Synthetic turbulent velocity field using statistical method

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Outline

What is turbulence?

- Turbulence: complex, aperiodic fluid motion with strong vortical structures.
- We study *Homogeneous and Isotropic Turbulence (HIT)*: statistics invariant under space translations and rotations.

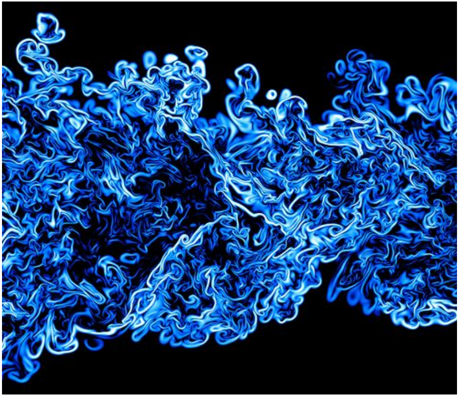


Figure: Slice through scalar dissipation (CNRS UMR 6614 CORIA and JSC).

Random Fourier model

- Build a synthetic velocity field by summing random Fourier modes.
- Frozen-turbulence assumption (no time dependence):

$$u^s(x) = 2 \sum_{n=1}^{N} \hat{u}_n \cos(\kappa^n \cdot x + \psi_n)\, \sigma^n \tag{1}$$

κ^n wave vector (random on a half-sphere to preserve isotropy)

σ^n direction (divergence-free: $\kappa^n \cdot \sigma^n = 0$)

ψ_n random phase (uniform $\mathcal{U}[0, 2\pi]$)

$\hat{u}_n$ amplitude linked to prescribed energy spectrum $E(\kappa_n)$

Coefficient sampling

- Wave vector components (spherical coordinates):

$$\kappa_1 = \sin(\theta)\cos(\varphi) \tag{2}$$
$$\kappa_2 = \sin(\theta)\sin(\varphi) \tag{3}$$
$$\kappa_2 = \cos(\theta) \tag{4}$$

- Sampling densities: $f_\theta(\theta) = \frac{\sin\theta}{2}$, $f_\varphi(\varphi) = \frac{1}{2\pi}$.
- Direction vector σ obtained by fixing an angle $\alpha \sim \mathcal{U}[0, 2\pi]$ and ensuring $\kappa \cdot \sigma = 0$.

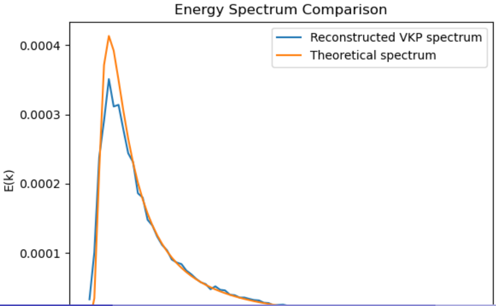
$$\sigma_1 = \cos(\varphi)\cos(\theta)\cos(\alpha) - \sin(\varphi)\sin(\alpha) \tag{5}$$
$$\sigma_2 = \sin(\varphi)\cos(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha) \tag{6}$$
$$\sigma_3 = -\sin(\theta)\cos(\alpha) \tag{7}$$

Energy spectrum: Kolmogorov and VKP

- Kolmogorov inertial-range law: $E(\kappa) = C_k \, \varepsilon^{2/3} \, \kappa^{-5/3}$ (valid in the inertial range).
- To capture large- and small-scale behavior use a full model such as von Kármán–Pao (VKP):

$$E_{\mathrm{VKP}}(\kappa) = \frac{2}{3} \alpha_e \, \kappa L_e \frac{(\kappa L_e)^4}{[(\kappa L_e)^2 + 1]^{17/6}} \exp\left(-2(\kappa L_\eta)^2 \right) \tag{8}$$

- Mode amplitudes: $\hat{u}_n = \sqrt{E(\kappa_n) \, \Delta\kappa_n}$ (log-spacing used for κ_n).



Metrics to assess synthetic field

- Energy spectrum reconstruction: match between theoretical and reconstructed $E(\kappa)$.
- Component means close to zero (homogeneity).
- RMS speed must match prescribed value.
- Velocity increments statistics (non-Gaussian heavy tails expected at small scales).

Empirical base metrics

Direction	Mean
x	-0.00269
y	-0.00010
z	-0.00126

Direction	RMS (expected: 0.222)
x	0.19583
y	0.17599
z	0.19307

Table: Velocity mean and RMS

Velocity increments & heavy tails

- Velocity increments: $\delta_r u = u(x + r) - u(x)$.
- In random Fourier base model increments are typically Gaussian (kurtosis $= 3$).
- Real turbulence: increments show heavy tails (increasing kurtosis when $r \to 0$).

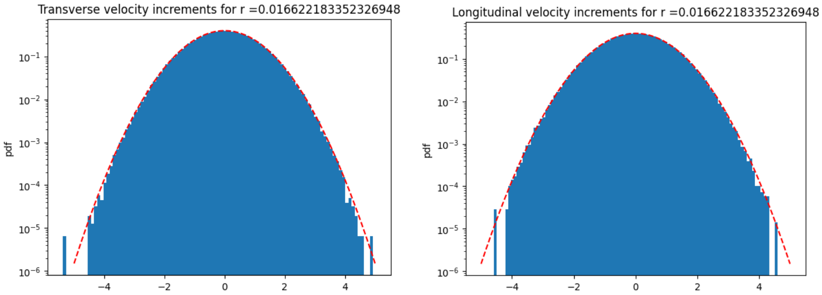


Figure: Transverse and longitudinal increments (model vs Gaussian)

Random Fourier Velocity field

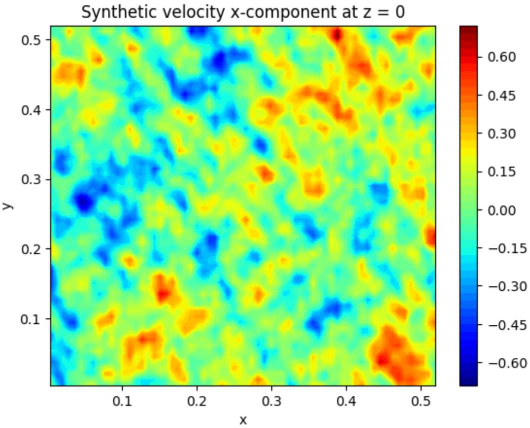


Figure: Synthetic velocity field obtain via random Fourier model

Goal and strategy

- **Goal:** produce heavy-tailed velocity increments while preserving energy spectrum and isotropy.
- **Strategy:** parameterize coefficients (angles, phases, amplitudes) as learnable parameters and optimize a loss.

Losses

Flatness (kurtosis) objective

- Use kurtosis (or "flatness" = kurtosis − 3) of velocity increments as a loss term to encourage heavy tails.

Energy-spectrum objective

- MSE between reconstructed and target spectrum, optionally with a smoothness regulariser (penalize high-frequency oscillations in $E(\kappa)$).

Combined loss: $\mathcal{L} = \lambda_F \mathcal{L}_F + \lambda_{ES} \mathcal{L}_{ES} +$ regularisers

Preliminary experiments

- Tested optimization toward Gaussian target flatness (flatness $= 0$) from different initialization.
- Starting from Gaussian increments: parameters remain close to initial Janin et al. choice.
- Starting from heavy-tailed increments: θ adapt to a flatter distribution.

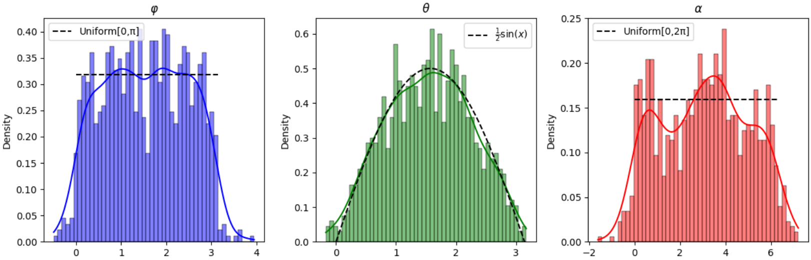


Figure: Angle distributions starting from Gaussian increments

Preliminary experiments

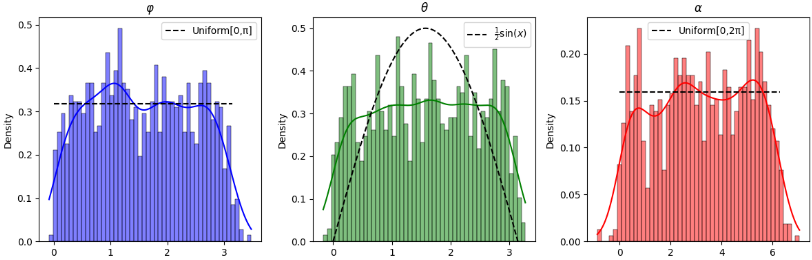


Figure: Angle distributions starting from heavy-tailed increments

- Observations: RMS and means largely preserved while increment statistics change.

Distribution shift

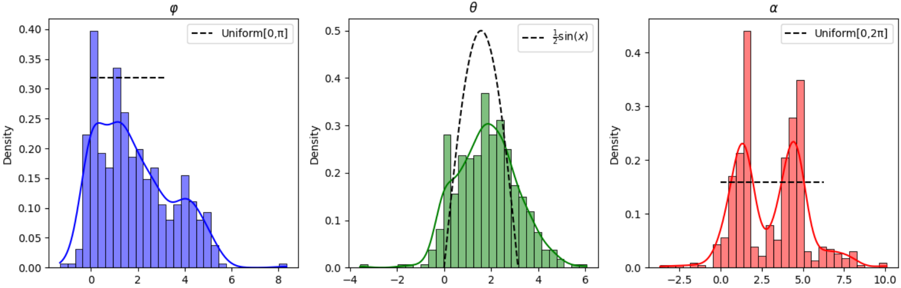


Figure: Learned angles distributions

Key observations

- The Random Fourier model can be steered (via angles) to modify increment statistics.
- Trade-offs observed: stronger heavy tails sometimes perturb RMS.

Parameters used

Parameter	Value	Unit
Length of box	$\frac{\pi}{6}$	m
Number of modes	250 or 500	–
RMS speed	0.222	$\mathrm{m\,s^{-1}}$
Integral length scale	0.024	m
Viscosity	1.8×10^{-5}	$\mathrm{m^2/s}$
Minimum wave number	$\frac{2\pi}{1.0}$	$\mathrm{m^{-1}}$
Maximum wave number	$\frac{2\pi}{0.01}$	$\mathrm{m^{-1}}$

References I