subject to the constraints :

$3x_1 + 10x_2 + 5x_3 \le 15$,

$x_1 + 2x_2 + x_3 \ge 4$,

$33x_1 - 10x_2 + 9x_3 \le 33$,

$x_1, x_2, x_3 \ge 0$.

[Gauhati (Stat.) 82]

subject to,

$x_1 + x_2 + x_3 = 100$

$12x_1 + 35x_2 + 15x_3 \ge 25$

$8x_1 + 3x_2 + 4x_3 \ge 6$;

$x_1, x_2, x_3 \ge 0$.

[Meerut (M.A.) 93; Karnal 84]

[**Ans.** There does not exist any feasible solution, because artificial variable is not removed in the problem]

9. A firm has an advertising budget of Rs. 7,20,000. It wishes to allocate this budget to two media : magazines and televisions, so that total exposure is maximized. Each page of magazine advertising is estimated to result in 60,000 exposures, whereas each spot on television is estimated to result in 1,20,000 exposures. Each page of magazine advertising costs Rs. 9,000 and each spot on television costs Rs. 12,000. An additional condition that the firm has specified is that at least two pages of magazine advertising be used and at least 3 spots on television. Determine the optimum media-mix for this firm. **(Delhi Univ. M.B.A. 1997)**

[**Hint.** The problem is :

Max. $z = 60{,}000x_1 + 12{,}000x_2$ s.t.

$9{,}000x_1 + 12{,}000x_2 \le 7{,}20{,}000$, $x_1 \ge 2$, $x_2 \ge 3$, $x_1, x_2 \ge 0$,

where x_1 = no. of pages of magazine

x_2 = no. of spots on television]

[**Ans.** $x_1 = 2$, $x_2 = 58 \cdot 5$ and max. z = 7,14,000]

5.5-4 Big-*M*-Method (Charne's Penalty Method)

[Kanpur (B.Sc.) 92, 91]

Computational steps of big-*M*-method are as stated below :

Step 1. Express the problem in the standard form.

Step 2. Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type (≥) and '='. When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists). On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by $-M$ for maximization problems ($+M$ for minimization problems), where $M > 0$.

Step 3. In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

Example 13. *Solve by using big-M method the following linear programming problem :*

Max. $z = -2x_1 - x_2$, *subject to* $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \ge 6$, $x_1 + 2x_2 \le 4$, *and* $x_1, x_2 \ge 0$.

[Bangalore B.E. (Ind. & Prod.) 85 ; Vikram (Math.) 84 ; Calcutta B.Sc. 83; Meerut (M.Sc.) 84, 82]

Solution.

Step 1. Introducing slack, surplus and artificial variables, the system of constraint equations become :

$$3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - x_3 + a_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$

ich can be written in the matrix form as :

$$\begin{array}{cccccc} \mathbf{X}_1 & \mathbf{X}_2 & \mathbf{X}_3 & \mathbf{X}_4 & \mathbf{A}_1 & \mathbf{A}_2 \end{array}$$
$$\begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

Step 2. Assigning the large negative price $-M$ to the artificial variables a_1 and a_2, the objective function becomes : Max. $z = -2x_1 - x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2$.

Step 3. Construct starting simplex table (Table 5·24)

Starting Simplex Table 5·24

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARAIBLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN. RATIO (X_B/X_1)
$\leftarrow a_1$	$-M$	3	$\boxed{3}$	1	0	0	1	0	$3/3 \leftarrow$
a_2	$-M$	6	4	3	-1	0	0	1	$6/4$
x_4	0	4	1	2	0	1	0	0	$4/1$
	$z = -9M$		$(2-7M)$ $\uparrow$	$(1-4M)$	M	0	0 $\downarrow$	0	$\leftarrow \Delta_j$

To apply optimality test, compute

$\Delta_1 = C_BX_1 - c_1 = (-M, -M, 0)(3, 4, 1) - (-2) = 2 + (-3M - 4M + 0) = 2 - 7M$

$\Delta_2 = C_BX_2 - c_2 = (-M, -M, 0)(1, 3, 2) - (-1) = 1 + (-M - 3M + 0) = 1 - 4M$

$\Delta_3 = C_BX_3 - c_3 = (-M, -M, 0)(0, -1, 0) + 0 = M$

$\therefore \quad \Delta_k = \min[\Delta_1, \Delta_2, \Delta_3] = \min[2-7M, 1-4M, M] = \Delta_1$. Therefore, X_1 will be entered.

Using minimum ratio rule, find the key element 3 which indicates that A_1 should be removed. Now the transformed table (*Table* 5·25) is obtained in usual manner.

First Improved Table 5·25

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN RATIO (X_B/X_2)
$\rightarrow x_1$	-2	1	1	$1/3$	0	0	$1/3$	0	$1/\frac{1}{3}$
$\leftarrow a_2$	$-M$	2	0	$\boxed{5/3}$	-1	0	$-4/3$	1	$2/\frac{5}{3} \leftarrow$
s_4	0	3	0	$5/3$	0	1	$-1/3$	0	$3/\frac{5}{3}$
	$z = -2-2M$		0	$(1-5M)/3$ $\uparrow$	M	0	$(-2+7M)/3$	0 $\downarrow$	$\leftarrow \Delta_j$

Again compute, $\Delta_2 = C_BX_2 - c_2 = (-2, -M, 0)(1/3, 5/3, 5/3) + 1 = (1-5M)/3$, and similar $\Delta_3 = M$, $\Delta_5 = (-2+7M)/3$.

Since *minimum* Δ_j *rule and minimum ratio rule* decide the key element $5/3$, so enter X_2 and remove

Therefore, the second improved table (Table 5.26) is formed.

Table 5·26

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN. RATIO
x_1	-2	$3/5$	1	0	$1/5$	0	$3/5$	$-1/5$	
x_2	-1	$6/5$	0	1	$-3/5$	0	$-4/5$	$3/5$	
x_4	0	1	0	0	1	1	1	-1	
	$z = C_BX_B = -12/5$		0	0	$1/5$	0	$M-2/5$	$M-1/5$	$\leftarrow \Delta_j \geq 0$

To test the solution for optimality, compute

$$\Delta_3 = C_BX_3 - c_3 = (-2, -1, 0)(1/5, -3/5, 1) - 0 = 1/5$$

$$\Delta_5 = C_BA_2 - c_5 = (-2, -1, 0)(3/5, -4/5, 1) + M = M - 2/5$$

$$\Delta_6 = C_BA_2 - c_6 = (-2, -1, 0)(-1/5, -3/5, -1) + M = M - 1/5.$$

Since M is as large as possible, $\Delta_3, \Delta_5, \Delta_6$ are all positive. Consequently, the optimal soluti

$x_1 = 3/5$, $x_2 = 6/5$, max $z = -12/5$.

Example 14. *Solve the following problem by Big-M-method : Max.* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to :*
$x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$, *and* $x_1, x_2, x_3, x_4 \geq 0$.

[IAS (Maths.) 95; Kanpur (B.Sc.) 92; Kerala (B.Sc.) 91; Meerut (B.Sc.) 90

Solution. Since the constraints of the given problem are equations, introduce the artificial variables $\geq 0, a_2 \geq 0$. The problem thus becomes :

Max. $z = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2$, subject to the constraints :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad + a_1 \quad &= 15 \\ 2x_1 + x_2 + 5x_3 \quad + a_2 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \quad &= 10 \\ \textit{and } x_1, x_2, x_3, x_4, a_1, a_2 \quad &\geq 0. \end{aligned}$$

Now applying the usual simplex method, the solution is obtained as given in the *Table 5·27.*

Table 5·27 (Example 14)

	$c_j \rightarrow$		1	2	3	-1	$-M$	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN RATIO (X_B/X_3)
a_1	$-M$	15	1	2	3	0	1	0	15/3
$\leftarrow a_2$	$-M$	20	2	1	$\boxed{5}$	0	0	1	20/5 $\leftarrow$
x_4	-1	10	1	2	1	1	0	0	10/1
	$z = (-35M - 10)$		$(-3M-2)$	$(-3M-2)$	$(-8M-4)$ $\uparrow$	0	0	0 $\downarrow$	$\leftarrow \Delta_j$
$\leftarrow a_1$	$-M$	3	$-1/5$	$\boxed{7/5}$	0	0	1	×	$\frac{3}{7/5} \leftarrow$
$\rightarrow x_3$	3	4	2/5	1/5	1	0	0	×	$\frac{4}{1/5}$
x_4	-1	6	3/5	9/5	0	1	0	×	$\frac{6}{9/5}$
	$z = (-3M+6)$		$(M-2)/5$	$-(7M-16)/5$ $\uparrow$	0	0	0 $\downarrow$	×	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	15/7	$-1/7$	1	0	0	×	×	—
x_3	3	25/7	3/7	0	1	0	×	×	25/3
$\leftarrow x_4$	-1	15/7	$\boxed{6/7}$	0	0	1	×	×	15/6 $\leftarrow$
	$z = 90/7$		$-6/7$* $\uparrow$	0	0	0 $\downarrow$	×	×	$\leftarrow \Delta_j$
x_2	2	15/6	0	1	0	1/6	×	×	
x_3	3	15/6	0	0	1	3/6	×	×	
$\rightarrow x_1$	1	15/6	1	0	0	7/6	×	×	
	$z = 15$		0	0	0	75/36	×	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been obtained as :

$$x_1 = x_2 = x_3 = \frac{15}{6} = \frac{5}{2}, \text{ max } z = 15.$$

Example 15. *Use penalty (Big-M) method to maximize :* $z = 3x_1 - x_2$ *subject to the constraints :*

$2x_1 + x_2 \geq 2$, $x_1 + 3x_2 \leq 3$, $x_2 \leq 4$, and $x_1, x_2 \geq 0$. **[Punjabi (Math.)**

Solution. By introducing the surplus variable $x_3 \geq 0$, artificial variable $a_1 \geq 0$, and slack varia $0, x_5 \geq 0$, the problem becomes : Max. $z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$2x_1 + x_2 - x_3 \quad + a_1 \quad = 2$$