_{-\boldsymbol{k}}$ is understood. It was vaguely suggested that there is some relationship between the phase sum and polyspectra based on a particular non-Gaussian model by Watts & Coles (2003). We now find the explicit relationship between them in general non-Gaussian fields.

If we further integrate all phases but one particular $\theta_{\boldsymbol{k}}$, the one-point probability function of a phase is uniform, $\mathcal{P}(\theta_{\boldsymbol{k}}) = 1/2\pi$, which is consistent with the previous N-body analysis (Suginohara & Suto 1991). This conclusion does not depend on the first-order approximation, since a single wavevector can not be closed unless $\boldsymbol{k} = 0$. Similarly, the two-point probability function $\mathcal{P}(\theta_{\boldsymbol{k}},\ \theta_{\boldsymbol{k}'})$ is also uniform unless $\boldsymbol{k} = 2\boldsymbol{k}'$. At first glance, this conclusion seems to contradict the reported nonuniform distribution of the phase difference of neighboring wavevectors $D_{\boldsymbol{k}} \equiv \theta_{\boldsymbol{k}+\Delta\boldsymbol{k}} - \theta_{\boldsymbol{k}}$ in N-body data (Scherrer et al. 1991; Coles & Chiang 2000; Chiang 2001; Chiang et al. 2002; Watts et al. 2003), where $\Delta\boldsymbol{k}$ is a fixed small vector. The same arguments are also applied to higher order approximations, so that the phase correlations between neighboring wavenumbers should not appear even in strongly non-Gaussian fields in a statistical sense.

To resolve this puzzle, it is useful to consider the conditional probability function given a Fourier coefficient of a small wavenumber $\alpha_{\Delta\boldsymbol{k}}$. In the first-order approximation, the joint probability of having phases $\theta_{\boldsymbol{k}}$,$\theta_{\boldsymbol{k}+\Delta\boldsymbol{k}}$ with fixed $\alpha_{\Delta\boldsymbol{k}}$ is given by

$$\mathcal{P}(\theta_{\boldsymbol{k}},\ \theta_{\boldsymbol{k}+\Delta\boldsymbol{k}}\ |\alpha_{\Delta\boldsymbol{k}}|) \propto 1 + \frac{\pi}{2}\ |\alpha_{\Delta\boldsymbol{k}}| \times \cos(\theta_{\boldsymbol{k}+\Delta\boldsymbol{k}} - \theta_{\boldsymbol{k}} - \theta_{\Delta\boldsymbol{k}}) p^{(3)}(\Delta\boldsymbol{k},\ \boldsymbol{k}), \tag{12}$$

which arise the nonuniform distribution pattern of the phase difference $D_{\boldsymbol{k}}$. The pattern depends on the fixed phase $\theta_{\Delta\boldsymbol{k}}$, which means the pattern varies from sample to sample, and this is exactly what is reported in the N-body analyses. The pattern of the phase difference should be significant for red power spectrum, which is also consistent with N-body analyses. The functional form of equation (12) also agrees with the N-body analysis (Watts et al. 2003). The statistics of phase difference is thus the manifestation of the large-scale patterns of individual realizations. The position of the trough in the distribution of the phase difference corresponds to the phase of the mode $\Delta\boldsymbol{k}$, and the degree of deviations from the uniform distribution depends on the specific amplitude of the mode $\Delta\boldsymbol{k}$ and also on the normalized bispectrum.

3. A NUMERICAL DEMONSTRATION

Equations (10) and (11) relate the bispectrum to the distribution of the phase sum $\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{k}'} - \theta_{\boldsymbol{k}+\boldsymbol{k}'}$. To see if this kind of phase information is practically robust, we numerically examine simple examples of non-Gaussian fields. Instead of examining cosmological simulations, the following simple example is enough to compare the numerical phase distributions and theoretical predictions. Series of non-Gaussian fields are simply generated by exponential mapping of a random Gaussian field:

$$f(\boldsymbol{x}) = \exp[g\phi(\boldsymbol{x}) - g^2/2] - 1, \tag{13}$$

where ϕ is a random Gaussian field with zero mean, unit variance, and g is the non-Gaussian parameter. We simply take a flat power spectrum for the Gaussian field ϕ. The field f has

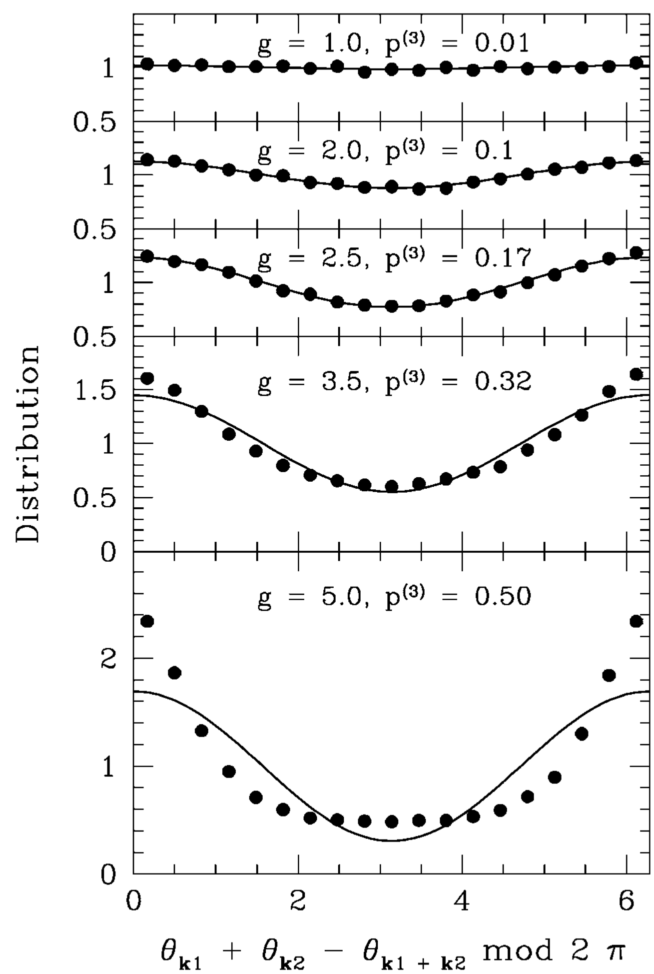

FIG. 1.—Distribution of the phase sum for a particular configuration of wavevectors. Five non-Gaussian fields are shown, where g is the non-Gaussian parameter and $p^{(3)}$ is the normalized bispectrum for the particular configuration. Theoretical predictions by normalized bispectra in the first-order approximation are shown by solid curves.

zero mean and variance $\langle f^2 \rangle = \exp(g^2) - 1$, and is called the lognormal field (Coles & Jones 1991). This field has quite similar statistical properties to gravitationally evolved non-Gaussian fields and approximately follows the hierarchical model of higher order correlations. The parameter g controls the non-Gaussianity, and the random Gaussian field is recovered by taking the limit $g \to 0$. The random field f is generated on 64^3 grids in a rectangular box with the periodic boundary condition.

In Figure 1, the distribution of the phase sum $\theta_{\boldsymbol{k}_1} + \theta_{\boldsymbol{k}_2} - \theta_{\boldsymbol{k}_1+\boldsymbol{k}_2}$ is plotted for a binned configuration of the wavevectors, $|\boldsymbol{k}_1| = [0.4, 0.5]$, $|\boldsymbol{k}_2| = [0.5, 0.6]$, $\theta_{12} = [50°, 60°]$, as an example, where θ_{12} is the angle between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, and the magnitudes of the wavenumber are in units of the Nyquist wavenumber.

The phase sum is averaged over the wavevectors in a configuration bin. The points represent the distributions of the phase sum in each realization. Poisson error bars are smaller than the size of the points. The normalized bispectra $p^{(3)}(\boldsymbol{k}_1, \boldsymbol{k}_2)$ are numerically evaluated from each realization, which are used to draw the theoretical curves in the first-order approximation of equation (11). There is not any fitting parameter at all. The agreement is remarkable in weakly non-Gaussian fields. When the non-Gaussianity becomes high, the data points deviate from the first-order approximation, and the distribution of the phase sum is sharply peaked at $\theta_{\boldsymbol{k}_1} + \theta_{\boldsymbol{k}_2} - \theta_{\boldsymbol{k}_1+\boldsymbol{k}_2} = 0 \bmod 2\pi$. Up to $g \sim 3.0$, or $\langle f^2 \rangle^{1/2} \simeq 100$, the distribution of the phase sum is accurately described by the first-order approximation and is determined only by the normalized bispectrum. Even though the non-Gaussianity $g \sim 3.0$ on scales of the Nyquist wavenumber is beyond the perturbative regime, the normalized bispectrum on scales of the presently tested configuration is still within the perturbative regime, $p^{(3)} \sim 0.25$. This means that the phase sum is well approximated by first-order formula of the present work even when the field is strongly nonlinear in dynamics, as long as the parameter $P(k)/V$ on the relevant scales is small. Increasing the power on relevant scales and/or decreasing the volume drive the phase correlation large, due to the fact that the phase correlations are particularly dependent on significant features in the sample.

4. SUMMARY

The structure of the phase correlations in non-Gaussian fields is elucidated. The method to relate the joint distribution of phases to polyspectra is newly found and developed. The distribution of the phase sum of closed wavevectors is represented by the polyspectra. We found the statistics of the phase difference reflect the particular phase of the mode within an individual sample. The distribution of the phase sum of three or more modes carries the statistically useful information. The understanding of the phase correlations in non-Gaussian fields is now reached unprecedented level in this Letter, so that many investigations to make use of the phase information will be followed, such as the analysis of the non-Gaussianity of all kinds of cosmic fields, the nonlinear gravitational evolution of the density fields, the biasing and redshift-space distortion effects on the galaxy clustering, the primordial non-Gaussianity from inflationary models, and so forth. One may also hope that phase information can be useful in statistical analyses of all kinds of non-Gaussian fields, from various phenomena of pattern formations to human brain mapping, etc.

I acknowledge support from grants MEXT 13740150.

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