Therefore, $k = 2$. So, the entering vector will be $\mathbf{a}_2$ corresponding to which $\mathbf{x}_2$ is given in *Table 8·9*.

To obtain the transformed table :

Key element is found to be $(-\frac{1}{2})$. So we get the transformed table as usual.

Table 8·10

		$c_j \rightarrow$	-2	0	-1	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	
x_2	0	14	5/2	1	0	-2	$-1/2$	
x_3	-1	9	3/2	0	1	-1	$-1/2$	
	$z=-9$		1/2	0	0	1	1/2	$\leftarrow \Delta_j$

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = (0\,,-1)\,(5/2\,,3/2) - 2 = \tfrac{1}{2}$$
$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (0\,,-1)\,(-2\,,-1) - 0 = 1$$
$$\Delta_5 = \mathbf{C_B X_5} - c_5 = (0\,,-1)\,(-\tfrac{1}{2}\,,-\tfrac{1}{2}) - 0 = \tfrac{1}{2}\,.$$

The associated basic solution is given by $x_1 = 0\,, x_2 = 14\,, x_3 = 9\,, z = -9$.

At this iteration, all the basic variables are non-negative, and consequently the calculations terminate with the above mentioned optimal solution.

Also, the optimal values of the dual variables $w_1\,, w_2$, as read from the final *Table* 8·10 , are given by

$$w_1 = \Delta_4 = 1\,,\ w_2 = \Delta_5 = \tfrac{1}{2}\,,$$

where the dual problem is given as follows :

Min. $z_w = 5w_1 + 8w_2$, subject to $w_1 + w_2 \le -2,\ w_1 - 2w_2 \le 0,\ -w_1 - 4w_2 \le -1$, *and* $w_1\,, w_2 \ge 0$.

Example 5. *Use dual simplex method to solve the following L.P.P. :*

$$\textit{Min.}\quad z = 6x_1 + 7x_2 + 3x_3 + 5x_4\,,\ \textit{subject to}$$
$$5x_1 + 6x_2 - 3x_3 + 4x_4 \ge 12,\ x_2 + 5x_3 - 6x_4 \ge 10,\ 2x_1 + 5x_2 + x_3 + x_4 \ge 8,\quad \textit{and}\quad x_1\,, x_2\,, x_3\,, x_4 \ge 0$$

[Meerut 91, 90; Mdurai B.Sc (Appl.) Math. 85; Punjabi MSc. (Math.) 85]

Solution. ***Step* 1.** The given L.P.P.is within in standard primal form as follows :

$$\begin{aligned} \textit{Max.}\quad & z' = -6x_1 - 7x_2 - 3x_3 - 5x_4,\ z' = -z \\ \textit{s.t.}\quad & -5x_1 - 6x_2 + 3x_3 - 4x_4 \le -12 \\ & -x_2 - 5x_3 + 6x_4 \le -10 \\ & -2x_1 - 5x_2 - x_3 - x_4 \le -8 \\ \textit{and}\quad & x_1\,, x_2\,, x_3\,, x_4 \ge 0\,. \end{aligned}$$

Since objective function is of maximization and all $c_j < 0$, we can solve this L.P.P. by dual simplex algorithm.

***Step* 2.** Introducing the slack variables $x_5\,, x_6$ and x_7 the constraints of the above problem reduce to the following equalities :

$$\begin{aligned} -5x_1 - 6x_2 + 3x_3 - 4x_4 + x_5 &= -12 \\ -x_2 - 5x_3 + 6x_4 + x_6 &= -10 \\ -2x_1 - 5x_2 - x_3 - x_4 + x_7 &= -8 \end{aligned}$$

$\therefore$ The starting basic solution to the primal is $x_1 = x_2 = x_3 = x_4 = 0,\ x_5 = -12\,, x_6 = -10\,, x_7 = -8$, which is infeasible.

The starting simplex table is as follows.

Table 8·11

Basic Var.	C_B	X_B	$c_j \rightarrow$ -6 X_1	-7 X_2	-3 X_3	-5 X_4	0 X_5 β_1	0 X_6 β_2	0 X_7 β_3	
x_5	0 →	-12	-5	$\boxed{-6}$	3	-4	1	0	0 →	
x_6	0	-10	0	-1	-5	6	0	1	0	
x_7	0	-8	-2	-5	-1	-1	0	0	1	
	$z'_x = 0$		6	7 ↑	3	5	0 ↓	0	0	← Δ_j

$$\Delta_1 = C_B X_1 - c_1 = 6\,,\ \Delta_2 = 7,\ \Delta_3 = 3\,,\ \Delta_4 = 5\,,\ \Delta_5 = 0 = \Delta_6 = \Delta_7\,.$$

Thus the starting basic solution is infeasible but optimal.

To determine the leaving vector (β_r) :

Since $\qquad x_{Br} = \text{Min}\,(\mathbf{x}_{Bi}\,,\, x_{Bi} < 0) = \text{Min}\,(-12\,,-10\,,-8) = -12 = x_{B1}$

$\therefore \quad r = 1$, *i.e.*, $\beta_1 (= \mathbf{X}_5)$ is the leaving vector.

To determine the entering vector $(\mathbf{a}_k)$ ***for predetermined value of*** r **(= 1) :**

$$\frac{\Delta_k}{x_{rk}} = \frac{\Delta_k}{x_{1k}} = \underset{j}{\text{Max}} \left\{ \frac{\Delta_j}{x_{1j}}\,,\, x_{1j} < 0 \right\} = \text{Max} \left\{ \frac{\Delta_1}{x_{11}}\,,\, \frac{\Delta_2}{x_{12}}\,,\, \frac{\Delta_4}{x_{14}} \right\}$$

$$= \text{Max} \left\{ \frac{6}{-5}\,,\, \frac{7}{-6}\,,\, \frac{5}{-4} \right\} = \frac{-7}{6} = \frac{\Delta_2}{x_{12}}\,.$$

Therefore, $\quad k = 2$, *i.e.*, $\mathbf{a}_2 (= \mathbf{X}_2)$ is the entering vector. Hence key element $= x_{12} = -6$.

Proceeding as usual, the second simplex table is as follows.

Table 8·12

Basic Var.	C_B	X_B	$c_j \rightarrow$ -6 X_1	-7 X_2	-3 X_3	-5 X_4	0 X_5	0 X_6	0 X_7	
x_2	-7	2	5/6	1	-1/2	2/3	-1/6	0	0	
x_6	0 →	$\boxed{-8}$	5/6	0	$\boxed{-11/2}$	20/3	-1/6	1	0 →	
x_7	0	2	13/6	0	-7/2	7/3	-5/6	0	1	
	$z'_x = -14$		1/6	0	13/2 ↑	1/3	7/6	0 ↓	0	← Δ_j

The solution given in Table (8·12) is : $x_1 = x_3 = x_4 = x_5 = 0\,,\ x_2 = 2\,,\ x_6 = -8\,,\ x_7 = 2$, which is infeasib
ptimal. Therefore, it can be improved further.

To determine the leaving vector (β_r) :

Since $\qquad x_{Br} = \text{Min}\,(x_{Bi}\,,\, x_{Bi} \le 0) = \text{Min}\,(x_{B2}) = \text{Min}\,(-8) = -8 = x_{B2}$

Therefore, $\quad r = 2$, *i.e.* $\beta_2 (= \mathbf{X}_6)$ is the leaving vector.

To determine the entering vector $(\mathbf{a}_k)$ ***for predetermined value of*** r **(= 2)**

$$\frac{\Delta_k}{x_{rk}} = \frac{\Delta_k}{x_{2k}} = \underset{j}{\text{Max}} \left\{ \frac{\Delta_j}{x_{2j}}\,,\, x_{2j} < 0 \right\} = \text{Max} \left\{ \frac{\Delta_3}{x_{23}}\,,\, \frac{\Delta_5}{x_{25}} \right\}$$

$$= \text{Max} \left\{ \frac{13/2}{-11/2}\,,\, \frac{7/6}{-1/6} \right\} = \text{Max} \left\{ \frac{-13}{11}\,,\, \frac{-7}{1} \right\} = \frac{-13}{11} = \frac{\Delta_3}{x_{23}}$$

Therefore, $\quad k = 3$, *i.e.* $\mathbf{a}_3 (= \mathbf{X}_3)$ is the entering vector. Hence key element $= x_{23} = -11/2$.

Proceeding as usual, the ***third simplex table*** is obtained as follows :

Table 8.13

		$c_j \rightarrow$	-6	-7	-3	-5	0	0	0	
Basic Var.	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	X_7	
x_2	-7	$30/11$	$25/33$	1	0	$2/33$	$-5/33$	$-1/11$	0	
x_3	-3	$16/11$	$-5/33$	0	1	$-40/33$	$1/33$	$-2/11$	0	
x_7	0	$78/11$	$18/11$	0	0	$-21/11$	$-8/11$	$-7/11$	1	
		$z' = -258/11$	$38/33$	0	0	$271/33$	$32/33$	$13/11$	0	Δ_j

The solution given in *Table 8.13* is : $x_1 = x_4 = x_5 = x_6 = 0$, $x_2 = 30/11$, $x_3 = 16/11$ and $x_7 = 78/11$, which is feasible and optimal.

Hence the optimal feasible solution of the given L.P.P. is :

$$x_1 = 0 \,,\, x_2 = 30/11 \,,\, x_3 = 16/11 \,,\, x_4 = 0 \text{ and Min. } z = -\text{Max. } z' = 258/11 \,.$$

Remark. It is interesting to note in the dual simplex method that we seek to maintain dual feasibility but remove the primal infeasibilities. The starting basic solution in this method is obviously *dual feasible.*

8·4. ADVANTAGE OF DUAL SIMPLEX METHOD OVER SIMPLEX METHOD

The main advantage of dual simplex method over the usual simplex method is that we do not require any artificial variables in the dual simplex method. Hence a lot of labour is saved whenever this method is applicable.

8·5. DIFFERENCE BETWEEN SIMPLEX AND DUAL SIMPLEX METHODS

The dual simplex method is similar to the standard simplex method except that in the latter the starting initia basic solution is feasible but not optimum while in the former it is infeasible but optimum or better tha optimum. The dual simplex method works towards feasibility while simplex method works towards optimality.

8·6. SUMMARY AND COMPUTER APPLICATIONS

The iterative procedure for dual simplex algorithm may be summarized as follows :

Step 1. First convert the minimization LPP into that of maximization, if it is given in the minimizati form.

Step 2. Convert the '$\geq$' type inequalities of given LPP, if any, into those of '$\leq$' type by multiplying corresponding constraints by -1.

Step 3. Introduce slack variables in the constraints of the given problem and obtain an initial basic solution. this solution in the starting dual simplex table.

Step 4. Test the nature of $z_j - c_j$ in the starting table.

(i) If all $z_j - c_j$ and x_{Bi} are non-negative for all i and j, then an optimum basic feasible solution been attained.

(ii) If all $z_j - c_j$ are non-negative and at least one basic variable, x_{Bi}, is negative, then go to ***step* 5.**

(iii) If at least one $z_j - c_j$ is negative, the method is not applicable to the given problem.

Step 5. Select the most negative x_{Bi}. The corresponding basis vector then leaves the basis set **B**. Let x the most negative basic variable so that β_r leaves the basis set **B**.

Step 6. Test the nature of x_{rj} , $j = 1, 2, \ldots, n$.

(i) If all x_{rj} are non-negative, there does not exist any feasible solution to the given problem.

(ii) If at least one x_{rj} is negative, compute the replacement ratios

$$\left\{ \frac{z_j - c_j}{x_{rj}},\, x_{rj} < 0 \right\},\, j = 1, 2, \ldots, n,$$

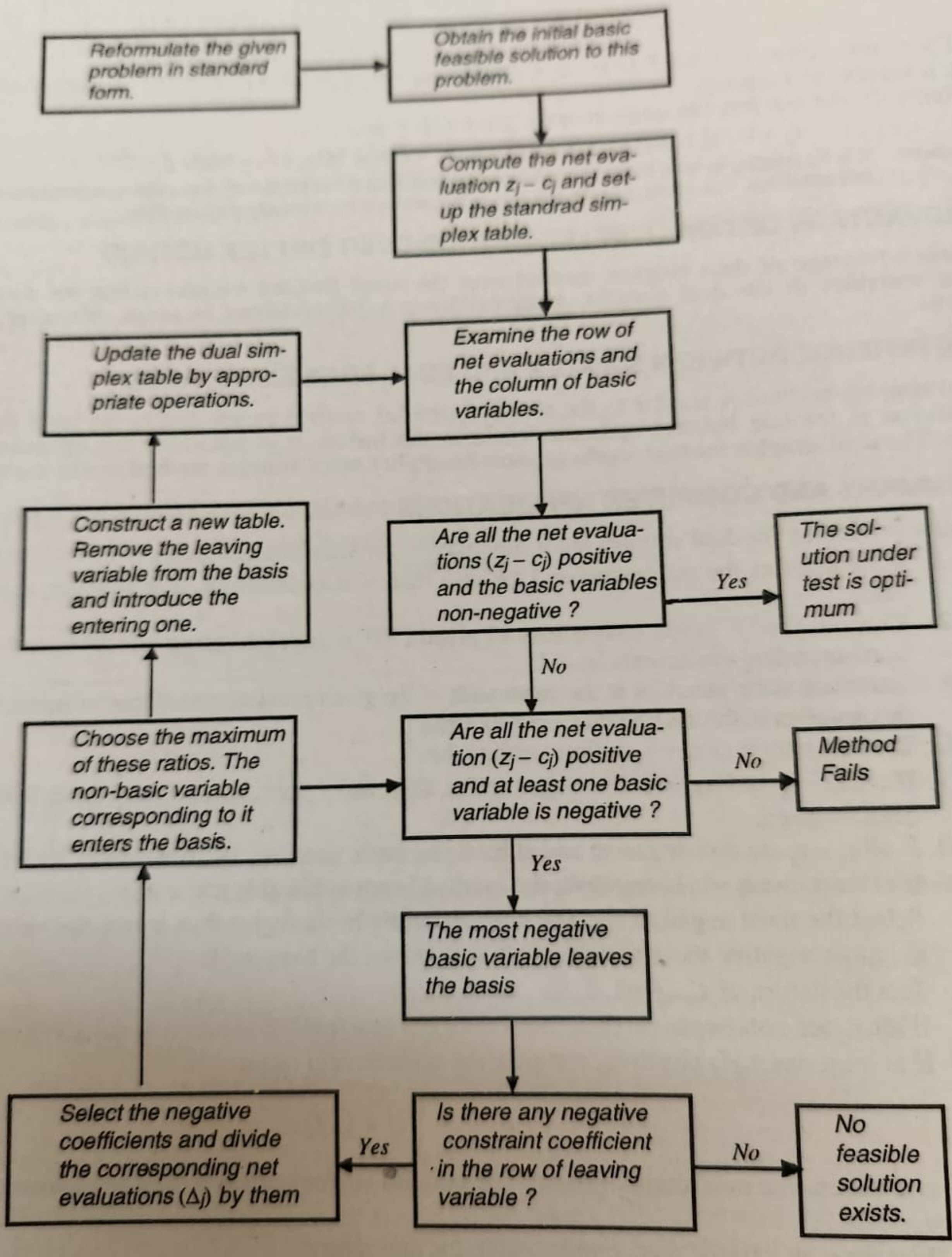
and choose the maximum of these. The column vector corresponding to x_{rk} then enters the ba **B**.

Step 7. Test the new iterated dual simplex table for optimality.

Repeat the entire procedure until either an optimum feasible solution has been attained in number of steps or there is an indication of the non-existence of a feasible solution.

For computer applications, flow-chart of dual simplex method is given as follows :

FLOWCHART OF DUAL SIMPLEX METHOD



... essential difference between regular simplex method and dual simplex method ? [Meerut M.Sc. (Met

2. Give the outlines of dual simplex method. [Meerut M.Sc. (Mat

3. What is dual simplex algorithm ? State various steps involved in the dual simplex algorithm. [Kuru

EXAMINATION PROBLEMS

Use dual simplex method to solve the following linear programming problems :

1. Max. $z = -3x_1 - 2x_2$
subject to
$x_1 + x_2 \geq 1$
$x_1 + x_2 \leq 7$
$x_1 + 2x_2 \leq 10$
$x_2 \leq 3$.
[Delhi (Math.) 72]
[Ans. $x_1 = 0$, $x_2 = 1$, Max. $z = -2$]

2. Max. $z = -2x_1 - 2x_2 - 4x_3$
subject to
$2x_1 + 3x_2 + 5x_3 \geq 2$
$3x_1 + x_2 + 7x_3 \leq 3$
$x_1, x_2, x_3 \geq 0$.
[Sambalpur (Math.) 77]
[Ans. $x_1 = 0$, $x_2 = 2/3$, $x_3 = 0$, $z^* = 4/3$]

3. Min. $x_1 + 2x_2 + 3x_3$,
subject to
$2x_1 - x_2 + x_3 \geq 4$
$x_1 + x_2 + 2x_3 \geq 8$
$x_2 - x_3 \geq 2$
$x_1, x_2, x_3 \geq 0$.
Start with a basic-nonfeasible sol
[Bambay B.Sc. (Math.) 81; I.I.Sc. (Appl. Ma
[Ans. $x_1 = 6$, $x_2 = 2$, $x_3 = 0$, min.

4. Min. $z = 6x_1 + x_2$,
subject to
$2x_1 + x_2 \geq 3$
$x_1 - x_2 \geq 0$
$x_1, x_2 \geq 0$.
[Delhi (Math.) 70]
[Ans. $x_1 = 1$, $x_2 = 1$, min. $z = 7$]

5. Min. $z = x_1 + x_2$,
subject to
$2x_1 + x_2 \geq 2$
$-x_1 - x_2 \geq 1$
$x_1, x_2 \geq 0$.
[Delhi (Math.) 72]
[Ans. Pseudo optimum basic feasible solution.]

6. Min. $z = 10x_1 + 6x_2 + 2x_3$,
subject to
$-x_1 + x_2 + x_3 \geq 1$
$3x_1 + x_2 - x_3 \geq 2$
$x_1, x_2, x_3 \geq 0$.
[Roorkee 73]
[Ans. $x_1 = 1/4$, $x_2 = 5/4$, x ... min.

7. Min. $z = 80x_1 + 60x_2 + 80x_3$,
subject to
$x_2 + 2x_2 + 3x_3 \geq 4$
$2x_1 + 3x_3 \geq 3$
$2x_1 + 2x_2 + x_3 \geq 4$
$4x_1 + x_2 + x_3 \geq 6$
$x_1, x_2, x_3 \geq 0$
[Ans. $x_1 = 16/13$, $x_2 = 6/13$, $x_3 = 8/13$, min. $z = 2280/13$]
[Madras BSc. (Math.) 80; Bombay (B.Sc. Stat. 75)]

8. Min. $z = 3x_1 + 2x_2 + x_3 + 4x_4$,
subject to
$2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$
$3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$
$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$
$x_1, x_2, x_3, x_4 \geq 0$.
[Meerut 88, 86, 85, 83 (P), 74]
[Ans. $x_1 = 65/23$, $x_2 = 0$, $x_3 = 20/23$, $x_4 = 0$, min. $z = 215/23$]

9. Min. $z = x_1 + 2x_2$
subject to
$2x_1 + x_2 \geq 4$
$x_1 + 2x_2 \geq 7$
$x_1, x_2 \geq 0$.
[Kerala M.Sc (Meth.) 84; Ku
[Ans. $x_1 = 0$, $x_2 = 2$, mi

10. Maximize $z = -4x_1 - 6x_2 - 18x_3$
subject to $x_1 + 3x_3 \geq 3$
$x_2 + 2x_3 \geq 5$
$x_1, x_2, x_3 \geq 0$
[Gauhati MSc (Stat.) 82]
[Ans. $x_1 = 0$, $x_2 = 3$, $x_3 = 1$, max. $z = -36$.]

11. Minimize $z = 2x_1 + 2x_2$
subject to $x_1 + 2x_2 \geq 1$
$2x_1 + x_2 \geq 1$
$x_1 \geq 0$ and $x_2 \geq 0$
[M.S. Baroda BSc. (Math) 82]
[Ans. $x_1 = 1/3$, $x_2 = 1/3$, min $z = 4/3$]

12. Use dual simplex method to obtain zeroth and first iteration for the problem :
$$-2x_1 - x_2 + 5x_3 \geq 2,\ 3x_1 + 2x_2 + 4x_3 \geq 16,\ 3x_1 + 5x_2 + 4x_3 = z \text{ (min.)},\ \text{and } x_1, x_2, x_3 \geq 0.$$
Write complementary basis corresponding to first iterate. Write the simplex multipliers with respect to the basis iterate. Verify these results. **[I.I. Sc. (Appl. Ma**
[Ans. $x_1 = x_2 = 0$, $x_3 = 4$, min $z = 16$]

13. (a) Show that the value of the objective function of the dual for any feasible solution is never less than the valu objective function of the primal corresponding to any feasible solution.
(b) Write the dual corresponding to $x + y + 2z \leq 120$, $3x - 2y - z \geq 90$, $2x + 4y + 2z = 10$,, $5x + 8y + 10z =$ ... $x, y, z \geq 0$.Use dual simplex or simplex method and obtain zeroth and first iterates of the dual. Write the multipliers corresponding to the basis of the first iterate. **[I.I. Sc. (Dip. Ind. M**

14. Show with the help of an example how when one solves an LP problem by simplex method going through infeas better than optimal solution, one indirectly goes through infeasible but better than optimal solution of the dual LP p How this fact is utilized in the solution of the dual. **[I.I. Sc. (Dip. Open. M**

15. What is the essential difference between regular simplex and dual simplex method ? **[Meerut**

16. Find optimum solution of the following problem by not using artificial variables :
Min. $z = 10x_1 + 10x_2$, s.t. $x_1 + x_2 \geq 10$, $3x_1 + 2x_2 \geq 24$, $x_1 \geq 0$, $x_2 \geq 0$, **[Delhi (MC**

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