10

Integer Linear Programming

(Gomory's Cutting Plane & Branch-and-Bound Methods)

10·1. INTRODUCTION

As the name implied *'Integer Linear Programming Problems'* are the special class of linear programming problems where *all* or *some* of the variables in the optimal solution are restricted to non-negative integer values. Such problems are called as ***'all integer'*** or ***'mixed integer'*** problems depending, respectively, on whether all or some of the variables are restriced to integer values.

One might think it sufficient to obtain an integer solution to this special class of linear programming problem by using regular simplex method and then rounding off the fractional values thus occuring in the optimal solution. But in some cases, the deviation from the "exact" optimal integer values (as a result of rounding) may become large enough to give an infeasible solution. Hence there was a need to develop a systematic procedure in order to identify the optimal integer solution to such problems.

In 1956, ***R. E. Gomory*** suggested first of all the systematic method to obtain an optimum integer solution to an *'all integer programming problem'*. Later, he extended the method to deal with the more complicated case of *'mixed integer programming problems'* when only some of the variables are required to be integer. These algorithms are proved to converge to the optimal integer solution in a finite number of iterations making use of familiar dual simplex method. This is called the *"cutting plane algorithm"* because it mainly introduces the clever idea of constructing *"secondary"* constraints which, when added to the optimum (non-integer) solution, will effectively cut the solution space towards the required result. Successive application of these constraints should gradually force the non-integer optimum solution toward the desired *"all-integer"* or *"mixed integer"* solution.

Another important approach, called the ***"branch-and-bound technique"*** for solving both the all-integer and the mixed-integer problems, has originated the straight forward idea of finding all feasible integer solutions. A general algorithm for solving ***'all integer'*** and ***'mixed integer'*** linear programming problems was developed by **A.H. Land** and **A.G. Doig** (1960). Also, **Egon Balas** (1965) introduced an interesting enumerative algorithm for L.P. problem with the variables having the value zero or *one,* called the ***zero one programming problem.***

Several algorithms have been developed so far for solving the integer programming problems. But, in this chapter, we shall discuss only two methods : (i) *Gomory's cutting plane method,* and (ii) *Branch-and-bound method.*

10·2. IMPORTANCE OF INTEGER PROGRAMMING PROBLEMS

We have already pointed out earlier that most industrial applications of large scale programming models are oriented towards planning decisions. There are several frequently occuring circumstances in business and industry that lead to planning models involving integer-valued variables. For example, in production, manufacturing is frequently scheduled in terms of batches, lots or runs. In allocation of goods, a shipment must involve a discrete number of trucks, freight cars or aircrafts. In such cases, the fractional value of variables may be meaningless in the context of the actual decision problem. For example, it is not possible to use 2.5 boilers in a thermal power station, 10.4 men in a project, or 5·6 lathes in a workshop.

Many other decision problems can necessitate integer programming models. One category of such problems deals with the sequencing, scheduling and routing decisions. An example is the *travelling salesman problem* which aims at finding a least distance route for a salesman who must visit each of n cities, starting and ending his journey at home city. Larger expenditures of capital and resources are required in *capital budgeting* decisions. This is the main reason why integer programming is so important for marginal decisions. An optimal solution to a capital budgeting problem may yield considerably more profit to a firm than will an approximate or guessed-at solution. For example, fertilizer manufacturing firm with 15 plants may be able to substantially increase profits by cutting back to 10 plants or less, provided this reduction is planned optimally.

10.3. WHY INTEGER PROGRAMMING IS NEEDED ?

We might think it sufficient to obtain an integer solution to the given LP problem by first obtaining the non-integer optimal solution using regular simplex method (or graphical method) and then rounding off the fractional values of the variables. But, in some cases, the deviation from the "exact" optimal integer solution (obtained as a result of rounding) may become large enough to give an infeasible solution. Hence it was necessary to develop a systematic procedure to determine the *optimal integer* solution to such problems. The following example will make the concept more clear.

The question *"why integer programming is needed ?"* can be more easily answered through the following illustrative example.

Consider a simple problem : Max. $z = 10x_1 + 4x_2$ *, subject to the constraints :*

$$3x_1 + 4x_2 \le 8\,,\ x_1 \ge 0,\ x_2 \ge 0, \text{ and } x_1\,,\ x_2 \text{ are integers.}$$

First, ignoring the integer valued restrictions, we obtain the optimal solution : $x_1 = 2\frac{2}{3}$, $x_2 = 0$, max. $z = 26\frac{2}{3}$, by using graphical method. Then, by rounding off the fractional value of $x_1 = 2\frac{2}{3}$, the optimum solution becomes $x_1 = 3$, $x_2 = 0$ with max. $z = 30$. But this solution does not satisfy the constraint $3x_1 + 4x_2 \le 8$ and thus this solution is not feasible.

Now, again, if we round off the solution to $x_1 = 2$, $x_2 = 0$, obviously this is the feasible solution and also integer valued. But, this solution gives $z = 20$ which is far away from the optimum value of $z = 26\frac{2}{3}$. So, this is another disadvantage of getting an integer valued solution by *rounding down* the exact optimum solution. Still there is no guarantee that the "rounding down" solution will be an optimal one because it may be far away from the optimum solution.

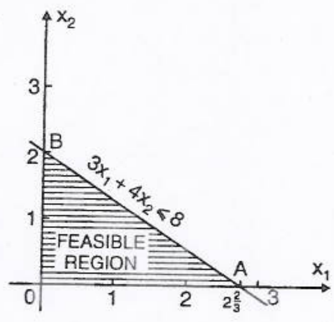


Fig. 10.1

Thus, a systematic procedure for obtaining an exact optimum integer solution to integer programming problems is needed.

We shall now give the formal definitions of integer programming problems.

10·4. DEFINITIONS

Definition 1. ***Integer Programming Problem (I.P.P.).*** *The linear programming problem : Max* $z = \mathbf{CX}$ *, subject to* $\mathbf{AX} = \mathbf{b}, \mathbf{X} \ge \mathbf{0}$ *and some* $x_j \in \mathbf{X}$ *are integers, where* $\mathbf{C}, \mathbf{X} \in R^n$, $\mathbf{b} \in R^m$ *and* $\mathbf{A}$ *is an* $m \times n$ *real matrix, is called an Integer Programming Problem, abbreviated as I.P.P.*

Definition 2. ***All Integer Programming Problem (All I.P.P.).*** An integer programming problem is said to be an *"All Integer Programming Problem"* if all $x_j \in \mathbf{X}$ are integers.

Definition 3. ***Mixed Integer Programming Problem (Mixed I.P.P.).*** An integer programming problem is said to be *'Mixed Integer Programming Problem'* if not all $x_j \in \mathbf{X}$ are integers.

Q. 1. State the general form of an integer programming problem.

[Meerut M.Sc. (Math) 93; Madurai B.Sc. (Appl. Math.) 85]

2. Distinguish between pure and mixed integer programming problems. **[Madurai B.Sc. (Appl. Math.) 83]**

I–Gomory's Cutting Plane Method

10·5. GOMORY'S ALL INTEGER PROGRAMMING TECHNIQUE

In this technique, we first find the optimum solution of the given I.P.P. by regular simplex method as discussed earlier, disregarding the integer condition of variables. Then, we observe the following :

(i) If all the variables in the optimum solution thus obtained have integer values, the current solution will be the desired optimum integer solution.

(ii) If not, the considered L.P.P. requires modification by introducing a *secondary* constraint (also called *Gomory's Constraint*) that reduces some of the non-integer values of variables to integer one, but does not eliminate any feasible integer.

(iii) Then, an optimum solution to this modified L.P.P. is obtained by using any standard algorithm. If all the variables in this solution are integers, then the optimum integer solution is obtained. Otherwise, another *secondary constraint* is added to the L.P.P. and the entire procedure is repeated.

In this way, the optimum integer solution will be obtained eventually after introducing the sufficient number of new constraints. Thus, it becomes specially important to discuss below— how the additional constraints (*Gomory's Constraints*) are constructed.

10·5–1. How to Construct Gomory's Constraint

The secondary constraints which will force the solution toward an all-integer point are constructed as follows.

Let the optimum non-integer solution to the maximization L.P.P. has been obtained. In our usual notations, this solution can be shown by the following optimal simplex table.

Table 10·1

BASIC VAR.	C_B	X_B	BASIC						NON BASIC			
			$X_1(\beta_1)$	$X_2(\beta_2)$	...	$X_i(\beta_i)$	...	$X_m(\beta_m)$	X_{m+1}	...	X_n	
x_1	c_{B1}	x_{B1}	1	0	...	0	...	0	$x_{1,m+1}$	...	x_{1n}	
x_2	c_{B2}	x_{B2}	0	1	...	0	...	0	$x_{2,m+1}$	...	x_{2n}	
:	:	:	:	:		:		:	:		:	
x_i	c_{Bi}	x_{Bi}	0	0	...	1	...	0	$x_{i,m+1}$	...	x_{in}	
:	:	:	:	:	:	:	:	:	:	:	:	
x_m	c_{Bm}	x_{Bm}	0	0	...	0	...	1	$x_{m,m+1}$	..	x_{mn}	
	$z = C_B X_B$		0	0	...	0	...	0	Δ_{m+1}	...	Δ_n	$\leftarrow \Delta_j$

In this table, the variable $(x_{Bi}, i = 1, 2, ..., m)$ represent the basic variables and the remaining $(n - m)$ variables $x_{m+1}, x_{m+2}, \dots, x_n$ are the non-basic variables. However, these variables have been arranged in this order, for our convenience.

Let the ith basic variable x_{Bi} possesses a non-integer value which is given by the constraint equation

$$x_{Bi} = 0x_1 + 0x_2 + \dots + 1x_i + \dots + 0x_m + x_{i,m+1}\, x_{m+1} + \dots + x_{in}\, x_n$$

or

$$x_{Bi} = x_i + \sum_{j=m+1}^{n} x_{ij}\, x_j \qquad \text{or} \qquad x_i = x_{Bi} - \sum_{j=m+1}^{n} x_{ij}\, x_j \qquad \text{...(10·1)}$$

Now, let $x_{Bi} = I_{Bi} + f_{Bi}$ and $x_{ij} = I_{ij} + f_{ij}$, where I_{Bi} and I_{ij} are the largest integral parts of x_{Bi} and x_{ij}, respectively, such that $I_{Bi} \leq x_{Bi}$ and $I_{ij} \leq x_{ij}$. It follows that $0 < f_{Bi} < 1$ and $0 \leq f_{ij} < 1$; that is, f_{Bi} is strictly positive fraction while f_{ij} is a non-negative fraction. For example,

x_a	I_a	$f_a = x_a - I_a$
$2\frac{1}{2}$	2	1/2
$-1\frac{1}{3}$	−2	2/3
−2	−2	0
−3/5	−1	2/5

Now substituting above values in the eqn. (10·1) for x_i, we get

$$x_i = (I_{Bi} + f_{Bi}) - \sum_{j=m+1}^{n} (I_{ij} + f_{ij})\, x_j \qquad \ldots(10{\cdot}2)$$

or

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j = x_i - I_{Bi} + \sum_{j=m+1}^{n} I_{ij}\, x_j\,. \qquad \ldots(10{\cdot}3)$$

Now for all the variables x_i $(i = 1, 2, \ldots, m)$ and x_j $(j = m+1, \ldots, n)$ to be integer valued, the right hand side of the above equation must be an integer. This implies that left-hand side

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j$$

must also be an integer. Since $0 < f_{Bi} < 1$ and $\sum_{j=m+1}^{n} f_{ij}\, x_j \geq 0$, it follows that the inequality condition is satisfied if

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j \leq 0\,. \qquad \ldots(10{\cdot}4)$$

This is true because $f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j \leq f_{Bi} < 1$.

But, since $f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j$ is an integer, then it can be either a zero or a negative integer.

Now the constraint (10·4) can be put in the form

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j + g_i = 0\,, \qquad \text{or} \qquad -f_{Bi} = -\sum_{j=m+1}^{n} f_{ij}\, x_j + g_i \qquad \ldots(10{\cdot}5)$$

where g_i is a non-negative *Gomorian slack variable* which by definition must also be an integer. The constraint equation (10·5) defines the so-called *Gomory's cutting plane.* From *Table 10·1*, the non-basic variables $x_j = 0$ $(j = m+1, \ldots, n)$ and thus by virtue of (8·5) $g_i = -f_{Bi}$ which is clearly infeasible. Thus in order to clear this infeasibility, we have no alternative except to use the *dual simplex method* (as described in *chapter 6*). Practically, this is equivalent to cutting of the solution space towards the optimal integer solution.

Now, after adding the *Gomory's Constraint* (10·5), the optimum simplex Table 10·1 takes the form :

Table 10·2 . Addition of Gomory's Constraint

BASIC VAR.	X_B	X_1 (β_1)	X_2 (β_2)	...	X_i (β_i)	...	X_m (β_m)	X_{m+1}	...	X_n	G_1 (β_{m+1})	
x_1	x_{B1}	1	0	...	0	...	0	$x_{1,\,m+1}$	...	x_{1n}	0	
x_2	x_{B2}	0	1	...	0	...	0	$x_{2,\,m+1}$	...	x_{2n}	0	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
x_i	x_{Bi}	0	0	..	1	...	0	$x_{i,\,m+1}$	...	x_{in}	0	
⋮	⋮	⋮	⋮					⋮		⋮	⋮	
x_m	x_{Bm}	0	0	...	0	...	1	$x_{m,\,m+1}$	...	x_{mn}	0	
g_i	$-f_{Bi}$	0	0	...	0	...	1	$-f_{B,\,m+1}$	...	$-f_{Bm}$	1	
	z	0	0	...	0	...	0	Δ_{m+1}	...	Δ_n	...0	$\leftarrow \Delta_j$

If the new solution (after applying the *dual simplex method*) is all-integer one, the process ends. Otherwise, *second Gomory's Constraint* is constructed from the resulting optimal table and the dual simplex method is again used to clear the infeasibility. This process is repeated so long as an all integer solution is obtained. However, if at any iteration, the dual simplex algorithm indicates that no feasible solution exists then the problem has no feasible integer solution.

10.5-2. Gomory's Cutting-plane (All I.P.P.) Algorithm.

The step-by-step procedure for the solution of all-integer programming problem is as follows :

Step 1. If the I.P.P. is in the minimization form, convert it to maximization form.

Step 2. Then convert the inequalities into equations by introducing *slack* and/or *surplus* variables (if necessary) and obtain the optimum solution of the L.P.P. (after ignoring the integer condition) by usual *simplex method.*

Step 3. Now, test the integerality of the optimum solution which is obtained in *Step* 2.

(i) If the optimum solution contains all integer values, then an optimum integer basic feasible solution has been achieved.

(ii) If not, go to next step.

Step 4. Examine the constraint equations corresponding to the current optimal solution. Let these constraints be expressed by $x_{Bi} = x_i + \sum_{j=m+1}^{n} x_{ij} x_j \quad (i = 1, 2, ..., m)$.

Select the largest fraction of x_{Bi}'s, *i.e.* find $\max_i [f_{Bi}]$. Let it be f_{Bk} for $i = k$.

Step 5. Express the negative fraction, if any in the kth row of the optimum simplex table, as the sum of a negative integer and a non-negative fraction.

Step 6. At this stage, construct the *Gomorian Constraint :*

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij} x_j \le 0,$$

as described in the preceding section, and then introduce the *Gomorian* equation

$$-f_{Bi} = -\sum_{j=m+1}^{n} f_{ij} x_j + g_i$$

to the current set of equality constraints.

Step 7. Starting with this new set of constraint equations, obtain the new optimum solution by using *dual simplex method* in order to clear infeasibility. The slack variable g_i will be the initial leaving basic variable.

Step 8. Now two possibilities may arise :

(i) If this new optimum solution for the *Modified* L.P.P. is an all-integer solution, it is also feasible and optimum for the given L.P.P.

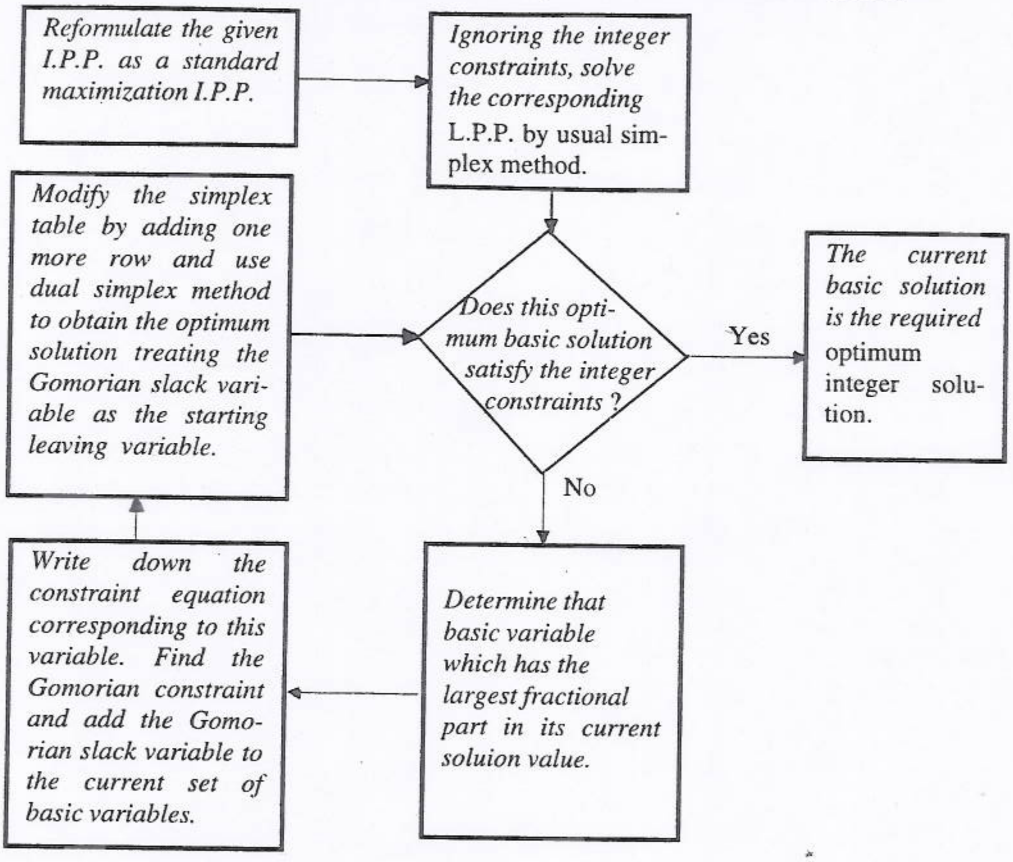
(ii) Otherwise, we return to *Step 4* and repeat the entire process until an optimum feasible integer solution is obtained.

All above steps of Gomory's algorithm can be more precisely demonstrated by the following flow chart :

Q. 1. Explain the concept of integer programming by a suitable example. Give any approach to solve an integer programming problem. **[Madurai B.Sc. (Comp. Sc.) 92]**

2. Explain the algorithm involved in the iterative solution to all I.P.P. **[Madras B.Sc. (Math.) 89]**

FLOW CHART OF GOMORY'S ALL I.P.P. ALGORITHM



10·5–3 Computational Demonstration of Gomory's Algorithm

Example 1. *Solve the integer programming problem :*

Max. $z = 7x_1 + 9x_2$, *subject to* $-x_1 + 3x_2 \le 6,\ 7x_1 + x_2 \le 35,\ x_1 \ge 0, x_2 \ge 0,$ *and integers.* **[Kanpur 96]**

Solution. Step 1. Since the problem is already given in standard maximization form, we go to the next step.

Step 2. Introducing the slack variables, we get the constraint equations

$$-x_1 + 3x_2 + x_3 = 6$$
$$7x_1 + x_2 + x_4 = 35.$$

Now ignoring the integer conditions and then using the regular simplex method we get the following set of tables. The optimum solution to the L.P.P. is given by *Table* 10·3 .

Table 10·3

		$c_j \rightarrow$	7	9	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	MIN. (X_B/X_k)
$\leftarrow x_3$	0	6	−1	$\boxed{3}$	1	0	6/3 ←
x_4	0	35	7	1	0	1	35/1
	$z = C_B X_B = 0$		−7	−9 ↑	0 ↓	0	$\leftarrow \Delta_j$
x_2	9	2	−1/3	1	1/3	0	—
$\leftarrow x_4$	0	33	$\boxed{22/3}$	0	−1/3	1	$33/\frac{22}{3}$
	$z = 18$		−10 ↑	0	3	0 ↓	$\leftarrow \Delta_j$

(Contd.)

x_2	9	$3\frac{1}{2}$	0	1	7/22	1/22	
x_1	7	$4\frac{1}{2}$	1	0	– 1/22	3/22	
	$z = \mathbf{C_B X_B} = 63$		0	0	28/11	15/11	$\leftarrow \Delta_j$

The optimum solution thus obtained is : $x_1 = 4\frac{1}{2}$, $x_2 = 3\frac{1}{2}$, $z = 63$.

Step 3. Since the optimum solution obtained as above is not an integer solution because of $x_1 = 4\frac{1}{2}$ and $x_2 = 3\frac{1}{2}$, we go to next step.

Step 4. We now select the constraint corresponding to max (f_{Bi}) = max. (f_{B1}, f_{B2}) .

Since $x_{B1} = I_{B1} + f_{B1} = 3 + \frac{1}{2}$, and $x_{B2} = I_{B2} + f_{B2} = 4 + \frac{1}{2}$, we have $f_{B1} = f_{B2} = \frac{1}{2}$.

Hence, max. $(f_{B1}, f_{B2}) = \text{max.} \left[\frac{1}{2}, \frac{1}{2}\right] = \frac{1}{2}$.

Thus, in this problem, since both the equations have the same value of f_{Bi} , that is, $f_{B1} = f_{B2} = \frac{1}{2}$, either one of the two equations can be used. Let us consider the x_2-equation, *i.e.*, first-row of optimum table.

Step 5. Negative fraction does not exist.

Step 6. ***To Construct the Gomorian Constraint.***

The Gomorian constraint is given by, $-f_{Bi} = -\sum_{j=m+1}^{n} f_{ij} x_j + g_i$

Here $m = 2$, $n = 4$, $i = 1$, $f_{B1} = \frac{1}{2}$. Thus above constraint becomes :

$$-f_{B1} = -\sum_{j=3}^{4} f_{ij} x_j + g_1 \quad \text{or} \quad -f_{B1} = -f_{13}x_3 - f_{14}x_4 + g_1 \qquad (\text{since } x_3, x_4 \text{ are slack variables})$$

Substituting the values : $f_{13} = 7/22$, $f_{14} = 1/22$, $f_{B1} = 1/2$, we get the required *Gomorian Constraint* as

$$-\frac{1}{2} = -\frac{7}{22} x_3 - \frac{1}{22} x_4 + g_1 \qquad (x_3 = x_4 = 0 \text{ , being non-basic})$$

Obviously, the coefficients of remaining variables x_1 and x_2 in the above *Gomorian constraint* will be taken 0. Thus complete *Gomorian Constraint* can be written as

$$-\frac{1}{2} = 0x_1 + 0x_2 - \frac{7}{22} x_3 - \frac{1}{22} x_4 + g_1$$

Adding this new constraint to the *Optimal Table* 10·3, we get the new *Table* 10·4 .

Table 10·4

		$c_j \rightarrow$	7	9	0	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{G_1}$	
x_2	9	$3\frac{1}{2}$	0	1	7/22	1/22	0	
x_1	7	$4\frac{1}{2}$	1	0	– 1/22	3/22	0	
g_1	0	$\rightarrow$ – 1/2	0	0	$\boxed{-7/22}$	– 1/22	1	
	$z = \mathbf{C_B X_B} = 63$		0	0	28/11 $\uparrow$	15/11	0 $\downarrow$	$\leftarrow \Delta_j$

Step 7. ***To apply dual simplex method.***

(i) leaving vector is $\mathbf{G_1}$, *i.e.*, β_3 . Therefore $r = 3$.

(ii) Entering vector is obtained by

$$\frac{\Delta_k}{x_{rk}} = \text{max.}\left[\frac{\Delta_3}{x_{33}}, \frac{\Delta_4}{x_{34}}\right] = \text{max}\left[\frac{28/11}{-7/22}, \frac{15/11}{-1/22}\right] = \text{max. } [-8, -30] = -8 = \frac{\Delta_3}{x_{33}} .$$

Therefore, $k = 3$. Hence we must enter the vector $\mathbf{a_3}$ corresponding to which $\mathbf{X_3}$ is given in the above table.

Thus, we get the following transformed table.

Table 10·5

		$c_j \rightarrow$	7	9	0	0	0
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	G_1
x_2	9	3	0	1	0	0	1
x_1	7	$4\frac{4}{7}$	1	0	0	1/7	− 1/7
x_3	0	$1\frac{4}{7}$	0	0	1	1/7	− 22/7
	$z = C_{X_B} = 59$		0	0	0	1	8 ← Δ_j

$$\Delta_4 = C_B X_4 - c_4 = (9\ ,7,0)\ (0, \tfrac{1}{7}, \tfrac{1}{7}) - 0 = (0 + 1 + 0) = 1$$

$$\Delta_5 = C_B G_1 - c_5 = (9, 7, 0)\ (1, -\tfrac{1}{7}, -\tfrac{22}{7}) - 0 = (9 - 1 + 0) = 8\ .$$

The non-integer optimum solution given by above table is : $x_1 = 4\frac{4}{7}$, $x_2 = 3$, $x_3 = 1\frac{4}{7}$, $z = 59$.

Step 8. The optimal solution as obtained above by dual simplex method is still non-integer. Thus, a new Gomory's constraint is to be constructed again. Selecting x_1-equation (*i.e.,* IInd row of above table) to generate the cutting plane (because largest fractional part can be $f_{B2} = f_{B3} = \frac{4}{7}$) , we get the Gomory's constraint as

$$-\tfrac{4}{7} = -\tfrac{1}{7}x_4 - \tfrac{6}{7}g_1 + g_2\ . \qquad \text{or} \qquad -\tfrac{4}{7} = 0x_1 + 0x_2 + 0x_3 - \tfrac{1}{7}x_4 - \tfrac{6}{7}g_1 + g_2$$

Adding this constraint to the above Table 10·5 , we get Table 10·6 .

Table 10·6

		$c_j \rightarrow$	7	9	0	0	0	0
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2
x_2	9	3	0	1	0	0	1	0
x_1	7	$4\frac{4}{7}$	1	0	0	1/7	− 1/7	0
x_3	0	$1\frac{4}{7}$	0	0	1	1/7	− 22/7	0
g_2	0	− 4/7	0	0	0	$\boxed{-1/7}$	− 6/7	1
	$z = C_B X_B = 59$		0	0	0	1 ↑	8	0 ↓ ← Δ_j

We again apply *dual simplex method.*

(i) Leaving vector is G_2 (*i.e.* β_4) . Therefore, $r = 4$.

(ii) Entering vector will be obtained by

$$\frac{\Delta_k}{x_{4k}} = \text{max.}\left[\frac{\Delta_4}{x_{44}}, \frac{\Delta_5}{x_{45}}\right] = \text{max.}\left[\frac{1}{-1/7}, \frac{8}{-6/7}\right] = \text{max.}\ [-7\ , -9\tfrac{1}{3}] = -7 = \frac{\Delta_4}{x_{44}}\ .$$

Therefore, $k = 4$. Hence we must enter $\mathbf{a}_4$ corresponding to which $\mathbf{X}_4$ given in the above table. Thus we get the transformed table as below :

Table 10·7

		$c_j \rightarrow$	7	9	0	0	0	0
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2
x_2	9	3	0	1	0	0	1	0
x_1	7	4	1	0	0	0	− 1	1
x_3	0	1	0	0	1	0	− 4	1
x_4	0	4	0	0	0	1	6	− 7
	$z = C_B X_B = 55$		0	0	0	0	2	7 ← Δ_j

$$\Delta_5 = \mathbf{C_B G_1} - c_5 = (9, 7, 0, 0)\,(1, -1, -4, 6) - 0 = (9 - 7 + 0 + 0) = 2$$
$$\Delta_6 = \mathbf{C_B G_2} - c_6 = (9, 7, 0, 0)\,(0, 1, 1, -7) - 0 = (0 + 7 + 0 + 0) = 7\,.$$

Thus, finally we get the optimal integer solution : $x_1 = 4$, $x_2 = 3$, max $z = 55$.

Verification by graphical method :

It can be easily verified that the addition of the above Gomory's constraints effectively 'cut' the solution space as desired. Thus the Gomory's first constraint :

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 + g_1 = -\frac{1}{2},$$

can be expressed in terms of x_1 and x_2 only by substituting :

$x_3 = 6 + x_1 - 3x_2$ and $x_4 = 35 - 7x_1 - x_2$

from the original constraint equations treating g_1 as a slack variable in *step* 2.

This gives $g_1 + x_2 = 3$ or $x_2 \le 3$, treating g_1 as a slack variable.

Similarly, for the Gomory's second constraint, $-\frac{1}{7}x_3 - \frac{6}{7}g_1 + g_2 = -\frac{4}{7}$,

the equivalent constraint in terms of x_1 and x_2 is obtained as $x_1 + x_2 \le 7$.

Now plotting the *Gomory's* constraints $x_2 \le 3$ and $x_1 + x_2 \le 7$ in addition to the constraints of the given problem, we find that it will result in the new (optimal) extreme point (4, 3) as shown in Fig. 10·2 .

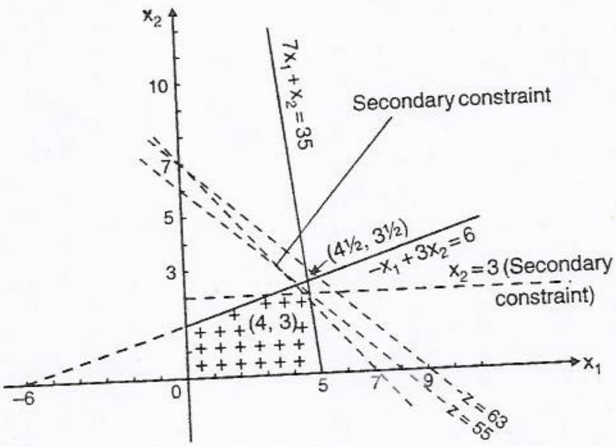


Fig. 10.2

Example 2. *Find the optimum integer solution to the following all I.P.P :*

Max. $z = x_1 + 2x_2$, *subject to the constraints* $2x_2 \le 7$, $x_1 + x_2 \le 7$, $2x_1 \le 11$, $x_1 \ge 0$, $x_2 \ge 0$, *and* x_1 , x_2 *are integers.* **[Vidyasagar 97; Shivaji M.Sc. (Math.) 85; Madras B.Sc. (Math.) 84, 81; Madurai B.Sc. (Appl. Math.) 83]**

Solution. Step 1. Introducing the slack variables, we get

$$\begin{aligned} 2x_2 + x_3 \qquad\quad &= 7 \\ x_1 + x_2 \quad + x_4 \quad &= 7 \\ 2x_1 \qquad\qquad + x_5 &= 11 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0\,. \end{aligned}$$

Step 2. Ignoring the integer condition, we get the initial simplex table as follows :

Table 10·8

		$c_j \rightarrow$	1	2	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	X_5	MIN RATIO (X_B/X_2)
$\leftarrow x_3$	0	7	0	$\boxed{2}$	1	0	0	$7/2 \leftarrow$
x_4	0	7	1	1	0	1	0	7/1
x_5	0	11	2	0	0	0	1	—
		$z = 0$	-1	-2 $\uparrow$	0 $\downarrow$	0	0	$\leftarrow \Delta_j$

Introducing $\mathbf{x_2}$ and leaving $\mathbf{x_3}$ from the basis, we get

Table 10·9

		$c_j \rightarrow$	1	2	0	0	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	MIN $(\mathbf{X_B}/\mathbf{X_k})$
x_2	2	$3\frac{1}{2}$	0	1	1/2	0	0	—
$\leftarrow x_4$	0	$3\frac{1}{2}$	$\boxed{1}$	0	$-1/2$	1	0	$3\frac{1}{2}/1$
x_5	0	11	2	0	0	0	1	11/2
		$z = \mathbf{C_B X_B} = 7$	-1 $\uparrow$	0	1	0 $\downarrow$	0	$\leftarrow \Delta_j$

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = (2, 0, 0)\ (0, 1, 2) - 1 = -1,\ \ \Delta_3 = \mathbf{C_B X_3} - c_3 = (2, 0, 0)\ (\tfrac{1}{2}, -\tfrac{1}{2}, 0) - 0 = 1\ .$$

Introducing $\mathbf{X_1}$ and leaving $\mathbf{X_4}$, we get the following optimum table.

Optimum Table 10·10

		$c_j \rightarrow$	1	2	0	0	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	
x_2	2	$3\frac{1}{2}$	0	1	1/2	0	0	
x_1	1	$3\frac{1}{2}$	1	0	$-1/2$	1	0	
x_5	0	4	0	0	1	-2	1	
		$z = 10\frac{1}{2}$	0	0	1/2	1	0	$\leftarrow \Delta_j$

$$\Delta_3 = \mathbf{C_B X_3} - c_3 = (2, 1, 0)\ (\tfrac{1}{2}, -\tfrac{1}{2}, 1) - 0 = (1 - \tfrac{1}{2} + 0) = \tfrac{1}{2}$$
$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (2, 1, 0)\ (0, 1, -2) - 0 = (0 + 1 + 0) = 1\ .$$

The optimum solution thus obtained is : $x_1 = 3\frac{1}{2}$, $x_2 = 3\frac{1}{2}$, $z = 10\frac{1}{2}$.

Step 3. Since the optimum solution obtained above is not an integer solution, we must go to next step.

Step 4. We now select the constraint corresponding to the criterion

$$\max_i (f_{Bi}) = \max\ (f_{B1}, f_{B2}, f_{B3}) = \max\ (\tfrac{1}{2}, \tfrac{1}{2}, 0) = \tfrac{1}{2}\ .$$

Since in this problem, the x_2-equation and x_1-equation both have the same value of f_{Bi}, *i.e.* $\frac{1}{2}$, either one of the two equations can be used. Let us consider the *first-row* of the above optimum table.
The *Gomory's* constraint to be added is therefore

$$-\sum_{j=3,\,4} f_{1j}\, x_j + g_1 = -f_{B1} \quad \text{or} \quad -f_{13}x_3 - f_{14}x_4 + g_1 = -f_{B1}$$

$$-\tfrac{1}{2}x_3 - 0x_4 + g_1 = -\tfrac{1}{2} \quad \text{or} \quad -\tfrac{1}{2}x_3 + g_1 = -\tfrac{1}{2}\ (x_3 = x_4 = 0)$$

Adding this new constraint to the optimal *Table 10·10*, we get the new *Table 10·11*.

Table 10·11 . New table after adding Gomory constraint

		$c_j \rightarrow$	1	2	0	0	0	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{G_1}$	
x_2	2	$3\frac{1}{2}$	0	1	1/2	0	0	0	
x_1	1	$3\frac{1}{2}$	1	0	$-1/2$	1	0	0	
x_5	0	4	0	0	1	-2	1	0	
g_1	0	$\rightarrow -1/2$	0	0	$\boxed{-1/2}$	0	0	1	
		$z = \mathbf{C_B X_B} = 10\frac{1}{2}$	0	0	1/2 $\uparrow$	1	0	0 $\downarrow$	$\leftarrow \Delta_j$

Step 5. ***To apply dual simplex method.*** Now, in order to remove the infeasibility of the optimum solution : $x_1 = 3\frac{1}{2}$, $x_2 = 3\frac{1}{2}$, $x_5 = 4$, $g_1 = -\frac{1}{2}$, we use the dual simplex method.

(i) Leaving vector is $\mathbf{G}_1$ (*i.e.*, β_4). Therefore, $r = 4$.

(ii) Entering vector is given by

$$\frac{\Delta_k}{x_{4k}} = \max.\left[\frac{\Delta_j}{x_{4j}},\ x_{4j} < 0\right] = \max.\left[\frac{\Delta_3}{x_{43}}\right] = \max.\left[\frac{\frac{1}{2}}{-\frac{1}{2}}\right] = \frac{\Delta_3}{x_{43}}.$$

Therefore, $k = 3$. So we must enter $\mathbf{a}_3$ corresponding to which $\mathbf{X}_3$ is given in the above table. Thus, dropping $\mathbf{G}_1$ and introducing $\mathbf{X}_3$ we get the following dual simplex table :

Table 10·12

		$c_j \rightarrow$	1	2	0	0	0	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{G_1}$	
x_2	2	3	0	1	0	0	0	1	
x_1	1	4	1	0	0	1	0	-1	
x_5	0	3	0	0	0	-2	1	2	
x_3	0	1	0	0	1	0	0	-2	
		$z = \mathbf{C_B X_B} = 10$	0	0	0	1	0	1	$\leftarrow \Delta_j$

$\Delta_4 = \mathbf{C_B X_4} - c_4 = (2, 1, 0, 0)\ (0, 1, -2, 0) - 0 = 1,\ \Delta_6 = \mathbf{C_B G_1} - c_6 = (2, 1, 0, 0)\ (1, -1, 2, -2) - 0 = 1$.

This shows that the optimum feasible solution has been obtained in integers. Thus, finally, we get the integer optimum solution to the given I.P.P. as : $x_1 = 4$, $x_2 = 3$, and max $z = 10$.

10·5-4. Short-cut Method for Constructing the Gomory's Constraint.

After obtaining the non-integer optimal solution by simplex method, we perform the following step-by-step procedure to construct the Gomory's constraint :

Step 1. In the optimal simplex table (with all $\Delta_j \geq 0$), first select the row corresponding to such basic variable which has the maximum fractional value. If more than one basic variables have the same maximum fractional value, then we can select the row corresponding to either of these basic variables.

In ***Example*** **1,** (see ***Table 10.3*** with all $\Delta_j \geq 0$) both the basic variables x_2 and x_1 have the same fractional value (*i.e.* $\frac{1}{2}, \frac{1}{2}$). so we can select either x_2-row or x_1-row. In this case, we have selected x_2-row, *i.e.*

$x_2 \rightarrow 3\frac{1}{2}$	0	1	7/22	1/22

Step 2. In the row selected above we express each number in two parts. First part must be an integer and second part must be a *non-negative* fraction. Thus applying this step to above selected row, we get

$x_2 \rightarrow 3 + (1/2)^*$	$0 + 0^*$	$1 + 0^*$	$0 + (7/22)^*$	$0 + (1/22)^*$

Here non-negative fractional values are marked with '*'

Step 3. Then, we write the *negative* of the fractional values which are marked with '*' in *step* 2. Thus, the new row corresponding to Gomorian slack variable g_1 becomes :

$g_1 \rightarrow -\frac{1}{2}$	0	0	$-7/22$	$-1/22$

This row can be directly added to the optimal simplex table with Gomorian slack variable g_1 as additional basic variable and immediately we increase the dimension of basis matrix by introducing one more unit matrix column $\mathbf{G_1}$.

Obviously, above constructed new row will give us the Gormorian constraint :

$$-\tfrac{1}{2} \geq 0x_1 + 0x_2 - \tfrac{7}{22}x_3 - \tfrac{1}{22}x_4 \quad \text{or} \quad 0x_1 + 0x_2 - \tfrac{7}{22}x_3 - \tfrac{1}{22}x_4 + g_1 = -\tfrac{1}{2}.$$

After introducing the new row corresponding to the *Gomorian* constraint, we apply usual dual simplex method to proceed further.

Above outlined prodcedure will be very convenient to apply directly (orally) whenever we need to construct a *Gomory's* constraint

Example 3. *Solve the following integer programming problem :*

Max. $z = 2x_1 + 20x_2 - 10x_3$, *subject to the constraints :*

$2x_1 + 20x_2 + 4x_3 \leq 15$, $\;6x_1 + 20x_2 + 4x_3 = 20$, *and* $x_1, x_2, x_3 \geq 0$; *and are integers.*

Solve the problem as a (continuous) linear program, then show that it is impossible to obtain feasible integer solution by using simple rounding. Solve the problem using any integer program algorithm.

[Sri Venkt. M.Sc. (Stat.) 77]

Solution. Introducing the slack variable $x_4 \geq 0$ and an artificial variable $a_1 \geq 0$, an initial basic feasible solution is $x_4 = 15$ and $a_1 = 20$.

Now computing the net-evaluations (Δ_j) and then using simplex method, the following optimum simplex table is obtained.

Optimal Simplex Table 10·13 .

		$c_j \rightarrow$	2	20	− 10	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	
x_2	20	5/8	0	1	1/5	3/40	
x_1	2	5/4	1	0	0	− 1/4	
	$z = 15$		0	0	14	1	$\leftarrow \Delta_j$

Thus the following non-integer optimum solution is obtained :

$$x_1 = 5/4,\ x_2 = 5/8,\ x_3 = 0,\ \max z = 15.$$

The rounded solution will be $x_1 = 1$, $x_2 = 0$, $x_3 = 0$.

Since this solution satisfies the first constraint only, it is not possible to obtain a feasible solution by using simple rounding. So to obtain the integer-valued solution, we proceed as follows :

$$\text{Max.}\ (f_{B1}, f_{B2}) = \text{Max.}\left(\frac{5}{8}, \frac{1}{4}\right) = \frac{5}{8}.$$

Therefore, from the first row of optimal table, we have

$$\tfrac{5}{8} = 0x_1 + x_2 + \tfrac{1}{5}x_3 + \tfrac{3}{40}x_4$$

or

$$(0 + \tfrac{5}{8}) = (0 + 0)\,x_1 + (1 + 0)\,x_2 + (0 + \tfrac{1}{5})\,x_3 + (0 + \tfrac{3}{40})\,x_4$$

The corresponding fractional cut will be $-\tfrac{5}{8} = 0x_1 + 0x_2 - \tfrac{1}{5}x_3 - \tfrac{3}{40}x_4 + g_1$

Now inserting the additional constraint in the optimum simplex table, the following modified table is obtained.

Table 10·14

		$c_j \rightarrow$	2	20	− 10	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	G_1	
x_2	20	5/8	0	1	1/5	3/40	0	
x_1	2	5/4	1	0	0	− 1/4	0	
g_1	0 $\rightarrow$	− 5/8	0	0	− 1/5	$\boxed{-3/40}$	1	
	$z = 15$		0	0	14	1 $\uparrow$	0 $\downarrow$	$\leftarrow \Delta_j$

First Iteration. Remove G_1 and insert X_4 by *dual simplex method.*

Table 10·15

BASIC VARIABLES	C_B	$c_j \rightarrow$ X_B	2 X_1	20 X_2	− 10 X_3	0 X_4	0 G_1
x_2	20	0	0	1	0	0	1
x_1	2	10/3	1	0	2/3	0	− 10/3
x_4	0	25/3	0	0	8/3	1	− 40/3
	$z = 20/3$		0	0	34/3	0	$40/3 \leftarrow \Delta_j$

Again, since the solution is non-integer one, insert one more fractional cut. From the third row of Table 10·15 ,

$$25/3 = 8/3\, x_3 + x_4 - 40/3\, g_1$$

or

$$(8 + 1/3) = (2 + 2/3)\, x_3 + (1 + 0)\, x_4 + (-14 + 2/3)\, g_1$$

The corresponding fractional cut will be $-1/3 = 0x_1 + 0x_2 - 2/3\, x_3 + 0x_4 - 2/3\, g_1 + g_2$

Inserting this constraint in Table 10·15 , the following modified table is obtained.

Table 10·16

BASIC VAR.	C_B	$c_j \rightarrow$ X_B	2 X_1	20 X_2	− 10 X_3	0 X_4	0 G_1	0 G_2
x_2	20	0	0	1	0	0	1	0
x_1	2	10/3	1	0	2/3	0	− 10/3	0
x_4	0	25/3	0	0	8/3	1	− 40/3	0
g_2	0	$\rightarrow$ − 1/3	0	0	$\boxed{-2/3}$	0	− 2/3	1
	$z = 20/3$		0	0	34/3 ↑	0	40/3	$0 \leftarrow \Delta_j$ ↓

Second Iteration. Using dual simplex method remove G_2 and introduce X_3 .

Table 10·17

BASIC VAR.	C_B	$c_j \rightarrow$ X_B	2 X_1	20 X_2	− 10 X_3	0 X_4	0 G_1	0 G_2
x_2	20	0	0	1	0	0	1	0
x_1	2	3	1	0	0	0	− 4	1
x_4	0	7	0	0	0	1	− 16	4
x_3	− 10	1/2	0	0	1	0	1	− 3/2
	$z = 1$		0	0	0	0	2	$17 \leftarrow \Delta_j$

Since the solution is still non-integer, a *third* fractional cut is required. From the last row of above table, we can construct the Gomorian constraint $-1/2 = -1/2\, g_2 + g_3$

Inserting this additional constraint in the above table, the modified simplex table becomes :

Table 10·18

BASIC VAR.	C_B	$c_j \rightarrow$ X_B	2 X_1	20 X_2	− 10 X_3	0 X_4	0 G_1	0 G_2	0 G_3	
x_2	20	0	0	1	0	0	1	0	0	
x_1	2	3	1	0	0	0	− 4	1	0	
x_4	0	7	0	0	0	1	− 16	4	0	
x_3	− 10	1/2	0	0	1	0	1	− 3/2	0	
g_3	0 $\rightarrow$	− 1/2	0	0	0	0	0	$\boxed{-1/2}$	1	
	$z = 1$		0	0	0	0	2	17 ↑	0 ↓	$\leftarrow \Delta_j$

Third Iteration. Using dual simplex method, remove G_3 and introduce G_2 .

Table 10·19

		$c_j \rightarrow$	2	20	− 10	0	0	0	0
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2	G_3
x_2	20	0	0	1	0	0	1	0	0
x_1	2	2	1	0	0	0	− 4	0	2
x_4	0	3	0	0	0	1	− 16	0	8
x_3	− 10	2	0	0	1	0	1	0	− 3
g_2	0	1	0	0	0	0	0	1	− 2
	$z = -16$		0	0	0	0	2	0	34 ← Δ_j

Thus an optimum integer solution is obained as : $x_1 = 2$, $x_2 = 0$, $x_3 = 2$, max. $z = -16$.

Example 4. *The owner of a ready-made garments store two types of shirts known as Zee-shirts and Button–down shirts. He makes a profit of Re. 1 and Rs. 4 per shirt on Zee-shirts and Button-down shirts respectively. He has two Tailors (A and B) at his disposal to stitch the shirts. Tailor A and Tailor B can devote at the most 7 hours and 15 hours per-day respectively. Both these shirts are to be stitched by both the tailors. Tailor A and Tailor B spend two hours and five hours respectively in stitching Zee-shirt, and four hours and three hours respectively in stitching a Button-down shirt. How many shirts of both the types should be stitched in order to maximize daily profit ?*

(a) Set-up and solve the linear programming problem.

(b) If the optimal solution is not integer-valued, use Gomory's technique to derive the optimal integer solution. **[Delhi (M.B.A.) 72]**

Formulation of the problem. Suppose the owner of ready-made garments decide to make x_1 Zee-shirts and x_2 Button-down shirts. Then the availability of time to tailors has the following restrictions :

$$2x_1 + 4x_2 \leq 7, \quad 5x_1 + 3x_2 \leq 15 , \quad \text{and} \quad x_1 , x_2 \geq 0 .$$

The problem of the owner is to find the values of x_1 and x_2 to maximize the profit $z = x_1 + 4x_2$.

Solution. Introducing the slack variables $x_3 \geq 0$, $x_4 \geq 0$ in the constraints of the given problem, we have an initial basic feasible solution : $x_3 = 7$, $x_4 = 15$.

Computing the net-evaluations Δ_j and using simplex method an optimun solution is obtained as given in the following table.

Table 10·20

		$c_j \rightarrow$	1	4	0	0
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4
x_2	4	→ 7/4	1/2	1	1/4	0
x_4	0	39/4	7/2	0	− 3/4	1
	$z = 7$		1	0	1	0 ← Δ_j

Thus a non-integer solution is obtained as : $x_1 = 0$, $x_2 = \frac{7}{4}$, $x_4 = 39/4$, $z = 7$.

To find the integer valued solution, add a fractional cut constraint in the optimum simplex table. Since the fractional parts of $\mathbf{X_B}$ are $\left[\frac{3}{4}, \frac{3}{4}\right]$, select the row arbitrarily. So $f_{B1} = \frac{3}{4}$. Then from the first row of the Table 10·20 , we have

$$(1 + \tfrac{3}{4}) = (0 + \tfrac{1}{2})\, x_1 + (1 + 0)\, x_2 + (0 + \tfrac{1}{4})\, x_3 + (0 + 0)\, x_4$$

The corresponding fractional cut is therefore given by

$$-\tfrac{3}{4} = -\tfrac{1}{2}\, x_1 + 0x_2 - \tfrac{1}{4}\, x_3 + 0x_4 + g_1$$

Now inserting this additional constraint in the optimum simplex table, the modified table becomes.

Table 10·21

		$c_j \rightarrow$	1	4	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	
x_2	4	$1\frac{3}{4}$	1/2	1	1/4	0	0	
x_4	0	$9\frac{3}{4}$	7/2	0	−3/4	1	0	
g_1	$0 \rightarrow$	−3/4	$\boxed{-1/2}$	0	−1/4	0	1	
	$z=7$		1 ↑	0	1	0	0 ↓	$\leftarrow \Delta_j$

First Iteration. Using dual simplex method, remove G_1 and insert X_1.

Table 10·22

		$c_j \rightarrow$	1	4	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	
x_2	4	1	0	1	0	0	1	
x_4	0	9/2	0	0	−5/2	1	7	
x_1	1	$\rightarrow$ 3/2	1	0	1/2	0	−2	
	$z=11/2$		0	0	1/2	0	2	$\leftarrow \Delta_j$

Again, since the solution is non-integer one, insert another fractional cut in *Table* 10·22 . From the third row of above table, we have $(1+\frac{1}{2}) = (1+0)\,x_1 + (0+0)\,x_2 + (0+\frac{1}{2})\,x_3 + (0+0)\,x_4 + (-2+0)\,g_1$

The corresponding fractional cut will be $-\frac{1}{2} = 0x_1 + 0x_2 - \frac{1}{2}x_3 + 0x_4 + 0 \cdot g_1 + g_2$.

Now inserting this additional constraint, the modified table becomes Table 10.23.

Table 10·23

		$c_j \rightarrow$	1	4	0	0	0		
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2	
x_2	4	1	0	1	0	0	1	0	
x_4	0	9/2	0	0	−5/2	1	7	0	
x_1	1	3/2	1	0	1/2	0	−2	0	
$\leftarrow g_2$	0	$\rightarrow$ −1/2	0	0	$\boxed{-1/2}$	0	0	1	
	$z=11/2$		0	0	1/2 ↑	0	2	0 ↓	$\leftarrow \Delta_j$

Second Iteration. Using dual simplex method, remove G_2 and insert X_3.

Table 10.24

		$c_j \rightarrow$	1	4	0	0	0	0
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2
x_2	4	1	0	1	0	0	1	0
x_4	0	7	0	0	0	1	7	−5
x_1	1	1	1	0	0	0	−2	1
x_3	0	1	0	0	1	0	0	−2
	$z=5$		0	0	0	0	2	$1 \leftarrow \Delta_j$

This gives us an optimum integer solution : $x_1 = 1$, $x_2 = 1$, and max $z = 5$.

Thus the owner of ready-made garments should produce one Zee-shirt and also one Button-down shirt in order to get the maximum profit of Rs. 5.

Example 5. *A manufacturer of baby-dolls makes two types of dolls : Doll X and Doll Y. Processing of these two dolls is done on two machines A and B. Doll X requires two hours on machine A and six hours on machine B. Doll Y requires five hours on machine A and also five hours on machine B. There are sixteen hours of time per day available on machine A and thirty hours on machine B. The profit gained on both the dolls is same, i.e., one rupee per doll. What should be daily production of each of the two dolls ?*

(a) *Set up and solve the linear programming problem.*

(b) *If the optimal solution is not integer valued, use Gomory's technique to derive the optimal solution.*

[Bharathiar M.Sc. (Math) 88; Delhi (M.B.A.) 73]

Formulation of the problem. Suppose the manufacturer decides to produce x_1 dolls of type X and x_2 dolls of type Y. Then availability of time on two machines has the following restrictions :

$$2x_1 + 5x_2 \le 16, \quad 6x_1 + 5x_2 \le 30, \quad \text{and} \quad x_1, x_2 \ge 0 .$$

The manufacturer wishes to determine the value of x_1 and x_2 so as to maximize the profit $z = \text{Rs. } (x_1 + x_2)$.

Solution. Introduce the slack variables $x_3 \ge 0$ and $x_4 \ge 0$ in the constraints of the given L.P. problem. An initial basic feasible solution is $x_3 = 16$ and $x_4 = 30$. Now using the simplex method, the optimum solution is obtained as given in the following table :

Table 10·25

		$c_j \rightarrow$	1	1	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	
x_2	1	$\rightarrow 9/5$	0	1	3/10	– 1/10	
x_1	1	7/2	1	0	– 1/4	1/4	
	$z = 53/10$		0	0	1/20	3/20	$\leftarrow \Delta_j$

This yields an optimum non-integer solution : $x_1 = \frac{7}{2}$, $x_2 = \frac{9}{5}$ and max $z = \frac{53}{10}$.

Since the fractional parts of $\mathbf{X_B}$ are $\left[\frac{4}{5}, \frac{1}{2}\right]$ and max $\left[\frac{4}{5}, \frac{1}{4}\right] = \frac{4}{5}$, therefore from the first row of above table,

$$(1 + \tfrac{4}{5}) = (0 + 0)\, x_1 + (1 + 0)\, x_2 + (0 + \tfrac{3}{10})\, x_3 + (-1 + \tfrac{9}{10})\, x_4$$

The corresponding fractional cut is given by

$$-\tfrac{4}{5} = 0x_1 + 0x_2 - \tfrac{3}{10} x_3 - \tfrac{9}{10} x_4 + g_1 .$$

Now inserting this additional constraint into the optimum simplex table, the modified table becomes,

Table 10·26

		$c_j \rightarrow$	1	1	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	
x_2	1	9/5	0	1	3/10	– 1/10	0	
x_1	1	7/2	1	0	– 1/4	1/4	0	
g_1	0	– 4/5	0	0	– 3/10	$\boxed{-9/10}$	1	
	$z = 53/10$		0	0	1/20	3/20 $\uparrow$	0 $\downarrow$	$\leftarrow \Delta_j$

First Iteration. Using *dual simplex method*, remove $\mathbf{G_1}$ and introduce $\mathbf{X_4}$.

Table 10·27

		$c_j \rightarrow$	1	1	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	
x_2	1	17/9	0	1	1/3	0	– 1/9	
x_1	1	59/18	1	0	– 1/3	0	5/18	
x_4	0	$\rightarrow 8/9$	0	0	1/3	1	– 10/9	
	$z = 31/6$		0	0	0	0	1/6	$\leftarrow \Delta_j$

Since solution is still non-integer, insert one more fractional cut in the above table. From the third row of above table, we have

$$\frac{8}{9} = (0+0)\,x_1 + (0+0)\,x_2 + \left(0+\frac{1}{3}\right)x_3 + (1+0)\,x_4 + \left(-2+\frac{8}{9}\right)g_1$$

The corresponding fractional cut becomes : $-\frac{8}{9} = 0x_1 + 0x_2 - \frac{1}{3}x_3 + 0x_4 - \frac{8}{9}g_1 + g_2$

Inserting this additional constraint, the modified table becomes :

Table 10·28

		$c_j \rightarrow$	1	1	0	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2	
x_2	1	17/9	0	1	1/3	0	− 1/9	0	
x_1	1	59/18	1	0	− 1/3	0	5/18	0	
x_4	0	8/9	0	0	1/3	1	− 10/9	0	
g_2	0	− 8/9	0	0	$\boxed{-1/3}$	0	− 8/9	1	
	$z = 31/6$		0	0	0 ↑	0	1/6	0 ↓	$\leftarrow \Delta_j$

Second Iteration. Using *dual simplex method*, remove G_2 and insert X_3 .

Table 10·29

		$c_j \rightarrow$	1	1	0	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2	
x_2	1	1	0	1	0	0	− 1	1	
x_1	1	25/6	1	0	0	0	7/6	− 1	
x_4	0	0	0	0	0	1	− 2	1	
x_3	0	→ 8/3	0	0	1	0	8/3	− 3	
	$z = 31/6$		0	0	0	0	1/6	0	$\leftarrow \Delta_j$

This solution is also non-integer one, so insert one more fractional cut. The fractional parts of X_B are $\left[\frac{1}{6}, \frac{2}{3}\right]$ and max $\left[\frac{1}{6}, \frac{2}{3}\right] = \frac{2}{3}$. Therefore, from the last row of the above table, we have

$$\left(2+\frac{2}{3}\right) = (0+0)\,x_1 + (0+0)\,x_2 + (1+0)\,x_3 + (0+0)\,x_4 + \left(2+\frac{2}{3}\right)g_1 + (-3+0)\,g_2$$

The corresponding fractional cut will be

$$-\frac{2}{3} = 0x_1 + 0x_2 + 0x_3 + 0x_4 - \frac{2}{3}g_1 + 0g_2 + g_3 .$$

Now inserting this constraint, the modified table becomes :

Table 10·30

		$c_j \rightarrow$	1	1	0	0	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2	G_3	
x_2	1	1	0	1	0	0	− 1	1	0	
x_1	1	25/6	1	0	0	0	7/6	− 1	0	
x_4	0	0	0	0	0	1	− 2	1	0	
x_3	0	8/3	0	0	1	0	8/3	− 3	0	
g_3	0	− 2/3	0	0	0	0	$\boxed{-2/3}$	0	1	
	$z = 31/6$		0	0	0	0	1/6 ↑	0	0 ↓	$\leftarrow \Delta_j$

Third Iteration. Using *dual simplex method*, remove G_3 and introduce G_1 .

Table 10·31

BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2	G_3
		$c_j \rightarrow$	1	1	0	0	0	0	0
x_2	1	2	0	1	0	0	0	1	− 3/2
x_1	1	3	1	0	0	0	0	− 1	7/4
x_4	0	2	0	0	0	1	0	1	− 3
x_3	0	0	0	0	1	0	0	− 3	4
g_1	0	1	0	0	0	0	1	0	− 3/2
	$z = 5$		0	0	0	0	0	0	1/4 ← Δ_j

This gives the optimum integer solution : $x_1 = 3$, $x_2 = 2$ and max $z = 5$.

Thus, the manufacturer should produce 3 dolls of type *X*, 2 dolls of type *Y* in order to get the maximum profit of Rs. 5.

Note. Alternative solutions are : $x_1 = 5$, $x_2 = 0$; and $x_1 = 4$, $x_2 = 1$.

10·6. GEOMETRICAL INTERPRETATION OF GOMORY'S CUTTING PLANE METHOD

The geometrical interpretation of cutting plane method can be easily understood through a practical example.

Let us consider the problem of ***Example* 5 :**

Max. $z = x_1 + x_2$, s.t. $2x_1 + 5x_2 \le 16$, $6x_1 + 5x_2 \le 30$, $x_1, x_2 \ge 0$.

The graphical solution of this problem is obtained in *Fig.* 10·3 with solution space represented by the convex region *OABC*. The optimum solution occurs at the extreme point *B*, *i.e.*

$$x_1 = 3.5 \, , \; x_2 = 1.8 \, , \; \max z = 5.3.$$

But, this solution is not integer-valued. While solving this problem by Gomory's method, we introduced the first Gomory's constraint :

$$-\tfrac{3}{10}x_3 - \tfrac{9}{10}x_4 \le -\tfrac{4}{5} \qquad \text{...(i)}$$

In order to express this constraint in terms of x_1 and x_2 , we make use of the constraint equations : $2x_1 + 5x_2 + x_3 = 16$ and $6x_1 + 5x_2 + x_4 = 30$,

where x_3 and x_4 are slack variables. From these, we get

$$x_3 = 16 - 2x_1 - 5x_2 \text{ and } x_4 = 30 - 6x_1 - 5x_2 \, ,$$

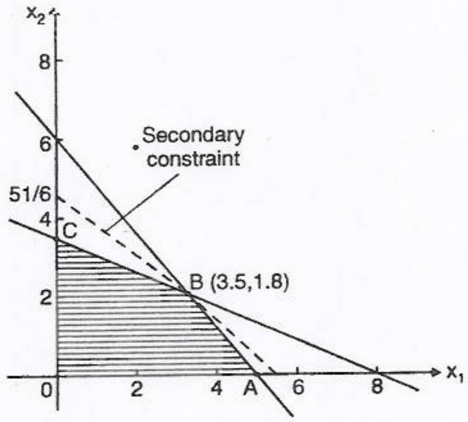


Fig. 10.3

The *Gomory's* constraint (i) then becomes

$$-\tfrac{3}{10}(16 - 2x_1 - 5x_2) - \tfrac{9}{10}(30 - 6x_1 - 5x_2) \le -\tfrac{4}{5}, \quad i.e. \quad x_1 + x_2 \le 5\tfrac{1}{6}.$$

This constraint cuts off the feasible region and now the feasible region is reduced to somewhat less than the previous one as shown in *Fig.* 10·3 .

Similarly, the second *Gomory's* constraint is $g_1 \ge 1$. But,

$$-\tfrac{3}{10}x_3 - \tfrac{9}{10}x_4 + g_1 = -\tfrac{4}{5}, \quad i.e. \quad g_1 = (\tfrac{3}{10}x_3 + \tfrac{9}{10}x_4) - \tfrac{4}{5}$$

Substituting the values of x_3 and x_4 from the constraint equations of the given problem, we immediately get

$g_1 = 31.8 - 6x_1 - 6x_2$. Therefore, $31{\cdot}8 - 6x_1 - 6x_2 \ge 1$ $(\because g_1 \ge 1)$ or $x_1 + x_2 \le 5{\cdot}103$.

This constraint also cuts off some space of the feasible region. Since this constraint very minutely cuts off the solution space, so it has not been plotted on the graph. Because of such cuttings, this method was named as *cutting plane method.*

EXAMINATION PROBLEMS

Find the optimum integer solution of the following all integer programming problems :

1. Max. $z = x_1 + x_2$, subject to
$3x_1 - 2x_2 \le 5$
$x_1 \le 2$
$x_1 \ge 0$, $x_2 \ge 0$, and are integers.
[Meerut M.Sc. (Math.) 94; Punjabi (Math.) 74]
[Ans. $x_1 = 3$, $x_2 = 2$, max $z = 5$]

2. Max $z = x_1 - 2x_2$, subject to
$4x_1 + 2x_2 \le 15$
$x_1 >\!<= 0$, $x_2 \ge 0$, and integers
[Madurai B.Sc. (Appl. Math.) 82]
[Ans. $x_1 = 3$, $x_2 = 0$, max $z = 3$]

3. Max $z = 3x_2$, subject to
$3x_1 + 2x_2 \le 7$
$x_1 - x_2 \ge -2$
x_1, $x_2 \ge 0$ and integers.
[Hint. Simplex method gives the integer solution.]
[Ans. $x_1 = 0$, $x_2 = 2$, max $z = 6$]

4. Max $z = x_1 + 5x_2$, subject to
$x_1 + 10x_2 \le 20$
$x_1 \le 2$
x_1, $x_2 \ge 0$ and integers.
[Ans. $x_1 = 2$, $x_2 = 1$, max $z = 7$]

5. Max $z = 2x_1 + 2x_2$, subject to the constraints :
$5x_1 + 3x_2 \le 8$
$x_1 + 2x_2 \le 4$
x_1, $x_2 \ge 0$ and are integers.
[Agra 99; Madurai M.Sc. (Appl. Math) 83]
[Ans. $x_1 = 1$, $x_2 = 1$, and max $z = 4$]

6. Max. $z = 4x_1 + 3x_2$, subject to the constraints
$x_1 + 2x_2 \le 4$
$2x_1 + x_2 \le 6$
x_1, $x_2 \ge 0$ and are integers.
[Agra 99]
[Ans. $x_1 = 3$, $x_2 = 0$, max $z = 12$]

7. Max $z = 3x_1 + 4x_2$, subject to the constraints :
$3x_1 + 2x_2 \le 8$
$x_1 + 4x_2 \ge 10$
x_1, $x_2 \ge 0$ and are integers.
[Ans. $x_1 = 0$, $x_2 = 4$, max. $z = 16$]

8. Max. $z = 11x_1 + 4x_2$, subject to the constraints :
$-x_1 + 2x_2 \le 4$
$5x_1 + 2x_2 \le 16$
$2x_1 - x_2 \le 4$
$x_1 \ge 0$, $x_2 \ge 0$ and are integers.
[Meerut M.Sc.93 ; Madras B.Sc. 85; IIT(M.Tech) 80]
[Ans. $x_1 = 2$, $x_2 = 3$, max $z = 34$]

9. Max $z = x_1 - x_2$, subject to the constraints :
$x_1 + 2x_2 \le 4$,
$6x_1 + 2x_2 \le 9$;
x_1, $x_2 \ge 0$, and are integers.
[Meerut M.Sc. (Math) 92; Madurai B.E. (Ind. Engg.) 82, B.Sc. (Math) 78]
[Ans. $x_1 = 1$, $x_2 = 0$; max. $z = 2$]

10. Max. $z = 3x_1 - 2x_2 + 5x_3$, subject to the constraints.
$5x_1 + 2x_2 + 7x_3 \le 28$
$4x_1 + 5x_2 + 5x_3 \le 30$
x_1, $x_2 < x_3 \ge 0$ and are integers.
[Madurai B.E. (Mech.) 76]
[Hint. Simplex method gives the integer solution]
[Ans. $x_1 = 0$, $x_2 = 0$, $x_3 = 4$, max. $z = 20$

II–Branch and Bound Method

10·7. THE BRANCH-AND-BOUND METHOD

This section deals with the algorithm given by *Land* and *Doig* for solving the ***all-integer*** and ***mixed-integer*** problems. Why this method is given the name '*branch-and-bound*' will be made clear in the following sections. This is the most general technique for the solution of an I.P.P. in which *a few* or *all* the variables are constrained by their upper or lower bounds or by both. This technique is now discussed below.

The general idea of the method is to solve the problem first as a continuous linear programming problem and then the original problem is partitioned (branched) into two sub-problems by imposing the integer conditions on one of its integer variables that currently has a fractional optimal value. Let x_j be an integer-constrained variable whose optimum continuous value $x_j{*}$ is obtained in terms of a fraction. Then clearly we shall have,

$$[x_j{*}] \le x_j \le [x_j{*}] + 1 .$$

Any feasible integer value, therefore, must satisfy one of the two conditions :

$$x_j \le [x_j{*}] \quad \text{or} \quad x_j \ge [x_j{*}] + 1 .$$

These two constraints are mutually exclusive and thus cannot be true simultaneously and hence both cannot be introduced in the integer programming problem simultaneously. By introducing these constraints one by one in the continuous linear programming problem, we shall have two sub-problems, both being integer-valued.

After branching in this manner, two sub-problems are constructed by inserting $x_j \le [x_j^*]$ and $x_j \ge [x_j^*] + 1$ one by one to the original set of constraints.

To be definite, let the mixed I.P.P. be :

$$\text{Max. } z = \sum_{j=1}^{n} c_j x_j \text{, subject to the constraints :} \qquad ...(10\cdot6)$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad \text{for } \; i = 1, 2, \ldots, m \qquad ...(10\cdot7)$$

where $\qquad x_j$ is integer valued for $j = 1, 2, \ldots, k \; (\le n)$, $\qquad ...(10\cdot8)$

and $\qquad x_j \ge 0 \;$ for $\; j = 1, 2, ..., k, k+1, \ldots, n$. $\qquad ...(10\cdot9)$

In addition to above, let us assume that for each integer-valued variable x_j lower and upper bounds can be assigned so that these bounds surely contain the optimal values

$$L_j \le x_j \le U_j \;\text{ for }\; j = 1, 2, \ldots, k. \qquad ...(10\cdot10)$$

The following principal idea is behind the *'branch-and-bound technique* we are looking for :

Let us consider any variable x_j and let I be some integer value such that $L_j \le I \le U_j - 1$. Then an optimum solution to the problem (10·6) through (10·9) also satisfies

either $\qquad$ *the linear constraint* $x_j \ge I + 1$ $\qquad ...(10\cdot11)$

or $\qquad$ *the linear constraint* $x_j \le I$. $\qquad ...(10\cdot12)$

To explain how this partitioning helps us, suppose we have overlooked the integer condition (10·8) and obtained an optimal solution to the L.P.P. consisting of (10·6), (10·7), (10·8) and (10·9) indicating $x_1 = 1\frac{3}{5}$ (for example). Obviously, $x_1 = 1\frac{3}{5}$ gives the range $1 < x_1 < 2$. Therefore, in an integer-valued solution, we must have either $x_1 \le 1$ or $x_1 \ge 2$.

Thus there will be no integer valued feasible solution in the region $x_1 = 1$ to $x_1 = 2$ as shown in the following figure.

Now our problem is to search for the optimum value of z either in the first region $(x_1 \le 1)$ or in the second region $(x_1 \ge 2)$.

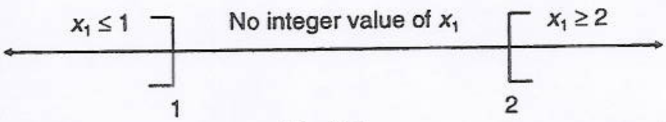


Fig. 10.4

Thus we formulate and solve the following two sub-problems separately :

***Sub-problem (1) : consisting of* (10·6), (10·7), (10·8) and $2 \le x_1 \le U_1$**

***Sub-problem (2) : consisting of* (10·6), (10·7), (10·8), and $L_1 \le x_1 \le 1$.**

If for any one of the sub-problems, optimum integer solution is obtained then that problem is not partitioned further. Sometimes, it may also be possible that the sub-problem has no solution at all. Such sub-problem is also discarded for ever. But, if any *sub-problem* involves some non-integer variable, then it is again partitioned and this process of partitioning continues so long as it is applicable until each sub-problem either possesses an integer-valued optimum solution or there is an indication that it cannot provide a better solution. The optimum integer-valued solution among all the sub-problems is finally selected which gives overall optimum value of the objective function.

We now discuss below the step-by-step procedure that specifies how the partitioning (10·11) and (10·12) can be applied systematically to eventually get an optimum integer-valued solution.

10·7-1 . Branch-and-Bound Algorithm

At the rth iteration we have available a *lower bound* (say, z_r) for the optimal value of the objective function. For convenience, we suppose that at the first iteration, z_1 is either strictly less than the optimal value, or equals the value of the objective function for a feasible solution that we have noted. In case, if we have no information about the problem we let $z_1 = -\infty$. In addition to a lower bound z_1 we also have a master list of linear programming problems to be solved differing only in the revision of the bounds (10·10). At the first iteration, the master list has only one problem consisting of (10·6), (10·7), (10·8) and (10·10).

The step-by-step procedure at this rth ($r = 0, 1, 2, ...$) iteration can be outlined as follows :

Step 1. Two possibilities may arise at the rth iteration :

(i) If the master list does not contain any linear programming problem (*i.e.*, empty), stop the computations.

(ii) Otherwise, go to ***step* 2** for removing a linear programming problem from the master list.

Step. 2. Solve the chosen problem to obtain the optimum solution by using bounded variable technique. Again, two possibilities may arise :

(i) If it has no feasible solution, or if the resulting optimal value of the objective function z is $\le z_r$, then let $z_{r+1} = z_r$ and return back to ***step 1.***

(ii) Otherwise, go to ***step 3.***

Step 3. *(i)* If the optimal solution to the linear programming problem thus obtained satisfies the integer condition, then record it, let z_{r+1} be associated optimal value of the objective function, and return back to *step 1.*

(ii) Otherwise, go to ***step 4.***

Step 4. Select any variable x_j, for $j = 1, 2, \ldots, k$, that does not have an integer value in the obtained optimal solution to the chosen linear programming problem. Let x_j^* denote this value, and $[x_j^*]$ stand for largest integer less than or equal to x_j^*. Now, include two linear programming problems in the master list. These two sub-problems are :

Sub-prob 1. Same as the problem chosen in ***step 1,*** except that the lower bound L_j on x_j is replaced by $[x_j^*] + 1$.

Sub-prob 2. Same as the problem chosen in ***step 1,*** except that the upper bound U_j on x_j is replaced by $[x_j^*]$.

Let $z_{r+1} = z_r$, and return back to ***step 1.*** At the termination of the process if we find a integer-valued feasible solution giving z_r, it will be optimal, otherwise no integer-valued feasible solution exists.

Q. 1. Describe any one method of solving mixed integer programming problem. **[Madras B.Sc. (Math.) 84]**
2. Sketch the branch-and-bound method in integer programming. **[Agra 99; Bharthidasan M.Sc. (Math.) 81]**
3. What is the main disadvantage of the branch and bound method ?
4. Explain with an example, how in some cases non-integer solution to a linear programming problem is meanigless. **[Madurai B.Sc. (Maths) 79]**

10·7-2. Computational Demonstration of Branch-and-Bound Method

The computational procedure of *Branch-and-Bound* algorithm is now explained below by solving a numerical example.

Example 6. *Use Branch-and-Bound technique to solve the following integer programming problem :*

$$\text{Max. } z = 7x_1 + 9x_2 \qquad \ldots(1)$$
$$\text{subject to } -x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35 \qquad \ldots(2)$$
$$(0 \le x_1, x_2 \le 7) \qquad \ldots(3)$$
$$\text{and } x_1, x_2 \text{ are integers.} \qquad \ldots(4)$$

[Agra 98; Banasthali (MSc) 93; Bharthidasan B.Sc. (Math.) 90, 85]

Solution. Step 1. At the initial iteration, we take $z^{(1)} = 0$ as the lower bound for z, since the solution $x_1 = x_2 = 0$ is feasible. The master list contains only the linear programming problem [(1), (2), (3)] which will be named as ***Sub-prob. 1.***

Step 2. Using graphical method, determine the optimal solution of ***Sub-prob.* 1** as $x_1 = 9/2$, $x_2 = 7/2$, $z^* = 63$. Since the solution is not integer-valued, go to ***step 3,*** and choose x_1. Since $[x_1^*] = [9/2] = 4$ add the following two sub-problems in the master list :

Sub-prob. 2 : (1), (2) and $5 \le x_1 \le 7$, $0 \le x_2 \le 7$

Sub-prob. 3 : (1), (2) and $0 \le x_1 \le 4$, $0 \le x_2 \le 7$

Returning to first step with $z^{(2)} = z^{(1)} = 0$, we select the ***Sub-prob.** 2*. Now, *step* 2 determines that *Sub-prob* 2 has the feasible solution

$$x_1 = 5\ , x_2 = 0\ , z^* = 35\ , \text{(Solution of Sub-prob. 2.)} \qquad ..(5)$$

Clearly, this solution satisfies the integer constraints. So we record it at this step, and take $z^{(3)} = 35$.

Again returning to *step 1* with $z^{(3)} = 35$, we have *Sub-Prob.* 3.

Step. 2. Immediately gives the optimum feasible solution to ***Sub Prob. 3*** as

$$x_1 = 4\ , x_2 = 10/3\ , z^* = 58\ . \text{[Solution to Sub-prob. 3]}$$

Since this solution is not integer-valued, go to *step 3.*

Step 3. Now consider x_2 . Since $[x_2^*] = [3\frac{1}{3}] = 3$, we add the following sub-problems to the master list :

Sub-prob. 4. (1), (2) and $0 \le x_1 \le 4,\ 4 \le x_2 \le 7$

Sub-prob. 5. (1), (2) and $0 \le x_1 \le 4\ ,\ 0 \le x_2 \le 3$.

Returning to *step 1* with $z^{(4)} = z^{(3)} = 35$, we select the *Sub-Prob. 4.* In *Step 2* we find that *Sub-Prob. 4* has no feasible solution. So we, again, return to *Step 1* with $z^{(5)} = z^{(4)} = 35$. Only *Sub-Prob. 5* is now available on the master list. Using *step 2,* we obtain the optimum solution to *Sub-Prob. 5.*

$$z^* = 55\ , x_1 = 4\ , x_2 = 3\ . \text{[Solution to Sub-prob. 5]} \qquad ...[6]$$

Clearly, this solution satisfies the integer conditions. So we record it at *step 3,* and let $z^{(6)} = 55$.

Again returning to *Step 1,* the master list becomes empty (*i.e.,* contains no sub-problem) and thus the process ends.

At the time of ending the process, we observe that only two feasible integer solutions (*5*) and (*6*) have been noted. The 'best one' of these two feasible integer solutions gives us the required optimum solution to the given integer programming problem.

Thus, finally, we get the optimum solution to the given I.P.P. as $z^* = 55\ , x_1 = 4\ , x_2 = 3$.

The tree-diagram corresponding to this problem is shown in the following figure.

The entire calculations of this tree-diagram may be summarised as shown in the following table.

Tree-Diagram of *Example 6*

Start

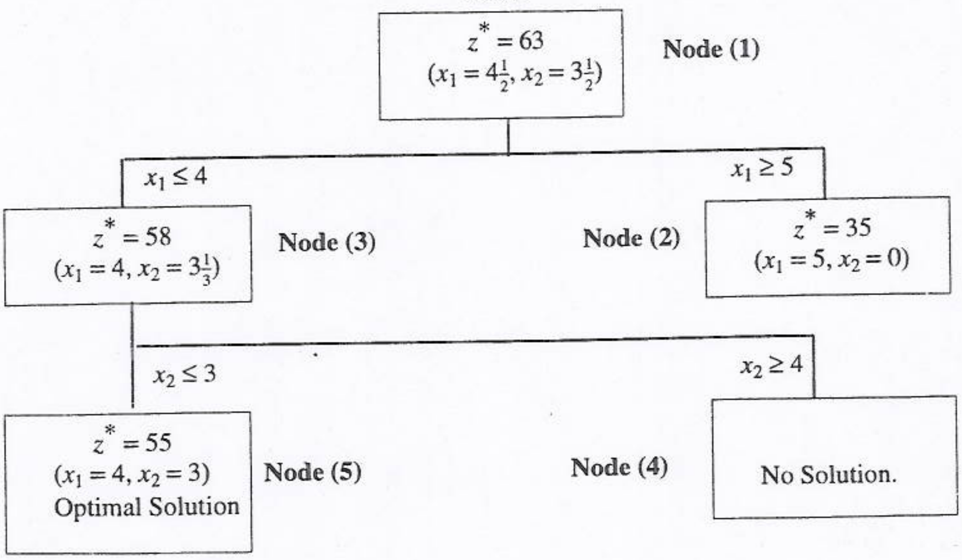


Fig. 10.4

Node	Solution			Additional Constraints	Type of solution
	x_1	x_2	z^*		
(1)	9/2	7/2	63	—	Non-integer (Original problem)
(2)	5	0	35	$x_1 \geq 5$	Integer $\leftarrow z^{*(1)}$
(3)	4	10/3	58	$x_1 \leq 4$	Non-integer
(4)	...	...	...	$x_1 \leq 4$, $x_2 \geq 4$	No Solution
(5)	4	3	55	$x_1 \leq 4$, $x_2 \leq 3$	Integer $\leftarrow z^{*(2)}$ (optimal)

Example 7. *Use Branch-and-Bound technique to solve the following problem.*

$$Max.\ z = 3x_1 + 3x_2 + 13x_3, \quad \textit{subject to} \qquad ...(1)$$

$$\begin{Bmatrix} -3x_1 + 6x_2 + 7x_3 \leq 8 \\ 6x_1 - 3x_2 + 7x_3 \leq 8 \end{Bmatrix} \qquad ...(2)$$

$$0 \leq x_j \leq 5 , \qquad ...(3)$$

$$\textit{and } x_j \textit{ are integers, for } j = 1, 2, 3 . \qquad ...(4)$$

Solution. First we find the optimal solution by inspection.

Iteration 1 :

Step 1. At the initial iteration, let the lower bound of z be $z^{(1)} = 0$, then $x_1 = x_2 = x_3 = 0$ is feasible. The master list consists of only the L.P.P. (1), (2) and (3), which is designated as *Sub-Problem 1.* Remove it in the *step 2.*

Step 2. Find the optimal solution of ***Sub-prob.*** *1* as $z^* = 16$, $x_1 = x_2 = 2\frac{2}{3}$, $x_3 = 0$.

Since the solution is not integer-valued, we proceed from *step 2* to *step 3,* and choose x_1 .

Step 3. Since $[x_1*] = [2\frac{2}{3}] = 2$, add the following two problems in the master list :

Sub-prob 2 : (1), (2), and $3 \leq x_1 \leq 5$, $0 \leq x_2 \leq 5$,$0 \leq x_3 \leq 5$.

Sub-prob 3 : (1), (2) and $0 \leq x_1 \leq 2$, $0 \leq x_2 \leq 5$, $0 \leq x_3 \leq 5$.

Iterations 2 and 3 :

Returning to *step 1* with $z^{(2)} = z^{(1)} = 0$, we remove *Sub-Prob 2.* It can be verified that *step 2* gives no feasible solution to *Sub-Prob 2.* Hence, put $z^{(3)} = z^{(2)} = 0$, and return to *step 1.*

In order to remove *Sub-Prob. 3* we obtain its optimal soluton in *step 2* as

$$x_1 = x_2 = 2 , x_3 = \frac{2}{7} , z* = 15\frac{5}{7} . \qquad \textbf{\textit{(Sol. of Sub-prob. 3)}}$$

Clearly, this solution is not integer-valued. So, we proceed from *step 2* to *step 4.*

Step 4. Since $[x_3^*] = [\frac{2}{7}] = 0$, and therefore include two sub-problems in the master list :

Sub-prob. 4 : (1), (2), and $1 \leq x_1 \leq 2$, $0 \leq x_2 \leq 5$, $1 \leq x_3 \leq 5$.

Sub-prob. 5 : (1), (2), and $0 \leq x_1 \leq 2$, $0 \leq x_2 \leq 5$, $0 \leq x_3 \leq 0$.

Here we observe that *Sub-Prob. 4* and *Sub-Prob. 5* differ from *Sub- Prob. 3* only in the bounds on x_3 .

Iteration 4 :

Now returning to *step 1* with $z^{(4)} = 0$, we remove *Sub-Prob 4.* The optimal solution is thus obtained as

$$x_1 = x_2 = \frac{1}{3} , x_3 = 1 , z* = 15 . \qquad \textbf{\textit{(Sol. of Sub-prob. 4)}}$$

This leads to *step 4* again; let us select x_2 , yielding, as a consequence, the following two sub-problems for including in the mater list.

Sub-prob. 6 : (1), (2) and $0 \leq x_1 \leq 2$, $1 \leq x_2 \leq 5$, $1 \leq x_3 \leq 5$.

Sub-prob. *7.* : (1), (2), and $0 \le x_1 \le 2$, $0 \le x_2 \le 0$, $1 \le x_3 \le 5$.

It is obvious that *Sub-prob. 6* and *Sub-prob. 7* differ from *Sub-prob. 4* only in the bounds on x_2 .

Iteration 5 :

Now returning to *step 1* with $z^{(5)} = 0$, we remove *Sub-prob. 6* and check that *Sub-prob. 5* and *Sub-prob. 7* still remain on the master list. We can find in *step 2* that *Sub-prob. 6* has no feasible solution.

Iteration 6 :

So we return to *step 1* with $z^{(6)} = 0$. We now remove *Sub-prob.* 7 whose optimal solution is obtained as

$$x_1 = x_2 = 0 \, , x_3 = 1\tfrac{1}{7} \, , z^* = 14\tfrac{6}{7} \qquad \textit{(Sol. of Sub-prob.7)}$$

Since x_3 is fractional, we again repeat the *step 4.* Let us select x_3 . Here $[x_3^*] = [1\tfrac{1}{7}] = 1$. So we add two more problems in the master list.

Sub-prob. 8 : (1), (2), and $0 \le x_1 \le 2$, $0 \le x_2 \le 0$, $2 \le x_3 \le 5$

Sub-prob. 9 : (1), (2), and $0 \le x_1 \le 2$, $0 \le x_2 \le 0$, $1 \le x_3 \le 1$.

Iterations 7, 8, 9 :

It can be easily verified that removal of *Sub-prob.* 8 at iteration 7 provides an indication of no feasible solution in *step 2,* and removal of *Sub-prob.* 9 at the 8th iteration yields in *step 2* :

$$x_1 = x_2 = 0 \, , x_3 = 1 \, , z^* = 13 \, . \qquad \textit{(Sol. of Sub-prob. 9)}$$

Therefore, at *step 3,* we record this optimal solution and let $z^{(9)} = 13$.

Returning to *step 1* again, we observe that only *Sub-prob.* 5 is now left on the master list whose optimal solution is :

$$x_1 = 2 \, , x_2 = 2\tfrac{1}{3} \, , x_3 = 0 \, , z^* = 13 \, . \qquad \textit{(Sol. of Sub-prob. 5)}$$

Since the value of objective function in the solution of *Sub-prob.* 9 and *Sub-prob.* 5 is the same and is equal to $z^* = 13$, we return to *step 1* and stop the computations because the master list is now empty.

Thus, finally, we get the optimal solution to the integer programming problem as recorded at the 8th iteration : $x_1 = x_2 = 0$, $x_3 = 1$, $z^* = 13$.

Remarks :

1. In the solution of above problem we have made arbitrary choices in the algorithm at two places :
   (i) Selection of the problem to remove in *Step 1.*
   (ii) Selection of the variable x_j to give us additional problem in *Step 4.* The number of iterations required to solve a problem can vary considerably depending on how these selections are actually done. For example, choice of *Prob. 4* instead of *Prob. 5* at the 4th iteration turned-out to be auspecious. Although, auxiliary numerical tests have been developed to help us in making these choices, but these are not discussed in this text, because they are specially useful to technical specialists.
2. The above algorithm can be demonstrated by means of a tree-like diagram as shown in Fig 8·5 and 8·6 . We have noted that each node in the tree diagram represents a problem on the master list, each branch is leading to one of the problems added to the master list in *Step 4.* On account of this graphical analogy the word 'branch' is used in the name of the algorithm 'Branch-and-Bound'. The word 'bound' is suggested by the test in *Step 2.*

Tree-Diagram of Example 7

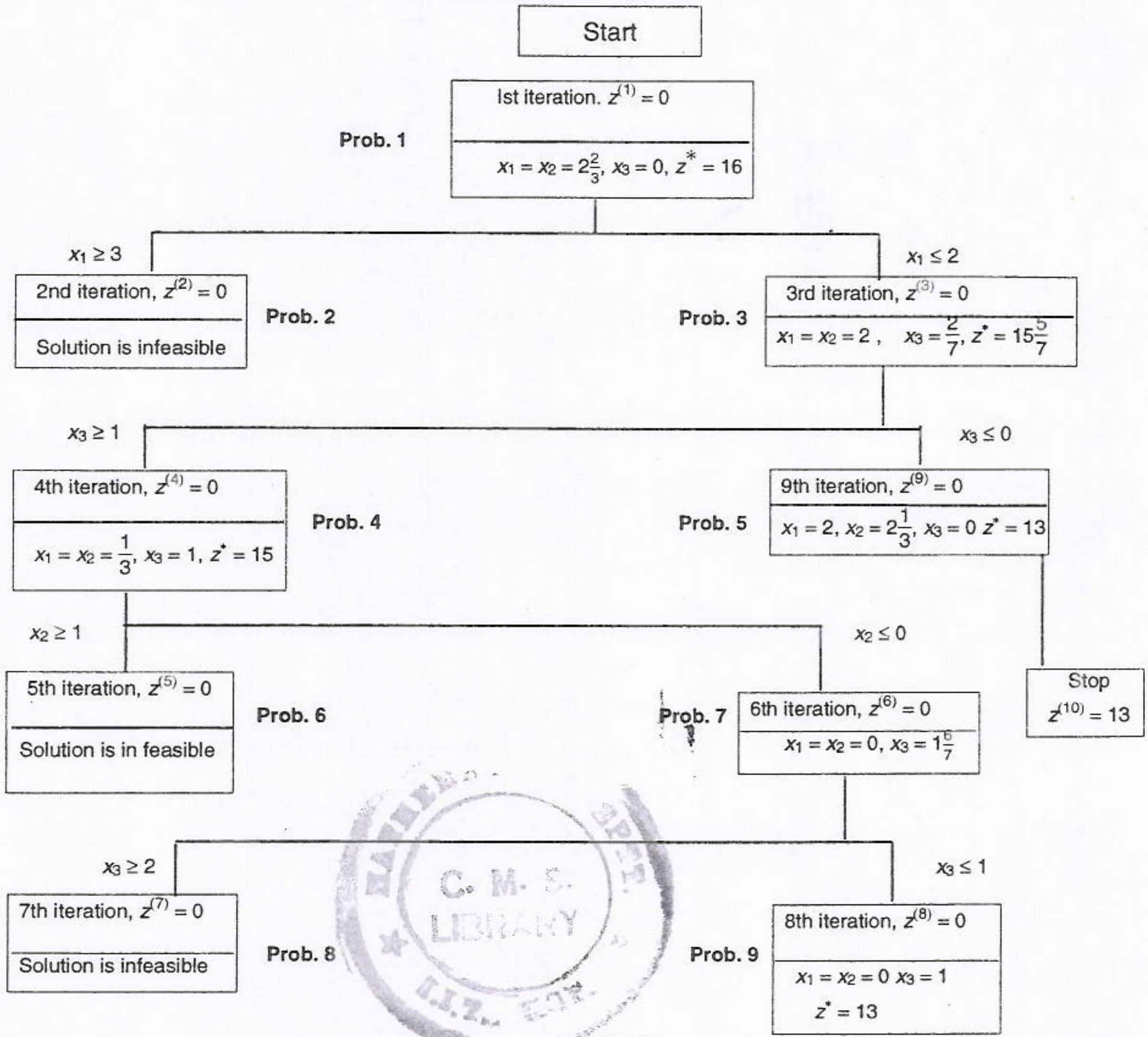


Fig. 10.5

10·8. GEOMETRICAL INTERPRETATION OF BRANCH-AND-BOUND METHOD

The geometrical interpretation of *Branch-and-Bound* method can be easily understood by the following practical example.

Example 8. *Explain the geometrical interpretation of Branch-and-Bound method by solving the following I.P.P. :*

Max. $z = x_1 + x_2$, *subject to the constraints :* $3x_1 + 2x_2 \leq 12$, $x_2 \leq 2$, $x_1 \geq 0$, $x_2 \geq 0$ *and are integers.*

Solution. Step 1. *To solve the problem by graphical methuod without intehger conditions.*

The graphical solution of ***Sub-problem 1*** : Max. $z = x_1 + x_2$, subject to

$3x_1 + 2x_2 = 12$, $x_2 \leq 2$, $x_2, x_2 \geq 0$ is shown by the convex region OABC in Fig. 10·7. The optimum solution occurs at the extreme point $B(x_1 = 8/3, x_2 = 2)$ with max. $z = 14/3$.

Step 2. Since the solution obtained above is not integer-valued, the given linear programming problem isbranched into two *sub-problems* as follows :

The non-integer value of $x_1 = 8/3$ gives the range $2 < 8/3 < 3$. Thus, two sub-problems are stated as follows :

Sub-prob 2 : Max. $z = x_1 + x_2$, s.t. $3x_1 + 2x_2 \le 12, 0 \le x_2 \le 2, 0 \le x_1 \le 2$

Sub-prob 3 : Max $z = x_1 + x_2$, s.t. $3x_1 + 2x_2 \le 12, 0 \le x_2 \le 2, x_1 \ge 3.$

The optimum solution of **sub-problem 2** is : $x_1 = 2, x_2 = 2$ and max. $z = 4$ as shown in Fig. 10·8. while the optimum solution of **sub-problem 3** is : $x_1 = 3, x_2 = 3/2$ and max. $z = 9/2$ as shown in Fig. 10.9.

In **sub-problem** 2. all the variables have integer values. So there is no need of further sub-division. But, **sub-problem-3** having non-integer solution needs further sub-division.

Step 3. In sub-problem-3, the non-integer value of $x_2 = 3/2$ gives the range $1 < x_2 < 2$. So we contruct two more sub-problems by adding the constraints $x_2 \le 1$ and $x_2 \ge 2$ one by one in ***sub-problem 3.*** Thus two additional sub-problems are :

Sub-problem 4 : Max. $z = x_1 + x_2$ s.t. $3x_1 + 2x_2 \le 12, 0 \le x_2 \le 2, x_1 \ge 3, \ \ 0 \le x_2 \le 1.$

Sub-problem 5 : Max. $z = x_1 + x_2$ s.t. $3x_1 + 2x_2 \le 12, 0 \le x_2 \le 2, x_1 \le 3, 0 \le x_2 \ge 2.$

In ***sub-problem - 4,*** the constraint $x_2 \le 2$ is redundant. The optimal solution to this sub-problem is obtained as $x_1 = 10/3, x_2 = 1$, and max. $z = 13/3$ as shown in Fig. 10·10. This solution is not integer valued.

Here it is clear that any further branching of **sub-problem 4** will not improve the value of the objective function because the next sub-division will impose the restrictions $x_1 \le 3$ and $x_1 \ge 4$. then the optimum solutions are obtained as $(x_1 \approx 3, x_2 = 1)$ and $(x_1 = 4, x_2 = 0)$ respectively. Both of these solutions give the maximum value of z equal to 4.

Further, it may be moted that there exists no feasible solution to *sub-problem* 5.

Step 4. Finally, maximum value of the objective function z is obtained as 4 and the integer valued solution is any of the following three :

$$(x_1 = 2, x_2 = 2) \quad \text{or} \quad (x_1 = 3, x_2 = 1) \quad \text{or} \quad (x_1 = 4, x_2 = 0)$$

EXAMINATION PROBLEMS

Use Branch-and-Bound technique to solve the following problems.

1. Max $\mathbf{z} = 3x_1 + 3x_2 + 13x_3$
subject to
$-3x_1 + 6x_2 + 7x_3 \le 8$
$5x_1 - 3x_2 + 7x_3 \le 8$
$0 \le x_j \le 5$
and all x_j are integers.

2. Max. $\mathbf{z} = 7x_1 + 9x_2$
subject to
$-x_1 + 3x_2 \le 6$
$7x_1 + x_2 \le 35$
$0 \le x_1, x_2 \ge 7$
[IGNOU (MCA II) 2000;
I.I. Sc. (Dip. Oper. Man.) 78]

3. Max. $z = 3x_1 + x_2$
subject to
$3x_1 - x_2 + x_3 \quad = 12$
$3x_1 + 11x_2 + x_4 = 66$
$x_j \ge 0$, $j = 1, 2, 3, 4$.

4. Max. $z = x_1 + x_2$
subject to
$4x_1 - x_2 \le 10$
$2x_1 + 5x_2 \le 10$
x_1 , $x_2 = 0, 1, 2, 3$.
[I. I. Sc. (Dip. Ind. Man.) 77]

5. Min. $z = -5x_1 + 7x_2 + 10x_3 - 3x_4 + x_5$
subject to the constraints
$x_1 + 3x_2 - 5x_3 + x_4 + 4x_5 \le 0$
$2x_1 + 6x_2 - 3x_3 + 2x_4 + 2x_5 \ge 4$
$x_2 - 2x_3 - x_4 + x_5 \le -2$
$x_i = 0, 1, (i = 1, 2, \ldots, 5)$.
[I.I. Sc. (Dip. Ind. Man.) 78]

6. $z = 21\,x_1 + 11\,x_2$
subject to
$7x_1 + 4x_2 + x_3 = 13$
$x_2 \le 5, x_1, x_2, x_3 \ge 0$
and integers
[Vidyasagar 97]

7. Min. $z = -4x_1 + 5x_2 + x_3 - 3x_4 + x_5$ subject to the constraints
$-x_1 + 2x_2 \ \ - x_4 \ - \ \ x_5 \le -2, -4x_1 + 5x_2 + x_3 - 3x_4 + x_5 \le -2, -x_1 - 3x_2 + 2x_3 + 6x_4 - 25x_5 \le 1$ every $x_j = 0$, 1 .
[Calcutta (Appl. Math.) 76]

10.9 APPLICATIONS OF INTEGER PROGRAMMING

We present in this section a number of applications of integer programming (all-integer and mixed). Some of these applications are connected with the direct formulation of the problem.

1. **Travelling Salesman Problem.** Let us asume that there are n towns with known distances between any pair of cities. A salesman wants to start from his home town; visit each town once, and then return to his starting point. The objective is to minimize the total travelling time (or cost or distance).
This problem can be formulated as zero-one integer programming problem. In a linear programming problem, if all the variables are restricted to take the values of 0 or 1 only, then such linear programming problem is called zero-one programming. The formulation of Travelling Salesman Problem is as follows :

$$\text{Min } z = \sum_i \sum_j \sum_k d_{ij}\, x_{ijk}, \; i \neq j,$$

where d_{ij} denotes the distance from town i to town j, and i, j, k are integers varying from 1 to n

$$x_{ijk} = \begin{cases} 1, \text{ if the } k\text{th directed arc is from town } i \text{ to town } j. \\ 0, \text{ if otherwise,} \end{cases}$$

The constraints are of the following type :

(i) $\displaystyle\sum_i \sum_{\substack{j \\ i \neq j}} x_{ijk} = 1, \; k = 1, 2, ..., n.$

This implies that only one directed arc may be assigned to a specific value of k.

(ii) $\displaystyle\sum_j \sum_k x_{ijk} = 1, i = 1, 2, ..., n.$

This implies that only other town may be reached from a specifiec town i.

(iii) $\displaystyle\sum_i \sum_k xsubijk = 1, j = 1, 2, ..., n.$

This implies that only one other town can initiate directed arc to a specified town j.

(iv) $\displaystyle\sum_{\substack{i \\ i \neq j.}} x_{ijk} = \sum_{\substack{r \\ r \neq j}} x_{jr(k+1)},$ for all j and k.

This constraint will ensure that the round trip will consist of connected arcs. It is given that the kth directed arc ends at some specific town j, the $(k + 1)$th directed are must start at the same town j.
This problem has several practical applications.

2. **Fixed Charge Problelm.** It is the problem where it is required to produce at least N units of a certain product on n different machines.
Let x_j be the number of units produced on machine $j, j = 1, 2, \ldots n$.
The production cost function for the jth machine is given by

$$c_j(x_j) = \begin{cases} k_j + c_j x_j, & x_j > 0, \\ \quad 0, & x_j = 0, \end{cases}$$

where k_j is the setup cost for machine j. Thus, the formulation of the problem is given by :

$$\text{Min. } z = \sum_{j=1}^{n} c_j(x_j), \text{ subject to } \sum_{j=1}^{n} x_j \geq N, \; x_j \geq 0 \text{ and integer.}$$

It is important ot note in the above formulation that the objective function is non-linear because of the presence of the fixed-charge k_j. This difficulty may be removed by using the mixed integer programming as follows :
Let M be a very large number exceeding the capacity of any of the machines and let $y_j = 0$ or 1 for all j.
The above formulation thus reduces to :

$$\text{Min. } z = \sum_{j=1}^{n} k_j y_j + \sum_{j=1}^{n} c_j\, x_j,$$

subject to $\displaystyle\sum_{j=1}^{n} x_j > N, \; x_j \leq M y_j$ for all j, $x_j \geq 0$ and integer, $y_j = 0 \; or \; 1$ for all j

Thus, we can solve this problem by usual techniques discussed in this chapter.

Q. 1. State the computational process for solving a linear programming problem with upper bound conditions. **[M.S. Baroda (Appl. Math.) 78]**

2. Discuss the importance of integer programming problem in optimization theory. Formate the travelling salesman problem as an integer programming problem. **[Madras B.Sc. (Maths.) 85]**

3. State the fixed charges problem. Show how to formulate this problem as a mixed integer programming problem. **[Madurai B.Sc. (Appl. Math.) 88]**

4. Explain gomory's method for solving an all integer linear programming problem. Formulate the travelling salesman problem as an integer programming problem. **[Madras B.Sc. (Math.) 81; I.I.T. (M. Tech.) 80; Andhra M.Sc. (Stat.) 79]**

10·10. ZERO-ONE (0-1) PROGRAMMING

If all the variables in a linear programming problem are restricted to take the value *zero* or *one* only, then such L.P.P. is known as *zero-one programming* problem. Various methods are available for solving the *zero-one programming problems.*

The study of zero-one programming problems is specially important because of two reasons :

(i) A certain class of *integer non-linear* programming problems can be converted into equivalent *zero-one* linear programming problems.

(ii) A large variety of management and industrial problems can also be formulated as zero-one programming problems.

The general integer programming methods such as Branch-and-Bound method can be used to solve a zero-one L.P.P. simply by introducing the additional constraints that all the variables must be less than or equal to one. The general integer programming methods were primarily developed for solving such type of problems, they do not take advantage of the special features of zero-one L.P.P. Thus a number of methods have been developed to solve zero-one linear programming problems more easily.

The theoretical development of these methods is beyond the scope of this book.

Q. 1. What is meant by zero-one programming problem ?

2. Write a short note on integer programming. **[Meerut M.Sc. (Math.) 84]**

SELF-EXAMINATION PROBLEMS

1. Derive the expression for Gomory-cut in the case of mixed integer linear programming problem. Apply it to obtain initial iterate to the following problem :

Min. $z = -110x_1 - 80x_2 - 60x_3 - 180x_4$, subject to the constraints

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 + x_6 = 50$$
$$x_1, x_2, x_3 = 0, 1, 2 \ldots, x_4, x_5, x_6 \geq 0 .$$

Following is the optimal iterate tableau of the corresponding linear programming problem of miximization.

		$c_j \rightarrow$	110	80	60	180	0	0	
Basic Var.	C_B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	
x_1	110	50/3	1	2/3	1/3	0	5/3	− 1/3	
x_4	180	10/3	0	1/3	2/3	1	− 2/3	1/3	
	$z = 2433\frac{1}{3}$		0	$-53\frac{1}{3}$	$-96\frac{2}{3}$	0	$-63\frac{1}{3}$	$-23\frac{1}{4}$	$\leftarrow \Delta_j$

[I.I. Sc. (Dip. Ind. Man) 78]

2. A company stocks an item that deteriorates with time as measured in weekly periods. The company has on hand four such items. The present ages of these items are A_1, A_2, A_3 and A_4. it has contracted to sell the stock as follows : it must deliver one item at each of weeks t_1, t_2, t_3 and t_4 from now : the revenue for an item is a function of its age at the time of delivery.

value

Formulate this optimization problem as a programming problem.

[Your answer should specifically indicate the feasible range of each variable involved.]

3. Suppose that three items are to be sequenced through n machines. Each item must be processed first on machine 1, then on machine 2, ..., and finally on machine n. The sequence of jobs may be different for each machine. Let t_{ij} be

the time required to perform the work on item i by machine j; assume each t_{ij} is an integer. The objective is to minimize the total make-span to complete all items. Formulate the problem as an integer programming model.

[Delhi (MBA) 76]

4. Formulate the following Capital Budgeting problem as a zero-one integer programming problem given in the following data. There are four projects under consideration. Assume that the project run into three years. Total available funds are Rs. 75,000 (to be used at the rate of Rs. 25,000/– each year). The expected profit and cost break-up is as follows :

Projects	Expected Profit	Cost		
		Year 1	Year 2	Year 3
1	90,000	8,000	10,000	12,000
2	60,000	2,000	5,000	8,000
3	1,80,000	15,000	10,000	5,000
4	1,00,000	10,000	5,000	5,000

[Punjab (MBA) 77]

5. Suppose five items are to be loaded on the vessel. The weight W, volume V and price p are tabulated below. The maximum cargo weight and cargo volume are $W = 112$, $V = 109$ respectively. Determine the most valuable cargo load in discrete unit of each item :

Item	1	2	3	4	5
W	5	8	3	2	7
V	1	8	6	5	4
Price (Rs.)	4	7	6	5	4

Formulate the problem as integer programming model and then solve. **[Roorkee (ME. Elec.) 77]**

6. (a) Suppose that salesman has to travel n cities where he starts from his home city and visits each of other $n-1$ cities once and only once and returns home city. Let d_{ij} be the distance between city i and city j. Formulate the problem as integer programming problem if he wishes to minimize the total distance travelled.

   (b) Describle the cutting plane method to solve integer programming problem.

7. Following is the optimal table of an L.P.P.

Basic V.	X_B	X_1	X_2	S_1	A_1	A_2	S_2	
x_1	3/5	1	0	1/5	3/5	− 1/5	0	
x_2	6/5	0	1	− 3/5	− 4/5	3/5	0	
s_2	0	0	0	1	1	− 1	1	
	$z = 12/5$	0	0	1/5	$M - \frac{2}{5}$	$M - 1/5$	0	$\leftarrow \Delta_j$

Find the optimal solution to the problem when x_2 is required to take an integer value. **[Roorkee M.Sc. I (OR) 96]**

8. Consider the following integer programming problem :

   Maximize $9x_1 + 7x_2$,

   Subject to $3x_1 - x_2 \leq 6$,

   $x_1 + 7x_2 \leq 35$

   where x_1, $x_2 \geq 0$ and are integers. **[IGNOU (MCA II) 2000]**

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