Stochastic maximum principle for optimal control of non exchangeable mean field systems.

Samy Mekkaoui

CMAP, École Polytechnique and LPSM

Advances in Financial Mathematics, Paris

Joint works with

Idris Kharroubi (LPSM, Sorbonne Université)
Huyên Pham (CMAP, École Polytechnique)
Xavier Warin (Senior Researcher, EDF)

27 January 2026

Contents

Motivations for optimal control of non exchangeable mean field systems

- Extend the known *MFC* theory to non exchangeable interactions. A lot of litterature has been developed recently within the graphons theory (see the works of Bayraktar et al. and Aurell et al. for instance) where an agent labeled by $u \in I := [0,1]$ interacts with the other agents through the probability measure

$$I \ni u \mapsto \frac{\int_{t_{i+1}} G(u,v)\mathbb{P}_{X_t^v}(\mathrm{d}x)\mathrm{d}v}{\int_{t_{i+1}} G(u,v)\mathrm{d}v}) \in \mathcal{P}(\mathbb{R}^d), \quad 0 \leqslant t \leqslant T, \quad u \in I. \tag{1}$$

.

Motivations for optimal control of non exchangeable mean field systems

- Extend the known *MFC* theory to non exchangeable interactions. A lot of litterature has been developed recently within the graphons theory (see the works of Bayraktar et al. and Aurell et al. for instance) where an agent labeled by $u \in I := [0,1]$ interacts with the other agents through the probability measure

$$I \ni u \mapsto \frac{\int_{t_{i+1}} G(u,v)\mathbb{P}_{X_t^v}(\mathrm{d}x)\mathrm{d}v}{\int_{t_{i+1}} G(u,v)\mathrm{d}v}) \in \mathcal{P}(\mathbb{R}^d), \quad 0 \leqslant t \leqslant T, \quad u \in I. \tag{1}$$

.

- Extend the framework without specifying the type of interaction. Dynamics are functions of the collection of laws $(\mathbb{P}_{X_t^v})_{v \in I}$.
  $\rightarrow$ Needs to define the maps on a suitable space. Formally, the study of such models is motivated by the limit of the following N-agent dynamical system :

$$\begin{cases} \mathrm{d}X_t^{i,N} & = b(\frac{i}{N}, X_t^{i,N}, \alpha_t^{i,N}, \frac{\frac{1}{N}\sum_{j=1}^N G(\frac{i}{N},\frac{j}{N})\delta_{X_t^{j,N}}}{\frac{1}{N}\sum_{j=1}^N G(\frac{i}{N},\frac{j}{N}}) \mathrm{d}t + \sigma(\frac{i}{N}, X_t^{i,N}, \alpha_t^{i,N}, \frac{\frac{1}{N}\sum_{j=1}^N G(\frac{i}{N},\frac{j}{N})\delta_{X_t^{j,N}}}{\frac{1}{N}\sum_{j=1}^N G(\frac{i}{N},\frac{j}{N}}) \mathrm{d}W_t^{i,N} \\ X_0^{i,N} & = \xi^{i,N}. \end{cases}$$

A strong formulation

Dynamics of the controlled state processes:

$$\begin{cases} dX_t^u &= b(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}t + \sigma(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}W_t^u, \quad 0 \leqslant t \leqslant T, u \in I, \\ X_0^u &= \xi^u. \end{cases} \tag{2}$$

Cost Functional : Aim to minimize over a collection of processes $\boldsymbol{\alpha} = (\alpha^u)_{u\in I}$ in a suitable class $\mathcal{A}$

$$J^S(\boldsymbol{\alpha}) = \int_{t_{i+1}} \mathbb{E}\Big[\int_0^T f(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}t + g(u, X_T^u, (\mathbb{P}_{X_T^v})_{v\in I})\Big]\mathrm{d}u \tag{3}$$

$\rightarrow$ Compute $V_0^S = J^S(\boldsymbol{\alpha}^\star)$ where $\boldsymbol{\alpha}^\star$ is a minimizer of J^S.

A strong formulation

Dynamics of the controlled state processes:

$$\begin{cases} dX_t^u &= b(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}t + \sigma(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}W_t^u, \quad 0 \leqslant t \leqslant T, u \in I, \\ X_0^u &= \xi^u. \end{cases} \tag{2}$$

Cost Functional : Aim to minimize over a collection of processes $\boldsymbol{\alpha} = (\alpha^u)_{u\in I}$ in a suitable class $\mathcal{A}$

$$J^S(\boldsymbol{\alpha}) = \int_{t_{i+1}} \mathbb{E}\Big[\int_0^T f(u, X_t^u, \alpha_t^u, (\mathbb{P}_{X_t^v})_{v\in I})\mathrm{d}t + g(u, X_T^u, (\mathbb{P}_{X_T^v})_{v\in I})\Big]\mathrm{d}u \tag{3}$$

$\rightarrow$ Compute $V_0^S = J^S(\boldsymbol{\alpha}^\star)$ where $\boldsymbol{\alpha}^\star$ is a minimizer of J^S.

• In the strong formulations, maps (b, σ, f, g) are defined over the space

$$L^2(I; \mathcal{P}_2(\mathbb{R}^d)) = \{u \to \mu^u \text{ is measurable and } \int_{t_{i+1}} \mathcal{W}_2(\mu^u, \delta_0)^2 \mathrm{d}u < +\infty\}.$$

• Due to the uncountable family of independent Brownian motions $\{W^u : u \in I\}$, this formulation lacks of joint measurability of the map $(u, \omega) \mapsto X^u(\omega)$ on the space product $(I \times \Omega, \mathcal{B}(I) \otimes \mathcal{A})$. However, from (3), the control problem is defined at the level of the law of the processes

$$u \mapsto \mathbb{P}_{(X^u, W^u)} \in \mathcal{P}_2(\mathcal{C}^d_{[0,T]} \times \mathcal{C}^n_{[0,T]}),$$

which can be shown to be measurable under some structural assumptions on $\boldsymbol{\alpha}$.

Since we are working at the level of the laws of the processes and for numerical purposes, we can relax the control problem formulation (2)-(3)

A label state formulation

Dynamics of the controlled state processes:

$$\begin{cases} dX_t &= b(U, X_t, \alpha_t, \mathbb{P}_{(U,X_t)})\mathrm{d}t + \sigma(U, X_t, \alpha_t, \mathbb{P}_{(U,X_t)})\mathrm{d}W_t, \quad 0 \leqslant t \leqslant T \\ X_0 &= \xi. \end{cases} \tag{4}$$

Cost Functional : Aim to minimize over α over a suitable class $\mathcal{A}$

$$J^W(\alpha) = \mathbb{E}\Big[\int_0^T f(U, X_t, \alpha_t, \mathbb{P}_{(U,X_t)})\mathrm{d}t + g(U, X_T, \mathbb{P}_{(U,X_T)})\Big] \tag{5}$$

$\rightarrow$ Compute $V_0^W = J^W(\alpha^\star)$ where $\alpha^\star$ is a minimizer of J^W.

Since we are working at the level of the laws of the processes and for numerical purposes, we can relax the control problem formulation (2)-(3)

A label state formulation

Dynamics of the controlled state processes:

$$\begin{cases} dX_t &= b(U, X_t, \alpha_t, \mathbb{P}_{(U,X_t)})\mathrm{d}t + \sigma(U, X_t, \alpha_t, \mathbb{P}_{(U,X_t)})\mathrm{d}W_t, \quad 0 \leqslant t \leqslant T \\ X_0 &= \xi. \end{cases} \tag{4}$$

Cost Functional : Aim to minimize over α over a suitable class $\mathcal{A}$

$$J^W(\alpha) = \mathbb{E}\Big[\int_0^T f(U, X_t, \alpha_t, \mathbb{P}_{(U,X_t)})\mathrm{d}t + g(U, X_T, \mathbb{P}_{(U,X_T)})\Big] \tag{5}$$

$\rightarrow$ Compute $V_0^W = J^W(\alpha^\star)$ where $\alpha^\star$ is a minimizer of J^W.

In the label-state formulation, maps (b, σ, f, g) are defined over the space

$$\mathcal{P}_2^\lambda(I \times \mathbb{R}^d) = \{\mu \in \mathcal{P}_2(I \times \mathbb{R}^d) : \mathrm{pr}_{1\sharp}\mu = \lambda\}.$$

Connection between strong and label-state formulation

- The connection between the 2 formulations comes from the disintegration theorem on Polish spaces. Given $\mu \in \mathcal{P}_2^{\lambda}(I \times \mathbb{R}^d)$, there exists an $\mathrm{d}u$ a.e unique family of measures $\mu^u \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mu(\mathrm{d}u, \mathrm{d}x) = \mu^u(\mathrm{d}x)\mathrm{d}u$. In our modeling, the fundamental link is

$$\mathbb{P}_{X_t^u} = \mathbb{P}_{X_t | U=u}, \quad \mathrm{d}u \text{ a .e,} \tag{6}$$

  when given the same control map $\hat{a}$.
- Label-state formulation is more suitable for numerical experiments but we lack pathwise interpretation of an optimal control. However, one can show that $V_0^S = V_0^W$ once, we can find an optimal control $\boldsymbol{\alpha}$ or α in the strong or the label-state formulation, formally by replacing u by U.

Connection between strong and label-state formulation

- The connection between the 2 formulations comes from the disintegration theorem on Polish spaces. Given $\mu \in \mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$, there exists an $\mathrm{d}u$ a.e unique family of measures $\mu^u \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mu(\mathrm{d}u, \mathrm{d}x) = \mu^u(\mathrm{d}x)\mathrm{d}u$. In our modeling, the fundamental link is

$$\mathbb{P}_{X_t^u} = \mathbb{P}_{X_t|U=u}, \quad \mathrm{d}u \text{ a .e,} \tag{6}$$

  when given the same control map $\hat{a}$.
- Label-state formulation is more suitable for numerical experiments but we lack pathwise interpretation of an optimal control. However, one can show that $V_0^S = V_0^W$ once, we can find an optimal control $\boldsymbol{\alpha}$ or α in the strong or the label-state formulation, formally by replacing u by U.

Objectives :

- Adapt the Pontryagin Maximum Principle to mean field control for non exchangeable mean field systems (NE-MFC) to find necessary and sufficient conditions for an admissible optimal control $\boldsymbol{\alpha}$.
- Propose an illustration in the Linear Quadratic (LQ) case with numerical illustrations.

Contents

A derivative in $L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$ (1)

(i) Given a function $v : \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \to \mathbb{R}$, we say that a measurable function

$$\frac{\delta}{\delta m} v : \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \times I \times \mathbb{R}^d \ \ni \ (\mu, u, x) \ \longmapsto \ \frac{\delta}{\delta m} v(\mu)(u, x) \in \mathbb{R} \tag{7}$$

is the linear functional derivative (or flat derivative) of v if

1. $(\mu, x) \mapsto \frac{\delta}{\delta m} v(\mu)(u, x)$ is continuous from $L^2(\mathcal{P}_2(\mathbb{R}^d)) \times \mathbb{R}^d$ to $\mathbb{R}$ for all $u \in I$;
2. for every compact set $K \subset \mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$ there exists a constant $C_K > 0$ such that
$$\left| \frac{\delta}{\delta m} v(\mu)(u, x) \right| \leqslant C_K \, (1 + |x|^2),$$
for all $u \in I$, $x \in \mathbb{R}^d$, $\mu \in K$;
3. we have
$$\begin{aligned} v(\nu) - v(\mu) \quad &= \quad \int_0^1 \langle \frac{\delta}{\delta m} v(\mu + \theta(\nu - \mu)), \nu - \mu \rangle \mathrm{d}\theta \\ &= \quad \int_0^1 \int_{t_{i+1}} \int_{\mathbb{R}^d} \frac{\delta}{\delta m} v(\mu + \theta(\nu - \mu)(u, x) \, (\nu^u - \mu^u)(\mathrm{d}x) \mathrm{d}u \mathrm{d}\theta \end{aligned}$$
for all $\mu, \nu \in \mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$.

A derivative in $\mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$ (2)

(ii) We say that the function v admits a continuously differentiable flat derivative if

1. v admits a flat derivative $\frac{\delta}{\delta m} v$ satisfying $x \mapsto \frac{\delta}{\delta m} v(\mu)(u,x)$ is Fréchet differentiable with Fréchet derivative denoted by $x \mapsto \partial \frac{\delta}{\delta m} v(\mu)(u,x)$ for all $(\mu, u) \in \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \times I$;
2. $(\mu, x) \mapsto \partial \frac{\delta}{\delta m} v(\mu)(u,x)$ is continuous from $\mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$ to $\mathbb{R}$ for all $u \in I$;
3. for every compact set $K \subset \mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$ there exists a constant $C_K > 0$ such that

$$\left| \partial \frac{\delta}{\delta m} v(\mu)(u,x) \right| \leqslant C_K \left(1 + |x|^2\right),$$

for all $u \in I$, $x \in \mathbb{R}^d$, $\mu \in K$.

Gateaux derivative on $\mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$

Let $f : I \times \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \to \mathbb{R}$ assumed to have a continuously differentiable linear functional derivative $\partial \frac{\delta}{\delta m} f$, i.e a map from $I \times \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \times I \times \mathbb{R}^d \to \mathbb{R}$. For $X, Y \in L^2\big(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^d\big)$ such that $\mathbb{P}_{(U,X)}, \mathbb{P}_{(U,Y)} \in \mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$ we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(f(u, x, \mathbb{P}_{(U,X+\epsilon Y)}) - f(u, x, \mathbb{P}_{(U,X)})\big) = \mathbb{E}\Big[\partial \frac{\delta}{\delta m} f(u, x, \mathbb{P}_{(U,X)})(U, X) \cdot Y)\Big] \tag{8}$$

Gateaux derivative on $\mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$

Let $f : I \times \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \to \mathbb{R}$ assumed to have a continuously differentiable linear functional derivative $\partial \frac{\delta}{\delta m} f$, i.e a map from $I \times \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \times I \times \mathbb{R}^d \to \mathbb{R}$. For $X, Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}^d)$ such that $\mathbb{P}_{(U,X)}, \mathbb{P}_{(U,Y)} \in \mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$ we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(f(u, x, \mathbb{P}_{(U,X+\epsilon Y)}) - f(u, x, \mathbb{P}_{(U,X)})\right) = \mathbb{E}\left[\partial \frac{\delta}{\delta m} f(u, x, \mathbb{P}_{(U,X)})(U, X) \cdot Y\right] \tag{8}$$

$\to$ Relation (8) is understood as a calculus of variation on $\mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$.

A notion of convexity in $\mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$

Let $f : I \times \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \to \mathbb{R}$. f is said to be convex if for every $u \in I$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$, we have :

$$\begin{aligned} f(u, x', \mu') - f(u, x, \mu) \geqslant\ & \partial_x f(u, x, \mu).(x' - x) \\ & + \mathbb{E}\Big[\partial \frac{\delta}{\delta m} f(u, x, \mu)(U, X).(X' - X)\Big]. \end{aligned} \tag{9}$$

where $(U, X') \sim \mu'$ and $(U, X) \sim \mu$.

A notion of convexity in $\mathcal{P}_2^\lambda(I \times \mathbb{R}^d)$

Let $f : I \times \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \to \mathbb{R}$. f is said to be convex if for every $u \in I$, $x, x' \in \mathbb{R}^d$, $\mu, \mu' \in L^2\big(\mathcal{P}_2(\mathbb{R}^d)\big)$, we have :

$$
\begin{aligned}
f(u, x', \mu') - f(u, x, \mu) \geqslant\ & \partial_x f(u, x, \mu).(x' - x) \\
& + \mathbb{E}\Big[\partial \frac{\delta}{\delta m} f(u, x, \mu)(U, X).(X' - X)\Big].
\end{aligned} \tag{9}
$$

where $(U, X') \sim \mu'$ and $(U, X) \sim \mu$.

- Can be extended to functions defined on $I \times \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \times A$:

$$
\begin{aligned}
f(u, x', \mu', a') - f(u, x, \mu, a) \geqslant\ & \partial_x f(u, x, \mu, a) \cdot (a' - a) + \partial_\alpha f(u, x, \mu, a) \cdot (a' - a) \\
& + \mathbb{E}\Big[\partial \frac{\delta}{\delta m} f(u, x, \mu, a)(U, X).(X' - X)\Big].
\end{aligned} \tag{10}
$$

Definition of the Hamiltonian H

The Hamiltonian $\mathbb{R}$-valued function H of the stochastic optimization problem is defined as :

$$H(u, x, \mu, y, z, a) = b(u, x, \mu, a) \cdot y + \sigma(u, x, \mu, a) : z + f(u, x, \mu, a) \tag{11}$$

where $(u, x, \mu, y, z, a) \in I \times \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \times A$.

Definition of the Hamiltonian H

The Hamiltonian $\mathbb{R}$-valued function H of the stochastic optimization problem is defined as :

$$H(u, x, \mu, y, z, a) = b(u, x, \mu, a) \cdot y + \sigma(u, x, \mu, a) : z + f(u, x, \mu, a) \tag{11}$$

where $(u, x, \mu, y, z, a) \in I \times \mathbb{R}^d \times \mathcal{P}_2^{\lambda}(I \times \mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^{d \times n} \times A$.

$\rightarrow$ Compute an optimality criterion involving the Hamiltonian H assuming differentiability and convexity as defined previously over the space $\mathcal{P}_2^{\lambda}(I \times \mathbb{R}^d)$
$\rightarrow$ In the following, A will denote a convex subset of $\mathbb{R}^m$ for $m \in \mathbb{N}^*$.

Adjoint Equations to X

We call adjoint processes of X any pair (Y, Z) in $\mathbb{S}^2([0,T];\mathbb{R}^d) \times \mathbb{H}^2([0,T];\mathbb{R}^{d\times n})$ such that (Y, Z) is solution to the adjoint equation

$$\begin{cases} dY_t = -\partial_x H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, Z_t, \alpha_t)\mathrm{d}t + Z_t \mathrm{d}W_t \\ \qquad\quad -\tilde{\mathbb{E}}\left[\partial \frac{\delta}{\delta m} H(\tilde{U}, \tilde{X}_t, \mathbb{P}_{(U,X_t)}, \tilde{Y}_t, \tilde{Z}_t, \tilde{\alpha}_t)(U, X_t)\right]\mathrm{d}t\ , \quad t \in [0,T]\ , \\ Y_T = \partial_x g(U, X_T, \mathbb{P}_{(U,X_T)}) + \tilde{\mathbb{E}}\left[\partial \frac{\delta}{\delta m} g(\tilde{U}, \tilde{X}_T, \mathbb{P}_{(U,X_T)})(U, X_T)\right]\ , \end{cases} \tag{12}$$

where $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\alpha})$ is an independent copy of (X, Y, Z, α) defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$

We now state the main results which are obtained under some standard regularity assumptions on b, σ, f and g.

Gâteaux derivative of J

For $\beta \in \mathcal{A}$ such that $\alpha + \epsilon\beta \in \mathcal{A}$ for $\epsilon > 0$ small enough, we have :

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(J^W(\alpha + \epsilon\beta) - J^W(\alpha)\right) = \mathbb{E}\Big[\int_0^T \Big(\partial_\alpha H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, Z_t, \alpha_t) \cdot \beta_t\Big)\mathrm{d}t\Big]$$

where X is given by (4), (Y, Z) are given by (12) and the Hamiltonian function H is given by (11).

We now state the main results which are obtained under some standard regularity assumptions on b, σ, f and g.

Gâteaux derivative of J

For $\beta \in \mathcal{A}$ such that $\alpha + \epsilon\beta \in \mathcal{A}$ for $\epsilon > 0$ small enough, we have :

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(J^W(\alpha + \epsilon\beta) - J^W(\alpha)\right) = \mathbb{E}\Big[\int_0^T \Big(\partial_\alpha H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, Z_t, \alpha_t) \cdot \beta_t\Big)\mathrm{d}t\Big]$$

where X is given by (4), (Y, Z) are given by (12) and the Hamiltonian function H is given by (11).

Necessary condition for optimality of α

Moreover, if we assume that H is convex in $a \in A$, that $\alpha = (\alpha_t)_{,0 \leqslant t \leqslant T}$ is optimal, that $X = (X_t)_{0 \leqslant t \leqslant T}$ is the associated optimal control state given by (4) and that $(Y, Z) = (Y_t, Z_t)_{0 \leqslant t \leqslant T}$ are the associated adjoint processes solving (12), then we have :

$$\forall a \in A, \quad H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, Z_t, \alpha_t) \leqslant H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, Z_t, a) \quad dt \otimes d\mathbb{P} \text{ a.e} \quad (13)$$

Sufficient condition for optimality of α

Let $\alpha \in \mathcal{A}$, $\mathbf{X}$ the corresponding controlled state process and $(\mathbf{Y}, \mathbf{Z})$ the corresponding adjoint processes.

$$(1) \quad \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \ni (x, \mu) \to g(U, x, \mu) \text{ is convex } \mathrm{d}\mathbb{P} \text{ a .e}$$

$$(2) \quad \mathbb{R}^d \times \mathcal{P}_2^\lambda(I \times \mathbb{R}^d) \times A \ni (x, \mu, a) \to H(U, x, \mu, Y_t, Z_t, a) \text{ is convex } dt \otimes d\mathbb{P} \text{ a.e}$$

If we assume also following the necessary condition for optimality :

$$H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, Z_t, \alpha_t) = \inf_{\beta \in A} H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, Z_t, \beta), \qquad dt \otimes d\mathbb{P} \text{ a.e}$$

Then, α is an optimal control in the sense that $J(\alpha) = \inf_{\alpha' \in \mathcal{A}} J(\alpha')$

Contents

Linear quadratic optimal control problem

We consider the following class of models (assuming for sake of simplicity σ constant and $A = \mathbb{R}^m$).

$$\begin{cases} dX_t & = \Big[\beta(U) + A(U)X_t + \tilde{\mathbb{E}}_{(\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}\big[G_A(U,\tilde{U})\tilde{X}_t\big] + B(U)\alpha_t\Big]\mathrm{d}t + \gamma(U)\mathrm{d}W_t, t \in [0,T] \\ X_0 & = \xi, \end{cases}$$

where $\beta \in L^\infty(I;\mathbb{R}^d)$, $\gamma \in L^\infty(I;\mathbb{R}^d)$, $A \in L^\infty(I,\mathbb{R}^{d\times d})$, $B \in L^\infty(I;\mathbb{R}^{d\times m})$ and $G_A \in L^\infty(I \times I;\mathbb{R}^{d\times d})$.

Linear quadratic optimal control problem

We consider the following class of models (assuming for sake of simplicity σ constant and $A = \mathbb{R}^m$).

$$\begin{cases} dX_t & = \Big[\beta(U) + A(U)X_t + \tilde{\mathbb{E}}_{(\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}\big[G_A(U,\tilde{U})\tilde{X}_t\big] + B(U)\alpha_t\Big]\mathrm{d}t + \gamma(U)\mathrm{d}W_t, t\in[0,T] \\ X_0 & = \xi, \end{cases}$$

where $\beta \in L^\infty(I;\mathbb{R}^d)$, $\gamma \in L^\infty(I;\mathbb{R}^d)$, $A \in L^\infty(I,\mathbb{R}^{d\times d})$, $B \in L^\infty(I;\mathbb{R}^{d\times m})$ and $G_A \in L^\infty(I\times I;\mathbb{R}^{d\times d})$.

Quadratic cost functional

$$\begin{aligned} J(\boldsymbol{\alpha}) = \mathbb{E}\Big[\int_0^T \Big[& Q(U)\big(X_t - \tilde{\mathbb{E}}_{[\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}]}[\tilde{G}_Q(U,\tilde{U})\tilde{X}_t]\big)\cdot\big(X_t - -\tilde{\mathbb{E}}_{[\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}[\tilde{G}_Q(U,\tilde{U})\tilde{X}_t] \\ & + \alpha_t\cdot R(U)\alpha_t + 2\alpha_t\cdot\Gamma(U)X_t + 2\alpha_t\cdot\tilde{\mathbb{E}}_{(\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}\big[G_{t_{i+1}}(U,\tilde{U})\tilde{X}_t\big]\Big]dt \\ & + H(U)\big(X_T - \tilde{\mathbb{E}}_{[\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}[\tilde{G}_H(U,\tilde{U})\tilde{X}_t]\big)\cdot\big(X_T - \tilde{\mathbb{E}}_{[\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}[\tilde{G}_H(U,\tilde{U})\tilde{X}_t]\big)\Big] \end{aligned} \quad (14)$$

Ansatz form for Y

We are looking for an ansatz $Y = (Y_t)_{0 \leqslant t \leqslant T}$ in the following form :

$$Y_t = 2\Big(K_t(U) X_t + \tilde{\mathbb{E}}_{(\tilde{U}, \tilde{X}_t) \sim \mathbb{P}_{(U, X_t)}} \big[\bar{K}_t(U, \tilde{U}) \tilde{X}_t \big] + \Lambda_t(U) \Big), \tag{15}$$

where $K \in C^1\big([0, T]; L^\infty(I; \mathbb{S}^d_+)\big)$, $\bar{K} \in C^1\big([0, T], L^2(I \times I; \mathbb{R}^{d \times d})\big)$ and $\Lambda \in C^1\big([0, T]; L^2(I; \mathbb{R}^d)\big)$ are to be determined through infinite dimensional Riccati equations.

Ansatz form for Y

We are looking for an ansatz $Y = (Y_t)_{0 \leqslant t \leqslant T}$ in the following form :

$$Y_t = 2\Big(K_t(U)X_t + \tilde{\mathbb{E}}_{(\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}\big[\bar{K}_t(U,\tilde{U})\tilde{X}_t\big] + \Lambda_t(U)\Big), \tag{15}$$

where $K \in C^1\big([0,T]; L^\infty(I;\mathbb{S}^d_+)\big)$, $\bar{K} \in C^1\big([0,T], L^2(I \times I;\mathbb{R}^{d\times d})\big)$ and $\Lambda \in C^1\big([0,T]; L^2(I;\mathbb{R}^d)\big)$ are to be determined through infinite dimensional Riccati equations.

→ We end up with a triangular Riccati system for K, $\bar{K}$ and Λ for which we can prove existence and uniqueness.

Ansatz form for Y

We are looking for an ansatz $Y = (Y_t)_{0 \leqslant t \leqslant T}$ in the following form :

$$Y_t = 2\Big(K_t(U)X_t + \tilde{\mathbb{E}}_{(\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}\big[\bar{K}_t(U,\tilde{U})\tilde{X}_t\big] + \Lambda_t(U)\Big), \tag{15}$$

where $K \in C^1\big([0,T]; L^\infty(I;\mathbb{S}^d_+)\big)$, $\bar{K} \in C^1\big([0,T], L^2(I \times I; \mathbb{R}^{d\times d})\big)$ and $\Lambda \in C^1\big([0,T]; L^2(I;\mathbb{R}^d)\big)$ are to be determined through infinite dimensional Riccati equations.

$\rightarrow$ We end up with a triangular Riccati system for K, $\bar{K}$ and Λ for which we can prove existence and uniqueness. $\rightarrow$ Finally, one can show existence and unicity to the associated forward SDE.

Proposition : Optimal control in the LQ case

In the Linear quadratic model, the unique optimal control $\hat{\alpha} = (\hat{\alpha}_t)_{0\leqslant t\leqslant T}$ is given by

$$\hat{\alpha}_t = S_t(U)X_t + \tilde{\mathbb{E}}_{(\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}\left[\bar{S}_t(U,\tilde{U})\tilde{X}_t\right] + \Gamma_t(U), \quad 0 \leqslant t \leqslant T, \tag{16}$$

where $X = (X_t)_{0\leqslant t\leqslant T}$ is the unique solution to the SDE obtained after replacing $\hat{\alpha}_t$ by (16) and where we denoted

$$\begin{cases} S_t(U) = -(R(U))^{-1}\Big((B(U))^\top K_t(U) + \Gamma(U)\Big), \\ \bar{S}_t(U,\tilde{U}) = -(R(U))^{-1}\Big((B(U)^\top \bar{K}_t(U,\tilde{U}) + G_{t_{i+1}}(U,\tilde{U})\Big) \\ \Gamma_t(U) = -(R(U))^{-1}(B(U))^\top \Lambda_t(U) \end{cases}$$

Systemic risk model with heterogeneous banks

Dynamics of the controlled state processes:

$$dX_t = \big[\kappa(X_t - \tilde{\mathbb{E}}_{(\tilde{U},\tilde{X}_t)\sim\mathbb{P}_{(U,X_t)}}\big[\tilde{G}_\kappa U, \tilde{U})\tilde{X}_t\big] + \alpha_t\big]dt + \sigma\, dW_t,$$
$$X_0 = \xi, \tag{17}$$

Cost Functional :

$$J(\alpha) = \mathbb{E}\Big[\int_0^T \Big[\eta(U)\big(X_t - \tilde{\mathbb{E}}\big[\tilde{G}_\eta(U,\tilde{U})\tilde{X}_t\big]\big)^2 + q(U)\alpha_t(X_t - \tilde{\mathbb{E}}\big[G_q(U,\tilde{U})\tilde{X}_t\big]) + \alpha_t^2\Big]\mathrm{d}t$$
$$+ r(U)\Big(X_T - \tilde{\mathbb{E}}\big[\tilde{G}_r(U,\tilde{U})\tilde{X}_T\big]\Big)^2\Big] \tag{18}$$

Numerical methods for learning the optimal feedback control map $\hat{\mathrm{a}}$.

Deep Graphon :
We parametrize directly the control in feedback form through a neural network taking as input the vector $(u, t, x, \mu) \in I \times [0, T] \times \mathbb{R} \times \mathcal{P}_2^\lambda(I \times \mathbb{R})$ and we try to learn the map $\hat{a}$ which minimizes the functional cost

$$J(\hat{\mathrm{a}}) = \inf_{a:L^0(I\times[0,T]\times\mathbb{R}^d\times\mathcal{P}_2^\lambda(I\times\mathbb{R}^d);A)} J(\mathrm{a}), \tag{19}$$

where the optimal control is then given by $\hat{\alpha}_t = \mathrm{a}(U, t, X_t, \mathbb{P}_{(U,X_t)})$ directly by gradient descent algorithm.

Numerical methods for learning the optimala control

Numerical methods for learning the optimal feedback control map $\hat{\mathrm{a}}$.

Deep Graphon :
We parametrize directly the control in feedback form through a neural network taking as input the vector $(u, t, x, \mu) \in I \times [0, T] \times \mathbb{R} \times \mathcal{P}_2^\lambda(I \times \mathbb{R})$ and we try to learn the map $\hat{a}$ which minimizes the functional cost

$$J(\hat{\mathrm{a}}) = \inf_{a:L^0(I\times[0,T]\times\mathbb{R}^d\times\mathcal{P}_2^\lambda(I\times\mathbb{R}^d);A)} J(\mathrm{a}), \tag{19}$$

where the optimal control is then given by $\hat{\alpha}_t = \mathrm{a}(U, t, X_t, \mathbb{P}_{(U,X_t)})$ directly by gradient descent algorithm.
Deep BSDE Graphon :
We use 2 neural networks $\mathcal{U}_\theta(\mu)(u, x)$ and $\mathcal{Z}_\theta(t, \mu)(u, x)$ to approximate the initial value of tbe backward component Y and the Z component at any time $t \in [0, T]$ and we look to minimize over θ the cost functional

$$\theta \mapsto L(\theta) = \mathbb{E}\Big[\big|Y_T^\theta - G(X_T^\theta, \mathbb{P}_{(U,X_T^\theta)})\big|^2\Big], \tag{20}$$

where we replaced $\hat{\alpha}_t = \mathrm{a}(U, t, X_t, \mathbb{P}_{(U,X_t)}, Y_t) = \underset{a\in A}{\arg\min}\, H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, a)$ and where we used starting from $Y_0^\theta = \mathcal{U}_\theta(\mathbb{P}_{(U,X_0)})(U, X_0)$

$$\begin{cases} X_{t_{i+1}}^\theta &= X_{t_{i+1}} + B(U, X_{t_{i+1}}^\theta, Y_{t_{i+1}}^\theta, \mathbb{P}_{(U,X_{t_{i+1}}^\theta)})\mathrm{d}t + \sigma\Delta W_{t_{i+1}} \\ Y_{t_{i+1}}^\theta &= Y_{t_{i+1}} + H(U, X_{t_{i+1}}^\theta, Y_{t_{i+1}}^\theta, \mathcal{Z}_\theta(t_{t_{i+1}}, \mathbb{P}_{(U,X_{t_{i+1}}^\theta)})(U, X_{t_{i+1}}^\theta), \mathbb{P}_{(U,X_{t_{i+1}}^\theta)}, \mathbb{P}_{(U,Y_{t_{i+1}}^\theta)})\Delta t + \mathcal{Z}_\theta(t_{t_{i+1}}, \end{cases} \tag{21}$$

for certains maps B, H depending on model coefficients.

Numerical methods for learning the optimal feedback control map $\hat{\mathrm{a}}$.

Deep Graphon :
We parametrize directly the control in feedback form through a neural network taking as input the vector $(u, t, x, \mu) \in I \times [0, T] \times \mathbb{R} \times \mathcal{P}_2^\lambda(I \times \mathbb{R})$ and we try to learn the map $\hat{a}$ which minimizes the functional cost

$$J(\hat{\mathrm{a}}) = \inf_{a:L^0(I\times[0,T]\times\mathbb{R}^d\times\mathcal{P}_2^\lambda(I\times\mathbb{R}^d);A)} J(\mathrm{a}), \tag{19}$$

where the optimal control is then given by $\hat{\alpha}_t = \mathrm{a}(U, t, X_t, \mathbb{P}_{(U,X_t)})$ directly by gradient descent algorithm.
Deep BSDE Graphon :
We use 2 neural networks $\mathcal{U}_\theta(\mu)(u, x)$ and $\mathcal{Z}_\theta(t, \mu)(u, x)$ to approximate the initial value of tbe backward component Y and the Z component at any time $t \in [0, T]$ and we look to minimize over θ the cost functional

$$\theta \mapsto L(\theta) = \mathbb{E}\Big[\big|Y_T^\theta - G(X_T^\theta, \mathbb{P}_{(U,X_T^\theta)})\big|^2\Big], \tag{20}$$

where we replaced $\hat{\alpha}_t = \mathrm{a}(U, t, X_t, \mathbb{P}_{(U,X_t)}, Y_t) = \underset{a\in A}{\arg\min}\ H(U, X_t, \mathbb{P}_{(U,X_t)}, Y_t, a)$ and where we used starting from $Y_0^\theta = \mathcal{U}_\theta(\mathbb{P}_{(U,X_0)})(U, X_0)$

$$\begin{cases} X_{t_{i+1}}^\theta &= X_{t_{i+1}} + B(U, X_{t_{i+1}}^\theta, Y_{t_{i+1}}^\theta, \mathbb{P}_{(U,X_{t_{i+1}}^\theta)})\mathrm{d}t + \sigma\Delta W_{t_{i+1}} \\ Y_{t_{i+1}}^\theta &= Y_{t_{i+1}} + H(U, X_{t_{i+1}}^\theta, Y_{t_{i+1}}^\theta, \mathcal{Z}_\theta(t_{t_{i+1}}, \mathbb{P}_{(U,X_{t_{i+1}}^\theta)})(U, X_{t_{i+1}}^\theta), \mathbb{P}_{(U,X_{t_{i+1}}^\theta)}, \mathbb{P}_{(U,Y_{t_{i+1}}^\theta)})\Delta t + \mathcal{Z}_\theta(t_{t_{i+1}}, \end{cases} \tag{21}$$

for certains maps B, H depending on model coefficients.

$\rightarrow$ At each timestep t_k, $k \in [\![1, N]\!]$, we need to compute quantities like $\tilde{\mathbb{E}}_{(\tilde{U},\tilde{X}_{t_k})\sim\mathbb{P}_{(U,X_{t_k})}}\big[\tilde{G}_\kappa U, \tilde{U})\tilde{X}_{t_k}\big]$ which is obtained by computing the conditional expectation $u \mapsto \mathbb{E}[X_{t_k}|U = u]$ from standard regression methods (polynomial regression).

Application with $\sigma = 1$, $\eta = 0.73$, $q = 0.8$, $r = 0.22$ and $\kappa = 0.62$:

Method	Ricatti	Deep Graphon	Deep BSDE Graphon
Value	0.58830	0.58826	0.58820

Table: Expected cost function using $M = 10000$ in simulation with $G(u, v) = e^{-uv}$.

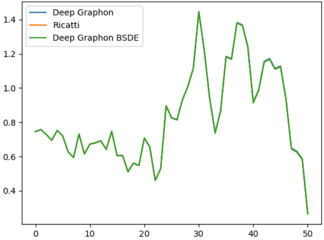

Figure: Optimal trajectory of X with $u = 0.708$

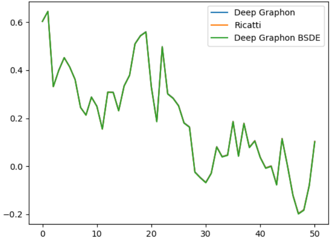

Figure: Optimal trajectory of X with $u = 0.599$

Figure: Comparison between NN solvers and the Riccati one with $G(u, v) = e^{-uv}$

A linear quadratic model

Numerical experiments

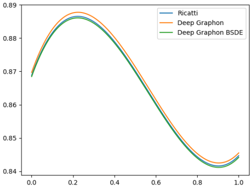

Figure: $t = 0.25$

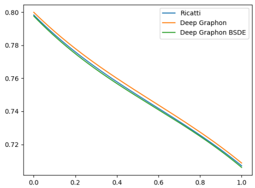

Figure: $t = 0.5$

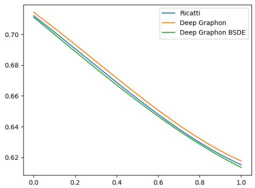

Figure: $t = 0.75$

Figure: Approximation of $u \mapsto \mathbb{E}[X_t | U = u]$ for different dates t with $G(u, v) = e^{-uv}$

Summary of the talk

- We introduced the case of non exchangeable mean field systems
- We present an application of our framework to the case of linear quadratic optimal control problem where we exhibit a new infinite-dimensional system of Riccati equations and we show numerically how to solve the optimal control problem through Deep learning algorithms.

Summary of the talk

- We introduced the case of non exchangeable mean field systems
- We present an application of our framework to the case of linear quadratic optimal control problem where we exhibit a new infinite-dimensional system of Riccati equations and we show numerically how to solve the optimal control problem through Deep learning algorithms.

Future Works

- In the present setting, agents interact through a specified graph/graphon structure but it could be interesting to add a control perspective on the agent's interactions.
- Adding some randomness in the graph structure would lead to the study of dynamical systems with random interactions $\implies$ Bridge with Random Matrix Theory and Operator-Theory.

References

- I. Kharroubi, S.Mekkaoui and H.Pham. *Stochastic maximum principle for optimal control problem of non exchangeable mean field systems, arXiv preprint arXiv:2506.05595*
- F.de Feo and S.Mekkaoui. *Optimal control of heterogeneous mean-field stochastic differential equations with common noise and applications to financial models, arXiv preprint arXiv:2511.18636*
- S. Mekkaoui, H.Pham and X.Warin *Learning mappings on labeled conditional distributions, Work in Progress*
- S.Mekkaoui, H.Pham *Analysis of Non-Exchangeable Mean Field Markov Decision Processes with common noise : From Bellman equation to quantitative propagation of chaos. Work in Progress*

THANK YOU FOR YOUR ATTENTION