Lotka-Volterra Predator-Prey Model: Simulation Problems

Problem 1: Classic Lotka-Volterra Model

Given the classic Lotka-Volterra predator-prey equations:

$$\frac{dN}{dt} = \alpha N - \beta NP$$
$$\frac{dP}{dt} = \delta NP - \gamma P$$

Where:

- N is the prey population,
- P is the predator population,
- $\alpha = 0.1$ (prey birth rate),
- $\beta = 0.02$ (predation rate),
- $\delta = 0.01$ (predator reproduction rate),
- $\gamma = 0.1$ (predator death rate).

Initial Conditions:

- Prey population $N(0) = 40$,
- Predator population $P(0) = 9$.

Tasks:

- Use numerical methods (e.g., Euler's method or Runge-Kutta method) to simulate the dynamics of the predator and prey populations over a period of 100 time units.
- Plot the populations of prey and predators over time.
- Examine whether the populations oscillate and interpret the dynamics of this predator-prey interaction.

Problem 2: Lotka-Volterra with Time-Dependent Parameters

In this variation, introduce time dependence in the birth rate of the prey population and the death rate of the predators.

$$\frac{dN}{dt} = \alpha(t)N - \beta NP$$
$$\frac{dP}{dt} = \delta NP - \gamma(t)P$$

Where:

- $\alpha(t) = 0.1 + 0.05\sin(0.1t)$ (time-dependent birth rate for prey),
- $\gamma(t) = 0.1 + 0.05\cos(0.1t)$ (time-dependent death rate for predators),
- $\beta = 0.02$,
- $\delta = 0.01$.

Initial Conditions:

- Prey population $N(0) = 40$,
- Predator population $P(0) = 9$.

Tasks:

- Simulate this system for $t \in [0, 100]$ with the given time-dependent parameters.
- Plot the prey and predator populations as a function of time and examine how the oscillations in the system change when parameters depend on time.
- Discuss how these time-dependent variations might model real-world scenarios where environmental conditions affect the growth and mortality rates of species.

Problem 3: Lotka-Volterra with Harvesting

This variation introduces a harvesting factor for the prey population (i.e., the prey is being harvested at a constant rate, which reduces their numbers).

$$\frac{dN}{dt} = \alpha N - \beta NP - hN$$
$$\frac{dP}{dt} = \delta NP - \gamma P$$

Where:

- $h = 0.1$ is the harvesting rate of the prey population,
- The other parameters (α, β, δ, γ) are the same as in the classic Lotka-Volterra model.

Initial Conditions:

- Prey population $N(0) = 40$,
- Predator population $P(0) = 9$.

Tasks:

- Simulate this model for $t \in [0, 100]$ with the harvesting term included.
- Plot the populations of prey and predators over time.
- Explore how the predator and prey populations change when the prey is harvested. Discuss whether the predator population becomes unstable or if the system reaches equilibrium.

Problem 4: Lotka-Volterra with Refuge for Prey

Introduce a "refuge" for the prey population, meaning that a certain proportion of the prey population is protected from predation, reducing the effective predation rate on them.

$$\frac{dN}{dt} = \alpha N - \beta N P(1 - r)$$
$$\frac{dP}{dt} = \delta N P - \gamma P$$

Where:

- $r = 0.3$ is the fraction of the prey population that is sheltered from predation,
- The other parameters (α, β, δ, γ) are the same as in the classic model.

Initial Conditions:

- Prey population $N(0) = 40$,
- Predator population $P(0) = 9$.

Tasks:

- Simulate this system over a time period of 100 time units.
- Plot the populations of prey and predators over time.
- Analyze how the protected portion of the prey population influences the dynamics. Does the refuge help stabilize the predator-prey relationship or does it lead to new dynamics?

Problem 5: Lotka-Volterra with Allee Effect on Prey

In this problem, the prey population has an **Allee effect**, meaning that at low population densities, the growth rate of the prey is negatively affected.

$$\frac{dN}{dt} = \alpha N \left(1 - \frac{N}{K}\right) - \beta NP$$

$$\frac{dP}{dt} = \delta NP - \gamma P$$

Where:

- The term $\alpha N \left(1 - \frac{N}{K}\right)$ includes the **Allee effect**, making the growth rate slower at low population densities (N),
- $K = 50$ is the carrying capacity of the prey population.

Initial Conditions:

- Prey population $N(0) = 10$,
- Predator population $P(0) = 5$.

Tasks:

- Simulate this system for $t \in [0, 100]$ and investigate the dynamics when the prey population is affected by the Allee effect.
- Plot the populations of prey and predators over time.
- Analyze how the Allee effect influences the persistence of the prey population and the stability of the predator-prey dynamics. Does the system reach a stable equilibrium or undergo extinction?