

11

TRANSPORTATION PROBLEMS

11.1. INTRODUCTION

As already defined and discussed earlier, the simplex procedure can be regarded as the most generalized method for linear programming problems. However, there is very interesting class of '*Allocation Methods*' which is applied to a lot of very practical problems generally called '*Transportation Problems*'. Whenever it is possible to place the given linear programming problem in the transportation frame-work, it is far more simple to solve it by '*Transportation Technique*' than by '*Simplex*'.

Let the nature of transportation problem be examined first. If there are more than one centres, called '*origins*', from where the goods need to be shipped to more than one places called '*destinations*' and the costs of shipping from each of the *origins* to each of the *destinations* being different and known, the problem is to ship the goods from various *origins* to different *destinations* in such a manner that the cost of shipping or transportation is minimum.

Thus, we can formally define the transportation problem as follows :

Definition. *The Transportation Problem is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.*

For example, a tyre manufacturing concern has m factories located in m different cities. The total supply potential of manufactured product is absorbed by n retail dealers in n different cities of the country. Then, transportation problem is to determine the transportation schedule that minimizes the total cost of transporting tyres from various factory locations to various retail dealers.

The various features of linear programming can be observed in these problems. Here the availability as well as the requirements of the various centres are finite and constitute the limited resources. It is also assumed that the cost of shipping is linear (for example, the costs of shipping of two objects will be twice that of shipping a single object). However, this condition is not often true in practical problems, but will have to be assumed so that the linear programming technique may be applicable to such problems. These problems thus could also be solved by '*Simplex*'. Mathematically, the problem may be stated as given in the following section.

Q. Define transportation problem.

[Bhubneshwar (IT) 2004]

11.2. MATHEMATICAL FORMULATION

Let there be m origins, i th origin possessing a_i units of a certain product, whereas there are n destinations (n may or may not be equal to m) with destination j requiring b_j units. Costs of shipping of an item from each of m origins (sources) to each of the n destinations are known either directly or indirectly in terms of mileage, shipping hours, etc. Let c_{ij} be the cost of shipping one unit product from i th origin (source) to j th destination., and ' x_{ij} ' be the amount to be shipped from i th origin to j th destination.

It is also assumed that total availabilities $\sum a_i$ satisfy the total requirements $\sum b_j$, i.e.,

$$\sum a_i = \sum b_j \quad (i = 1, 2, \dots, m; j = 1, 2, \dots, n) \quad \dots(11.1)$$

(In case, $\sum a_i \neq \sum b_j$, some manipulation is required to make $\sum a_i = \sum b_j$, which will be shown later).

The problem now is to determine non-negative (≥ 0) values of ' x_{ij} ' satisfying both, the availability constraints :

$$\sum_{j=1}^n x_{ij} = a_i \quad \text{for } i = 1, 2, \dots, m \quad \dots(11.2)$$

as well as the requirement constraints :

$$\sum_{i=1}^m x_{ij} = b_j \quad \text{for } j = 1, 2, \dots, n \quad \dots(11.3)$$

and minimizing the total cost of transportation (shipping)

$$z = \sum_{i=1}^m \sum_{j=1}^n x_{ij} c_{ij} \quad (\text{objective function}). \quad \dots(11.4)$$

It may be observed that the constraint equations (11.2), (11.3) and the objective function (11.4) are all linear in x_{ij} , so it may be looked like a linear programming problem.

This special type of LPP will be called a **Transportation Problem (T.P.)**.
Remark : By requiring strict inequalities $a_i > 0$ and $b_j > 0$ we are not restricting anything. Since all $x_{ij} \geq 0$, it follows that each $a_i \geq 0$ and each $b_j \geq 0$. Moreover any $a_k = 0 \Rightarrow x_{kj} = 0$ and thus can be eliminated from the problem.

- Q. 1. Explain Transportation problem and show that it can be considered as L.P.P.
 2. Formulate transportation problem as a L.P.P.
 3. Specify a transportation problem (TP). Is this an LPP ?
 4. Explain the difference between a transportation problem and an assignment problem. Explain situations where an assignment problem can arise. [Meerut (Maths) 99]
 5. Show that assignment problem is the special case of the transportation problem. [IAS (Main) 88]
 6. Give the mathematical formulation and difference between 'Transportation' and 'Assignment' problems.
 [Agra 99; Kanpur 96; Meerut (IPM) 91; 90]

11.3. MATRIX FORM OF TRANSPORTATION PROBLEM

Consider the transportation problem as mathematically formulated above. The set of constraints $\sum_{j=1}^n x_{ij} = a_i$ ($i = 1, 2, \dots, m$) and $\sum_{i=1}^m x_{ij} = b_j$ ($j = 1, 2, \dots, n$) represent $m+n$ equations in mn non-negative variables x_{ij} . Each variable x_{ij} appears in exactly two constraints, one associated with the i th origin O_i and the other with the j th destination D_j . In the above ordering of constraints, first we write the origin-equations and then destination-equations. Then the transportation problem can be restated in the matrix form as :

Minimize $z = CX$, $X \in \mathbb{R}^{mn}$, subject to the constraints $AX = b$, $X \geq 0$, $b \in \mathbb{R}^{m+n}$, where $X = [x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}]$, C is the cost vector, $b = [a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n]$ and A is an $(m+n) \times mn$ real matrix containing the coefficients of constraints.

It is worth noting that the elements of A are either 0 or 1. Thus the general LPP can be reduced to *transportation problem* if

(i) a_{ij} 's are restricted to the values 0 and 1; and (ii) the units among the constraints are homogeneous.

For example, if $m = 2$, $n = 3$, the matrix A is given by

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_{23}^{(1)} & e_{23}^{(2)} \\ I_3 & I_3 \end{bmatrix}$$

And, therefore, for a general transportation problem, we may write

$$A = \begin{bmatrix} e_{mn}^{(1)} & e_{mn}^{(2)} & \dots & e_{mn}^{(m)} \\ I_n & I_n & \dots & I_n \end{bmatrix}$$

where $e_{mn}^{(i)}$ is an $m \times n$ matrix having a row of unit elements as its i th row and 0's everywhere else, and I_n is the $n \times n$ identity matrix.

If a_{ij} denotes the column vector of A associated with any variable x_{ij} , then it can be easily verified that
 $a_{ij} = e_i + e_{m+j}$, where $e_i, e_{m+j} \in \mathbb{R}^{m+n}$ are unit vectors.

- Q. 1. Show how a 2×3 transportation problem may be transformed into a special network termed bipartite network.
 2. If a transportation problem has p factories and 2 retail shops, what is the number of variables and what is the number of constraints?
 [IGNOU 99, 96]

11.4. FEASIBLE SOLUTION, BASIC FEASIBLE SOLUTION, AND OPTIMUM SOLUTION

The terms *feasible solution*, *basic feasible solution* and *optimum solution* may be formally defined with reference to the transportation problem (T.P.) as follows:

- (i) **Feasible Solution (FS).** A set of non-negative individual allocations ($x_{ij} \geq 0$) which simultaneously removes deficiencies is called a *feasible solution*.
- (ii) **Basic Feasible Solution (BFS).** A feasible solution to a m -origin, n -destination problem is said to be *basic* if the number of positive allocations are $m+n-1$, i.e., one less than the sum of rows and columns (see Theorem 11.2).
 If the number of allocations in a basic feasible solution are less than $m+n-1$, it is called *degenerate BFS* (otherwise, *non-degenerate BFS*).
- (iii) **Optimum Solution.** A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

- Q. Define the terms (a) Feasible solution (b) Basic feasible solution (c) Optimum in solution.

11.4-1. Existence of Feasible Solution

Theorem 11.1. (Existence of Feasible Solution). A necessary and sufficient condition for the existence of feasible solution of a transportation problem is $\sum a_i = \sum b_j$ ($i = 1, \dots, m$; $j = 1, \dots, n$). [Rewa (M.P.) 93]

Proof. The condition is necessary. Let there exist a feasible solution to the transportation problem. Then,

$$\sum_{i=1}^m \sum_{j=1}^n x_{ij} = \sum_{i=1}^m a_i, \quad \sum_{j=1}^n \sum_{i=1}^m x_{ij} = \sum_{j=1}^n b_j \Leftrightarrow \sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

The condition is sufficient. Let $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j = k$ (say).

If $\lambda_i \neq 0$ be any real number such that $x_{ij} = \lambda_i b_j$ for all i and j , then λ_i is given by

$$\sum_{j=1}^n x_{ij} = \sum_{j=1}^n \lambda_i b_j = \lambda_i \sum_{j=1}^n b_j = k \lambda_i \Rightarrow \lambda_i = \frac{1}{k} \sum_{j=1}^n x_{ij} = \frac{a_i}{k}.$$

Thus, $x_{ij} = \lambda_i b_j = \frac{a_i b_j}{k} \geq 0$, since $a_i > 0, b_j > 0$ for all i and j

Hence a feasible solution exists.

11.4-2 . Basic Feasible Solution of Transportation Problem

It has been observed that a *transportation problem* is a special case of a *linear programming problem*. So a basic feasible solution of a transportation problem has the same definition as earlier given for L.P.P. (in Sec 3.8, page 93). However, we observe that in the case of a T.P., there are only $m+n-1$ basic variables out of mn unknowns. This happens due to redundancy in the constraints of the transportation problem. This can be easily justified by proving the following theorem.

Theorem 11.2. The number of basic variables in a transportation problem are at the most $m+n-1$.

Proof. To prove this, consider the first $m+n-1$ constraints of the transportation problem as

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, \dots, n-1 \quad (n-1 \text{ equations}) \quad \dots(1)$$

and $\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, \dots, m.$ (m equations) ... (2)

Now adding $(n - 1)$ destination-constraints (1), we get

$$\sum_{j=1}^{n-1} \sum_{i=1}^m x_{ij} = \sum_{j=1}^{n-1} b_j \quad ... (3)$$

Also, adding m origin-constraints (2), we get

$$\sum_{i=1}^m \sum_{j=1}^n x_{ij} = \sum_{i=1}^m a_i \quad ... (4)$$

Then, subtracting (3) from (4), we get

$$\sum_{i=1}^m \left(\sum_{j=1}^n x_{ij} - \sum_{j=1}^{n-1} x_{ij} \right) = \sum_{i=1}^m a_i - \sum_{j=1}^{n-1} b_j \quad ... (5)$$

or $\sum_{i=1}^m \left(\sum_{j=1}^n x_{ij} - \sum_{j=1}^{n-1} x_{ij} \right) = \sum_{j=1}^n b_j - \sum_{j=1}^{n-1} b_j (\because \sum_i a_i = \sum_j b_j)$

or $\sum_{i=1}^m \left(x_{in} + \sum_{j=1}^{n-1} x_{ij} - \sum_{j=1}^{n-1} x_{ij} \right) = b_n + \sum_{j=1}^{n-1} b_j - \sum_{j=1}^{n-1} b_j$

or $\sum_{i=1}^m x_{in} = b_n$ (which is exactly the last (n th) destination-constraint)

This obviously indicates that if the first $m + n - 1$ constraints are satisfied then $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ ensures that the $(m + n)$ th constraint will be automatically satisfied.

Thus, out of $m + n$ equations, one (any) is redundant and remaining $m + n - 1$ equations form a linearly independent set. Hence the theorem is proved.

It is concluded that *a basic feasible solution will consist of at most $m + n - 1$ positive variables, others being zero. In the degenerate case, some of the basic variables will also be zero, i.e., the number of positive variables will now become less than $m + n - 1$. By fundamental theorem of linear programming, one of the basic feasible solutions will be the optimal solution.*

11.4-3. Existence of Optimal Solution

Theorem 11.3. (Existence of an optimal solution). *There always exists an optimal solution to a balanced transportation problem.*

Proof. Let $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, so that a feasible solution x_{ij} exists. It, therefore, follows from the constraints of the problem that each x_{ij} is bounded, viz., $0 \leq x_{ij} \leq \min(a_i, b_j)$.

Thus the feasible region of the problem is *closed, bounded* and *non-empty*, and hence there exists an optimal solution.

Note. In future discussion we shall assume that the above condition holds for the transportation problem without mentioning it.

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- Q. 1. Prove that the transportation problem always possesses a feasible solution.
 - 2. If all the sources are emptied and all the destinations are filled, show that $\sum a_i = \sum b_j$ is a necessary and sufficient condition for the existence of a feasible solution to the transportation problem. [Delhi B.Sc. (Maths.) 90]
 - 3. Prove that the solution of the transportation problem is invariant under the addition (subtraction) of the same constant to (from) any row or column of the unit cost matrix of the problem.
 - 4. Derive a mathematical model for a cost-minimizing 'Transportation Problem'. Show that every transportation problem has a feasible solution.
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11.5. TABULAR REPRESENTATION

Suppose there are m factories and n warehouses. The transportation problem is usually represented in a tabular form (*Table 11.1*). Calculations are made directly on the 'transportation arrays' which give the current trial solution.

Table 11.1

Warehouse → Factory ↓	W_1	W_2	...	W_j	...	W_n	Factory Capacities
F_1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}	a_1
F_2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}	a_2
:	:	:		:		:	:
F_i	c_{i1}	$\dots c_{i2}$...	c_{ij}	...	c_{in}	a_i
:	:	:		:		:	:
F_m	c_{m1}	c_{m2}	...	c_{mj}	...	c_{mn}	a_m
Warehouse requirements	b_1	b_2	...	b_j	...	b_n	$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$

Table 11.2

Warehouse → Factory ↓	W_1	W_2	...	W_j	...	W_n	Factory Capacities
F_1	x_{11}	x_{12}	...	x_{1j}	...	x_{1n}	a_1
F_2	x_{21}	x_{22}	...	x_{2j}	...	x_{2n}	a_2
:	:	:		:		:	:
F_i	x_{i1}	$\dots x_{i2}$...	x_{ij}	...	x_{in}	a_i
:	:	:		:		:	:
F_m	x_{m1}	x_{m2}	...	x_{mj}	...	x_{mn}	a_m
Warehouse requirements	b_1	b_2	...	b_j	...	b_n	$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$

In general, *Tables 11.1 and 11.2* are combined by inserting each unit cost c_{ij} together with the corresponding amount x_{ij} into the cell (i, j) . The product $x_{ij} (c_{ij})$ gives the net cost of shipping x_{ij} units from factory F_i to warehouse W_j .

Note. Whenever the amount x_{ij} and the corresponding unit cost c_{ij} are entered in the cell (i, j) , there may be a confusion to distinguish between them. Therefore, in order to remove such confusion the quantities in parenthesis will denote the unit cost c_{ij} .

- Q. 1. Describe the transportation table.
- 2. Describe the matrix form of the transportation problem. Illustrate with 2 origins and 3 destinations.

11.6. SPECIAL STRUCTURE OF TRANSPORTATION PROBLEM

The transportation problem has a *triangular basis*, i.e. the system of equations is represented in terms of basic variables only; non-basic variables are considered to be zero. The matrix of coefficients of the variables is triangular. In other words, there is an equation in which only one basic variable occurs ; in a second equation not more than two basic variables occur, in a third equation not more than three basic variables occur, and so on.

Equations (11.2) and (11.3) may be called the row and column equations, respectively.

Theorem 11.4. *The transportation problem has a triangular basis.*

Proof. To prove this theorem, consider equations (11.2) and (11.3) written row-wise and column-wise in the tabular form (*Table 11.2*).

There cannot be an equation in which no basic variable exists, otherwise the equation cannot be satisfied, for $a_i \neq 0$ or $b_j \neq 0$. The theorem will be proved by contradiction.

Suppose every equation has at least two basic variables. Then there will be at least two basic variables in each row, and the total number of basic variables will be at least $2m$. Also each column equation will have at least two basic variables, and hence in all there will be at least $2n$ variables. Let the total number of basic variables be N . Thus, $N \geq 2m$, $N \geq 2n$. Now, three cases may arise.

Case 1. If $m > n$, then $N \geq 2m$ becomes $N > m + n$.

Case 2. If $m < n$, then $N < 2n$ becomes $N < n + m$.

Case 3. If $m = n$, then $N \geq 2m$ becomes $N = m + n$.

Thus, it is observed that in every case $N \geq m + n$. But $N = m + n - 1$, which is a contradiction. Thus, the assumption of existing at least two basic variables in each row and each column is wrong. Therefore, at least one such row or column equation exists having one basic variable only.

Let x_{rc} be the only variable in the r th row and the c th column. Then, $x_{rc} = a_r$. Then equation can be eliminated from the system by deleting the r th row equation and substituting $x_{rc} = a_r$ in the c th column equation. Thus, r th row now stands cancelled, and b_c is replaced by $b'_c = b_c - a_r$.

The resulting system now consists of $m - 1$ row equations and n column equations of which $m + n - 2$ are linearly independent. Thus, there are $m + n - 2$ basic variables in this system. Repeat the process and it is concluded that there is an equation in the reduced system which has only one basic variable. If this equation happens to be the c th column equation in the original system, the c th column equation now contains two basic variables. Thus the original system has an equation which has at most two basic variables. Continue this process and ultimately it can be shown that there is an equation which has at most three basic variables, and so on.

Thus the theorem is now completely proved.

- Q.** 1. Prove that there are only $m + n - 1$ independent equations in a transportation problem, m and n being the number of origins and destinations, and any one equation can be dropped as the redundant equation.
 2. What do you mean by the triangular form of a system of linear equations? When we can say that a system of linear equations has a triangular basis.
 3. Show that all bases for transportation problem are triangular.
 4. What do you mean by non-degenerate basic feasible solution of a transportation problem.
 5. State a transportation problem. When does it have a unique solution? Explain.

11.7. LOOPS IN TRANSPORTATION TABLE AND THEIR PROPERTIES

Loop. Def. In a transportation table, an ordered set of four or more cells is said to form a **LOOP** if,

- (i) any two adjacent cells in the ordered set lie either in the same row or in the same column; and
 (ii) any three or more adjacent cells in the ordered set do not lie in the same row or in the same column.

The first cell of the set is considered to follow the last one in the set.

If we join the cells of a loop by horizontal and vertical lineup segments, we get a closed path satisfying the above conditions (i) and (ii). Let us denote the (i, j) th cell of the transportation table by (i, j) . Then it can be observed from the diagrammatic illustration in Fig. 11.1, that the set $L = \{(1, 1), (4, 1), (4, 4), (2, 4), (2, 3), (1, 3)\}$ form a loop; and on the other hand, the set $L' = \{(3, 2), (3, 5), (2, 5), (2, 4), (2, 3), (1, 3), (1, 2)\}$ does not form a loop, because three cell entries $(2, 3), (2, 4)$ and $(2, 5)$ lie in the same row (second).

(1, 1)		(1, 3)	
		(2, 3)	(2, 4)
(4, 1)			(4, 4)

(i) Loop L

	(1, 2)	(1, 3)		
		(2, 3)	(2, 4)	(2, 5)
	(3, 2)			(3, 5)

(ii) Non-loop L'

Fig. 11.1

Theorem 11.5. Every loop has an even number of cells.

Proof. For any loop, we can always choose arbitrarily a starting point and a direction by an arrow mark (\rightarrow). We consider a loop formed by n number of cells which are consecutively numbered from 1 to n . Now assume that cell 1 and 2 exist in the same column. Thus the step from cell 1 to cell 2 involves a row change. Obviously, step from cell 2 to cell 3 must involve a column change, from cell 3 to cell 4 a row change, and so on; in general, the step to cell k involves a row change, if and only if, k is even. Since the step to cell 2 involved a row change, the step from cell n to cell 1 must be a column change and the step from cell $n - 1$ to cell n a row change. Hence n will be even.

Set containing a loop. Def. A set X of cells of a transportation table is said to contain a loop if the cells of X or of a subset of X can be sequenced (ordered) so as to form a loop.

Theorem 11.6 (Linear Dependence and Loops). Let X be a set of column vectors of the coefficient matrix of a T.P.. Then, a necessary and sufficient condition for vectors in X to be linearly dependent is that the set of their corresponding cells in the transportation table contains a loop.

Proof. Let us consider an m -origin, n -destination T.P. expressed in its matrix form :

Minimize $z = CX$; $C, X \in \mathbb{R}^{mn}$, subject to the constraints : $AX = b$, $x \geq 0$, $b \in \mathbb{R}^{m+n}$
where $b = (a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n)$, A is an $(m+n) \times mn$ real matrix containing the coefficients of constraints and C is the cost vector.

To prove, the condition is sufficient :

Let us assume that the cells associated with the vectors of X contain a loop

$$L = \{(i, j), (i, k), (l, k), \dots, (p, o), (p, j)\}$$

If a_{ij} denotes the column vector of matrix A associated with the variable x_{ij} [the cell (i, j)], then it follows from the discussion in sec. 11.3 that $a_{ij} = e_i + e_{m+j}$, where $e_i, e_{m+j} \in \mathbb{R}^{m+n}$ are unit vectors. Thus X includes the column vectors :

$$a_{ij} = e_i + e_{m+j}, a_{ik} = e_i + e_{m+k}, a_{lk} = e_l + e_{m+k}, a_{lm} = e_l + e_{m+m}, \dots, a_{po} = e_p + e_{m+o},$$

and

$$a_{pj} = e_p + e_{m+j}.$$

Hence by successive addition and subtraction, we get $a_{ij} - a_{ik} + a_{lk} - a_{lm} + \dots + a_{po} - a_{pj} = 0$
(by Theorem 11.5, a loop contains an even number of cells)

Therefore, this particular subset of X , and hence X itself, is a linearly dependent set.

To prove, the condition is necessary :

Let us assume that X is a linearly dependent set. Then, there must exist scalars λ_{ij} not all zero such that

$$\sum \lambda_{ij} a_{ij} = 0, \text{ where } a_{ij} \in X.$$

For simplification, remove all those vectors from X for which $\lambda_{ij} = 0$.

Now we choose arbitrarily a vector from the remaining vectors in X . Let it be $a_{ij} = e_i + e_{m+j}$. We claim that X must contain at least one more vector whose second subscript is j . Suppose to the contrary that it does not, then since $\lambda_{ij} \neq 0$, the $(m+j)$ th component of the vector equation $\sum \lambda_{ij} a_{ij} = 0$ gives $\lambda_{ij} \cdot 1 = 0$ implying $\lambda_{ij} = 0$, a contradiction. So X contains at least one more vector with second subscript j .

Suppose that this vector is $a_{kj} = e_k + e_{m+j}$. By similar reasoning, we conclude that there must be at least one more vector in X with the first subscript k ; say, $a_{kl} = e_k + e_{m+l}$. By same argument once again, X must contain at least one vector with the second subscript l . Let it be, say, $a_{il} = e_i + e_{m+l}$.

Thus we have determined four vectors in X , namely a_{ij}, a_{kj}, a_{kl} and a_{il} whose corresponding cells, form a loop. Thus the proof is complete.

If the last vector is $a_{il} = e_i + e_{m+l}$ instead of a_{il} , then as explained just before there must exist at least one more vector with first subscript n . If it is a_{nj} , a loop is complete, if not, let it be $a_{n0} = e_n + e_{m+0}$. X must contain at least one more vector with second subscript 0. Now two cases will arise :

- (1) The first subscript of newly discovered vector is one that has already been identified. In this case a loop has been completed.
- (2) The first subscript of the newly discovered vector is also new. In this case, since the number of vectors in X is finite (by extending the above reasoning), we conclude that eventually a loop must be formed.

Corollary. A feasible solution to a transportation problem is basic if, and only if, the corresponding cells in the transportation table do not contain a loop.

Proof: Left as an exercise.

This corollary provides us a method to verify whether the current feasible solution to the transportation problem is basic or not.

- Q. 1. A feasible solution to a transportation problem is basic, if and only if, the corresponding cells in the transportation table do not contain.....
2. With reference to a transportation problem define the following terms :
 (i) Feasible solution (ii) Basic feasible solution, (iii) Optimal solution, (iv) Non-degenerate basic feasible solution.
3. Define 'loop' in a transportation table. What role do they play ? [Madurai B.Sc (Math.) 94]
4. In the classical transportation problem explain as to how many independent equations are there when there are m -origins and n -destinations. What happens and how to handle the solution, when the initial assignment in the problem gives less than this number of occupied cells ?

Initial Basic Feasible Solution

11.8. THE INITIAL BASIC FEASIBLE SOLUTION TO TRANSPORTATION PROBLEM

Methods of finding an *optimal solution* of the transportation problem will consist of two main steps :

(i) *To find an initial basic feasible solution :*

(ii) *To obtain an optimal solution by making successive improvements to initial basic feasible solution until no further decrease in the transportation cost is possible.*

There will be fewer improvements to make if initially we start with a better initial basic feasible solution. So first we shall discuss below the methods for obtaining initial basic feasible solution of a T.P.

Remark : Although the transportation problem can be solved using the regular simplex method, its special properties provide a more convenient method for solving this type of problems. This method is based on the same theory of simplex method. It makes use, however, of some shortcuts which provide a less burdensome computational scheme.

11.8-1 Methods for Initial Basic Feasible Solution

Some simple methods are described here to obtain the initial basic feasible solution of the transportation problem. These methods can be easily explained by considering the following numerical example. However, the relative efficiency of these methods is still unanswerable.

Example 1. Find the initial basic feasible solution of the following transportation problem.

Table 11.3

Warehouse → Factory ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution.

First Method : North-West Corner Rule (Stepping Stone Method). [Kanpur (B.Sc.) 2003; IAS (Main) 96 Type]

In this method, first construct an empty 3×4 matrix complete with row and column requirements (Table 11.4).

Table 11.4

	W_1	W_2	W_3	W_4	Available
F_1					7
F_2					9
F_3					18
Requirements →	5	8	7	14	

Insert a set of allocations in the cells in such a way that the total in each row and each column is the same as shown against the respective rows and columns. Start with cell (1, 1) at the north-west corner (upper left-hand

corner) and allocate as much as possible there. In other words, $x_{11} = 5$, the maximum which can be allocated to this cell as the total requirement of this column is 5. This allocation ($x_{11} = 5$) leaves the surplus amount of 2 units for row 1 (Factory F_1), so allocate $x_{12} = 2$ to cell (1, 2). Now, allocations for first row and first column are complete, but there is a deficiency of 6 units in column 2. Therefore, allocate $x_{22} = 6$ in the cell (2, 2). Column 1 and column 2 requirements are satisfied, leaving a surplus amount of 3 units for row 2. So allocate $x_{23} = 3$ in the cell (2, 3), and column 3 still requires 4 units. Therefore, continuing in this way, from left to right and top to bottom, eventually complete all requirements by an allocation $x_{34} = 14$ in the *south-east* corner. Table 11.5 shows the resulting feasible solution.

Table 11.5

5 (19)	2 (30)				7
	6 (30)	3 (40)			9
		4 (70)	14 (20)		18
5	8	7	14		

On multiplying each individual allocation by its corresponding unit cost in '()' and adding, the total cost becomes $= 5(19) + 2(30) + 6(30) + 3(40) + 4(70) + 14(20) = \text{Rs. } 1015$.

Q. Explain the application of North-West Corner Rule with an example.

Second Method : *The Row Minima Method*.

Step 1. The transportation table of the given problem has 12 cells. Following the *row minima method*, since $\min(19, 30, 50, 10) = 10$, the first allocation is made in the cell (1, 4), the amount of the allocation is given by $x_{14} = \min(7, 14) = 7$. This exhausts the availability from factory F_1 and thus we cross-out the first row from the transportation table (Table 11.6).

Table 11.6

	W_1	W_2	W_3	W_4
F_1	(19)	(30)	(50)	7
F_2	(70)	(30)	(40)	(60)
F_3	(40)	(80)	(70)	(20)
	5	8	7	7

Table 11.7

	W_1	W_2	W_3	W_4
F_1				7
F_2	(70)	8	(40)	(60)
F_3	(40)		(70)	(20)
	5	8	7	7

Step 2. In the resulting transportation table (Table 11.7), since $\min(70, 30, 40, 60) = 30$, the second allocation is made in the cell (2, 2), the amount of allocation being $x_{22} = \min(9, 8) = 8$. This satisfies the requirement of warehouse W_2 and thus we cross-out the second column from the transportation table obtaining new Table 11.8.

Step 3. In Table 11.8, since $\min(70, 40, 60) = 40$, the third allocation is made in the cell (2, 3), the amount being $x_{23} = \min[1, 7] = 1$. This exhausts the availability from factory F_2 ,

Table 11.8

	W_1	W_2	W_3	W_4
F_1				7
F_2	(70)	8	1	
F_3	(40)		(70)	(20)
	5	x	6	7

Table 11.9

	W_1	W_2	W_3	W_4
F_1				7
F_2	(70)	8	1	(60)
F_3	(40)		(70)	(20)
	5	x	6	7

and thus we cross-out the second row from the table 11.8 getting the *Table 11.9*,

Step 4. The next allocation is made in the cell (3,4), since $\min(40, 70, 20) = 20$, the amount of allocation being $x_{34} = \min(7, 18) = 7$. This exhausts the requirement of warehouse W_4 and thus we cross-out the fourth column from the *Table 11.9*.

Step 5. The next allocation is made in the cell (3, 1), since $\min(40, 70) = 40$, the amount of allocation being $x_{31} = \min(5, 11) = 5$. This satisfies the requirement of warehouse W_1 and so we cross-out the first column W_1 to get new *Table 11.11*.

Table 11.10

	W_1	W_2	W_3	W_4
F_1				7
F_2		8	1	
F_3	(40)		(70)	7
	5	x	6	x

x II

Table 11.11

	W_1	W_2	W_3	W_4
F_1				7
F_2		8	1	
F_3	5		6	7
	x	x	x	x

x x x

Step 6. The last allocation of amount $x_{33} = 6$ is obviously made in the cell (3, 3). This exhausts the availability from factory F_3 and requirement of warehouse W_3 simultaneously. So we cross-out third row and third column to get the final solution *Table 11.12*.

Since the basic cells indicated by (*) do not form a loop, an initial basic feasible solution has been obtained. The solution is displayed in *Table 11.12*.

Table 11.12

	W_1	W_2	W_3	W_4
F_1				7 •
F_2		8	1	
F_3	5		6	7
	x	x	x	x

x x x

The transportation cost is given by

$$z = 7 \times 10 + 8 \times 30 + 1 \times 40 + 5 \times 40 + 6 \times 70 + 7 \times 20$$

$$= \text{Rs. } 1110.$$

Third Method : The Column Minima Method.

This method is similar to **row-minima method** except that we apply the concept of minimum cost on columns instead of rows. So, the reader can easily solve the above problem by column minima method also.

Fourth Method : Lowest Cost Entry Method (Matrix Minima Method).

The initial basic feasible solution obtained by this method usually gives a lower beginning cost. In this method, first write the cost and requirements matrix (*Table 11.13*).

Start with the lowest cost entry (8) in the cell (3, 2) and allocate as much as possible, i.e., $x_{32} = 8$. The next lowest cost (10) lies in the cell (1, 4), so allocate $x_{14} = 7$. The next lowest cost (19) lies in the cell (1, 1), so make no allocation, because the amount available from factory F_1 was already used in the cell (1, 4). Next lowest cost entry is (20) in the cell (3, 4) where at the most it is possible to allocate $x_{34} = 7$ in order to complete the requirements of 7 units in column 4.

Further, next lowest cost is (30) in cells (2, 2) and (1, 2) so no allocation is possible, because the requirement of column 2 has already been exhausted. This way, required feasible solution is obtained (*Table 11.13*).

This feasible solution results in lower transportation cost, i.e.,

$$2(70) + 3(40) + 8(8) + 8(40) + 7(10) + 7(20) = \text{Rs. } 814.$$

This cost is less by Rs. 201, i.e., Rs. (1015–814) as compared to the cost obtained by *north-west corner rule*.

Table 11-13

	• (19)	• (30)	• (50)	7(10)	Available 7
	2(70)	• (30)	7(40)	• (60)	9
	3(40)	8(8)	• (70)	7(20)	18
Requirements	5	8	7	14	

Q. Explain the application of Matrix-Minimum method with an example.

Fifth Method. Vogel's Approximation Method (Unit Cost Penalty Method).

[Banasthali (M.Sc.) 93]

Step 1. In lowest cost entry method, it is not possible to make an allocation to the cell (1, 1) which has the second lowest cost in the matrix. It is trivial that allocation should be made in at least one cell of each row and each column.

Table 11-14

	W ₁	W ₂	W ₃	W ₄	Available
F ₁	(19)	(30)	(50)	(10)	7
F ₂	(70)	(30)	(40)	(60)	9
F ₃	(40)	(8)	(70)	(20)	18
Requirement	5	8	7	14	

Step 2. Next enter the *difference between the lowest and second lowest cost* entries in each column beneath the corresponding column, and put the difference between the lowest and second lowest cost entries of each row to the right of that row. Such individual differences can be thought of a **penalty** for making allocations in second lowest cost entries instead of lowest cost entries in each row or column. For example, allocate one unit in the second lowest cost cell (3, 1) instead of cell (1, 1) with lowest unit cost (19). There will be a loss (penalty) of Rs 21 per unit. In case, the lowest and second lowest costs in a row/ column are equal, the penalty will be taken zero.

Table 11-15

	W ₁	W ₂	W ₃	W ₄	Available	Penalties
F ₁	*(19)	*(30)	*(50)	*(10)	7	(9)
F ₂	*(70)	*(30)	*(40)	*(60)	9	(10)
F ₃	*(40)	8 (8)	*(70)	*(20)	18/10	(12)
Requirements :	5	8/0	7	14		
Penalties :	(21)	(22) ↑	(10)	(10)		

Step 3. Select the row or column for which the **penalty** is the largest, i.e., (22) (Table 11-15), and allocate the maximum possible amount to the cell (3, 2) with the lowest cost (8) in the particular column (row) making $x_{32} = 8$. If there are more than one largest penalty rows (columns), select one of them arbitrarily.

Table 11-16

	W ₁	W ₃	W ₄	Available	Penalties
F ₁	5(19)	*(50)	*(10)	7	(9)
F ₂	*(70)	*(40)	*(60)	9	(20)
F ₃	*(40)	*(70)	*(20)	10 (Note)	(20)
Requirements	5/0	7	14		
Penalties	(21) ↑	(10)	(10)		

Step 4. Cross-out that column (row) in which the requirement has been satisfied. In this example, second column has been crossed-out. Then find the corresponding penalties correcting the amount available from factory F₃. Construct the first reduced penalty matrix Table 11-16.

Step 5. Repeat steps 3 and 4 till all allocations have been made. Successive reduced penalty matrices are obtained. Since the largest penalty (21) is now associated with the cell (1, 1), so allocate $x_{11} = 5$. This allocation ($x_{11} = 5$) eliminates the column 1 giving the second reduced matrix (Table 11-17).

Table 11-17

	W_3	W_4	Available	Penalty
F_1	*(50)	*(10)	2 (Note)	(40)
F_2	*(40)	*(60)	9	(20)
F_3	*(70)	10(20)	10/0	(50) ←
Requirements :	7	14/4		
Penalties :	(10)	(10)		

Table 11-18

	W_3	W_4	Available	Penalties
F_1	*(50)	2(10)	2/0	(40)
F_2	7(40)	2(60)	9/0	(20)
Requirements :	7	4/0 (Note)		
Penalties :	(10)	(50) ↑		

Now, allocate according to the largest penalty (50) as $x_{14} = 2$ and remaining $x_{24} = 2$. Then allocate $x_{23} = 7$.

Step 6. Finally, construct Table 11-19 for the required feasible solution.

Table 11-19

The total cost is :

$$5(19) + 8(8) + 2(10) + 2(60) + 10(20) + 7(40) = \text{Rs. } 779.$$

This cost is Rs. 35 less as compared to the cost obtained by *Lowest Cost Entry Method*.

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
Requirements :	5	8	7	14	

In order to reduce large number of steps required to obtain the optimal solution, it is advisable to proceed with the initial feasible solution which is close to the optimal solution. Vogel's method often gives the better initial feasible solution to start with. Although Vogel's method takes more time as compared to other two methods, but it reduces the time in reaching the optimal solution.

Short-cut. After a little practice, students may prefer to perform the entire procedure of Vogel's method within the original cost requirement Table 11-14. It needs merely to cross-out rows and columns as and when they are completed and to revise requirements, available supplies and penalties as shown below.

	W_1	W_2	W_3	W_4	Available	Penalties
F_1	5 • (19)	(30)	(50)	(10)	7/2/0	(9/9/40/40)
F_2			7 • (40)	(60)	9/0	(10/20/20/20)
F_3		8 • (40)	(70)	(20)	18/10/0	(12/20/50)
Required	5/0	8/0	7/0	14/4/0		
Penalties	(21/21)	(22) ↑	(10/10/10/10)	(10/10/10/50)		

Example 2. Obtain an initial basic feasible solution to the following transportation problem :

	Stores				
	I	II	III	IV	Availability
A	7	3	5	5	34
B	5	5	7	6	15
C	8	6	6	5	12
D	6	1	6	4	19
Demand	21	25	17	17	80

[Kanpur (B.Sc.) 2003; M.G. Univ., (M. Com.) 98]

[Ans. By Vogel's (Penalty) Method, the initial solution is :

$x_{12} = 6, x_{13} = 7, x_{14} = 5, x_{21} = 15, x_{34} = 12, x_{42} = 19$; Total transportation cost = Rs. 324.]

- Q. Explain the use of Vogel's Approximation Method (VAM) with an example.

11-8-2. Summary of Methods for Initial BFS

The methods for obtaining an initial basic feasible solution to a transportation problem can be summarized as follows :

I. North-West Corner Rule (Stepping Stone Method)

Step 1. The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is, $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1, 1) of the transportation table.

- Step 2.** (i) If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).
(ii) If $b_1 < a_1$, move horizontally right-side to the second column and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).
(iii) If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude $x_{12} = \min(a_1 - a_1, b_2) = 0$ in the cell (1, 2) or $x_{21} = \min(a_2, b_1 - b_1) = 0$ in the cell (2, 1).
Step 3. Start from the new north-west corner of the transportation table and repeat steps 1 and 2 until all the requirements are satisfied.

II. The Row Minima Method

Step 1. The smallest cost in the first row of the transportation table is determined. Let it be c_{ij} . Allocate as much as possible amount $x_{1j} = \min(a_1, b_j)$ in the cell $(1, j)$, so that either the capacity of origin O_1 is exhausted, or the requirement at destination D_j is satisfied or both.

- Step 2.** (i) If $x_{1j} = a_1$ so that the availability at origin O_1 is completely exhausted, cross-out* the first row of the table and move down to the second row.
(ii) If $x_{1j} = b_j$ so that the requirement at destination D_j is satisfied, cross-out the j th column and re-consider the first row with the remaining availability of origin O_1 .
(iii) If $x_{1j} = a_1 = b_j$, the origin capacity a_1 is completely exhausted as well as the requirement at destination D_j is completely satisfied. An arbitrary tie-breaking choice is made. Cross-out the j th column and make the second allocation $x_{1k} = 0$ in the cell $(1, k)$ with c_{1k} being the new minimum cost in the first row. Cross-out the first row and move down to the second row.

Step 3. Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied.

III. The Column Minima Method

Step 1. Determine the smallest cost in the first column of the transportation table. Let it be c_{i1} . Allocate $x_{i1} = \min(a_i, b_1)$ in the cell $(i, 1)$.

* By saying "crossout a row or a column" we shall mean that no cell from that row or column can be chosen for the basis entry at a later step.

- Step 2.** (i) If $x_{ij} = b_j$, cross-out the first column of the transportation table and move towards right to the second column.
(ii) If $x_{ij} = a_i$, cross-out the i th row of the transportation table and reconsider the first column with the remaining demand.
(iii) If $x_{ij} = b_j = a_i$, cross-out the i th row and make the second allocation $x_{k1} = 0$ in the cell $(k, 1)$ with c_{k1} being the new minimum cost in the first column. Cross-out the column and move towards right to the second column.
- Step 3.** Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied.

IV. Lowest Cost Entry Method (LCEM) or Matrix Minima Method

- Step 1.** Determine the smallest cost in the cost matrix of the transportation table. Let it be (c_{ij}) . Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) .
- Step 2.** (i) If $x_{ij} = a_i$, cross-out the i th row of the transportation table and decrease b_j by a_i . Go to step 3.
(ii) If $x_{ij} = b_j$, cross out the j th column of the transportation table and decrease a_i by b_j . Go to step 3.
(iii) If $x_{ij} = a_i = b_j$, cross-out either the i th row or j th column but not both.
- Step 3.** Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

V. Vogel's Approximation Method (VAM)

[JNTU (MCA III) 2004]

- Step 1.** For each row of the transportation table identify the *smallest* and *next-to-smallest* cost. Determine the difference between them for each row. These are called '*penalties*'. Put them along side the transportation table by enclosing them in the parentheses against the respective rows. Similarly, compute these penalties for each column.
- Step 2.** Identify the row or column with the largest penalty among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the largest penalty correspond to i th row and let c_{ij} be the smallest cost in the i th row. Allocate the largest possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross-out the i th row or the j th column in the usual manner.
- Step 3.** Again compute the column and row penalties for the reduced transportation table and then go to step 2. Repeat the procedure until all the requirements are satisfied.

Note. In 1989, a new method for initial solution of transportation problem was developed by the Author of this book. The brief outlines of this new method are given in the Appendix. This method provides the initial solution very near to optimal solution. In most of the cases, the initial solution obtained by this new method proves to be optimum.

- Q. 1. Explain with an example the North-West corner rule, the least cost method, and the Vogel's Approximation method for obtaining an initial basic feasible solution of a transportation problem. [C.A. (Nov.) 91]
2. Explain Vogel's Approximation Method of solving a transportation problem.
3. Explain the lowest cost entry method for obtaining an initial basic solution of a transportation problem. [Madurai B.Sc. (Comp. Sc.) 92]
4. List the various methods that can be used for obtaining an initial basic feasible solution for a transportation problem and describe any one of them. [Garhwal M.Sc. (Math.) 95; Delhi B.Sc (Math.) 93]
5. Discuss the algorithm of stepping stone method. [VTU 2002]
6. Write the steps to find the initial basic feasible solution by North-West corner rule. [Bhubneshwar (IT) 2004]

EXAMINATION PROBLEMS

1. Determine an initial basic feasible solution to the following transportation problem using the north-west corner rule, where O_i and D_j represent i th origin and j th destination respectively.

(i)

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	35

(ii)

	I	II	III	IV	Supply
From	A	13	11	15	20
B	17	14	12	13	6,000
C	18	18	15	12	7,000
Demand	3,000	3,000	4,000	5,000	

[IAS (Main) 89]

[Bharathidasan B.Sc (Math.) 90]

[Ans. (i) $x_{11} = 6, x_{12} = 8, x_{22} = 2, x_{23} = 14, x_{33} = 1, x_{34} = 4$, cost = Rs. 128]

(ii) $x_{11} = 2, x_{21} = 1, x_{22} = 3, x_{23} = 2, x_{33} = 2, x_{34} = 5$]

(iii)

		To					Available	
		1	2	3	4	5		
From	1	9	12	9	8	4	3	5
	2	7	3	6	8	9	4	8
	3	4	5	6	8	10	14	6
	4	7	3	5	7	10	9	7
	5	2	3	8	10	2	4	3
Require	3	4	5	7	6	4		

[Ans. $x_{11} = 3, x_{12} = 2, x_{22} = 2, x_{23} = 5, x_{34} = 1, x_{44} = 0, x_{45} = 6, x_{46} = 1, x_{56} = 3$.]

2. Determine an initial basic feasible solution to the following T.P. using the row/column minima method.

(i)

From	To			Availability
	A	B	C	
I	50	30	220	1
II	90	45	170	4
III	250	200	50	4
Require	4	2	3	

(ii)

From	To				Available
	A	B	C	D	
I	6	3	5	4	22
From II	5	9	2	7	15
III	5	7	8	6	8
Demand	7	12	17	9	

[Bharathidasan B.Sc (Math.) 90]

[Ans. (i) $x_{12} = 1, x_{21} = 3, x_{22} = 1, x_{31} = 1, x_{33} = 3$

(ii) $x_{12} = 12, x_{13} = 1, x_{14} = 9, x_{23} = 15, x_{31} = 7, x_{33} = 1$]

3. Obtain an initial basic feasible solution to the following T.P. using he matrix minima method.

	D ₁	D ₂	D ₃	D ₄	Capacity
O _i	1	2	3	4	
O ₂	4	3	2	0	
O ₃	0	2	2	1	
Demand	4	6	8	6	

where O_i and D_j denote ith origin and jth destination respectively.

[Ans. $x_{12} = 6, x_{23} = 2, x_{24} = 6, x_{31} = 4, x_{32} = 0, x_{33} = 6$]

4. Find the initial basic feasible solution of the transportation problem where cost-matrix is given below :

Destination

Origin	Destination				Supply
	A	B	C	D	
I	1	5	3	3	34
II	3	3	1	2	15
III	0	2	2	3	12
IV	2	7	2	4	19
Demand	21	25	17	17	

[Ans. $x_{11} = 9, x_{12} = 8, x_{14} = 17, x_{23} = 15, x_{31} = 12, x_{42} = 17, x_{43} = 2$;
cost = Rs. 238, using 'lowest cost entry method'].

5. Determine an initial basic feasible solution using (i) Vogel's method and (ii) Row minima method, by considering the following transportation problem :

Source	Destination				Supply
	1	2	3	4	
1	21	16	15	13	11
2	17	18	14	23	13
3	32	27	18	41	19
Demand	6	10	12	15	43

[JNTU (MCA III) 2004; VTU (BE Mech.) 2002; Gauhati (MCA) 92]
[Ans. $x_{14} = 11, x_{21} = 6, x_{22} = 3, x_{24} = 4, x_{32} = 7, x_{33} = 12$, cost = Rs. 686]

6. Determine an initial basic feasible solution to the following T.P. using : (a) North-west corner rule, and (b) Vogel's method.

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		Destination					Supply
Origin	Demand	A ₁	B ₁	C ₁	D ₁	E ₁	
		2	11	10	3	7	4
		1	4	7	2	1	8
A	C	3	9	4	8	12	9
		3	3	4	5	6	21

[JNTU 2000]

[Ans. **North west corner Rule**: $x_{11} = 3, x_{12} = 1, x_{22} = 2, x_{23} = 4, x_{24} = 2, x_{34} = 3, x_{35} = 6$, cost = Rs. 153.]

Vogel's Method: $x_{14} = 4, x_{22} = 2, x_{35} = 6, x_{34} = 3, x_{32} = 1, x_{33} = 4, x_{34} = 1$, cost = Rs. 68].

7. Use north west corner rule to determine an initial basic feasible solution to the following T.P. when does it have a unique solution ?

		To				Supply
From	Demand	A	B	C		
		a	2	7	4	5
		b	3	3	1	8
		c	5	4	7	7
		d	1	6	2	14
		7	9	18		34

[Meerut B.Sc. (Math.) 90]

Does the use of matrix minima method give a better (improved) basic feasible solution ? Why ?

[Ans. $x_{11} = 5, x_{21} = 2, x_{22} = 6, x_{32} = 3, x_{33} = 4, x_{43} = 14$, cost = Rs. 102; Yes]

8. Determine an initial basic feasible solution to the following T.P. using : (a) matrix minima method, (b) Vogel's approx. method.

		Destination				Supply
O ₁	O ₂	D ₁	D ₂	D ₃	D ₄	
		1	2	1	4	30
		3	3	2	1	50
O ₃	O ₃	4	2	5	9	20
Demand	Demand	20	40	30	10	100

[Meerut 2002; IAS (Main) 88]

[Ans. For (a) and (b) bold : $x_{11} = 20, x_{13} = 10, x_{22} = 20, x_{23} = 20, x_{24} = 20, x_{34} = 10, x_{32} = 20$, cost = Rs. 180]

9. (i) Explain vogel's method by obtaining initial BFS of the following transportation problem :

		D ₁	D ₂	D ₃	Supply
O ₁	O ₂	13	15	16	17
		7	11	2	12
		19	20	9	16
Demand	Demand	14	8	23	

[Ans. $x_{11} = 9, x_{12} = 3, x_{21} = 5, x_{23} = 7, x_{33} = 16$.]

		To				Supply
From	Demand	I	II	III		
		A	50	30	200	1
		B	90	45	170	3
C	250		200	50	4	
Demand	4		2	2		

[Bharthidasan B.Sc (Math.) 90]

[Ans. $x_{11} = 1, x_{21} = 3, x_{31} = 0, x_{32} = 2, x_{33} = 2$]

10. Determine an initial feasible solution to the following T.P. using (a) (North-West corner rule, and (b) Vogel's approximation method :

Origin	Destination				Supply
	D ₁	D ₂	D ₃	D ₄	
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	950

[Ans. $x_{11} = 200, x_{12} = 50, x_{22} = 175, x_{24} = 125, x_{33} = 275, x_{34} = 125$.]

[Bhubneshwar (IT) 2004]

11. Determine the initial bfs to the following TP by matrix method.

		To			
		A	B	C	a_i
From	I	14	15	10	20
	II	21	13	19	24
	III	17	26	9	12
	b_j	28	22	6	56

[AIMS Bangalore (MBA) 2002]

Optimum Solution

11.9. MOVING TOWARDS OPTIMALITY

After obtaining an initial basic feasible solution to a given transportation problem, the next question is 'how to arrive at the optimum solution'. The basic steps for reaching the optimum solution are the same as given for simplex method, namely :

- Step 1.** Examination of initial basic feasible solution for *non-degeneracy*. If it is *degenerate*, some modification is required to make it non-degenerate (as discussed in Sec. 11.11).
- Step 2.** (i) Determination of net-evaluations (cost-difference) for empty cells.
(ii) Optimality test of current solution.
- Step 3.** Selection of the entering variable, provided Step 2(ii) indicates that the current solution can be improved.
- Step 4.** Selection of the leaving variable.
- Step 5.** Finally, repeating the *steps 1 through 4* until an optimum solution is obtained.

11.9-1. To Examine the Initial Basic Feasible Solution for Non-degeneracy.

A basic feasible solution of an $m \times n$ transportation problem is said to be *non-degenerate*, if it has the following two properties :

- (1) *Initial BFS must contain exactly $m + n - 1$ number of individual allocations.*

For example, in 3×4 transportation problem, the number of individual allocations in BFS obtained by any one of the methods discussed so far is equal to 6, i.e., $3 + 4 - 1$, which can be easily verified from **Tables 11-5, 11-12, 11-13 and 11-19**.

- (2) *These allocations must be in 'independent positions.'*

Independent positions of a set of allocations mean that it is always impossible to form any closed loop through these allocations. **Tables 11-20, 11-21** show the non-independent, and **11-23** independent positions by '•'.

Table 11-20
Non-Independent Positions

•	•		
		↗	↘
↖	↙		

closed loop

Table 11-21
Non-Independent Positions

↑			↗	
↖	
			↓	
		↖	•

Table 11.22
Non-Independent Positions

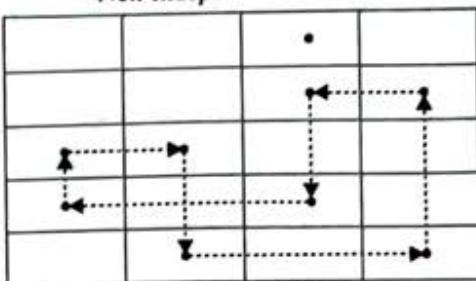


Table 11.23
Independent Positions.

*				*
			*	*
		*		*

In above allocation patterns of different problems, the dotted lines constitute what are known as *loops*. A loop may or may not involve all allocations. It consists of (at least 4) horizontal and vertical lines with an allocation at each corner which, in turn, is a join of a horizontal and vertical lines. At this stage, loop of *Table 11.22* should be particularly noted. Here two lines intersect each other at cell (4, 2) and do not simply join; therefore this is not to be regarded as a corner. Such allocations in which a loop can be formed are known as non-independent positions whereas those (of *Table 11.23*) in which a loop cannot be formed are regarded as independent.

11.9.2 . Determination of Net-Evaluations (u, v method).

Unlike the simplex method, the net-evaluations for a transportation problem can be determined more easily by using the properties of the *primal* and *dual* problems.

Let us consider the following m -origin, n -destination transportation problem :

Determine x_{ij} so as to minimize $z = \sum_{i=1}^m \sum_{j=1}^n x_{ij} (c_{ij})$, subject to the constraints :

$$\sum_{j=1}^n x_{ij} = a_i \text{ or } a_i - \sum_{j=1}^n x_{ij} = 0, \quad \text{for } i = 1, 2, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j \text{ or } b_j - \sum_{i=1}^m x_{ij} = 0, \quad \text{for } j = 1, 2, \dots, n$$

and $x_{ij} \geq 0$, for all i and j .

Let u_1, u_2, \dots, u_m and v_1, v_2, \dots, v_n be the dual variables associated with the above *origin* and *destination* constraints, respectively. Since the above *primal* T.P. has $m+n$ constraint equations in mn number of variables, so the *dual* of above problem will contain mn constraints in $m+n$ dual variables in the form :

$u_i + v_j \leq c_{ij}$ and u_i, v_j are unrestricted for all i and j (\because constraints in the primal are equations)

From *chapter 7*, we recall that for any standard primal L.P.P. with basis B and associated cost vector C_B , the associated solution to its dual is given by $W_B = C_B B^{-1}$. Thus, if a_j is the j th column of the primal constraint matrix, then an expression for evaluating the net-evaluation for minimization problem is given by

$$c_j - z_j = c_j - C_B (B^{-1} a_j) = c_j - W_B a_j \quad \text{for all } j.$$

But, in the present case of *rectangular* transportation problem (which is the special case of L.P.P.), the dual solution can be represented by

$$(u, v) = (u_1, \dots, u_m, v_1, \dots, v_n)$$

and therefore the net evaluations are analogously obtained by simply replacing $c_j \rightarrow c_{ij}$, $W_B \rightarrow (u, v)$, $a_j \rightarrow a_{ij}$ in the above formula to get

$$\begin{aligned} c_{ij} - z_{ij} &= c_{ij} - (u, v) a_{ij} = c_{ij} - (u_1, \dots, u_m, v_1, \dots, v_n) [e_i + e_{m+j}] \\ &= c_{ij} - (u_i + v_j); i = 1, \dots, m; j = 1, \dots, n, \end{aligned}$$

where a_{ij} is the column vector of the constraint matrix associated with the *rectangular* variable x_{ij} . For simplicity, we shall denote the net evaluation $c_{ij} - z_{ij}$ by d_{ij} in all further discussions.

Now, since the net evaluations must vanish for the basic variables it follows that $d_{ij} = c_{ij} - (u_i + v_j)$ for all non-basic cells (i, j) where u_i and v_j satisfy the relation $c_{rs} = u_r + v_s$ for all basic cells (r, s) . Except for the degeneracy case, there are $m+n-1$ dual equations in $m+n$ dual unknowns for the $m+n-1$ basic cells. We

can arbitrarily assign the value to one of these unknowns u_i and v_j , and solve uniquely for the remaining $m + n - 1$ variables. After this arbitrary assignment, say $u_1 = 0$, the rest of the values are obtained by simple addition and subtraction. Once we determine all the u_i and v_j , the net evaluations for all the non-basic cells are easily determined by the relation $d_{ij} = c_{rs} - (u_i + v_j)$.

Alternative Method to Determine Net-Evaluations.

The necessary condition for optimality can also be established in the form of the following theorem.

Theorem 11.7. If we have a feasible solution consisting of $m + n - 1$ independent allocations, and if numbers u_i and v_j satisfying $c_{rs} = u_i + v_j$, for each occupied cell (r, s) , then the evaluation d_{ij} corresponding to each empty cell (i, j) , is given by $d_{ij} = c_{ij} - (u_i + v_j)$.

Proof. The transportation problem is to find $x_{ij} \geq 0$ in order to minimize

$$z = \sum_{i=1}^m \sum_{j=1}^n x_{ij} (c_{ij}) \quad \dots(11.6)$$

subject to the restrictions

$$\sum_{j=1}^n x_{ij} = a_i, i = 1, 2, 3, \dots, m \quad \dots(11.7)$$

$$\sum_{i=1}^m x_{ij} = b_j, j = 1, 2, 3, \dots, n \quad \dots(11.8)$$

and

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j.$$

The restrictions (11.7) and (11.8) may be written as

$$0 = a_i - \sum_{j=1}^n x_{ij}, i = 1, 2, 3, \dots, m \quad \dots(11.9)$$

$$0 = b_j - \sum_{i=1}^m x_{ij}, j = 1, 2, 3, \dots, n. \quad \dots(11.10)$$

Any multiple of each of these restrictions [(11.9) and (11.10)] can be legally added to the objective function (11.6) to try to eliminate the basic variable. These multiples are denoted by u_i ($i = 1, 2, \dots, m$) and v_j ($j = 1, 2, \dots, n$), respectively. Thus,

$$z = \sum_{i=1}^m \sum_{j=1}^n x_{ij} (c_{ij}) + \sum_{i=1}^m u_i (a_i - \sum_{j=1}^n x_{ij}) + \sum_{j=1}^n v_j (b_j - \sum_{i=1}^m x_{ij}) \quad \dots(11.11a)$$

$$\text{or } z = \sum_{i=1}^m \sum_{j=1}^n [c_{ij} - (u_i + v_j)] x_{ij} + \sum_{i=1}^m u_i a_i + \sum_{j=1}^n v_j b_j. \quad \dots(11.11b)$$

The necessary condition for a coefficient of zero is

$$c_{rs} = u_r + v_s \quad \dots(11.12)$$

for each basic variable x_{rs} , i.e., for each occupied cell (r, s) . Since there are $m + n - 1$ number of equations of the form (11.12) in $(m + n)$ number of unknowns (u_i and v_j), so if assignment is made to an arbitrary value of one of the u_i or v_j , then rest of the $(m + n - 1)$ unknowns can be easily solved algebraically. One reasonable and convenient rule, which will be adopted here, is to select the u_i which has the largest number of allocations in its row, and assign it the value of zero. And $c_{ij} = u_i + v_j$ immediately yields v_j for columns containing those allocations.

To prove the required result, first suppose that the empty cell (i, j) be connected to occupied cells by a closed loop (Table 11.24).

First, allocate +1 unit to the empty cell (i, j) , and in order to balance the total requirement of warehouse W_j , add -1 unit to occupied cell (r, j) . Consequently, the total amount available from factory F_r will be balanced by adding +1 unit to occupied cell (r, s) , which in turn causes column W_s to become unbalanced. So balance the column W_s by adding -1 unit to the occupied cell (i, s) .

Table 11.24

	W_1	W_2	W_3	W_4	W_5	W_n	Available
F_1							a_1
F_2							a_2
F_i			(c_{ij})		(c_{is})		a_i
F_r			$\downarrow + 1$		$\downarrow - 1$		a_r
F_m			$\downarrow - 1$		$\downarrow + 1$		a_m
Required.	b_1	b_2	\dots	b_j	\dots	b_s	\dots

This process will give the *cost difference* d_{ij} [called the *empty cell evaluation* for (i, j)] between the new solution and the original solution.

$$\text{Thus, } d_{ij} = c_{ij} - c_{rj} + c_{rs} - c_{is}. \quad \dots(11.13)$$

Using the result (11.12) for all occupied cells such as $(r, j), (r, s)$ and (i, s) ,

$$d_{ij} = c_{ij} - (u_r + v_j) + (u_r + v_s) - (u_i + v_s) = c_{ij} - (u_i + v_j). \quad \dots(11.14)$$

This proves the result for a loop of square shape connecting the empty cell (i, j) to occupied cells. In a similar fashion, generalize the empty cell (i, j) to occupied cells.

This completes the proof of the theorem.

11.9-3. The Optimality Test

If the cost difference $d_{ij} \geq 0$ (which implies increase in cost for each empty cell), then the BFS under test must be optimal. Otherwise, if $d_{ij} < 0$ (because negative difference implies decrease in cost) for one or more empty cells, then it would be better to reduce the cost more by allocating as much as possible to the cell with the largest negative (smallest) value of d_{ij} . This way, it is possible to improve the BFS successively for reduced cost till the optimal solution is obtained for which $d_{ij} \geq 0$ for each empty cell.

The optimality test for given BFS of the transportation problem may be summarized as follows :

1. Start with a basic feasible solution consisting of $m + n - 1$ allocations in independent positions.
2. Determine a set of $m + n$ numbers u_i ($i = 1, 2, 3, \dots, m$) and v_j ($j = 1, 2, 3, \dots, n$) such that for each occupied cell (r, s) $c_{rs} = u_r + v_s$.
3. Calculate cell evaluations (unit cost differences) d_{ij} for each empty cell (i, j) by using the formula

$$d_{ij} = c_{ij} - (u_i + v_j).$$
4. Finally, examine the matrix of cell evaluations d_{ij} for negative entries and conclude that –
 - (i) solution under test is optimal, if none is negative ;
 - (ii) alternative optimal solutions exist, if none is negative but any is zero ;
 - (iii) solution under test is not optimal, if any is negative, then further improvement is required by repeating the above process.

We now proceed to answer the question : how to improve the current BFS if it is not optimal, i.e. if all $d_{ij} \leq 0$.

11.9-4. Selection of Entering Variable

Here our aim is to minimize the cost of transportation. So the current basic feasible solution will not be optimum so long as any of the net evaluation d_{ij} is negative. Thus if all d_{ij} are non-negative, the current solution is an optimum one, otherwise using simplex like criterion we select such variable x_{rs} to enter the basis for which the net evaluation $d_{rs} = \min_{i,j} \{d_{ij} < 0\}$.

11.9-5. Selection of Leaving Variable

Our next step will be to determine the leaving basic variable and then to determine the new improved basic solution. The simplex like leaving criterion in the notations of transportation problem states that if the variable x_{rs} is selected to enter the basis, then the basic variable x_{Bi} corresponding to the *minimum ratio* :

$\min \left\{ \frac{x_{Bi}}{y_{irs}}, y_{irs} > 0 \right\}$, will leave the basis. However, due to special structure of the transportation problem the above criterion has been simplified to a great extent.

Theorem 11.8. Let $\{b_1, b_2, \dots, b_{m+n-1}\}$ be a basis set for the column vectors of the coefficient matrix A of m-origin, n-destination transportation problem. In the representation of any non-basic a_{rs} as a linear combination:

$$a_{rs} = \sum_{i=1}^{m+n-1} y_{irs} b_i,$$

of basis vectors, every scalar element (y_{irs}) is either -1 or +1.

Proof. Since $\{a_{rs}, b_1, \dots, b_{m+n-1}\}$ is a linearly dependent set (by Theorem 11.6), the set of the associated cells contains a loop. So the cell (r, s) must be in a loop. The vectors b_i 's are, of course, some column vectors a_{ij} 's of A.

Suppose the set of associated cells contains the following loop:

$$L = \{(r, s), (r, t), (p, t), (p, q), \dots, (u, v), (u, s)\},$$

where $a_{rs}, a_{rt}, \dots, a_{us}$ are the given basis vectors ($\leq m+n-1$).

Now, since $a_{ij} = e_i + e_{m+j}$ for all i and j , and because the number of cells in a loop is always even, we have

$$a_{rs} - a_{rt} + a_{pt} - a_{pq} + \dots + a_{uv} - a_{us} = 0$$

which yields, $a_{rs} = a_{rt} - a_{pt} + a_{pq} - \dots - a_{uv} + a_{us}$.

This is the unique representations of a_{rs} as a linear combination of basis vectors; and hence the y_{irs} elements associated with the basis vector in the above representation are +1, -1, ..., -1 and +1.

This completes the proof of the theorem.

Thus if x_{rs} is the entering variable, the basic variable x_{Bi} will leave the basis if $x_{Bi} = \min \{x_{Bi}\}$, since positive y_{irs} is +1 for all basic variables.

11.9-6. Determination of New (Improved) Basic Feasible Solution

After the entering and leaving variables are determined, all that remains to determine is '**the new (improved) basic solution**'. The usual transformation formulae for obtaining the new basic solution, in the simplex transportation notation, are given by

$$\hat{x}_{Bi} = \frac{x_{Bi}}{y_{irs}}, \quad y_{irs} > 0 \quad \text{and} \quad \hat{x}_{Bi} = x_{Bi} - \frac{x_{Bi}}{y_{irs}} y_{irs} \quad \text{for all } i \neq t$$

Since $y_{irs} = +1$, $y_{irs} = \pm 1$ for basic variables, and $y_{irs} = 0$ for non-basic variables, the above transformations get simplified to

$$\hat{x}_{Bi} = x_{Bi},$$

$$\hat{x}_{Bi} = x_{Bi} - y_{irs} x_{Bi} = x_{Bi} \pm x_{Bi} \quad \text{for all } x_{Bi} \in X, i \neq t$$

and $\hat{x}_{Bi} = x_{Bi}$, for all $x_{Bi} \notin X$,

where X is the set of those basic variables whose corresponding cells are included in the loop as identified in the preceding theorem.

In practice, however, this improvement procedure is quite simple. The working-rule is outlined in the following steps :

11.9-7. Working Rule to Obtain Leaving Variable and Improved Basic Feasible Solution

Step 1. After identifying the entering variable x_{rs} , describe a loop which starts and ends at the non-basic cell (r, s) connecting only the basic cells. Such a closed path exists and is unique for any non-degenerate basic solution.

Step 2. The amount (say θ) to be allocated to the entering variable is interchangably subtracted from and added to the successive end points of the closed loop so that the supply and demand constraints always remain satisfied.

Step 3. Then the minimum value of θ , which will render non-negative values for all the basic variables in the new solution, is obtained. This consequently, determines the leaving variable.

For illustration, see **Example 2** in Sec 11.10-2 .

- Q. 1. Describe any two methods of constructing a basic feasible solution for the transportation problem. How would you test for optimality of a given solution for the transportation problem ? [Meerut (Stat.) 92]
2. Show that the transportation and assignment problem can be regarded as the particular cases of L.P.P. [Meerut (Stat.) 98]

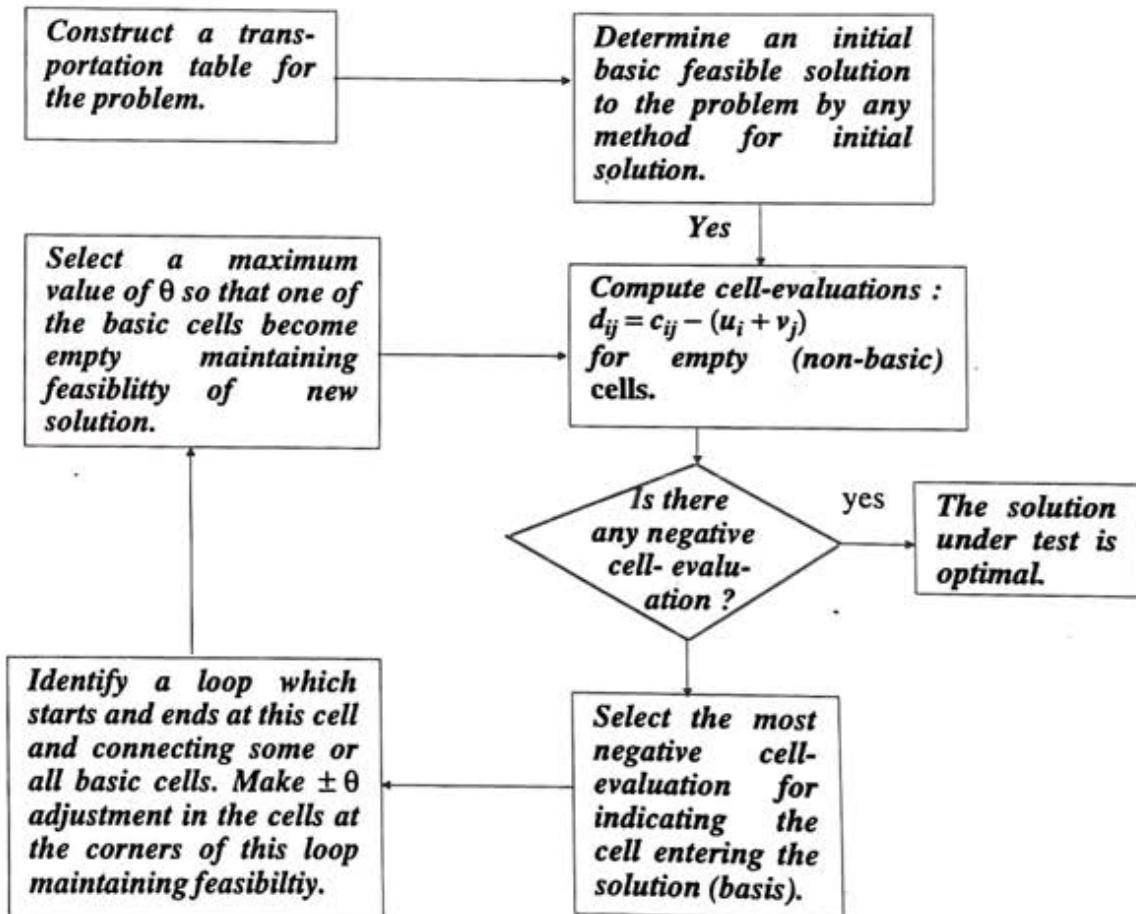
11.10. TRANSPORTATION ALGORITHM FOR MINIMIZATION PROBLEM

The transportation algorithm for minimization problem can be summarized in the following steps.

The Algorithm :

- Step 1.** First, construct a transportation table entering the origin capacities a_i , the destination requirements b_j and the costs c_{ij} , as shown in *Table 11.14* (page 272)
- Step 2.** Find an initial basic feasible solution by Vogel's method or by any of the given methods. Enter the solution at the centre (•) of basic cells.
- Step 3.** For all the basic variables x_{ij} , solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$ and entering successively the values of u_i and v_j on the transportation table as shown in *Table 11.26*. (page 285)
- Step 4.** Compute the cost differences $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic cells and enter them in the upper right corners of the corresponding cells.

Flowchart of Transportation Algorithm



Step 5. Apply optimality test by examining the sign of each d_{ij} :

(i) If all $d_{ij} \geq 0$, the current basic feasible solution is an optimum one.

(ii) If at least one $d_{ij} < 0$ (negative), select the variable x_{rs} (having the most negative d_{rs}) to enter the basis.

Step 6. Let the variable x_{rs} enter the basis. Allocate an unknown quantity say θ , to the cell (r, s) . Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7. Assign the largest possible value to θ in such a way that the value of at least one basic variable becomes zero and other basic variables remain non-negative (≥ 0). The basic cell whose allocation has been made zero will leave the basis.

Step 8. Now, return to **step 3** and then repeat the process until an optimum basic feasible solution is obtained.

The above iterative procedure determines an optimum solution in a finite number of steps. This method is called **MODIMETHOD** and can be easily remembered with the help of the following FLOW-CHART.

Q. 1. Give an algorithm for solving transportation problem.

2. State the transportation problem. Describe clearly the steps involved in solving the problem.

3. Describe the transportation problem. Give a method of finding an initial feasible solution. Explain what is meant by an optimality test? Give the method of improving over the initial solution to reach the optimal feasible solution. [Meerut 94]

4. Assume that in a transportation problem the demand and supply levels are all positive and integral. Show that there exists an integral optimal solution if the total demand equals total supply levels.

5. Describe the computational procedure of optimality test in a transportation problem.

6. Explain briefly the step-wise description of the computational procedure for solving the transportation problem. [Delhi B.Sc. (Math.) 91]

7. Develop mathematical model of a balanced transportation problem. Prove that it always has a feasible solution. [IAS (Main) 99]

8. How do you diagnose that the given transportation problem is having more than one optimal alternate optimal solution. [AIMS (Ind.) 2002]

11.10-1. Computational Demonstration of Optimality Test

Example 3. (a) Obtain an initial basic feasible solution to the transportation problem of **Example 1**. Is this solution an optimal solution? If not, obtain the optimal solution.

[IGNOU 2001; JNTU (Mach.) 99; Gauhati (MCA) 91]

(b) If a company is spending Rs. 1000 on transportation of its units to four warehouses from three factories. What can be the maximum saving by optimal scheduling.

Solution. (a) Computational demonstration for optimality is performed by taking the initial basic feasible solution of **Example 1** with $m+n-1$ allocations in independent positions with transportation cost of Rs 779 obtained (by *Vogel's Method*). This initial basic feasible solution is given in Table 11.25.

Table 11.25

	W ₁	W ₂	W ₃	W ₄	Available
F ₁	5(19)			2(10)	7
F ₂			7(40)	2(60)	9
F ₃		8(8)		10(20)	18
Required	5	8	7	14	

Step 1. The initial BFS has $m+n-1$ allocations, that is, $3+4-1=6$ allocations in independent positions. Therefore, condition (1) of optimality test [in sec. 11.9-3] is satisfied.

Step 2. Since u_i ($i = 1, 2, 3$) and v_j ($j = 1, 2, 3, 4$) are to be determined by means of unit cost in the respective occupied cells only, assign a u -value of any particular amount (conveniently zero) to any particular row (convenient rule is to select the u_i which has the largest number of allocations in its row). Since all rows contain the same number of allocations, take any of the u_i (say u_3) equal to zero.

When $u_3 = 0, v_4 = 20$ (since $c_{34} = u_3 + v_4; c_{34} = 20$). Similarly, $c_{32} = u_3 + v_2$ or $8 = 0 + v_2$ or $v_2 = 8$. Again, $c_{14} = u_1 + v_4$ or $10 = u_1 + 20$ (since $c_{14} = 10$), then $u_1 = -10$. In the same way $60 = 20 + u_2$, which gives $u_2 = 40$; $19 = u_1 + v_1$ or $19 = -10 + v_1$, which gives $v_1 = 29$; $40 = u_2 + v_3$ or $40 = 40 + v_3$, which gives $v_3 = 0$. This completes the set of u_i ($i = 1, 2, 3$) and v_j ($j = 1, 2, 3, 4$) as shown in Table 11.26.

Step 3. To compute the matrix of cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells, it is convenient to write a matrix $[c_{ij}]$ for empty cells and the matrix of numbers $[u_i + v_j]$ for empty cells only, then subtract the later matrix from the former one.

Table 11.27 (from Table 11.3)
Matrix $[c_{ij}]$ for empty cells

•	(30)	(50)	•
(70)	(30)	•	•
(40)	•	(70)	•

Now, subtracting the matrix $[u_i + v_j]$ from the matrix $[c_{ij}]$, i.e., (Table 11.27 – Table 11.28), the following matrix $[c_{ij} - (u_i + v_j)]$ of cell evaluations is obtained.

Table 11.29 gives the empty cell evaluations : $d_{12} = 32, d_{13} = 60$,

$d_{21} = 1, d_{22} = -18, d_{31} = 11$ and $d_{33} = 70$. The largest negative cell evaluation (marked \checkmark) is $d_{22} = -18$ which indicates that allocation of one unit to this empty cell (2, 2) will reduce the achieved cost of Rs 779 by Rs. 18. So allocate (say, θ) to cell (2, 2) as much as possible, followed by alternately subtracting and adding the amount of this allocation to other corners of the loop in order to restore

feasibility (non-negativity of allocations). For this purpose, the initial basic feasible solution can be read from Table 11.30. It is easily seen by the following rule that at the most $\theta = 2$ units can be allocated from cell (2, 4) to cell (2, 2) still satisfying the row and column total and non-negativity restrictions on the allocations.

Required 5 8 7 14

Table 11.29

•	32	60	•
1	-18	•	•
11	•	70	•

Table 11.30

Available	7	9	18
5	•		2
(19)			(10)
	+θ	7	2-θ
	✓	(40)	(60)
	8-θ		10+θ
(8)			(20)

A Rule to Determine θ : Reallocation is done by transferring the maximum possible amount θ in the marked (\checkmark) cell. The value of θ , in general, is obtained by equating to zero the minimum of the allocations containing $-\theta$ (not $+\theta$) at the corners of the closed loop. That is, in Table 11.30, $\min[8-\theta, 2-\theta] = 0$ or $2-\theta = 0$ or $\theta = 2$ units. Thus improved basic feasible solution is given in Table 11.31.

The cost for this solution becomes

$$= 5(19) + 2(10) + 2(30) + 7(40) + 6(8) + 12(20) = \text{Rs } 743.$$

The cost of Rs 743 is Rs $(2 \times 18 = 36)$ less than Rs 779 which was expected also.

Table 11.26

u_i	-10	40	0
• (19)		• (40)	• (60)
		• (8)	• (20)
v_j	29	8	0

Table 11.31

Available	7	9	18
5(19)			2(10)
	2(30)	7(40)	
	6(8)		12(20)

Step 4. Test this improved solution (*Table 11.31*) for optimality by repeating steps 1, 2 and 3. In each step, following matrices are obtained.

Table 11.32

Matrix $[c_{ij}]$ for empty cells			
•	(30)	(50)	
(70)	•	•	(60)
(40)	•	(70)	•

Table 11.34
Matrix $[u_i + v_j]$ for empty cells

•	-2	8	•
51	•	•	42
29	•	18	•

Since none of the cell evaluations is negative, i.e., $d_{12} = 32$, $d_{13} = 42$, $d_{21} = 19$, $d_{24} = 18$, $d_{31} = 11$ and $d_{33} = 52$, the solution given in *Table 11.31* is optimal with minimum cost of Rs. 743.

(b) Maximum saving = Rs. 1000 – Rs. 743 = Rs. 257. **Ans.**

11.10.2. More Solved Examples

Example 4. Solve the following transportation problem in which cell entries represent unit costs.

Table 11.36

		To			Available
		2	7	4	
From	3	3	1	8	11
	5	4	7	7	14
	1	6	2	2	34
	Required	7	9	18	

Solution. By Vogel's method, the following initial basic feasible solution having the transportation cost of Rs. 80 is obtained. To test the solution for optimality, required tables are given below.

Table 11.38
[Matrix for set of u_i and v_j]

(2)		
		(1)
	(4)	
(1)	(6)	(2)

Table 11.40

[Matrix $(u_i + v_j)$ for empty cells]		
•	7	3
0	5	•
-1	•	0
•	•	•

Table 11.33

Matrix for u_i and v_j				u_i
• (19)				• (10)
	• (30)	• (40)		22
	• (8)		• (20)	0

 $v_j \rightarrow$

29 8 18 20

The solution can be improved as shown in *Tables 11-42* and *11-43*.

Table 11-42

	5		
(2)			
	$\sqrt{+ \theta}$	$8 - \theta$	(1)
	7		
	(4)		
2	$2 - \theta$	$10 + \theta$	(2)
(1)	(6)		
7	9	18	

Here $\min [8 - \theta, 2 - \theta] = 0$ or $2 - \theta = 0$ or $\theta = 2$ units.

Table 11-44

Matrix for set of u_i and v_j		
(2)		
	(3)	(1)
	(4)	
(1)		(2)
1	4	2

u_i
1
-1
0
0

Table 11-46
Matrix $[u_i + v_j]$ for empty cells

•	5	3
0	•	•
1	•	2
•	4	•

Since all empty cell evaluations in *Table 11-47* are positive, the solution in *Table 11-43* is optimal.

The minimum cost for this solution is $= 5(2) + 2(3) + 6(1) + 7(4) + 2(1) + 12(2) = \text{Rs. } 76$.

Example 5. Solve the transportation problem :

	D_1	D_2	D_3	D_4	Available
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Required	20	40	30	10	100 Total

Solution. By '*Lowest Cost Entry Method*', the starting basic feasible solution is obtained having the transportation cost Rs 180.

Table 11-48

20(1)		10(1)		Available
	20(3)	20(2)	10(1)	30
	20(2)			50

Requirement 20 40 30 To test this solution for optimality, following tables are obtained :

Table 11.49
Matrix for set of u_i and v_j

(1)		(1)		u_i
		(2)	(1)	0
	(3)			-1
v_j	2	3	2	1

Table 11.51
Matrix $(u_i + v_j)$ for empty cells

•	2		0
2	•	•	•
1	•	1	0

Table 11.52 shows that all empty cell evaluations are non-negative. Hence the solution under test is optimal.

Furthermore, the cell evaluation $d_{12} = 0$ indicates that alternative optimal solutions also exist.

Example 6. Determine the optimum basic feasible solution to the following transportation problem.

To

	A	B	C	Available
I	50	30	220	1
	90	45	170	3
III	250	200	50	4
Required	4	2	2	

[I.A.S. (Main) 91]

Solution. The initial basic feasible solution can be easily obtained by two different methods as follows:

By Lowest Cost Method

	1 (30)	1
2 (90)	1 (45)	3
2 (250)		2 (50)
4	2	2

$$\text{Cost } z = 1 \times 30 + 2 \times 90 + 1 \times 45 + 2 \times 250 + 2 \times 50$$

$$= \text{Rs. } 855.$$

By Vogel's Method

1 (50)	(30)	220	1
3 (90)	(45)	(170)	3
(250)	2 (200)	2 (50)	4

$$\text{Cost } z = 1 (50) + 3 (90) + 2 (200) + 2 (50)$$

$$= \text{Rs. } 820.$$

Proceeding to test the initial solution (by *lowest cost method*) for optimality, following tables are obtained :

Matrix for u_i and v_j

	(30)	
(90)	(45)	
(250)		(50)
v_j	90	45
		-110

Matrix $(u_i + v_j)$ for empty cells

75	•	-125
•	•	-110
•	205	•

u_i

-15
0
160

Matrix $[c_{ij}]$ for empty cells

(50)	•	(220)
•	•	(170)
•	(200)	•

u_i

-15
0
160

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

-25	•	345
•	•	280
•	-5	•

Since cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ are not all non-negative, the solution under test is not optimal.

First Iteration. Since the most negative cell evaluation is $d_{11} = -25$, so allocate as much as possible to cell (1, 1). This necessitates the shifting of 1 unit to this cell (1, 1) from cell (1, 2) as directed by the closed loop in the following table. Thus, cell (1, 1) enters the solution, while cell (1, 2) leaves the solution, i.e., it becomes empty.

Table 11-53

Here $\min [1 - \theta, 2 - \theta] = 0$ or $1 - \theta = 0$ or $\theta = 1$ unit.

+θ ✓	1 - θ ↓	
(30)		
2 - θ ↓	1 + θ →	
(90)	(45)	
2 ●		2 ●
(250)		(50)

1

3

4

4 2 2

To test the improved solution for optimality :

Matrix for u_i and v_j

(50)			50
(90)	(45)		90
(250)		(50)	250

 v_j 0 -45 -200Matrix $(u_i + v_j)$ for empty cells

•	5	-150	50
•	•	-110	90
•	205	•	250

 v_j 0 -45 -200

Since the cell evaluation, $d_{32} = -5$ (negative), the improved solution under test is not optimal. So proceed to second iteration.

Second Iteration. Since the largest negative cell evaluation is $d_{32} = -5$, so allocate as much as possible to cell (3, 2). Thus, the cell (3, 1) leaves the solution while (3, 2) enters the solution. Cell (2, 2) may also be selected for leaving.

Here $2 - \theta = 0$ or $\theta = 2$ units.

1 ● (50)		
1 + θ ↔	2 - θ ↓	
(90)	(45)	

1

3

4

4 2 2

Improved solution ($z = \text{Rs. } 820$)

1(50)			1
3(90)	0(45)		3
	2(200)	2(50)	4

Again, proceed as earlier to test the next improved solution for optimality. It has been observed that all cell evaluations are now non-negative. Hence the solution under test is optimal.

$$z = 1 \times 50 + 3 \times 90 + 0 \times 45 + 2 \times 200 + 2 \times 50 = \text{Rs. } 820.$$

This solution was initially obtained by *Vogel's method*.

Example 7. Determine the optimum basic feasible solution to the following transportation problem :

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24 (Total)

where O_i and D_j denote i th origin and j th destination, respectively.

Solution. The initial solution to this problem can be easily obtained. In order to make the number of allocations equal to $m + n - 1$, allocate 0 amount to cell $(3, 2)$, otherwise it is not possible to test for optimality. Also see sec 11.11 on **degeneracy** for handling such situations. $z = 12 + 4 + 0 + 12 + 0 + 0 = \text{Rs. } 28$.

	6(2)			6
		2(2)	6(0)	8
4(0)	0(2)	6(2)		10

To test the initial solution for optimality (short-cut way). Compute the following table. Once the process is understood carefully, it will be easy to compute all the necessary information in one table only, instead of computing four tables for $[(u_i, v_j), (c_{ij}), (u_i + v_j), c_{ij} - (u_i + v_j)]$.

	1	6	1	4	u_i		
(1)	0+0	(2)	(3)	0+2	(4)	0+0	0
4		1	2		6		0
(4)	0+0	(3)	0+2	(2)	(0)		0
4		0	6			1	0
(0)	0	(2)	(2)	(1)	0+0		0

 v_j

Each cell of above table is read as follows :

1. The numbers written at the centre of the (*occupied*) cells are the allocations x_{ij} in the current solution. Occupied cells are also called **basic-cells** indicated by '*' at the centre of the basic cells.
2. **Down-left corner.** The numbers in the parenthesis written at the down-left corners of the cells are the unit transportation costs (c_{ij}).
3. **Down-right corner.** The numbers written at the down-right corners of (*empty*) cells are the sum of numbers u_i and v_j [i.e., $u_i + v_j$]. The numbers u_i and v_j are calculated so that $c_{ij} = u_i + v_j$ for each *occupied* cell.
4. **Upper-right corner.** The numbers written at the upper-right corners of each *empty* cell are the net evaluations $d_{ij} = (c_{ij}) - (u_i + v_j)$ which are simply obtained by subtracting the number $u_i + v_j$ (at the down-right corner) from the corresponding unit cost (c_{ij}) written at the down-left corner. The cell evaluations are computed as follows :

$$\begin{array}{ll} d_{11} = c_{11} - (u_1 + v_1) = (1) - (0+0) = 1 & d_{14} = c_{14} - (u_1 + v_4) = (4) - (0+0) = 4 \\ d_{21} = c_{21} - (u_2 + v_1) = (4) - (0+0) = 4 & d_{22} = c_{22} - (u_2 + v_2) = (3) - (0+2) = 1 \\ d_{13} = c_{13} - (u_1 + v_3) = (3) - (0+2) = 1 & d_{34} = c_{34} - (u_3 + v_4) = (1) - (0+0) = 1 \end{array}$$

These calculations can be done within the table orally.

From these calculations, it is observed that all cell evaluations (d_{ij}) are positive.

Hence the solution under test is optimal.

Example 8. Given the following data.

The cost of shipment from third source to the third destination is not known. How many units should be transported from sources to the destinations so that the total cost of transporting all the units to their destinations is a minimum.

		Destinations			
		1	2	3	Capacity
Sources	1	2	2	3	10
	2	4	1	2	15
	3	1	3	x	40
Demand		20	15	30	

Solution. Since the cost c_{33} is unknown, assign a large cost, say M , to this cell. Now using Vogel's method an initial basic feasible solution is obtained.

Table 11.54
Initial B.F.S.

(2)	(2)	10(3)
(4)	(1)	15(2)
20(1)	15(3)	5(M)

Table 11.55
($u_i + v_j$) Matrix

4 - M	6 - M	• (3)	3 - M
3 - M	5 - M	• (2)	2 - M
• (1)	• (3)	• (M)	0

v_j 1 3 M

Table 11.56

Matrix [$c_{ij} - (u_i + v_j)$]

2 - (4 - M) +ive	2 - (6 - M) +ive	•	3 - M
4 - (3 - M) +ive	1 - (5 - M) +ive	•	2 - M
•	•	•	0

v_j 1 3 M

Since all the net evaluations for empty cells are positive, the current solution is optimum. But, it is to be noticed here that cell (3, 3) also appears in the solution for which the cost for shipment is not known. Hence, there exists a **pseudo optimum basic feasible solution** : $x_{13} = 10$, $x_{23} = 15$, $x_{31} = 20$, $x_{23} = 15$ and $x_{33} = 5$.

Example 9. Is $x_{13} = 50$, $x_{14} = 20$, $x_{21} = 55$, $x_{31} = 30$, $x_{32} = 35$, $x_{34} = 25$ an optimum solution of the following transportation problem?

		To				Available units
		6	1	9	3	70
From	11	5	2	8	55	
	10	12	4	7	90	
Required units		85	35	50	45	

If not, modify it to obtain a between feasible solution.

[Meerut (Maths) 91]

Solution. The initial feasible solution is given as in the following table with cost of Rs. 1460.

		50(9)	20(3)
55(11)			
30(10)	35(12)		25(7)

Matrix for u_i and v_j				u_i
		• (9)	• (3)	-4
• (11)				1
• (10)	• (12)		• (7)	0

Starting Table. Vacate the cell (3, 4) and occupy (2, 3).
 $\theta = \min (50, 55, 25) = 25$

First Iteration Table. Vacate the cell (1, 3) and occupy (1, 2). Here $\theta = \min [35, 30, 25] = 25$.

	0	-7	50 - θ	20 + θ	u_j
(6)	6	(1)	8 (9)	(3)	-4
(11)	55 - θ	-8	-12	0	1
(10)	30 + θ	35	-9	25 - θ	0
v_j	10	12	13	7	

Second Iteration Table. Vacate cell (2, 1) and occupy (2, 2). Here $\theta = \min(5, 10) = 5$.

	-12	+θ -19	25 - θ	45	u_i
(6)	18	(1)	20 (9)	(3)	0
(11)	30 - θ	-8	25 + θ		12
(10)	55 + θ	35 - θ	3	3	12
v_j	18	20	9	3	

Third Iteration Table. Vacate the cell (3, 2) and occupy (3, 4). Here $\theta = \min[45, 5] = 5$.

	7	25	19	45	u_j
(6)	-1	(1)	(9)	(3)	0
(11)	5 - θ	0 - 8	50	-7	12
(10)	80 + θ	10 - θ	3	-7	11
v_j	-1	1	-10	3	

	7	11	45 - θ	u_i
(6)	-1	(1)	(9)	0
(11)	8	-2	(3)	1
(10)	3 (5)	(2)	(8)	4
v_j	-1	1	-2	3

Since all the net evaluations are non-negative, an optimum basic feasible solution is obtained as :

$$\begin{aligned}x_{12} &= 30, x_{14} = 40, x_{22} = 5, x_{23} = 50, \\x_{31} &= 85, \text{ and } x_{34} = 5.\\z &= 30 \times 1 + 40 \times 3 + 5 \times 5 + 50 \times 2 + 85 \times 10 \\&= \text{Rs. } 1160.\end{aligned}$$

Optimal Iteration Table					
	0	30	11	40	u_i
(6)	6	(1)	(9)	-2	0
(11)	10	(5)	(2)	(8)	1
(10)	85	7	2	5	4
v_j	6	1	-2	3	

Example 10. Solve the transportation problem where all entries are unit costs.

	D_1	D_2	D_3	D_4	D_5	a_j
O_1	73	40	9	79	20	8
O_2	62	93	96	8	13	7
O_3	96	65	80	50	65	9
O_4	57	58	29	12	87	3
O_5	56	23	87	18	12	5
b_j	6	8	10	4	4	

Solution. Using '*Lowest Cost Entry Method*', the initial basic feasible solution having transportation cost Rs. 1123 is obtained as below.

		8(9)		
3(62)			4(8)	
2(96)	7(65)			
1(57)		2(29)		
	1(23)			4(12)
6	8	10	4	4

Starting Iteration Table

v_j Here $\theta = \min [3, 4, 7] = 3$.

First Iteration Table. Vacate the cell $(2, 1)$ and occupy $(2, 5)$.

(73)	37 (40)	6 (9)	8 •	(79)	-10	(20)	-5	-18
(62)	55 (93)	24 (96)	27 (8)			(13)	3 •	0
(96)	5 •	4 •		+	+			+
(57)	(65)	(80)	68 (50)	49 (65)				54 41
(56)	1 •	+	2 •	+	+			+
	(58)	26 (29)	(12)	10	(87)			15 2
	55	24	27	26 (18)	8	7 (12)	13	-1

Since all the net evaluations at the upper-right corners of all empty cells are non-negative, the solution under test is optimal. Hence optimum allocation is given by, $x_{13} = 8, x_{24} = 4, x_{25} = 3, x_{31} = 5, x_{32} = 4, x_{41} = 1, x_{43} = 2, x_{52} = 4$ and $x_{55} = 1$, and minimum transportation cost

$$z = 8 \times 9 + 4 \times 8 + 3 \times 13 + 5 \times 96 + 4 \times 65 + 1 \times 57 + 2 \times 29 + 4 \times 23 + 1 \times 12 = \text{Rs. } 1102.$$

Market

Example 11. The following table shows all the necessary information on the available supply to each warehouse, the requirement of each market and the unit transportation cost from each warehouse to each market.

	I	II	III	IV	
Warehouse A	5	2	4	3	Supply 22
Warehouse B	4	8	1	6	15
Warehouse C	4	6	7	5	15
Requirement	7	12	17	9	8

The shipping clerk has worked-out the following schedule from experience : 12 units from A to II, 1 unit from A to III, 9 units from A to IV, 15 units from B to III, 7 units from C to I, and 1 unit from C to III.

- Check and see if the clerk has the optimal schedule,
- Find the optimal schedule and minimum total shipping cost.
- If the clerk is approached by a carrier of route C to II who offers to reduce his rate in the hope of getting some business, by how much must the rate be reduced before the clerk should consider giving him an order.

[VTU (BE Mech.) 2003]

Solution. The basic feasible solution is given as follows :

	12 (2)	1 (4)	9 (3)	
		15 (1)		22
7 (4)		1 (7)		15
7	12	17	9	8

(a) Starting Iteration Table.

(b) Optimal Table

	+	12	1 + θ	9 - θ	u_i
(5)	1 (2)	(4)	(3)		0
6	-			6	
(4)	-2 (8)	-1 (1)	• 15	(6) 0	-3
7	•	+	1 - θ	+ θ ✓	3
(4)	(6)	5 (7)	(5)	6	
v_j	1	2	4	3	

	+	12	2	8	u_i
(5)	2 (2)	(4)	(3)		0
+	-		15		6
(4)	-1 (8)	-1 (1)	•	(6) 0	-3
7	•	+	+	1	2
(4)	(6)	4 (7)	6 (5)		
v_j	2	2	4	3	

Here $\theta = \min [9, 1] = 1$.

Since all the net evaluations (at the upper-right corners) of empty cells are non-negative, the current solution is optimal.

Hence the optimal schedule is : $x_{12} = 12, x_{13} = 2, x_{14} = 8, x_{23} = 15, x_{31} = 7, x_{34} = 1$.

Minimum total shipping cost will be $z = 12 \times 2 + 2 \times 4 + 8 \times 3 + 15 \times 1 + 7 \times 4 + 1 \times 5 = \text{Rs. } 104$.

- If the clerk decides to transport at the most 8 units from C to II (instead of 7 to I and 1 to IV), then II may reduce his cost from $c_{32} = 6$ to at least 4 in order to have the improved cost. So according to the given proposal, the total minimum transportation cost will become Rs. 103.

Example 12. The following table gives the cost for transporting material from supply points A, B, C and D to demand points E, F, G, H, and J.

	To				
	E	F	G	H	J
A	8	10	12	17	15
B	15	13	18	11	9
C	14	20	6	10	13
D	13	19	7	5	12

The present allocation is as follows :

A to E 90 ; A to F 10 ; B to F 150 ; C to F 10 ; C to G 50 ; C to J 120 ; D to H 210 ; D to J 70.

(a) Check if this allocation is optimum. If not, find an optimum schedule.

(b) If in the above problem, the transportation cost from A to G is reduced to 10, what will be the new optimum schedule ? [IAS (Main) 91]

Solution. The given allocation has the transportation cost = Rs. 6930. Now test the given solution for optimality as follows :

Starting Iteration Table

							u_j
	90 - 0	10 + 0					+
(8)			(12)	- 4	(17)	- 3	(15) 3
	+ 11	(13) 150		+ (18)	- 1 11	0 9	+ 6
(15)							
	- 4 18	10 - 0 (20)	50		(10) 7	(13) 120	20
(14)							
	- 4 17	(19) 0 19	(7) 5	(6) 210	(12) 70		19
(13)							
	v_j - 2	0		- 14	- 13	- 7	

Here $\theta = \min [90, 10] = 10$.

First Iteration Table. Vacate the cell (3, 2) and occupy (3, 1).

							u_j
	80 - 0	20 + 0					+
(8)			(12)	0	(17)	1 15	7
	+ 11	(13) 150 - 0		+ (18)	3 11	4 (9) $\sqrt{+ \theta}$ 10	- 1
(15)							
	10 + 0		50		(10) 7	(13) 120 - 0	0
(14)							
	0 13	(19) 16 15 (7)	5 (6)	(6) 210	(12) 70		- 1
(13)							
	v_j 14	16	6	7	13		

Here $\theta = \min [120, 80, 150] = 80$,

Second Iteration (Optimal) Table

								u_i	
		+	100 •		+		+	-3	
(8)	7	(10)		(12)	-1	(17)	0	(15)	6
		+	70 •		+	<td>+</td> <td>80 •</td> <td>0</td>	+	80 •	0
(15)	10	(13)		(18)	2	(11)	3	(9)	
		+	90 •		50 •		+	40 •	4
(14)	0	(20)	17 •	(6)		(10)	7	(13)	
		+			+		210 •	70 •	3
(13)	13	(19)	16 •	(7)	5	(6)		(12)	9
v_j	10	13	13	2	5	3			

Since all the net-evaluations are non-negative, the following optimum solution is obtained :

$$x_{12} = 100, x_{22} = 70, x_{25} = 80, x_{31} = 90, x_{33} = 50, x_{35} = 40, x_{44} = 210, x_{45} = 70,$$

and minimum cost = Rs. 6810.

Example 13. Hindustan Construction Company needs 3, 3, 4 and 5 million cubic feet of fill at four earthen dam-sites in Punjab. It can transfer the fill from three mounds A, B and C where 2, 6 and 7 million cubic feet of fill is available respectively. Costs of transporting one million cubic feet of fill from mounds to the four sites in lakhs are given in the table.

(a) Solve the problem using transportation algorithm for minimum cost :

(b) Formulate the problem as L.P.P.

		To				
		I	II	III	IV	a_i
From	A	15	10	17	18	2
	B	16	13	12	13	6
	C	12	17	20	11	7
b_j	3	3		4	5	

Solution. Using "Vogel's Method" the initial B.F.S with cost Rs. 190 is obtained as follows :

		2(17)		
	3(13)	2(12)		1(13)
3(12)			4(11)	

Starting Iteration Table

	-4	+θ	-8	2-θ	0	u_i
(15)	19	(10)	18	(17)	(18)	18
	+	3-θ		2+θ	1	5
(16)	14	(13)		(12)	(13)	0
(12)	3	(17)	11	(20)	10	4
v_j	14	13	12	11	13	-2

$$\text{Hence } \theta = \min [3, 2] = 2.$$

Since all the net-evaluations are not non-negative, the current solution is not optimal.

First Iteration Table. Vacate the cell (1, 3) and occupy (1, 2).

						u_i
	+	*		+		+
	(15)	11 (10)		(17)	9 (18)	10
	+	2 •				-3
	(16)	14 (13)	1 •		4 •	1 •
			(12)		(13)	0
	3 •		+		+	-2
v_j	14	(17)	13	11 (20)	12	13
				10 (11)		

Since all the net-evaluations are non-negative, the optimum solution is

$$x_{12} = 2, x_{22} = 1, x_{23} = 4, x_{24} = 1, x_{31} = 3, x_{34} = 4,$$

and minimum cost $z = \text{Rs. } 174$.

L.P. Formulation. Let x_{ij} be the amount of fill transferred from mounds to dam-sites. Then formulation of the problem becomes :

$\text{Min } z = (15x_{11} + 10x_{12} + 17x_{13} + 18x_{14}) + (16x_{21} + 13x_{22} + 12x_{23} + 13x_{24}) + (12x_{31} + 17x_{32} + 20x_{33} + 11x_{34})$
subject to the constraints :

$$x_{11} + x_{12} + x_{13} + x_{14} = 2$$

$$x_{12} + x_{22} + x_{32} = 3$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 6$$

$$x_{13} + x_{23} + x_{33} = 4$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 7$$

$$x_{14} + x_{24} + x_{34} = 5$$

$$x_{11} + x_{21} + x_{31} = 3$$

$$\text{and all } x_{ij} \geq 0 \text{ (i = 1, 2, 3; j = 1, 2, 3, 4)}$$

EXAMINATION PROBLEMS

1. Find the optimum solution to the following transportation problem for which the cost, origin-availabilities, and destination-requirements are as given below :
(i)

		To					
		A	B	C	D	E	a_i
<i>From</i>	<i>I</i>	3	4	6	8	8	20
	<i>II</i>	2	10	1	5	30	30
	<i>III</i>	7	11	20	40	15	15
	<i>IV</i>	2	1	9	14	18	13
$b_j \rightarrow$	40	6	8	18	6		

[Meerut M.Sc (Maths.) Jan. 98 BP, 94]

[Ans. $x_{11} = 14, x_{15} = 6, x_{21} = 4, x_{23} = 8, x_{24} = 18, x_{31} = 15, x_{41} = 7, x_{42} = 6$; min cost = Rs. 321.]

(ii)

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	175 Total

[Hint. Find the initial solution by 'Lowest Cost Entry Method'. Although, "Vogel's method" will also yield the same initial solution, but will be longer one. One improvement is only required to reach the optimal solution.]

[Ans. $x_{11} = 20, x_{13} = 10, x_{23} = 20, x_{24} = 10, x_{25} = 20, x_{32} = 40, x_{35} = 10, x_{36} = 25, x_{45} = 20$, min cost = Rs. 430.]

(iii)

	Store			a_i	
	A	B	C		
Factory	I	10	9	8	8
	II	10	7	10	7
	III	11	9	7	9
	IV	12	14	10	4
	$b_j \rightarrow$	10	10	8	

[Hint. Find initial BFS by 'Lowest Cost Entry Method' and prove it to be optimal]

[Ans. $x_{11} = 6$, $x_{12} = 2$, $x_{22} = 7$, $x_{32} = 1$, $x_{33} = 8$, $x_{41} = 4$, min. cost = Rs. 240]

(v)

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
	$b_j \rightarrow$	16	18	31	25

[Hint. Find initial BFS by 'VAM' and prove it to be optimal]

[Ans. $x_{12} = 18$, $x_{13} = 1$, $x_{21} = 12$, $x_{24} = 25$, $x_{31} = 4$, $x_{33} = 30$, min. cost = Rs. 355]

(vii)

	To			a_i	
	1	2	3		
From	I	2	7	4	5
	II	3	3	7	8
	III	5	4	1	7
	IV	1	6	2	14
	$b_j \rightarrow$	7	9	18	

[Meerut (B. Sc.) 90]

[Hint. Find initial solution by VAM and and revise once optimum.]

[Ans. $x_{11} = 5$, $x_{22} = 8$, $x_{32} = 1$, $x_{33} = 6$, $x_{41} = 2$, $x_{43} = 12$, min. cost = Rs. 70]

(ix)

	D_1	D_2	D_3	D_4	Supply
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Demand	22	35	25	41	123 (Total)

[Hint. Find initial solution by VAM, it will be proved optimum]

[Ans. $x_{14} = 30$, $x_{21} = 5$, $x_{22} = 35$, $x_{31} = 17$, $x_{33} = 25$, $x_{34} = 11$, cost = Rs. 1921]

(xi)

	D_1	D_2	D_3	D_4	s_i
O_1	3	8	9	16	8
O_2	6	11	14	6	9
O_3	5	13	10	12	13
	d_j	6	7	7	10

[Ans. $x_{11} = 4$, $x_{12} = 4$, $x_{24} = 9$, $x_{31} = 2$, $x_{32} = 3$, $x_{33} = 7$, $x_{34} = 1$, min cost = 256]

(iv)

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	-2	3	70
O_2	2	4	0	1	38
O_3	1	2	-2	5	32
Demand	40	28	30	32	

[Hint. Use 'VAM' to find initial BFS with cost Rs. 86.]

[Ans. $x_{11} = 40$, $x_{12} = 26$, $x_{14} = 4$, $x_{24} = 38$, $x_{32} = 2$ and $x_{33} = 30$; min cost = Rs. 86].

(vi)

	1	2	3	4	5	Supply
From	2	3	5	7	5	17
	4	1	2	1	6	13
	3	2	8	6	1	16
	5	3	7	2	4	20

[Hint. By VAM we get degeneracy. Find initial solution by Lowest Cost Entry Method, then prove optimum.]

(viii)

	1	2	3	4	Supply
From	5	3	6	4	30
	3	4	7	8	15
	9	6	5	8	15
Demand	10	25	18	7	60 (Total)

[Hint. Find initial solution by VAM and it will be proved optimum. to get Alternative solutions also exist.]

[Ans. (i) $x_{12} = 20$, $x_{13} = 3$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 5$, $x_{33} = 15$, min. cost = Rs. 231 (ii) $x_{12} = 23$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 2$, $x_{23} = 3$, $x_{33} = 15$

(x)

	D_1	D_2	D_3	D_4	Supply
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand	16	18	31	25	

[Mudurai BSc (Comp. Sc.) 92]

[Ans. $x_{12} = 18$, $x_{13} = 1$, $x_{21} = 12$, $x_{24} = 25$, $x_{31} = 4$, $x_{33} = 33$, min cost = 355]

.

(xii)

	I	II	III	IV	Supply
A	6	3	5	4	22
B	5	9	2	7	15
C	5	7	8	6	8
Requirement	7	12	17	9	

[Ans. $x_{12} = 12$, $x_{13} = 2$, $x_{14} = 8$, $x_{23} = 15$, $x_{31} = 7$, $x_{34} = 1$, min. cost = 149]

2. Goods have to be transported from factories F_1, F_2, F_3 , to ware-house W_1, W_2, W_3 and W_4 . The transportation cost per unit, capacities and requirement of the warehouse are given in the following table.

	W_1	W_2	W_3	W_4	Capacity
F_1	95	105	80	15	12
F_2	115	180	40	30	7
F_3	195	180	95	70	5
Requirement	5	4	4	11	

Find the distribution with minimum cost.

3. Solve the following cost-minimizing transportation problem.

	D_1	D_2	D_3	D_4	D_5	D_6	Available
O_1	2	1	3	3	2	5	50
O_2	3	2	2	4	3	4	40
O_3	3	5	4	2	4	1	60
O_4	4	2	2	1	2	2	30
Required	30	50	20	40	30	10	180

[JNTU (B. Tech.) 2003; Delhi B.Sc. (Maths.) 91]

4. Consider four bases of operations B_i and three targets T_j . The tons of bombs per aircraft from any base that can be delivered to any target are given in the table below.

	T_1	T_2	T_3
B_1	8	6	5
B_2	6	6	6
B_3	10	8	4
B_4	8	6	4

The daily sortie capacity of each of the four bases is 150 sorties per day. The daily requirement in sorties over each individual target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all three targets explaining each step.

5. There are three sources or origins which store a given product. These sources supply these products to four dealers. The capacities of the sources and the demands are as given below :

$$S_1 = 150, S_2 = 40, S_3 = 80, D_1 = 90, D_2 = 70, D_3 = 50, D_4 = 60.$$

The cost of transporting the product from various sources to various dealers is shown in the table below.

	D_1	D_2	D_3	D_4
S_1	27	23	31	69
S_2	10	45	40	32
S_3	30	54	35	57

Find out the optimum solution for transporting the products at a minimum cost.

[Hint. Find initial BFS by VAM and prove it to be optimal.]

$$[Ans. x_{11} = 30, x_{12} = 70, x_{13} = 50, x_{14} = 40, x_{21} = 60, x_{24} = 20, \text{min cost} = 8190]$$

6. A firm manufacturing a single product has plants I, II, and III. The three plants have produced 60, 35 and 40 units respectively during this month. The firm had made a commitment to sell 2 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D, and 30 units to customer E. Find the minimum possible transportation cost of shipping the manufactured product to five customers. The net per unit cost of transporting from the three plants to five customers is given in the table.

[JNTU (Mech. & Prod.) 2004]

	Customer				
	A	B	C	D	E
I	4	1	3	4	4
Plant II	2	3	2	2	3
III	3	5	2	4	4

[Hint. Find initial BFS by VAM and prove it to be optimal.]

$$[Ans. x_{12} = 45, x_{15} = 15, x_{21} = 17, x_{24} = 18, x_{31} = 5, x_{33} = 20, x_{35} = 15, \text{min. cost} = \text{Rs. } 290].$$

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7. The figures in the body of the table below are proportional to the cost of transportation of the tonne of food grain from the port given by the row heading to the destination given by column heading.

Ports	Delhi	Hyderabad	Mysore	Nagpur	stock (in thousand tonnes)
Bombay	9	5	8	5	225
Calcutta	9	10	13	7	75
Madras	14	5	3	7	100
Requirements (thousand tonnes)	125	80	95	100	400

Plan a transportation scheme satisfying the requirements of each destination and at the same time minimizing the total transportation cost.

[Hint. Find the initial BFS by VAM and prove it to be optimal. Alternative solutions exists.]

[Ans. $x_{11} = 50$, $x_{12} = 75$, $x_{14} = 100$, $x_{21} = 75$, $x_{32} = 5$, $x_{33} = 95$, min. cost = 2310]

8. An oil corporation has got three refineries P, Q, and R and it has to send petrol to four different depots A, B, C and D. The cost of shipping 1 gal. of petrol and the available petrol at the refineries are given in the table. The requirement of the depots and the available petrol at the refineries are also given. Find the minimum cost of shipping after obtaining an initial solution by VAM.

		Depot				
		A	B	C	D	Available
Refinery	P	10	12	15	8	130
	Q	11	11	9	10	150
	R	20	9	7	18	170
Required		90	100	140	120	

[Hint. Find initial BFS by VAM and prove it to be optimal.]

[Ans. $x_{11} = 90$, $x_{14} = 40$, $x_{23} = 70$, $x_{24} = 80$, $x_{32} = 100$, $x_{33} = 70$; min. cost = Rs. 3460]

9. King Mahendra of the Pallavedynasty Red sculptors stationed at Kanchipuram, Mallapuram and Tiruchirapalli. There were 15 sculptors at Kanchipuram, 30 at Mallapuram and 5 at Tiruchirapalli. Rock-cut shrines had to be excavated and 12 sculptors were required at Mandagapattu, 20 at Pallavapuram, 8 at Seevamangalam and 10 at Mahendravadi, give a solution which minimizes the cost of transport of the sculptors assuming that the cost of transport per mile is the same for all the routes. The distance in miles between the different places is given in the following table.

From	To			
	Mandagapattu	Pallavapuram	Seevaman-galam	Mahendravadi
Kanchipuram	60	40	50	20
Mallapuram	70	30	65	55
Tiruchirapalli	100	200	120	210

[Ans. $x_{13} = 5$, $x_{14} = 10$, $x_{21} = 7$, $x_{22} = 20$, $x_{23} = 3$, $x_{31} = 5$; min distance = 2235 miles.]

10. A company has factories at F_1 , F_2 and F_3 which supply warehouses at W_1 , W_2 and W_3 . Weekly factory capacities are 200, 160 and 90 units respectively. Weekly warehouses requirements are 180, 120 and 150 units respectively. Unit shipping costs (in rupees) are as follows :

Factory	Warehouse			Supply
	W_1	W_2	W_3	
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Demand	180	120	150	350

Determine the optimum distribution for this company to minimize shipping costs.

[Kerala (M. Com.) 97]

[Ans. $F_1 : x_{12} = 140$, $x_{13} = 60$, $F_2 : x_{21} = 40$, $x_{22} = 120$, $x_{23} = 0$

$F_3 : x_{31} = 0$, $x_{32} = 0$, $x_{33} = 90$ Total transportation cost = Rs. 5,920]

11. A wholesale company has three warehouses from which supplies are drawn for four retail customers. The company deals in a single product, the supplies of which at each warehouse are :

Warehouse no.	Supply (units)	Customer no.	Demand (units)
1	20	1	15
2	28	2	19
3	17	3	13
		4	18

Conveniently, total supply at the warehouses is equal to total demand from the customers. The following table gives the transportation costs per unit shipment from each warehouse to each customer.

Warehouse	Customer			
	1	2	3	4
1	3	6	8	5
2	6	1	2	5
3	7	8	3	9

Determine what supplies to despatch from each of the warehouses to each customer so as to minimize overall transportation cost.

[Ans. $x_{11} = 15, x_{14} = 5, x_{22} = 19, x_{24} = 9, x_{33} = 13, x_{34} = 4$. Total transportation cost = Rs. 209.]

12. A manufacturer has distribution centres at X, Y, and Z. These centres have availability 40, 20 and 40 units of his product. His retail outlets at A, B, C, D and E requires 25, 10, 20, 30 and 15 units respectively. The transport cost (in rupees) per unit between each centre outlet is given below :

Distribution centre	Retail outlets				
	A	B	C	D	E
X	55	30	40	50	50
Y	35	30	100	45	60
Z	40	60	95	35	30

Determine the optimal distribution to minimize the cost of transportation.

[Poone (M.B.A.) 97]

13. ABC limited has three production shops supplying a product of five warehouses. The cost of production varies from shop to shop and cost of transportation from one shop to a warehouse also varies. Each shop has a specific production capacity and each warehouse has certain amount of requirement. The cost of transportation are as given below :

Shop	Warehouse					Capacity
	I	II	III	IV	V	
A	6	4	4	7	5	100
B	5	6	7	4	8	125
C	3	4	6	3	4	175
Requirement	60	80	85	105	70	400

The cost of manufacture of the product at different production shops is :

Shop	Variable cost	Fixed cost
A	14	7,000
B	16	4,000
C	15	5,000

Find the optimum quantity to be supplied from each shop to different warehouses at minimum total cost.

[Nagpur (M.B.A.) Nov. 98]

14. A company has three factories at Amethi, Baghpat and Gwalior; and four distribution centres at Allahabad, Bombay, Calcutta and Delhi. With identical cost of production at the three factories the only variable cost involved is transportation cost. The production at the three factories is 5,000 tonnes; 6,000 tonnes; 2,500 tonnes respectively. The demand at four distribution centres is 6,000 tonnes; 4,000 tonnes; 2,000 tonnes and 1,500 tonnes respectively. The transportation costs per tonne from different factories to different centres are given below :

Factory	Distribution Centre			
Amethi	3	2	7	6
Baghpat	7	5	2	3
Gwalior	2	5	4	5

Suggest the optimum transportation schedule and find the minimum cost of transportation.

[Gujarat (MBA) 96; Delhi (M.Com.) 95]

[Ans. $x_{11} = 3500, x_{12} = 1500, x_{22} = 2500, x_{23} = 2000, x_{24} = 1500$ and $x_{31} = 2500$, Min. transportation cost = Rs. 39,500.]

15. A company manufacturing air-coolers has two plants located at Bombay and Calcutta with a weekly capacity of 200 units and 100 units, respectively. The company supplies air coolers to its 4 show rooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a demand of 75, 100, 100 and 30 units, respectively. The cost of transportation per unit (in Rs.) is shown in the following table :

	Ranchi	Delhi	Lucknow	Kanpur
Bombay	90	90	100	100
Calcutta	50	70	130	85

Plant the production programme so as to minimize the total cost of transportation.

[Rajasthan (M.Com.) 97]

[Ans. $x_{12} = 75$, $x_{13} = 95$, $x_{14} = 30$, $x_{21} = 75$, $x_{22} = 25$, $x_{31} = 30$, $x_{32} = 10$, $x_{33} = 0$.]

Min. transportation cost = Rs. 24750. Alternative optimum solution also exists.]

16. The ABC Tool Company has a sales force of 25 men who work out from three regional offices. The company produces four basic product lines of hand tools. Mr. Jain, the sales manager feels that 6 salesmen are needed to distribute product line 1, 10 salesmen to distribute product line 2, 4 salesmen for product line 3 and 5 salesmen for product line 4. The cost (in Rs.) per day of assigning salesmen from each of the offices for selling each of the product lines are as follows :

	Product lines			
	1	2	3	4
Regional office A	Rs. 20		Rs. 16	Rs. 18
Regional office B	17	28	14	16
Regional office C	29	23	19	20

At the present time, 10 salesmen are allocated to office A, 9 salesmen to office B, and 7 salesmen to office C. How many salesmen should be assigned from each office to sell each product line in order to minimize costs ? Identify alternate optimum solutions, if any.

[Delhi (MBA.) April 98]

- [Ans. $x_{12} = 4$, $x_{13} = 1$, $x_{14} = 5$, $x_{21} = 6$, $x_{23} = 3$, $x_{32} = 6$, $x_{35} = 1$, Min. transportation cost = Rs. 472.]

17. The Purchase Manager, Mr. Shah, of the State Road Transport Corporation must decide on the amounts of fuel to buy from three possible vendors. The corporation refuels its buses regularly at the four depots within the area of its operations.

The three oil companies have said that they can furnish up to the following amounts of fuel during the coming month : 275,000 litres by oil company 1 : 50,000 litres by oil company 2; and 660,000 litres by oil company 3. The required amount of the fuel is 110,000 litres by depot 1; 20,000 litres at depot 2; 330,000 litres at depot 3; and 440,000 litres at depot 4.

When the transportation costs are added to the bid price per litre supplied, the combined cost per litre for fuel from each vendor servicing a specific depot is shown below :

	Company 1	Company 2	Company 3
Depot 1	5.00	4.75	4.25
Depot 2	5.00	5.50	6.75
Depot 3	4.50	6.00	5.00
Depot 4	5.50	6.00	4.50

Determine the optimum transportation schedule.

[Delhi (M.B.A.) Nov. 97]

[Ans. $x_{13} = 110$, $x_{21} = 55$, $x_{22} = 165$, $x_{31} = 220$, $x_{33} = 110$, $x_{43} = 440$, $x_{52} = 385$, Min. transportation cost = Rs. 51,70,000.]

11.11. DEGENERACY IN TRANSPORTATION PROBLEMS

The solution procedure for non-degenerate basic feasible solution with exactly $m + n - 1$ strictly positive allocations in *independent positions* has been discussed so far. However, sometimes it is not possible to get such initial feasible solution to start with. Thus degeneracy occurs in the transportation problem whenever a number of occupied cells is less than $m + n - 1$.

We recall that a basic feasible solution to an m -origin and n -destination transportation problem can have at most $m + n - 1$ number of positive (non-zero) basic variables. If this number is exactly $m + n - 1$, the BFS is said to be *non-degenerate*; and if less than $m + n - 1$ the basic solution degenerates. It follows that whenever the number of basic cells is less than $m + n - 1$, the transportation problem is a degenerate one.

Degeneracy in transportation problems can occur in two ways :

1. Basic feasible solutions may be degenerate from the initial stage onward.
2. They may become degenerate at any intermediate stage.

- Q. 1. What is degeneracy problem in transportation ? What is its cause ? How it can be overcome.

[JNTU (Mech. & Prod.) 2004]

2. Explain briefly about unbalanced transportation problem and degenerate case in transportation problem.

[VTU (BE Mech.) 2002]

11-11-1. Resolution of Degeneracy During the Initial Stage

To resolve degeneracy, allocate an extremely small amount of goods (close to zero) to *one or more* of the empty cells so that a number of occupied cells becomes $m + n - 1$. The cell containing this extremely small allocation is, of course, considered to be an occupied cell.

- Rule :** The extremely small quantity usually denoted by the Greek letter Δ (delta) [also sometimes by ϵ (epsilon)] is introduced in the *least cost* independent cell subject to the following assumptions. If necessary, two or more Δ 's can be introduced in the least and second least cost independent cells.
1. $\Delta < x_{ij}$ for $x_{ij} > 0$.
 2. $x_{ij} + \Delta = x_{ij} - \Delta, x_{ij} > 0$.
 3. $\Delta + 0 = \Delta$.
 - *4. If there are more than one Δ 's in the solution, $\Delta < \Delta'$, whenever Δ is above Δ' .
If Δ and Δ' are in the same row, $\Delta < \Delta'$ when A is to the left of Δ' .

Q. What do you mean by non-degenerate basic feasible solution of a transportation problem.

For example, $50 + \Delta = 50$, and $200 - \Delta = 200$. Of course, if Δ is subtracted from itself the result is assumed to be zero.

Following example will make the procedure clear.

Example 14. A company has three plants A, B and C and three warehouses X, Y and Z. Number of units available at the plants is 60, 70 and 80, respectively. Demands at X, Y and Z are 50, 80 and 80, respectively. Unit costs of transportation are as follows :

Table 11-56

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

What would be your transportation plan ? Give minimum distribution cost.

[Garhwal 97]

Solution. Step 1. First, write the given cost-requirement Table 11-56 in the following manner :

Q. State the transportation problem in general terms and explain the problem of degeneracy.

[IAS (Main) 97]

Table 11-57

	X	Y	Z	Available
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80

Requirement 50 80 80 210 (Total)

Step 2. Using either the "Lowest Cost Entry Method" or "Vogel's Approximation Method", obtain the initial solution.

Step 3. Since the number of occupied cells in Table 11-58 is 4, which does not equal to $m + n - 1$ (that is 5), the problem is degenerate at the very beginning, and so the attempt to assign u_i and v_j values to Table 11-58 will not succeed. However, it is possible to resolve this degeneracy by addition of Δ to some suitable cell.

Table 11-58

		60(3)	Available
50(3)		20(9)	60
	80(3)		70

Requirement 50 80 80

Generally, it is not possible to add Δ to any empty cell. But, Δ must be added to one of those empty cells which make possible the determination of a unique set of u_i and v_j . So choose such empty cell with careful judgement, for if the empty cell (1,1) is made an occupied cell (by addition of Δ) it will not be possible to assign u_i and v_j values. The allocations in cell (1, 1), (1, 3), (2, 1), and (2, 3) will become in non-independent (rather than independent) positions.

On the other hand, the addition of Δ to any of the cells (1, 2), (2, 2), (3, 1) and (3, 3) will enable us to resolve the degeneracy and allow to determine a unique set of u_i and v_j values. So proceed to resolve the degeneracy by allocating Δ to least cost independent empty cell (3, 3) as shown in Table 11-59.

Now proceed to test this solution for optimality.

Step 4. Values of u_i and v_j will be obtained as shown in Table 11-60.

Once a unique set of u_i and v_j values has been determined, various steps of the transportation algorithm can be applied in a routine manner to obtain an optimal solution.

Table 11-59

		Requirement	Available
		50	60
		80	70
		80 + $\Delta = 80$	80 + $\Delta = 80$

Table 11-60

		u_i
		-2
	*	4
		0

Table 11-61
Matrix $[c_{ij}]$ for empty cells only

(8)	(7)	*
*	(8)	*
(11)	*	*

Table 11-62
Matrix $[u_i + v_j]$ for empty cells

-3	1	*
*	7	*
-1	*	*

Since all cell-evaluations in Table 11-63 are positive, the solution under test is optimal. The real total cost of subsequent solutions did not happen to change in the example after Δ was introduced. In general, this will not be the case. In as much as the infinitesimal quantity Δ plays only an auxiliary role and has no significance, it is removed when the optimal solution is obtained. Hence, the final answer is given in Table 11-64

Table 11-63

Matrix $[c_{ij} - (u_i + v_j)]$

11	6	*
*	1	*
12	*	*

Table 11-64

		60(3)
50(3)		20(9)
	80(3)	

Thus minimum cost is : $180 + 150 + 180 + 240 = \text{Rs. } 750$.

11.11-2. Resolution of Degeneracy During the Solution Stages

The transportation problem may also become degenerate during the solution stages. This happens when most favourable quantity is allocated to the empty cell having the largest negative cell-evaluation resulting in simultaneous vacation of two or more of currently occupied cells. To resolve degeneracy, allocate Δ to one or more of recently vacated cells so that the number of occupied cells is $m + n - 1$ in the new solution. This type of degeneracy can be explained below in Example 15.

Example 15. The cost-requirement table for the transportation problem is given as below :

Table 11-65

	W_1	W_2	W_3	W_4	W_5	Available
F_1	4	3	1	2	6	40
F_2	5	2	3	4	5	30
F_3	3	5	6	3	2	20
F_4	2	4	4	5	3	10
Required	30	30	15	20	5	

Solution. By 'North-West-Corner Rule', the non-degenerate initial solution is obtained in Table 11-66 :

Table 11-66

	W_1	W_2	W_3	W_4	W_5	Available
F_1	30(4)	10(3)				40
F_2		20(2)	10(3)			30
F_3			5(6)	15(3)		20
F_4				5(5)	5(3)	10
Required	30	30	15	20	5	

Now, test this solution for optimality to get the following table in usual manner.

Table 11-67

Matrix for set of u_i and v_j					
v_j	4	3	2	3	
	4	3	2	3	
4					
3					
2					
1					
-1					

Table 11-69
Matrix $(u_i + v_j)$ for empty cells

*	*	4	1	-1
3	*	*	0	-2
6	5	*	*	1
8	7	8	*	*

 u_i Matrix $[c_{ij}]$ for empty cells

0	*	*	(1)	(2)	(6)
-1	(5)	*	*	(4)	(5)
2	(3)	(5)	*	*	(2)
4	(2)	(4)	(4)	*	*

Table 11-68

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

*	*	-3	1	7
2	*	*	4	7
-3	0	*	*	1
-6	-3	-4	*	*

Since, the largest negative cell evaluation is $d_{41} = -6$ (marked \checkmark), allocate as much as possible to this cell (4, 1). This necessitates shifting of 5 units to this cell (4, 1) as directed by the closed loop in Table 11-71.

Table 11-71

	$30 - \theta$	$10 + \theta$				Available
(4)						40
						30
						20
						10
Required	30	30	15	20	5	

Here maximum possible value of θ is obtained by usual rule :

$$\min. [30 - \theta, 20 - \theta, 5 - \theta, 5 - \theta] = 0 \text{ i.e., } 5 - \theta = 0 \text{ or } \theta = 5 \text{ units.}$$

From Table 11-71, the revised solution becomes :

Table 11-72

	25(4)	15(3)				Available
		15(2)	15(3)			40
			0*(6)	20(3)		30
	5(2)			0*(5)	5(3)	20
Required	30	30	15	20	5	10

In this solution, the number of allocations becomes less than $m + n - 1$ on account of simultaneous vacation of two cells [(3, 3), (4, 4), as indicated by *]. Hence this is a degenerate solution.

Now, this degeneracy may resolve by adding Δ to one of the recently vacated cells [(3, 3) or (4, 4)]. But in minimization problem, add Δ to recently vacated cell (4, 4) only, because it has the lowest shipping cost of Rs 5 per unit.

The rest of the procedure will be exactly the same as explained earlier. This way, the optimal solution can be obtained.

Table 11.73

					Available
Required	30	30	15	20	5
	5(4)		15(1)	20(2)	
		30(2)			
	15(3)				5(2)
	10(2)				

Note. Here the aim of finding an initial solution by 'North-West Corner Rule' is only to show as to how degeneracy may arise during the solution stages. Otherwise, the optimal solution can be obtained immediately by using the initial solution by Vogel's method.

Example 16. Solve the following transportation problem :

	D_1	D_2	D_3	D_4	D_5	D_6	Available	
From	O_1	9	12	9	6	9	10	5
	O_2	7	3	7	7	5	5	6
	O_3	6	5	9	12	3	11	2
	O_4	6	8	11	2	2	10	9
		4	4	6	2	4	2	22 (Total)

[Meerut (Maths.) 97P, (Stat.) 90]

Solution. Using 'VAM' the initial basic feasible solution having the transportation cost Rs. 112 is given below :

Initial BFS

		5(9)					5
		4(3)	Δ (7)			2(5)	6 + Δ = 6
	1(6)		1(9)				2
	3(6)			2(2)	4(2)		9
	4	4	6 + Δ = 6	2	4	2	

Since the number of allocations (= 8) is less than $m + n - 1$ (= 9), a very small quantity Δ may be introduced in the independent cell (2, 3), although least cost independent cell is (2, 5).

Optimum Table

	+	+	5	+	+	+	u_i
(9)	6 (12)	5 (9)	*	(6)	2 (9)	2 (10)	7
(7)	+	4	*	+	+	2	2
(6)	1	0	1	+	+	+	0
(6)	(5)	5 (9)	(12)	2 (3)	2 (11)	7	2
4	(8)	5 (11)	9 (2)	2	4	(10)	2

Since all the net evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is : $x_{13} = 5, x_{22} = 4, x_{26} = 2, x_{31} = 1, x_{41} = 3, x_{44} = 2, x_{45} = 4, x_{33} = 1$.

The optimum transportation cost is given by

Note. In above optimum table, Δ may also be introduced in least cost independent cell (2, 5).

Example 17. Solve the following transportation problem (cell entries represent unit costs) :

						Available
						3
						4
						2
						8
Required						17 (Total)
5	3	7	3	8	5	
5	6	12	5	7	11	
2	1	3	4	8	2	
9	6	10	5	10	9	
3	3	6	2	1	2	

[Meerut (M.Com.) Jan. 98 (BP), (M.Sc.) 93 P]

Soluton. Using 'VAM', an initial BFS having the transportation cost Rs. 103 is given below :

Initial BFS

	1(3)				2(5)	
3(5)			$\Delta(5)$	1(7)		
		2(3)				
	2(6)	4(10)	2(5)			

3 3 6 $2 + \Delta = 2$ 1 2

.. Since the number of allocations (8) in the initial BFS is less than $m + n - 1 (= 9)$, introduce negligible quantity Δ in the independent cell (2, 4).

							u_i
(5)	+	1	0	+	+	2	-3
2	(3)	(7)	7	(3)	2	(5)	
3	0	+	+	Δ	1	(11)	0
(5)	(6)	6	(12)	10	(5)	(7)	
+	+	+	2	+	+	+	
(2)	-2	(1)	-1	(3)	(4)	-2	-7
5	2	4	5	2	7	7	0
(9)	(6)	(10)	(5)	(10)	(9)		
v_j	5	6	10	5	7	8	

Since all the net-evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is given by : $x_{12} = 1, x_{16} = 2, x_{21} = 3, x_{25} = 1, x_{33} = 2, x_{42} = 2, x_{43} = 4$ and $x_{44} = 2$.

The optimum transportation cost is

$$z = 1(13) + 2(5) + 3(5) + \Delta(5) + 1(7) + 2(3) + 2(6) + 4(10) + 2(5) = \text{Rs. } 103, \text{ as } \Delta \rightarrow 0.$$

Example 18. A company has 4 warehouses and 6 stores; the cost of shipping one unit from warehouse i to store j is c_{ij} .

If $C = (c_{ij}) = \begin{pmatrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & 8 \end{pmatrix}$, and the requirements of the six stores are 4, 4, 6, 2, 4, 2 and quantities at

warehouse are 5, 6, 2, 9. Find the minimum cost solution.

[IAS (Main) 95]

Solution. Using 'lowest cost entry method', an initial solution having the transportation cost Rs 70 is obtained as below :

		5(7)						
	4(1)	Δ (5)					2(3)	
2(4)								
2(4)		1(9)		2(0)		4(0)		
4	4	6 + Δ = 6		2		4	2	

Since the number of allocations (8) in the initial BFS is less than $m + n - 1 (= 9)$, introduce a negligible quantity Δ in the independent empty cell (2, 3).

Starting Iteration Table

	+	+	5	+	+	+	+	u_i
v_j	7	2	(10)	3	(7)	(4)	-2	5
			4 - θ		Δ + θ			2
		0	(1)		(5)	(5)	-4	(3)
	2	-θ		-2		-2		+0
	(4)		(3)	5	(7)	9	(9)	7
	2	+θ			1 - θ		2	7
	(4)		(6)	5	(9)	(0)	5	0
	4	5	9	0	0	0	4	0
							(8)	7

$$\text{Min. } [4 - \theta, 2 - \theta, 1 - \theta] = 0 \Rightarrow \theta = 1.$$

Since all the net-evaluations for the non-basic (empty) cells are not non-negative, the initial BFS is not optimal. The empty cell (3, 2) must be allocated the maximum possible amount $\theta = 1$ to this cell. Consequently, cell (4, 3) becomes empty.

First Iteration Table. Vacate the cell (4, 3) and occupy the cell (3, 2).

	+	+	5	+	+	+	+	u_i
v_j	(7)	4	(10)	3	(7)	(4)	0	5
			3		1			2
		2	(1)		(5)	(5)	-2	(3)
	1				0			0
	(4)		(3)		(7)	7	(9)	5
	3				+	+	2	2
	(4)		(6)	3	(9)	7	(0)	5
	2	1	5	7	0	-2	-2	3
								2

Since all the net evaluations are non-negative, the current solution is optimum. Hence the optimum solution is given by $x_{13} = 5, x_{22} = 3, x_{23} = 1, x_{26} = 2, x_{31} = 1, x_{32} = 1, x_{41} = 3, x_{44} = 2, x_{45} = 4$.

The optimum transportation cost is given by

$$z = 5(7) + 3(1) + 1(5) + 2(3) + 1(4) + 3(4) + 1(3) + 2(0) + 4(0) = \text{Rs. 68.}$$

- Q. 1. Explain "Degeneracy" in a transportation problem. [Bharathidasan B.Sc (Math) 90; Delhi B.Sc (Math.) 90]
 2. How does the problem of degeneracy arise in a transportation Problem ? Explain how does one overcome it. [Meerut 94]
 3. Explain how to solve the degeneracy in transportation problems.

EXAMINATION PROBLEMS

(ON DEGENERACY)

1. Explain : (i) a method of constructing a basic solution of a transportation problem, (ii) the technique of improvement of the constructed basic solution. Solve the following transportation problem :

Cost-matrix

From	To			Available
	0	2	0	
Required	70	50	30	150

[Hint. Use 'VAM' to find initial BFS and prove it to be optimal.]

[Ans. $x_{11} = 20, x_{12} = 50, x_{23} = 30, x_{31} = 50$; min. cost = Rs. 100].

2. Determine the optimal solution to each of the following degenerate transportation problem :

(i)

	D_1	D_2	D_3	D_4	D_5	$a_i \downarrow$
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
$b_j \rightarrow$	8	3	7	2	2	

[Ans. $x_{11} = 1, x_{13} = 1, x_{21} = 7, x_{15} = 2, x_{32} = 3, x_{33} = 6, x_{44} = 2$; min. cost = 56].

(ii)

Demand points (iii)

	D_1	D_2	D_3	D_4	Supply
Sources	2	3	11	7	6
S_1	1	0	6	1	1
S_2	5	8	15	10	10
Demand	7	5	3	2	17

[Ans. $x_{12} = 5, x_{13} = 1, x_{24} = 1, x_{31} = 7, x_{33} = 2, x_{34} = 1$.

min cost = Rs. 102. Alternative solutions also exist.]

From	To				$a_i \downarrow$
	10	7	3	6	
	1	6	7	3	5
	7	4	5	3	7
$b_j \rightarrow$	3	2	6	4	

[Ans. $x_{13} = 3, x_{21} = 3, x_{24} = 2, x_{32} = 2, x_{33} = 3, x_{34} = 2$

min. cost Rs. 47].

3. A manufacturer has distribution centres located at Agra, Allahabad and Calcutta. These centres have available 40, 20 and 40 units of his product. His retail outlets require the following number of units : A, 25; B, 10; C, 20; D, 30; E, 15. The shipping cost per unit in rupees between each centre and outlet is given in the following table :

Distribution Centres	Retail Outlets				
	A	B	C	D	E
Agra	55	30	40	50	40
Allahabad	35	30	100	45	60
Calcutta	40	60	95	35	30

Determine the optimal shipping cost.

[Ans. $x_{12} = 10, x_{13} = 20, x_{15} = 10, x_{21} = 20, x_{31} = 5, x_{34} = 30, x_{35} = 5$, min. cost = Rs. 3600].

4. A manufacturer wants to ship 8 loads of his product as shown in the table. The matrix gives the mileage from origin O to destination D. Shipping costs are Rs. 10 per load per mile. What shipping schedule should be used.

[Hint. Find the initial solution by using Vogel's method. In this solution, number of allocations is less than $m + n - 1$ (i.e., 5). Hence resolve the degeneracy by introducing Δ to one of the empty cells [say, (2, 2)]. Then this initial solution will be optimal with minimum mileage 820 or cost Rs. 8200].

[Ans. $x_{11} = 1, x_{21} = 3, x_{32} = 2, x_{33} = 2$]

	D_1	D_2	D_3	Available
Q_1	50	30	220	1
Q_2	90	45	170	3
Q_3	250	200	50	4
Required	4	2	2	

[IAS (Main) 91]

5. (i) Using North-West Corner Rule for initial basic feasible solution, obtain an optimum basic feasible solution to the following degenerate transportation problems :

	To			Available
From	7	3	4	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

(iii)

	D_1	D_2	D_3	a_i
O_1	0	2	1	5
O_2	2	1	5	10
O_3	2	4	3	5
$b_j \rightarrow$	5	5	10	

[Ans. $x_{13} = 2, x_{22} = 1, x_{23} = 2, x_{31} = 4, x_{33} = 1$, min. cost = 33]

6. Solve the transportation problem whose cost matrix

[Ans. $x_{21} = 5, x_{22} = 5, x_{23} = 5, x_{33} = 5$ and min. cost = 35]

7. Obtain the optimal solution by using the best starting solution.

	To				Available
From	5	5	4	7	5
	6	4	1	2	5
	5	9	1	4	6
	8	3	2	4	4
Required	5	8	3	10	6

	To				a_i
From	10	20	5	7	10
	13	9	12	8	20
	4	15	7	9	30
	14	7	1	0	40
	3	12	6	19	50
$b_j \rightarrow$	60	60	20	10	

[Ans. $x_{11} = 2, x_{12} = 3, x_{22} = 1, x_{24} = 4, x_{31} = 3, x_{33} = 3, x_{42} = 4, x_{54} = 6$, [Ans. $x_{13} = 10, x_{22} = 20, x_{31} = 30, x_{42} = 20, x_{43} = 10$, and min. cost = 73]

and min. cost = 830]

8. A company has 4 warehouses and 6 stores., the surplus in the warehouses, the requirements of the stores and costs (in Rs) of transporting one unit of the commodity from warehouse i to the store j are given below. How should the commodity be transported so that the total transportation cost is a minimum ? Obtain the initial program by applying the north-west corner rule :

Warehouse	Store						Surplus
	1	2	3	4	5	6	
1	7	5	9	5	10	7	30
2	7	8	24	7	9	13	40
3	4	10	5	6	10	4	20
4	11	8	12	7	12	11	80
Requirement	30	30	60	20	10	20	170

[Delhi B.Sc. (Math.) 90]

[Ans. $x_{12} = 10, x_{16} = 20, x_{21} = 30, x_{25} = 10, x_{33} = 20, x_{42} = 20, x_{43} = 40, x_{44} = 20$, and min. cost = Rs. 1370]

9. Given below is the unit costs array with supplies $a_i, i = 1, 2, 3$ and demands $b_j, j = 1, 2, 3, 4$.

source	Sink				a_i
	1	2	3	4	
1	8	10	7	6	50
2	12	9	4	0	40
3	9	11	10	8	30
b_j	25	32	40	23	120

Find the optimal solution to the above Hitchcock problem.

[Meerut 2002]

11.12. UNBALANCED TRANSPORTATION PROBLEMS

So far we have discussed the *balanced* type of transportation problems where the total destination requirement equals the total origin capacity (*i.e.*, $\sum a_i = \sum b_j$). But, sometimes in practical situations, the demand may be more than the availability or *vice versa* (*i.e.*, $\sum a_i \neq \sum b_j$).

Thus, if in a transportation problem, the sum of all available quantities is not equal to the sum of requirements, that is, $\sum_{i=1}^m a_i \neq \sum_{j=1}^n b_j$, then such problem is called an *unbalanced transportation problem*.

11.12-1. To Modify Unbalanced T.P. to Balanced Type

An unbalanced T.P. may occur in two different forms. (i) *Excess of availability*, (ii) *Shortage in availability*.

We now discuss these two cases by considering our usual m -origin, n -destination T.P. with the condition that $\sum_{i=1}^m a_i \neq \sum_{j=1}^n b_j$.

Case 1. (Excess Availability, i.e. $\sum a_i \geq \sum b_j$).

The general T.P. may be stated as follows :

$$\text{Minimize } z = \sum_{i=1}^m \sum_{j=1}^n x_{ij} c_{ij}, \text{ subject to the constraints}$$

$$\sum_{j=1}^n x_{ij} \leq a_i, \quad i = 1, 2, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, \dots, n$$

and

$$x_{ij} \geq 0, \quad i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n.$$

The problem will possess a feasible solution if $\sum a_i \geq \sum b_j$. In the first constraint, the introduction of slack variable $x_{i,n+1}$ ($i = 1, 2, \dots, m$) gives

$$\sum_{j=1}^n x_{ij} + x_{i,n+1} = a_i; \quad i = 1, 2, \dots, m$$

$$\text{or } \sum_{i=1}^m \left(\sum_{j=1}^n x_{ij} + x_{i,n+1} \right) = \sum_{i=1}^m a_i \quad \text{or} \quad \sum_{j=1}^n \left(\sum_{i=1}^m x_{ij} \right) + \sum_{i=1}^m x_{i,n+1} = \sum_{i=1}^m a_i$$

$$\text{or } \sum_{i=1}^m x_{i,n+1} = \sum_{i=1}^m a_i - \sum_{j=1}^n b_j = \text{Excess of Availability.} \quad \left(\because \sum_{i=1}^m x_{ij} = b_j \right)$$

If this *excess availability* is denoted by b_{n+1} , the modified general T.P. can be reformulated as :

$$\text{Minimize } z = \sum_{i=1}^m \sum_{j=1}^{n+1} x_{ij} (c_{ij}), \text{ subject to the constraints :}$$

$$\sum_{j=1}^{n+1} x_{ij} + x_{i,n+1} = a_i, \quad i = 1, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, \dots, n+1$$

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j,$$

$$\text{where } c_{i,n+1} = 0 \text{ for } i = 1, 2, \dots, m \text{ and } \sum_{i=1}^m a_i = \sum_{j=1}^{n+1} b_j.$$

This is clearly the *balanced T.P.* and thus can be easily solved by transportation algorithm.

Working Rule : Whenever $\sum a_i \geq \sum b_j$, we introduce a dummy destination-column in the transportation table. The unit transportation costs to this dummy destination are all set equal to zero. The requirement at this dummy destination is assumed to be equal to the difference $\sum a_i - \sum b_j$.

Case 2. (Shortage in Availability, i.e. $\sum a_i \leq \sum b_j$).

In this case, the general T.P. becomes :

$$\text{Minimize } z = \sum_{i=1}^m \sum_{j=1}^n x_{ij} (c_{ij}) \text{ subject to the constraints :}$$

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, \dots, m$$

$$\sum_{i=1}^m x_{ij} \leq b_j, \quad j = 1, \dots, n$$

$$x_{ij} \geq 0, \quad i = 1, \dots, m; j = 1, \dots, n.$$

Now, introducing the slack variable $x_{m+1,j}$ ($j = 1, \dots, n$) in the second constraint, we get

$$\sum_{i=1}^m x_{ij} + x_{m+1,j} = b_j, \quad j = 1, \dots, n$$

or

$$\sum_{j=1}^n \left(\sum_{i=1}^m x_{ij} + x_{m+1,j} \right) = \sum_{j=1}^n b_j \quad \text{or} \quad \sum_{i=1}^m \left(\sum_{j=1}^n x_{ij} \right) + \sum_{j=1}^n x_{m+1,j} = \sum_{j=1}^n b_j$$

or

$$\sum_{j=1}^n x_{m+1,j} = \sum_{j=1}^n b_j - \sum_{i=1}^m a_i = \text{shortage in availability } a_{m+1}, \text{ say.}$$

Thus the modified general T.P. in this case becomes :

$$\text{Minimize } z = \sum_{i=1}^{m+1} \sum_{j=1}^n x_{ij} c_{ij},$$

subject to the constraints :

$$\sum_{j=1}^n x_{ij} = a_i, \quad i = 1, \dots, m+1$$

$$\sum_{i=1}^m x_{ij} + x_{m+1,j} = b_j, \quad j = 1, \dots, n$$

$$x_{ij} \geq 0, \quad i = 1, \dots, m+1; j = 1, \dots, n.$$

where $c_{m+1,j} = 0$ for $j = 1, \dots, n$ and $\sum_{i=1}^{m+1} a_i = \sum_{j=1}^n b_j$.

Working Rule : Whenever $\sum a_i \leq \sum b_j$, introduce a dummy source in the transportation table. The cost of transportation from this dummy source to any destination are all set equal to zero. The availability at this dummy source is assumed to be equal to the difference $(\sum b_j - \sum a_i)$.

Thus, an unbalanced transportation problem can be modified to balanced problem by simply introducing a fictitious sink in the first case and a fictitious source in the second. The inflow from the source to a fictitious sink represents the surplus at the source. Similarly, the flow from the fictitious source to a sink represents the unfilled demand at that sink. For convenience, costs of transporting a unit item from fictitious sources or to fictitious sinks (as the case may be) are assumed to be zero. The resulting problem then becomes balanced one and can be solved by the same procedure as explained earlier. The method for dealing with such type of problems will be clear in Example 19.

- Q. 1. What is unbalanced transportation problem? How do you start in this case?
- 2. Explain the technique used to solve the transportation problem with the following restrictions imposed separately in each problem :
 - (i) it is required to have all basic solutions non-degenerate,
 - (ii) it is required to have no allocation in the (i, j) th cell,
 - (iii) it is required to have some positive allocation in the (i, j) th cell.
 - (iv) it is found that the total units ready to ship be less than the total units required.

Example 19. XYZ tobacco company purchases tobacco and stores in warehouses located in the following four cities :

Warehouse location	Capacity (tonnes)
City A	90
City B	50
City C	80
City D	60

The warehouses supply tobacco to cigarette companies in three cities that have the following demand :

Cigarette company	Demand (tonnes)
Bharat	120
Janta	100
Red Lamp	110

The adjoining railroad shipping costs per tonne (in hundred rupees) have been determined :

To	Bharat	Janta	Red Lamp
From			
A	7	10	5
B	12	9	4
C	7	3	11
D	9	5	7

Because of railroad construction, shipments are temporarily prohibited from warehouse at city A to Bharat Cigarette Company.

(i) Find the optimum distribution for XYZ Tobacco Company.

(ii) Are there multiple optimum solutions ? If there are alternative optimum solutions, identify them.

(iii) Write the dual of the given transportation problem and use it for checking the optimum solution.

[Punjab (M.B.A.,) 97]

Solution. (i) The given information is exhibited in the following table. Further, the problem is balanced by adding a dummy row and the cost element (of 7) in the given table corresponding to the cell (A—Bharat) is replaced by M , since the route is prohibited. Using VAM, the initial feasible solution is obtained and which, when tested for optimality is found to be optimum.

Initial Soln. is Optimal

	Bharat	Janta	Red Lamp	Supply u_i
A	+ ($M - 13$)	+ 1	90	90 1
(M)	13 (10)	9 (5)		
B	30 (12)	+ 1 (9)	20 (4)	50 0
C	0 (7)	80 (3)	+ 12 (11)	80 - 5
D	40 (9)	20 (5)	- 1 (7)	60 - 3
Dammy	50 (0)	+ 4 (0)	+ 8 (0)	50 - 12
Demand	120	100	110	
$v_j \rightarrow$	12	8	4	

Since opportunity cost in all the unoccupied cells is positive, initial solution shown in the above table is also an optimum solution. The total transport cost associated with this solution is :

$$5 \times 90 + 12 \times 30 + 4 \times 20 + 3 \times 80 + 9 \times 40 + 5 \times 20 = 1,59,000$$

(b) Since opportunity cost in cell (C, Bharat) $\Delta_{31} = 0$, there exists an alternative optimum solution :

$$x_{13} = 90, x_{21} = 30, x_{23} = 20, x_{31} = 40, x_{42} = 60, \text{total cost} = \text{Rs. } 1,59,000.$$

(c) The dual of the given problem is :

Maximize $Z = (90 u_1 + 50 u_2 + 80 u_3 + 60 u_4 + 50 u_5) + (120 v_1 + 100 v_2 + 110 v_3)$
 subject to the constraints

$$\begin{array}{lll} u_1 + v_1 \leq M, & u_2 + v_3 \leq 4 & u_4 + v_2 \leq 5 \\ u_1 + v_2 \leq 10 & u_3 + v_1 \leq 7 & u_4 + v_3 \leq 7 \\ u_1 + v_3 \leq 5 & u_3 + v_2 \leq 3 & \\ u_2 + v_1 \leq 12 & u_3 + v_3 \leq 11 & \\ u_2 + v_2 \leq 9 & u_4 + v_1 \leq 9 & \end{array}$$

u_i, v_j unrestricted in sign ($i = 1, 2, 3$, and $j = 1, 2, 3, 4$). Using optimum values of u_i 's and v_j 's in the objective function, we get

$$90(1) + 50(0) + 80(-5) + 60(-3) + 50(-12) + 120 \times 12 + 100 \times 8 + 110 \times 4 = \text{Rs. } 1,59,000,$$

which is the same value as obtained earlier.

Example 20. A company has four terminals U, V, W and X. At the start of a particular day 10, 4, 6 and 5 trailors are available at these terminals. During the previous night 13, 10, 6 and 6, trailors respectively were loaded at plants A, B, C and D. The company despatcher has come up with the costs between the terminals and plants as follows :

The sale price in rupees per unit and the demand in kg. per unit time are as follows :

Sales centre	Sale price (Rs.) per unit	Demand (kg.) per unit
1	15	120
2	14	140
3	16	60

[Delhi (MCI) 2000; C.A., Nov. 97]

Solution : The profit matrix is obtained along with demand and supply and the balanced matrix is given below :

Table 11.73 : Profit Matrix

Factory	Sales Centre			Supply
	1	2	3	
A	3	2	4	100
B	0	-1	1	20
C	4	3	5	60
D	2	1	3	80
E (Dummy)	0	0	0	60
Demand	120	140	60	320

The table 11.73 representing profit can be converted to an equivalent minimization of loss by subtracting all the profit values in the table from the highest profit value (i.e., 5). The initial basic feasible solution can be found by using Vogel's method. Then obtain optimum solution.

[Ans. $x_{11} = 4$, $x_{13} = 6$, $x_{22} = 3$, $x_{24} = 1$, $x_{32} = 6$, $x_{44} = 5$, (Dummy) $x_{51} = 9$, $x_{52} = 1$. Min. transportation cost is Rs. 555.]

Example 21. A multi-plant company has three manufacturing plants, A, B and C and two markets X and Y. Production cost at A, B and C is Rs. 1,500, 1,600 and 1,700 per piece respectively. Selling prices in X and Y are Rs. 4,400 and Rs. 4,700 respectively. Demands in X and Y are 3,500 and 3,600 pieces respectively. Production capacities at A, B and C are 2,000, 3,000 and 4,000 pieces respectively. Transportation costs are as shown in the adjacent table. Build a mathematical model.

Solution. Here three plants differ in production cost and we are given the selling prices also. Therefore, our problem is to determine the schedule of production which may result in the maximum profit. The various profits per item are as shown in the adjacent table.

Plant	Market	
	X	Y
A	1,000	1,500
B	2,000	3,000
C	1,500	2,500

Plant	Market	
	X	Y
A	1,900	1,700
B	800	100
C	1,200	500

The profit (selling price – production cost – transportation cost) data from plants to markets are shown below :

From A to X : $4400 - 1500 - 1000 = 1900$; from A to Y : $4700 - 1500 - 1500 = 1700$;

from B to X : $4400 - 1600 - 2000 = 800$; and so on.

Further, total production at A, B and C plants is $2,000 + 3,000 + 4,000 = 9,000$ units while total requirement at X and Y is $3,500 + 3,600 = 7,100$ units. Hence this is an unbalanced transportation problem. By introducing a dummy market Z to receive an excess production of $9,000 - 7,100 = 1,900$ units, the complete relevant information is summarized in the following table :

	Market			Supply
	X	Y	Dummy	
Plant A	(1900)	(1700)	(0)	2000
Plant B	(800)	(100)	(0)	3000
Plant C	(1200)	(500)	(0)	4000
Demand	3500	3600	1900	9000

Let x_{ij} be the quantity to be transported from plant i ($i = 1, 2, 3$) to market j ($j = 1, 2, 3$).

Now the mathematical model based on the given data can be formulated as follows :

Maximize (total profit) $Z = 1900x_{11} + 1700x_{12} + 800x_{21} + 100x_{22} + 1200x_{31} + 500x_{32}$
subject to the constraints

$$\left. \begin{array}{l} x_{11} + x_{12} + x_{13} = 2000 \\ x_{21} + x_{22} + x_{23} = 3000 \\ x_{31} + x_{32} + x_{33} = 4000 \end{array} \right\} \text{(Supply constraints)} \quad \left. \begin{array}{l} x_{11} + x_{21} + x_{31} = 3500 \\ x_{12} + x_{22} + x_{32} = 7600 \\ x_{13} + x_{23} + x_{33} = 1900 \end{array} \right\} \text{(Demand constraints)}$$

and $x_{ij} \geq 0$ for i and j .

Example 22. A steel company has three open hearth furnaces and five rolling mills. Transportation cost (rupees per quintal) for shipping steel from furnaces to rolling mills are shown in the following table :

Table 11-74
Mills

	F_1	M_1	M_2	M_3	M_4	M_5	Capacities (in quintals)
		4	2	3	2	6	
Furnaces	F_2	5	4	5	2	1	12
	F_3	6	5	4	7	3	14
Requirement (in quintals)		4	4	6	8	8	

[JNTU (B. Tech.) 2003; Meerut (Maths.) 2003, 91]

What is the optimal shipping schedule ?

Solution. Since the total requirements of mills are 30 quintals and the total capacities of all furnaces are 34 quintals, the problem is of unbalanced type. Therefore, the problem can be modified as follows.

Step 1. (Modifying the given problem to balanced type).

Since the capacities are four quintals more than the total requirements, consider a fictitious mill requiring four quintals of steel. Thus the modified (balanced) transportation cost matrix becomes :

Table 11-75

	M_1	M_2	M_3	M_4	M_5	M_f	Capacities
	4	2	3	2	6	0	
F_1	5	4	5	2	1	0	12
F_2	6	5	4	7	3	0	14
Required	4	4	6	8	8	4	34

Step 2. (To find the initial solution).

Applying the Vogel's method in the usual manner, the initial solution is obtained as given below.

Table 11.76

	M_1	M_2	M_3	M_4	M_5	M_f
F_1		4(2)		4(2)		
F_2				4(2)	8(1)	
F_3	4(6)		6(4)			4(0)

This gives the transportation cost = $4(2) + 4(2) + 4(2) + 4(2) + 8(1) + 4(6) + 6(4) + 4(0) = \text{Rs. } 80$.

Step 3. (To test the initial solution for optimality).

Since the total number of allocations is 7 (instead of $6 + 3 - 1 = 8$), this is a degenerate basic feasible solution. Therefore, allocate an infinitesimal quantity Δ to empty cell (1, 1). Then, proceeding in the usual manner, following tables for testing the optimality of the solution are obtained.

Table 11.77

		u_i and v_j					u_i
		•(4)	•(2)	•(2)	•(1)	•(0)	0
		4	2	2	1	-2	v_j
							0
							0
							2

Table 11.78

$(u_i + v_j)$ for empty cells							0
•	•	2	•	1	-2		0
4	2	2	•	•	-2		0
•	4	•	4	3	•		2
4	2	2	2	1	-2		

Table 11.79

$d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells

$d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells						
•	•	1	•	5	8	
1	2	3	•	•	2	
•	1	•	3	0	•	

Since all $d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells are non-negative, the solution under test is optimal. Further, 0 in the cell (3,5) indicates that alternative solutions will also exist.

Example 23. A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500, and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in Rs.) are given below :

Determine an optimum distribution for the company in order to minimize the total transportation cost.

		To	D	E	F	G	H
From		A	5	8	6	6	3
	B	4	7	7	6	5	5
	C	8	4	6	6	4	900

[JNTU 2003, 02, 98; AIMS 2002; Kerala B.Sc (Math.) 90]

Solution. In this problem, the total warehouse requirements (= 2500 units) is greater than the total plant capacity (= 2200 units). Therefore, the problem is of unbalanced type. So introduce a dummy plant P having all transportation costs equal to zero and having the plant availability equal to $(2500 - 2200) = 300$ units. The modified transportation table is thus obtained as below :

		To	D	E	F	G	H	Plant Capacity
From		A	5	8	6	6	3	800
	B	4	7	7	6	5	500	
	C	8	4	6	6	4	900	
	P	0	0	0	0	0	300	
Requirements	400	400	500	400	800			

Using 'VAM' the following initial BFS is obtained :

	D	E	F	G	H	
A			500(6)		300(3)	800
B	400(4)			100(6)	$\Delta(5)$	500 + $\Delta = 500$
C		400(4)			500(4)	900
P				300(0)		300
400	400	500	400	800 + $\Delta = 800$		

Since the number of basic cell allocations ($= 7$) is less than $m + n - 1$ ($= 8$), the solution is degenerate. To make the number of allocations equal to 8, introduce a negligible small positive quantity Δ in the independent cell (2, 5). Now test the current solution for optimality.

Starting Table*

	+	+	$500 - \theta$	+	$300 + \theta$	u_i
(5)	2 (8)	3 (6)		(6)	4 (3)	-2
(4)	400		+ -1	100 + θ	$\Delta - \theta$	0
(8)		400	-1		(5)	0
(0)	3 (4)	(6)	7 (6)	5 (4)		-1
$v_j \rightarrow$	4	5	8	6	5	-6

Here $\theta = \min [500, \Delta, 300] = \Delta$. So enter the non-basic cell (4, 3) and leave the basic cell (2, 5).

First Iteration Table. Vacate the cell (2, 5) and occupy the cell (4, 3).

	+	+	$500 - \theta$	0	$300 + \theta$	u_i
(5)	4 (8)	3 (6)		(6)	6 (3)	0
(4)	400		+ +	100		+
(8)		400	-1	-1	500 - θ	0
(0)	5 (4)	(6)	7 (6)	7 (4)		1
v_j	4	3	6	6	3	-6

Here $\min [500 - \theta, 500 - \theta, 300 - \theta] = 0$ or $300 - \theta = 0$ or $\theta = 300$.

So introduce the cell (3, 4) and drop the cell (4, 4) in the next iteration.

Second Iteration Table. Vacate the cell (4, 4) and occupy the cell (3, 4).

	+	+	$200 - \theta$	+	$600 + \theta$	u_i
(5)	3 (8)	3 (6)		(6)	5 (3)	-1
(4)	400		0	100		+
(8)		400	-1	300	200 - θ	0
(0)	4 (4)	(6)	7 (6)	(4)		0
$v_j \rightarrow$	4	4	7	6	4	-7

$$\text{Here } \min [200 - \theta, 200 + \theta] = 0 \Rightarrow \theta = 200.$$

So introduce the cell $(3, 3)$ and drop the cell $(1, 3)$ or $(3, 5)$ in the next iteration.

Third Iteration Table. Vacate the cell $(1, 3)$ or $(3, 5)$ and occupy the cell $(3, 3)$.

Optimum Table

Optimum Table						
	+	+	0	0	800	*
(5)	4	(8)	4	(6)	(6)	6 (3)
	400*		+	+	100*	+
(4)		(7)	4	(7)	6 (6)	(5) 3
	+	400*		200*	300*	+
(8)	4	(4)		(6)	(6)	(4) 3
	+		+		0	+
(0)	-2	(0)	-2	(0)	(0)	0 (0) -3
	4	4	6	6	3	-6

Since all the net evaluations are non-negative, the optimum solution is :

$$x_{11} = 0, x_{12} = 800, x_{21} = 400, x_{22} = 100, x_{32} = 400, x_{33} = 200, x_{34} = 300, x_{43} = 300.$$

The optimum transportation cost is given by

$$z = 0(6) + 800(3) + 400(4) + 100(6) + 400(4) + 200(6) + 300(6) + 300(0) = \text{Rs. } 9200.$$

Example 24. The Bombay Transport Company has trucks available at four different sites in the following numbers :

<i>Site</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>No. of Trucks</i>	5	10	7	3

Customers W, X, and Y require trucks as shown:

Customer : W X Y
 No. of Trucks : 5 8 10

Variable costs of getting trucks to the customers are:

From A to W → Rs 7, to X → Rs 3, to Y → Rs 6; From B to W → Rs 4, to X → Rs 6, to Y → Rs. 8;

From A to W → Rs 7, to X → Rs 4, to Y → Rs 4; From C to W → Rs 5, to X → Rs 8, to Y → Rs. 4; From D to W → Rs 8, to X → Rs 4, to Y → Rs. 3.

Solve the above transportation problem.

Solution. Since the availability of trucks is greater than the requirement of trucks, the problem is of unbalanced type. Therefore, a dummy requirement column having all the transportation costs equal to zero has been introduced with $25 - 23 (= 2)$ trucks as its requirement. The following balanced transportation table is then constructed :

		To				
		W	X	Y	Z	Available
From	A	7	3	6	0	5
	B	4	6	8	0	10
	C	5	8	4	0	7
	D	8	4	3	0	3
Requirement		5	8	10	2	

Initial BFS Table

	5(3)		
5(4)	3(6)	2(8)	
		5(4)	2(0)
		3(3)	

Now, using 'VAM', the initial basic feasible solution having transportation cost Rs 58 is obtained:

Starting Table. The solution under test is not optimum. The most negative cell evaluation is -4 , so introduce the cell $(2, 4)$.

Here, $\min[2 - \theta, 2 - \theta] = 0 \Rightarrow \theta = 2$.

Therefore, drop either the cell $(2, 3)$ or $(3, 4)$. The resulting solution degenerates.

	+ 5 +	5 • 3 +	+	-1 $\sqrt{-4}$ + θ 2 - θ 5 + θ 2 - θ	u_j -3 0 -4 +
(7)	1	(3)	(6)	5 (0)	-3
(4)	5	3	2 - θ	$\sqrt{-4}$	0
(5)	0	(8)	2 (4)	(0)	-4
(8)	-1	(4)	1 (3)	1 (0)	-5
v_i	4	6	8	4	

First Iteration Table. Vacate the cell $(3, 4)$ and occupy the cell $(2, 4)$ with $\theta = 2$.

Since all the net-evaluations are non-negative, the current solution is optimum. The optimum solution is given by $x_{12} = 5, x_{21} = 5, x_{22} = 3, x_{23} = 0, x_{24} = 2, x_{33} = 7, x_{43} = 3$.

The optimum transportation cost is given by $z = 5(3) + 5(4) + 3(6) + 2(0) + 7(4) + 3(3) = \text{Rs. } 90$.

	+ 5 • 3 0 7 3 +	+	+ 2 • 0 • 7 • 0	u_i ↓ -3 0 -4 -
(7)	1	(3)	(6)	5 (0) -3
(4)	5	3	0	2 •
(5)	0	(8)	2 (4)	(0) -4
(8)	-1	(4)	1 (3)	(0) -5
v_i →	4	6	8	0

Example 25. Consider the following unbalanced transportation problem :

Since there is not enough supply, some of the demands at these destinations may not be satisfied. Suppose there are penalty costs for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations 1, 2 and 3 respectively. Find the optimal solution.

	To			Supply
From	1	2	3	
1	5	1	7	10
2	6	4	6	80
3	3	2	5	15
Demand	75	20	50	

Solution. In this problem, demand exceeds the supply. The problem is of unbalanced type. Therefore, introduce a 'dummy source' whose transportation costs are given as 5, 3 and 2 respectively, and supply $145 - 105 = 40$. The modified transportation table is then constructed as follows :

Using 'VAM', an initial basic feasible solution having the transportation cost Rs. 595 is obtained as given below :

	To			Supply
From	1	2	3	
1	5	1	7	10
2	6	4	6	80
3	3	2	5	15
4	5	3	2	40
Demand	75	20	50	

Initial BFS Table			10
	10(1)		10
60(6)	10(4)	10(6)	80
15(3)			15
		40(2)	40
75	20	50	

Optimum Table			u_i
	+	10	+
(5)	3	(1)	(7) 3
(6)	60	10	10
(3)	15		+
(5)	2	(3)	40
v _j	6	4	6

Since all the net-evaluations are non-negative, the optimum solution is given by

$$x_{12} = 10, x_{21} = 60, x_{22} = 10, x_{23} = 10, x_{31} = 15 \text{ (allocation in dummy row is not considered)}$$

$$\text{The optimum transportation cost is given by } z = 0(1) + 60(6) + 10(4) + 10(6) + 15(3) = \text{Rs. 515.}$$

Example 26. A company has received a contract to supply gravel for three new construction projects located in towns A, B and C. Construction engineers have estimated required amount of gravel which will be needed at these construction projects :

Project location	:	A	B	C
Weekly requirement (truck loads)	:	72	102	41

The company has three gravel pits in towns W, X and Y. The gravel required by the construction projects can be supplied by three pits. The amount of gravel which can be supplied by each pit is as follows :

Plant	:	W	X	Y
Amount available (truck loads)	:	76	82	77

The company has computed the delivery cost from each pit to each project site. These costs (in Rs) are shown in the following table :

Schedule the shipment from each pit to each project in such a manner so as to minimize the total transportation cost within the constraints imposed by pit capacities and project requirements. Find the minimum cost. Is the solution unique ? If it is not, find alternative schedule with the same minimum cost.

Solution. In this problem, total availability exceeds the total requirement of trucks. So the problem is of unbalanced type. Therefore, introduce a dummy project location D where all the transportation costs are zero and the requirement of the new project location is equal to the difference $(235 - 215) = 20$ units. Then the modified (balanced) transportation table becomes :

Project Location			
	A	B	C
W	4	8	8
X	16	24	16
Y	8	16	24

	A	B	C	D	Available
W	4	8	8	0	76
X	16	24	16	0	62
Y	8	16	24	0	77
Requirement	72	102	41	20	

Initial BFS			
	35(8)	41(8)	
	62(24)		20(0)
72(8)	5(16)		

Now using 'VAM', the initial BFS having the transportation cost of Rs. 2752 is obtained as below.

Starting Iteration. Since all the net-evaluations are not non-negative, the initial B.F.S. is not optimum and can be improved further.

Here $\min [62 - \theta, 41 - \theta] = 0 \Rightarrow \theta = 41$. So the cell (1, 3) should leave the basis and the non-basic cell (2, 3) must enter the basis.

	+ 0	35 + θ (8)	41 - θ (8)	(0) - 16	u_i ↓ 8
(4)	0	62 - θ (21)	-8 (16)	20 (0)	24
(16)	16	5 (16)	(24)	16 (0)	16
v_j → -8	0	0	0	-24	

First Iteration Table. Vacate the cell (1, 3) and occupy the cell (2, 3).

Since all the $d_{ij} \geq 0$, the optimum solution is : $x_{12} = 76, x_{22} = 21, x_{23} = 41, x_{31} = 72, x_{32} = 5$, ($x_{24} = 20$ is dummy so it is neglected). This solution gives the minimum cost of Rs. 2424. Further, since $d_{21} = 0$ for the empty cell (2, 1), the solution obtained above is not unique. If we again transfer $\theta = 21$ units to empty cell (2, 1) and vacate the cell (2, 2), we at once obtain the alternative schedule : $x_{12} = 76, x_{21} = 21, x_{23} = 41, x_{31} = 51, x_{32} = 26$ with the same minimum cost Rs. 2424.

Example 27. A company produces a small component for all industrial products and distributes it to five wholesalers at a fixed delivered price of Rs. 2.50 per unit. Sales forecasts indicate that monthly deliveries will be 3000, 3000, 10000, 5000, 4000 units to wholesalers 1, 2, 3, 4 and 5 respectively. The monthly production capacities are 5000, 10000 and 12500 at plants 1, 2 and 3 respectively. The direct costs of production of each unit are Rs. 1, Rs. 0.90 and Re 0.80 at plants 1, 2 and 3 respectively. The transportation costs of shipping a unit from a plant to a whole-saler are given below :

Find how many components of each plant supplies to each wholesaler in order to maximize profit.

[CA (May) 2000]

Solution. Since the total capacity of plants is more than the supply to the wholesalers by a quantity $27,500 - 25,000 = 2,500$ units, so the problem is of unbalanced type. Introduce a dummy wholesaler supplying 2500 units with all the transportation costs from the plants to this destination are assumed to be zero. Also the direct costs of production of each unit are given as Re 1, Re. 0.90 and Rs. 0.80 at plants 1, 2 and 3 respectively. The modified balanced transportation problem is now obtained as follows :

Plant	Wholesaler	Wholesaler				
		1	2	3	4	5
1	·05	·07	·10	·15	·15	
2	·08	·06	·09	·12	·14	
3	·10	·09	·08	·10	·15	

Plant	Wholesaler	1	2	3	4	5	6	Supply
		1.05	1.07	1.10	1.15	1.15	0	
1								5,000
2								10,000
3								12,500
		3,000	3,000	10,000	5,000	4,000	2,500	

For simplicity in computation, multiply all the transportation costs in the table by 100, and consider 100 units = 1 unit of items. So, simplified transportation table becomes :

		Wholesalers						Capacity
Plant	Supply	1	2	3	4	5	6	50
		105	107	110	115	115	0	
		98	96	99	102	104	0	100
		90	89	88	90	95	0	125
		30	30	100	50	40	25	

Using 'VAM', the initial BFS having transportation cost of Rs 23,730 is obtained as follows :

25(105)					25(0)	50
5(98)	30(96)	25(99)		40(104)		100
		75(88)	50(90)			125
	30	30	100	50	40	25

Find the net-evaluations in the usual manner as shown in the following starting table.

		+		+		+		$\downarrow u_i$
25 • (105)	(107)	103	(110)	106	(115)	108	(115)	25 • 7
5 • (98)	30 • (96)	25 • (99)			(102)	101	(104)	0 + 0 -7 + -11
(90)	+ 87 (89)	+ 85 (88)	75 • (90)	50 • (95)			93 (0)	7 -4
$v_j \rightarrow$	98	96	99	101		104		

Since all the net-evaluations in the empty cells are non-negative, the optimum solution is given by

$$x_{11} = 2500, x_{21} = 500, x_{22} = 3000, x_{23} = 2500, x_{25} = 4000, x_{33} = 7500, x_{34} = 5000.$$

The optimum transportation cost is given by

$$z = 2500(1.05) + 500(-.98) + 3000(-.96) + 2500(-.99) + 4000(1.04) + 7500(-.88) + 5000(-.90) = \text{Rs. } 23730.$$

Total 25000 units are supplied to the wholesalers at the fixed rate of Rs. 2.50 per unit,

∴ Total Sale = $(25000 \times 2.50) = \text{Rs. } 62,500.$

Total production cost of three plants at the rate of Rs. 1, Rs. 0.90 and Rs. 0.80, respectively, becomes
 $= \text{Rs. } (5000 \times 1 + 10,000 \times .90 + 12,000 \times .80) = \text{Rs. } 23,600.$

Hence the net maximum profit to the manufacturer becomes = $\text{Rs. } 62,500 - (\text{Rs. } 23,730 + \text{Rs. } 23,600)$
 $= \text{Rs. } 15,170.$

Example 28. A manufacturer of jeans is interested in developing an advertising campaign that will reach four different age groups. Advertising campaigns can be conducted through T.V., Radio and Magazines. The following table gives the estimated cost in paisa per exposure for each age group according to the medium employed. In addition, maximum exposure levels possible in each of the media, namely T.V., Radio and Magazines are 40, 30 and 20 millions respectively. Also the minimum desired exposures within each age group, namely 13–18, 19–25, 26–35, 36 and older are 30, 25, 15 and 10 millions. The objective is to minimize the cost of attaining the minimum exposure level in each age group.

Media	Age Groups			
	13–18	19–25	26–35	36 and older
T.V.	12	7	10	10
Radio	10	9	12	10
Magazine	14	12	9	12

(i) Formulate the above as a transportation problem, and find the optimal solution.

(ii) Solve this problem if the policy is to provide at least 4 million exposures through T.V. in the 13–18 age group and at least 8 million exposures through T.V. in the age group 19–25.

[C.A. (May) 91]

Solution. (i) Formulation as a transportation problem.

Media	Age Groups				Max. exposure available (in millions)
	13-18	19-25	26-35	36 and older	
T.V.	12	7	10	10	40
Radio	10	9	12	10	30
Magazine	14	12	9	12	20
Minimum number of exposures required	30	25	15	10	80/90

Since this transportation problem is of unbalanced type, it can be made balanced by introducing a dummy category before applying Vogel's Approximation Method.

Initial Solution Table

Age Groups Media	13-18	19-25	26-35	36 and older	Dummy	Max. exposure (in million)
T.V.		25 (7)	5 (10)	10 (10)		40
Radio	30 (10)					30
Magazine			10 (9)		10 (0)	20
Min. exposures required (million)	30	25	15	10	10	

The initial solution given by VAM is degenerate since there are only 6 allocations. We put a Δ in the least cost independent cell to proceed for optimality let $u_1 = 0$ and we calculate the remaining u_i and v_j 's.

Age Groups

		13 - 18	19 - 25	26 - 35	36 and older	Dummy	$u_i \downarrow$
Media		11	7	10	10	1	0
	T.V.	1 (12)	25 (7)	5 - θ (10)	10 (10)	$\sqrt{\theta}$ (0)	-1
Radio		30 (10)	3	3	1	1	0
	Magazine	4 (14)	6 (9)	9 (12)	9 (10)	9 (0)	-1
		10	10	10	10	10 (0)	-1
		$v_j \rightarrow$	11	7	10	10	1

Min. [5 - θ , 10 - θ] = 0 gives 5 - θ = 0 or $\theta = 5$.

Improved Solution Table

		2	25	1	10	5	$u_i \downarrow$
Media		10 (12)	(7)	9 (10)	(10)	(0)	0
	T.V.	30 (10)	2	3	0	Δ (0)	0
Radio		4 (14)	5 (9)	15 (12)	2 (12)	5 (0)	0
	Magazine	10 (12)	7 (9)	10 (10)	10 (12)	10 (0)	0
		$v_j \rightarrow$	10	7	9	10	0

Since all d_{ij} 's are non-negative, the improved solution is optimal.

Through T.V., 25 million people must be reached in the age-group 19-25 and 10 million people in the age group 36 & older.

Through Radio, 30 million people must be reached in the age group 13-18.

Through Magazines, 15 million people must be reached in the age group 26–35. Total minimum cost of attaining the minimum exposure level is Rs. 71 lakhs. Since $d_{24} = 0$, this solution is not unique. Alternative solutions also exist.

(ii) The required solution is given by

4 • (12)	25 • (7)		10 • (10)	1 • 0	40
26 • (10)	(9)	(12)	(10)	0	30
(14)	(12)	(9)	15 • (12)	0	20

30 25 15 10 10

Total cost for this allotment is Rs. 71.8 lakhs.

Example 29. A Company wishes to determine an investment strategy for each of the next four years. Five investment types have been selected, investment capital has been allocated for each of the coming four years, and maximum investment levels have been established for each investment type. An assumption is that amounts invested in any year will remain invested until the end of the planning horizon of four years. The following table summarizes the data for this problem. The values in the body of the table represent net return on investment of one rupee upto the end of the planning horizon. For example, a rupee invested in investment type B at the beginning of year will grow to Rs. 1.90 by the end of the fourth year, yielding a net return of Rs. 0.90.

Investment made at the beginning of year	Investment Type					Rupees available (in 000's)
	A	B	C	D	E	
	NET RETURN DATA					
1	0.80	0.90	0.60	0.75	1.00	500
2	0.55	0.65	0.40	0.60	0.50	600
3	0.30	0.25	0.30	0.50	0.20	750
4	0.15	0.12	0.25	0.35	0.10	800
Maximum Rupees Investment (in 000's)	750	600	500	800	1000	.

The objective in this problem is to determine the amount to be invested at the beginning of each year in an investment type so as to maximize the net rupee return for the four year period.

Solve the above transportation problem and get an optimal solution. Also calculate the net return on investment for the planning horizon of four-year period. [C.A. (May) 93]

Solution. We observe that this transportation problem is of unbalanced type and it is a maximization problem. The step-by-step procedure is as follows :

Step 1. We balance the transportation problem by introducing a dummy year with net return 0 assigned to A, B, C, D and E .

Investment Type/Net Return Data

Type Years \	A	B	C	D	E	Available Rs. (in 000's)
1	0.80	0.90	0.60	0.75	1.00	500
2	0.55	0.65	0.40	0.60	0.50	600
3	0.30	0.25	0.30	0.50	0.20	750
4	0.15	0.12	0.25	0.35	0.10	800
Dummy	0	0	0	0	0	1000
Max. Inv. (in 000's)	750	600	500	800	1000	3650

Step 2. Now convert the above *profit matrix* into a *loss matrix* by subtracting all the elements from the largest element Re 1.00 in the table.

		Investment Type				
		A	B	C	D	E
Years	Type	0.20	0.10	0.40	0.25	0
	1	0.45	0.35	0.60	0.40	0.50
2	0.70	0.75	0.70	0.50	0.80	
3	0.85	0.88	0.75	0.65	0.90	
Dummy	1.00	1.00	1.00	1.00	1.00	1.00

Step 3. For convenience, we express the net loss data in above table in paise. Thereafter, we obtain the initial solution by VAM and apply transportation algorithm in the following table. Since all d_{ij} 's are not non-negative, we assign θ to most negative d_{ij} cell (5, 2).

		20	$\Delta - \theta$	50	45	$500 + \theta$	$u_i \downarrow$
		(20)	0 (10)	(40)	-10 (25)	-20 (0)	-85
20	20	600		45	35		25
(45)	25 (35)			15 (40)	5 (50)		25
0	70 (75)	-5		10	750		10
(70)	80 (70)	80		60 (50)	(80)		70
250	(88)	-7	500	50	(90)		5
(85)	95 (75)	95		(65)			85
500	(100)	-10		10	20	500 - θ	0
(100)	(100)	110	(100)	90 (100)	80 (100)		15
$v_j \rightarrow$	85	95	75	65	85		

$$\text{Min } [500 - \theta, \Delta - \theta] = 0 \text{ gives } \theta = \Delta.$$

Step 4. Putting $\theta = \Delta$, we get the revised solution and again apply optimality test in the following table.

		20	10	50	45	500	$u_i \downarrow$
		(20)	0 (10)	0 (40)	-10 (25)	-20 (0)	-85
10	10	600		35	25		15
(45)	35 (35)	*	(60)	25 (40)	15 (50)		-50
0	70 (75)	70 (70)		60 (50)	(80)		-15
(70)	85 (75)	3	500	50	(90)		5
250	(88)	85 (75)		(65)			85
500	(100)	Δ	(100)	10	20	500	0
(100)	(100)	(100)	(100)	90 (100)	80 (100)		15
$v_j \rightarrow$	85	85	75	65	85		

Since all d_{ij} are non-negative, the solution under test is optimal. The optimal allocation is given below.

Year	1	2	3	4	5	6
Investment type	E	B	D	A	C	D
Amount (in 000's)	500	600	750	250	500	50

The net return on investment for the planning horizon of four years is given by
 $500 \times 1.0 + 600 \times 0.65 + 750 \times 0.50 + 250 \times (0.15) + 500 \times 0.25 + 50 \times 0.35 = \text{Rs. } 1445 \text{ thousands.}$

Example 30. A leading firm has three auditors. Each auditor can work upto 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours, project 2 will take 140 hours, the project 3 will take 160 hours. The amount per hour that can be billed for assigning each auditor to each project is given in the table.

Auditor	Project			3 (Rs.)
	1 (Rs.)	2 (Rs.)	3 (Rs.)	
1	1,200	1,500	1,900	1,900
2	1,400	1,300	1,200	1,200
3	1,600	1,400	1,500	1,500

Formulate this as a transportation problem and find the optimal solution. Also find out the maximum total billings during the next month. [C.A. (May) 95]

Solution. Formulation. The given problem can be put in the following tabular form of T.P.

Auditor	Project			Available
	1	2	3	
1	1,200	1,500	1,900	160
2	1,400	1,300	1,200	160
3	1,600	1,400	1,500	160
Required	130	140	160	

The given problem is of unbalanced type. So we introduce a dummy project to balance it, as follows :

Here the problem is to maximize the total billing amount of the auditors. So first we convert this maximization problem into a minimization problem by subtracting all the elements of the above payoff matrix from the highest payoff, i.e. Rs. 1900. Thus the minimization T.P. becomes :

Auditor	Project			Dummy	Available
	1	2	3	Dummy	
1	1,200	1,500	1,900	0	160
2	1,400	1,300	1,200	0	160
3	1,600	1,400	1,500	0	160
Required	130	140	160	50	480

Auditor	Project			Dummy	Available
	1	2	3	Dummy	
1	700	400	0	1,900	160
2	500	600	700	1,900	160
3	300	500	400	1,900	160
Required	130	140	160	50	480

Now apply VAM for finding the initial feasible solution. Since it is a degenerate solution, we introduce a very small quantity Δ in the least cost independent cell (3, 3) and then apply optimality test in the usual manner. For convenience, we take figures of payoff's matrix in hundreds of rupees (Rs. 00's).

	8	3	160	5	u_i
(7)	-1	(4)	1 (0)	(19)	-4
(5)	1	110		2	
4	(6)	*	(7)	5 (19)	
(3)	130	30	Δ	(19)	1
	*	*	*	(19)	0
$v_j \rightarrow$	3	5	4	18	

Since all d_{ij} 's for non-basic cells are positive, the initial solution obtained above by VAM is optimal.

The optimal allocation of projects to auditors and their billing amount is given below. Here an auditor may involve in more than one project as it is clear from the following :

Auditor	1	2	3	3
Project	3	2	1	2
Billing Amount (Rs.)	160 × Rs. 1900 (Rs. 3,04,000)	110 × Rs. 1300 (Rs. 1,43,000)	130 × Rs. 1600 (Rs. 2,08,000)	30 × Rs. 1400 (Rs. 42000)

Hence, the maximum total billing during the next month will be Rs. 6,97,000.

Example 31. A particular product is manufactured in factories A, B, C and D; and is sold at centres 1, 2 and 3. The cost (in rupees) of product per unit and capacity (in kg.) per unit time of each plant is given below :

Factory	Cost (Rs.) per unit	Capacity (kg.) per unit
A	12	100
B	15	20
C	11	60
D	18	80

The sale price in rupees per unit and the demand in kg. per unit time are as follows :

Sales centre	Sale price (Rs.) per unit	Demand (kg.) per unit
1	15	120
2	14	140
3	16	60

Find the optimum sales distribution.

[C.A., Nov. 97]

Solution. The profit matrix is obtained along with demand and supply and the balanced matrix is given below :

Factory	Profit Matrix			Supply
	Sales Centre			
	1	2	3	
A	3	2	4	100
B	0	-1	1	20
C	4	3	5	60
D	2	1	3	80
E (Dummy)	0	0	0	60
Demand	120	140	60	320

The table representing profit can be converted to an equivalent minimization of loss by subtracting all the profit values in the table from the highest profit value (i.e., 5). The initial basic feasible solution is found by using Vogel's method.

Since the number of occupied cells are 6 which is one less than the required number $m + n - 1 = 7$, the solution is degenerate and after introducing an allocation of Δ to the least cost independent cell (A, 3), the initial solution is tested for optimality in table using MODI method.

Sales Centres						
Factory	1	2	3		Supply	$u_i \downarrow$
	100 (2)	0 (3)	0 (1)	Δ	100	-1
B	0 (5)	20 (6)	3 (4)	0	20	2
	0 (1)	0 (2)	0 (0)	60	60	-2
C	20 (3)	60 (4)	0 (2)	0	80	0
	+1 (5)	4 (5)	60 (2)	2	60	1
Demand	120	140	60	320		
$v_j \rightarrow$	3	4	2			

Since there is no negative opportunity cost in the unoccupied cells in table, the solution is optimum. The total maximum profit associated with this solution is as follows:

$$100 \times 3 + 20 \times (-1) + 60 \times 5 + 20 \times 2 \times 60 \times 1 + 60 \times 0 = \text{Rs. } 680.$$

EXAMINATION PROBLEMS

1. Solve the following unbalanced transportation problem (symbols have their usual meanings).

[IGNOU 2000]

[Ans. $x_{12} = 5, x_{21} = 8, x_{23} = 4$, and min. cost = 23]

	D_1	D_2	D_3	a_i
O_1	4	3	2	10
O_2	2	5	0	13
O_3	3	8	6	12
b_j	8	5	4	

2. Find an optimum basic feasible solution to the following transportation problem (use dummy destinations, if needed):

[Ans. $x_{12} = 400, x_{24} = 350, x_{25} = 50, x_{31} = 450, x_{33} = 200, x_{35} = 250$ and min cost = 6,100.]

Warehouse	Destination					Capacity
	5	4	8	6	5	
	5	4	8	6	5	600
	4	5	4	3	2	400
	3	6	5	8	4	1000
	450	400	200	250	300	

3. A company has three factories I, II, III and four warehouses 1, 2, 3 and 4. The transportation cost (in Rs.) per unit from each factory to each warehouse, the requirements of each warehouse, and the capacity of each factory are given below:

Find the minimum cost transportation schedule.

[Ans. $x_{14} = 300, x_{21} = 200, x_{22} = 300, x_{33} = 500, x_{34} = 100$ and min cost = 15,500.]

Requirement 300 300 500 500

4. Consider the following transportation problem with the cost matrix:

	1	2	3	4	5	6	Available
1	5	1	5	8	9	7	30
2	4	3	1	9	2	2	40
3	2	1	3	2	8	2	10
4	1	0	2	8	6	3	110
Required	50	20	10	35	15	50	

- (i) Determine a shipping scheme by any method.
(ii) Test the above solution for optimality.
(iii) If the above solution is not optimal, find a better solution (you need not to find the optimal solution)

[Ans. (i) $x_{13} = 10, x_{14} = 10, x_{25} = 15, x_{26} = 25, x_{34} = 10, x_{41} = 50, x_{42} = 20, x_{44} = 15, x_{46} = 25$.

(ii) Not optimal.
(iii) $x_{14} = 20, x_{23} = 10, x_{25} = 15, x_{26} = 15, x_{34} = 10, x_{44} = 50, x_{42} = 20, x_{44} = 5, x_{46} = 35$.]

5. Consider the transportation problem with the following cost matrix:

Origins	Destinations				Available
	P	Q	R	S	
A	4	6	8	13	50
B	13	11	10	8	70
C	14	4	10	13	30
D	9	11	13	8	50
Required	25	35	105	20	—

- (i) Determine a shipping scheme by the north-west corner rule.
(ii) Test the above solution for optimality.
(iii) If the above solution is not optimal, find a better one.

[Ans. (i) $x_{11} = 25, x_{12} = 25, x_{22} = 10, x_{23} = 60, x_{33} = 30, x_{43} = 15, x_{44} = 20$.

(ii) No.

(iii) $x_{11} = 25, x_{12} = 5, x_{13} = 20, x_{23} = 70, x_{32} = 30, x_{43} = 15, x_{44} = 20$; and min. cost = 1465.]

6. XYZ & Co. has provided the following data seeking your advice on optimum investment strategy :

Investment made at the beginning of year	Net Return Data (in Paise) of Selected Investments				Amount Available (lacs)
	P	Q	R	S	
1	95	80	70	60	70
2	75	65	60	50	40
3	70	45	50	40	90
4	60	40	40	30	30
Max. Investment (lacs)	40	50	60	60	-

The following additional information are also provided :

—P, Q, R and S represent the selected investments.

—The company has decided to have four years investment plan.

—The policy of the company is that amount invested in any year will remain so until the end of fourth year.

—The values (paise) in the table represent net return on investment of one rupee till the end of the planning horizon (for example, a rupee in investment P at the beginning of year will grow to Rs. 1.95 by the end of the fourth year, yielding a return of 95 paise).

—Using the above, determine the optimum investment strategy.

[C.A. (Nov.) 96]

7. The following table gives unit cost of transporting a certain material from supply points A, B, C to demand points 1, 2, 3, 4 and supply capacities at A, B, C and demands at 1, 2, 3, and 4.

From \ To	1	2	3	4	Supply Capacity
A	5	1	7	10	10
B	6	4	6	5	80
C	3	2	5	8	15
Demand	75	20	50	25	—

Due to previous contractual obligation, a minimum of 5 units of the material must be supplied from A to 4. As there is more demand than supply capacity, demands at some of the demand points cannot be satisfied. There are penalty costs for every unsatisfied demand unit which are given by 5, 3, 2 and 4 for demand points 1, 2, 3 and 4 respectively. Find the optimum schedule of transportation and associated transportatiol cost. Is the optimum schedule obtained unique?

[VTU (BE Mech.) 2002]

8. In the state of Behar, in particular region there are five coal mines which produce the following output at the indicated production cost.

Mine	Output m. tons/day	Production cost units of 100 Rs. per metric ton
1	120	25
2	150	29
3	80	34
4	160	26
5	140	28

Before the coal can be sold to the steel making units, it must be cleaned and graded at one of the coal preparation plants, the capacities and operating cost of these 3 plants are as following :

Plants	Capacities metric tons/day	Operating cost units of 100 Rs. per metric ton
A	300	2
B	2000	3
C	200	3

All coal is transforted by rail at a cost of Rs. 50 per metric ton kilometer and the distance in kilometers from each mine to the transfortation plants are indicated below :

	Distance Kilometer to mines				
	1	2	3	4	5
A	22	44	26	52	24
B	18	16	24	42	48
C	44	32	16	16	22

[Hint : (i) Using transportation model determine how the output of each mine should be allocated to the three preparation plants to opmize cost.

- (ii) Are alternative approaches possible ? If so, what is the logic of 1st allocation in these alternatives ?
 (iii) What is degeneracy and when can it happen ?
 [Hint. Using Transportation Model]

[IES (Mech. 1998)]

Cost matrix per metric tonne is given as (Rs.)

Preparation plant mine	A	B	C	Supply
1	$(25 \times 100 + 2 \times 100 + 22 \times 50) = 3800$	$(2500 + 300 + 18 \times 50) = 3700$	$2500 + 300 + 44 \times 50 = 5000$	120
2	$(29 \times 100 + 200 + 44 \times 50) = 5300$	$(2900 + 300 + 16 \times 50) = 4000$	$2900 + 300 + 32 \times 50 = 4800$	150
3	$3400 + 200 + 26 \times 50 = 4900$	$(3400 + 300 + 42 \times 50) = 4900$	$3400 + 300 + 16 \times 50 = 4500$	80
4	$2600 + 200 + 52 \times 50 = 5400$	$(2600 + 300 + 42 \times 50) = 5000$	$2600 + 300 + 16 \times 50 = 3700$	160
5	$2800 + 200 + 24 \times 50 = 4200$	$(2800 + 300 + 48 \times 50) = 5500$	$2800 + 300 + 22 \times 50 = 4200$	140
Demand	300	200	200	700

Here Demand > supply, hence add dummy source with zero cost.

Cost metric in hundred's of rupees with dummy source and using least cost method

Source \ Plant	A	B	C	Supply
1	(38)	(37)	120 (50)	120/0
2	(53)	70 (40)	80 (48)	150/70/0
3	(49)	80 (49)	(45)	80/0
4	(54)	(50)	(37)	160/0
5	(42)	100 (55)	(42)	140/100/0
Dummy	(0)	50 (0)	(0)	50/0
Demand	300/200/50/0	200/80/0	200/40/0	700

11.13. TIME-MINIMIZING TRANSPORTATION PROBLEMS

In time-minimizing transportation problems, the objective is to minimize the time of transportation rather than the cost of transportation. For example in military, the time of supply is considered more valuable than the cost of transportation and, therefore, it is always preferred to minimize the total time of supply and not the cost. Such problems are often encountered in emergency services like military services, hospital management, fire services, etc.

In fact, the time-minimization transportation problems are similar to the cost-minimization transportation problems, except that the unit transportation cost c_{ij} is replaced by the unit time t_{ij} required to transport the items from i th origin to j th destination.

The *feasible plan (initial basic feasible solution)* for this problem can also be obtained by using any of the methods discussed in sec. 11.8. Now, if T_j is the largest transportation time associated with the j th feasible plan, then we have to find such plan that gives minimum (T_j).

The *step-by-step* procedure for the solution of such problems is given below.

11.13-1. Solution Procedure for Time-Minimization T.P.

- Step 1.** First, find an initial basic feasible solution by using any of the methods discussed in sec. 11.8. Enter the solution at the centres of the basic cells.
- Step 2.** Compute T_j for this basic feasible solution and cross out all the non-basic cells for which $t_{ij} \geq T_j$.

Step 3. Now construct a loop for the basic cells corresponding to T_j in such a way that when the values at the centre of the cells are shifted around, the value at this cell tends towards (not-necessarily) zero and no variable becomes zero. If no such closed path is possible, the solution under test is optimal, otherwise go to **step 2**.

Step 4. Repeat the procedure until an optimum basic feasible solution is attained.

- Q. 1. What are the least time transportation problem?
2. What are the areas, where the time minimization problems are applied?
3. State the characteristics of a "loop" of a least-time transportation problems.
4. Give the solution procedure for solving the least-time transportation problems.

11.13-2. Computational Demonstration of Solution Procedure

The computational procedure is best explained by the following example.

Example 32. If the matrix elements represent the time, solve the following transportation problem:

	To					
	D_1	D_2	D_3	D_4	Available	
From	O_1	10	0	20	11	15
	O_2	1	7	9	20	25
	O_3	12	14	16	18	5
Required		12	8	15	10	45

Solution.

First Iteration :

Step 1. Using Vogel's Approximation Method, find an initial basic feasible solution as given in the **Table 11.80**. The numbers written in the down-left corners of each cell represent the corresponding times.

Step 2. The times for this feasible plan are : $t_{12} = 0$, $t_{14} = 11$, $t_{21} = 1$, $t_{23} = 9$, $t_{33} = 16$, and $t_{34} = 18$.

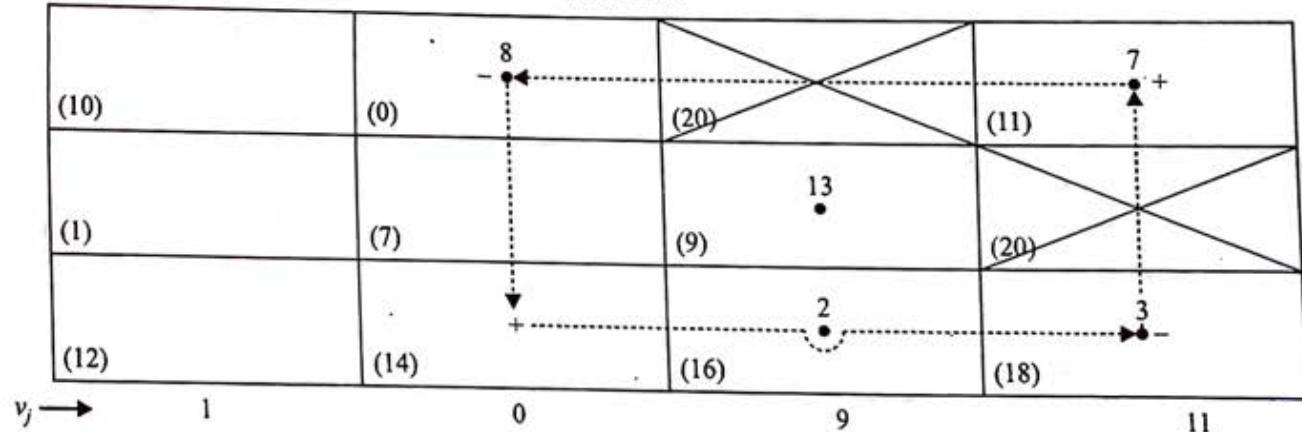
$$\therefore T_1 = \max [0, 11, 1, 9, 16, 18] = 18.$$

Obviously, all the shipments for this plan are to be completed within the time $T_1 = 18$. So we cross-out the cell (1, 3) and (2, 4) because $t_{13} > T_1$ and $t_{34} > T_1$.

Table 11.80

	D_1	D_2	D_3	D_4
O_1		8 (0)		
O_2	12 (1)		13 (9)	
O_3			2 (16)	3 (18)

Table 11.81



Step 3. Now the closed path (loop) for the cell (3, 4) for which $t_{34} = T_1 = 18$ is shown in **Table 11.81**. It is evident from **Table 11.81** that only 3 units can be shifted around.

Second Iteration :

Step 1. The revised feasible plan is shown in **Table 11.82**

Table 11.82

(10)	(0)	8		10
(1)	(7)		(20)	(11)
(12)	(14)	3	(9)	(20)
			(16)	(18)

Step 2. Here $T_2 = \max\{t_{12}, t_{14}, t_{21}, t_{22}, t_{32}, t_{33}\} = \max\{0, 11, 1, 9, 14, 16\} = 16$

As all the shipments for this feasible solution are completed within time $T_2 = 16$, we cross-out the cell (3, 4) also since $t_{34} > T_2$.

Step 3. The closed loop starts from cell (3, 3) as shown in Table 11.82 above. Clearly, only 2 units can be shifted around.

Third Iteration

Step 1. The revised feasible plan is shown in Table 11.83

Table 11.83

(10)	(0)	5		10
(1)	(7)	+		(20)
(12)	(14)	3	(9)	(20)

Step 2. Here $T_3 = \max\{t_{12}, t_{14}, t_{21}, t_{23}, t_{31}, t_{32}\} = \max\{0, 11, 1, 9, 12, 14\} = 14$.

As all the shipments for this feasible plan are completed within time $T_3 = 14$, we cross-out the cell (3, 3) also, since $t_{33} > T_3$.

Step 3. Since $t_{32} = T_3 = 14$, The closed loop starts from the cell (3, 2) as shown in Table 11.83. Clearly, 3 units can be shifted around.

Fourth Iteration :

Step 1. The next revised feasible plan is shown in Table 11.84

Table 11.84

(10)	(0)	5		10
(1)	(7)	3		(20)
(12)	(14)		(9)	(20)

Step 2. Here $T_4 = \max\{t_{12}, t_{14}, t_{21}, t_{22}, t_{23}, t_{31}\} = \max\{0, 11, 1, 7, 9, 12\} = 12$.

As all the shipments for this feasible plan are completed within time $T_4 = 12$, we cross-out the cell (3, 2) also since $t_{32} > 12$.

- Step 3.** Now we cannot form any closed loop without increasing the present minimum shipping time. Hence the feasible plan at this stage is optimal.
Thus, all the shipments can be made within 12 time units.

EXAMINATION PROBLEM

1. Solve the following transportation problem, the matrix represents the times t_{ij} :

		To				Available
		D_1	D_2	D_3	D_4	
From	O_1	6	7	3	4	5
	O_2	7	9	1	2	7
	O_3	6	5	16	7	8
	O_4	18	9	10	2	10
Required		10	5	10	5	30

[JNTU (Mech. & Prod.) 2004]

[Ans. Total shipment time is 9 units. Details of plan are :

$x_{11} = 2$ with $t_{11} = 6$; $x_{13} = 3$ with $t_{13} = 3$; $x_{23} = 7$ with $t_{23} = 1$; $x_{31} = 8$ with $t_{31} = 6$; and $x_{42} = 5$ with $t_{42} = 9$.]

11.14. TRANSHIPMENT PROBLEMS

Definition. A transportation problem in which available commodity frequently moves from one source to another source or destination before reaching its actual destination is called a transhipment problem.

11.14.1. Main Characteristics of Transhipment Problems

Following are the main characteristics of transhipment problems :

1. The number of sources and destinations in the transportation problem are m and n respectively. But in transhipment problems, we have $m + n$ sources and destinations.
2. If S_i denotes the i th source and D_j denotes the j th destination, then commodity can move along the route $S_i \rightarrow D_i \rightarrow D_j$, $S_i \rightarrow S_j \rightarrow D_i \rightarrow D_j$, $S_i \rightarrow D_i \rightarrow S_j \rightarrow D_j$, or in various other ways. Clearly, transportation cost from S_i to S_i is zero and the transportation costs from S_i to S_j or S_i to D_i do not have to be symmetrical, i.e., in general, $S_i \rightarrow S_j \neq S_j \rightarrow S_i$.
3. While solving the transhipment problem, we first obtain the optimum solution to the transportation problem, and then proceed in the same manner as in solving the transportation problems.
4. The basic feasible solution contains $2m + 2n - 1$ basic variables. If we omit the variables appearing in the $(m + n)$ diagonal cells, we are left with $m + n - 1$ basic variables.

- Q. 1. Explain transhipment problem.
 2. What are the main characteristics of a transhipment problem ?
 3. Explain the method of solving the transhipment problem.
 4. Indicate how a transhipment problem can be solved as a transportation problem.
 5. What is transhipment problem ? Show how a transhipment problem can be modelled as a transportation problem.
 6. Define a Transhipment problem. How does it differ from a transportation problem. How is it solved ?

[Garhwal 97]

[Delhi (MBA) 95]

11.14.2. Computational Demonstration of Solution Procedure

The computational procedure for solving transhipment problems is best explained by the following example.

Example 33. Consider the following transhipment problem with two sources and two destinations, the costs for shipment in rupees are given below. Determine the shipping schedule :

	S_1	S_2	D_1	D_2	
S_1	0	1	3	4	5
S_2	1	0	2	4	25
D_1	3	2	0	1	
D_2	4	4	1	0	30
			20	10	

Solution. Step 1. (To get modified transportation problem). In the transhipment problem, each given source and destination can be considered a source or a destination. If we now take the quantity available at each of the sources D_1 and D_2 to be zero and also at each of the destinations S_1 and S_2 the requirement to be zero, then to have a supply and demand from all the points (sources or destinations) a fictitious supply and demand quantity termed as 'buffer stock' is assumed and is added to both supply and demand of all the points. Generally, this buffer stock is chosen equal to $\sum a_i$ or $\sum b_j$. In our problem, the buffer-stock comes-out to be 30 units.

Step 2. (To find initial solution of modified problem)

By adding 30 units of commodity to each point of supply and demand, an initial basic feasible solution is obtained in Table 11.86 by using Vogel's Approximation method.

Starting Table 11.86

	Modified Table 11.85				
	S_1	S_2	D_1	D_2	Available
S_1	0	1	3	4	35
S_2	1	0	2	4	55
D_1	3	2	0	1	30
D_2	4	4	1	0	30
Required	30	30	50	40	

	a_i
(0)	30
(1)	30
(2)	20
(3)	5
(4)	30
	35
	55
	30
	30

b_j	30	30	50	40	
(0)	30	30	50	40	
(1)		30	20	5	
(2)			30		
(3)				30	
(4)					30

Step 3. (To apply optimality test)

The variables u_i ($i = 1, 2, 3, 4$) and v_j ($j = 1, 2, 3, 4$) have been determined by using successively the relations $u_i + v_j = c_{ij}$ for all the basic (occupied) cells. These values are then used to compute the net-evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic (empty) cells. Clearly $d_{34} (= -1)$ is the only negative quantity. Hence an unknown quantity θ is assigned to this cell (3, 4). After identifying the loop, we find that $\theta = 5$ and that the cell (2, 4) leaves the basis (i.e., becomes empty).

Table 11.87

	u_i
(0)	0
(1)	0
(2)	0
(3)	-1
(4)	-2
	0
	0
	-1
	-2
	-4

v_j	30	1	1	5	
(0)	30	0 + 0	(3)	0 + 2	(4)
(1)	1	30	20 + θ	5 - θ	
(2)	0 + 0	(0)	(2)		
(3)	5	4	30 - θ	+ θ	-2 + 4
(4)	-2 + 0	(2)	-2 + 0	(0)	
	8	8	3	30	
	-4 + 0	(4)	-4 + 0	(1)	
	0	0	2	-4 + 2	(0)
	0	0	2	4	

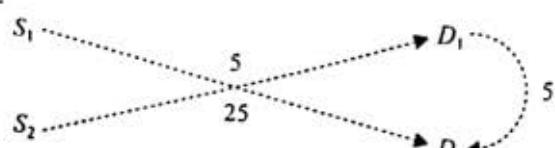
Step 4. Introduce the cell (3, 4) into the basis and drop the cell (2, 4) from the basis. Then, again test the optimality of the revised solution.

Table 11.88

Since all the current net evaluations are non-negative, the current solution is an optimum one. It is shown in Table 12.88. The minimum transportation cost is:

$$z^* = 5 \times 4 + 25 \times 2 + 5 \times 1 = 75.$$

and the optimum transportation route is as shown below.



					u_i
	30	0	0	5	0
(0)	30	-1	3	4	1
(1)	2	30	25		
(2)	-1	(0)	(2)	(4)	3
(3)	6	4	25	5	-3
(4)	-3	(2)	(0)	(1)	
(5)	8	7	2	30	-4
(6)	-4	(4)	(1)	(0)	
v_j	0	1	3	4	

EXAMINATION PROBLEMS

1. Given the following data, find the optimum transportation routes :

	S_1	S_2	D_1	D_2	Capacity
S_1	0	2	2	1	8
S_2	1	0	2	3	3
D_1	2	2	0	2	
D_2	1	3	2	0	
Demand			7	4	

[Ans. $(S_1, D_1) = 4$ units, $(S_1, D_2) = 4$ units, $(S_2, D_1) = 3$ units; min. cost = 18.]

2. The unit cost of transportation from site i to site j are given below. At site $i = 1, 2, 3$, stocks of 150, 200, 170 units respectively, are available. 300 units are to be sent to site 4 and rest to site 5. Find the cheapest way of doing this :

	1	2	3	4	5
1	-	3	4	13	7
2	1	-	2	16	6
3	7	4	-	12	13
4	8	3	9	-	5
5	2	1	7	5	-

[Ans. $x_{14} = 130$, $x_{15} = 20$, $x_{25} = 200$, $x_{34} = 170$; and min. cost = 5070.]

11.15. GENERALIZED TRANSPORTATION PROBLEM

A transportation problem is called generalized transportation problem if the constraints of the general transportation problem are expressed as

$$\sum_{j=1}^n a_{ij} x_{ij} = a_i \quad (1 \leq i \leq m), \quad \sum_{i=1}^m b_{ij} x_{ij} = b_j \quad (1 \leq j \leq n) \quad \text{and} \quad \sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

with $x_{ij} \geq 0$, ($i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$). In the transportation problem, linearly independent equations were $m+n-1$ in number, and the fact led to the simplicity with which an initial basic feasible solution could be found and tested for optimality. This advantage disappears in the present case. Fortunately, it is still possible to establish an algorithm on the similar lines as that for transportation problem, but is not so simple in practice. Recently, various algorithms have been developed for the solution of generalized problems by many research workers in this field, which are too extensive to be included here.

EXAMINATION QUESTIONS

- Q. 1. Indicate how a transhipment problem can be solved as a transportation problem.
 2. M sources have $s_i, i = 1, \dots, M$, units of a commodity while demand at retail shop j is $d_j, j = 1, \dots, N$. Profit per unit supplied from source i to shop j is $p_{ij}, i = 1, \dots, M; j = 1, \dots, N$. Indicate how the profit-maximizing problem with the above data can be converted to an equivalent cost-minimizing problem, stating clearly any more assumptions one can make in this regard.
 3. What is meant by balanced transportation problem? Give a method of solving transportation problem with capacities.
 4. Explain in detail, any one method for solving a transportation problem. Would you recommend this method to solve an assignment problem.
 5. What is degeneracy in transportation problem? How transportation problems is solved when demand and supply are not equal.
 6. State the transportation problem in general terms and explain the problem of degeneracy. How one does overcome it?
 7. What is the total number of constraint equations in a general transportation model with m sources and n destinations? How many of these are independent? Justify your answer. Name the methods for obtaining an initial solution. Which is the best one and why? Also describe it.
- [IAS (Maths.) 90]
[Meerut 2003]

EXAMINATIONS REVIEW PROBLEMS

1. Formulate the following problem as a Transportation Problem. Do not solve it.

A production manager has a product, which must be produced to meet a fluctuating demand. He knows the monthly requirements are 900 for the 1st month, 700 for the 2nd month, 1100 for the 3rd month, and 1000 for the 4th month. The product can be produced either on regular time or on overtime. However, there are two restrictions imposed by technical conditions: regular production cannot exceed 900 items per month and overtime production cannot exceed 500 items per month. Manufacturing cost per item in normal working time varies each month: Rs. 3 in 1st month, Rs. 4 in 2nd month, Rs. 2.50 in 3rd month, and Rs. 3 in 4th month. The cost varies due to the anticipated volume for each month. The manufacturing costs for overtime are Rs. 4 in 1st month, Rs. 5 in 2nd month, Rs. 3.50 in the 3rd month and Rs. 4 in 4th month. Monthly storage cost is Rs. 2 per unit. Items manufactured but not distributed during the month are kept in stock and distributed the following month. Since the time period is limited, no inventory is to remain at the end of the period (4th month). The number of units produced must equal to the number demanded and distributed.

2. Solve the following transportation problem:

		Consumers			
		A	B	C	Available
Suppliers	I	6	8	4	14
	II	4	0	8	12
	III	1	2	6	5
Required		6	10	15	31

[Hint. Use 'VAM' to find an initial BFS and prove it to be optimal.]

[Ans. $x_{12} = 14, x_{21} = 6, x_{22} = 5, x_{32} = 1, x_{33} = 5$, min. cost = Rs. 143].

3. A company has factories at A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 160, 150 and 190 units respectively. Monthly warehouse requirements are 80, 90, 110 and 160 units respectively. Unit costs.

		To				
		D	E	F	G	
From	A	42	48	38	37	
	B	40	49	52	51	
	C	39	38	40	43	

[Meerut 2005]

[Ans. $x_{14} = 160, x_{21} = 80, x_{22} = 10, x_{32} = 80, x_{33} = 110$, min. cost = Rs. 17050. Alternate solution exists].

4. Given below the unit costs array with supplies $a_i, i = 1, 2, 3$; and demands $b_j, j = 1, 2, 3, 4$. Find the optimum solution to this Hitchcock problem.

		Sink				
		1	2	3	4	a_i
Source	1	8	10	7	6	50
	2	12	9	4	7	40
	3	9	11	10	8	30
$b_j \rightarrow$		25	32	40	23	120

[Ans. $x_{11} = 25, x_{12} = 2, x_{13} = 5, x_{25} = 40, x_{14} = 23, x_{32} = 30$, min. cost = Rs. 840]

5. A company has three factories F_i ($i = 1, 2, 3$) from which it transports the product to four warehouses W_j ($j = 1, 2, 3, 4$). The unit cost of production at the three factories are Rs. 4, 3, 5 respectively. Given the following information on unit costs of transportation, capacities at the three factories and requirements at the four warehouses. Find the optimum allocations.

Unit cost of Production	Transportation cost				Available	
	W_1	W_2	W_3	W_4		
F_1	4	5	7	3	8	300
F_2	3	4	6	9	5	500
F_3	5	2	6	4	5	200
Required	200	300	400	100		1,000

6. A company has four factories F_1, F_2, F_3 and F_4 and four warehouses, W_1, W_2, W_3 and W_4 . The warehouses are located at varying distances from the factories, from where the supplies are transported to them; the transportation costs from the factories to the warehouses thus naturally vary from Rs. 2 to Rs. 6 per unit and the company desires to minimize these transportation costs. In the form of a matrix, the costs from the factories to the warehouses are as shown in the table.

		Warehouses				Capacity
		W_1	W_2	W_3	W_4	
Factories	F_1	2	3	4	5	400
	F_2	3	2	3	4	500
	F_3	4	3	3	4	600
	F_4	6	4	4	5	700
Required		700	600	500	400	

Assign factory capacities to warehouse requirements so as to minimize the costs of transportation by making use of the technique of linear programming.

[Ans. $x_{13} = 300, x_{21} = 100, x_{22} = 300, x_{24} = 100, x_{31} = 100, x_{33} = 100$, min. cost = Rs. 3200].

7. Origins O_1, O_2, O_3 and O_4 have surplus of 30, 50, 75 and 20 empty freight cars respectively, and the destinations D_1, D_2, D_3, D_4, D_5 and D_6 are in need of 20, 40, 30, 10, 50 and 25 empty cars respectively. The cost per unit of moving an empty car from origins to destinations is given below :

		Destination					
		D_1	D_2	D_3	D_4	D_5	D_6
Origin	O_1	1	2	1	4	5	2
	O_2	2	3	2	1	4	3
	O_3	4	2	5	9	6	2
	O_4	3	1	7	3	4	6

Determine the minimum possible cost of transporting empty cars from 'excess' origins to 'deficit' destinations.

8. XYZ transportation company has four terminals A, B, C and D . At the start of a particular day, there are 8, 8, 6 and 3 tractors available at terminals A, B, C and D , respectively. During the previous night, trailers were loaded at plants R, S, T and U with quantities of 2, 12, 5 and 6 respectively. The company despatcher has come up with the distances (in kms) between the terminals and plants which appear in the table. Based upon the foregoing information, what tractors should the despatcher send to which plants in order to minimize total distances ?

	R	S	T	U
A	22	46	16	40
B	41	15	50	40
C	82	32	48	60
D	40	40	36	30

9. General Electrodes is a big electrode manufacturing company. It has two plants and three main distribution centres in three cities. The supply and demand of transportation for units of electrodes (truck load) are given below along with unit cost of transportation. How the trips be scheduled, so that the cost of transportation is maintained ?

The present cost of transportation is around Rs 3100/- per month. What can be maximum saving by proper scheduling ?

		A	B	C	Capacity
Plants	X	25	35	10	
	Y	20	5	80	100
Requirement		50	50	150	

10. At the beginning of next week there will be a surplus of 6, 9, 7 and 5 trailers in cities 1, 2, 3 and 4 respectively. Cities A, B and C will have a deficit of 8, 7 and 9 trailers, respectively. The cost for moving from each surplus city to each deficit city is given in rupee in the table.

		Deficit		
		A	B	C
Surplus	1	26	32	28
	2	19	27	16
	3	39	21	32
	4	18	24	23

Find the Minimum Cost Transportation Schedule.

11. A company has decided to manufacture some or all of five new products at three of their plants. The production capacity of each of these three plants is as follows :

	Plant No. :	1	2	3
Production capacity in total number of units	:	40	60	90

Sales potential of the five products is as follows :

	Product No. :	1	2	3	4	5
Market potential in units	:	30	40	70	40	60

Plant No. 3 cannot produce No. 5. The variable cost per unit for the respective plant and product combination is given in the table.

		Product				
		1	2	3	4	5
Plant	1	20	19	14	21	16
	2	15	20	13	19	16
	3	1	15	18	20	-

Based on the above data, determine the optimum product to plant combination by linear programming.

12. A department store wishes to purchase the following quantities of dresses :

Dress type :	A	B	C	D	E
Quantity :	150	100	75	250	200

Tenders are submitted by four different manufacturers who undertake to supply not more than the quantities indicated below :

Manufacturer :	W	X	Y	Z
Total dress quantity :	300	250	150	200

The store estimates that its profit (in Rs.) per dress will vary with the manufacturer as shown in the following table :

	A	B	C	D	E
W	2.75	3.50	4.25	2.25	1.50
X	3.00	3.25	4.50	1.75	1.00
Y	2.50	3.50	4.75	2.00	1.25
Z	3.25	2.75	4.00	2.50	1.75

13. The following information is available concerning the operation of the XYZ manufacturing company :

Period	Units on order	Production Capacity		Excess cost per unit over time (Rs.)	Storage cost per unit (Rs.)
		Regular time	Overtime		
Month 1	800	920	920	1.25	0.50
Month 2	1400	250	250	1.25	0.50

Formulate the above problem as a transportation problem.

[Ans.

	1	2	3'	a_i
Regular time	1	1	1.50	0
(Overtime)	1'	2.25	2.75	0
Regular time	2	∞	1	0
(Overtime)	2'	∞	2.25	0
	b_j	800	1400	140

14. A company has factories A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are 250, 300 and 400 units respectively for regular production. If overtime production is utilized, factories A and B can produce 50 and 75 additional units respectively at overtime incremental costs of Rs. 4 and Rs. 5 respectively. The current warehouse requirement are 200, 225, 275 and 300 units respectively. Unit transportation costs (in Rs.) from factories to the warehouses are as below.

To

	D	E	F	G
A	11	13	17	14
B	16	18	14	10
C	21	24	13	10

Determine the optimum distribution for this company to minimize costs.

[Ans.

	D	E	F	G	H	a_i
A	11	13	17	14	0	250
B	16	28	14	10	0	300
C	21	24	13	10	0	400
A'	15	17	21	18	0	50
B'	21	23	19	15	0	75
b_j	200	225	275	300	75	

15. A manufacturer must produce a certain product in sufficient quantity to meet contracted sales in the next four months. The production facilities available for this product are limited, but by different amounts in respective months. The unit cost of production also varies accordingly to the facilities and personnel available. The product may be produced in one month and then held for sale in a later month, but an estimated storage cost of Re. 1 per unit per month. No storage cost is incurred for goods sold in the same month in which they are produced. There is presently no inventory of this product and none is desired at the end of 4 months. Given the following table show how much to produce in each of four months in order to minimize total cost.

Month	Contracted sales (in units)	Maximum production (in units)	Unit cost of production (Rs.)	Unit storage cost per month (Rs.)
1	20	40	14	1
2	30	50	16	1
3	50	30	15	1
4	40	50	17	1

Formulate the problem as a transportation problem and hence solve it.

[Ans.

	1	2	3	4	5	a_i
	14	15	16	17	0	40
1	0	16	17	18	0	50
2	0	0	15	16	0	30
3	0	0	0	17	0	50
$b_j \rightarrow$	20	30	50	40	30	

16. Company has four warehouses a, b, c, d. It is required to deliver a product from these warehouses to three customers A, B, and C. The warehouses have the following amounts in stock :

Warehouse	a	b	c	d
No. of units	15	16	12	13

and the customer's requirements are :

Customer	A	B	C
No. of units	18	20	18

The table below shows the costs of transporting one unit from warehouse to the customer.

	a	b	c	d
A	8	9	6	3
B	6	11	5	10
C	3	8	7	9

Solve the problem.

[Hint. Use 'VAM' to find initial BFS and prove it to be optimal.]

[Ans. $x_{12} = 5, x_{14} = 13, x_{22} = 8, x_{23} = 12, x_{31} = 15, x_{32} = 3$, min. cost = Rs. 331]

17. Solve the following transportation problem.

21	16	25	13
17	18	14	23
32	27	18	41

11
13
19
43

18. Goods have to be transported from factories F_1, F_2, F_3 , to warehouses W_1, W_2, W_3 , and W_4 . The transportation costs per unit, capacities of the factories and requirements of the warehouses are given in the following table. Find the distribution with minimum cost.

	W_1	W_2	W_3	W_4	Capacity
F_1	15	24	11	12	5000
F_2	25	20	14	16	4000
F_3	12	16	22	13	4000
Requirement	3000	2500	3500	4000	

[Ans. $x_{13} = 2,500, x_{14} = 2,500, x_{23} = 1000, x_{21} = 3000, x_{32} = 2500, x_{34} = 1500$ and min. $Z = 1,67,000$.]

19. What is an assignment problem? Show that an assignment problem can be reduced to transportation problem and vice-versa.
20. What is degeneracy in Transportation Problems? How transportation problem is solved when demand and supply are not equal?
21. A company has four Factories from which it ships its product units to four warehouses W_1, W_2, W_3 and W_4 which are the distribution centres. Transportation costs per unit between various combinations of factories (F_1, F_2, F_3 and F_4) and warehouses are as:

	W_1	W_2	W_3	W_4	Available
F_1	48	60	56	58	140
F_2	45	55	53	60	260
F_3	50	65	60	62	360
F_4	52	64	55	61	220
Required	200	320	250	210	

Find the transportation schedule which minimizes the distribution cost.

[Hint. Find the initial solution by VAM as $(F_1, W_2) = 60, (F_1, W_3) = 30, (F_1, W_4) = 50, (F_2, W_2) = 260, (F_3, W_1) = 200, (F_3, W_4) = 160, (F_4, W_3) = 220$ units.

and prove it to be optimum. Since d_{33} will be zero, there will exist alternative solutions also.]

22. Solve the following transportation problem using north-west corner rule for initial feasible solution.

A company has 3 Plants P_1, P_2, P_3 each producing 50, 100 and 150 units of a similar product. There are five warehouses W_1, W_2, W_3, W_4 and W_5 having demand of 100, 70, 50, 40 and 40 units respectively.

The cost of sending a unit from various plants to the warehouses differs as given by the cost matrix below. Determine a transportation schedule so that cost is minimized.

	W_1	W_2	W_3	W_4	W_5	a_i
P_1	20	28	32	55	70	50
P_2	48	36	40	44	25	100
P_3	35	55	22	45	48	150
b_j	100	70	50	40	40	300/300

[Ans. $(P_1, W_1) = 40, (P_1, W_2) = 10, (P_2, W_2) = 60, (P_2, W_5) = 40, (P_3, W_1) = 60, (P_3, W_3) = 50, (P_3, W_4) = 40$, min. cost = Rs. 9240].

23. A transportation problem with 3 sources and 4 destinations has :

$C_{11} = 2, C_{12} = 3, C_{13} = 11, C_{14} = 7, C_{21} = 1, C_{22} = 0, C_{23} = 6, C_{24} = 1, C_{31} = 5, C_{32} = 8, C_{33} = 15, C_{34} = 9$
 $S_1 = 6, S_2 = 1, S_3 = 10, D_1 = 7, D_2 = 5, D_3 = 3, D_4 = 2$

Solve this T.P.

[Meerut (Math.) 96]

24. (a) Give the difference between transportation and assignment problems.

(b) Explain in brief with examples : (i) North west corner rule (ii)Vogel's approximation method.

25. State the Transportation problem in general terms and explain the problem of degeneracy. How does one overcome it ? [IAS (Main) 90]

26. Four intravenous fluid manufacturing plants are located at S_1, S_2, S_3 and S_4 which supply the Hospitals located at H_1, H_2, H_3, H_4 and H_5 . Daily plant capacities are 80, 120, 125 and 75 units respectively. Daily hospital requirements are 30, 70, 150, 50 and 100 units, respectively. Unit transportation costs (in Rs.) are given below :

		To				
		H_1	H_2	H_3	H_4	H_5
From	S_1	6	8	15	17	9
	S_2	11	13	7	4	16
	S_3	13	15	8	6	11
	S_4	10	5	9	3	6

Determine the optimum distribution for the plants in order to minimize the total transportation cost. [Delhi (MBA) 94]

27. A company has three warehouses W_1, W_2 and W_3 . It is required to deliver a product from these warehouses to three customers A, B and C . The warehouses have the following units in stock.

Warehouse	W_1	W_2	W_3
No. of units	65	42	43

and customer requirements are :

Customer	A	B	C
No. of units	70	30	50

The table below shows the costs of transporting one unit from warehouse to the customer :

		Warehouse		
		W_1	W_2	W_3
Customer	A	5	7	8
	B	4	4	6
	C	6	7	7

(C.A., May 99)

Find the optimum transportation route.

28. (a) A company is spending Rs. 1,000 on transportation of its units from three plants to four distribution centres. The supply and demand of units with unit cost of transportation are given as :

Plant	Distribution centre				Capacity (tonnes)
	D_1	D_2	D_3	D_4	
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Demand	5	8	7	15	35

What can be the maximum saving by optimum scheduling ?

(AIMA, (P.G. Dip. in Management) June 96)

- (b) A company has three cement factories A, B and C , and four area distributors W, X, Y and Z . With identical costs of production at the three factories, the only variable cost involved is the transportation cost. The monthly production capacity (in tonne) of the three factories, monthly demand of the four distributors, and the transportation cost per tonne (in rupees) from the different factories to different distribution centres, are given below :

Factory	Distributor				Monthly availability
	W	X	Y	Z	
A	20	25	50	10	45,000
B	45	50	15	40	50,000
C	22	10	45	35	55,000
Monthly demand	50,000	40,000	30,000	30,000	

Suggest an optimal transport schedule and find the minimum transportation cost.

(Delhi (M.Com.) 99)

[Hint : Take dummy refinery with zero cost and capacity 100 to make the problem balanced. Initial solution by VAM is optimal.]

Ans. $x_{11} = 300, x_{12} = 200, x_{14} = 200, x_{22} = 400, x_{33} = 700, x_{34} = 100, x_{44} = 100$, Min. cost = Rs. 700.]

29. A state has four government hospitals A, B, C and D . Their monthly requirements of medicines, etc. are met by four distribution centres X, Y, Z and W . The data in respect of a particular item *via-a-vis* availabilities at the centres, requirements at the hospitals and distribution cost per unit (in paise) is given in the following table :

Distribution centre	Hospital				Availability
	A	B	C	D	
X	44	84	84	80	2,000
Y	92	30	64	80	12,000
Z	32	100	96	72	5,000
W	80	36	120	60	6,000
Requirement	8,000	8,000	6,000	3,000	25,000

Determine the optimum distribution.

[Hint : First find initial solution by VAM.]

Ans. $x_{11} = 100, x_{23} = 200, x_{33} = 450, x_{41} = 400, x_{53} = 200, x_{62} = 350, x_{71} = 100, x_{72} = 50, x_{73} = 150.$

(Delhi (MBA) Nov. 98)

30. A company must ship from 3 factories to 7 warehouses. The transportation cost per unit from each factory to each warehouse, the requirement of each warehouse, and the capacity of each factory are :

Warehouse	Factory			Warehouse requirement
	1	2	3	
A	6	11	8	100
B	7	3	5	200
C	5	4	3	450
D	4	5	6	400
E	8	4	5	200
F	6	3	8	350
G	5	2	4	300
Factory capacity	600	400	1000	2000

(i) Find the minimum cost schedule.

(ii) Suppose warehouse B goes out of business. This means that there is now excess capacity of 200 units. Find the minimum cost transportation schedule now. (Kurukshetra (M.B.A) 96)

31. A firm manufacturing a single product has three plants I, II and III. They have produced 60, 35 and 40 units respectively during this month. The firm had made a commitment to sell 22 units to customer A , 45 units to customer B , 20 units to customer C , 18 units to customer D and 30 units to customer E . Find the minimum possible transportation cost of shifting the manufactured product to the five customers. The net unit cost (in Rs.) of transporting from the three plants to the five customers is given below :

Plants	Customers			
	A	B	C	D
I	4	1	3	4
II	2	3	2	3
III	3	5	2	2

(Delhi (M.B.A.) March 99)

32. Four gasoline dealers A, B, C and D require 50, 40, 60 and 40 KL of gasoline respectively. It is possible to supply these from locations 1, 2 and 3 which have 80, 100 and 50 KL respectively. The cost (in Rs.) for shipping every KL is shown in the table below :

Location	A	B	C	D
	1	7	6	6
2	5	7	6	7
3	8	5	8	6

Determine the most economical supply pattern.

[Hint : Add a dummy dealer to make the problem balanced.]

Ans. $x_{13} = 10, x_{14} = 30, x_{21} = 50, x_{23} = 50, x_{32} = 40, x_{34} = 10$ and min. cost = Rs. 1,050.]

(Jammu (M.B.A.) 96)

33. Priyanshu Enterprise has three factories at locations A, B and C which supplies three warehouse located at D, E and F. Monthly factory capacities are 10, 80 and 15 units respectively. Monthly warehouse requirements are 75, 20 and 50 units respectively. Unit shipping costs (in Rs.) are given in the following table :

		Warehouse		
		D	E	F
Factory	A	5	1	7
	B	6	4	6
	C	3	2	5

The penalty costs for not satisfying demand at the warehouses D, E and F are Rs. five, Rs. three, and Rs. two per unit respectively. Determine the optimum distribution for Priyanshu, using any of the known algorithms write the dual of this problem.

(Delhi (M.B.A.) March 99)

[Hint : Demand (145) > Supply (105), Add dummy source with supply 40 and transportation costs 5, 3 and 2 for destination 1, 2 and 3 respectively.]

Ans. $x_{12} = 0$, $x_{21} = 60$, $x_{22} = 10$, $x_{23} = 10$, $x_{31} = 15$ and $x_{43} = 40$, min. cost = Rs. 514. Penalty for transporting 40 units to destination 3 at the cost of Rs. 2 per unit is Rs. 80.]

34. The Purchase Manager, Mr. Shah, of the State Road Transport Corporation must decide on the amounts of fuel to buy from three possible vendors. The corporation refuels its buses regularly at the four depots within the area of its operations.

The three oil companies have said that they can furnish up to the following amounts of fuel during the coming month : 275,000 litres by oil company 1 : 50,000 litres by oil company 2; and 660,000 litres by oil company 3. The required amount of the fuel is 110,000 litres by depot 1; 20,000 litres at depot 2; 330,000 litres at depto 3; and 440,000 litres at depot 4.

When the transportation costs are added to the bid price per litre supplied, the combined cost per litre for fuel from each vendor servicing a specific depot is shown below :

	Company 1	Company 2	Company 3
Depot 1	5.00	4.75	4.25
Depot 2	5.00	5.50	6.75
Depot 3	4.50	6.00	5.00
Depot 4	5.50	6.00	4.50

Determine the optimum schedule.

[Delhi (MBA) Nov. 97]

[Hint : Supply (1485000) > Demand (1100000), add dummy column with demand of 385000 either of oil.

Ans. $x_{13} = 110$, $x_{21} = 55$, $x_{22} = 165$, $x_{31} = 220$, $x_{33} = 110$, $x_{43} = 440$, $x_{52} = 385$, min. transportation cost = Rs. 51,700. Thus required cost = Rs. 515 + 80 + Rs. 595.]

35. (a) A company has three warehouses A, B and C and four stores W, X, Y and Z, the warehouses have altogether a surplus of 150 units of a given commodity as follows :

A	50
B	60
C	40

The four stores need the following amounts :

W	20
X	70
Y	50
Z	10

Cost (in rupees) of shipping one unit of commodity from various warehouses to different stores is as follows :

		Store			
		W	X	Y	Z
Warehouse	A	50	150	70	60
	B	80	70	90	10
	C	15	87	79	81

Find the optimum distribution schedule and the associated transportation cost.

(Madras (MBA,) Nov. 97)

(b) A distribution system for a particular kind of medicine has the following characteristics :

Warehouse	Availability (in cases)	Dispensary	Requirement (in cases)
A	55	I	35
B	25	II	65
C	50	III	30

The transportation costs per case (in rupees) associated with each route are :

	I	To	(Delhi (MBA) April 98)
		II	
From	A	15	9
		7	12
		10	8

Find the optimum distribution schedule.

36. An oil company has three refineries that produce total which is then transported to four distribution centres. The total quantity produced by each refinery and the total requirement of each distribution centre and the associated transportation cost per 1000 barrels are given below :

Refinery	Distribution centre				Supply (barrels)
	W	X	Y	Z	
A	80	70	50	60	16,000
B	60	90	40	80	20,000
C	50	50	95	90	14,000
Demand (barrels)	10,00	10,000	12,000	18,000	50,000

(i) Suggest the transportation schedule that minimizes the total transportation cost.

(ii) If the company wants that at least 5,000 barrels of oil are transported from Refinery C to distribution Centre W, will the optimum transportation schedule change ? If so, what will be the new schedule ?

(iii) If the transportation cost from A to W increases by Rs. 10/- will the solution in (i) change ? Why ?

(Delhi (M.Com.) 94)

37. A cement company has three factories which manufacture cement which is then transported to four distribution centres. The quantity of monthly production of each factory, the demand of each distribution centre and the associated transportation cost per quintal are given below :

Factory	Distribution centres				Monthly production (in quintals)
	W	X	Y	Z	
A	10	8	5	4	7,000
B	7	9	15	8	8,000
C	6	10	14	8	10,000
Monthly demand (in quintals)	6,000	6,000	8,000	5,000	25,000

(i) Suggest the optimum transportation schedule.

(ii) Is there any other transportation schedule which is equally attractive ? If so, write that.

(iii) If the company wants that at least 5,000 quintals of cement are transported from factory C to distribution centre Y, will the transportation schedule be any different ? If so, what will be the new optimum schedule and the effect on cost ?

(Delhi (M. Com.) 97)

38. A company has three plants of cement which has to be transported to four distribution centres. With identical costs of production at the three plants, the only variable costs involved are transportation costs. The monthly demand at the four distribution centres and the distance from the plants to the distribution centres (in kms) are given below :

Plant	Distribution centre				Monthly production (tonnes)
	W	X	Y	Z	
A	500	1,000	150	800	10,000
B	200	700	500	100	12,000
C	600	400	100	900	8,000
Monthly demand (tonnes)	9,000	9,000	10,000	4,000	

The transport charges are Rs. 10 per tonne per kilometer. Suggest optimum transportation schedule indicate the total minimum transportation cost. If, for certain reasons, route from plant C to distribution centre X is closed down, will the transportation scheme change? If so, suggest the new schedule and effect on total cost. (Delhi (M. Com.) 98)

39. A company has three factories A, B and C, and four distribution centres X, Y, Z and W. With identical costs of production at the three factories, the only variable costs involved are transportation costs. The monthly production at the three factories and demand at the four distribution centres is 6,000 tonnes, 4,000 tonnes, 2,000 tonnes and 15,000 tonnes, respectively. The transportation cost per tonne from different factories to different centres are given below :

Factory	Distribution Centre			
	W	X	Y	Z
A	3	2	7	6
B	7	5	2	3
C	2	5	4	5

- (i) Suggest the optimum transportation schedule. What will be the minimum transporation cost?
- (ii) If the transportation cost from factory C to centre Y increases by Rs. 2 per tonne, will the optimum transportation schedule change? Why?
- (iii) If the company wants to transport at least 1,000 tonnes of the product from factory A to centre Z, will the solution in (i) above change? If so, what will be the new schedule and the effect on total transportation cost?

[Delhi (M. Com.), 96]

40. (a) Solve the following transportation problem to maximize profit and give criteria for optimality:

Origin	Profit (Rs.)/Unit Destination				Supply
	1	2	3	4	
A	42	27	24	35	200
B	46	37	32	32	60
C	40	40	30	32	140
Demand	80	40	120	60	

[Sardar Patel (MBA) 98]

[Hint : Add dummy destination with zero cost and demand of 100 units.

Ans. Max. profit = (40) (42) + (60) (35) + (35) (40) + 46 + (20) (32) + (40) (40) + (100) (30) = Rs. 10,860.]

- (b) Solve the following transportation problem to maximize profit and give criteria for optimality. Use Vogel's approximation method for maximization.

Origin	Profit (Rs./unit) Destination				Supply
	1	2	3	4	
A	40	25	22	33	100
B	44	35	30	30	30
C	38	38	28	30	70
Demand	40	20	60	30	200
					150

41. A departmental store wishes to purchase the following types of sarees :

Types of sarees :	A	B	C	D	E
Quantity :	150	100	75	250	200

Tenders are submitted by four different manufacturers who undertake to supply not more than the quantities mentioned below (all types of sarees combined) :

Manufacturer :	W	X	Y	Z
Total quantity :	300	250	150	200

The store estimates that its profit per saree will vary with the manufacturer as shown below :

Manufacturer	Saree			
	A	B	C	D
W	275	350	425	225
X	300	325	450	175
Y	250	350	475	200
Z	325	275	400	250

Use transportation technique to determine how the orders should be placed ? What is the maximum profit ?
(Punjabi (MBA) 96)

[Hint : First add dummy column with demand 125 sarees and subtract all elements of the profit matrix from the highest element 475.]

Ans. $x_{12} = 25, x_{14} = 50, x_{15} = 200, x_{16} = 25, x_{21} = 150, x_{27} = 100, x_{32} = 75, x_{33} = 75, x_{44} = 200.$

42. A company produces three kinds of dolls A, B and C. The monthly production is 1000 units; 2,000 units and 3,000 units respectively. Those dolls are sold through three departmental stores X, Y and Z. 1,500 dolls are to be supplied to each store every month. However, store Z does not want any doll of type A. Profit per unit of dolls sold to each of the stores is given below :

Dolls	A	Stores		
		X	Y	Z
	A	5	10	—
	B	16	8	9
	C	12	9	11

(Kerala (M.Com.) 97)

Suggest optimum policy schedule and the profit element.

43. The City Super Market keeps five different patterns of a particular size of readymade garments for sales. There are four manufacturers available to the Manager of the market to order the required quantity. The demand for the coming season and the maximum quantity that can be produced by a manufacturer is as follows :

Pattern	1	2	3	4	5
Quantity required	U. Cut	P. Form	Maxi	Mini	Midi
Manufacturer	A	B	C	D	
Capacity	200	300	250	250	

The following quotations have been submitted by the manufacturers for different patterns :

Manufacturer	1	2	3	4	5
A	Rs. 110	Rs. 130	Rs. 160	Rs. 70	Rs. 140
B	100	120	150	75	130
C	115	140	155	80	120
D	125	145	140	60	125

The Super Market is selling different patterns at the following prices :

Pattern :	1	2	3	4	5
Selling Price (Rs.) :	120	150	170	80	140

How should the Super Market Manager place the order ?

44. A leading firm has three auditors. Each auditor can work up to 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours, project 2 will take 140 hours, and project 3 will take 160 hours. The account per hour that can be billed for assigning each auditor to each project is given below :

Auditor	Project		
	1	2	3
	Rs.	Rs.	Rs.
1	1,200	1,500	1,900
2	1,400	1,300	1,200
3	1,600	1,400	1,500

FORMULATE this as a transportation problem and find the optimum solution. Also find out the maximum total billings during the next month. (C.A., May 95)

[Hint : First convert the problem from maximization to minimization by subtracting all elements from the highest pay off Rs. 1900.]

Ans. Here an auditor may be involved in more than one project as is apparent from the following :

Auditor	Project	Rate	Hours	Billings (Rs.)
1	3	1900	160	304000
2	2	1300	110	143000
3	1	1600	130	208000
3	2	1400	30	42000
				697000

45. A company has four factories situated in four different locations in the country and four sales agencies located in four other locations in the country. The cost of production (Rs. per unit), the sale price, capacities and monthly requirement are given below :

Factory	Sales Agency				Monthly capacity (units)	Cost of production
A	7	5	6	4	10	10
B	3	5	4	2	15	15
C	4	6	4	5	20	16
D	8	7	6	5	15	15
Monthly requirement (units)	8	12	18	22		
Sales price	20	22	25	18		

Find the monthly production and distribution schedule which will maximize profit.

(C.A., May 96)

[Hint : Profit = Sales price – Production cost – Shipping cost.]

Ans. The allocation of factories to sales agencies and their profit amount is given below :

From	To	Quantity	Profit	Total Profit
A	2	10	7	70
B	4	15	1	15
C	1	8	0	0
C	3	12	5	60
D	2	2	0	0
D	3	6	4	24
D	4	7	(-2)	-14

Alternative solutions also exist Rs. 155.

46. A company has three plants in which it produces a standard product. It has four agencies in different parts of the country where this product is sold. The production cost varies from factory to factory and the selling price from market to market. The shipping cost per unit of the product from each plant to each of the agencies is known and is stable. The relevant data are given in the following table :

(a)	Plant	Weekly production capacity		Unit production cost (in Rs.)
		(Units)		
	1	400		18
	2	300		24
	3	800		20

(b) Shipping cost (in Rs.) per unit :

		To Agency		
		1	2	3
From Plant	1	2	5	7
	2	8	4	6
	3	3	4	4

(c)	Agency	Demand (units)	Selling price (Rs.)
	1	300	32
	2	400	35
	3	300	31
	4	500	36

(Delhi (MBA) Nov. 97)

- Determine the optimum plan so as to maximize the profits.
47. ABC Enterprises is having three plants manufacturing dry-cells, located at different locations. Production cost differs from plant to plant. There are five sales offices of the company located in different regions of the country. The sales prices can differ from region to region. The shipping cost from each plant to each sales office and other data are given by following table :

PRODUCTION DATA TABLE

Production cost per unit	Max. capacity in number of units	Plant number
20	150	1
22	200	2
18	125	3

Shipping costs :

	Sales Office 1	Sales Office 2	Sales Office 3	Sales Office 4	Sales Office 5
Plant 1	1	1	4	9	4
Plant 2	9	7	8	3	3
Plant 3	4	5	3	2	7

Demand & Sales prices

Demand	80	100	75	45	125
Sales Price	30	32	31	34	29

(C.A., Nov. 98)

Find the production and distribution schedule most profitable to the company.

[Hint : Convert into loss matrix by subtracting each element from largest element 14. Add dummy column with demand 50.]

Ans.

Plant	Sales Office	Units	Profit	Total Profit (Rs.)
P_1	S_1	50	9	450
P_1	S_2	100	11	1100
P_2	S_4	25	9	425
P_2	S_5	125	1	125
P_3	S_1	30	8	240
P_3	S_3	75	10	750
P_3	S_4	20	14	280
Total				3170

48. XYZCo. has provided the following data seeking your advice on optimum investment strategy :

Investment made at the beginning of year	Net return data (in Paise) of selected investments				Amount available (lacs)
	P	Q	R	S	
1	95	80	70	60	70
2	75	65	60	50	40
3	70	45	50	40	90
4	60	40	40	30	30
Maximum Investment (Lacs)	40	50	50	50	—

The following additional information are also provided :

- P, Q, R and S represent the selected investments.
- The company has decided to have four-year investment plan.
- The policy of the company is that amount invested in any year will remain so until the end of the fourth year.
- The values (paise) in the table represent net return on investment of one rupee till the end of the planning horizon. (For example, a rupee invested in investment P at the beginning of year 1 will grow to Rs. 1.95 by the end of the fourth year, yielding a return of 95 paise.)

using the above, determine the optimum investment strategy.

[Ans. The optimum allocations given below.]

(C.A., Nov. 96)

Year	Investments (Rs.)	Net return (Rs. Lacks)
1	40 lacs to P, 30 lacs to Q	$0.95 \times 40 = 38, 0.80 \times 30 = 24$
2	20 lacs to Q, 20 lacs to R	$0.65 \times 20 = 13, 0.60 \times 20 = 12$
3	40 lacs to R, 50 lacs to S	$0.50 \times 40 = 20, 0.40 \times 50 = 20$
4	50 lacs to S	$0.30 \times 10 = 3$

49. A company has three factories, A, B and C located in different parts of the country. It has four supply points : D, E, F and G, also located in different parts of the country. Monthly factory production capacities are 2500, 3000 and 4000 units respectively with regular production. If overtime facilities are utilized, capacities of factories A and B can be increased by 20 per cent and 25 per cent additional cost of overtime production is 40 paise and 50 paise in factory A and B respectively. The current supply-point requirements are 2000, 2250, 2750 and 3000 units respectively. Unit production and transportation costs (in Rs.) from factories to supply points are as follows :

Factory	Supply points			
	D	E	F	G
A	1.10	1.30	1.70	1.40
B	1.60	1.80	1.40	1.00
C	2.10	2.40	1.30	1.00

Determine the optimum distribution for this company to minimize total costs.

(Delhi (MBA) Dec. 96)

50. ABC company wishes to develop a monthly production schedule for the next three months. Depending upon the sales commitments, the company can either keep the production constant, allowing fluctuations in inventory, or inventories can be maintained at a constant level, with fluctuating production. Fluctuating inventories result in inventory carrying cost of Rs. 2 per unit. If the company fails to fulfill its sales commitment, it occurs a shortage cost of Rs. 4 per unit per month. Assume that the shortages are back ordered. The production capacities for the next three months are shown below :

Month	Production Capacity		
	Regular	Overtime	Sales
1	50	30	60
2	50	0	120
3	60	50	40

Determine the optimum production schedule.

(Delhi (MBA) April 98)

51. The following information is available concerning the operations of the XYZ manufacturing company :

	Month 1		Month 2	
	Units on order	800	Production capacity .	1,400
Regular time	920		920	
Overtime	250		250	
Excess cost/unit-overtime	Rs. 1.25		Rs. 1.25	
Storage cost/unit	Rs. 0.50		Rs. 0.50	

Formulate and solve the above problem as transportation problem.

(Delhi (MBA) Nov. 95)

52. Stronghold Construction Company is interested in taking loans from banks for some of its projects P,Q,R,S and T. The rates of interest and the lending capacity differ from bank to bank. All these projects are to be completed. The relevant details are provided in the following table :

Bank	Interest rate in percentage for project					Maximum Credit (Rs.'000)
	P	Q	R	S	T	
Private Bank	20	18	18	17	17	Any amount
Nationalised Bank Co-operative Bank	16	16	16	15	16	400
	15	15	15	13	14	250
Amount required(Rs.'000)	200	150	200	125	75	

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Assuming the role of consultant adviser of the company as to how it should take the loans so that the total interest payable will be the least. Are there alternate optimum solutions? If so, indicate one such solution. [Delhi (MCI) 2000]

53. A baking firm can produce a speciality bread in either of its two plants as follows :

Plant	Production Capacity Loaves	Production cost Rs./Loaf
A	2,500	2.30
B	2,100	2.50

Four restaurant chains are willing to purchase this bread; their demand and the prices they are willing to pay are as follows :

Chain	Maximum Demand Loaves	Price Offered Rs./Loaf
1	1,800	3.90
2	2,300	3.70
3	550	4.00
4	1,750	3.60

The cost in paisa of shipping a loaf from a plant to a restaurant chain is

Plant	Chain 1	Chain 2	Chain 3	Chain 4
A	6	8	11	9
B	12	6	8	5

Determining a delivery schedule for the baking firm that will maximize its profit from this bread. Also, write the dual of this transportation problem and use it for checking the optimal solution to the given problem. [Delhi (MBA) 2000]

54. A manufacturer has distribution centres at Delhi, Kolkata and Chennai. These centres have available 30, 50 and 70 units of product. His four retail outlets require the following number of units.

A, 30; B, 20; C, 60; D, 40;

The transportation cost per unit in rupees between each centre and outlet is given in the following table :

Distribution Centres	Retail Outlets			
	A	B	C	D
Delhi	10	7	3	6
Kolkata	1	6	7	3
Chennai	7	4	5	3

[IAS (Main) 2001]

Determine the minimum transportation cost.

55. A company has factories at four different places which supply to warehouses A, B, C, D and E. Monthly factory capacities are 200, 175, 150 and 325 respectively. Unit shipping costs in rupees are given below :

		Warehouses				
		A	B	C	D	E
Factories	1	13	—	31	8	20
	2	14	9	17	6	10
	3	25	11	12	17	15
	4	10	21	13	—	17

Shipping from 1 to B and from 4 to D is not possible. Determine the optimum distribution to minimize the shipping cost. [AIMS (BE Ind.) Bangl. 2002]

56. The following information is available concerning the operations of the xyz manufacturing company :

Type	Month 1	Month 2
Units on order	800	1,400
Production capacity regular time	920	920
Production capacity overtime	250	250
Excess cost/unit (overtime)	Rs. 1.25	Rs. 1.25
Storage cost/unit	Rs. 0.50	Rs. 0.50

Formulate and solve the above problem as transportation problem.

[Delhi (MBA) 2002]

57. What is the total number of constraint equations in a general transportation model with m sources and n destinations? How many of these are independent? Justify your answer. Name the methods for obtaining an initial basic solution. Which is the best one, and why? Also describe it. [Meerut (OR) 2003]

58. Solve the following transportation problem by finding the initial solution by using VAM :

Source \ Destination	D ₁	D ₂	D ₃	Supply
S ₁	8	7	4	60
S ₂	3	8	9	70
S ₃	11	3	5	80
Demand	50	80	80	210

[JNTU (MCA III) 2004]

59. Consider four basis of operation B_i and three targets T_j. The tons of bombs per aircraft from any base that can be delivered to any target are given in the side table :

The daily sortie capacity of each of the four bases is 150 sorties per day. The daily requirement in sorties over each individual target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all three targets.

[JNTU (Mech. & Prod.) May 2004]

	T ₁	T ₂	T ₃
B ₁	8	8	5
B ₂	6	6	6
B ₃	10	8	4
B ₄	8	6	10

OBJECTIVE QUESTIONS

- The initial solution of a transportation problem can be obtained by applying any known method. However, the only condition is that
 - the solution be optimal.
 - the rim conditions are satisfied.
 - the solution not be degenerate.
 - all of the above.
- The dummy source or destination in a transportation problem is added to
 - satisfy rim conditions.
 - prevent solution from becoming degenerate.
 - ensure that total cost does not exceed a limit.
 - none of the above.
- The occurrence of degeneracy while solving a transportation problem means that
 - total supply equals total demand.
 - the solution so obtained is not feasible.
 - the few allocations become negative.
 - none of the above.
- An alternative optimal solution to a minimization transportation problem exists whenever opportunity cost corresponding to unused route of transportation is:
 - positive and greater than zero.
 - positive with at least one is equal to zero.
 - negative with at least one is equal to zero.
 - none of the above.
- One disadvantage of using North-West Corner Rule to find initial solution to the transportation problem is that
 - it is complicated to use.
 - it does not take into account the cost of transportation.
 - it leads to a degenerate initial solution.
 - all of the above.
- The solution to a transportation problem with m -rows (supplies) and n -columns (destinations) is feasible if number of positive allocations are
 - $m + n$.
 - $m \times n$.
 - $m + n - 1$.
 - $m + n + 1$.
- The number of non-negative variables in a basic feasible solution to a $m \times n$ transportation problem is :
 - mn .
 - $m + n$.
 - $m + n + 1$.
 - $m + n - 1$.
- The calculation of opportunity cost in the MODI method is analogous to a
 - $z_j - c_j$ value for non-basic variable columns in the simplex method.
 - value of a variable in X_B -column of the simplex method.
 - variable in the B -column in the simplex method.
 - none of the above.
- An unoccupied cell in the transportation method is analogous to a
 - $z_j - c_j$ value in the simplex method.
 - variable in the B -column in the simplex method.
 - variable not in the B -column in the simplex method.
 - value in the X_B -column in the simplex method.
- If we were to use opportunity cost value for an unused cell to test optimality, it should be
 - equal to zero.
 - most negative number.
 - most positive number.
 - any value.
- During an iteration while moving from one solution to the next, degeneracy may occur when
 - the closed path indicates a diagonal move.
 - two or more occupied cells are on the closed path but neither of them represents a corner of the path.
 - two or more occupied cells on the closed path with minus sign are tied for lowest circled value.
 - either of the above.

Answers

1. (b) 2. (a) 3. (b) 4. (b) 5. (b) 6. (c) 7. (d) 8. (a) 9. (c) 10. (b) 11. (c).

