subject to the constraints :

$x_1 + 2x_2 + 2x_3 \le 10$
$2x_1 + 4x_2 + 3x_3 \le 15$
$x_1 , x_2 < x_3 \ge 0$

[Bangalore B.E. (Mech.) 78]
[**Ans.** One iteration only.
$x_1 = 15/2 , x_2 = x_3 = 0$, max. $z = 45/2$]

$6x_1 + 5x_2 + 3x_3 \le 52$
$6x_1 + 2x_2 + 5x_3 \le 14$.
$x_1 , x_2 , x_3 \ge 0$.

[Mysore (Math.) 81]
[**Ans.** Two iterations. $x_1 = 0$, $x_2 = 7$, $x_3 = 0$, max. $z = 161$]

29. Max. $z = 7x_1 + x_2 + 2x_3$,
subject to the constraints :
$x_1 + x_2 - 2x_3 \le 10$
$4x_1 + x_2 + x_3 \le 20$
$x_1 , x_2 , x_3 \ge 0$.

[Delhi M.A. (Bus. Econ.) 82]
[**Ans.** Two iterations. $x_1 = x_2 = 0$, $x_3 = 20$
max. $z = 40$]

30. Max. $R = 2x - 3y + z$
subject to the constraints :
$3x + 6y + z \le 6$
$4x + 2y + z \le 4$
$x - y + z \le 3$
$x , y , z \ge 0$

[Ranchi (Stat.) 80]
[**Ans.** Two iterations. $x = 1/3$,
$y = 0$, $z = 8/3$, max. $R = 10/3$]

31. Max. $R = 2x + 4y + 3z$
subject to the constraints :
$3x + 4y + 2z \le 60$
$2x + y + 2z \le 40$
$x + 3y + 2z \le 80$
$x , y , z \ge 0$.

[Madras B.E. (Prod. Engg.) 81]
[**Ans.** Two iterations. $x = 0$, $y = 20/3$,
$z = 50/7$, max. $R = 250/3$.]

32. Max. $z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$
subject to the constraints :
$x_1 + x_4 + 6x_6 = 9$
$3x_1 + x_2 - 4x_3 + 2x_6 = 2$
$x_1 + 2x_3 + x_5 + 2x_6 = 6$
$x_i \ge 0$, $i = 1$, 2, 3, 4, 5, 6.

[Marathwada (Math.) 82]
[**Ans.** One iteration. $x_1 = 2/3$, $x_2 = x_3 = 0$,
$x_4 = 25/3$, $x_5 = 16/3$, $x_6 = 0$, max $z = 43/3$]

33. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yield a profit of Rs. 30. An acre of wheat cost Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans cost Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8,000 man-days of work, how masny acres should be allocated to each crop to maximize profit ? **(Jammu Univ. M.B.A., Feb. 1996)**

[**Hint.** Formulation of the problem is :

Max. $z = 30x_1 + 40x_2 + 20x_3$, s.t.
$10x_1 + 12x_2 + 7x_3 \ge 10{,}000$; $7x_1 + 10x_2 + 8x_3 \le 8{,}000$
$x_1 + x_2 + x_3 \le 1{,}000$; $x_1, x_2, x_3 \ge 0$.]

[**Ans.** Acreage for carn, wheat and soyabeans are 250, 625 and respectively with max. profit of Rs. 32,500]

5·5. ARTIFICIAL VARIABLE TECHNIQUES

5·5– 1. Two Phase Method

[Garhwal 97; Kanpur (B.Sc.) 90; Rohil. 90; Shivaji 77

Linear programming problems, in which constraints may also have '$\ge$' and '=' signs after ensuring that all b are ≥ 0 , are considered in this section. In such problems, basis matrix is not obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable, called, the ***artificial variable.*** These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. Artificial variables can be eliminated from the simplex table as and when they become zero (non-basic). The process of eliminating artificial variables is performed in ***Phase I*** of the solution and ***Phase II*** is used to get an optimal solution. Since the solution of the LP problem is completed in two phases it is called ***'Two Phase Simplex Method'*** due to **Dantzig, Orden** and **Wolfe.**

Remarks :

1. **The objective of Phase I is to search for a B.F.S. to the given problem It ends up either giving a B.F.S. or indicating that the given L.P.P. has no feasible solution at all.**

2. **The B.F.S. obtained at the end of Phase 1 provides a starting B.F.S. for the given L.P.P. Phase II is then just the application of simplex method to move towards optimality.**
3. **In Phase II, care must be taken to ensure that an artifical variable is never allowed to become positive, if were present in the basis. Moreover, whenever some artifical variable happens to leave the basis, its column must be deleted from the simplex table altogether.**

Q. Explain the term 'Artificial variable' and its use in linear programming. [C.A. (May) 87 , (Nov.) 82]

This technique is well explained by the following example.

Example 10. *Solve the problem : Minimize* $z = x_1 + x_2$, *subject to* $2x_1 + x_2 \geq 4, x_1 + 7x_2 \geq 7$, *and* $x_1 , x_2 \geq 0$.

[Delhi B.Sc. (Math.) 91, 88; Bharthidasan B.Sc. (Math.) 90; Raj. 87; Bombay B.Sc. (Stat.) 84; Baroda (B.Sc.) 82]

Solution. First convert the problem of minimization to maximization by writing the objective function as :

$$\text{Max} \quad (-z) = -x_1 - x_2 \quad \text{or} \quad \text{Max. } z' = -x_1 - x_2 \text{, where } z' = -z.$$

Since all b_i 's (4 and 7) are positive, the 'surplus variables' $x_3 \geq 0$ and $x_4 \geq 0$ are introduced, then constraints become :

$$\begin{aligned} 2x_1 + x_2 - x_3 \quad &= 4 \\ x_1 + 7x_2 \quad - x_4 &= 7 . \end{aligned}$$

But the basis matrix **B** would not be an identity matrix due to negative coefficients of x_3 and x_4 . Hence the starting basic feasible solution cannot be obtained.

On the other hand, if so-called 'artificial variables' $a_1 \geq 0$ and $a_2 \geq 0$ are introduced, the constraint equations can be written as

$$\begin{aligned} 2x_1 + x_2 - x_3 \quad + a_1 \quad &= 4 \\ x_1 + 7x_2 \quad - x_4 \quad + a_2 &= 7 . \end{aligned}$$

It should be noted that $a_1 < x_3$, $a_2 < x_4$, otherwise the constraints of the problem will not hold.

Phase I. Construct the first table (*Table 5·14*) where $\mathbf{A}_1$ and $\mathbf{A}_2$ denote the artificial column-vectors corresponding to a_1 and a_2, respectively.

Table 5·14

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{A}_1$	$\mathbf{A}_2$
a_1	4	2	1	−1	0	1	0
a_2	7	1	$\boxed{7}$	0	−1	0	1
			↑	×	×	×	↓

Now remove each artificial column vector $\mathbf{A}_1$ and $\mathbf{A}_2$ from the basis matrix. To remove vector $\mathbf{A}_2$ first, select the entering vector either $\mathbf{A}_1$ or $\mathbf{A}_2$, being careful to choose any one that will yield a non-negative (feasible) revised solution. Take the vector $\mathbf{X}_2$ to enter the basis matrix. It can be easily verified that if the vector $\mathbf{A}_2$ is entered in place of $\mathbf{X}_1$, the resulting solution will not be feasible. Thus transformed table (Table 5·15) is obtained.

Table 5·15

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{A}_1$	$\mathbf{A}_2$
a_1	3	13/7	0	−1	$\boxed{1/7}$	1	−1/7
x_2	1	1/7	1	0	−1/7	0	1/7
					↑	↓	

(Delete column $\mathbf{A}_2$ for ever at this stage)

This table gives the solution : $x_1 = 0 , x_2 = 1 , x_3 = 0 , x_4 = 0, a_1 = 3 , a_2 = 0$. When the artificial variable a_2 becomes zero (non-basic), we forget about it and never consider the corresponding vector $\mathbf{A}_2$ again for re-entry into the basis matrix.

Similarly, remove $\mathbf{A}_1$ from the basis matrix by introducing it in place of $\mathbf{X}_4$ by the same method. Thus Table 5·16 is obtained.

Table 5·16

BASIC VARAIBLES	X_B	X_1	X_2	X_3	X_4	A_1
x_4	21	13	0	−7	1	7
x_2	4	4	1	−1	0	1

(Delete column A_1 for ever at this stage)

This table gives the solution : $x_1=0, x_2=4, x_3=0, x_4=21, a_1=0$. Since the artificial variable a_1 becomes zero (non-basic), so drop the corresponding column A_1 from this table. Thus, the solution $(x_1=0, x_2=4, x_3=0, x_4=21)$ is the basic feasible solution and now usual simplex routine can be started to obtain the required optimal solution.

Phase II. Now in order to test the starting above solution for optimality, construct the starting simplex Table 5.17

Table 5·17

		$c_j \rightarrow$	−1	−1	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	Min. Ratio (X_B/X_1)
$\leftarrow x_4$	0	21	$\boxed{13}$	0	−7	1	21/13 ←
x_2	−1	4	2	1	−1	0	4/2
	$z'=C_BX_B$ $=-4$		−1 ↑	0	1	0 ↓	$\leftarrow \Delta_j$

Compute $\Delta_1=-1, \Delta_3=1$.

Key element $\boxed{13}$ indicates that X_4 should be removed from the basis matrix. Thus, by usual transformation method Table 5·18 is formed.

Table 5·18

		$c_j \rightarrow$	−1	−1	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	MIN. RATIO COLUMN
$\rightarrow x_1$	−1	21/13	1	0	−7/13	1/13	
x_2	−1	10/13	0	1	1/13	−2/13	
	$z'=-31/13$		0	0	6/13	1/13	$\leftarrow \Delta_j \geq 0$

Also, verify that

$$\Delta_3 = C_BX_3 - c_3 = (-1, -1)(-7/13, 1/13) = 6/13$$

$$\Delta_4 = C_BX_4 - c_4 = (-1, -1)(1/13, -2/13) = 1/13.$$

Since all $\Delta_j \geq 0$, the required optimal solution is :

$$x_1 = 21/13, x_2 = 10/13 \text{ and min. } z = 31/13 \text{ (because } z=-z').$$

5.5·2. Simple Way for Two-Phase Simplex Method

Phase I : Table 5·19.

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	4	2	1	−1	0	1	0
$\leftarrow a_2$	7	1	$\boxed{7}$ ↑	0	−1	0	1 ↓
$\leftarrow a_1$	3	13/7	0	−1	$\boxed{1/7}$	1	−1/7
$\rightarrow x_2$	1	1/7	1	0	−1/7 ↑	0 ↓	1/7
$\leftarrow x_4$	21	13	0	−7	1	7	×
x_2	4	2	1	−1	0	1	×

Thus, initial basic feasible solution is : $x_1 = 0$, $x_2 = 4$, $x_3 = 0$, $x_4 = 21$. Now start to improve this solution in Phase II by usual simplex method.

Note.

1. Remove the artificial vector $\mathbf{A_2}$ and insert it anywhere such that $\mathbf{X_B}$ remains feasible (≥ 0) .
2. As soon as $\mathbf{A_2}$ is removed from the basis by matrix transformation or otherwise, delete $\mathbf{A_2}$ for ever.
3. Similar process is adopted to remove other artificial vectors one by one from the basis.
4. Purpose of introducing artificial vectors is only to provide an initial basic feasible solution to start with simplex method in Phase II. So, as soon as the artificial variables become non-basic (*i.e.* zero), delete artificial vectors to enter Phase II.
5. Then, start Phase II, which is exactly the same as original simplex method.

Phase II. Table 5·20

		$c_j \rightarrow$	-1	-1	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	MIN. RATIO $(\mathbf{X_B}/\mathbf{X_k})$
$\leftarrow x_4$	0	21	$\boxed{13}$	0	-7	1	$21/13$
x_2	-1	4	2	1	-1	0	$4/2$
	$z' = -4$		-1^* $\uparrow$	0	1	0 $\downarrow$	$\leftarrow \Delta_j$
$\rightarrow x_1$	-1	$21/13$	1	0	$-7/13$	$1/13$	
x_2	-1	$10/13$	0	1	$1/10$	$2/13$	
	$z' = -31/13$		0	0	$6/13$	$1/13$	$\leftarrow \Delta_j \geq 0$

Thus, the desired solution is obtained as : $x_1 = 21/13$, $x_2 = 10/13$, max. $z = 31/13$.

5.5–3. Alternative Approach of Two-phase Simplex Method

The two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase I. In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1. Assign a cost -1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function.

Step 2. Construct the auxiliary linear programming problem in which the new objective function z^* is to be maximized subject to the given set of constraints.

Step 3. Solve the auxiliary problem by simplex method until either of the following three possibilities do arise :

(i) Max $z^* < 0$ and at least one artificial vector appear in the optimum basis at a positive level. In this case given problem does not possess any feasible solution.

(ii) Max $z^* = 0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to *Phase–II.*

(iii) Max $z^* = 0$ and no artificial vector appears in the optimum basis. In this case also proceed to *Phase–II.*

Phase II. Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. That is, simplex method is applied to the modified simplex table obtained at the end of *Phase–I* , until an optimum basic feasible solution (if exists) has been attained.The artificial variables which are non-basic at the end of *Phase–I* are removed.

Q. 1. What are artificial variables ? Why do we need them ? Describe briefly the two-phase method of solving a L.P. problem with artificial variables. **[Meerut M.Sc. (Math.) 93; Delhi B.Sc. (Math) 85, (Special Course) 83]**

2. What do you mean by two phase method for solving a given L.P.P. ? Why is it used ? **[Bharthiar B.Sc. (Math.) 86; Madras B.Sc. (Math.) 85]**

The following examples will make the *alternative* two-phase method clear.

Example 11. *Use two-phase simplex method to solve the problem : Minimize* $z = x_1 - 2x_2 - 3x_3$, *subject to the constraints* :$-2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$, *and* $x_1, x_2, x_3 \geq 0$,

[Meerut (Maths) 91 ; Bombay (M.Com.) 74]

Solution. First convert the objective function into maximization form ;

$$\text{Max } z' = -x_1 + 2x_2 + 3x_3, \text{ where } z' = -z.$$

Introducing the artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the constraints of the given problem become,

$$-2x_1 + x_2 + 3x_3 + a_1 \quad = 2$$
$$2x_1 + 3x_2 + 4x_3 \quad + a_2 = 1$$
$$x_1, x_2, x_3, a_1, a_2 \geq 0.$$

Phase I. Auxiliary L.P. problem is : *Max.* $z'^* = 0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$ *subject to above given constraints.*

The following solution table is obtained for auxiliary problem.

Table 5·21

		$c_j \rightarrow$	0	0	0	-1	-1	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	A_1	A_2	MIN. RATIO (X_B/X_k)
a_1	-1	2	-2	1	3	1	0	2/3
$\leftarrow a_2$	-1	1	2	3	$\boxed{4}$ $\uparrow$	0	1	$1/4 \leftarrow$
	$z'^* = 3$		0	-4	-7^* $\uparrow$	0	0 $\downarrow$	$\leftarrow \Delta_j$
a_1	-1	5/4	$-7/2$	$-5/4$	0	1	$-3/4$	
$\rightarrow x_3$	0	1/4	1/2	3/4	1	0	1/4	
	$z'^* = -5/4$		7/4	5/4	0	0	3/4	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution to the auxiliary L.P.P. has been attained. But at the same time max. z'^* is negative and the artificial variable a_1 appears in the basic solution at a positive level. Hence the original problem does not possess any feasible solution. Here there is no need to enter *Phase II*.

Example 12. *Use two-phase simplex method to solve the problem :*

Minimize $z = 15/2\ x_1 - 3x_2$, *subject to the constraints* :

$$3x_1 - x_2 - x_3 \geq 3,\ x_1 - x_2 + x_3 \geq 2,\ \textit{and}\ x_1, x_2, x_3 \geq 0.$$

[Roorkee (Appl. Math.) 74]

Solution. Convert the objective function into the maximization form : Maximize $z' = -15/2\ x_1 + 3x_2$.

Introducing the surplus variables $x_4 \geq 0$ and $x_5 \geq 0$, and artificial variables $a_1 \geq 0$, $a_2 \geq 0$, the constraints of the given problem become

$$3x_1 - x_2 - x_3 - x_4 \quad + a_1 \quad = 3$$
$$x_1 - x_2 + x_3 \quad - x_5 \quad + a_2 = 2$$
$$x_1, x_2, x_3, x_4, a_1, a_2 \geq 0.$$

Phase I. Assigning a cost -1 to artificial variables a_1 and a_2 and cost 0 to all other variables, the new objective function for auxiliary problem becomes : Max. $z'^* = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 - 1a_1 - 1a_2$, subject to the above given constraints.

Now apply simplex method in usual manner, (see *Table 5·22*).

Phase I : Table 5.22

		$c_j \rightarrow$	0	0	0	0	0	-1	-1	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	A_2	MIN RATIO (X_B/X_k)
$\leftarrow a_1$	-1	3	$\boxed{3}$	-1	-1	-1	0	1	0	$3/3 \leftarrow$
a_2	-1	2	1	-1	1	0	-1	0	1	$2/1$
	$z'^* = -5$		-4^* $\uparrow$	2	0	1	1	0 $\downarrow$	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	0	1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	—
$\leftarrow a_2$	-1	1	0	$-2/3$	$\boxed{4/3}$	$1/3$	-1	$1/3$	1	$3/4 \leftarrow$
	$z'^* = -1$		0	$2/3$	$-4/3^*$ $\uparrow$	$-1/3$	1	$2/3$	0 $\downarrow$	$\leftarrow \Delta_j$
x_1	0	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	
x_3	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	$1/4$	$3/4$	
	$z'^* = 0$		0	0	0	0	0	1	1	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ and no artificial variable appears in the basis, an optimum solution to the auxiliary problem has been attained.

Phase 2. In this phase, now consider the actual costs associated with the original variables, the objective function thus becomes : Max. $z' = -15/2\, x_1 + 3x_2 + 0x_4 + 0x_5$

Now apply simplex method in the usual manner.

Phase 2 : Table 5.23

		$c_j \rightarrow$	$-15/2$	3	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	MIN RATIO (X_B/X_k)
x_1	$-15/2$	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	
x_3	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	
	$z' = -75/8$		0	$3/4$	0	$15/8$	$15/8$	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been attained.

Hence optimum solution is : $x_1 = 5/4$, $x_2 = 0$, $x_3 = 3/4$, min $z = 75/8$.

Solve the following LP problems by two-phase method :

1. Max. $z = 3x_1 - x_2$
subject to the constraints :
$2x_1 + x_2 \geq 2$
$x_1 + 3x_2 \leq 2$
$x_2 \leq 4$
and $x_1 , x_2 \geq 0$.

[Delhi (Math.) 76]
[**Ans.** $x_1 = 2$, $x_2 = 0$ Max $z = 6$]

2. Max. $z = 5x_1 + 8x_2$
subject to the constraints :
$3x_1 + 2x_2 \geq 3$
$x_1 + 4x_2 \geq 4$
$x_1 + x_2 \leq 5$
and $x_1 , x_2 \geq 0$.

[Roorkee (M.E. Elect.) 77]
[**Ans.** $x_1 = 0$, $x_2 = 5$, max. $z = 40$]

3. Max $z = x_1 + 1.5x_2 + 2x_3 + 5x_4$
with the conditions :
$3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$
$2x_1 + x_2 + x_3 + 5x_4 \leq 4$
$2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$
$x_1 + 3x_2 - 4x_3 + 3x_4 = 0$
$x_i \, (i = 1 , 2 , 3 , 4) \geq 0$

[Cochin M.Sc. (Maths.) 85]
[**Ans.** $x_1 = 1.2$, $x_2 = 0$, $x_3 = 0.9$
$x_4 = 0$, max. $z = 19.8$]

4. Minimize $z = x_1 - 2x_2 - 3x_3$, subject to
$-2x_1 + x_2 + 3x_3 = 2$
$2x_1 + 3x_2 + 4x_3 = 1$,
$x_j \geq 0 , j = 1 , 2 , 3$,
[M.S. Baroda (B.Sc. Math.) 81 ; Bombay (M.Com.) 74]

[**Ans.** Here all $\Delta_j \geq 0$, but at the

5. Max. $z = 3x_1 + 2x_2 + x_3 + 4x_4$
subject to
$4x_1 + 5x_2 + x_3 - 3x_4 = 5$
$2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$
$x_1 + 4x_2 + 2.5x_3 - 4x_4 = 6$
$x_1 , x_2 , x_3 \geq 0$

[Meerut 83, 80]

6. Max $z = 5x_1 - 2x_2 + 3x_3$
subject to
$2x_1 + 2x_2 - x_3 \geq 2$
$3x_1 - 4x_2 \quad \leq 3$
$x_2 + 3x_3 \leq 5$
$x_1 , x_2 , x_3 , x_4 \geq 0$.

[Delhi M.Sc. (Math.) 82]