Week 6, November 4th: Inequalies, Moment Generating Function, Cumulative Ditribut

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1 Important exercises

Exercise 1. The goal of this exercise is to review basic properties of exponential random variables and justify the fact that we can view exponential random variables as appropriate limits of scaled geometric random variables with a small probability of success.

(i) Let $X_n \sim \mathrm{Ge}(p/n)$. show that for any $x \in \mathbb{R}$ we have

$$\lim_{n\to+\infty} F_{n^{-1}X_n}(x) = F_X(x)$$

where $X \sim \mathrm{Exp}(p)$. Therefore, nX_n is almost an exponential random variable for n large.

(ii) We observe that the moment generating function $M_X(t)$ of an absolutely continuous random variable can be defined in analogy with the discrete case as follows

$$M_X(t) := \mathbb{E}[\exp(tX)] \in (0,+\infty].$$

Compute this for $X \sim \mathrm{Exp}(p)$

(iii) Deduce from the previous calculation the value of $\mathbb{E}[X^2]$.

Exercise 2. Consider now a Gaussian random variable $X \sim \mathrm{N}(0,\sigma^2)$.

(i) Prove that for all $t \in \mathbb{R}$ we have

$$M_X(t) = \exp(\frac{\sigma^2 t^2}{2})$$

(*Hint: use Gauss integral formula*)

(ii) Deduce the Chernoff bound

$$\forall x \geq 0, \quad \mathbb{P}[X \geq x] \leq \exp(-\frac{x^2}{2\sigma^2}).$$

Exercise 3. Let X, Y be two independent random variables with $X, Y \sim \mathrm{U}([0,1])$.

(i) Compute the density of $Z = X + Y$.

(ii) Using point (i) and the results of the course, compute expectation and variance of Z.

(iii) Propose an alternative method to compute expectation and variance of Z using the independence of X and Y.

Exercise 4. Let X be a real valued absolutely continuous random variable such that $\mathbb{P}[X \in (a,b)] = 1$ for some $-\infty \le a < b \le +\infty$. Assume in addition that F_X is piecewise $\mathcal{C}^1$. Let φ be a function of class $\mathcal{C}^1$ on (a,b) and such that $\varphi'(x) > 0$ for all $x \in (a,b)$. The goal of the exercise is to show that the random variable $Y = \varphi(X)$ is absolutely continuous with density given by

$$f_Y(y) = \frac{f_X(\varphi^{-1}(y))}{\varphi'(\varphi^{-1}(y))}\mathbf{1}_{(c,d)}(y),$$

where $(c,d) = \varphi((a,b))$ is the image of (a,b) through φ.

(i) Show that the cumulative distribution function F_Y of Y is given by

$$F_Y(y) = \begin{cases} 0 & \text{if } y \le c \\ F_X(\varphi^{-1}(y)) & \text{if } c < y \le d \\ 1 & \text{if } y > d \end{cases}$$

(ii) Show that F_Y is piecewise $\mathcal{C}^1$ and conclude using the results of the course.

Exercise 5. For any $a, b > 0$ define

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$$

(i) Show the relations

$$a\beta(a,b+1) - b\beta(a+1,b) = 0,$$

and

$$\beta(a,b) - \beta(a+1,b) = \beta(a,b+1).$$

(ii) Using the conclusion of (i), show that if $X \sim Beta(a,b)$, then

$$\mathbb{E}[X] = a/(a+b), \quad \mathrm{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$$

Exercise 6. Let $X \sim \mathrm{U}([0,1])$. What is the law of $Y = \ln(1/X)$? Describe in words how to use this result for simulation purposes.

2 Optional exercises

Exercise 7. Let X be a random number in $(0,1)$ (not necessarily uniformly distributed). X divides the interval $(0,1)$ into two segments. We call $Y \ge 1$ the ratio between the longer and the shorter segment.

(i) Express Y as a function of X .
(*Hint: Use the events* $\{X < 1/2\}$ *and* $\{X \ge 1/2\}$)

(ii) Assume that $X \sim \mathrm{U}(0,1)$. Compute the cumulative distribution function and the density of Y

(iii) Show that $\mathbb{E}[Y] = +\infty$

(iv) Assume that X is an absolutely continuous random variable, whose density f_X satisfies:

$$f_X(x) + f_X(1-x) = 2, \forall x \in (0,1).$$

Show that the distribution of Y is the same found for the case when $X \sim \mathrm{U}([0,1])$.
(*Hint: It may help to show that* $F_X(z) - F_X(1-z) = 2z - 1 \; \forall z \in (0,1)$.)

Exercise 8. Let X be a Gaussian random variables with $X \sim \mathrm{N}(0, \sigma_X^2)$

(i) Show that $\mu_{-X} = \mu_X$, i.e. the law of the random variable $-X$ is $\mathrm{N}(0, \sigma_X^2)$

(ii) Show that $\mu_{\alpha X} = \mathrm{N}(0, \alpha^2 \sigma_X^2)$.

(iii) Prove by induction that for all $n \geq 1$

$$\mathbb{E}[X^n] = \begin{cases} \sigma_X^n (n-1)!! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Exercise 9. Let $X \sim \mathrm{U}([0,1])$.

(i) Compute the density f_X of X. Draw an analogy with discrete uniform random variables.

(ii) Compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

(iii) Is $Y = X^2$ an absolutely continuous random variable? If yes, compute its density f_Y.

Exercise 10. A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over $(0, 100)$. There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located $25, 50$, and 75 miles, respectively, from A. Do you agree? Why?