Illustrative example 1

*Blue Ridge Hot Tubs manufactures and sells two models of hot tubs: the **Aqua-Spa** and the **Hydro-Lux**.*
Blue Ridge buys prefabricated fiberglass hot tub shells from a local supplier and adds the pump and tubing to the shells to create his hot tubs. (This supplier has the capacity to deliver as many hot tub shells as needed)

*Blue Ridge installs the same type of pump into both hot tubs. They will have **only 200 pumps available during their next production cycle**.*

Illustrative example 1

The main difference between the two models of hot tubs is the amount of tubing and labor required.

- *Each Aqua-Spa requires 9 hours of **labor** and 12 feet of **tubing**.*
- *Each Hydro-Lux requires 6 hours of **labor** and 16 feet of **tubing**.*
- *The firm expects to have 1,566 production labor hours and 2,880 feet of tubing available during the next production cycle.*
- *Each Aqua-Spa unit sold earns a **profit** of $350 and each Hydro-Lux sold earns a **profit** of $300.*

Historically, all the hot tubs are sold and no inventory needs to be carried forward.

Identify the decision variables

- How many Aqua-Spas and Hydro-Luxes should be produced?
  - Let X_1 and X_2 represent the number of Aqua-Spas and Hydro-Luxes respectively, to produce.

State the objective function as a linear combination of the decision variables

- The company earns a profit of \$350 on each Aqua-Spa (X_1) sold and \$300 on each Hydro-Lux (X_2) sold.
- The objective of maximizing the profit earned is stated mathematically as:

$$\text{MAX} \quad (350X_1 + 300X_2)$$

State the constraints as linear combinations of the decision variables.

- Only 200 pumps are available and each hot tub requires one pump

Constraint : $1X_1 + 1X_2 \leq 200$

- Only 1,566 labor hours available during the next production cycle.

- Each Aqua-Spa built (each unit of X_1) requires 9 labor hours and each Hydro-Lux (each unit of X_2) requires 6 labor hours

Constraint : $9\ X_1 + 6\ X_2 \leq 1{,}566$

- Each Aqua-Spa requires 12 feet of tubing, and each Hydro-Lux produced requires 16 feet of tubing

Constraint : $12X_1 + 16X_2 \leq 2{,}880$

Hidden constraints

- There are simple lower bounds of zero on the variables X1 and X2 because it is impossible to produce a negative number of hot tubs.

$X1 \geq 0; X2 \geq 0$

$$
\begin{array}{lrcrcr}
\text{MAX:} & 350X_1 & + & 300X_2 & & \\
\text{Subject to:} & 1X_1 & + & 1X_2 & \leq & 200 \\
 & 9X_1 & + & 6X_2 & \leq & 1{,}566 \\
 & 12X_1 & + & 16X_2 & \leq & 2{,}880 \\
 & 1X_1 & & & \geq & 0 \\
 & & & 1X_2 & \geq & 0
\end{array}
$$

Run the above Linear Program

Scenario 1 : Original values

Type 1 more profitable, involves more labor but less material as compared to Type 2

$$
\begin{aligned}
\max \quad & 350x_1 + 300x_2 \\
\text{subject to :} \quad & x_1 + x_2 \leq 200 \\
& 9x_1 + 6x_2 \leq 1566 \\
& 12x_1 + 16x_2 \leq 2880 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

LP Formulation

x_1 = , x_2 =

Total profit =

Solution

What if : Scenario 2

Tubing values interchanged :
Type 1 more profitable, but involves more labor and material, all other values remain unchanged.

$$
\begin{aligned}
\max \quad & 350x_1 + 300x_2 \\
\text{subject to :} \quad & x_1 + x_2 \leq 200 \\
& 9x_1 + 6x_2 \leq 1566 \\
& 16x_1 + 12x_2 \leq 2880 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

LP Formulation

x_1 =, x_2 =   Total profit =

Solution

What if : Scenario 3

Both labor hours and tubing interchanged
i.e. Type 1 is more profitable, involves less material and labor

$$
\begin{array}{ll}
\max & 350x_1 + 300x_2 \\
\text{subject to :} & x_1 + x_2 \leq 200 \\
 & 6x_1 + 9x_2 \leq 1566 \\
 & 12x_1 + 16x_2 \leq 2880 \\
 & x_1 \geq 0, x_2 \geq 0
\end{array}
$$

LP Formulation

x_1 =, x_2 =

Total profit =

Solution

What if : Scenario 4

Type 1 more profitable but consumes more labor and significantly more material, also available labor hours = 1700

$$
\begin{array}{ll}
\max & 350x_1 + 300x_2 \\
\text{subject to :} & x_1 + x_2 \le 200 \\
 & 9x_1 + 6x_2 \le 1700 \\
 & 36x_1 + 16x_2 \le 2880 \\
 & x_1 \ge 0, x_2 \ge 0
\end{array}
$$

LP Formulation

x_1 = , x_2 =

Total profit =

Solution

What if : Scenario 5

Labor and tubing interchanged, total labor hours = 1700
Type 1 more profitable, consumes less labor but more material than Type 2.

$$
\begin{aligned}
\text{max} \quad & 350x_1 + 300x_2 \\
\text{subject to :} \quad & x_1 + x_2 \le 200 \\
& 6x_1 + 9x_2 \le 1700 \\
& 16x_1 + 12x_2 \le 2880 \\
& x_1 \ge 0, x_2 \ge 0
\end{aligned}
$$

LP Formulation

x_1 =, x_2 =

Total profit =

Solution

What if : Scenario 6

Tubing interchanged :

Type 1 more profitable, consumes more labor and more material than Type 2, also available labor hours = 1700

$$
\begin{aligned}
\text{max} \quad & 350x_1 + 300x_2 \\
\text{subject to :} \quad & x_1 + x_2 \leq 200 \\
& 9x_1 + 6x_2 \leq 1700 \\
& 16x_1 + 12x_2 \leq 2880 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

LP Formulation

x_1 =, x_2 =

Total profit =

Solution

What if : Scenario 7

All other values unchanged, only labor hours changed to 1700.

$$
\begin{aligned}
\max \quad & 350x_1 + 300x_2 \\
\text{subject to :} \quad & x_1 + x_2 \leq 200 \\
& 9x_1 + 6x_2 \leq 1700 \\
& 12x_1 + 16x_2 \leq 2880 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

LP Formulation

x_1 =, x_2 =

Total profit =

Solution

- Suppose, for example, that Blue Ridge Hot Tubs has only 1,520 hours of labor and 2,650 feet of tubing available during its next production cycle. The company might be interested in solving the following ILP problem

$$
\begin{array}{llrl}
\text{MAX:} & 350X_1 + 300X_2 & & \}\ \text{profit} \\
\text{Subject to:} & 1X_1 + 1X_2 \le & 200 & \}\ \text{pump constraint} \\
 & 9X_1 + 6X_2 \le & 1{,}520 & \}\ \text{labor constraint} \\
 & 12X_1 + 16X_2 \le & 2{,}650 & \}\ \text{tubing constraint} \\
 & X_1, X_2 \ge 0 & & \}\ \text{nonnegativity conditions} \\
 & X_1, X_2 \text{ must be integers} & & \}\ \text{integrality conditions}
\end{array}
$$

Solve the Linear Program

An Assignment problem

- A 400-meter medley relay involves four different swimmers, who successively swim 100 meters of the backstroke, breaststroke, butterfly and freestyle. A coach has six very fast swimmers whose expected times (in seconds) in the individual events are given in following table.

Which swimmers must the coach assign for which style so that the overall team swimming time is reduced?

	Event 1 (backstroke)	Event 2 (breaststroke)	Event 3 (butterfly)	Event 4 (freestyle)
Swimmer 1	68	73	63	57
Swimmer 2	67	70	64	58
Swimmer 3	68	72	69	55
Swimmer 4	67	75	70	59
Swimmer 5	71	69	75	57
Swimmer 6	69	71	66	59

A “Cost matrix” C where C(i,j) denotes the cost of assigning swimmer i to event j

Seeking a solution in the form of a matrix (say X) which is the same size as cost matrix where :

$$X(i,j) = \begin{cases} 1 & \text{if swimmer } i \text{ is assigned to event } j \\ 0 & \text{otherwise} \end{cases}$$

	Event 1	**Event 2**	**Event 3**	**Event 4**
Swimmer 1	X(1,1)	X(1,2)		
Swimmer 2				
Swimmer 3		…		
Swimmer 4				
Swimmer 5				
Swimmer 6	X(6,1)			X(6,4)

Total number of variables being solved for = size of cost matrix = 24

Total cost = sum of individual costs.
Hence objective function is

$$\sum_{i=1}^{6}\sum_{j=1}^{4} C(i,j)X(i,j)$$

Constraints

$\sum_{j=1}^{4} X(i,j) \leq 1$	$1 \leq i \leq 6$	row sums must not exceed 1 since any swimmer can be assigned to at most one event
$\sum_{i=1}^{6} X(i,j) = 1$	$1 \leq j \leq 4$	column sums must be exactly one since each event is assigned to exactly one swimmer
$\sum_{j} X(i,j) \geq 0$	$1 \leq i \leq 6$	row sums must be ≥ 0 since swimmer assignments cannot be negative numbers
$0 \leq X(i,j) \leq 1$	$1 \leq i \leq 6$ $1 \leq j \leq 4$	condition that $X(i,j) \in \{0,1\}$ relaxed to $X(i,j)$ lies between 0 and 1