Synthetic turbulence generation using statistical methods

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What is turbulence?

- Turbulence: complex, aperiodic fluid motion with strong vortical structures.
- We study *Homogeneous and Isotropic Turbulence (HIT)*: statistics invariant under space translations and rotations.

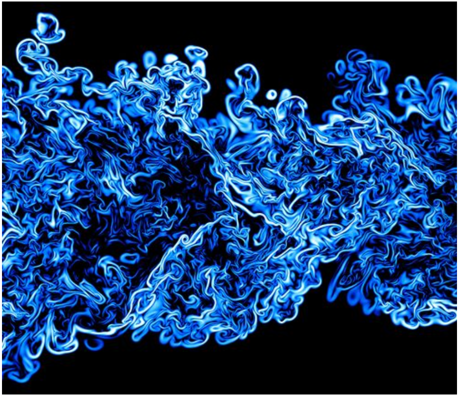

Figure: Slice through scalar dissipation (CNRS UMR 6614 CORIA and JSC).

Random Fourier model

- Build a synthetic velocity field by summing random Fourier modes.
- Frozen-turbulence assumption (no time dependence):

$$u^s(x) = 2\sum_{n=1}^{N} \hat{u}_n \cos(\kappa^n \cdot x + \psi_n)\,\sigma^n \tag{1}$$

κ^n wave vector (random on a half-sphere to preserve isotropy)

σ^n direction (divergence-free: $\kappa^n \cdot \sigma^n = 0$)

ψ_n random phase (uniform $\mathcal{U}[0, 2\pi]$)

$\hat{u}_n$ amplitude linked to prescribed energy spectrum $E(\kappa_n)$

Coefficient sampling

- Wave vector components (spherical coordinates):

$$\kappa_1 = \sin(\theta)\cos(\varphi) \tag{2}$$
$$\kappa_2 = \sin(\theta)\sin(\varphi) \tag{3}$$
$$\kappa_2 = \cos(\theta) \tag{4}$$

- Sampling densities: $f_\theta(\theta) = \frac{\sin\theta}{2}$, $f_\varphi(\varphi) = \frac{1}{2\pi}$.
- Direction vector σ obtained by fixing an angle $\alpha \sim \mathcal{U}[0, 2\pi]$ and ensuring $\kappa \cdot \sigma = 0$.

$$\sigma_1 = \cos(\varphi)\cos(\theta)\cos(\alpha) - \sin(\varphi)\sin(\alpha) \tag{5}$$
$$\sigma_2 = \sin(\varphi)\cos(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha) \tag{6}$$
$$\sigma_3 = -\sin(\theta)\cos(\alpha) \tag{7}$$

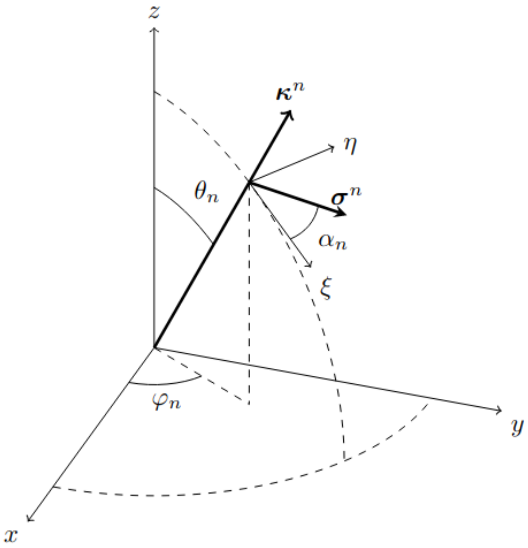

Figure: Wave vector geometry

Energy spectrum

- von Kármán–Pao (VKP) energy spectrum:
  $E_{\mathrm{VKP}}(\kappa) = \frac{2}{3}\alpha_e\,\kappa L_e \frac{(\kappa L_e)^4}{[(\kappa L_e)^2+1]^{17/6}} \exp\left(-2(\kappa L_\eta)^2\right)$

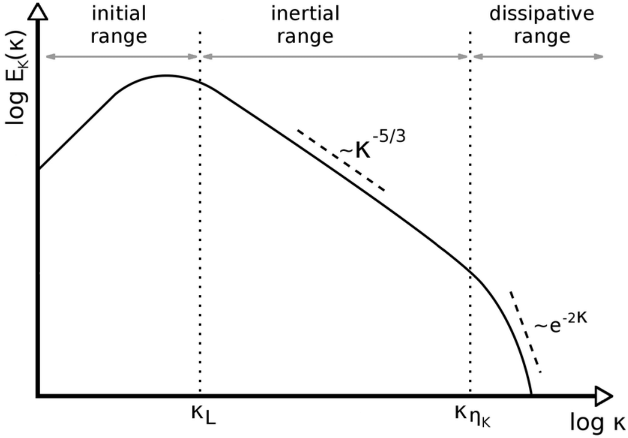

Figure: Energy spectrum of a turbulent flow

Losses

Flatness (excess kurtosis) objective

- Use flatness (= kurtosis − 3) of velocity increments as a loss term to encourage heavy tails.

$$\mathcal{L}_F = \frac{1}{n}\sum_{i=1}^{n}(kurt - 3 - F_{\text{target}}) \tag{8}$$

Energy-spectrum objective

- MSE between reconstructed and target spectrum, optionally with a smoothness regulariser (penalize high-frequency oscillations in $E(\kappa)$).

$$L_{ES} = \frac{1}{n}\sum_{i=1}^{n}(E_{\text{rec},i} - E_{\text{theory},i})^2 \tag{9}$$

Combined loss: $\mathcal{L} = \mathcal{L}_F + 10^3\mathcal{L}_{ES}$

Angles

- Shift the focus solely on the angles $(\varphi, \theta, \alpha)$.
- Ensure the condition $\kappa \cdot \sigma = 0$.

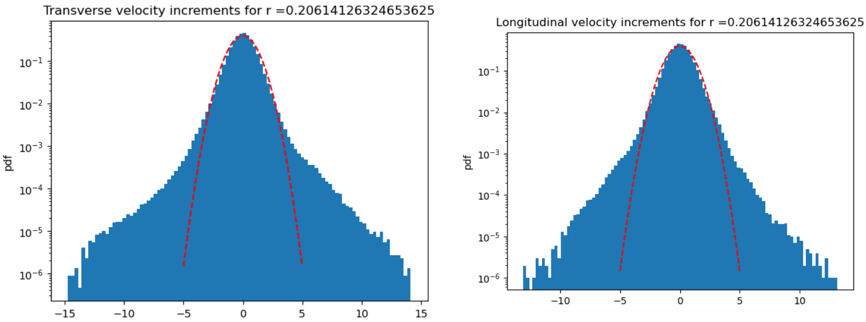

Figure: Velocity increments after learning on $(\varphi, \theta, \alpha)$

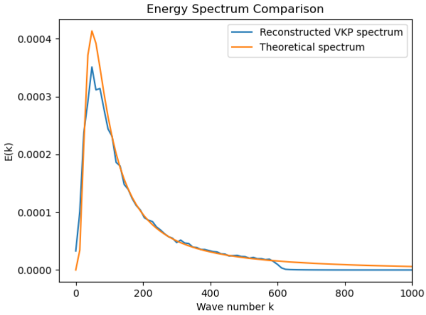

Figure: Energy spectrum reconstructed with the learned $(\varphi, \theta, \alpha)$

Direction	Mean
x	0.002564
y	-0.00652
z	-0.00093

Direction	RMS (expected: 0.222)
x	0.12904
y	0.31651
z	0.11406

Table: Velocity mean and RMS

Distribution shift

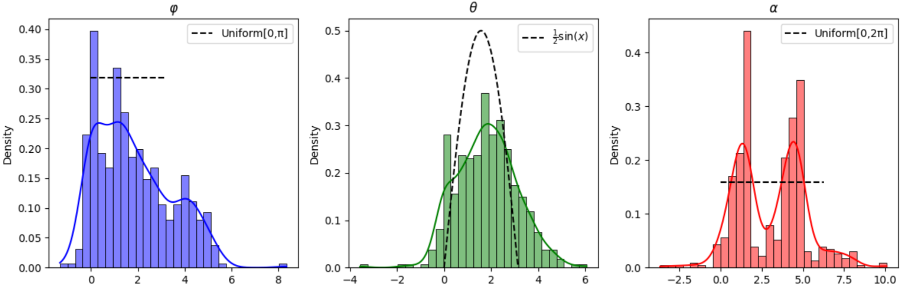

Figure: Learned angles distributions

Conclusion

- The Random Fourier model can be steered (via angles) to modify increment statistics while preserving spectral fidelity.
- Trade-offs observed: stronger heavy tails sometimes perturb RMS.

Future work :

- Incorporate time dependence (non-frozen turbulence).
- Change the turbulence hypothesis.