17. A factory manufactures three products which are processed through three different production stages. The time required to manufacture, one unit of each of the three products and their daily capacity of the stages are given in the following table :

Stage	Time per unit in minutes			Stage capacity (in minutes)
	Product 1	Product 2	Product 3	
1	1	1	1	430
2	3	—	2	460
3	1	4	—	420
Profit per unit				—

(i) Set the data in simplex table.
(ii) Find the table for optimum solution.
(iii) State from the table-min. profit, production pattern and surplus capacity at any stage.
(iv) What is the meaning of shadow price ? Where is it shown in the table ? Explain it in respect of resource of stages having shadow price.
(v) How many units of other resources will be required so as to completely utilise the surplus resource ?

[Osmania (MBA) Feb. 97]

18. Ashok Chemicals Co. manufactures two chemicals *A* and *B* which are sold to the manufacturers of soaps and detergents. On the basis of the next month's demand, the management has decided that the total production for chemicals *A* and *B* should be at least 350 kilograms. Moreover, a major customer's order for 125 kgs. of product *A* must also be supplied. Product *A* requires 2 hours of processing time per kg. and product *B* requires one hour of processing time per kg. For the coming month, 600 hours of processing time are available. The company wants to meet the above requirements at a minimum total production cost. The production costs are Rs. 2/– per kg. for product *A* and Rs. 3/– per kg for product *B*.

Ashok Chemicals Co. wants to determine its optimum productwise and the total minimum cost relevant thereto.
(i) Formulate the above as a linear programming problem.
(ii) Solve the problem with the simplex method.
(iii) Does the problem have multiple optimum solutions ?

[Delhi (M. Com.) 98]

19. A firm manufacturing office furniture provides you the following information regarding resource consumption and availability and profit contribution :

Resources	Usage per unit			Daily availability
	Tables	Chairs	Bookcases	
Timber (cu. ft)	8	4	3	640
Assembly department (man-hours)	4	6	2	540
Finishing department (man-hours)	1	1	1	100
Profit contribution per unit (Rs.)	30	20	12	
Minimum production requirement	0	50	0	

The firm wants to determine its optimal product mix.
(i) Formulate the linear programming problem with the help of the above data.
(ii) Solve the problem with the Simplex Method and find the optimal product mix and the total maximum profit contribution.
(iii) Identify the shadow prices of the resources.
(iv) What other information can be obtained from the optimal solution of the problem ?

[Delhi (M. Com.) 97]

20. Use penalty (Big M) method to solve the following LP problem :

Min. $z = 5x_1 + 3x_2$, s.t. $2x_1 + 4x_2 \leq 12$, $2x_1 + 2x_2 = 10$, $5x_1 - 2x_2 \geq 10$, and $x_1, x_2 \geq 0$

[IPM (PGDBM) 2000]

Problem of Degeneracy (Tie for Minimum Ratio)

5.7. WHAT IS DEGENERACY PROBLEMS ?

At the stage of improving the solution during simplex procedure, minimum ratio $\mathbf{X_B}/\mathbf{X_k}$ ($\mathbf{X_k} > 0$) is determined in the last column of simplex table to find the key row (*i.e.*, a row containing the *key element*). But, sometimes this ratio may not be unique, *i.e.*, the key element (hence the variable to leave the basis) is not uniquely determined or at the very first iteration, the value of one or more basic variables in the $\mathbf{X_B}$ column become equal to zero, this causes the problem of degeneracy.

However, if the minimum ratio is zero for two or more basic variables, degeneracy may result the simplex routine to cycle indefinitely. That is, the solution which we have obtained in one iteration may repeat again after few iterations and therefore no optimum solution may be obtained under such circumstances. Fortunately, such phenomenon very rarely occurs in practical problems.

5.7–1. Method to Resolve Degeneracy (Tie)

The following systematic procedure can be utilised to avoid cycling due to degeneracy in L.P. problems.

Step 1. First pick up the rows for which the min. non-negative ratio is same (tied). To be definite, suppose such rows are first, third, etc., for example.

Step 2. Now rearrange the columns of the usual simplex table so that the columns forming the original unit matrix come first in proper order.

Step 3. Then find the minimum of the ratio :

$$\left[\frac{\textbf{elements of first column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique. That is, for the rows *first, third,* etc. as picked up in step 1. (*key column* is that one for which Δ_j is minimum).

(i) If this minimum is attained for third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is also not unique, then go to next step.

Step 4. Now compute the minimum of the ratio :

$$\left[\frac{\textbf{elements of second column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique in *Step* 3.

(i) If this min. ratio is unique for the first row (say), then this row will determine the key element by intersecting the key column.

(ii) If this minimum is still not unique then go to next step.

Step 5. Next compute the *minimum* of the ratio :

$$\left[\frac{\textbf{elements of third column of } \textbf{\textit{unit matrix}}}{\textbf{corresponding elements of } \textbf{\textit{key column}}}\right],$$

only for the rows for which min. ratio was not unique in *Step* 4.

(i) If this min. ratio is unique for the third row (say), then this row will determine the key element by intersecting the key column.

(ii) If this min. is still not unique, then go on repeating the above outlined procedure till the unique min. ratio is obtained to resolve the degeneracy. After the resolution of this tie, simplex method is applied to obtain the optimum solution. Following example will make the procedure clear.

Q. What do you understand by degeneracy ? Discuss a method to resolve degeneracy in a LPP.

[Meerut (L.P.) 89; (Maths) 85, 82; Delhi (O.R.) 79]

Example 19. *Maximize* $z = 3x_1 + 9x_2$, *subject to the constraints* : $x_1 + 4x_2 \le 8$, $x_1 + 2x_2 \le 4$, and $x_1, x_2 \ge 0$.

[Shivaji M.Sc. (Math.) 76]

Solution. Introducing the slack variables $s_1 \ge 0$ and $s_2 \ge 0$, the problem becomes :

$$\text{Max. } z = 3x_1 + 9x_2 + 0s_1 + 0s_2$$

subject to the constrains :

$$\begin{aligned} x_1 + 4x_2 + s_1 \quad &= 8 \\ x_1 + 2x_2 \quad + s_2 &= 4 \\ x_1, x_2, s_1, s_2 &\ge 0. \end{aligned}$$

Table 5.32. Starting Simplex Table

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	3 X_1	9 X_2	0 S_1	0 S_2	MIN. RATIO (X_B/X_k)
s_1	0	8	1	4	1	0	$\left.\begin{matrix}8/4=2\\4/2=2\end{matrix}\right\}$ Tie
s_2	0	4	1	2	0	1	
	$z=0$		-3	-9	0	0	$\leftarrow \Delta_j$

Since min. ratio 2 in the last column of above table is not unique, both the slack variables s_1 and s_2 may leave the basis. This is an indication for the existence of degeneracy in the given LP problem. So we apply the above outlined procedure to resolve degeneracy (tie).

First arrange the columns X_1, X_2, S_1 and S_2 in such a way that the initial identity (basis) matrix appears first. Thus the initial simplex table becomes :

Table 5.33

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	0 S_1	0 S_2	3 X_1	9 X_2	MIN RATIO (S_1/X_2)
S_1	0	8	1	0	1	4	1/4
$\leftarrow S_2$	0	4	0	1	1	$\leftarrow\boxed{2}$	$0/2 \leftarrow$
	$z=0$		0	0 $\downarrow$	-3	-9 $\uparrow$	$\leftarrow \Delta_j \geq 0$

Now using the *step 3* of the procedure for resolving degeneracy, we find

$$\min\left[\frac{\text{elements of first column } (S_1)}{\text{corres. elements of } \textit{key column } (X_2)}\right]=\min\left[\frac{1}{4},\frac{0}{2}\right]=0$$

which occurs for the second row. Hence S_2 must leave the basis, and the key element is 2 as shown above.

First Iteration. By usual matrix transformation introduce X_2 and leave S_2.

Table 5.34. First Improvement Table

BASIC VARIABLES	C_B	$C_j \rightarrow$ X_B	0 S_1	0 S_2	3 X_1	9 X_2	MIN RATIO
S_1	0	0	1	-2	-1	0	
$\rightarrow X_2$	9	2	0	1/2	1/2	1	
	$z=18$		0	9/2	3/2	0	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal solution has been reached. Hence the optimum basic feasible solution is : $x_1=0, x_2=2$, max. $z=18$.

Example 20. *Max.* $z=2x_1+x_2$, *subject to* $4x_1+3x_2 \leq 12$, $4x_1+x_2 \leq 8$, $4x_1-x_2 \leq 8$, *and* $x_1, x_2 \geq 0$

Solution. Introducing the slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$, and proceeding in the usual manner, the starting simplex table is given below :

Table 5.35

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	2 X_1	1 X_2	0 S_1	0 S_2	0 S_3	MIN. RATIO (X_B/X_k)
s_1	0	12	4	3	1	0	0	12/4
s_2	0	8	4	1	0	1	0	$\left\{\begin{matrix}8/4\\8/4\end{matrix}\right\}$
s_3	0	8	4	-1	0	0	1	
	$z=0$		-2	-1	0	0	0	$\leftarrow \Delta_j$

Since min. ratio in the last column of above table is 2 which is same for *second* and *third* rows. This is an indication of degeneracy. So arrange the columns in such a way that the initial identity (basis) matrix comes first. Then starting simplex table becomes.

Table 5.36

BASIC VARIABLES	C_B	X_B	S_1	S_2	S_3	X_1	X_2	MIN (S_1/X_1)	MIN (S_2/X_1)
s_1	0	12	1	0	0	4	3	—	—
s_2	0	8	0	1	0	4	1	0/4	1/4
s_3	0	8	0	0	1	$\boxed{4}$	−1	0/4	0/4←
	$z=0$		0	0	0 ↓	−2 ↑	−1	← Δ_j	

Using the procedure of degeneracy, compute

$$\left[\frac{\text{elements of first column } (S_1) \text{ of unit matrix}}{\text{corres. elements of key column } (X_1)}\right],$$

only for second and third rows. Therefore, min $[-, \frac{0}{4}, \frac{0}{4}]$ which is not unique.

So again compute

$$\min\left[\frac{\text{element of second column } (S_2) \text{ of unit matrix}}{\text{corres. element of key column } (X_1)}\right],$$

only for second and third rows. Therefore, min $[-, \frac{1}{4}, \frac{0}{4}] = 0$ which occurs corresponding to the third row. Hence the key element is 4.

Now improve the simplex **Table 5.36** in the usual manner to get Table 5.37.

Table 5.37

		$c_j \rightarrow$	0	0	0	2	1	
BASIC VARIABLES	C_B	X_B	S_1	S_2	S_3	X_1	X_2	MIN. (X_B/X_k)
s_1	0	4	1	0	−1	0	$\boxed{4}$	4/4
s_2	0	0	0	1	−1	0	2	0/2 ←
x_1	2	2	0	0	1/4	1	−1/4	—
	$z=4$		0	0 ↓	1/2	0	−3/2 ↑	← Δ_j
s_1	0	4	1	−2	1	0	0	4/1←
x_2	1	0	0	1/2	−1/2	0	1	—
x_1	2	2	0	1/8	1/8	1	0	$2/\frac{1}{8}$
	$z=4$		0 ↓	3/4	−1/4 ↑	0	0	← Δ_j
s_3	0	4	1	−2	1	0	0	
x_2	1	2	1/2	−1/2	0	0	1	
x_1	2	3/2	−1/8	3/8	0	1	0	
	$z=5$		1/4	1/4	0	0	0	← $\Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum solution is obtained as : $x_1 = 3/2$, $x_2 = 2$, max $z = 5$.

Example 21. *Max.* $z = 5x_1 - 2x_2 + 3x_3$, *subject to* $2x_1 + 2x_2 - x_3 \geq 2$, $3x_1 - 4x_2 \leq 3$, $x_2 + 3x_3 \leq 5$, *and* $x_1, x_2, x_3 \geq 0$. **[Kanpur 96; Madras (Appl. Math.) 78; Gauhati (Math.) 75; Punjabi (Math.) 75]**

Solution. Introducing the surplus variable $s_1 \geq 0$, slack variables $s_2 \geq 0$, $s_3 \geq 0$ and an artificial variable $a_1 \geq 0$, the constraints of the problem become :

$$\begin{aligned} 2x_1 + 2x_2 - x_3 - s_1 \qquad\qquad + a_1 &= 2 \\ 3x_1 - 4x_2 \qquad\qquad + s_2 \qquad\qquad &= 3 \\ x_2 + 3x_3 \qquad\qquad + s_3 \qquad &= 5\,. \end{aligned}$$

and using big-M technique objective function becomes :

$$\text{Max. } z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - Ma_1.$$

In the usual manner, the starting simplex table is obtained as below :

Table 5.38

		$c_j \rightarrow$	5	−2	3	0	0	0	−M	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	A_1	MIN. RATIO (X_B/X_k)
$\leftarrow a_1$	−M	2	$\boxed{2}$	2	−1	−1	0	0	1	2/2←
s_2	0	3	3	−4	0	0	1	0	0	3/3
s_3	0	5	0	1	3	0	0	1	0	–
		$z = -2M$	−2M − 5 ↑	−2M + 2	M − 3	M	0	0	0 ↓	$\leftarrow \Delta_j$

Net evaluations Δ_j are computed by the formula $\Delta_j = c_B X_j - c_j$ in the usual manner. Since Δ_1 is the most negative, X_1 enters the basis. Further, since the min. ratio in the last column of above table is 1 for both the first and second rows, therefore either A_1 or S_2 tends to leave the basis. This is an indication of the existence of degeneracy. But, A_1 being an artificial vector will be prefered to leave the basis. *Note that there is no need to apply the procedure for resolving degeneracy under such circumstances.*

Continuing the simplex routine, the computations are presented in the following tabular form.

Table 5.39

		$c_j \rightarrow$	5	−2	3	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	MIN. RATIO (X_B/X_k)
$\rightarrow x_1$	5	1	1	1	−1/2	−1/2	0	0	–
$\leftarrow s_2$	0	0	0	−7	$\boxed{3/2}$	3/2	1	0	$0/\frac{3}{2}$ ←
s_3	0	5	0	1	3	0	0	1	5/3
		$z = 5$	0	7	−11/2 ↑	−5/2	0 ↓	0	$\leftarrow \Delta_j$
x_1	5	1	1	−4/3	0	0	1/3	0	–
$\rightarrow x_3$	3	0	0	−14/3	1	1	2/3	0	–
$\leftarrow s_3$	0	5	0	$\boxed{15}$	0	−3	−2	1	5/15←
		$z = 5$	0	−56/3 ↑	0	3	11/3	0 ↓	$\leftarrow \Delta_j$
x_1	5	13/9	1	0	0	−4/15	7/45	4/45	–
$\leftarrow x_3$	3	14/9	0	0	1	$\boxed{1/15}$	2/45	14/45	70/3←
$\rightarrow x_2$	−2	1/3	0	1	0	−1/5	−2/15	1/15	–
		$z = 101/9$	0	0	0 ↓	−11/15 ↑	53/45	56/45	$\leftarrow \Delta_j$
x_1	5	23/3	1	0	4	0	1/3	4/3	
$\rightarrow s_1$	0	70/3	0	0	15	1	2/3	14/3	
x_2	−2	5	0	1	3	0	0	1	
		$z = 85/3$	0	0	11	0	5/3	14/3	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, optimum solution is : $x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$, max. $z = 85/3$.

Q. 1. What is degeneracy ? Discuss a method to resolve degeneracy in L.P. problems.

[Meerut (Math.) 85, 82; Delhi (OR) 79, 76; Punjab (Math.) 74]

2. Explain what is meant by degeneracy and cycling in linear programming. How their effects overcome ?

[Meerut (L.P.) 90; Kuruk. (M. Stat.) 78]

EXAMINATION PROBLEMS

Solve the following LP problems :

1. Max. $z = 5x_1 + 3x_2$
subject to
$x_1 + x_2 \leq 2$
$5x_1 + 2x_2 \leq 10$
$3x_1 + 8x_2 \leq 12$
$x_1, x_2, \geq 0.$
[Rohil. 85; Meerut (Math.) 74]
[Ans. $x_1 = 2, x_2 = 0, z = 10$]

2. Max. $R = 22x + 30y + 25z$
subject to
$2x + 2y \leq 100$
$2x + y + z \leq 100$
$x + 2y + 2z \leq 100$
$x, y, z, \geq 0$.
[Meerut (Math.) 74]
[Ans. $x = 100/3, y = 50/3, z = 50/3,$ $R = 1650$]

3. Max. $z = 2x_1 + 3x_2 + 10x_3$
subject to
$x_1 + 2x_3 = 0$
$x_2 + x_3 = 1$
$x_1, x_2, x_3 \geq 0$.
[Meerut B.Sc. (Hons.) 70]
[Ans. $x_1 = 0, x_2 = 1, x_3 = 0$ and max. $z = 3$]

4. Max. $z = 3x_1 + 5x_2$
subject to the constraints
$x_1 + x_3 = 4, x_2 + x_4 = 6,$
$3x_1 + 2x_2 + x_5 = 12$, and
$x_1, x_2, x_3, x_4, x_5 \geq 0$
Does the degeneracy occur in this problem ?
[Ans. $x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 0, x_5 = 0,$
$z^* = 30$. Yes, degeneracy occurs.]

5. Max. $z = 2x_1 + x_2$
subject to the constraints
$x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 6,$
$x_1 - x_2 \leq 2, x_1 + 2x_2 \leq 1,$
$2x_1 - 3x_2 \leq 1$, and $x_1, x_2 \geq 0$.
[Ans. $x_1 = 5/7, x_2 = 1/7$
max. $z = 11/7$]

6. Max. $z = 3/4\, x_1 - 150\, x_2 + 1/50\, x_3 - 6x_4,$
subject to the constraints
$1/4\, x_1 - 60\, x_2 - 1/26\, x_3 + 9\, x_4 \leq 0,$
$1/2\, x_1 - 90\, x_2 - 1/50\, x_3 + 3x_4 \leq 0,$
$x_3 \leq 1$ and $x_1, x_2, x_3, x_4 \geq 0$.
[Ans. $x_1 = 1/25, x_2 = 0, x_3 = 1$
and $x_4 = 0$, max. $z = 1/20$]

7. Max. $z = 2x_1 + 3x_2 + 10x_3$, subject to
$x_1 + 2x_3 = 1, x_2 + x_3 = 1,$
and $x_1, x_2, x_3 \geq 0$.
[Meerut (Maths.) 70]
[Ans. $x_1 = 0, x_2 = 1/2, x_3 = 1/2$, max. $z = 13/2$]

8. Min. $z = -3/4\, x_1 + 20x_2 - 1/2\, x_3 + 6x_4$, subject to
$1/4\, x_1 - 8x_2 - x_3 + 9x_4 \leq 0,\ 1/4\, x_1 - 12x_2 - 1/2\, x_3 + 3x_4 \leq 0$
and $x_1, x_2, x_3, x_4, \geq 0$.
[Meerut (Math.) 79]
[Ans. Unbounded solution.]

5·8. SPECIAL CASES : ALTERNATIVE SOLUTIONS, UNBOUNDED SOLUTIONS, NON-EXISTING SOLUTIONS

In this section, some important cases (except degeneracy) are discussed which are very often encountered during simplex procedure. The properties of these cases have already been visualised in the graphical solution of two variable LP problems.

5·8-1 Alternative Optimum Solutions

Example 22. *Use penalty (or Big-M) method to solve the problem :*

Max. $z = 6x_1 + 4x_2$, *subject to* $2x_1 + 3x_2 \leq 30,\ \ 3x_1 + 2x_2 \leq 24,\ \ x_1 + x_2 \geq 3$, and $x_1, x_2 \geq 0$.

Is the solution unique ? If not, give two different solutions. **[Bombay B.Sc. (Stat.) 73]**

Solution. Introducing the slack variables $x_3 \geq 0, x_4 \geq 0$, surplus variable $x_5 \geq 0$, and artificial variable $a_1 \geq 0$, the problem becomes :

Max. $z = 6x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} 2x_1 + 3x_2 + x_3 \qquad\qquad &= 30 \\ 3x_1 + 2x_2 \qquad + x_4 \qquad &= 24 \\ x_1 + x_2 \qquad\qquad - x_5 + a_1 &= 3 \\ x_1, x_2, x_3, x_4, x_5, a_1 &\geq 0. \end{aligned}$$

Now the solution is obtained as follows :

Table 5·40

BASIC VARIABLES	C_B	X_B	X_1 (6)	X_2 (4)	X_3 (0)	X_4 (0)	X_5 (0)	A_1 ($-M$)	MIN RATIO (X_B/X_k)
x_3	0	30	2	3	1	0	0	0	30/2
x_4	0	24	3	2	0	1	0	0	24/3
$\leftarrow a_1$	$-M$	3	$\boxed{1}$	1	0	0	-1	1	$3/1 \leftarrow$
	$z=-3M$		$(-M-6)$ ↑	$(-M-4)$	0	0	M	0 ↓	$\leftarrow \Delta_j$
x_3	0	24	0	1	1	0	2	×	24/2
$\leftarrow x_4$	0	15	0	-1	0	1	$\boxed{3}$	×	$15/3 \leftarrow$
$\rightarrow x_1$	6	3	1	1	0	0	-1	×	—
	$z=18$		0	2	0	0 ↓	-6 ↑	×	$\leftarrow \Delta_j$
$\leftarrow x_3$	0	14	0	$\boxed{5/3}$	1	$-2/3$	0	×	$\frac{14}{5/3}=42/5 \leftarrow$
$\rightarrow x_5$	0	5	0	$-1/3$	0	1/3	1	×	—
x_1	6	8	1	2/3	0	1/3	0	×	$\frac{8}{2/3}=12$
	$z=48$		0	0* ↑	0 ↓	2	0	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, optimum solution is obtained as : $x_1 = 8$, $x_2 = 0$, max $z = 48$.

Alternative Solutions. Since Δ_2 corresponding to non-basic variable x_2 is obtained zero, this indicates that the *alternative solutions* also exist. Therefore, the solution as obtained above is not unique.

Thus we can bring $\mathbf{X_2}$ into the basis in place of $\mathbf{X_3}$. Therefore, introducing $\mathbf{X_2}$ into the basis in place of $\mathbf{X_3}$, the new optimum simplex table is obtained as follows :

Table 5·41

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/X_k)
x_2	4	42/5	0	1	3/5	$-2/5$	0	×	
x_5	0	39/5	0	0	1/5	1/5	1	×	
x_1	6	12/5	1	0	$-2/5$	3/5	0	×	
	$z=48$		0	0	0	2	0	×	$\leftarrow \Delta_j \geq 0$

From this table we get a different optimum solution : $x_1 = 12/5$, $x_2 = 42/5$, max. $z = 48$.

Thus, if two alternative optimum solutions can be obtained, then any number of optimum solutions can be obtained, as given below :

Variables	First Sol.	Second. Sol.	General Solution
x_1	8	12/5	$x_1 = 8\lambda + (12/5)(1-\lambda)$
x_2	0	42/5	$x_2 = 0\lambda + (42/5)(1-\lambda)$
x_3	14	0	$x_3 = 14\lambda + 0(1-\lambda)$
x_4	0	0	$x_4 = 0\lambda + 0(1-\lambda)$
x_5	5	39/5	$x_5 = 5\lambda + (39/5)(1-\lambda)$
a_1	0	0	$a_1 = 0\lambda + 0(1-\lambda)$

For any arbitrary value of λ, same optimal value of z will be obtained.

Note. If two optimum solutions of an LP problem are obtained. thus the mean of these two solutions will give us the third optimum solution. This process can be continued indefinitely to get as many alternative solutions as we want.

Example 23. *Maximize* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to the constraints :*

$x_1+2x_2+3x_3=15,\ 2x_1+x_2+5x_3=20,\ x_1+2x_2+x_3+x_4=10$, *and* $x_1, x_2, x_3, x_4 \geq 0$. **[Meerut 83, 82]**

Solution. Introducing artificial variables a_1 and a_2 in the *first* and *second* constraint equations, respectively, and the *original variable* x_4 can be treated to work as an artificial variable for the *third* constraint equation to obtain :

$$\begin{aligned} x_1+2x_2+3x_3 \quad +a_1 \quad &= 15 \\ 2x_1+x_2+5x_3 \quad\quad +a_2 &= 20 \\ x_1+2x_2+x_3+x_4 \quad\quad &= 10\,. \end{aligned}$$

Phase 1 : Table 5·42

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	15	1	2	3	0	1	0
a_2	20	2	1	5	0	0	1
$\leftarrow x_4$	10	$\boxed{1}$ ↑	2	1	1 ↓	0	0

By the same arguments as given in the previous examples of two-phase method insert X_4 in place of X_1. The transformed table (*Table 3·43*) is obtained by applying row transformations $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 2R_3$.

Table 5·43

BASIC VARIABES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
a_1	5	0	0	2	−1	1	0
$\leftarrow a_2$	0	0	−3	$\boxed{3}$	−2	0	1
$\to x_1$	10	1	2	1 ↑	1	0	0 ↓

In spite of the fact that the artificial variable x_4 has served its purpose, the column X_4 cannot be deleted from *Table 5·43*, because x_4 is the original variable also. Although the value of the artificial variable a_2 also becomes zero at this stage, the column A_2 cannot be deleted unless it is inserted at one of the places X_2 or X_3 or X_4 (wherever it is possible). Now, it is observed that A_2 can be inserted in place of X_3. Hence transformation *Table* 5·44 is obtained by applying the row transformations : $R_2 \to \frac{1}{3}R_2$, $R_1 \to R_1 - \frac{2}{3}R_2$, $R_3 \to R_3 - \frac{1}{3}R_2$.

Table 5·44

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1	A_2
$\leftarrow a_1$	5	0	2	0	1/3	1	−2/3
$\to x_3$	0	0	−1	1	−2/3	0	1/3
x_1	10	1	3	0	5/3	0	−4/3
			↑			↓	

Now removing A_1 and inserting it in the suitable position of X_2, the next transformed *Table 5·45* is obtained by row transformations : $R_1 \to \frac{1}{2}R_1$, $R_2 \to R_2 + \frac{1}{2}R_1$, $R_3 \to R_3 - \frac{3}{2}R_1$.

Table 5·45

BASIC VARIABLES	X_B	X_1	X_2	X_3	X_4	A_1
x_2	5/2	0	1	0	1/6	1/2
x_3	5/2	0	0	1	−1/2	1/2
x_1	5/2	1	0	0	7/6	−3/2

Delete column A_1 $(a_1=0)$. The starting basic feasible solution is obtained : $x_1=x_2=x_3=5/2$, $x_4=0$. Further, proceed to test this solution for optimality in *Phase II*. For this, compute

$$\Delta_4 = C_B X_4 - c_4 = (2\,,3\,,1)\,(1/6\,,-1/2\,,7/6) - 0 = 0\,.$$

Phase II. Table 5·46

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	Min. Ratio
x_2	2	5/2	0	1	0	1/6	
x_3	3	5/2	0	0	1	– 1/2	
x_1	1	5/2	1	0	0	7/6	
	$z = C_B X_B = 15$		0	0	0	0^*	$\leftarrow \Delta_j$

Since all Δ_j 's are zero, the solution : $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$, is optimal to give us $z^* = 15$. Further, Δ_4 being zero indicates that *alternative* optimal solutions are also possible.

Note. Here Δ_j corresponding to nonbasic vector X_4 also becomes zero. This indicates that alternative optimum solutions are possible. However, the other optimal solutions can be obtained as : $x_1 = 0$, $x_2 = 15/7$, $x_3 = 25/7$, $x_4 = 0$, max. $z = 15$.

Now, given the *two* alternative basic solutions ;

(i) $x_1 = x_2 = x_3 = 5/2$, $x_4 = 0$ (ii) $x_1 = 0$, $x_2 = 15/7$, $x_3 = 25/7$, $x_4 = 0$

an infinite number of *non-basic* solutions can be obtained and by realizing them any weighted average of these two basic solutions is also an alternative optimum solution.

To verify this, *third* solution will be obtained as :

$$x_1 = \frac{5/2 + 0}{2}, \; x_2 = \frac{5/2 + 15/7}{2}, \; x_3 = \frac{5/2 + 25/7}{2}, \; x_4 = \frac{0+0}{2}$$

i.e.,
$$x_1 = 5/4 , \; x_2 = 65/28 , \; x_3 = 85/28 , \; x_4 = 0 ,$$

yielding the maximum value of $z = 15$.

Note. Also see example 14 page 2.81.

Example 24. *Following is the LP problem : Maximize* $z = x_1 + x_2 + x_4$, *subject to the constraints* :

$x_1 + x_2 + x_3 + x_4 = 4$, $x_1 + 2x_2 + x_3 + x_5 = 4$, $x_1 + 2x_2 + x_3 = 4$, $x_1, x_2, x_3, x_4, x_5 \geq 0$. **[I.S.I. (Dip.) 74]**

(i) *Find out all the optimal basic feasible solutions by using penalty (or Big-M) method.*

(ii) *Write-down the general form of an optimal solution.*

Solution. Since the constraints of the given problem are already equations, only artificial variables are required to form the basis matrix. In order to bring the basis matrix as unit matrix, only artificial variable $a_1 \geq 0$ is needed in the third constraint. So the problem may be re-written in the form :

Max. $z = x_1 + x_2 + 0x_3 + x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 \qquad\qquad &= 4 \\ x_1 + 2x_2 + x_3 \quad + x_5 \quad &= 4 \\ x_1 + 2x_2 + x_3 \qquad + a_1 &= 4 \\ x_1, x_2, \ldots, x_5, a_1 &\geq 0 \end{aligned}$$

These constraints may be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}.$$

Applying the usual simplex method, the solution is obtained as follows :

Table 5·47

		$c_j \rightarrow$	1	1	0	1	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	MIN. RATIO (X_B/Y_k)
x_4	1	4	1	1	1	1	0	0	4/1
x_5	0	4	1	2	1	0	1	0	4/2
$\leftarrow a_1$	$-M$	4	1	$\boxed{2}$	1	0	0	1	4/2 ←(Note)
	$z = -4M + 4$		$-M$	$-2M$ ↑	$-M+1$	0	0	0 ↓	$\leftarrow \Delta_j$

Note. Here it is observed that the minimum 4/2 occurs at two places (2nd and 3rd) in the last column. Although one of these two may be chosen by degeneracy rule (see 3.7, page 2.86), but minimum at 3rd place has been chosen to remove artificial basis vector A_1 from the basis matrix.

Table 5·48

BASIC VARIABLES	C_B	$c_j \to$ X_B	1 X_1	1 X_2	0 X_3	1 X_4	0 X_5	$-M$ A_1	MIN. RATIO (X_B/X_k)
x_4	1	2	1/2	0	1/2	1	0	×	
x_5	0	0	0	0	0	0	1	×	
$\leftarrow x_2$	1	2	1/2	1	1/2	0	0	×	
	$z=4$		0*	0	1	0	0	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal basic feasible solution has been attained. Thus the optimum solution is given by

$$x_1=0\,,\, x_2=2\,,\, x_3=0\,,\, x_4=2\,,\, x_5=0\,,\ \text{max. } z=4\,.$$

Since $\Delta_1=0$, alternative optimum solutions also exist.

5·8–2 . Unbounded Solutions

The case of unbounded solutions occurs when the feasible region is unbounded such that the value of the objective function can be increased indefinitely. It is not necessary, however, that an unbounded feasible region should yield an unbounded value for the objective function. The following examples will illustrate these points.

Example 25. (Unbounded Optimal Solution)

Max. $z=2x_1+x_2$, *subject to* : $x_1-x_2 \leq 10,\ 2x_1-x_2 \leq 40$, *and* $x_1 \geq 0\,,\, x_2 \geq 0$.

Solution. The starting simplex table is as follows :

BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1 (β_1)	S_2 (β_2)
s_1	0	10	1	-1	1	0
s_2	0	40	2	-1	0	1
	$z=C_BX_B=0$		-2	-1	0	0

It can be seen from the starting simplex table that the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are candidates for the entering vector. Since Δ_1 has the minimum value, $\mathbf{X}_1$ should be selected as the entering vector. It is noticed, however, that if $\mathbf{X}_2$ is selected as the entering vector, the value of x_2 (and hence the value of z) can be increased indefinitely without affecting the feasibility of the solution (since it has all x_{i2} negative). It is thus concluded that the problem has no bounded solution. This can also be seen from the graphical solution of the problem in Fig. 5.1.

In general, *an unbounded solution can be detected if, at any iteration, any of the candidates for the entering vector* $\mathbf{X}_k$ *(for which* $\Delta_k<0$, *i.e.* $z_k-c_k<0$) *has all* $x_{ik} \leq 0$, *for* $i=1, 2, \ldots, m$, *i.e., all elements of the entering column are* ≤ 0.

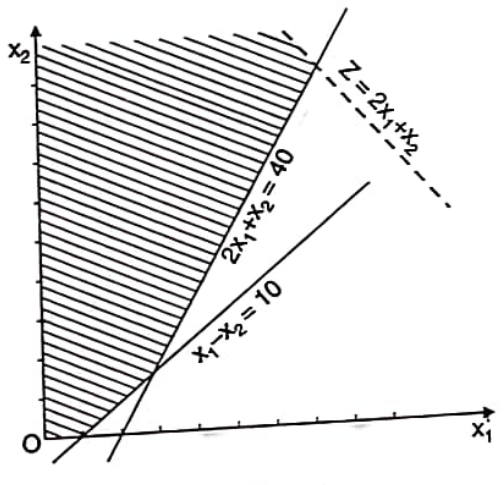

Fig. 5.1

Example 26. (Unbounded Solutions)

Maximize $z=107x_1+x_2+2x_3$, *subject to* :

$$14x_1+x_2-6x_3+3x_4=7,\ 16x_1+\tfrac{1}{2}x_2-6x_3 \leq 5,\ 3x_1-x_2-x_3 \leq 0,\ \textit{and}\ x_1\,,x_2\,,x_3 \geq 0\,.$$

[M.S. Baroda B.E. (Chem.) 78]

Solution. By introducing slack variables, $x_5 \geq 0\,,\, x_6 \geq 0$, the set of constraints is converted into the system of equations :

$$\left\{\begin{array}{l} 14x_1 + x_2 - 6x_3 + 3x_4 \qquad\qquad = 7 \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 \qquad + x_5 \quad = 5 \\ 3x_1 - x_2 - x_3 \qquad\qquad + x_6 = 0 \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} \frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 \qquad\qquad = 7/3 \\ 16x_1 + \frac{1}{2}x_2 - 6x_3 \qquad + x_5 \quad = 5 \\ 3x_1 - x_2 - x_3 \qquad\qquad + x_6 = 0 \end{array}\right.$$

or
$$\begin{bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}$$

Here original variable x_4 has been treated as slack variable as its coefficient in the objective function is zero,

i.e., *Maximize* $z = 107x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

Now start simplex method as follows :

Table 5·50

		$c_j \rightarrow$	107	1	2	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO
x_4	0	7/3	14/3	1/3	−2	1	0	0	7/14
x_5	0	5	16	1/2	−6	0	1	0	5/16
x_6	0	0	$\boxed{3}$	−1	−1	0	0	1	0/3 ←
	$z = 0$		−107 ↑	−1	−2	0	0	0 ↓	← Δ_j
x_4	0	7/3	0	17/9	−4/9	1	0	−14/9	
x_5	0	5	0	35/6	−2/3	0	1	−16/3	
x_1	107	0	1	−1/3	−1/3	0	0	1/3	
	$z = 0$		0	$-\frac{110}{3}$	$-\frac{113}{3}$	0	0	$\frac{107}{3}$	← Δ_j

Since corresponding to negative Δ_3 , all elements of X_3 column are negative, so X_3 cannot enter into the basis matrix. Consequently, this is an indication that there exists an *unbounded solution* to the given problem.

Example 27. ***(Unbounded feasible region but bounded optimal solution)***

Max. $z = 6x_1 - 2x_2$, *subject to* $2x_1 - x_2 \leq 2, x_1 \leq 4,$ *and* $x_1 , x_2 \geq 0$.

Solution. We only give the successive tables here. Students are advised to fill up the details.

Table 5·51. Starting Simplex Table

		$c_j \rightarrow$	6	−2	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3 (β_1)	X_4 (β_2)	MIN. RATIO (X_B/X_1)
x_3	0	2	2	−1	1	0	2/2 ←
x_4	0	4	1	0	0	1	4/1
	$z = C_BX_B = 0$		−6 ↑	2	0 ↓	0	← Δ_j

First Improvement. We enter X_1 and remove β_1 .

Table 5·52

		$c_j \rightarrow$	6	−2	0	0	
BASIC VARIABLES	C_B	X_B	X_1 (β_1)	X_2	X_3	X_4 (β_2)	MIN. RATIO (X_B/X_2)
x_1	6	1	1	− 1/2	1/2	0	—
x_4	0	3	0	$\boxed{1/2}$	−1/2	1	$3/\frac{1}{2}$
	$z = C_BX_B = 6$		0	−1 ↑	3	0 ↓	← Δ_j

Second Improvement. Enter $\mathbf{X}_2$ and remove β_2 .

Table 3·53

BASIC VARABLES	C_B	X_B	X_1 (β_1)	X_2 (β_2)	X_3	X_4	Min. Ratio
x_1	6	4	1	0	0	1	
x_2	–2	6	0	1	–1	2	
	$z = C_BX_B = 12$		0	0	2	2	$\leftarrow \Delta_j$

The optimal solution is : $x_1 = 4$, $x_2 = 6$, and $z = 12$.

It is now interesting to note from starting table that the elements of $\mathbf{X}_2$ are negative or zero (–1 and 0). This is an immediate indication that the feasible region is not bounded (see Fig. 5·2). From this, we conclude that a problem may have unbounded feasible region but still the optimal solution is bounded.

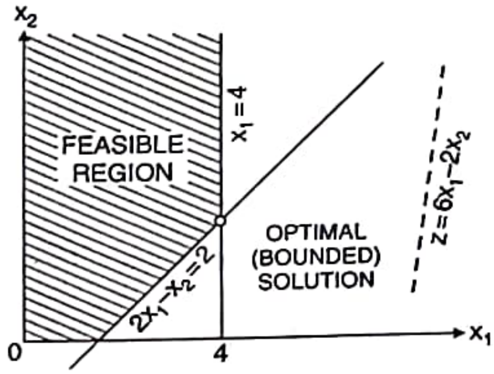

Fig. 5.2

5·8–3 . Non-existing feasible solutions

In this case, the feasible region is found to be empty which indicates that the problem has no feasible solution. The following example shows how such a situation can be detected by simplex method.

Example 28. (*Problem with no feasible solution*).

Max. $z = 3x_1 + 2x_2$, *subject to* $2x_1 + x_2 \le 2$, $3x_1 + 4x_2 \ge 12$, *and* x_1 , $x_2 \ge 0$.

[Garhwal 97; Meerut (O.R.) 90; M.S. Baroda B.Sc. (Math.) 80]

Solution. Introducing slack variable x_3 , surplus variable x_4 together with the artificial variable a_1 , the constraints become :

$$2x_1 + x_2 + x_3 = 2$$
$$3x_1 + 4x_2 - x_4 + a_1 = 12 .$$

Here we use M-technique for dealing with artificial variable a_1 . For this, we write the objective function as

$$\text{Max. } z = 3x_1 + 2x_2 + 0x_3 + 0x_4 - Ma_1 .$$

The starting simplex table will be as follows.

Table 5·54

		$c_j \rightarrow$	3	2	0	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3 (β_1)	X_4	A_1 (β_2)	MIN. RATIO (X_B/X_K)
$\leftarrow x_3$	0	2	1	$\boxed{1}$	1	0	0	$2/1 \leftarrow$
a_1	$-M$	12	3	4	0	–1	1	12/4
	$z = C_BX_B = -12M$		$(-3M-3)$	$(-4M-2)$ $\uparrow$	0 $\downarrow$	M	0	$\leftarrow \Delta_j$

$$\Delta_1 = C_BX_1 - c_1 = (0 , -M)\ (2, 3) - 3 = (0 - 3M) - 3 = -3 - 3M$$
$$\Delta_2 = C_BX_2 - c_2 = (0 , -M)\ (1 , 4) - 2 = (0 - 4M) - 2 = -2 - 4M$$
$$\Delta_4 = C_BX_4 - c_4 = (0 , -M)\ (0 ,- 1) - 0 = M .$$

First improvement. Inserting $\mathbf{X}_2$ and removing β_1 , *i.e.* $\mathbf{X}_3$

Table 5·55

		$c_j \rightarrow$	3	2	0	0	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	S_1	S_2	A_1	
x_2	2	2	2	1	1	0	0	
a_1	$-M$	4	–5	0	–4	–1	1	
	$z = C_BX_B = 4 - 4M$		$(1 + 5M)$	0	$(2 + 4M)$	M	0	$\leftarrow \Delta_j$

$$\Delta_1 = \mathbf{C_B Y_1} - c_1 = (2\,,-M)\,(2\,,-5) - 3 = (4+5M) - 3 = (1+5M)$$
$$\Delta_3 = \mathbf{C_B Y_3} - c_3 = (2\,,-M)\,(1\,,-4) - 0 = (2+4M) - 0 = (2+4M)$$
$$\Delta_4 = \mathbf{C_B Y_4} - c_4 = (2\,,-M)\,(0\,,-1) - 0 = (0+M) = M\,.$$

Here all Δ_j are positive since $M > 0$. So according to the optimality condition, this solution is optimal.

Note. Here we should, however, note that the optimal (basic) solution :

$x_1 = 0\,,\ x_2 = 2\,,\ x_3 = 0\,,\ x_4 = 0\,,\ a_1 = 4\,,$

includes the artificial variable a_1 with positive value 4. This immediately indicates that the problem has no feasible solution, because the positive value of a_1 violates the second constraint of given problem. This situation can be observed by the graphical representation of this example in Fig. 5·3.

Such solution may be called "*pseudo-optimal*", since (as clear from the Figure 5·3) it does not satisfy all the constraints, but it satisfies the optimality condition of the simplex method. **[JNTU (B.Tech) 98]**

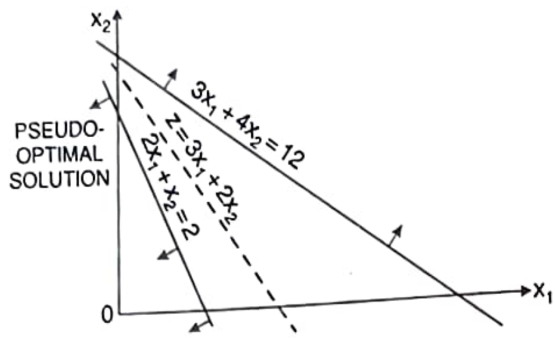

Fig. 5.3.

5·9. SOLUTION OF SIMULTANEOUS EQUATIONS BY SIMPLEX METHOD

For the solution of n simultaneous linear equations in n variables a *dummy* objective function is introduced as

$$\text{Max.}\quad z = 0\,\mathbf{X} - 1\,\mathbf{X_a}$$

where $\mathbf{X}_a$ are artificial variables, and $x_r = x_r' - x_r''$, such that $x_r' \geq 0\,,\ x_r'' \geq 0\,.$

The reformulated linear programming problem is then solved by simplex method. The optimal solution of this problem gives the values of the variables (**X**).

The following example will illustrate the procedure.

Example 29. *Use simplex method to solve the following system of linear equations :*

$$x_1 - x_3 + 4x_4 = 3\,,\ 2x_1 - x_2 = 3\,,\ 3x_1 - 2x_2 - x_4 = 1\,,\ \text{and}\ x_1\,,x_2\,,x_3\,,x_4 \geq 0\,.$$

[Meerut M. Com. Jan. 98 (BP), M.Sc. (OR) 86; Delhi (OR) 79, B.Sc. (Math.) 75]

Solution. Since the objective function for the given constraint equation is not prescribed, so a dummy objective function is introduced as :

Max. $z = 0x_1 + 0x_2 + 0x_3 + 0x_4 - 1a_1 - 1a_2 - 1a_3$, where $a_1 \geq 0\,, a_2 \geq 0\,, a_3 \geq 0$ are artificial variables. Introducing artificial variables, the given equations can be written as :

$$\begin{aligned} x_1 \quad - x_3 + 4x_4 + a_1 \qquad &= 3\\ 2x_1 - x_2 \qquad\qquad + a_2 \quad &= 3\\ 3x_1 - 2x_2 \quad - x_4 \qquad + a_3 &= 1\,. \end{aligned}$$

Now apply simplex method to solve the reformulated problem as shown in *Table 5·56.*

Table 5·56

		$c_j \rightarrow$	0	0	0	0	−1	−1	−1	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	A_3	MIN RATIO (X_B/X_k)
a_1	−1	3	1	0	−1	4	1	0	0	3/1
a_2	−1	3	2	−1	0	0	0	1	0	3/2
$\leftarrow a_3$	−1	1	$\boxed{3}$	−2	0	−1	0	0	1	1/3 ←
	$z = -10$		−6 ↑	3	1	−3	0	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	−1	8/3	0	2/3	−1	$\boxed{13/3}$	1	0	×	8/13 ←
a_2	−1	7/3	0	1/3	0	2/3	0	1	×	7/2
$\rightarrow x_1$	0	1/3	1	−2/3	0	−1/3	0	0	×	—
	$z = -5$		0	−1	1	−5 ↑	0 ↓	0	×	$\leftarrow \Delta_j$

Contd.