6

Revised Simplex Method

6·1. INTRODUCTION

The usual simplex method described so far is a straight forward algebraic procedure. But the examination of the sequence of calculations in the usual simplex method, however, leads to the following disadvantages :

(i) It is very time-consuming even when considered on the time scale of electronic digital computers. Hence it is not an efficient computational procedure.

(ii) In the usual simplex method, many numbers are computed and stored which are either never needed at the current iteration or are needed only in an indirect way.

(iii) It does not give the inverse and simplex multipliers. Although it is possible to modify the ordinary simplex method to give the inverse and simplex multipliers, but this would in general increase the computational effort.

Keeping this in view, a *revised simplex method* has been developed to overcome these disadvantages, which consequently speed up the calculations by reducing the required amount of computational effort. In general, approach of the revised simplex method is identical to that on which the ordinary simplex method is based.

Proceeding from one iteration to the other in the simplex method, it was unnecessary to transform all the $\mathbf{X}_j$, $\mathbf{X}_\mathbf{B}$, $z_j - c_j$ and z at each iteration. If fact, all new quantities ($\mathbf{B}^{-1}$, $\mathbf{X}_\mathbf{B}$, $\mathbf{C}_\mathbf{B}\,\mathbf{B}^{-1}$, z) can be computed directly from their definitions, provided $\mathbf{B}^{-1}$ is known ; that is if, only the basis inverse is transformed and only such $\mathbf{X}_j$ is determined at each iteration for which the vector is entered in basis. Thus only the parts of information relevent at each iteration are :

(i) coefficients of non-basic variables in the objective function $z = \mathbf{CX}$;
(ii) coefficient of the entering basic variable in the system of constraint equations $\mathbf{AX} = \mathbf{b}$; and
(iii) right side of the equation $\mathbf{AX} = \mathbf{b}$, that is, the vector $\mathbf{b}$.

6·2. STANDARD FORMS FOR REVISED SIMPLEX METHOD

There are two *standard forms* for the revised simplex method :

Standard Form I. In this form, it is assumed that an identity (basis) matrix is obtained after introducing lack variables only.

Standard Form II. If artificial variables are needed for an initial identity (basis) matrix, then *two-phase ethod* of ordinary simplex method is used in a slightly different way to handle artificial variables.

The *revised simplex method* is now discussed in above two standard forms separately.

Revised Simplex Method in Standard Form–I

3. FORMULATION OF LP PROBLEM IN STANDARD FORM-I

[Meerut 70, 69 (Summer)]

A linear programming problem in standard form is :

$$\text{Max. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \ldots \, 0x_{n+m}, \text{ subject to} \quad \ldots(6\cdot1)$$

$$\left.\begin{array}{llllll} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n+x_{n+1} & & & =b_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n & +x_{n+2} & & =b_2 \\ \vdots \quad \vdots \quad \vdots & \vdots & & \vdots \\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n & & +x_{n+m} & =b_m \end{array}\right\}$$ ...(6·2)

and

$$x_1, x_2, \ldots, x_{n+m} \geq 0,$$ ...(6·3)

where the starting basis matrix **B** is an $m \times m$ identity matrix.

In the revised simplex form, the objective function (6·1) is also considered as if it were another constraint in which z is as large as possible and unrestricted in sign.

Thus, (6·1) and (6·2) may be written in a compact form as:

$$\left.\begin{array}{lllll} z-c_1x_1 \;-c_2x_2-\ldots-\; c_nx_n+0x_{n+1}+0\,x_{n+2}+\ldots+0x_{n+m} & & & =0 \\ a_{11}x_1\;+a_{12}x_2+\ldots+a_{1n}x_n\;+x_{n+1} & & & =b_1 \\ a_{21}x_1\;+a_{22}x_2+\ldots+a_{2n}x_n & +x_{n+2} & & =b_2 \\ \vdots \quad \vdots \quad \vdots & \vdots & & \vdots \\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n & & +x_{n+m} & =b_m \end{array}\right\}$$ ...(6·4

which can be considered as a system of $m+1$ simultaneous equations in $(n+m+1)$ number of variable $(z, x_1, x_2, \ldots, x_{n+m})$. Here our aim is to find the solution of the system (6·4) such that z is as large as possible an unrestricted in sign.

Now, the system (6·4) may be re-written as follows :

$$\left.\begin{array}{l} 1.x_0+a_{01}x_1\;+a_{02}x_2\;+\ldots+a_{0n}x_n\;+a_{0,\,n+1}x_{n+1}+\ldots+a_{0,\,n+m}x_{n+m}=0 \\ 0.x_0+a_{11}x_1\;+a_{12}x_2\;+\ldots+a_{1n}x_n\;+\quad 1.x_{n+1}+\ldots+\quad 0.x_{n+m}=b_1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ 0.x_0+a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n\;+\quad 0x_{n+1}+\ldots+\quad 1.x_{n+m}=b_m \end{array}\right]$$ ...(6

where $z=x_0$ and $-c_j=a_{0j}$ $(j=1, 2, \ldots, n+m)$.

Again, writing the system (6·5) in matrix form,

$$\begin{bmatrix} 1 & : & a_{01} & a_{02}\ldots a_{0n} & a_{0,\,n+1} & \ldots a_{0,\,n+m} \\ \ldots\ldots & : & \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & : & a_{11} & a_{12}\ldots a_{1n} & 1 & 0 \\ \vdots & : & \vdots & \vdots \quad \vdots & \vdots & \vdots \\ 0 & : & a_{m1} & a_{m2}\ldots a_{mn} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$$ ...(

Using the partitioning of a matrix,

$$\begin{bmatrix} \mathbf{1} & \mathbf{a_0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$ ...

where $\mathbf{a_0}=(a_{01}, a_{02}, \ldots, a_{0n}, \ldots, a_{0,\,n+m})$ and the remaining symbols have their usual meanings.

The matrix equation (6·7) can be expressed in the original notation form as

$$\begin{bmatrix} \mathbf{1} & -\mathbf{C} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{X} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$ ..

Equation (6·7) or (6·7)′ is referred to as *standard form–I* for the revised simplex method.

6·4. NOTATIONS FOR STANDARD FORM–I

It has been observed earlier that all the vectors have $(m+1)$ components instead of m. Hence superscr used for all vectors to show that they have $(m+1)$ components in standard form–I.

(I) Corresponding to each $\mathbf{a_j}$ in **A**, a new $(m+1)$-component vector is represented by $\mathbf{a_j}^{(1)}$ as :

$$\mathbf{a_j}^{(1)}=[-c_j, a_{1j}, a_{2j}, \ldots, a_{mj}], j=1, 2, \ldots, n+m$$

or

$$\mathbf{a_j}^{(1)}=[a_{0j}, a_{1j}, \ldots, a_{mj}], j=1, 2, \ldots, n+m$$

or

$$\mathbf{a_j}^{(1)}=[a_{0j}, \mathbf{a_j}].$$

(II) Similarly, corresponding to m-component vector $\mathbf{b}$ in $\mathbf{AX} = \mathbf{b}$, we shall represent the $(m+1)$ component vector by $\mathbf{b}^{(1)}$ given by

$$\mathbf{b}^{(1)} = [0, b_1, b_2 \dots, b_m] = [0, \mathbf{b}] \qquad \text{...(6·9)}$$

(III) The column vector corresponding to z (or x_0) is the $(m+1)$ component unit vector which is usually denoted by $\mathbf{e}_1$ and will always be in the first column of the basis matrix $\mathbf{B}_1$ (the subscript 1 will show that it is of order $(m+1) \times (m+1)$) whose remaining m columns are any $\mathbf{a}_j^{(1)}$ such that the corresponding $\mathbf{a}_j$ are linearly independent and denoted by $\beta_i^{(1)}$, $i = 1, 2, \dots, m$ (in some order).

Therefore, $$\mathbf{B}_1 = [\mathbf{e}_1, \beta_1^{(1)}, \dots, \beta_m^{(1)}] = [\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)}] \qquad \text{...(6·10)}$$

If the basis matrix $\mathbf{B}$ for $\mathbf{AX} = \mathbf{b}$ be represented by

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2m} \\ \dots & \dots & \dots & \dots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mm} \end{bmatrix},$$

then, from equation (6·10),

$$\begin{array}{cc} & \begin{array}{cccccc} e_1 & & \beta_1^{(1)} & \beta_2^{(1)} & \dots & \beta_m^{(1)} \end{array} \\ \mathbf{B}_1 = & \left[\begin{array}{cccccc} 1 & : & -c_{B1} & -c_{B2} & \dots & -c_{Bm} \\ \hdashline 0 & : & \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ 0 & : & \beta_{21} & \beta_{22} & \dots & \beta_{2m} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & : & \beta_{m1} & \beta_{m2} & \dots & \beta_{mm} \end{array}\right] \end{array} \qquad \text{...(6·11)}$$

where $-c_{Bi}$ $(i = 1, 2, \dots, m)$ are the coefficients of x_{Bi} $(i = 1, 2, \dots, m)$ in the equations

$$z - c_1 x_1 - c_2 x_2 - \dots - c_n x_n - 0x_{n+1} - \dots - 0x_{n+m} = 0, \text{ and } \mathbf{C_B} = [c_{B1}, c_{B2}, \dots, c_{Bm}].$$

Thus, the basis matrix $\mathbf{B}_1$ [in equation (6·11)] can be represented in the partitioned form as

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{1} & -\mathbf{C_B} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}. \qquad \text{...(6·12)}$$

Now the right side of (6·12) can be frequently used to obtain the basis matrix $\mathbf{B}_1$ in revised simplex method for *standard form–I*.

(IV) To compute $\mathbf{B}_1^{-1}$.

Since it is very essential to find $\mathbf{B}_1^{-1}$, compute this by applying the following rule of matrix algebra.

If $$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{Q} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}, \qquad \text{...(6·13}$$

where $\mathbf{R}^{-1}$ exists and is known, then inverse of matrix $\mathbf{M}$ is computed by the formula

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{QR}^{-1} \\ \mathbf{0} & \mathbf{R}^{-1} \end{bmatrix}. \qquad \text{...(6·14}$$

Now, to apply this rule to compute $\mathbf{B}_1^{-1}$, compare the matrices $\mathbf{B}_1$ (6·12) and $\mathbf{M}$ (6·13) to get

$$\mathbf{I} = [\mathbf{1}], \qquad \mathbf{Q} = -\mathbf{C_B} \text{ and } \mathbf{R} = \mathbf{B}.$$

Substituting these values of $\mathbf{I}, \mathbf{Q}, \mathbf{R}$ in the formula (6·14) for matrix inverse, we get

$$\mathbf{B}_1^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{C_B B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix}. \qquad \text{...(6·1}$$

(V) Any $\mathbf{a}_j^{(1)}$ (not in the basis matrix $\mathbf{B}_1$) can be expressed as the linear combination of column vectors

$$(\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \dots, \beta_m^{(1)})$$

in B_1. Therefore,

$$a_j^{(1)} = x_{0j}\,\beta_1^{(1)} + x_{1j}\,\beta_1^{(1)} + \ldots + x_{mj}\,\beta_m^{(1)} = (x_{0j}, x_{1j}, \ldots, x_{mj})\,(\beta_0^{(1)}, \beta_1^{(1)}, \ldots, \beta_m^{(1)}) = \mathbf{X}_j^{(1)}\,\mathbf{B}_1, \quad \text{[from (6·10)]}$$

which yields

$$\mathbf{X}_j^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_j^{(1)}. \qquad \ldots(6\cdot16)$$

(VI) A very interesting result can be obtained by using the formula (6·15) and (6·16). Substituting $\mathbf{B}_1^{-1}$ from (6·15) in (6·16),

$$\mathbf{X}_j^{(1)} = \begin{bmatrix} 1 & \mathbf{C_B}\,\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix}\begin{bmatrix} -c_j \\ \mathbf{a_j} \end{bmatrix} = \begin{bmatrix} -c_j + \mathbf{C_B}\mathbf{B}^{-1}\,\mathbf{a_j} \\ 0 + \mathbf{B}^{-1}\,\mathbf{a_j} \end{bmatrix} = \begin{bmatrix} -c_j + z_j \\ \mathbf{X_j} \end{bmatrix} = \begin{bmatrix} z_j - c_j \\ \mathbf{X_j} \end{bmatrix} = \begin{bmatrix} \Delta_j \\ \mathbf{X_j} \end{bmatrix} \qquad \ldots(6\cdot17)$$

It is interesting to note from result (6·17) that the first component of $\mathbf{x}_j^{(1)}$ is $(z_j - c_j)$ or (Δ_j), which is always used to decide the optimality.

Note. The greatest advantage of treating the objective function as one of the constraints is that, $z_j - c_j$ or (Δ_j) for any $\mathbf{a_j}$ not in the basis can be easily computed by taking the product of first row of $\mathbf{B}_1^{-1}$, with $\mathbf{a}_j^{(1)}$ not in the basis, that is,

$$\Delta_j = z_j - c_j = (\text{first row of } \mathbf{B}_1^{-1}) \times \mathbf{a}_j^{(1)} \text{ not in the basis.}$$

(VII) The $(m+1)$-component solution vector $\mathbf{X}_\mathbf{B}^{(1)}$ is given by

$$\mathbf{X}_\mathbf{B}^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{b}^{(1)} \qquad \ldots(6\cdot18)$$

or

$$\mathbf{X}_\mathbf{B}^{(1)} = \begin{bmatrix} 1 & \mathbf{C_B}\,\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 \times 0 + \mathbf{C_B}\,(\mathbf{B}^{-1}\,\mathbf{b}) \\ 0 \times 0 + \mathbf{B}^{-1}\,\mathbf{b} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{C_B}\,\mathbf{X_B} \\ \mathbf{X_B} \end{bmatrix} = \begin{bmatrix} z \\ \mathbf{X_B} \end{bmatrix} \qquad [\text{because } \mathbf{X_B} = \mathbf{B}^{-1}\,\mathbf{b},\ \mathbf{C_B}\,\mathbf{X_B} = z]$$

Thus,

$$\mathbf{X}_\mathbf{B}^{(1)} = \begin{bmatrix} \mathbf{C_B}\mathbf{X_B} \\ \mathbf{X_B} \end{bmatrix} = \begin{bmatrix} z \\ \mathbf{X_B} \end{bmatrix} \qquad \textbf{(Note)} \qquad \ldots(6\cdot19)$$

In (6·19), it has been observed that $\mathbf{X}_\mathbf{B}^{(1)}$ is a basic solution (not necessarily feasible, because z may be negative also) for the matrix equation (6·7)′ corresponding to the basis matrix $\mathbf{B}_1$. Also, the first component of $\mathbf{X}_\mathbf{B}^{(1)}$ immediately gives the value of the objective function while the second component $\mathbf{X_B}$ gives exactly the basic feasible solution to original constraint system $\mathbf{AX} = \mathbf{b}$ corresponding to its basis matrix $\mathbf{B}$. Thus the result (6·19) is of great importance.

Now the results of this section are applied for computational procedure of revised simplex method.

6·5. TO OBTAIN INVERSE OF INITIAL BASIS MATRIX AND INITIAL BASIC FEASIBLE SOLUTION

6·5·1. When No Artificial Variables are Needed.

As discussed in section 6·4, the inverse of initial basis matrix $\mathbf{B}_1$ is given by

$$\mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C_B}\,\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \qquad \ldots(6\cdot20)$$

But, the initial basis matrix $\mathbf{B}$ for the original problem is always $(m \times m)$ identity matrix $(\mathbf{I}_m)$. It should be noted that $\mathbf{I}_m$ always appears in $(\mathbf{AX} = \mathbf{b})$ (if it is not so, it can be made to appear in $\mathbf{A}$ by introducing the artificia variables).

Since $\quad \mathbf{B} = \mathbf{I}_m = \mathbf{B}^{-1}, \quad \mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C_B}\,\mathbf{I}_m \\ 0 & \mathbf{I}_m \end{bmatrix} \quad$ or $\quad \mathbf{B}_1^{-1} = \begin{bmatrix} 1 & \mathbf{C_B} \\ 0 & \mathbf{I}_m \end{bmatrix}$

Furthermore, if after ensuring that all $b_i \geq 0$, only the slack variables are needed and the initial basis matri $\mathbf{B} = \mathbf{I}_m$ appears, then

$$c_{B1} = c_{B2} = c_{B3} = \ldots = c_{Bm} = 0, \text{ i.e. } \mathbf{C_B} = \mathbf{0}.$$

Thus, (6·20) becomes

$$\mathbf{B}_1^{-1}=\begin{bmatrix}\mathbf{1} & : & \mathbf{0}\\ \cdots & \cdots & \cdots\\ \mathbf{0} & : & \mathbf{I}_m\end{bmatrix}=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}=\mathbf{I}_{m+1}$$

Thus, it can be concluded that the inverse of the initial basis matrix $\mathbf{B}$ will be $\mathbf{B}_1^{-1}=\mathbf{B}_1=\mathbf{I}_{m+1}$ to start with the revised simplex procedure.

Then, the initial basic solution becomes ...(6·21)

$$\mathbf{X}_B^{(1)}=\mathbf{B}_1^{-1}\,\mathbf{b}^{(1)}=\mathbf{I}_{m+1}\,\mathbf{b}^{(1)}=\begin{bmatrix}0\\ \mathbf{b}\end{bmatrix}$$

which is feasible.

After obtaining the initial basis matrix inverse $\mathbf{B}_1^{-1}=\mathbf{I}_{m+1}$ and an initial basic feasible solution to start with the revised simplex procedure, we need to construct the starting revised simplex table.

6.5·2. To Construct the Starting Table in Standard Form –I.

Since $x_0\,(=z)$ should always be in the basis, the first column $\beta_0^{(1)}\,(=\mathbf{e}_1)$ of initial basis matrix inverse $\mathbf{B}_1^{-1}=\mathbf{I}_{m+1}$ will not be removed at any subsequent iteration. The remaining column vectors of $\mathbf{B}_1^{-1}$ will be $\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_m^{(1)}$.

The last column in the revised simplex table will be $\mathbf{X}_k^{(1)}=\begin{bmatrix}z_k-c_k\\ \mathbf{X}_k\end{bmatrix}=\begin{bmatrix}\Delta_k\\ \mathbf{X}_k\end{bmatrix}$ where k is predetermined by the formula

$$\Delta_k=\min\,\Delta_j\ (\text{for those } j \text{ for which } \mathbf{a}_j \text{ is not in } \mathbf{B}_1).$$

Note. If there is a tie, we can use smallest index *j* which is an arbitrary rule but computationally strong.

Finally, it is concluded that only the column vectors $\mathbf{e}_1, \beta_1^{(1)}, \beta_2^{(1)} \ldots, \beta_m^{(1)}$ of $\mathbf{B}_1^{-1}$, $\mathbf{X}_B^{(1)}$ and $\mathbf{X}_k^{(1)}$ will be needed to construct the revised simplex table.

Now the starting table for revised simplex method can be constructed as follows. Also, for convenience, form an additional table for those $\mathbf{a}_j^{(1)}$ which are not in the basis and will be useful to determine the required Δ_j's.

Starting Table in Standard Form–I

Table 6·1

Variables in the basis	e_1	$\mathbf{B}_1^{-1}$					
		$\beta_1^{(1)}$	$\beta_2^{(1)}$	...	$\beta_m^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
z	1	0	0	...	0	0	z_k-c_k
x_{B1}	0	1	0	...	0	b_1	x_{1k}
x_{B2}	0	0	1		0	b_2	x_{2k}
:	:	:	:		:	:	:
:	:	:	:	:	:	:	:
x_{Bm}	0	0	0	...	1	b_m	x_{mk}

Table (6·1)′

Additional table for those $\mathbf{a}_j^{(1)}$ which are not included in the $\mathbf{B}_1^{-1}$ of starting table.

We now proceed to demonstrate how the computational procedure discussed so far can be applied to solve the practical problems.

Q. Describe the revised simplex procedure for solving a L.P.P.

[Meerut (L.P.) 90; Madras (B.Sc Meth.) 85; Madurai (B. Sc Math.) 81, 78]

6·6. Application of Computational Procedure : Standard Form–I

Now apply the computational procedure of revised simplex method to solve numerical problems of linear programming. All necessary steps involved in this procedure can be easily understood by solving a simple type of problem. All the necessary steps are explained in a systematic order by applying each of them to the following illustrative example so that each step could be followed more easily without any trouble.

Illustrative Example

Example 1. *Solve the following simple linear programming problem by revised simplex method.*
Max $z = 2x_1 + x_2$, *subject to* $3x_1 + 4x_2 \le 6$, $6x_1 + x_2 \le 3$, *and* $x_1, x_2 \ge 0$.

[Kanpur 96; Delhi (B. Sc. Math.) 93, 79, 78; Madurai (BSc. Math.) 84; Kerala (MSc. Math.) 84; Madras (BSc. Math) 83; Meerut (MSc. Math.) 80]

Solution. Step 1. ***Express the given problem in Standard Form–I.***

After ensuring that all $b_i \ge 0$ and transforming the objective function of original problem for maximization of z (if necessary), introduce non-negative slack variables to convert the *restrictive inequalities to equations*. It should be remembered that the objective function is also treated as if it were the first constraint equation.

Thus, the given problem is transformed to the following suitable form,

$$\begin{aligned} z - 2x_1 - x_2 \qquad\qquad &= 0 \\ 3x_1 + 4x_2 + x_3 \qquad &= 6 \\ 6x_1 + x_2 \qquad + x_4 &= 3 \end{aligned} \qquad \text{...(i)}$$

Step 2. ***Construct the starting table in revised simplex form.***

Now proceed to obtain the initial basis matrix $\mathbf{B}_1$ as an identity matrix and complete all the columns o starting revised simplex table except the last column $\mathbf{X}_k^{(1)}$ (which can be filled up in *Step 5* only).

Applying this step, the system (i) of constraint equations can be expressed in the following matrix form.

$$\begin{array}{ccccc} \mathbf{e}_1\ (= \beta_0^{(1)}) & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)}\ (= \beta_1^{(1)}) & \mathbf{a}_4^{(1)}\ (= \beta_2^{(1)}) \end{array}$$

$$\left[\begin{array}{ccccc} 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \end{array}\right] \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$$

Here the columns $\beta_0^{(1)}$, $\beta_1^{(1)}$ and $\beta_2^{(1)}$ will constitute the basis matrix $\mathbf{B}_1$ (whose inverse is also $\mathbf{B}_1$, because $\mathbf{B}_1 =$ here). Now starting revised simplex table can be constructed as follows :

Table 6·2

Variables in the basis	$\mathbf{B}_1^{-1}$			$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
	e_1 (z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$		
z	1	0	0	0	
$x_{B1} = x_3$	0	1	0	6	
$x_{B2} = x_4$	0	0	1	3	

Additional Table 6·2′

$\mathbf{a}_1^{(1)}$	$\mathbf{a}_2^{(1)}$
−2	−1
3	4
6	1

First Iteration

Step 3. ***Computations of*** $\Delta_j = z_j - c_j$ ***for*** $\mathbf{a}_1^{(1)}$ ***and*** $\mathbf{a}_2^{(1)}$ **.**

Applying the formula : $\Delta_j =$ (first row of $\mathbf{B}_1^{-1}$) $\times$ ($\mathbf{a}_j^{(1)}$ not in the basis),

$\Delta_1 =$ (first row of $\mathbf{B}_1^{-1}$) $\times \mathbf{a}_1^{(1)} = (1, 0, 0)(-2, 3, 6) = [1 \times (-2) + 0 \times 3 + 0 \times 6] = -2$

$\Delta_2 =$ (first row of $\mathbf{B}_1^{-1}$) $\times \mathbf{a}_2^{(1)} = (1, 0, 0)(-1, 4, 1) = [1 \times (-1) + 0 \times 4 + 0 \times 1] = -1$.

Remark. Instead of computing each required Δ_j separately, we can also compute them simultaneously in single step as follows :

$$(\Delta_1, \Delta_2) = (\text{first row of } \mathbf{B}_1^{-1})\,[\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}] = [1, 0, 0]\begin{bmatrix} -2 & -1 \\ 3 & 4 \\ 6 & 1 \end{bmatrix}$$

or

$$(\Delta_1, \Delta_2) = \begin{bmatrix} 1 \times (-2) + 0 \times 3 + 0 \times 6 \\ 1 \times (-1) + 0 \times 4 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = (-2, -1)$$

which gives the values $\Delta_1 = -2$, $\Delta_2 = -1$ as obtained earlier.

Step 4. ***Apply test of optimality.***

Now apply usual simplex rule to test the starting solution $(x_1 = x_2 = 0, x_3 = 6, x_4 = 3)$ for optimality.

Since Δ_1, Δ_2 obtained in *step 3* are both negative, so the starting basic feasible solution is not optimal. Hence we must proceed to determine the entering vector $\mathbf{a}_k^{(1)}$.

Step 5. ***Determination of the 'entering vector' $a_k^{(1)}$.***

To determine the vector $\mathbf{a}_k^{(1)}$ entering the basis matrix at the subsequent iteration, find such value of k for which the criterion : $\Delta_k = \min. \{\Delta_j\}$ *for those j for which* $\mathbf{a}_j^{(1)}$ *are not in the basis is satisfied*

So, in this example, we have $\Delta_k = \min. [\Delta_1, \Delta_2] = \min [-2, -1] = -2 = \Delta_1$

$$\therefore \qquad \Delta_k = \Delta_1 \Rightarrow k = 1.$$

Hence $\mathbf{a}_1^{(1)}$ enters the basis. This indicates that the corresponding variable x_1 will enter the solution.

Now, in order to find the leaving vector in ***Step 7,*** first compute $\mathbf{X}_k^{(1)}$ for $k = 1$ in the next step.

Step 6. ***Compute column vector $\mathbf{X}_k^{(1)}$ (for $k = 1$).***

Since $\mathbf{X}_k^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_k^{(1)} = \mathrm{I}_{m+1}\,\mathbf{a}_k^{(1)}$ therefore, $\mathbf{X}_1^{(1)} \equiv \mathbf{a}_1^{(1)} = (-2, 3, 6)$.

Now complete the last column $\mathbf{X}_k^{(1)}$ of starting *Table 6·2* by writing $\mathbf{X}_1^{(1)} = \mathbf{a}_1^{(1)} = (-2, 3, 6)$ in that column. So the starting Table 6·2 grows to the following form.

Table 6·3

Variables in the basis	$\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $\mathbf{a}_3^{(1)}$	$\beta_2^{(1)}$ $\mathbf{a}_4^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$
z	1	0	0	0	−2
x_3	0	1	0	6	3
x_4	0	0	1	3	6

Step 7. ***Determination of the leaving vector $\beta_r^{(1)}$, given the entering vector $a_1^{(1)}$.***

The vector $\beta_r^{(1)}$ to be removed from the basis is determined by using the **minimum ratio rule** (similar to hat of ordinary simplex method) to find the value of suffix r for predetermined value of k $(= 1)$. *i.e.*,

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0 \text{ for } k = 1\right]. = \min_i \left[\frac{x_{Bi}}{x_{i1}}, x_{i1} > 0\right] = \min\left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{6}{3}, \frac{3}{6}\right] = \frac{3}{6}.$$

$$\therefore \quad \frac{x_{Br}}{x_{r1}} = \frac{x_{B2}}{x_{21}} \Rightarrow r = 2 \text{ (Equating the suffixes on both sides } ({}_{r1} = {}_{21}) \text{ find } r = 2\text{ .)}$$

The value of r thus obtained shows that the vector $\beta_2^{(1)}$ must leave the basis.

Table 6·4

Variables in the basis	e_1	$\beta_1^{(1)}$ (S_1)	$\beta_2^{(1)}$ (S_2)	$X_B^{(1)}$	$X_1^{(1)}$	Min. ratio rule : min. $\left(\frac{X_B}{X_1}\right)$
z	1	0	0	0	−2	
$x_{B1} = x_3$	0	1	0	6	3	6/3
$x_{B2} = x_4$	0	0	1 ↓	3	$\boxed{6}$ ↑	3/6 ←

Leaving vector $\beta_2^{(1)}$ Key column

Note. It is interesting to note that the entire process of *Step 7* can be more conveniently performed by adding one more column in *Table 6·3*, for 'minimum ratio rule' (as we have seen in ordinary simplex method). *In table 6.4*, we observe that the number 6 in the column $X_1^{(1)}$ comes out to be the *'key element* or *pivot element*'. So we must bring unity at its place and zero at all other places of this column $X_1^{(1)}$ in order to determine the transformed table from which the new (improved) solution can be read off.

Remark. If the $\min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0\right]$ is attained for more than one value of i, the resulting basic feasible solution will be degenerate. So, in order to ensure that cycling will never occur, we shall use our usual techniques to resolve the degeneracy.

Step 8. *Determination of the improved solution by transforming Table 6·4.*

In order to bring uniformity with the ordinary simplex method, adopt the simple matrix transformation rules which are easier for hand computations. Here the intermediate coefficient matrix can be written as :

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$X_B^{(1)}$	$X_1^{(1)}$
R_1	0	0	0	−2
R_2	1	0	6	3
R_3	0	1 ↓	3	$\boxed{6}$

[It should be remembered that the column $\mathbf{e}_1$ *will never change. So there is no need to write the column* $\mathbf{e}_1$ *in the above intermediate coefficient matrix. Also, because the vector* $\mathbf{X}_1^{(1)}$ *is going to be replaced by the outgoing vector* $\beta_2^{(1)}$*, the column* $\mathbf{X}_1^{(1)}$ *is placed outside the rectangular boundary*].

Now, divide the row R_3 by key element 6. Then add twice of third row to first, and subtract 3 times of third row from second. In this way, obtain the next matrix. Now the vector $\beta_2^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has been thrown out of the basis matrix and it has entered in place of $\mathbf{X}_1^{(1)}$. In this way, the process of entering $\mathbf{a}_1^{(1)}$ and removing $\beta_2^{(1)}$ (*i.e.*, $\mathbf{a}_4^{(1)}$) from the basis is now complete. Accordingly, write the column $\mathbf{a}_4^{(1)}$ in the *additional* table given below.

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$X_B^{(1)}$	
0	1/3	1	0
1	−1/2	9/2	0
0	1/6	1/2	1

Thus, the following table is obtained to start with the second iteration.

Table 6·5

Basic Var.	e_1 (z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$X_B^{(1)}$	$X_k^{(1)}$ $(k=2)$	Min. Ratio Rule min. (X_B/X_2)
z	1	0	1/3	1	−2/3	
x_3	0	1	−1/2	9/2	$\boxed{7/2}$	$\frac{9/2}{7/2}$ ←
→ x_1	0	0 ↓	1/6	1/2	1/6 ↑	$\frac{1/2}{1/6}$

$\underbrace{\qquad\qquad}_{B_1^{-1}}$

Additional Table

$a_4^{(1)}$	$a_2^{(1)}$
0	−1
0	4
1	1

The improved solution is read from this table as :

$$z = 1 \,,\, x_3 = 9/2 \,,\, x_1 = 1/2 \,,\, x_2 = x_4 = 0 \,.$$

The last column of this table will be complete only when the further improvement in this solution is possible. This completes the first iteration. Repeat the entire procedure starting from *Step 3* to *Step 8* (if necessary) to obtain an optimum solution with a finite or infinite value of objective function.

Second Iteration

Step 9. ***Computation of Δ_j for $\mathbf{a}_4^{(1)}$ and $\mathbf{a}_2^{(1)}$**, i.e.* $(\Delta_4 \,,\, \Delta_2)$.

$$\{\Delta_4 \,,\, \Delta_2\} = (\text{first row of } \mathbf{B}_1^{-1})\,(\mathbf{a}_4^{(1)} \,,\, \mathbf{a}_2^{(1)}) = (1 \,,\, 0 \,,\, \tfrac{1}{3}) \begin{bmatrix} 0 & -1 \\ 0 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1\times0+0\times0+\frac{1}{3}\times1 \\ 1\times(-1)+0\times4+\frac{1}{3}\times1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}.$$

Thus, we get $\Delta_4 = \frac{1}{3} \,,\, \Delta_2 = -\frac{2}{3}$. Since Δ_2 is still negative, the solution under test can be further improved.

Step 10. ***Determination of the entering vector $\mathbf{a}_k^{(1)}$.***

To find the value of k , we have $\Delta_k = \min\,[\Delta_4 \,,\, \Delta_2] = \min\,[\frac{1}{3} \,,\, -\frac{2}{3}] = \Delta_2$. Hence $k = 2$.

So $\mathbf{a}_2^{(1)}$ should enter the solution, means that the variable x_2 will enter the basic solution.

Step 11. ***Determination of the leaving vector, given the entering vector $\mathbf{a}_2^{(1)}$.***

Compute the vector $\mathbf{X}_2^{(1)}$ so that the column $\mathbf{X}_k^{(1)}$ for $k = 2$ in *Table 6·5* may be complete at this stage.

$$\mathbf{X}_2^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_2^{(1)} = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+0+1/3 \\ 0+4+-1/2 \\ 0+0+1/6 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 7/2 \\ 1/6 \end{bmatrix}.$$

Now, instead of preparing a fresh table for performing necessary steps in second iteration, increase on more column for '*minimum ratio rule*' in Table 6·5 (which is the last table of first iteration).

The '*minimum ratio rule*' shows that 7/2 is the key element.

So remove the vector $\beta_1^{(1)}$ from the basis, to bring it in place of $\mathbf{X}_2^{(1)}$ by matrix transformation.

Step 12. ***Determination of new table for improved solution.***

For this, the intermediate coefficient matrix is :

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_2^{(1)}$
R_1	0	1/2	1	−2/3
R_2	1	−1/2	9/2	$\boxed{7/2}$
R_3	0	1/6	1/2	1/6
	↓			↑

Applying the operations : $R_2 \to \frac{2}{7} R_2 \,,\, R_1 \to R_1 + \frac{2}{3}\left(\frac{2}{7} R_2\right)$, and $R_3 \to R_3 - \frac{1}{6}\left(\frac{2}{7} R_2\right)$, we get

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	
4/21	5/21	13/7	0
2/7	−1/7	9/7	1
−1/21	8/42	2/7	0

Now, the table for improved solution is as follows :

Table 6·6

Variables in the basis	z $\mathbf{e}_1$	$\mathbf{X}_2^{(1)}$ $\beta_1^{(1)}$	$\mathbf{X}_1^{(1)}$ $\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
z	1	4/21	5/21	13/7	
$x_2 = x_{B1}$	0	2/7	−1/7	9/7	
$x_1 = x_{B2}$	0	−1/21	4/21	2/7	

$\underbrace{\qquad\qquad\qquad\qquad}_{\mathbf{B}_1^{-1}}$

Additional Table

$\mathbf{a}_4^{(1)}$	$\mathbf{a}_3^{(1)}$
0	0
0	1
1	0

First Iteration

Step 1. ***Compute*** Δ_j ***for*** $\mathbf{a}_1^{(1)}$ ***and*** $\mathbf{a}_2^{(1)}$, ***i.e.*** (Δ_1, Δ_2).

$$\{\Delta_1 , \Delta_2\} = (\text{first row of } \mathbf{B}_1^{-1}) \times (\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}) = (1, 0, 0, 0)\begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} = \{-1 , -2\}$$

Hence $\Delta_1 = -1$, $\Delta_2 = -2$. Since Δ_1 and Δ_2 both are negative, the solution $x_3 = 3$, $x_4 = 5$, $x_5 = 6$, $z = 0$ is not optimal. Therefore, we proceed to obtain the next improved solution.

Step 2. ***Determination of entering vector*** $\mathbf{a}_k^{(1)}$:

To find the entering vector $\mathbf{a}_k^{(1)}$, apply the rule : $\Delta_k = \min\,[\Delta_1 , \Delta_2] = \min\,[-1 , -2] = -2 = \Delta_2$

Hence $k = 2$. So the vector $\mathbf{a}_2^{(1)}$ must enter the basis. This shows that x_2 will enter the basic feasible solution.

Step 3. ***Determination of the leaving vector*** $\beta_r^{(1)}$, ***given the entering vector*** $\mathbf{a}_2^{(1)}$.

Compute the column $\mathbf{X}_2^{(1)}$ corresponding to vector $\mathbf{a}_2^{(1)}$

$$\mathbf{X}_2^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_2^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Apply the minimum ratio rule by increasing one more column in *Table 6·7*. This rule shows that [2] is the 'key element' corresponding to which $\beta_2^{(1)}$ must leave the basis matrix. Hence x_4 will be the outgoing variable.

Step 4. ***Determination of the improved solution.***

From *Table 6·7*, the intermediate coefficient matrix is :

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_2^{(1)}$
0	0	0	0	−2
1	0	0	3	1
0	1	0	5	$\boxed{2}$
0	0	1	6	1
	↓			↑

Apply usual rules of transformation to obtain

0	2	0	5	0
1	−1/2	0	1/2	0
0	1/2	0	5/2	1
0	−1/2	1	7/2	0

then construct *Table 6·8* for improved solution.

Table 6·8

Variables in the basis	$\mathbf{B}_1^{-1}$				$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
	e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$		
z	1	0	1	0	5	
$x_3 = x_{B1}$	0	1	−1/2	0	1/2	
$x_2 = x_{B2}$	0	0	1/2	0	5/2	
$x_5 = x_{B3}$	0	0	−1/2	1	7/2	

Additional Table

$\mathbf{a}_1^{(1)}$	$\mathbf{a}_4^{(1)}$
−1	0
1	0
1	1
3	0

he improved solution now becomes : $z = 5$, $x_3 = 1/2$, $x_2 = 5/2$, $x_5 = 7/2$.

Second Iteration

tep 5. ***Computations of*** Δ_j ***for*** $\mathbf{a}_1^{(1)}$ ***and*** $\mathbf{a}_4^{(1)}$, ***i.e.,***

$$(\Delta_1 , \Delta_4) = (1 , 0 , 1 , 0)\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \{0 , 1\}$$

Hence $\Delta_1 = 0$, $\Delta_4 = 1$. Since Δ_1 and Δ_4 both are ≥ 0, the solution under test is optimal.

Furthermore, $\Delta_1 = 0$ shows that the problem has alternative optimum solutions. Thus, the required optimal solution is $x_1 = 0, x_2 = 5/2$, max $z = 5$.

Alternative solution can also be obtained as $x_1 = 1, x_2 = 2$, max. $z = 5$.

Example 3. *Solve by revised simplex method :*

Max. $z = 6x_1 - 2x_2 + 3x_3$ *subject to* $2x_1 - x_2 + 2x_3 \leq 2, x_1 + 4x_3 \leq 4$ and $x_1 , x_2 , x_3 \geq 0$.

[Kanpur BSc. 95; Madurai (MSc. Appl. Sc.) 83

Solution. The given problem in the revised simplex form may be expressed by introducing the slacl variables x_4 and x_5 as

$$\begin{aligned} z - 6x_1 + 2x_2 - 3x_3 \qquad\qquad &= 0 \\ 2x_1 - x_2 + 2x_3 + x_4 \quad &= 2 \\ x_1 \qquad + 4x_3 \qquad + x_5 &= 4. \end{aligned}$$

The system of constraint equations may be represented in the following matrix form :

$$\begin{array}{cccccc} \mathbf{e}_1 & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)} & \mathbf{a}_4^{(1)} & \mathbf{a}_5^{(1)} \\ \beta_0^{(1)} & & & & \beta_1^{(1)} & \beta_2^{(1)} \\ \end{array}$$

$$\begin{bmatrix} 1 & -6 & 2 & -3 & 0 & 0 \\ 0 & 2 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

The starting revised simplex table is given below in *Table 6·9*.

Table 6·9

Variables in the Basis	$\mathbf{e}_1$	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)} = \mathbf{X}_1^{(1)}$	Min. $(\mathbf{X}_B/\mathbf{X}_1)$
z	1	0	0	0	-6	↓
$x_4 = x_{B1}$	0	1	0	$\boxed{2}$	2	2/2 ←
$x_5 = x_{B2}$	0	0 ↓	1	4 ↑	1	4/1

$\mathbf{B}_1^{-1}$

Additional Table

$\mathbf{a}_1^{(1)}$	$\mathbf{a}_2^{(1)}$	$\mathbf{a}_3^{(1)}$
-6	2	-3
2	-1	2
1	0	4

The starting solution is : $x_1 = x_2 = x_3 = 0 , x_4 = 2 , x_5 = 4 , z = 0$.

First Iteration

Step 1. ***Computations of*** Δ_j ***for*** $\mathbf{a}_1^{(1)}$, $\mathbf{a}_2^{(1)}$ ***and*** $\mathbf{a}_3^{(1)}$, *i.e.,* $(\Delta_1 , \Delta_2 , \Delta_3)$.

$$\{\Delta_1 , \Delta_2 , \Delta_3\} = (\text{first row of } \mathbf{B}_1^{-1})\,(\mathbf{a}_1^{(1)} , \mathbf{a}_2^{(1)} , \mathbf{a}_3^{(1)}) = (1 , 0 , 0)\begin{bmatrix} -6 & 2 & -3 \\ 2 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix} = \{-6 , 2 , -$$

Hence $\Delta_1 = -6 , \Delta_2 = 2 , \Delta_3 = -3$.

Since Δ_1 and Δ_3 are negative, the solution under test is not optimal.

Step 2. ***Determination of the entering vector*** $\mathbf{a}_k^{(1)}$.

$$\Delta_k = \min. [\Delta_1 , \Delta_2 , \Delta_3] = \min \{-6 , 2 , -3\} = -6 = \Delta_1 .$$

Hence $k = 1$.

So the entering vector is found to be $\mathbf{a}_1^{(1)}$. This also means that the variable x_1 will enter the basic solution.

Step 3. ***Determination of the leaving vector*** $\beta_r^{(1)}$, ***given the entering vector*** $\mathbf{a}_1^{(1)}$.

First we need to compute the column $\mathbf{X}_1^{(1)}$ corresponding to the entering vector $\mathbf{a}_1^{(1)}$.

$$\mathbf{X}_1^{(1)} = \mathbf{B}_1^{-1}\mathbf{a}_1^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}$$

Now apply the min. ratio rule by increasing one more column in Table 6·9. This rule indicates that [2] is the 'key element' corresponding to which $\beta_1^{(1)}$ must leave the basis matrix. Hence x_4 will be the outgoing variable.

Step 4. ***Determination of the first improved solution.***

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$
0	0	0	−6
1	0	2	$\boxed{2}$
0	1	4	1

To transform the *Table 6·9*, transform the above intermediate coefficient matrix.

Apply usual rules of matrix transformation to obtain

3	0	6	0
1/2	0	1	1
−1/2	1	3	0

Now construct the transformed Table 6·10 for second iteration.

Table 6·10

Variables in the Basis	$\mathbf{B}_1^{-1}$			$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)} = \mathbf{X}_2^{(1)}$	Min. $(\mathbf{X}_B/\mathbf{X}_2)$ ↓
	e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
z	1	3	0	6	−1	
$x_1 = x_{B1}$	0	1/2	0	1	−1/2	—
$x_5 = x_{B2}$	0	−1/2	1 ↓	3	$\boxed{1/2}$	$3/\frac{1}{2}$ ←

Additional Table

$\mathbf{a}_4^{(1)}$	$\mathbf{a}_2^{(1)}$	$\mathbf{a}_3^{(1)}$
0	2	−3
1	−1	2
0	0	4

The improved solution is : $z = 6 , x_1 = 1 , x_2 = x_3 = x_4 = 0 , x_5 = 3$.

Second Iteration

Step 5. ***Computations of*** Δ_j ***for*** $\mathbf{a}_4^{(1)}, \mathbf{a}_2^{(1)}$, ***and*** $\mathbf{a}_3^{(1)}$ (*i.e.*, $\Delta_4 , \Delta_2 , \Delta_3$).

$$\{\Delta_4 , \Delta_2 , \Delta_3\} = (\text{first row of } \mathbf{B}_1^{-1})\,(\mathbf{a}_4^{(1)} , \mathbf{a}_2^{(1)} , \mathbf{a}_3^{(1)}) = (1 , 3 , 0)\begin{bmatrix} 0 & 2 & -3 \\ 1 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \{3 , -1 , 3\}$$

Hence $\Delta_4 = 3 , \Delta_2 = -1 , \Delta_3 = 3$. Since Δ_2 is still negative, the solution under test is not optimal. Henc rther improvement is possible. So we proceed to find the '*entering*' and '*leaving*' vectors in the next step.

Step 6. ***Determination of the entering vector*** $\mathbf{a}_k^{(1)}$.

Here, we have $\Delta_k = \min. [\Delta_4 , \Delta_2 , \Delta_3] = \min. [3 , -1 , 3] = -1 = \Delta_2$. Hence $k = 2$.

Therefore, $\mathbf{a}_2^{(1)}$ must enter the basis. The entering vector $\mathbf{a}_2^{(1)}$ indicates that the variable x_2 must enter th w solution.

Step 7. ***Determination of the leaving vector $\beta_r^{(1)}$, given the entering vector $\mathbf{a}_2^{(1)}$.***

First compute the column $\mathbf{X}_2^{(1)}$ corresponding to vector $\mathbf{a}_2^{(1)}$.

$$\mathbf{X}_2^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_2^{(1)} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Now complete the column $\mathbf{X}_k^{(1)} = \mathbf{X}_2^{(1)}$ of Table 6·10.

The *'min ratio rule'* in the additional column of *Table 6·10* indicates that 1/2 is the key element corresponding to which the vector $\beta_2^{(1)}$ must leave the basis. Hence x_5 will be the outgoing variable.

Step 8. ***Determination of the next improved solution.***

Transform the Table 6·10 into Table 6·11 from which the next improved solution can be easily read.

Table 6·11

Variables in the Basis	$\mathbf{B}_1^{-1}$				
	$\mathbf{e}_1$	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
z	1	2	2	12	
$x_1 = x_{B1}$	0	0	1	4	
$x_2 = x_{B2}$	0	−1	2	6	

Additional Table

$\mathbf{a}_4^{(1)}$	$\mathbf{a}_5^{(1)}$	$\mathbf{a}_3^{(1)}$
0	0	−3
1	0	2
0	1	4

The next improved solution from *Table 6·11* is : $z = 12$, $x_1 = 4$, $x_2 = 6$, $x_3 = x_4 = x_5 = 0$.

Third Iteration

Step 9. ***Computations of Δ_j for $\mathbf{a}_4^{(1)}$, $\mathbf{a}_5^{(1)}$ and $\mathbf{a}_3^{(1)}$, i.e. $(\Delta_4, \Delta_5, \Delta_3)$.***

$$\{\Delta_4, \Delta_5, \Delta_3\} = (\text{first row of } \mathbf{B}_1^{-1})\,(\mathbf{a}_4^{(1)}, \mathbf{a}_5^{(1)}, \mathbf{a}_3^{(1)}) = (1, 2, 2)\begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \{2, 2, 9\}$$

Hence $\Delta_4 = 2$, $\Delta_5 = 2$, $\Delta_3 = 9$.

The solution under test is optimal because Δ_4, Δ_5, Δ_3 are all positive. Thus, the required optimal solution is :

$$x_1 = 4,\ x_2 = 6,\ x_3 = 0,\ \text{max. } z = 12. \quad \textbf{Ans.}$$

Example 4. *Solve the following L.P.P. by revised simplex method.*

$$\text{Max} \quad z = 3x_1 + x_2 + 2x_3 + 7x_4, \textit{ subject to the constraints :}$$

$$2x_1 + 3x_2 - x_3 + 4x_4 \le 40, \quad -2x_1 + 2x_2 + 5x_3 - x_4 \le 35, \; x_1 + x_2 - 2x_3 + 3x_4 \le 100, \textit{ and}$$

$$x_1 \ge 2,\ x_2 \ge 1,\ x_3 \ge 3,\ x_4 \ge 4.$$

[Meerut 82]

Solution. Step 1. In order to make the lower bounds of the variables zero, we substitute $x_1 = y_1 + 2$, $x_2 = y_2 + 1$, $x_3 = y_3 + 3$, $x_4 = y_4 + 4$ in the given LPP and obtain the following modified problem :

$$\begin{aligned} \text{Maximize} \quad & z' = 3y_1 + y_2 + 2y_3 + 7y_4, \text{ where } z' = z - 41 \\ \text{subject to} \quad & 2y_1 + 3y_2 - y_3 + 4y_4 \le 20 \\ & -2y_1 + 2y_2 + 5y_3 - y_4 \le 26 \\ & y_1 + y_2 - 2y_3 + 3y_4 \le 91 \end{aligned}$$

and

$$y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0,\ y_4 \ge 0.$$

Step 2. ***To express the modified LPP in revised simplex form.***

$$\begin{aligned} \text{Max.} \quad & z' = 3y_1 + y_2 + 2y_3 + 7y_4, \text{ subject to} \\ & z' - 3y_1 - y_2 - 2y_3 - 7y_4 = 0 \\ & 2y_1 + 3y_2 - y_3 + 4y_4 + y_5 = 20 \\ & -2y_1 + 2y_2 + 5y_3 - y_4 + y_6 = 26 \\ & y_1 + y_2 - 2y_3 + 3y_4 + y_7 = 91, \end{aligned}$$

$y_i \ge 0$ $(i = 1, 2, \ldots, 7)$, and z' is unrerstricted in sign.

Clearly, the problem is of *standard form–I.*

In matrix form, the system of constraint equations can be written as :

$$\begin{array}{cccccccc} \beta_0^{(1)} & & & & & \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ \mathbf{e}_1 & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)} & \mathbf{a}_4^{(1)} & \mathbf{a}_5^{(1)} & \mathbf{a}_6^{(1)} & \mathbf{a}_7^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -3 & -1 & -2 & -7 & 0 & 0 & 0 \\ 0 & 2 & 3 & -1 & 4 & 1 & 0 & 0 \\ 0 & -2 & 2 & 5 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z' \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ 26 \\ 91 \end{bmatrix}$$

Step 3. ***To find initial basic solution and the basis matrix*** $\mathbf{B}_1$.

Here $\mathbf{X}_B^{(1)} = (0, 20, 26, 91)$ is the initial BFS and basis matrix $\mathbf{B}_1$ is given by $\mathbf{B}_1 = [\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)}, \beta_3^{(1)}] = \mathbf{I}_4$ (unit matrix). So $\mathbf{B}_1^{-1} = I_4$.

Step 4. ***To construct the starting simplex table.***

Table 6·12

Variables in the basis	$\mathbf{B}_1^{-1}$				Solution $\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)} = \mathbf{X}_4^{(1)} = \mathbf{B}_1^{-1}\mathbf{a}_4^{(1)}$	Min. Ratio $(\mathbf{X}_B/\mathbf{X}_4)$
	$\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $\mathbf{a}_5^{(1)}$	$\beta_2^{(1)}$ $\mathbf{a}_6^{(1)}$	$\beta_3^{(1)}$ $\mathbf{a}_7^{(1)}$			
z'	1	0	0	0	0	−7	
y_5	0	1	0	0	20	$\boxed{4}$	5 ← (min.)
y_6	0	0	1	0	26	−1	—
y_7	0	0	0	1	91	3	91/3

↓ **Outgoing vector** ↑ **incoming vector**

Step 5. ***Test for optimality.*** Compute Δ_j for all $\mathbf{a}_j^{(1)}, j = 1, 2, 3, 4$ not in the basis.

$$(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = (\text{first row of } \mathbf{B}_1^{-1})\ [\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}, \mathbf{a}_3^{(1)}, \mathbf{a}_4^{(1)}]$$

$$= (1, 0, 0, 0)\begin{bmatrix} -3 & -1 & -2 & -7 \\ 2 & 3 & -1 & 4 \\ -2 & 2 & 5 & -1 \\ 1 & 1 & -2 & 3 \end{bmatrix} = (-3, -1, -2, -7)$$

Since all Δ_j's are ≤ 0, the solution is not optimal.

Step 6. ***To find incoming and outgoing vectors.***

Incoming vector. $\Delta_k = \min\limits_j \Delta_j = -7 = \Delta_4, \therefore k = 4$,

Thus $\mathbf{a}_4^{(1)}$ is the vector entering the basis. So the column vector $\mathbf{X}_4^{(1)}$ coresponding to $\mathbf{a}_4^{(1)}$ is given by

$$\mathbf{X}_4^{(1)} = \mathbf{B}_1^{-1}\mathbf{a}_4^{(1)} = \mathbf{I}_4(-7, 4, -1, 3) = [-7, 4, -1, 3]$$

Outgoing vector. Since $\dfrac{x_{Br}}{x_{r4}} = \min\left[\dfrac{20}{4}, -, \dfrac{91}{3}\right] = \dfrac{20}{4} = \dfrac{x_{B1}}{x_{14}}$, so $r = 1$ and hence $\beta_1^{(1)} = a_5^{(1)}$ is the outgo ctor.

$\therefore$ Key element $= x_{14} = 4$, by min. ratio rule.

Step 7. ***To find the improved solution.***

In order to bring $\mathbf{a}_4^{(1)}$ in place of $\beta_1^{(1)} (= \mathbf{a}_5^{(1)})$ in $\mathbf{B}_1^{-1}$, we divide second row by 4 and then add 7, 1 and es in first, third and fourth rows, respectively to get the revised simplex *Table 6·13.*

Table 6·13

Variables in the basis	B_1^{-1}				Solution $X_B^{(1)}$	$X_k^{(1)} = X_3^{(1)}$ $= B_1^{-1} a_3^{(1)}$	Min. Ratio X_B/X_3
	$\beta_0^{(1)}$ e_1	$\beta_1^{(1)}$ $a_4^{(1)}$	$\beta_2^{(1)}$ $a_6^{(1)}$	$\beta_3^{(1)}$ $a_7^{(1)}$			
z'	1	7/4	0	0	35	− 15/4	
y_4	0	1/4	0	0	5	− 1/4	—
y_6	0	1/4	1	0	31	$\boxed{19/4}$	124/19 ←
y_7	0	− 3/4	0	1	76	− 5/4	—

↓ Outgoing vector

↑ Incoming vector

Step 8. ***Test of optimality for the revised solution Table 6·13 .***

We compute $(\Delta_1 , \Delta_2 , \Delta_3 , \Delta_5) =$ (first row of $\mathbf{B}_1^{-1}$) $(\mathbf{a}_1^{(1)} , \mathbf{a}_2^{(1)} , \mathbf{a}_3^{(1)} , \mathbf{a}_5^{(1)})$.

$$= (1 , 7/4 , 0 , 0) \begin{bmatrix} -3 & -1 & -2 & 0 \\ 2 & 3 & -1 & 1 \\ -2 & 2 & 5 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = [1/2 , 17/4 , - 15/4 , 7/4]$$

Since $\Delta_3 = - 15/4$ is still negative, the solution under test is not optimal. So we proceed to improve the solution in the next step.

Step 9. ***To find entering and outgoing vectors.***

As in ***step 6,*** we find the entering vector $\mathbf{a}_3^{(1)}$. The column vector $\mathbf{X}_3^{(1)}$ corresponding to $\mathbf{a}_3^{(1)}$ is given by

$$\mathbf{X}_3^{(1)} = \mathbf{B}_1^{(1)} \mathbf{a}_3^{(1)} = [- 15/4 , - 1/4 , 19/4 , - 5/4] .$$

By *min. ratio rule*, we find the outgoing vector is $\beta_2^{(1)} = \mathbf{a}_6^{(1)}$. So the key element will be 19/4 .

Step 10. ***To find the revised solution.***

In order to bring $\mathbf{a}_3^{(1)}$ in place of $\beta_2^{(1)}$ $(= \mathbf{a}_6^{(1)})$ in the basis $\mathbf{B}_1^{-1}$, we divide the third row by 19/4 and then add its 15/4 , 1/4 and 5/4 times in first, second and fourth rows respectively to obtain the next revised *Table 6·14* .

Table 6·14

Variables in the basis	B_1^{-1}				Solution $X_B^{(1)}$	$X_k^{(1)} = X_1^{(1)}$ $= B_1^{-1} a_1^{(1)}$	Min ratio X_B/X_1
	$\beta_0^{(1)}$ e_1	$\beta_1^{(1)}$ $a_4^{(1)}$	$\beta_2^{(1)}$ $a_3^{(1)}$	$\beta_3^{(1)}$ $a_7^{(1)}$			
z'	1	37/19	15/19	0	1130/19	− 13/19	
y_4	0	5/19	1/19	0	126/19	$\boxed{8/19}$	63/4 ←
y_3	0	1/19	4/19	0	124/19	− 6/19	—
y_7	0	− 13/19	5/19	1	1599/19	− 17/19	—

↓ Outgoing vector

↑ Incoming vector

Step 11. ***To test the optimality for the revised solution Table 6·14 .***

We compute, $[\Delta_1 , \Delta_2 , \Delta_5 , \Delta_6] =$ (first row of $\mathbf{B}_1^{-1}$) $[\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)} , \mathbf{a}_5^{(1)} , \mathbf{a}_6^{(1)}]$

$$= \left[1 , \frac{37}{19} , \frac{15}{19} , 0\right] \begin{bmatrix} -3 & -1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \left[\frac{-13}{19} , \frac{122}{19} , \frac{37}{19} , \frac{15}{19}\right]$$

Since $\Delta_1 < 0$, the solution under test is not optimal. So we proceed to revise the solution in the next step.

Step 12. ***To find entering and outgoing vectors.***

As in ***step 6,*** we find the entering vector $\mathbf{a}_1^{(1)}$. The column vector corresponding to $\mathbf{a}_1^{(1)}$ is given by

$$\mathbf{X}_1^{(1)} = \mathbf{B}_1^{-1} \mathbf{a}_1^{(1)} = \left[\frac{-13}{19} , \frac{8}{18} , \frac{-6}{19} , \frac{-17}{19}\right]$$

By **min ratio rule,** we find the outgoing vector is $\beta_1^{(1)} = \mathbf{a}_4^{(1)}$. So the key element is 8/19.

Step 13. ***To find the improved solution.***

In order to bring $\mathbf{a}_1^{(1)}$ in place of $\beta_1^{(1)} (= \mathbf{a}_4^{(1)})$, we divide second row by 8/19, then add its 13/19, 6/19 and 17/19 times in first, third and fourth rows respectively to obtain the next improved solution Table 6·15.

Table 6·15

Varaibles in the basis	B_1^{-1}				**Solution** $\mathbf{x}_B^{(1)}$
	$\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $\mathbf{a}_1^{(1)}$	$\beta_2^{(1)}$ $\mathbf{a}_3^{(1)}$	$\beta_3^{(1)}$ $\mathbf{a}_7^{(1)}$	
z'	1	19/8	7/8	0	281/4
y_1	0	5/8	1/8	0	63/4
y_3	0	1/4	1/4	0	23/2
y_7	0	−1/8	3/8	1	393/4

Step 14. ***To test the optimality of the improved solution Table 6·15.***

We compute, $(\Delta_2, \Delta_4, \Delta_5, \Delta_6) = (\text{first row of } \mathbf{B}_1^{-1})\,(\mathbf{a}_2^{(1)}, \mathbf{a}_4^{(1)}, \mathbf{a}_5^{(1)}, \mathbf{a}_6^{(1)})$

$$= \left(1, \frac{19}{8}, \frac{7}{8}, 0\right) \begin{bmatrix} -1 & 7 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} = \left(\frac{63}{8}, \frac{13}{8}, \frac{19}{8}, \frac{7}{8}\right)$$

Since all $\Delta_j > 0$, the solution under test is optimal. So the optimal solution of modified LPP is,

$$y_1 = 63/4,\ y_2 = 0,\ y_3 = 23/2,\ y_4 = 0 \text{ and } \max z' = 281/4.$$

Transforming this solution for the original LPP, we get the desired solution as,

$x_1 = y_1 + 2 = 71/4,\ x_2 = y_2 + 1 = 1,\ x_3 = y_3 + 3 = 29/2,\ x_4 = y_4 + 4 = 4$

and $\max z = \max (z' + 41) = 445/4.$ **Ans.**

6·8 SUMMARY OF REVISED SIMPLEX METHOD IN STANDARD FORM–I (COMPUTATIONAL PROCEDURE)

[Meerut 90; Raj 81]

The computational procedure of revised simplex method in *standard form*-I (when no artificial variables are needed) may be more conveniently out-lined as follows :

Step 1. ***If the problem is of minimization; convert it into the maximization problem.***

Step 2. Express the given problem in Standard Form–I.

After ensuring that all $b_i \geq 0$, express the given problem in revised simplex form-I as explained in **section 6·3**.

Step 3. Find the initial basic feasible solution and the basis matrix $\mathbf{B}_1$.

In this step, we proceed to obtain the initial basis matrix $\mathbf{B}_1$ as an identity matrix. Thus the initial solution is given by $\mathbf{x}_B^{(1)} = (0, b_1, b_2, \ldots, b_m)$.

Step 4. Construct the starting table for revised simplex method as explained in section 6·6.

Step 5. Test the optimality of current BFS.

This is done by computing $\Delta_j = z_j - c_j$ for all $\mathbf{a}_j^{(1)}$ not in the basis $\mathbf{B}_1$ by the formula :

$$\Delta_j = (\text{first row of } \mathbf{B}_1^{-1}) \times (\mathbf{a}_j^{(1)} \text{ not in this basis})$$

The BFS is optimal only when all $\Delta_j \geq 0$.

If current BFS is neither optimal nor unbounded, proceed to improve it in the next step.

Step 6. Improve the BFS.

In this step, we first find the ***incoming*** (entering) vector and the ***leaving*** (outgoing) vector to obtain the key element. Then we determine the improved solution like regular simplex method as follows :

(i) ***To find in-coming vector.*** The incoming vector will be taken as $\mathbf{a}_k^{(1)}$ if $\Delta_k = \min(\Delta_j)$ for those j for which $\mathbf{a}_j^{(1)}$ are not in the basis $\mathbf{B}_1$.

(ii) ***To find out-going vector.*** For this, first we compute $\mathbf{X}_k^{(1)}$ by the formula :

$$\mathbf{X}_k^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_k^{(1)} = [\Delta_k, x_{1k}, x_{2k}, \ldots, x_{mk}]$$

The vector $\beta_r^{(1)}$ to be removed from the basis is determined by using the *minimum ratio rule*. That is, it is selected corresponding to such value of r for which

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Bi}}{x_{ik}}, x_{ik} > 0\right]$$

Note. Here $\mathbf{a}_k^{(1)}$ is the in-coming vector and $\mathbf{X}_k^{(1)}$ is the column vector corresponding to $\mathbf{a}_k^{(1)}$.

(iii) ***To find the key element.*** When $\mathbf{a}_k^{(1)}$ is the in-coming vector and $\beta_r^{(1)}$ is the out-going vector, the ***key-element*** x_{rk} is situated at the intersection of rth row and kth column of the matrix.

(iv) ***To transform the revised simplex table.***

In order to bring $\mathbf{a}_k^{(1)}$ in place to $\beta_r^{(1)}$, we proceed similarly as in ordinary simplex method and then construct the new (revised) simplex table.

In this manner, we obtain the improved BFS.

Step 7. Now again test the optimality of above improved BFS as in *Step 5*

If this solution is not optimal, then repeat ***step 6*** until an optimal solution is finally obtained.

Q. Give a brief outline for the standard form I of the revised simplex method. **[Delhi BSc. (Maths) 93, 91, 90]**

EXAMINATION PROBLEMS

se revised simplex method to solve the following linear programming problems :

. Max. $z = x_1 + x_2$
subject to the constraints :
$3x_1 + 3x_2 \le 6$
$x_1 + 4x_2 \le 4$
$x_1, x_2 > 0$.
[Meerut (Math.) 74]
[Ans. $x_1 = \frac{8}{5}, x_2 = \frac{3}{5}$, max. $z = \frac{11}{5}$]

2. Max. $z = x_1 + 2x_2$
subject to
$x_1 + 2x_2 \le 3$
$x_1 + 3x_2 \le 1$
$x_1, x_2 \le 0$.
[Delhi 69]
[Ans. $x_1 = 1, x_2 = 0, z^* = 1$]

3. Max. $z = 5x_1 + 3x_2$
subject to
$3x_1 + 5x_2 \le 15$
$3x_1 + 2x_2 \le 10$
$x_1, x_2 \ge 0$.
[Ans. $x_1 = \frac{22}{19}, x_2 = \frac{45}{19}, z^* = \frac{285}{19}$]

Max. $z = 3x_1 + 2x_2 + 5x_3$
subject to
$x_1 + 2x_2 + x_3 \le 430$
$3x_1 \quad + 2x_3 \le 460$
$x_1 + 4x_2 \le 420$
$x_1, x_2, x_3 \ge 0$.
[Shivaji (M.Sc. Math.) 76]
[Ans. $x_1 = 0, x_2 = 100, x_3 = 230, z^* = 1350$]

5. Max. $z = x_1 + x_2 + 3x_3$
subject to the constraints :
$3x_1 + 2x_2 + x_3 \le 3$
$2x_1 + x_2 + 2x_3 \le 2$
$x_1, x_2, x_3 \ge 0$.
[Meerut (Math.) 77]
[Ans. $x_1 = 0, x_2 = 0, x_3 = 1, z'' = 3$]

6. Max. $z = 30x_1 + 23x_2 + 29x_3$
subject to the constraints :
$6x_1 + 5x_2 + 3x_3 \le 26$
$4x_1 + 2x_2 + 5x_3 \le 7$
and $x_1, x_2, x_2 \ge 0$
[Meerut M.A. (P) 93]
[Ans. $x_1 = 0, x_2 = 7/2, x_3 = 0,$
$z^* = 161/2$]

Max. $z = x_1 + x_2$,
s.t. $x_1 + 2x_2 \le 2$
$4x_1 + x_2 \le 4$
$x_1, x_2 \ge 0$
[Delhi (BSc. Math.) 79]
[Ans. $x = 6/7, x_2 = 4/7$, max $z = 10/7$]

8. Max. $z = 2x_1 + 3x_2$
s.t. $x_2 - x_1 \ge 0, x_1 \le 4$, and
$x_1, x_2 \ge 0$
[Meerut (MSc. Math.) 81]
[Ans. Unbounded sol.]

Explain the revised simplex method and compare it with the **simplex** method. **[Meerut (L.P.) 89]**

The improved solution is : $z = 13/7$, $x_2 = 9/7$, $x_1 = 2/7$.

Third Iteration

Step 13. ***Computation of Δ_4 for $\mathbf{a}_4^{(1)}$ and Δ_3 for $\mathbf{a}_3^{(1)}$.***

$$\{\Delta_4, \Delta_3\} = (\text{first row of } \mathbf{B}_1^{-1})\,(\mathbf{a}_4^{(1)}, \mathbf{a}_3^{(1)}) = (1, 4/21, 5/21)\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or
$$\{\Delta_4, \Delta_1\} = \begin{bmatrix} 1 \times 0 + 4/21 \times 0 + 5/21 \times 1 \\ 1 \times 0 + 4/21 \times 1 + 5/21 \times 0 \end{bmatrix} = \begin{bmatrix} 5/21 \\ 4/21 \end{bmatrix} \quad \therefore \ \Delta_4 = 5/21 \,;\, \Delta_3 = 4/21 .$$

The positive values of Δ_4 and Δ_3 indicate that the optimal solution is : $z = 13/7$, $x_2 = 9/7$, $x_1 = 2/7$.

Remark. While solving the numerical problems by revised simplex method, the students need not give full explanation of each step. Here, we have given the detailed explanation of each step, so that the students may be able to follow each step correctly.

6·7. MORE EXAMPLES ON STANDARD FORM–I

Example 2. *Solve the following problem by revised simplex method :*

$$\text{Max. } z = x_1 + 2x_2, \text{ subject to}$$

$$x_1 + x_2 \le 3,\ x_1 + 2x_2 \le 5,\ 3x_1 + x_2 \le 6, \text{ and } x_1, x_2 \ge 0 .$$

[Garhwal 97; Meerut M.Sc. (L.P.) 94; 90; (B.A. Pvt.) 90; Gauhati (M.C.A.) 92]

Solution. First express the given problem in revised simplex form :

$$\begin{aligned} z - x_1 - 2x_2 \qquad\qquad &= 0 \\ x_1 + x_2 + x_3 \qquad &= 3 \\ x_1 + 2x_2 \quad + x_4 \quad &= 5 \\ 3x_1 + x_2 \qquad\qquad + x_5 &= 6 . \end{aligned}$$

Then express the system of constraint equations in the following matrix form :

$$\begin{array}{cccccc} \mathbf{e}_1 & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)} & \mathbf{a}_4^{(1)} & \mathbf{a}_5^{(1)} \\ \beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 6 \end{bmatrix}$$

Now form the revised simplex table for the first iteration.

Table 6·7

Variables in the basis	$\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $(\mathbf{a}_3^{(1)})$	$\beta_2^{(1)}$ $(\mathbf{a}_4^{(1)})$	$\beta_3^{(1)}$ $(\mathbf{a}_5^{(1)})$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$ $(k = 2)$	Min. $(\mathbf{X_B}/\mathbf{X}_2)$ ↓
z	1	0	0	0	0	−2	
$x_3 = x_{B1}$	0	1	0	0	3	1	3/1
$x_4 = x_{B2}$	0	0	1	0	5	$\boxed{2}$	5/2 ←
$x_5 = x_{B3}$	0	0	0	1	6	1	6/1
			↓			↑	

$\underbrace{\qquad\qquad\qquad\qquad}_{\mathbf{B}_1^{-1}}$

Additional Table

$\mathbf{a}_1^{(1)}$	$\mathbf{a}_2^{(1)}$
−1	−2
1	1
1	2
3	1