2. Max. $z = x_1 - x_2 + 3x_3$, subject to the constraints :
$x_1 + x_2 + x_3 \le 10$, $2x_1 - x_3 \le 2$, $2x_1 - 2x_2 + 3x_3 \le 6$ ; $x_1, x_2, x_3 \ge 0$. **[Delhi M. Com. 76]**
**[Ans.** Min. $z_w = 10w_1 + 2w_2 + 6w_2$, s.t. $w_1 + 2w_2 + 2w_3 \ge 1$,
$w_1 - 2w_3 \ge -1$, $w_1 - w_2 + 3w_3 \ge 3$ ; $w_1, w_2, w_3 \ge 0$].
3. Max. $z = 3x_1 + x_2 + 4x_3 + x_4 + 9x_5$, subject to the constraints :
$4x_1 - 5x_2 - 9x_3 + x_4 - 2x_5 \le 6$ ; $2x_1 + 3x_2 + 4x_3 - 5x_4 + x_5 \le 9$; $x_1 + x_2 - 5x_3 - 7x_4 + 11x_5 \le 10$, $x_1$, and $x_2, x_3, x_4, x_5 \ge 0$.
**[Madurai B.Sc. (Appl. Math) 83; Madras BSc. (Math.) 83; Bombay (DIM) 75]**
**[Ans.** Min. $z_w = 6w_1 + 9w_2 + 10w_3$, s.t. $4w_1 + 2w_2 + w_3 \ge 3$ ; $-5w_1 + 3w_2 + w_3 \ge 1$,
$-9w_1 + 4w_2 - 5w_3 \ge 4$, $w_1 - 5w_2 - 7w_3 \ge 1$, $-2w_1 + w_2 + 11w_3 \ge 9$ ; $w_1, w_2, w_3 \ge 0$]
4. Min. $z = 7x_1 + 3x_2 + 8x_3$, subject to the constraints $8x_1 + 2x_2 + x_3 \ge 3$ ; $3x_1 + 6x_2 + 4x_3 \ge 4$, $4x_1 + x_2 + 5x_3 \ge 1$,
$x_1 + 5x_2 + 2x_3 \ge 7$ ; $x_1, x_2, x_3 \ge 0$. **[Delhi B. Sc. (Math.) 74]**
**[Ans.** Max. $z_w = 3w_1 + 4w_2 + w_3 + 7w_4$, s.t. $8w_1 + 3w_2 + 4w_3 + w_4 \le 7$
$2w_1 + 6w_2 + w_3 + 5w_4 \le 3$ ; $w_1 + 4w_2 + 5w_3 + 2w_4 \le 8$ ; $w_1, w_2, w_3, w_4 \ge 0$].
5. Max. $z = 3x_1 + x_2 + 2x_3 - x_4$, subject to the constraints :
$2x_1 - x_2 + 3x_3 + x_4 = 1$, $x_1 + x_2 - x_3 + x_4 = 3$ ; $x_1, x_2, x_3 \ge 0$, and $x_4$ is unrestricted. **[Delhi 77]**
**[Ans.** Min. $z_w = w_1 + 3w_2$, s.t. $2w_1 + w_2 \ge 3$, $-w_1 + w_2 \ge 1$,
$3w_1 - w_2 = 2$, $w_1 + w_2 = -1$ ; $w_1$ and $w_2$ are unrestricted].
6. Max. $z = 3x_1 + x_2 + x_3 - x_4$, subject to the constraints :
$x_1 + 5x_2 + 3x_3 + 4x_4 \le 5$, $x_1 + x_2 = -1$, $x_3 - x_4 \ge -5$ ; $x_1, x_2, x_3, x_4 \ge 0$. **[Bombay B. Sc. (Stat.) 77, 76, 75]**
**[Ans.** Min. $z_w = 5w_1 - w_2 + 5w_3$, s.t. $w_1 + w_2 \ge 3$, $5w_1 + w_2 \ge 1$,
$3w_1 - w_3 \ge 1$, $4w_1 + w_3 \ge -1$ ; $w_1, w_3 \ge 0$ and $w_2$ is unrestricted].
7. Min. $z = x_3 + x_4 + x_5$, subject to the constraints :
$x_1 - x_3 + x_4 + x_5 = -2$, $x_2 - x_3 - x_4 + x_5 = 1$, $x_j \ge 0$ ($j = 1, 2, ..., 5$). **[Gauhati (M.Sc. Stat.) 75]**
**[Ans.** Max. $z_w = -2w_1 + w_2$, s.t. $w_1 + w_2 \le 1$, $w_1 - w_2 \le 1$, $-w_1 + w_2 \le 1$ and $w_1, w_2$ both are unrestricted].
8. Min. $z = x_1 + x_2 + x_3$, subject to the constraints :
$x_1 - 3x_2 + 4x_3 = 5$, $x_1 - 2x_2 \le 3$, $2x_2 - x_3 \ge 4$ ; $x_1, x_3 \ge 0$, and $x_2$ is unrestricted.
**[Delhi M.A. (Bus. Eco.) 78, B. Sc. (Math.) 75]**
**[Ans.** Max. $z_w = 5w_1 + 3w_2 + 4w_3$, s.t. $w_1 - w_2 \le 1$, $-3w_1 + 2w_2 + 2w_3 \le 1$,
$4w_1 - w_3 = 1$, $w_3 \ge 0$, $w_2 \ge 0$, and $w_1$ is unrestricted].
9. Max. $z = 6x_1 + 4x_2 + 6x_3 + x_4$, subject to the constraints :
$4x_1 + 4x_2 + 4x_3 + 8x_4 = 21$, $3x_1 + 17x_2 + 80x_3 + 2x_4 \le 48$, $x_1, x_2 \ge 0$, and $x_3, x_4$ are unrestrieted.
**[Delhi (Math.) 1972]**
**[Ans.** Min. $z_w = 21w_1 + 48w_2$, s.t. $4w_1 + 3w_2 \ge 6$, $4w_1 + 17w_2 \ge 4$,
$4w_1 + 80w_2 = 6$, $8w_1 + 2w_2 = 1$, $w_2 \ge 0$ and $w_1$ is unrestricted]
0. Min. $z = 10x_1 + 6x_2 + 2x_3$, subject to $-x_1 + 5x_2 + x_3 \ge 1$, $3x_1 + x_2 - x_3 \ge 2$; $x_1, x_2, x_3 \ge 0$. **[ICWA (June) 86]**

## UALITY THEOREMS

ual problem is itself a linear programming problem, the dual of the dual problem (7·2) can also be cted. For this, we shall prove the following theorem : *'The dual of a dual is the primal.'* Further, in the ng section, we shall prove some important theorems showing the *fundamental properties of dual ns.*

ll now start to prove a number of fundamental theorems to describe the relationships between the prima dual. These relationships continue to be useful in the development of mathematical programming.

**eorem 7·1 .** *The dual of the dual of a given primal is the primal.*
**[Raj. 85; Kerala (M.Sc. Stat.) 83; Meerut 84, (B.Sc. Hons.) 8**

**of.** Re-writing the *primal* and *dual* problems (7·1) and (7·2) respectively, we have

*nal.* Max. $z_x = c_1x_1 + c_2x_2 + \ldots + c_nx_n$, subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \end{array}\right\} \quad ..(7\cdot$$

$$
\begin{aligned}
-x_1+3x_2-4\,(x_3'-x_3'') &\le -5 \\
x_1-2x_2 \qquad\qquad &\le 3 \\
-2x_2+\quad (x_3'-x_3'') &\le -4 \\
x_1,\,x_2,\,x_3',\,x_3'' &\ge 0 .
\end{aligned}
$$

Let $w_1'$, $w_1''$, $w_2$, $w_3$ be the dual variables. The dual problem of above standard primal is obtained as :

$$\text{Min.}\quad z_w' = 5\,(w_1'-w_1'')+3w_2-4w_3,$$

subject to the constraints :

$$
\begin{aligned}
(w_1'-w_1'')+w_2+0w_3 &\ge -1 \\
-3\,(w_1'-w_1'')-2w_2-2w_3 &\ge -1 \\
4\,(w_1'-w_1'')+0w_2+w_3 &\ge -1 \\
-4\,(w_1'-w_1'')+0w_2-w_3 &\ge 1 \\
w_1',\,w_1'',\,w_2,\,w_3 &\ge 0 .
\end{aligned}
$$

This dual can be written in more compact from as : Max. $z_w=-5w_1-3w_2+4w_3$, subject to the constraints :

$$-w_1-w_2\le 1,\quad 3w_1+2w_2+2w_3\le 1,\quad -4w_1-w_3=1$$

$w_2$, $w_3 \ge 0$, and $w_1$ is unrestricted.

**Example 6.** *Give the dual of the linear programming problem :* Max.$z=3x_1-2x_2$, *subject to*

$x_1\le 4,\ x_2\le 6,\ x_1+x_2\le 5\ \ -x_2\le -1,$ *and* $x_1,\ x_2\ge 0$

or $x_1+x_2\le 5,\ x_1\le 4,\ 1\le x_2\le 6;$ *and* $x_1,\ x_2\ge 0.$ **[Meerut (L.P.) 89, (Math.) 77, Kuruk. 76]**

**Solution.** Since the given problem is already present in the standard primal form, we apply rules of ***Section* 7·3** to get the following dual problem : Min.$z=4w_1+6w_2+5w_3-1w_4$, subject to,

$$
\left.\begin{aligned}
1w_1+0w_2+1w_3+0w_4 &\ge 3 \\
0w_1+1w_2+1w_3-1w_4 &\ge -2 \\
w_1,\,w_2,\,w_3,\,w_4 &\ge 0 .
\end{aligned}\right]
\quad\text{or}\quad
\begin{bmatrix}1&0&1&0\\0&1&1&-1\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\\w_4\end{bmatrix}\ge\begin{bmatrix}3\\-2\end{bmatrix}
$$

**Example 7.** *Convert the following problem into its dual* :

*Min.* $z=2x_1+2x_2+4x_3$, *subject to*

$2x_1+3x_2+5x_3\ge 2,\quad 3x_1+x_2+7x_3\le 3,\quad x_1+4x_2+6x_3\le 5\ ;\quad x_1,\,x_2,\,x_3\ge 0.$ **[Meerut 71 (S)]**

**Solution.** Using the rules of ***Section* 7·3**, we get the standard primal form :

$$\text{Max.}\quad z'=-2x_1-2x_2-4x_3,\ \text{where } z'=-z$$

subject to,

$$
\left.\begin{aligned}
-2x_1-3x_2-5x_3 &\le -2 \\
3x_1+\ x_2+7x_3 &\le 3 \\
x_1+4x_2+6x_3 &\le 5 \\
x_1,\,x_2,\,x_3 &\ge 0 .
\end{aligned}\right\}
\quad\text{or}\quad
\begin{bmatrix}-2&-3&-5\\3&1&7\\1&4&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\le\begin{bmatrix}-2\\3\\5\end{bmatrix}.
$$

Applying the usual rules, we get the corresponding dual problem :

$$\text{Min.}\quad z'=-2w_1+3w_2+5w_3$$

subject to

$$
\left.\begin{aligned}
-2w_1+3w_2+1w_3 &\ge -2 \\
-3w_1+1w_2+4w_3 &\ge -2 \\
-5w_1+7w_2+6w_3 &\ge -4 \\
w_1,\,w_2,\,w_3 &\ge 0 .
\end{aligned}\right]
\quad\text{or}\quad
\begin{bmatrix}-2&3&1\\-3&1&4\\-5&7&6\end{bmatrix}\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}\ge\begin{bmatrix}-2\\-2\\-4\end{bmatrix}.
$$

## EXAMINATION PROBLEMS

***Obtain the dual of the following linear programming problems :***

1. Max. $3x_1+4x_2$, subject to $2x_1+6x_2\le 16$, $5x_1+2x_2\ge 20$ ; $x_1,\,x_2\ge 0$.
   **[Ans.** Min. $z_w=16w_1-20w_2$, s.t. $2w_1-5w_2\ge 3$, $6w_1-2w_2\ge 4$]. **[C.A. (Nov.)**

$4x_1 + 3x_2 + x_3 = 6,\ x_1 + 2x_2 + 5x_3 = 4,$ *and* $x_1, x_2, x_3 \geq 0$

**[Kanpur BSc. 95; Meerut 90; Madurai BSc. (Appl. Math) 85, 83; Calcutta Bsc. (Math.) 80; Raj (M.Sc. 79)]**

**Solution.** The given problem is first written in the standard primal form

Max. $z_x = 2x_1 + 3x_2 + x_3$, subject to the constraints

$$\begin{aligned} 4x_1 + 3x_2 + x_3 &\leq 6 \\ -4x_1 - 3x_2 - x_3 &\leq -6 \\ x_1 + 2x_2 + 5x_3 &\leq 4 \\ -x_1 - 2x_2 - 5x_3 &\leq -4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Following the rules, its dual is obtained as follows :

Minimize $z_w = 6(w_1 - w_2) + 4(w_3 - w_4)$

subject to the constraints :

$$\begin{aligned} 4(w_1 - w_2) + (w_3 - w_4) &\geq 2 \\ 3(w_1 - w_2) + 2(w_3 - w_4) &\geq 3 \\ (w_1 - w_2) + 5(w_3 - w_4) &\geq 1 \\ w_1, w_2, w_3, w_4 &\geq 0. \end{aligned}$$

Again, the dual can also be written as :

Minimize $z_w = 6y_1 + 4y_2$

subject to the constraints :

$$\begin{aligned} 4y_1 + y_2 &\geq 2 \\ 3y_1 + 2y_2 &\geq 3 \\ y_1 + 5y_2 &\geq 1, \end{aligned}$$

$y_1, y_2$ are unrestricted.

**Example 4.** *Give the dual of the LP problem : Min.* $z = 2x_1 + 3x_2 + 4x_3$, *subject to the constraints :*

$2x_1 + 3x_2 + 5x_3 \geq 2,\quad 3x_1 + x_2 + 7x_3 = 3,\quad x_1 + 4x_2 + 6x_3 \leq 5,\quad x_1, x_2 \geq 0$ *and* $x_3$ *is unrestricted.*

**[Delhi (M. Com.) 78 ; Gujrat (Stat.)**

**Solution.** Since the variable $x_3$ is unrestricted in sign, the given LP problem can be transformed i standard primal form by substituting $x_3 = x_3' - x_3''$, where $x_3' \geq 0, x_3'' \geq 0$. Therefore, standard pri becomes :

$$\text{Max.}\quad z_x' = -2x_1 - 3x_2 - 4(x_3' - x_3'')$$

subject to the constraints :

$$\begin{aligned} -2x_1 - 3x_2 - 5(x_3' - x_3'') &\leq -2 \\ 3x_1 + x_2 + 7(x_3' - x_3'') &\leq 3 \\ -3x_1 - x_2 - 7(x_3' - x_3'') &\leq -3 \\ x_1 + 4x_2 + 6(x_3' - x_3'') &\leq 5 \\ x_1, x_2, x_3', x_3'' &\geq 0. \end{aligned}$$

The dual of the given standard primal is,

Min. $z_w' = -2w_1 + 3(w_2' - w_2'') + 5w_3$

subject to the constraints :

$$\begin{aligned} -2w_1 + 3(w_2' - w_2'') + w_3 &\geq -2 \\ -3w_1 + (w_2' - w_2'') + 4w_3 &\geq -3 \\ -5w_1 + 7(w_2' - w_2'') + 6w_3 &\geq -4 \\ 5w_1 - 7(w_2' + w_2'') - 6w_3 &\geq 4 \\ w_1, w_2', w_2'', w_3 &\geq 0 \end{aligned}$$

*OR*

Min. $z_w' = -2w_1 + 3w_2 + 5w_3$,

subject to the constraints :

$$\begin{aligned} -2w_1 + 3w_2 + w_3 &\geq -2 \\ -3w_1 + w_2 + 4w_3 &\geq -3 \\ 5w_1 - 7w_2 - 6w_3 &= 4 \end{aligned}$$

$w_1, w_3 \geq 0$ and $w_2$ is unrestricted

**Example 5.** *Obtain the dual of the LP problem :*

*Min.* $z = x_1 + x_2 + x_3$. *subject to the constraints :*

$x_1 - 3x_2 + 4x_3 = 5,\ x_1 - 2x_2 \leq 3,\ 2x_2 - x_3 \geq 4;\ x_1, x_2 \geq 0$ *and* $x_3$ *is unrestricted.*

**[JNTU (B. Tech) 98;Garhwal 97; Meerut M.Sc. (Math.) 94, (TDC) 90; Bharthidasan BSc. (M**

**Solution.** Transform the given LP problem into the standard primal form by substituting $x_3 =$ where $x_3' \geq 0, x_3'' \geq 0$.

$$\text{Max. } z_x' = -x_1 - x_2 - (x_3' - x_3''),\quad z_x = -z$$

subject to the constraints : $\quad x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$

$$\left\{\begin{array}{l} x_1 - x_2 + 3x_3 \le 4 \\ x_1 - x_2 + 3x_3 \ge 4 \end{array}\right\} \text{ or } \left\{\begin{array}{l} x_1 - x_2 + 3x_3 \le 4 \\ -x_1 + x_2 - 3x_3 \le -4 \end{array}\right\}$$

***Step 4.*** Thus, original problem now becomes of the standard primal form :

Max. $z_x = 0x_1 - 2x_2 - 5x_3$, subject to

$$\left.\begin{array}{ll} -x_1 - x_2 & \le -2 \\ 2x_1 + x_2 + 6x_3 & \le \ \ 6 \\ x_1 - x_2 + 3x_3 & \le \ \ 4 \\ -x_1 + x_2 - 3x_3 & \le -4 \\ x_1, x_2, x_3 & \ge \ \ 0. \end{array}\right\} \qquad ...(7.5)$$

***Step 5.*** Thus, by using rules of *Sec.* 7·3, the required dual is given by :

Min. $z'_w = -2w_1 + 6w_2 + 4w_3 - 4w_4$, subject to

$$\left.\begin{array}{ll} -w_1 + 2w_2 + w_3 - w_4 & \ge \ \ 0 \\ -w_1 + w_2 - w_3 + w_4 & \ge -2 \\ 6w_2 + 3w_3 - 3w_4 & \ge -5 \\ w_1, w_2, w_3, w_4 & \ge \ \ 0 \end{array}\right\} \qquad ...(7.6)$$

**Note.** It is interesting to note that the primal (7·5) and its dual (7·6) both can be conveniently remembered at the same time by using the following tabular form :

$$\begin{array}{ccccc} & (x_1, \ x_2, \ x_3) & & \text{Min.} \\ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} & \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 6 \\ 1 & -1 & 3 \\ -1 & 1 & -3 \end{bmatrix} & \le & \begin{bmatrix} -2 \\ 6 \\ 4 \\ -4 \end{bmatrix}. \\ & \ge & & \\ & \text{Max. } (0, -2, -5) & & \end{array}$$

Reading horizontally, we have the primal problem (7·5) and reading vertically, we have the corresponding dual problem (7·6) .

## 7·4. MORE ILLUSTRATIVE EXAMPLES

**Example 2.** *Write the dual of the following LP problem : Min.* $z = 3x_1 - 2x_2 + 4x_3$, *subject to the constraints*

$$3x_1 + 5x_2 + 4x_3 \ge 7, \quad 6x_1 + x_2 + 3x_3 \ge 4, \quad 7x_1 - 2x_2 - x_3 \le 10,$$
$$x_1 - 2x_2 + 5x_3 \ge 3, \quad 4x_1 + 7x_2 - 2x_3 \ge 2, \text{ and } x_1, x_2, x_3 \ge 0.$$

**[Madras B.E. (Civil) 91; ICWA (June) 91; Madurai BSc (Appl.) Math. 84; Madras Bsc. (Math) 83 Dibrugarh (Stat.) 74; Meerut B. Sc. (Math.) 70]**

**Solution.** The given problem can be written in the standard primal form as :

$$\text{Max. } z_x' = -3x_1 + 2x_2 - 4x_3, \text{ where } z'_x = -z$$

ubject to the constraints :

$$\begin{array}{rl} -3x_1 - 5x_2 - 4x_3 & \le -7 \\ -6x_1 - x_2 - 3x_3 & \le -4 \\ 7x_1 - 2x_2 - x_3 & \le \ \ 10 \\ -x_1 + 2x_2 - 5x_3 & \le -3 \\ -4x_1 - 7x_2 + 2x_3 & \le -2 \\ x_1, x_2, x_3 & \ge 0. \end{array}$$

Following the rules of *Sec.* 7·3, the dual of this problem becomes :

Min. $z_w' = -7w_1 - 4w_2 + 10w_3 - 3w_4 - 2w_5$, subject to the constraints :

$$\begin{array}{rl} -3w_1 - 6w_2 + 7w_3 - w_4 - 4w_5 & \ge -3 \\ -5w_1 - w_2 - 2w_3 + 2w_4 - 7w_5 & \ge \ \ 2 \\ -4w_1 - 3w_2 - w_3 - 5w_4 + 2w_5 & \ge -4 \\ w_1, w_2, w_3, w_4, w_5 & \ge \ \ 0. \end{array}$$

**Example 3.** *Obtain the dual of the following LP problem : Max.* $z = 2x_1 + 3x_2 + x_3$, subject to

**Dual Problem.** *Find a column vector* $\mathbf{w} \in R^m$, *which minimizes* $z_w = \mathbf{b}'\mathbf{w}$, *subject to* $A'\mathbf{w} \geq \mathbf{c}'$.

*Here it is worthnoting that the dual variables are unrestricted in sign.*

*Now the problem is 'what will be the rules and tricks to obtain a dual problem for such linear programming problem which is not given in the standard primal form (7.1) considered in* ***subsection 7.2.1.*** *'*

The following section is devoted to answer this question.

---

**Q.** 1. Define the dual of a linear programming problem. **[Meerut M.Sc. (Math.) 94; Calicut 78, 76]**

2. What is dual ? **[Bharthiar BSc. (Stat.) 94; Madras BSc. (Math) 93]**

3. What do you mean by primal and dual problems ? Is the number of constraints in the primal and dual the same ?

**[Kanpur 96; Madras MSc. (Math.) 90]**

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## 3. GENERAL RULES FOR CONVERTING ANY PRIMAL INTO ITS DUAL

If the system of constraints in a given LPP consists of a *mixture of equations, inequalities* ($\leq$ or $\geq$), *non-negative variables or unrestricted variables*, then the dual of the given problem can be obtained by reducing to standard primal form by adopting the following alogrithm.

***Step 1.*** First convert the objective function to maximization form, if not.

***Step 2.*** If a constraint has inequality sign $\geq$, then multiply both sides by $-1$ and make the inequality sign $\leq$.

***Step 3.*** If a constraint has an equality sign $(=)$, then it is replaced by two constraints involving the inequalities going in opposite directions, simultaneously.

*For example*, an equation, $x_1 + 2x_2 = 4$, is replaced by two opposite inequalities ($\leq$ and $\geq$) constraints :

$$x_1 + 2x_2 \leq 4 \quad \text{and} \quad x_1 + 2x_2 \geq 4 .$$

The second inequality with $\geq$ sign, can be further written as $-x_1 - 2x_2 \leq -4$.

***Step 4.*** Every unrestricted variable is replaced by the difference of two non-negative variables.

**[Note.** The dual variables that correspond to primal equality constraints must be unrestricted in sign, and those associated with the primal inequalities must be non-negative.]

***Step 5.*** We get the *standard primal form*, of given LPP in which —

*(i)* all the constraints have '$\leq$' sign, where the objective function is of maximization form ; or

*(ii)* all the constraints have '$\geq$' sign, where the objective function is of minimization form.

***Step 6.*** Finally, the dual of the given problem is obtained by :

*(i)* transposing the rows and columns of constraint coefficients;

*(ii)* transposing the coefficients $(c_1, c_2, \ldots, c_n)$ of the objective function and the right side constants $(b_1, b_2, \ldots, b_m)$;

*(iii)* changing the inequalities from '$\leq$' to '$\geq$' sign; and

***(iv)*** **minimizing the objective function instead of maximizing it.**

---

**1. State the general rules for converting any primal LPP into its dual.** **[Madras BSc (Math.)]**

**2. Set up the dual when its primal is given in canonical form.** **[Bharthidasan B.Sc. (Math.)]**

**3. Write a note on duality in linear programming problem.** **[Madras BSc. (Appl. Math.)]**

---

**Example 1.** *Find the dual of the following primal problem :*

*Min.* $z_x = 2x_2 + 5x_3$, *subject to* $x_1 + x_2 \geq 2,\ 2x_1 + x_2 + 6x_3 \leq 6,\ x_1 - x_2 + 3x_3 = 4$, *and* $x_1, x_2, x_3 \geq 0$.

**[Kanpur 2000, 96; Meerut (MSc) 84, (BSc Hons.)**

**Solution.** First, convert the problem into *standard primal form*, as follows :

***Step 1.*** *Change the objective function of minimization into maximization one*, that is,

$$\max.\ z'_x = -2x_2 - 5x_3 ,\ \text{where } z'_x = -z_x .$$

***Step 2.*** The inequality $x_1 + x_2 \geq 2$ can be written as $-x_1 - x_2 \leq -2$.

***Step 3.*** The equation $x_1 - x_2 + 3x_3 = 4$ can be expressed as a pair of inequalities :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\le b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\le b_m \\ \text{and} \quad x_1, x_2, \ldots, x_n &\ge 0 \end{aligned} \qquad \ldots(7\cdot1)$$

*where the sign of all parameters (a, b, c's) are arbitrary.*

The dual of the above problem is obtained by,

(i) *transposing the coefficient matrix;*

(ii) *interchanging the role of constant terms and the coefficients of the objective function;*

(iii) *reverting the inequalities;*

(iv) *minimizing the objective function instead of maximizing it.*

***Dual Problem*** : *Find* $w_1, w_2, w_3, \ldots, w_m$, *which minimize* $z_w = b_1w_1 + b_2w_2 + \ldots + b_mw_m$, *subject to*

$$\left.\begin{aligned} a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m &\ge c_1 \\ a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m &\ge c_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m &\ge c_n, \\ \text{and} \qquad w_1, w_2, \ldots, w_m &\ge 0. \end{aligned}\right\} \qquad \ldots(7\cdot2)$$

Thus, by definition, (7·2) is the dual of (7·1), and $w_1, w_2, w_3, \ldots, w_m$ are called the *dual variables.*

The primal-dual relationship may be remembered more conveniently by using the following table :

$$\begin{array}{ccc} & (x_1, \ldots, x_n) & \text{Min.} \\ \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} & \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \le & \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \\ & \ge & \\ \text{Max.} & (c_1, \ldots, c_n) & \end{array}$$

Primal constraints should be read across the table while dual constraints should be read down the columns.

An example of a symmetric primal and its dual is given below :

***Primal Problem*** : *Max.* $z_x = 3x_1 + 5x_2$, *subject to* $x_1 \le 4$, $x_2 \le 6$, $3x_1 + 2x_2 \le 18$, *and* $x_1, x_2 \ge 0$.

The corresponding dual problem is the following :

***Dual Problem*** : Min. $z_w = 4w_1 + 6w_2 + 18w_3$, *subject to* $w_1 + 3w_3 \ge 3$, $w_2 + 2w_3 \ge 5$ and $w_1, w_2, w_3 \ge 0$.

## 2-2 Matrix Form of Symmetric Primal and Its Dual

***Primal Problem.*** Find a column vector $\mathbf{X} \in R^n$, which maximizes $z_x = \mathbf{CX}$, $\mathbf{C} \in R^n$ (*primal objectiv* *nction*) subject to

$$\mathbf{AX} \le \mathbf{b}, \mathbf{b} \in R^m, \mathbf{X} \ge \mathbf{0}. \qquad \ldots(7\cdot$$

here $\mathbf{A}$ is an $m \times n$ real matrix.

***Dual Problem.*** Find a column vector $\mathbf{W} \in R^m$, which minimizes $z_w = \mathbf{b}^T\mathbf{W}$, $\mathbf{b} \in R^m$ (*dual object* *nction*) subject to

$$\mathbf{A}^T\mathbf{W} \ge \mathbf{C}^T, \mathbf{C} \in R^n, \ \mathbf{W} \ge \mathbf{0}, \qquad \ldots($$

ere $\mathbf{W} = (w_1, w_2, \ldots, w_m)$ and $\mathbf{A}^T$, $\mathbf{b}^T$, $\mathbf{C}^T$ are the transpose of $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{C}$ (given in the primal) respectiv

## -3 Unsymmetric Primal-Dual Problems

***Primal Problem.*** Find a column vector $\mathbf{X} \in R^n$, which maximizes $z_x = \mathbf{CX}$, $\mathbf{C} \in R^n$, subject to

$$\mathbf{AX} = \mathbf{b}, \mathbf{X} \ge \mathbf{0}, \mathbf{b} \in R^m$$

ere $\mathbf{A}$ is an $m \times n$ real matrix.

[illegible handwriting]



Now associated with the above problem, we can consider a different problem.

*Suppose there is a wholesale dealer selling two vitamins $v_1$ and $v_2$ along with some other commodities. The retailers purchase the vitamins from him and from the two foods $F_1$ and $F_2$ (as given in above table). The dealer knows very well that the foods $F_1$ and $F_2$ have their market values only because of their vitamin contents. The problem of the dealer is to fix up the maximum per unit selling prices for the two vitamins $v_1$ and $v_2$ in such a manner that the resulting prices of foods $F_1$ and $F_2$ do not exceed their existing market prices.*

To formulate this problem mathematically, let the dealer decide to fix up two prices $w_1$ and $w_2$ per unit respectively. The dealer's problem is to determine the values of $w_1$ and $w_2$ so as

**To maximize** $z_w = 80\,w_1 + 100\,w_2$
**subject to the constraints :**
$$5w_1 + 6w_2 \le 10$$
$$7w_1 + 11w_2 \le 15$$
**and** $w_1, w_2 \ge 0$.

This associated LPP is considered as the *dual* of the given primal.

We observe that both the above problems are symmetrical in the following sense :

(i) The costs associated with the objective function of one problem are just the requirements in the other's set of constraints.

(ii) The constraint coefficient matrix associated with one problem is simply the transpose of the constraint coefficient matrix associated with the other.

However, one of the problems is a maximization problem while the other is a *minimization* problem.

The above primal dual construction relationship can be more easily understood by the following diagram :

| ***Primal Problem*** | ***Dual Problem*** |
|---|---|
| To minimize $10x_1 + 15x_2$ | To maximize $80w_1 + 100w_2$ |
| subject to the constraints : | subject to the constraints : |
| $5x_1 + 7x_2 \ge 80$ | $5w_1 + 6w_2 \le 10$ |
| $6x_1 + 11x_2 \ge 100$ | $7w_1 + 11w_2 \le 15$ |
| $x_1, x_2 \ge 0$ | $w_1, w_2 \ge 0$ |

**Q. 1. Explain the concept of duality.** **[Madurai B.Sc. (Appl. Math) 84, 82**
**2. Discuss relationship between primal and its dual.** **[Madurai B.Sc. (Appl. Math.) 85; Karnal N.L. Dairy (May) 84**

The concept of a dual problem formulation has often proved useful in science and engineering. Circu theory, economics, and game theory are other examples of such cases. The dual linear programming problem h been, and continues to be, a prowerful tool in the analysis of linear programming and related areas.

## 7·2. DEFINITION OF PRIMAL-DUAL PROBLEMS

### 7·2-1 Symmetric Primal-Dual Problems

Let us consider a linear programming problem in the following form, which may be called the *symme primal problem.*

***Primal Problem :*** *Find* $x_1, x_2, x_3, \ldots, x_n$, *which maximize* $z_x = c_1x_1 + c_2x_2 + \ldots + c_nx_n$, *subject to*

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# Duality in Linear Programming

## 7.1. INTRODUCTION : CONCEPT OF DUALITY

One of the most important discoveries in the early development of linear programming was the concept of duality and its division into important branches. The discovery disclosed the fact that every linear programming problem has associated with it another linear programming problem. The original problem is called the "primal" while the other is called its "*dual*". It is important to note that, in general, either problem can be considered the primal, with the remaining problem its dual. The relationship between the '*primal*' and '*dual*' problems is actually a very intimate and useful one. The optimal solution of either problem reveals information concerning the optimal solution of the other. If the optimal solution to one is known, then the optimal solution of the other is readily available. This fact is important because the situation can arise where the dual is easier to solve than the primal.

### 7.1-1. Concept of Duality in Linear Programming.

In order to make the concept of duality clear, we consider the following diet problem of our common interest.

*The amounts of two vitamins* $v_1$ and $v_2$ *per unit present in two different foods* $F_1$ and $F_2$ *respectively are given in the following table :*

| Vitamin | Food | | Minimum Daily Requirement (units) |
|---|---|---|---|
| | $F_1$ | $F_2$ | |
| $v_1$ | 5 | 7 | 80 |
| $v_2$ | 6 | 11 | 100 |
| Cost per unit | Rs. 10 | Rs. 15 | |

*The problem is to determine the minimum quantities of two foods* $F_1$ and $F_2$ *so that the minimum daily requirement of two vitamins is met and that at the same time, the cost of purchasing these quantities of* $F_1$ and $F_2$ *is minimum.*

To formulate this problem mathematically, let $x_1$ and $x_2$ be the number of units of food $F_1$ and $F_2$ to be purchased respectively. **The problem is to find the values of** $x_1$ and $x_2$ so as :

**To minimize** $z_x = 10x_1 + 15x_2$

**subject to the constraints :**

$$5x_1 + 7x_2 \geq 80$$
$$6x_1 + 11x_2 \geq 100$$

**and** $x_1, x_2 \geq 0$

Here in the formulation of the problem, we have assumed that taking more than the minimum requirement is not harmful, and purchase of negative quantity is meaningless. This LPP will be considered as the *primal problem.*