Optimization Technique Lab
Assignment 1: Gauss-Seidel Method

1. Complete three iterations of Gauss-Seidel method for the following system of equations, starting with any initial point of your choice. For reference, you may see a book or information about the Gauss-Seidel method on the next page.

$$\begin{aligned} 7x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 8x_2 - 3x_3 &= 10 \\ -5x_1 - x_2 + 9x_3 &= -6 \end{aligned}$$

2. Write a program in C/C^{++} to solve the following system of linear equations by the Gauss-Seidel method.

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned}$$

Given the values of $a_{ij}, b_i,\ i, j = 1, 2 \cdots n$, and unknown quantities $x_1 \ldots x_n$.

3. Using your C/C^{++}-program, solve the following problems and verify your answer by solving this in MATLAB/PYTHON using inbuilt codes.

(a)

$$\begin{aligned} -7x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 8x_2 - 3x_3 &= 10 \\ -5x_1 - x_2 + 9x_3 &= -6 \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 3 \\ -x_1 + 5x_2 + 2x_3 + x_4 &= 6 \\ 2x_1 - 3x_2 + 9x_3 + 2x_4 &= 9 \\ x_1 + x_2 - 3x_3 + 7x_4 &= 6 \end{aligned}$$

(c)

$$\begin{aligned} 10x_1 + 2x_2 - x_3 + x_4 + 2x_5 - x_6 + x_7 &= 14 \\ x_1 + 8x_2 + x_3 + x_5 - x_6 &= 10 \\ x_1 + x_2 + 9x_3 + 2x_4 + x_7 &= 14 \\ -x_2 + 5x_3 + x_4 + 11x_5 - x_6 + x_7 &= 16 \\ 2x_3 - x_4 + 10x_5 &= 11 \\ 2x_1 + x_2 - x_3 + x_4 - 2x_5 + 10x_6 + x_7 &= 11 \\ x_1 + x_2 - x_3 - x_4 + x_5 - x_6 + 9x_7 &= 9 \end{aligned}$$

Reference Material:

G-S method is used to solve a system of linear equations $Ax = b$, under the sufficient conditions.

" A is either a diagonally dominant matrix or a symmetric positive definite matrix of order $n \times n$."

The diagonal elements of A are non-zero i.e $a_{ii} \neq 0 \ \forall i = 1, 2, \ldots, n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \ b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$G - S$ is an iterative process, which generates a sequence of points $\left\{x^k\right\}$, which converges to a solution of $Ax = b$.

Iterative scheme is:

$$x_i^{k+1} = \frac{1}{a_{ii}} \left(b - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^k \right),$$

where $x^k = \left(x_1^k, x_2^k, \cdots x_n^k\right)^T$ is the k^{th} iterate.

Stopping condition:

Either $\left\|x^k - x^{k+1}\right\| < \epsilon$ for very small ϵ.

You may take $\epsilon = .001$ and $\left\|x^k - x^{k+1}\right\| = \sqrt{\sum_{j=1}^{n} \left(x_j^k - x_j^{k+1}\right)^2}$

Or

Take number of iterations as N, very large number. Stop if $k = N$.

Steps:

Input A, b, ϵ or N
Write the iterative formula
Take the initial point
Compute the iteration and continue until the stopping condition holds.

Example:

$$2x_1 + 7x_2 + 3x_3 = 12$$
$$-x_1 + 3x_2 + x_3 = 3$$
$$x_1 - 8x_2 - 5x_3 = -6$$

Diagonally dominant: $|a_{ii}| \geq \sum\limits_{\substack{j=1 \\ i \neq j}}^{n} | a_{ij}|$

This is a diagonally dominant system.

Initial pont $x^0 = \begin{pmatrix} {x_1}^0 \\ {x_2}^0 \\ x_3 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ say.

$$x^1 = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(12 - 7x_2^0 - 3x_3^0\right) \\ \frac{1}{3}\left(3 + x_1^0 - x_3^0\right) \\ -\frac{1}{5}\left(-6 - x_1^0 + 8x_2^0\right) \end{pmatrix}_{(0,0,0)} = \begin{pmatrix} 6 \\ 1 \\ 6/5 \end{pmatrix}$$

$$x^2 = \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(12 - 7x_2^1 - 3x_3^1\right) \\ \frac{1}{3}\left(3 + x_1^1 - x_3^1\right) \\ -\frac{1}{5}\left(-6 - x_1^1 + 8x_2^1\right) \end{pmatrix}_{(6,1,6/5)} = \begin{pmatrix} -16 \\ 2 \\ 0 \end{pmatrix}$$

Continue this process till a stopping condition holds.