Inspire…Educate…Transform.

Statistics and Probability in Decision Modeling

Multiple Linear Regression

Dr. Venkatesh Sunkad

MATERIAL CONTENT FROM Dr. SRIDHAR PAPPU

Jan 5 2019

Degrees of Freedom [EXCEL "Degree of Freedom"]

Degrees of freedom, ν: # of independent observations for a source of variation minus the number of independent parameters estimated in computing the variation.*

When sample size is considered, degrees of freedom are *n-1*.

* Roger E. Kirk, *Experimental Design: Procedures for the Behavioral Sciences*. Belmont, California: Brooks/Cole, 1968.

CSE 7302c

Influential Observations

An observation which, when not included, greatly alters the predicted scores of other observations.

Cook's D is a measure of the influence and is proportional to the sum of the squared differences between predictions made with all observations in the analysis and predictions made leaving out the observation in question.

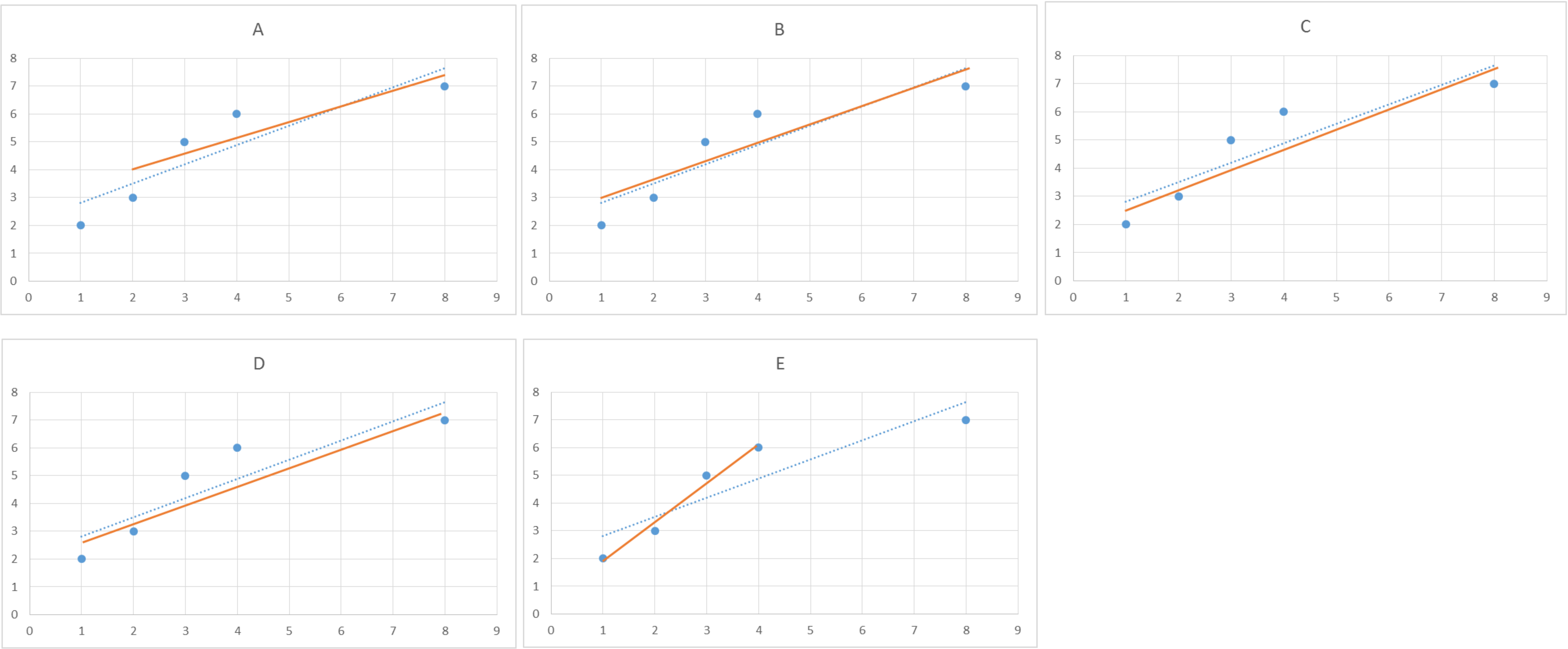
Influence is a function of **leverage** and **distance (or 'residuality' or 'outlierness')**.

Influential Observations

ID	X	Y	h	R	D
A	1	2	0.39	-1.02	0.4
B	2	3	0.27	-0.56	0.06
C	3	5	0.21	0.89	0.11
D	4	6	0.2	1.22	0.19
E	8	7	0.73	-1.68	8.86
h is the leverage, R is the studentized residual, and D is Cook's measure of influence.					

Source: http://onlinestatbook.com/2/regression/influential.html
Last accessed: June 30, 2017

Influential Observations



Influential Observations – Rules of Thumb

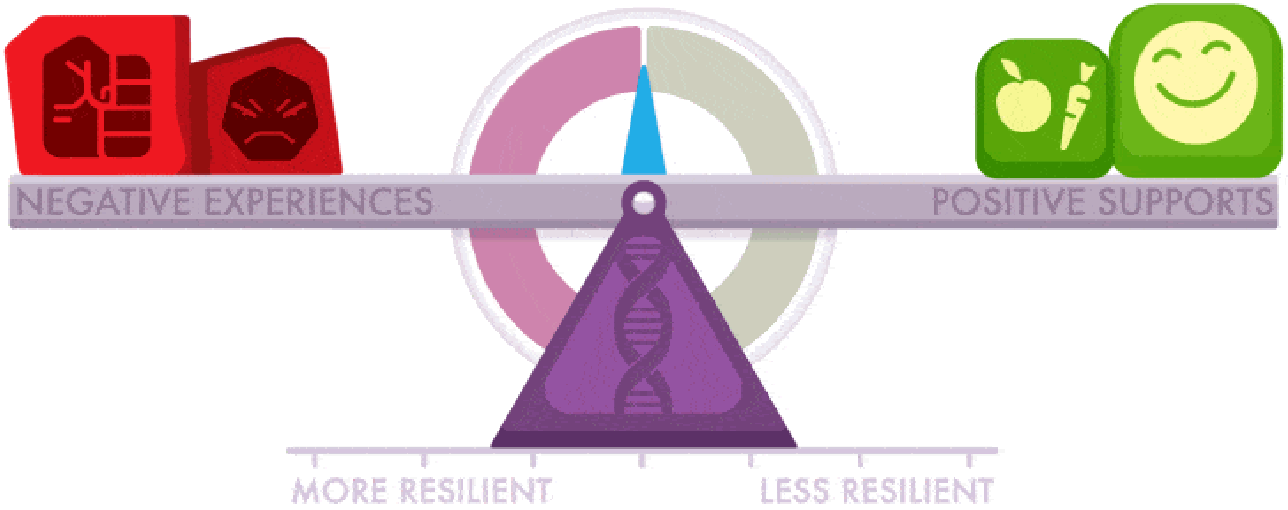
- If Cook's D of any observation (D_i) > 1, that observation can be considered as having too much influence, but investigate values greater than 0.5 also.

- ***Relative size interpretation*: In general, investigate any value that is very different from the rest.**

Influential Observations - Leverage

How much the observation's value on the **predictor variable** differs from the mean of the **predictor variable**.

That is, it tells us about extreme x values, which have the potential to highly influence the regression in certain conditions. *Remember Eric McCoo.*



Influential Observations – Leverage [Excel "Rsquared-Significance"]

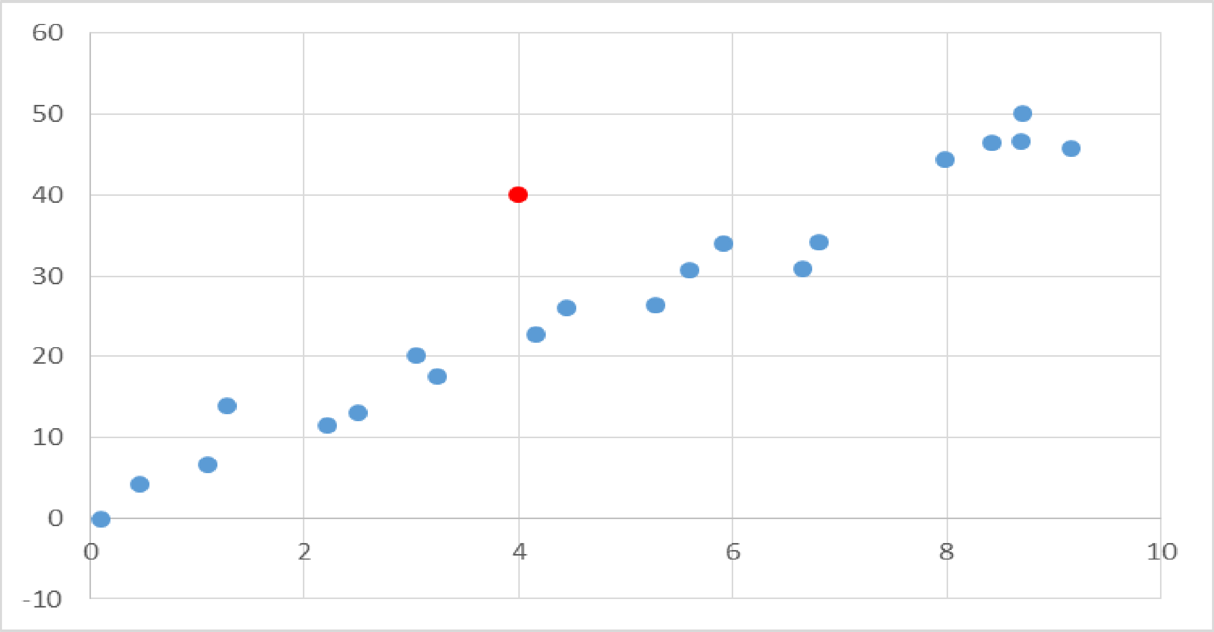
Leverage of the i^{th} data point is given by:

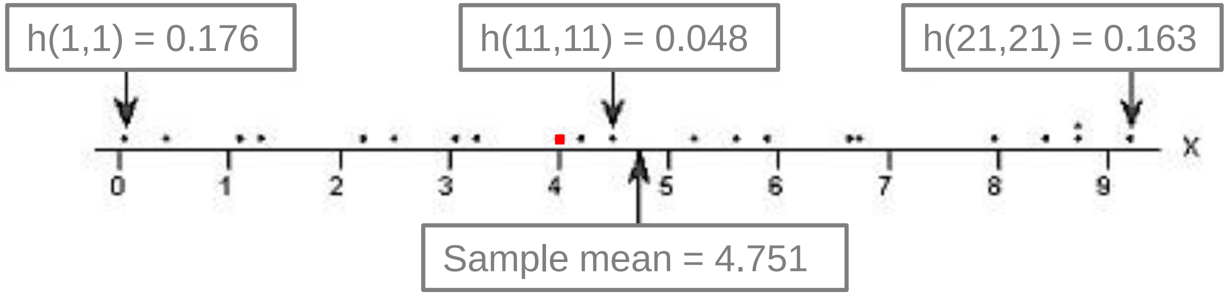
$$h_i = \frac{1+z^2}{n}$$

The sum of leverages = # of parameters, *p* (regression coefficients **including intercept**).

EXCEL ACTIVITY

Influential Observations - Leverage





Flag observations whose h > 3* avg(h) or h > 2* avg(h)

$$Avg(h) = \frac{sum(h)}{n} = \frac{p}{n}$$

CSE 7302c

Influential Observations - Distance

Based on error of prediction and is measured by Studentized Residual. This is calculated on the **dependent** variable and is a measure of ‘outlierness’.

Recall Student’s t-test. So, Studentizing is related to calculating the t-statistic of the metric in question, i.e., it is related to error of prediction of that observation divided by the standard deviation of the errors of prediction.

Influential Observations - Distance [Excel "RSquared-Significance" Influence Tab]

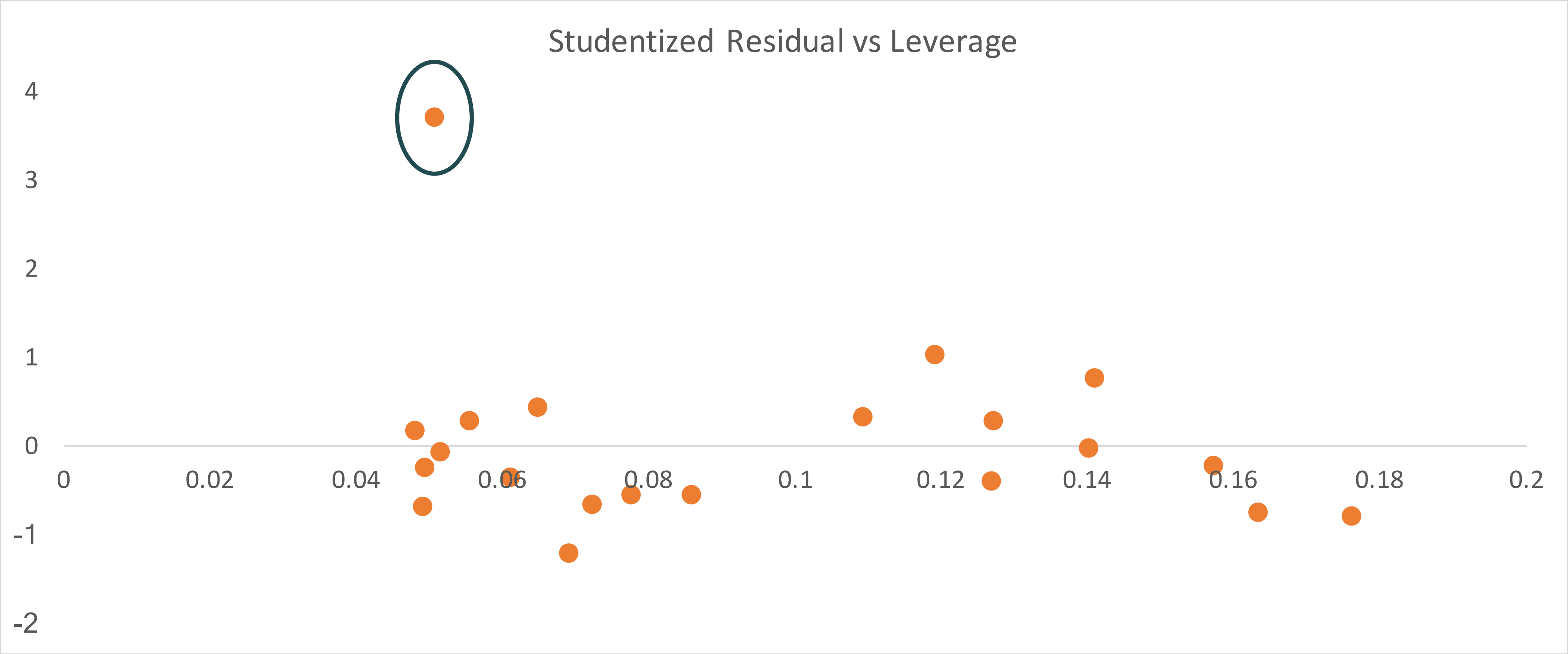
$$stdres_i = \frac{e_i}{\sqrt{MSE(1 - h_i)}}$$

<u>Investigate</u> observations with internally studentized residuals smaller than -2 or larger than 2.

Recall the empirical rule for normal distribution and the assumption that residuals follow normal distribution.

EXCEL ACTIVITY

Influential Observations - Distance

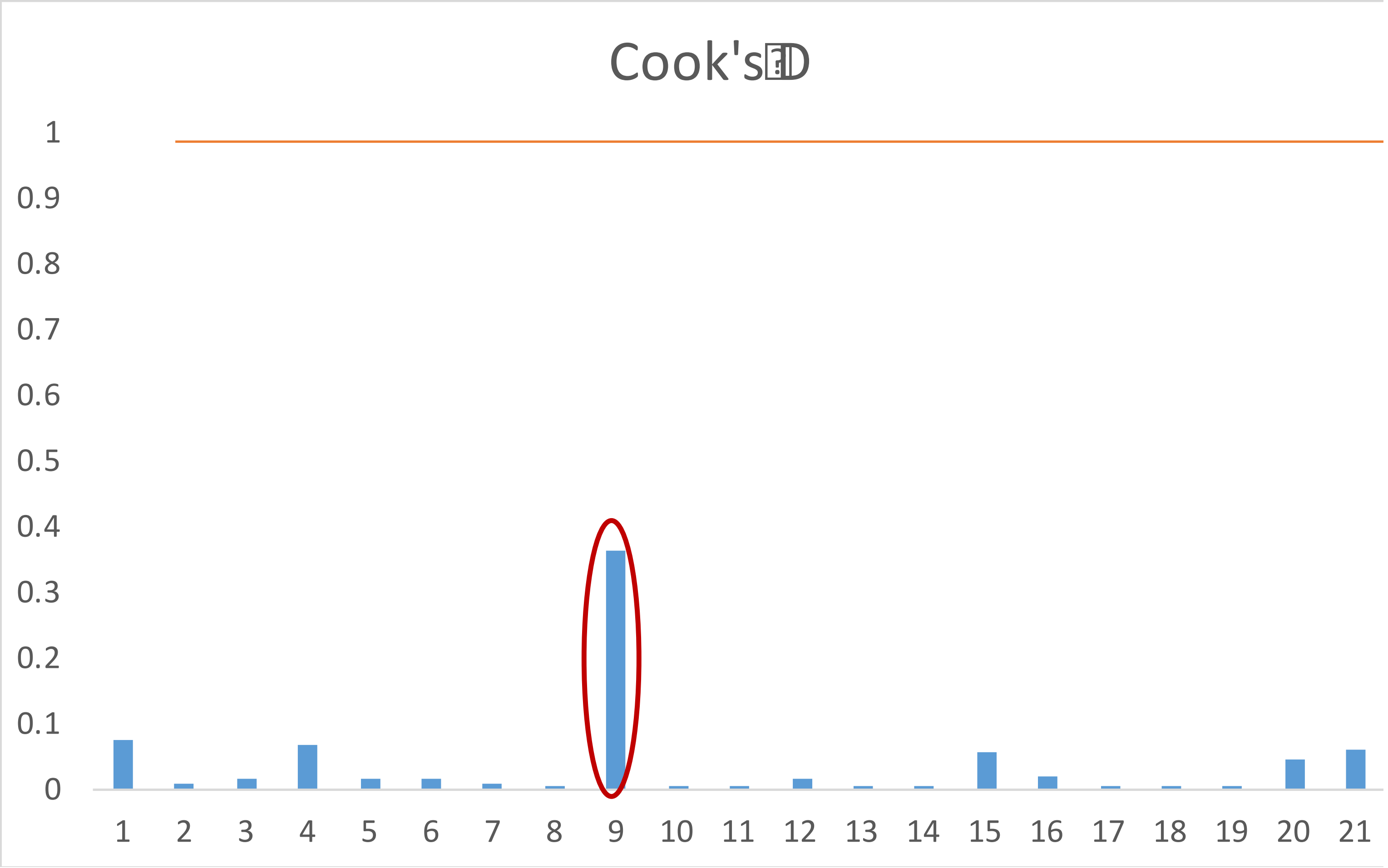


Influential Observations – Cook's D

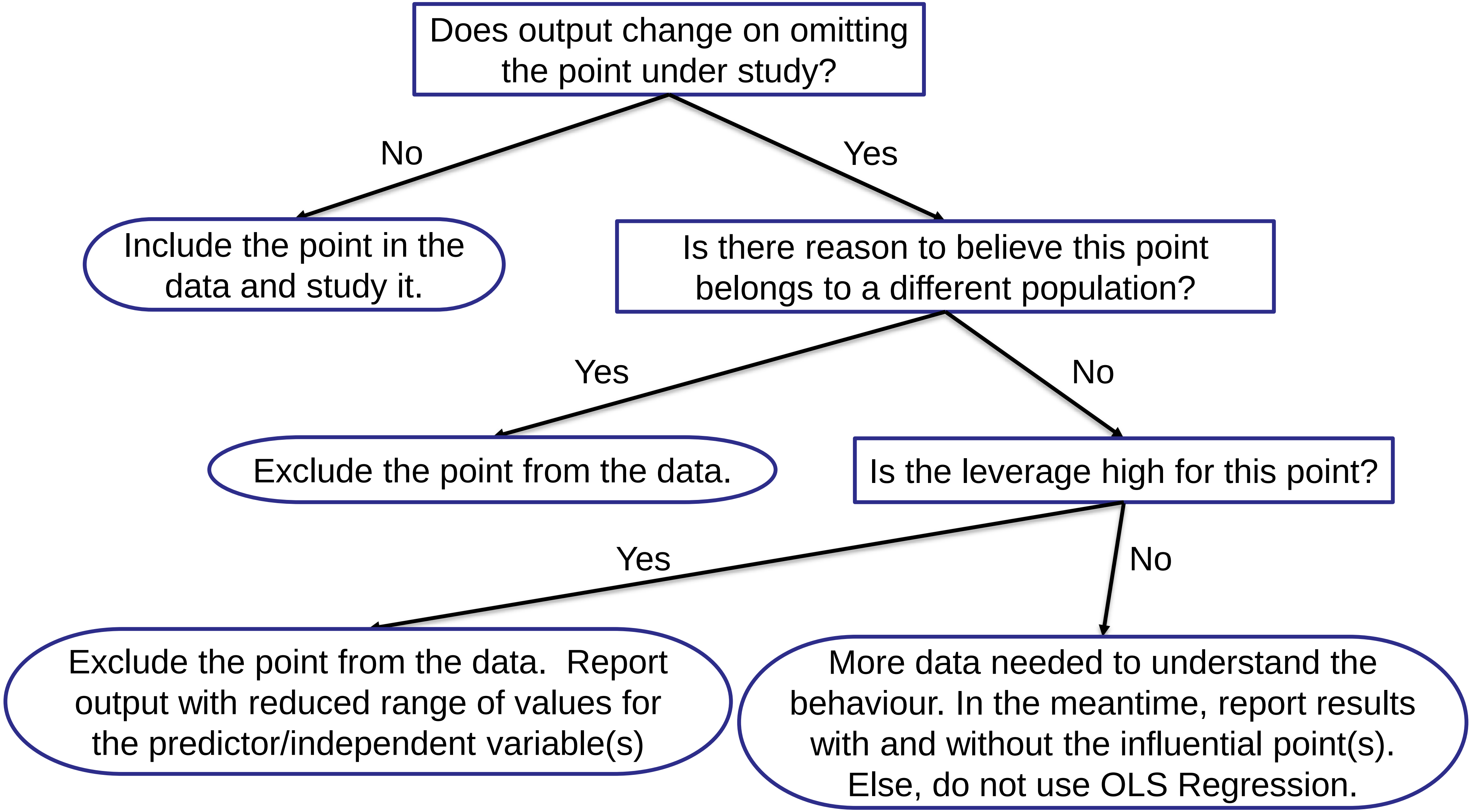
Measures overall influence of an observation by seeing the impact on the regression coefficients when this observation is omitted. It accounts both for **leverage** and **residual**.

$$D_i = \frac{1}{p}(stdres_i)^2\left(\frac{h_i}{1-h_i}\right)$$

Influential Observations – Cook's D



Handling Influential Observations



Caution – R^2 in Regression Through Origin

- Temptation to drop intercept if it is **not significant**.

SUMMARY OUTPUT								
Regression Statistics								
Multiple R	0.717055011							
R Square	0.514167888							
Adjusted R Square	0.494734604							
Standard Error	4.21319131							
Observations	27							
ANOVA								
	df	*SS*	*MS*	*F*	*Significance F*			
Regression	1	469.6573265	469.6573265	26.4581054	2.57053E-05			
Residual	25	443.7745253	17.75098101					
Total	26	913.4318519						
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 99.0%*	*Upper 99.0%*
Intercept	-4.154014573	2.447784673	-1.697050651	0.102104456	-9.195321476	0.88729233	-10.97705723	2.669028089
Big Mac Price ($)	3.547427488	0.689658599	5.143744297	2.57053E-05	2.127049014	4.967805962	1.625048409	5.469806567

Caution – R^2 in Regression Through Origin

- Physical process makes intuitive sense for *y*=0 when *x*=0. For example, if the speed of the car = 0 mph, the distance travelled before it comes to a stop = 0 ft. However if you do not have sufficient data around origin do not drop the intercept.

Multiple Linear Regression

THE OUTPUT

Multiple Linear Regression

- Simple Linear Regression models the effect of one independent variable, *x*, on one dependent variable, *y*
- Multiple Regression models the effect of several independent variables, x_1, x_2 etc., on one dependent variable, *y*
- The different x variables are combined in a linear way and each has its own regression coefficient:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots .. + \beta_n x_n + \varepsilon$$

- The β parameters reflect the **independent contribution** of each independent variable, *x*, to the value of the dependent variable, *y*.

Interpreting Regression Coefficients

A coefficient is the slope of the linear relationship between the dependent variable (DV) and the **independent contribution** of the independent variable (IV), i.e., that part of the IV that is independent of (or uncorrelated with) all other IVs.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.89666084
R Square	0.804000661
Adjusted R Square	0.750546296
Standard Error	2.90902388
Observations	15

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	381.8467141	127.282238	15.04087945	0.00033002
Residual	11	93.08661926	8.462419933		
Total	14	474.9333333			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($)	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($)	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286

Assumptions of Multiple Linear Regression

- Same as simple linear regression
  - Linearity
  - Independence of errors
  - Homoscedasticity (constant variance)
  - Normality of errors

- Methods of checking assumptions are also the same

Determining the Multiple Regression Equation

- *k+1* equations to solve for *k* independent variables and the intercept.
- In solving for intercept and slope in a simple linear regression model, we needed $\sum x$, $\sum y$, $\sum xy$, and $\sum x^2$.
- For multiple regression model with 2 independent variables, we need $\sum x_1$, $\sum x_2$, $\sum y$, $\sum {x_1}^2$, $\sum {x_2}^2$, $\sum x_1 x_2$, $\sum x_1 y$, and $\sum x_2 y$.

Determining the Multiple Regression Equation – Excel ["Regression" Multiple Regression Tab]

In a real estate study, multiple variables were explored to determine the price of a house.

- # of bedrooms
- # of bathrooms
- Age of the house
- # of square feet of living space
- Total # of square feet of space
- # of garages

Find the equation if you want to predict the price of the house by total square feet and age of the house.

CSE 7302c

Determining the multiple regression equation – Interpreting the output

What is the equation?

$$\hat{y} = 57.35 + 0.0177Area - 0.666Age$$

Are the coefficients and the model significant?

Yes

SUMMARY OUTPUT						
Regression Statistics						
Multiple R	0.860872681					
R Square	0.741101773					
Adjusted R Square	0.715211951					
Standard Error	11.96038667					
Observations	23					
ANOVA						
	df	*SS*	*MS*	*F*	*Significance F*	
Regression	2	8189.723012	4094.861506	28.62521631	1.35298E-06	
Residual	20	2861.016988	143.0508494			
Total	22	11050.74				
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	57.35074586	10.00715186	5.73097587	1.31298E-05	36.47619286	78.22529885
Area (sq ft) (x1)	0.017718036	0.00314562	5.632605205	1.63535E-05	0.011156388	0.024279685
Age of House (years) (x2)	-0.666347946	0.227996703	-2.922620973	0.008417613	-1.141940734	-0.190755157

Residuals – Practice Assignment

Residuals are determined the same way as in simple linear regression. The predicted value is calculated by substituting the predictor values of interest. The residual is again the difference between the observed and the predicted values, $y - \hat{y}$.

SSE and Standard Error of the Estimate, SE – Practice Assignment

$$SSE = \sum (y - \hat{y})^2$$

$$SE = \sqrt{\frac{SSE}{n - k - 1}}$$

k = Number of independent variables

Coefficient of Multiple Determination, R^2 – Practice Assignment

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

Adjusted R^2 - Excel

As additional independent variables are added to the regression model, the value of R^2 increases.

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

However, sometimes these variables are insignificant and add no real value, yet inflating the R^2 value.

Adjusted R^2 takes into consideration both the additional information and the changed degrees of freedom.

$$Adjusted\ R^2 = 1 - \frac{\frac{SSE}{(n-k-1)}}{\frac{SST}{n-1}} = R^2 - \left(1 - R^2\right)\frac{k}{n-k-1} = 1 - \frac{MSE}{MST}$$

BREAK

Sample R Output

```
Call:
lm(formula = ToxinConc$Toxin ~ ToxinConc$Rain + ToxinConc$NoonTemp +
    ToxinConc$Sunshine + ToxinConc$WindSpeed, data = ToxinConc)

Residuals:
      1       2       3       4       5       6       7       8
-1.8818  2.0498 -0.6314  0.4787 -0.5805  1.2508 -0.1921 -0.1813
      9      10
-1.1552  0.8429

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)          31.6084     7.1051   4.449  0.00671 **
ToxinConc$Rain        7.0676     1.0031   7.046  0.00089 ***
ToxinConc$NoonTemp   -0.4201     0.2413  -1.741  0.14215
ToxinConc$Sunshine   -0.2375     0.5086  -0.467  0.66018
ToxinConc$WindSpeed  -0.7936     0.2977  -2.666  0.04458 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.574 on 5 degrees of freedom
Multiple R-squared:  0.9186,     Adjusted R-squared:  0.8535
F-statistic: 14.11 on 4 and 5 DF,  p-value: 0.006232
```

Multiple Linear Regression

HANDLING SPECIAL SITUATIONS

Nonlinear Models – Polynomial Regression

For example, $y = \beta_0 + \beta_1 x_1 + \beta_2 {x_1}^2 + \epsilon$

How is this a special case of the general linear model?

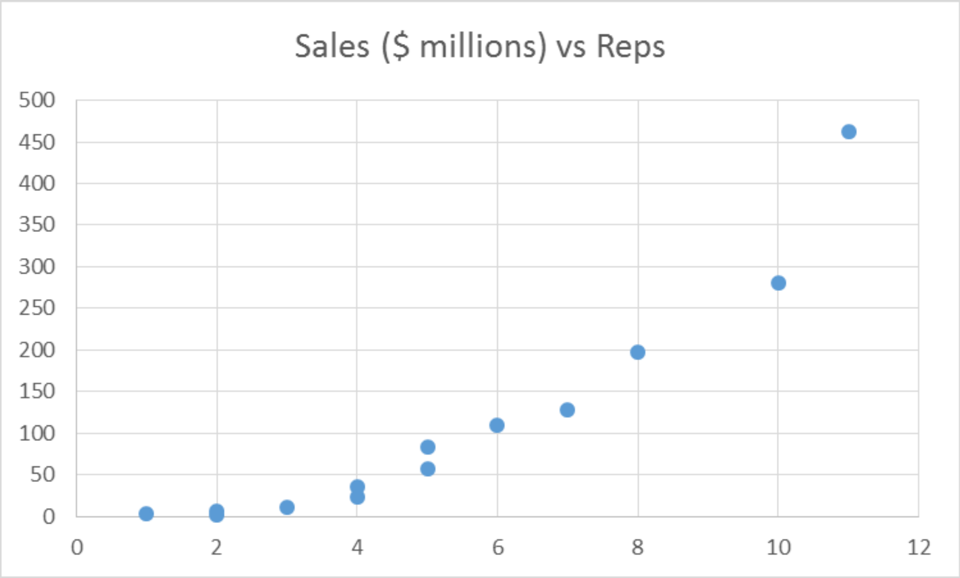
Replace ${x_1}^2$ with x_2, so that $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$

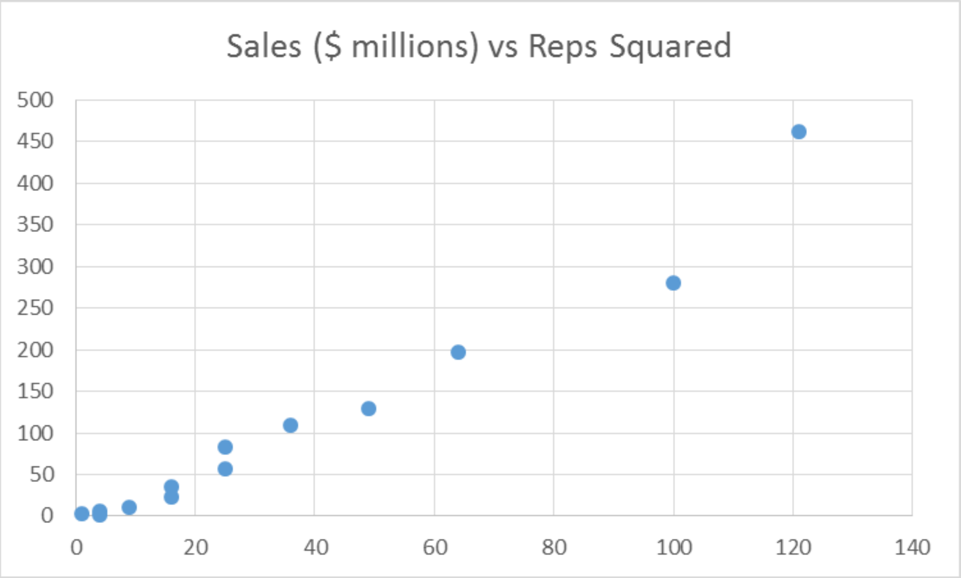
Multiple linear regression assumes a linear fit of the regression coefficients and regression constant, but not necessarily a linear relationship of the independent variable values.

CSE 7302c

Nonlinear Models – Polynomial Regression - Excel

Sales volume versus # of sales reps and # of sales reps squared

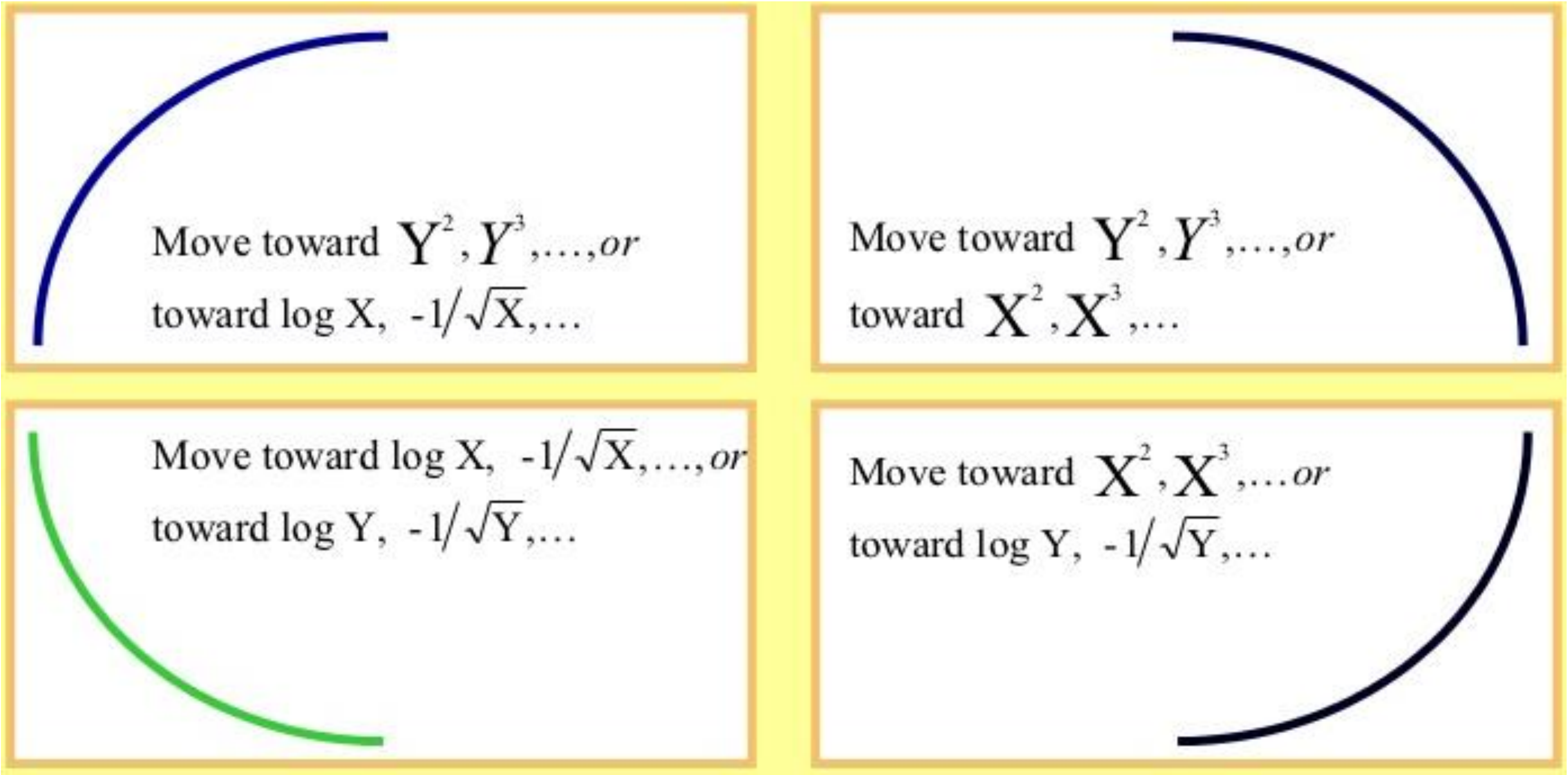




Tukey's Ladder of Transformations

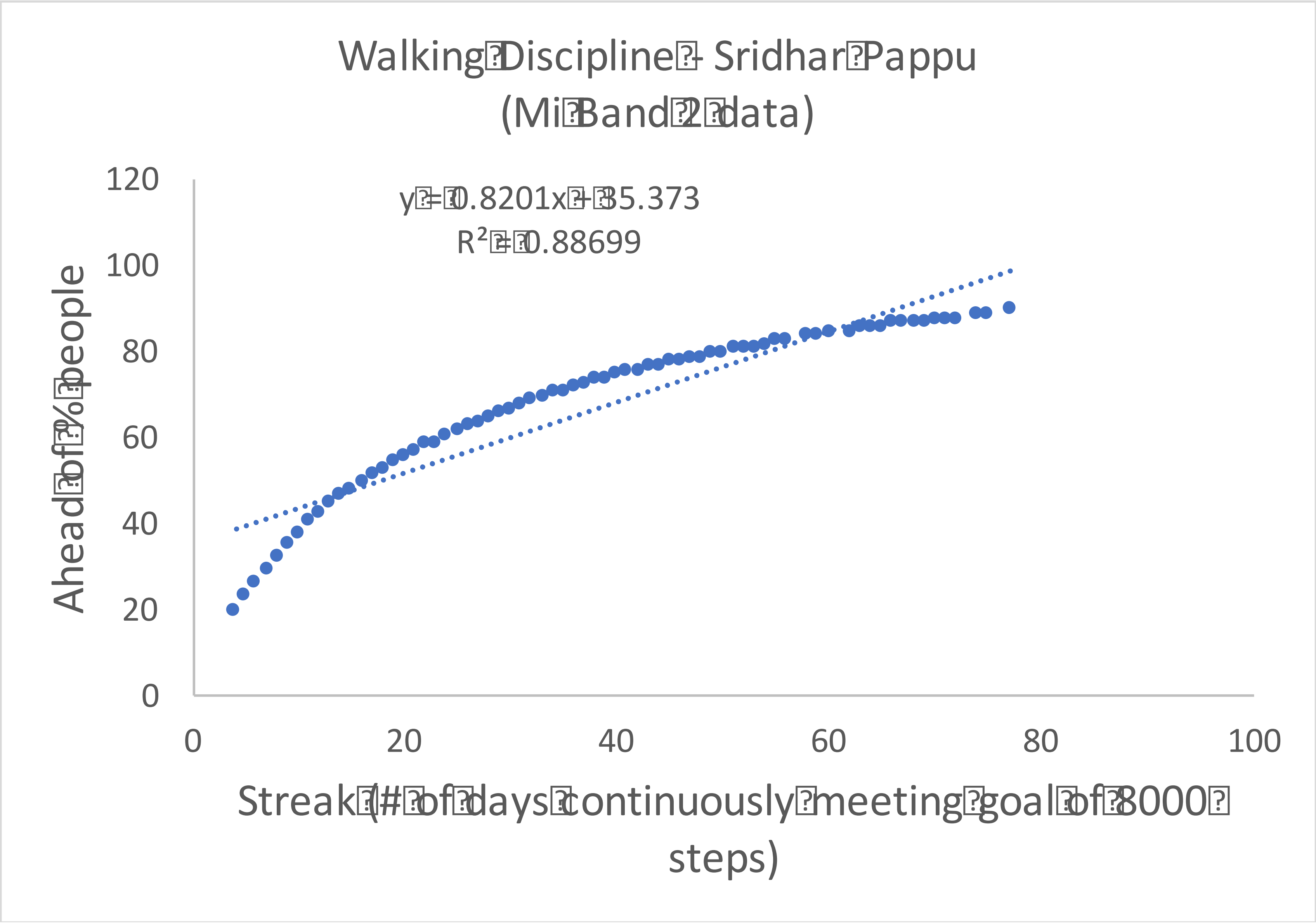
Ladder for x		
Up ladder	Neutral	Down ladder
$\ldots, x^4, x^3, x^2, x$	$\sqrt{x}, x, logx$	$-\frac{1}{\sqrt{x}}, -\frac{1}{x}, -\frac{1}{x^2}, -\frac{1}{x^3}, \ldots$
Ladder for y		
Up ladder	Neutral	Down ladder
$\ldots, y^4, y^3, y^2, y$	$\sqrt{y}, y, logy$	$-\frac{1}{\sqrt{y}}, -\frac{1}{y}, -\frac{1}{y^2}, -\frac{1}{y^3}, \ldots$

Tukey’s Four-Quadrant Approach

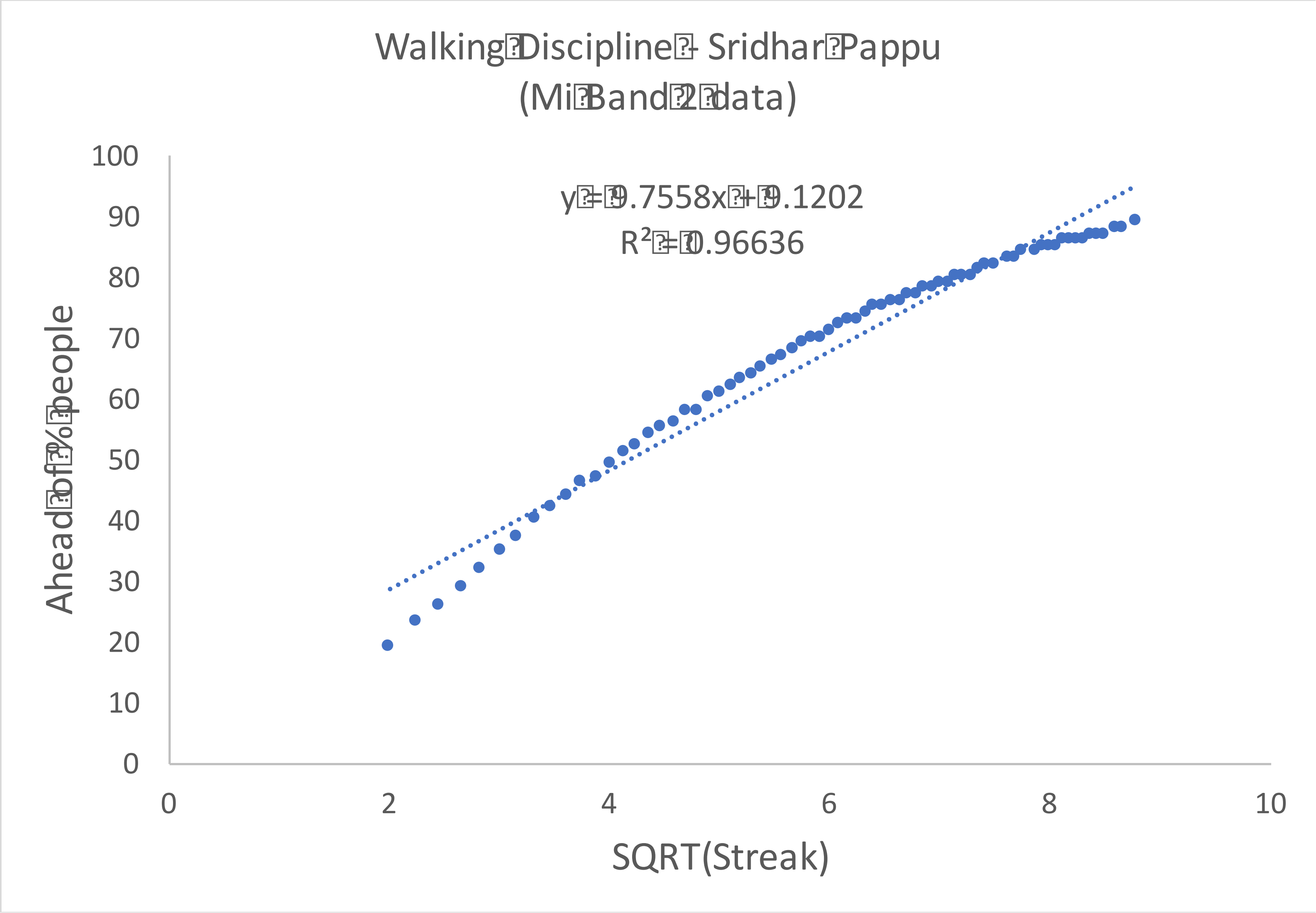


CSE 7302c

Based on Tukey's 4-Quadrant Approach, what transformation do you recommend?

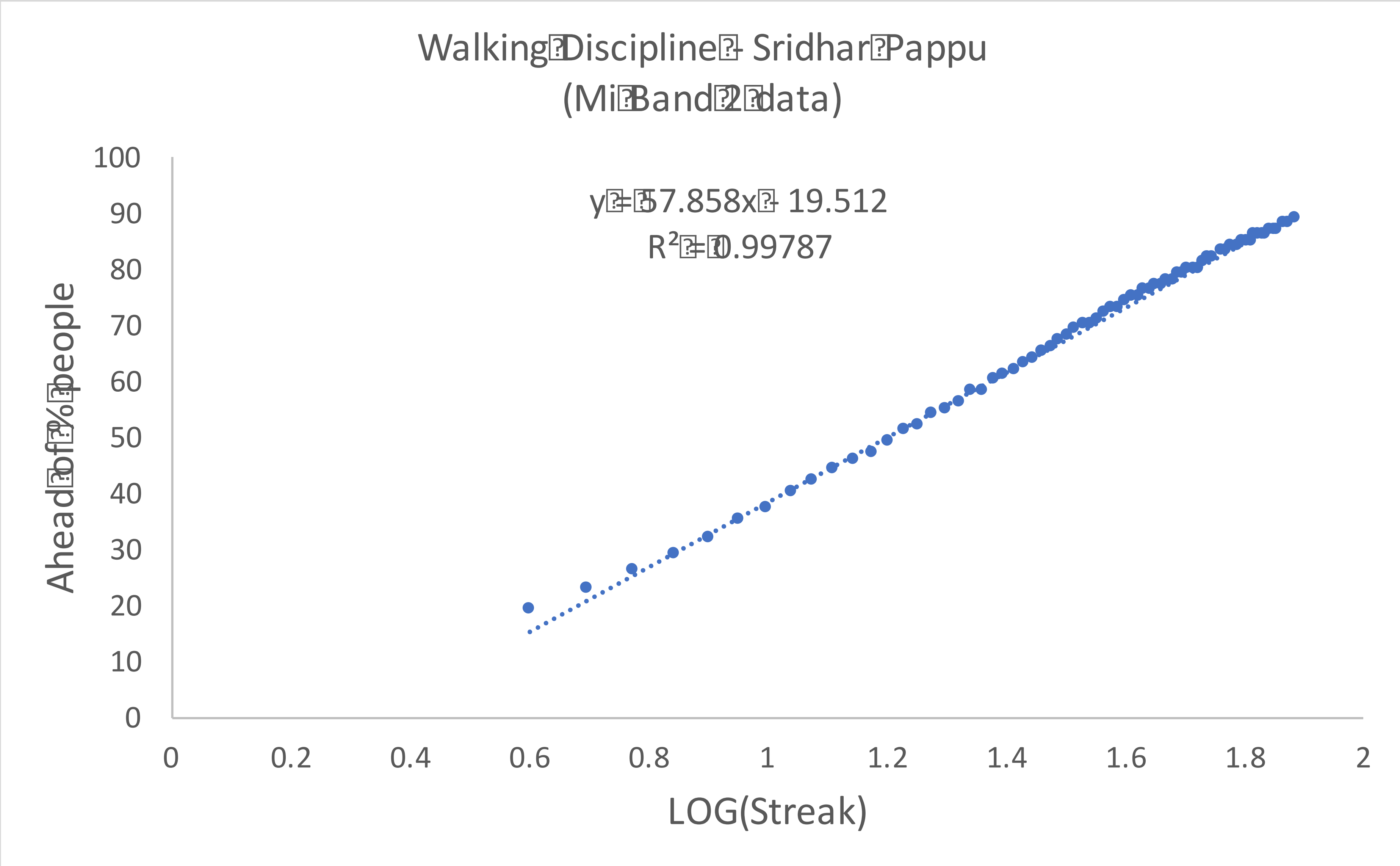


SQRT Transformation on X

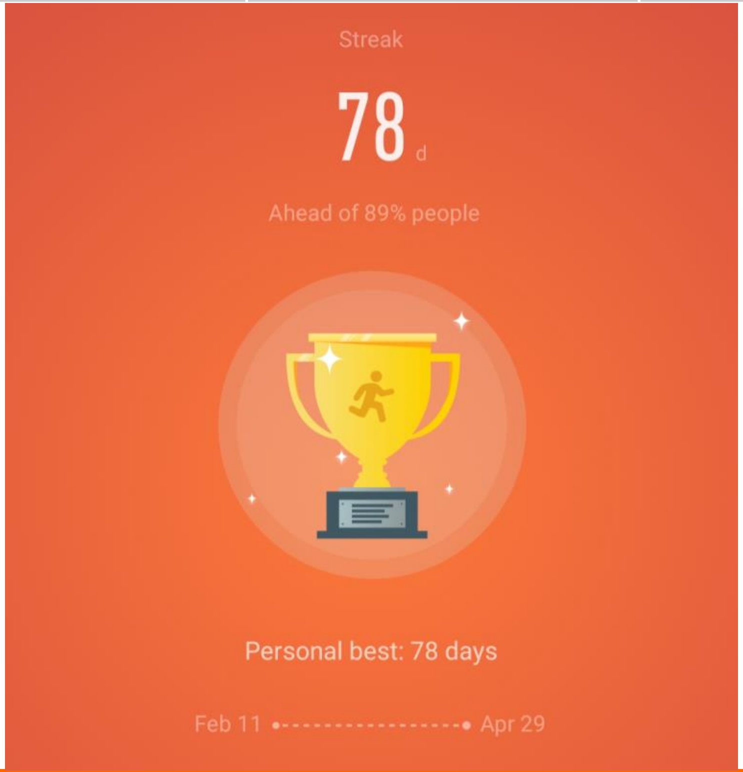


CSE 7302c

LOG Transformation on X



Data	Equation	R-Squared	Ahead of % People (Prediction for Day 78)
Original	0.8201x + 35.373	88.7%	99.34
Square Root on X	9.7558x + 9.1202	96.6%	95.28
Log on X	57.858x – 19.512	99.8%	89.96

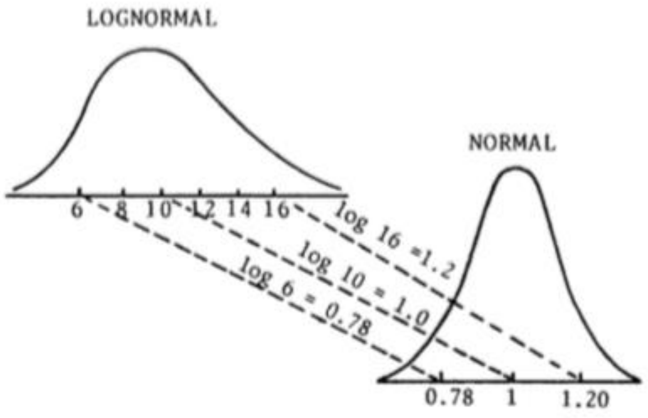


More thoughts on Transformations

DATA TRANSFORMATION

As suggested by Tabachnick and Fidell (2007) and Howell (2007), the following guidelines (including SPSS compute commands) should be used when transforming data.

If your data distribution is…	Use this transformation method.
Moderately positive skewness	Square-Root NEWX = SQRT(X)
Substantially positive skewness	Logarithmic (Log 10) NEWX = LG10(X)
Substantially positive skewness (with zero values)	Logarithmic (Log 10) NEWX = LG10(X + C)
Moderately negative skewness	Square-Root NEWX = SQRT(K – X)
Substantially negative skewness	Logarithmic (Log 10) NEWX = LG10(K – X)



C = a constant added to each score so that the smallest score is 1.
K = a constant from which each score is subtracted so that the smallest score is 1; usually equal to the largest score + 1.

Source: http://oak.ucc.nau.edu/rh232/courses/eps625/handouts/data%20transformation%20handout.pdf
Last accessed: May 12, 2016

CSE 7302c

Approach to determine whether to transform X or Y to achieve **linearity**, **homoscedasticity** and **normality**:

1. Often, a transformation that fixes one, fixes all.
2. In general, transforming both is not required, although sometimes it is.
3. A general rule of thumb:
    1. Transform Y first to remove heteroscedasticity and non-normality.
    2. Then transform X to remove non-linearity.

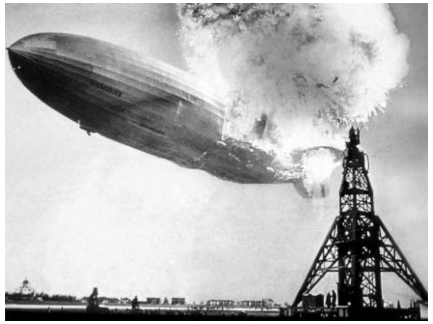
Nonlinear Models – With Interaction

Interaction can be examined as a separate independent variable in regression.

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \varepsilon$$

For example,

- Individually each of two drugs might improve symptoms, but when taken together, they may interact and cause a decline in health.
- Fire increases a balloon's levity (hot air balloon). Hydrogen also increases levity as in the Zeppelins. But fire and hydrogen dramatically reduce the levity.



Nonlinear Models – Without Interaction – Excel["Regression" Regression with Interaction Tab]

SUMMARY OUTPUT						
Regression Statistics						
Multiple R	0.687213365					
R Square	0.47226221					
Adjusted R Square	0.384305911					
Standard Error	4.570195728					
Observations	15					
ANOVA						
	df	*SS*	*MS*	*F*	*Significance F*	
Regression	2	224.2930654	112.1465327	5.369282452	0.021602756	
Residual	12	250.6402679	20.88668899			
Total	14	474.9333333				
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	50.85548009	3.790993168	13.41481713	1.38402E-08	42.59561554	59.11534464
Stock 2 ($)	-0.118999968	0.19308237	-0.616317112	0.54919854	-0.539690313	0.301690376
Stock 3 ($)	-0.07076195	0.198984841	-0.35561478	0.728301903	-0.504312675	0.362788775

Model is significant but neither of the variables is.

Nonlinear Models – With Interaction - Excel

SUMMARY OUTPUT	
Regression Statistics	
Multiple R	0.89666084
R Square	0.804000661
Adjusted R Square	0.750546296
Standard Error	2.90902388
Observations	15

- One of the earlier insignificant variables along with the interaction term are now significant.
- Model remains significant.
- Adjusted R-sq doubled.

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	381.8467141	127.282238	15.04087945	0.00033002
Residual	11	93.08661926	8.462419933		
Total	14	474.9333333			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($)	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($)	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3	-0.009984949	0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

Indicator (Dummy) Variables

Categorical variables such as gender, geographic region, occupation, marital status, level of education, economic class, religion, buying/renting a home, etc. can also be used in multiple regression analysis.

If there are *n* levels in a category, *n-1* dummy variables need to be inserted into the regression analysis replacing that category.

Indicator (Dummy) Variables

If a survey question asks about the region of country your office is located in, with North, South, East and West as the options, the **recoding** can be done as follows:

Region
North
East
North
South
West
West
East

North	West	South
1	0	0
0	0	0
1	0	0
0	0	1
0	1	0
0	1	0
0	0	0

Indicator (Dummy) Variables - Excel

Consider the issue of gender discrimination in the salary earnings of workers in some industries. If there is discrimination, how much is one gender earning more than the other?

CSE 7302c

Indicator (Dummy) Variables – [Excel “Regression”- Significance” Dummy Variables in Regression Tab]

BREAK

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.933293402
R Square	0.871036574
Adjusted R Square	0.869727301
Standard Error	0.095635901
Observations	200

Monthly Salary = 1.821901302 + 0.083754451*Age + 0.467628629*Gender

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	12.16964651	6.084823253	665.2824405	2.40412E-88
Residual	197	1.801806432	0.009146226		
Total	199	13.97145294			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*ower 95.0%*	*Jpper 95.0%*
Intercept	1.821901302	0.059565421	30.58655988	9.46086E-77	1.704433585	1.939369019	1.704434	1.939369
Age (10 years)	0.083754451	0.018135789	4.618186202	6.97762E-06	0.047989241	0.11951966	0.047989	0.11952
Gender (1=Male, 0=Female)	0.467628629	0.014321506	32.65219766	2.00282E-81	0.439385488	0.49587177	0.439385	0.495872

Separate equation for each gender

Indicator (Dummy) Variables – Interpreting Coefficients and Relationship to ANOVA Excel [Regression" Multiple Regression Tab 2"]

ANOVA

Anova: Single Factor

SUMMARY

Groups	Count	Sum	Average	Variance
Exp-Fresher	55	119.7279	2.176871	0.096379
Exp-Low	70	168.6399	2.409142	0.045699
Exp-Med	75	179.2534	2.390045	0.049032

ANOVA

Source of Variation	SS	df	MS	F	P-value	F crit
Between Groups	1.985371	2	0.992685	16.31551	2.78E-07	3.04175303
Within Groups	11.98608	197	0.060843			
Total	13.97145	199				

OLS

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.376964139
R Square	0.142101962
Adjusted R Square	0.133392337
Standard Error	0.246663853
Observations	200

ANOVA

	df	SS	MS	F	Significance F
Regression	2	1.985370871	0.992685	16.31551	2.77596E-07
Residual	197	11.98608207	0.060843		
Total	199	13.97145294			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	2.176871087	0.033260147	65.44983	1.3E-135	2.111279448	2.242463
Exp-Low	0.232270784	0.044445741	5.22594	4.4E-07	0.14462027	0.319921
Exp-Med	0.213174092	0.043789018	4.868209	2.3E-06	0.126818687	0.299529

- Mean of the reference group in ANOVA is the intercept in OLS.
- Differences between means of groups are the coefficients in OLS.

Indicator (Dummy) Variables – Interpreting Coefficients and Relationship to ANOVA

Choice of reference group is not important; end results remain the same.

What will be the salary of a fresher in the two cases below where *Fresher* is the reference group in the 1st case and *Low experience* is the reference group in the 2nd?

Multiple R	0.376964139
R Square	0.142101962
Adjusted R Square	0.133392337
Standard Error	0.246663853
Observations	200

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	1.985370871	0.992685	16.31551	2.77596E-07
Residual	197	11.98608207	0.060843		
Total	199	13.97145294			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	2.176871087	0.033260147	65.44983	1.3E-135	2.111279448	2.242463
Exp-Low	0.232270784	0.044445741	5.22594	4.4E-07	0.14462027	0.319921
Exp-Med	0.213174092	0.043789018	4.868209	2.3E-06	0.126818687	0.299529

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.376964139
R Square	0.142101962
Adjusted R Square	0.133392337
Standard Error	0.246663853
Observations	200

ANOVA

	df	*SS*	*MS*	*F*	*ignificance F*
Regression	2	1.985370871	0.992685	16.31551	2.78E-07
Residual	197	11.98608207	0.060843		
Total	199	13.97145294			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	2.409141871	0.029481969	81.71577	6.3E-154	2.351001	2.467283
Exp-Fresher	-0.232270784	0.044445741	-5.22594	4.4E-07	-0.31992	-0.14462
Exp-Med	-0.019096692	0.040993015	-0.46585	0.641836	-0.09994	0.061745

p-values here indicate if the level (or group) is significantly different from the reference level (or group).

What might you do if there is no significant difference as is the case between low and medium experience? Also, check the averages in ANOVA output.

A possible action could be to combine Low and Medium groups into a single group

Indicator (Dummy) Variables – Interpreting Coefficients and Relationship to ANOVA

Interpret the coefficients of the numeric and categorical variables below.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.948085877
R Square	0.898866831
Adjusted R Square	0.896792304
Standard Error	0.08512366
Observations	200

ANOVA

	df	SS	MS	F	ignificance F
Regression	4	12.55848	3.139619	433.2877	8.41E-96
Residual	195	1.412977	0.007246		
Total	199	13.97145			

	Coefficients	*andard Errc*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	1.631967642	0.059023	27.64988	2.15E-69	1.515563	1.748372
Age (10 years)	0.122503981	0.016996	7.20789	1.22E-11	0.088985	0.156023
Gender (1=Male, 0=Female)	0.430437318	0.013721	31.37032	3.96E-78	0.403376	0.457498
Exp-Low	0.114744786	0.016665	6.885566	7.7E-11	0.081879	0.147611
Exp-Med	0.100583631	0.016081	6.254777	2.47E-09	0.068868	0.132299

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.948085877
R Square	0.898866831
Adjusted R Square	0.896792304
Standard Error	0.08512366
Observations	200

ANOVA

	df	SS	MS	F	ignificance F
Regression	4	12.55848	3.139619	433.2877	8.41E-96
Residual	195	1.412977	0.007246		
Total	199	13.97145			

	Coefficients	*andard Errc*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	1.746712428	0.054233	32.20771	5.32E-80	1.639754	1.85367
Age (10 years)	0.122503981	0.016996	7.20789	1.22E-11	0.088985	0.156023
Gender (1=Male, 0=Female)	0.430437318	0.013721	31.37032	3.96E-78	0.403376	0.457498
Exp-Fresher	-0.114744786	0.016665	-6.88557	7.7E-11	-0.14761	-0.08188
Exp-Med	0.014161156	0.01419	-0.99797	0.319532	-0.04215	0.013824

Y(salary) = 1.631967642+0.122503981 *Age+0.430437318*Gender+0.114744786*Exp-low+0.100583631*Exp-Med

Y(salary) = 1.746712428+0.122503981 *Age+0.430437318*Gender-0.114744786*Exp-Fresher+0.014161156*Exp-Med

- **Numeric:** For unit change in Age (numeric), Salary **increases** by 0.1225 (x 1000 $).
- **Categorical (Dummy):** If a person is a fresher, (s)he makes 0.1147 (x 1000$) **less** than a person with low experience.

Multiple Linear Regression

MODEL BUILDING METHODS

CSE 7302c

Model Building: Search Procedures

Suppose a model to predict the world crude oil production (barrels per day) is to be developed and the predictors used are:

- US energy consumption (BTUs)
- Gross US nuclear electricity generation (kWh)
- US coal production (short-tons)
- Total US dry gas (natural gas) production (cubic feet)
- Fuel rate of US-owned automobiles (miles per gallon)

What does your intuition say about how each of these variables would affect the oil production?

CrudeOilOutput

WorldOil	USEnergy	USAutoFuelRate	USNuclear	USCoal	USDryGas
55.7	74.3	13.4	83.5	598.6	21.7
55.7	72.5	13.6	114	610	20.7
52.8	70.5	14	172.5	654.6	19.2
57.3	74.4	13.8	191.1	684.9	19.1
59.7	76.3	14.1	250.9	697.2	19.2
60.2	78.1	14.3	276.4	670.2	19.1
62.7	78.9	14.6	255.2	781.1	19.7
59.6	76	16	251.1	829.7	19.4
56.1	74	16.5	272.7	823.8	19.2
53.5	70.8	16.9	282.8	838.1	17.8
53.3	70.5	17.1	293.7	782.1	16.1
54.5	74.1	17.4	327.6	895.9	17.5
54	74	17.5	383.7	883.6	16.5
56.2	74.3	17.4	414	890.3	16.1
56.7	76.9	18	455.3	918.8	16.6
58.7	80.2	18.8	527	950.3	17.1
59.9	81.4	19	529.4	980.7	17.3
60.6	81.3	20.3	576.9	1029.1	17.8
60.2	81.1	21.2	612.6	996	17.7
60.2	82.2	21	618.8	997.5	17.8
60.2	83.9	20.6	610.3	945.4	18.1
61	85.6	20.8	640.4	1033.5	18.8
62.3	87.2	21.1	673.4	1033	18.6
64.1	90	21.2	674.7	1063.9	18.8
66.3	90.6	21.5	628.6	1089.9	18.9
67	89.7	21.6	666.8	1109.8	18.9

Model Building: Search Procedures

Two considerations in model building:

- Explaining most variation in dependent variable
- Keeping the model simple AND economical

Quite often, the above two considerations are in conflict of each other.

If 3 variables can explain the variation nearly as well as 5 variables, the simpler model is better. Search procedures help choose the more attractive model.

Search Procedures: All Possible Regressions

All variables used in all combinations. For a dataset containing *k* independent variables, 2^k-1 models are examined. In the example of the oil production, 31 models are examined.

Tedious, Time-Consuming, Inefficient, Overwhelming.

Search Procedures: Stepwise Regression - R

AIC (Akaike's Information Criterion)

AIC = $2k + n\ln(RSS/n)$ where RSS is Residual Sum of Squares or SSE.

k is the number of parameters including intercept.

Sum of Sq is the additional reduction in SSE due to the addition of a variable or additional increase in SSE due to the removal of a variable.

```
> stepAICOil <- stepAIC(CrudeOilOutputlm, direction = "both")
Start:  AIC=15.29
CrudeOilOutput$WorldOil ~ CrudeOilOutput$USEnergy + CrudeOilOutput$USAutoFuelRate +
    CrudeOilOutput$USNuclear + CrudeOilOutput$USCoal + CrudeOilOutput$USDryGas

                                Df Sum of Sq    RSS    AIC
- CrudeOilOutput$USDryGas        1     0.151 29.661 13.425
- CrudeOilOutput$USNuclear       1     0.651 30.161 13.860
<none>                                       29.510 15.293
- CrudeOilOutput$USAutoFuelRate  1     2.640 32.150 15.521
- CrudeOilOutput$USCoal          1     2.683 32.193 15.555
- CrudeOilOutput$USEnergy        1    31.720 61.231 32.270

Step:  AIC=13.42
CrudeOilOutput$WorldOil ~ CrudeOilOutput$USEnergy + CrudeOilOutput$USAutoFuelRate +
    CrudeOilOutput$USNuclear + CrudeOilOutput$USCoal

                                Df Sum of Sq     RSS    AIC
- CrudeOilOutput$USNuclear       1     0.583  30.243 11.931
<none>                                        29.661 13.425
- CrudeOilOutput$USCoal          1     4.296  33.956 14.941
- CrudeOilOutput$USAutoFuelRate  1     4.575  34.236 15.154
+ CrudeOilOutput$USDryGas        1     0.151  29.510 15.293
- CrudeOilOutput$USEnergy        1   137.158 166.818 56.329

Step:  AIC=11.93
CrudeOilOutput$WorldOil ~ CrudeOilOutput$USEnergy + CrudeOilOutput$USAutoFuelRate +
    CrudeOilOutput$USCoal

                                Df Sum of Sq     RSS    AIC
<none>                                        30.243 11.931
- CrudeOilOutput$USCoal          1     3.997  34.240 13.158
+ CrudeOilOutput$USNuclear       1     0.583  29.661 13.425
+ CrudeOilOutput$USDryGas        1     0.082  30.161 13.860
- CrudeOilOutput$USAutoFuelRate  1    13.531  43.774 19.545
- CrudeOilOutput$USEnergy        1   195.845 226.088 62.234
```

Multiple Linear Regression

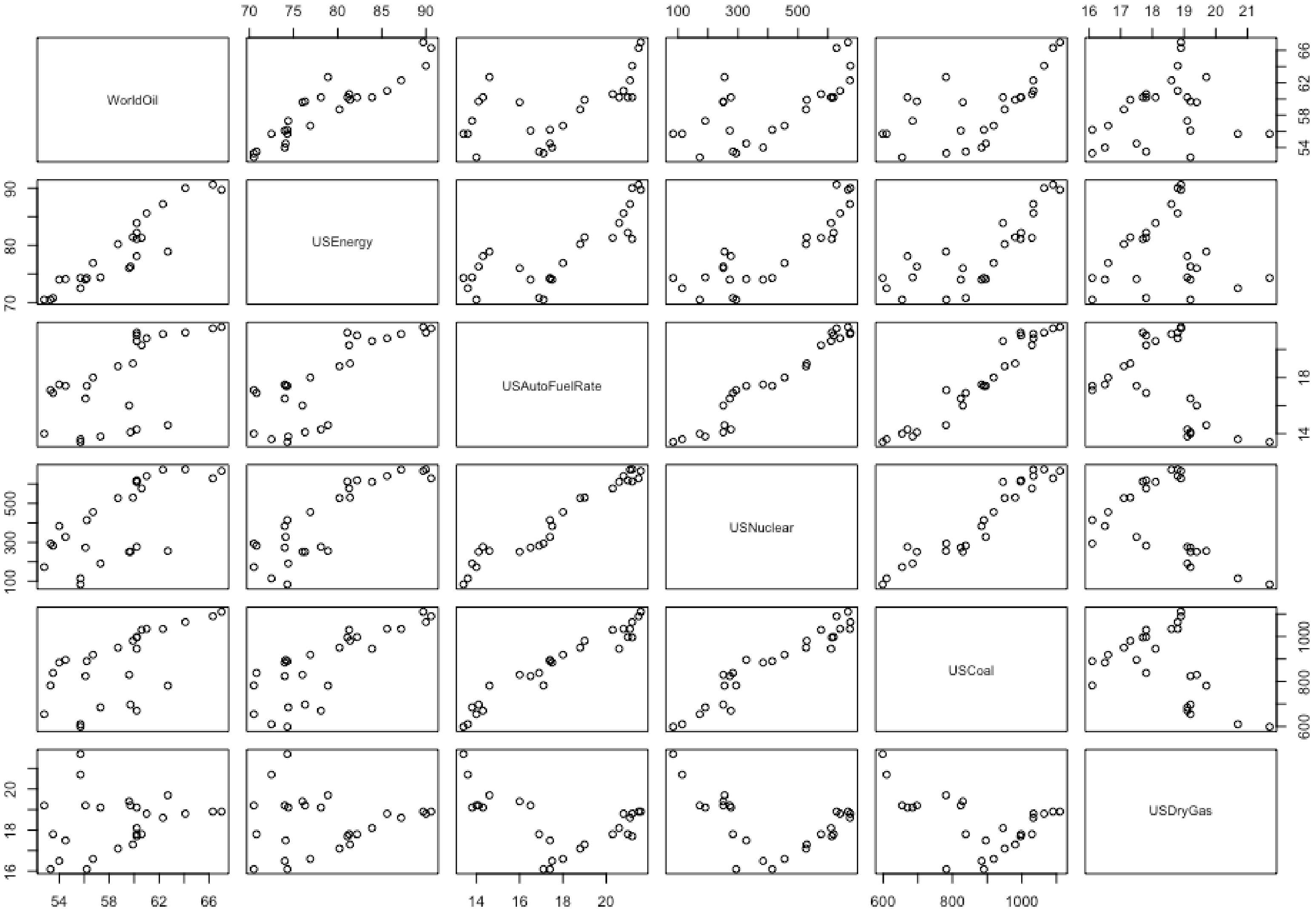
HANDLING MULTICOLLINEARITY

Multicollinearity - R

Two or more **independent variables** are highly correlated.

	Energy consumption	Nuclear	Coal	Dry gas	Fuel rate
Energy consumption	1				
Nuclear	0.856	1			
Coal	0.791	0.952	1		
Dry gas	0.057	-0.404	-0.448	1	
Fuel rate	0.791	0.972	0.968	-0.423	1

Multicollinearity - R



Multicollinearity

Sign of estimated regression coefficient when interacting may be opposite of the signs when used as individual predictors.

For example, fuel rate and coal production are highly correlated (0.968).

$$\hat{y} = 44.869 + 0.7838(fuel\ rate)$$

$$\hat{y} = 45.072 + 0.0157(coal)$$

$$\hat{y} = 45.806 + 0.0277(coal) - 0.3934(fuel\ rate)$$

Multicollinearity

Multicollinearity can lead to a model where the model (*F* value) is significant but all individual predictors (*t* values) are insignificant.

(Recall the with- and without-interaction example)

Correlation between stock 2 and stock 3 is 0.96

SUMMARY OUTPUT						
Regression Statistics						
Multiple R	0.687213365					
R Square	0.47226221					
Adjusted R Square	0.384305911					
Standard Error	4.570195728					
Observations	15					
ANOVA						
	df	*SS*	*MS*	*F*	*Significance F*	
Regression	2	224.2930654	112.1465327	5.369282452	0.021602756	
Residual	12	250.6402679	20.88668899			
Total	14	474.9333333				
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	50.85548009	3.790993168	13.41481713	1.38402E-08	42.59561554	59.11534464
Stock 2 ($)	-0.118999968	0.19308237	-0.616317112	0.54919854	-0.539690313	0.301690376
Stock 3 ($)	-0.07076195	0.198984841	-0.35561478	0.728301903	-0.504312675	0.362788775

Multicollinearity

- Variance Inflation Factor (VIF): A regression analysis is conducted to predict an independent variable by the other independent variables. The independent variable being predicted becomes the dependent variable in this analysis.

$$VIF = \frac{1}{1 - {R_i}^2}$$

- VIF quantifies how much the variance of an estimated coefficient gets inflated in the presence of correlated predictors, compared to the baseline variance when only that one variable is present.

Recall the $Standard\ Error\ of\ the\ Slope = \frac{SE}{\sqrt{SS_{xx}}}$ where $SS_{xx} = \sum(x - \bar{x})^2$ and hence the baseline **variance** of the slope (coefficient) is $\frac{\sigma^2}{\sum(x-\bar{x})^2}$

Multicollinearity - VIF

- VIF > 4 (${R_i}^2$>0.75), 5 (${R_i}^2$>0.80) and 10 (${R_i}^2$>0.90) are commonly used as rules of thumb to indicate severe multicollinearity.
- In practical situations, sometimes even 1.5 is considered as large VIF.
- Remove such variables, rebuild models and compare with earlier model. Make decision based on whether **accuracy of prediction** is more important to the business or **interpretation of the model and the coefficients**.
- Let us look at 2 cases to understand why blindly using the rules of thumb for VIF may be impractical. Stepwise regression prevents multicollinearity to a great extent.

Case 1: Motor Trend Car Road Tests – mtcars dataset in R

Data was extracted from the *Motor Trend* US magazine with a goal to predicting the fuel consumption (mpg) using 10 variables dealing with automobile design and performance.

	mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb
Mazda RX4	21.0	6	160.0	110	3.90	2.620	16.46	0	1	4	4
Mazda RX4 Wag	21.0	6	160.0	110	3.90	2.875	17.02	0	1	4	4
Datsun 710	22.8	4	108.0	93	3.85	2.320	18.61	1	1	4	1
Hornet 4 Drive	21.4	6	258.0	110	3.08	3.215	19.44	1	0	3	1
Hornet Sportabout	18.7	8	360.0	175	3.15	3.440	17.02	0	0	3	2
Valiant	18.1	6	225.0	105	2.76	3.460	20.22	1	0	3	1
Duster 360	14.3	8	360.0	245	3.21	3.570	15.84	0	0	3	4
Merc 240D	24.4	4	146.7	62	3.69	3.190	20.00	1	0	4	2
Merc 230	22.8	4	140.8	95	3.92	3.150	22.90	1	0	4	2
Merc 280	19.2	6	167.6	123	3.92	3.440	18.30	1	0	4	4
Merc 280C	17.8	6	167.6	123	3.92	3.440	18.90	1	0	4	4
Merc 450SE	16.4	8	275.8	180	3.07	4.070	17.40	0	0	3	3
Merc 450SL	17.3	8	275.8	180	3.07	3.730	17.60	0	0	3	3
Merc 450SLC	15.2	8	275.8	180	3.07	3.780	18.00	0	0	3	3
Cadillac Fleetwood	10.4	8	472.0	205	2.93	5.250	17.98	0	0	3	4
Lincoln Continental	10.4	8	460.0	215	3.00	5.424	17.82	0	0	3	4
Chrysler Imperial	14.7	8	440.0	230	3.23	5.345	17.42	0	0	3	4
Fiat 128	32.4	4	78.7	66	4.08	2.200	19.47	1	1	4	1
Honda Civic	30.4	4	75.7	52	4.93	1.615	18.52	1	1	4	2
Toyota Corolla	33.9	4	71.1	65	4.22	1.835	19.90	1	1	4	1
Toyota Corona	21.5	4	120.1	97	3.70	2.465	20.01	1	0	3	1
Dodge Challenger	15.5	8	318.0	150	2.76	3.520	16.87	0	0	3	2
AMC Javelin	15.2	8	304.0	150	3.15	3.435	17.30	0	0	3	2
Camaro Z28	13.3	8	350.0	245	3.73	3.840	15.41	0	0	3	4
Pontiac Firebird	19.2	8	400.0	175	3.08	3.845	17.05	0	0	3	2
Fiat X1-9	27.3	4	79.0	66	4.08	1.935	18.90	1	1	4	1

|	mpg	Miles/(US) gallon
|	cyl	Number of cylinders
|	disp	Displacement (cu.in.)
|	hp	Gross horsepower
|	drat	Rear axle ratio
|	wt	Weight (1000 lbs)
|	qsec	1/4 mile time
|	vs	V/S
|	am	Transmission (0 = automatic, 1 = manual)
|	gear	Number of forward gears
|	carb	Number of carburetors

Case 1: mtcars – Model Building

```
Call:
lm(formula = mpg ~ ., data = mtcars)

Residuals:
    Min      1Q  Median      3Q     Max 
-3.4506 -1.6044 -0.1196  1.2193  4.6271 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)  
(Intercept) 12.30337   18.71788   0.657   0.5181  
cyl         -0.11144    1.04502  -0.107   0.9161  
disp         0.01334    0.01786   0.747   0.4635  
hp          -0.02148    0.02177  -0.987   0.3350  
drat         0.78711    1.63537   0.481   0.6353  
wt          -3.71530    1.89441  -1.961   0.0633 .
qsec         0.82104    0.73084   1.123   0.2739  
vs           0.31776    2.10451   0.151   0.8814  
am           2.52023    2.05665   1.225   0.2340  
gear         0.65541    1.49326   0.439   0.6652  
carb        -0.19942    0.82875  -0.241   0.8122  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.65 on 21 degrees of freedom
Multiple R-squared:  0.869,	Adjusted R-squared:  0.8066 
F-statistic: 13.93 on 10 and 21 DF,  p-value: 3.793e-07
```

- Very good Adjusted R^2
- No significant variable at 5% significance level
- Model is significant
- Indicates multicollinearity

```
> vif(mtcarslm)
      cyl      disp        hp      drat        wt      qsec 
15.373833 21.620241  9.832037  3.374620 15.164887  7.527958 
       vs        am      gear      carb 
 4.965873  4.648487  5.357452  7.908747 
```

- Rules of thumb indicate almost everything is highly collinear
- Let’s run StepAIC

Case 1: mtcars – Model Building

```
> mtcarsStepAIC <- stepAIC(mtcarslm)
Start:  AIC=70.9
mpg ~ cyl + disp + hp + drat + wt + qsec + vs + am + gear + carb

       Df Sum of Sq    RSS    AIC
- cyl   1    0.0799 147.57 68.915
- vs    1    0.1601 147.66 68.932
- carb  1    0.4067 147.90 68.986
- gear  1    1.3531 148.85 69.190
- drat  1    1.6270 149.12 69.249
- disp  1    3.9167 151.41 69.736
- hp    1    6.8399 154.33 70.348
- qsec  1    8.8641 156.36 70.765
<none>              147.49 70.898
- am    1   10.5467 158.04 71.108
- wt    1   27.0144 174.51 74.280

Step:  AIC=68.92
mpg ~ disp + hp + drat + wt + qsec + vs + am + gear + carb

       Df Sum of Sq    RSS    AIC
- vs    1    0.2685 147.84 66.973
- carb  1    0.5201 148.09 67.028
- gear  1    1.8211 149.40 67.308
- drat  1    1.9826 149.56 67.342
- disp  1    3.9009 151.47 67.750
- hp    1    7.3632 154.94 68.473
<none>              147.57 68.915
- qsec  1   10.0933 157.67 69.032
- am    1   11.8359 159.41 69.384
- wt    1   27.0280 174.60 72.297

Step:  AIC=66.97
mpg ~ disp + hp + drat + wt + qsec + am + gear + carb

       Df Sum of Sq    RSS    AIC
- carb  1    0.6855 148.53 65.121
- gear  1    2.1437 149.99 65.434
```

```
> mtcarsStepAIC

Call:
lm(formula = mpg ~ wt + qsec + am, data = mtcars)

Coefficients:
(Intercept)           wt         qsec           am  
      9.618       -3.917        1.226        2.936  
```

- StepAIC identified 3 variables as significant
- Let us build the model with these 3

Case 1: mtcars – Model Building

```
Call:
lm(formula = mpg ~ am + qsec + wt, data = mtcars)

Residuals:
    Min      1Q  Median      3Q     Max 
-3.4811 -1.5555 -0.7257  1.4110  4.6610 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   9.6178     6.9596   1.382 0.177915    
am            2.9358     1.4109   2.081 0.046716 *  
qsec          1.2259     0.2887   4.247 0.000216 ***
wt           -3.9165     0.7112  -5.507 6.95e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.459 on 28 degrees of freedom
Multiple R-squared:  0.8497,	Adjusted R-squared:  0.8336 
F-statistic: 52.75 on 3 and 28 DF,  p-value: 1.21e-11
```

```
> vif(mtcarslm2)
      am     qsec       wt 
2.541437 1.364339 2.482952 
```

- Adjusted R^2 improved
- All variables are significant
- Model is significant
- VIF values are around 2.5 or less

Case 2: Predicting Fungal Toxin Contamination

A drug precursor molecule is extracted from a type of nut, which is commonly contaminated by a fungal toxin that is difficult to remove during the purification process. The suspected predictors of the amount of fungus are:

- Rainfall (cm/week)
- Noon temperature (^{o}C)
- Sunshine (h/day)
- Wind speed (km/h)

The fungal toxin concentration is measured in µg/100 g.

FungalToxinContamination

Toxin	Rain	NoonTemp	Sunshine	WindSpeed
18.1	1.3	20.9	6.23	13.3
28.6	2.28	25.4	8.13	10.8
15.9	1.11	28.2	10.21	10.9
19.2	0.74	23.7	6.96	8.2
19.3	1.32	26.5	9.04	9.8
14.8	0.51	23.9	7.84	12.3
21.7	1.56	26.7	6.69	10
16.5	1.32	30	8.3	12.2
23.8	2.05	24.9	9.22	10.7
19	1.37	22	8.37	15

Case 2: Model Building

```
Call:
lm(formula = ToxinConc$Toxin ~ ToxinConc$Rain + ToxinConc$NoonTemp +
    ToxinConc$Sunshine + ToxinConc$WindSpeed, data = ToxinConc)

Residuals:
      1       2       3       4       5       6       7       8
-1.8818  2.0498 -0.6314  0.4787 -0.5805  1.2508 -0.1921 -0.1813
      9      10
-1.1552  0.8429

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)          31.6084     7.1051   4.449  0.00671 **
ToxinConc$Rain        7.0676     1.0031   7.046  0.00089 ***
ToxinConc$NoonTemp   -0.4201     0.2413  -1.741  0.14215
ToxinConc$Sunshine   -0.2375     0.5086  -0.467  0.66018
ToxinConc$WindSpeed  -0.7936     0.2977  -2.666  0.04458 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.574 on 5 degrees of freedom
Multiple R-squared:  0.9186,    Adjusted R-squared:  0.8535
F-statistic: 14.11 on 4 and 5 DF,  p-value: 0.006232
```

Multiple regression tends to remove correlated pairs of IVs, as in the case of Noon Temperature and Sunshine here.

Case 2: Model Building – R - VIF

```
> vif(ToxinConclm)
    ToxinConc$Rain  ToxinConc$NoonTemp  ToxinConc$Sunshine ToxinConc$WindSpeed
          1.031045            1.616535            1.415269            1.209717
```

```
> correlation
                Toxin         Rain    NoonTemp    Sunshine    WindSpeed
Toxin      1.00000000  0.868734134 -0.07319548 -0.05169949 -0.270555628
Rain       0.86873413  1.000000000  0.11691043  0.16841144 -0.002180167
NoonTemp  -0.07319548  0.116910426  1.00000000  0.50082303 -0.368972511
Sunshine  -0.05169949  0.168411437  0.50082303  1.00000000 -0.018439486
WindSpeed -0.27055563 -0.002180167 -0.36897251 -0.01843949  1.000000000
```

There doesn't appear to be any strongly correlated variables either using correlation values or the VIF, although in some situations, a VIF of 1.5 is considered high.

It may be worthwhile to build another model keeping one of the correlated variables in the model. The more significant can be preferred but business intuition may be cautiously used to include other statistically insignificant variable(s).

Let us do StepAIC first.

Case 2: Model Building – R - StepAIC

```
> ToxinConclm1 <- stepAIC(ToxinConclm, direction = "both")
Start:  AIC=12.14
ToxinConc$Toxin ~ ToxinConc$Rain + ToxinConc$NoonTemp + ToxinConc$Sunshine + 
    ToxinConc$WindSpeed

                      Df Sum of Sq     RSS    AIC
- ToxinConc$Sunshine   1     0.540  12.927 10.567
<none>                              12.387 12.141
- ToxinConc$NoonTemp   1     7.510  19.897 14.880
- ToxinConc$WindSpeed  1    17.603  29.990 18.983
- ToxinConc$Rain       1   122.991 135.378 34.055

Step:  AIC=10.57
ToxinConc$Toxin ~ ToxinConc$Rain + ToxinConc$NoonTemp + ToxinConc$WindSpeed

                      Df Sum of Sq     RSS    AIC
<none>                              12.927 10.567
+ ToxinConc$Sunshine   1     0.540  12.387 12.141
- ToxinConc$NoonTemp   1    13.417  26.344 15.686
- ToxinConc$WindSpeed  1    19.688  32.615 17.822
- ToxinConc$Rain       1   122.830 135.757 32.083
```

Case 2: Model Building – R

```
Call:
lm(formula = ToxinConc$Toxin ~ ToxinConc$Rain + ToxinConc$NoonTemp +
    ToxinConc$WindSpeed, data = ToxinConc)

Residuals:
    Min      1Q  Median      3Q     Max 
-1.6394 -0.9308  0.1394  0.6545  2.0909 

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)    
(Intercept)          31.5651     6.6253   4.764  0.00311 ** 
ToxinConc$Rain        7.0108     0.9285   7.551  0.00028 ***
ToxinConc$NoonTemp   -0.4790     0.1919  -2.495  0.04682 *  
ToxinConc$WindSpeed  -0.8218     0.2718  -3.023  0.02331 *  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.468 on 6 degrees of freedom
Multiple R-squared:  0.915,	Adjusted R-squared:  0.8726 
F-statistic: 21.54 on 3 and 6 DF,  p-value: 0.001298
```

```
> vif(ToxinConclm1)
     ToxinConc$Rain  ToxinConc$NoonTemp ToxinConc$WindSpeed 
           1.015857            1.175947            1.159879 
```

Toxin concentrations increase with increasing rainfall and decrease in drier climates characterized by higher temperatures and wind speeds.

The business can take a decision to rent farms in drier climates if the cost benefits of saved nuts versus higher rents are high.

Multiple Linear Regression

RECAP - OUTPUT ANALYSIS

Output Analysis - Recap

What is the total variation and its explainable and unexplainable components?

$$SST = SSR + SSE$$

$$SST = \sum (y_i - \bar{y})^2 \qquad SSR = \sum (\hat{y}_i - \bar{y})^2 \qquad SSE = \sum (y_i - \hat{y}_i)^2$$

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.89666084
R Square	0.804000661
Adjusted R Square	0.750546296
Standard Error	2.90902388
Observations	15

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	381.8467141	127.282238	15.04087945	0.00033002
Residual	11	93.08661926	8.462419933		
Total	14	474.9333333			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($)	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($)	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3	-0.009984949	0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

Output Analysis - Recap

How much of total variation can be explained by variation in independent variables?

$$\frac{SSR}{SST} = \frac{381.85}{474.93}$$ or R^2

SUMMARY OUTPUT						
Regression Statistics						
Multiple R	0.89666084					
R Square	0.804000661					
Adjusted R Square	0.750546296					
Standard Error	2.90902388					
Observations	15					
ANOVA						
	df	*SS*	*MS*	*F*	*Significance F*	
Regression	3	381.8467141	127.282238	15.04087945	0.00033002	
Residual	11	93.08661926	8.462419933			
Total	14	474.9333333				
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($)	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($)	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3	-0.009984949	0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

Output Analysis - Recap

How much of total variation can be explained by variation in independent variables (IVs) that *actually affect* the Dependent Variable DV? Don't forget that this does not mean those are not important or that they don't have *practical* significance.

$$R^2 - (1 - R^2)\frac{k}{n-k-1}$$

$$1 - \frac{MSE}{MST}$$

Remember

$$MST = \frac{SST}{n-1}$$

Regression Statistics						
Multiple R	0.89666084					
R Square	0.804000661					
Adjusted R Square	0.750546296					
Standard Error	2.90902388					
Observations	15					
ANOVA						
	df	*SS*	*MS*	*F*	*Significance F*	
Regression	3	381.8467141	127.282238	15.04087945	0.00033002	
Residual	11	93.08661926	8.462419933			
Total	14	474.9333333	33.923809521			
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($)	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($)	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3	-0.009984949	0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

Output Analysis - Recap

What is the average of the squared errors?

SUMMARY OUTPUT						
Regression Statistics						
Multiple R	0.89666084					
R Square	0.804000661					
Adjusted R Square	0.750546296					
Standard Error	2.90902388					
Observations	15					
ANOVA						
	df	*SS*	*MS*	*F*	*Significance F*	
Regression	3	381.8467141	127.282238	15.04087945	0.00033002	
Residual	11	93.08661926	8.462419933			
Total	14	474.9333333				
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($)	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($)	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3	-0.009984949	0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

$$MSE = \frac{SSE}{df_{error}}$$

Output Analysis - Recap

Is the model significant?

SUMMARY OUTPUT						
Regression Statistics						
Multiple R	0.89666084					
R Square	0.804000661					
Adjusted R Square	0.750546296					
Standard Error	2.90902388					
Observations	15					
ANOVA						
	df	*SS*	*MS*	*F*	*Significance F*	
Regression	3	381.8467141	127.282238	15.04087945	0.00033002	
Residual	11	93.08661926	8.462419933			
Total	14	474.9333333				
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($)	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($)	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3	-0.009984949	0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

$$F = \frac{MSR}{MSE}$$

R code for critical F: $qf(0.05,3,11) = 3.5874$

R code for Significance F or p value : $pf(15.04087945,3,11) = 0.00033002$

Output Analysis – Recap

What do regression coefficients mean?

A coefficient is the slope of the linear relationship between the dependent variable (DV) and the **independent contribution** of the independent variable (IV), i.e., that part of the IV that is independent of (or uncorrelated with) all other IVs.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.89666084
R Square	0.804000661
Adjusted R Square	0.750546296
Standard Error	2.90902388
Observations	15

ANOVA

	df	SS	MS	F	Significance F
Regression	3	381.8467141	127.282238	15.04087945	0.00033002
Residual	11	93.08661926	8.462419933		
Total	14	474.9333333			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($)	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($)	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3	-0.009984949	0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

Output Analysis - Recap

Are the coefficients significant? How do I calculate the "t" values

$t = \frac{b_i - \beta_{i_{null}}}{SE_{b_i}}$ $\beta_{i_{null}} = 0$

R code for critical t: $qt(0.025,11) = 2.201$

SUMMARY OUTPUT						
Regression Statistics						
Multiple R	0.89666084					
R Square	0.804000661					
Adjusted R Square	0.750546296					
Standard Error	2.90902388					
Observations	15					
ANOVA						
	df	*SS*	*MS*	*F*	*Significance F*	
Regression	3	381.8467141	127.282238	15.04087945	0.00033002	
Residual	11	93.08661926	8.462419933			
Total	14	474.9333333				
	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept b_0	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($) b_1	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($) b_2	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3 b_3	0.009984949	0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

Output Analysis - Recap

What are the confidence intervals for the coefficients?

$$b_i - t_{\left(\frac{\alpha}{2},\nu\right)} * SE_{b_i} \leq \beta_i \leq b_i + t_{\left(\frac{\alpha}{2},\nu\right)} * SE_{b_i}$$

R code for critical t: $t_{\left(\frac{\alpha}{2},\nu\right)} = qt(0.025,11) = 2.201$

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.89666084
R Square	0.804000661
Adjusted R Square	0.750546296
Standard Error	2.90902388
Observations	15

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	381.8467141	127.282238	15.04087945	0.00033002
Residual	11	93.08661926	8.462419933		
Total	14	474.9333333			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept b_0	12.04617703	9.312399791	1.29356313	0.222319528	-8.450276718	32.54263077
Stock 2 ($) $\boxed{b_1}$	0.878777607	0.26187309	3.355738482	0.006412092	0.302398821	1.455156393
Stock 3 ($) b_2	0.220492727	0.143521894	1.536300286	0.152714573	-0.095396832	0.536382286
Stock 2*Stock 3 b_3	-0.009984949	-0.002314083	-4.314862356	0.00122514	-0.015078211	-0.00489169

Multiple Linear Regression

CASE - MONEYBALL

Case – Oakland A's 2002 Success (Moneyball)

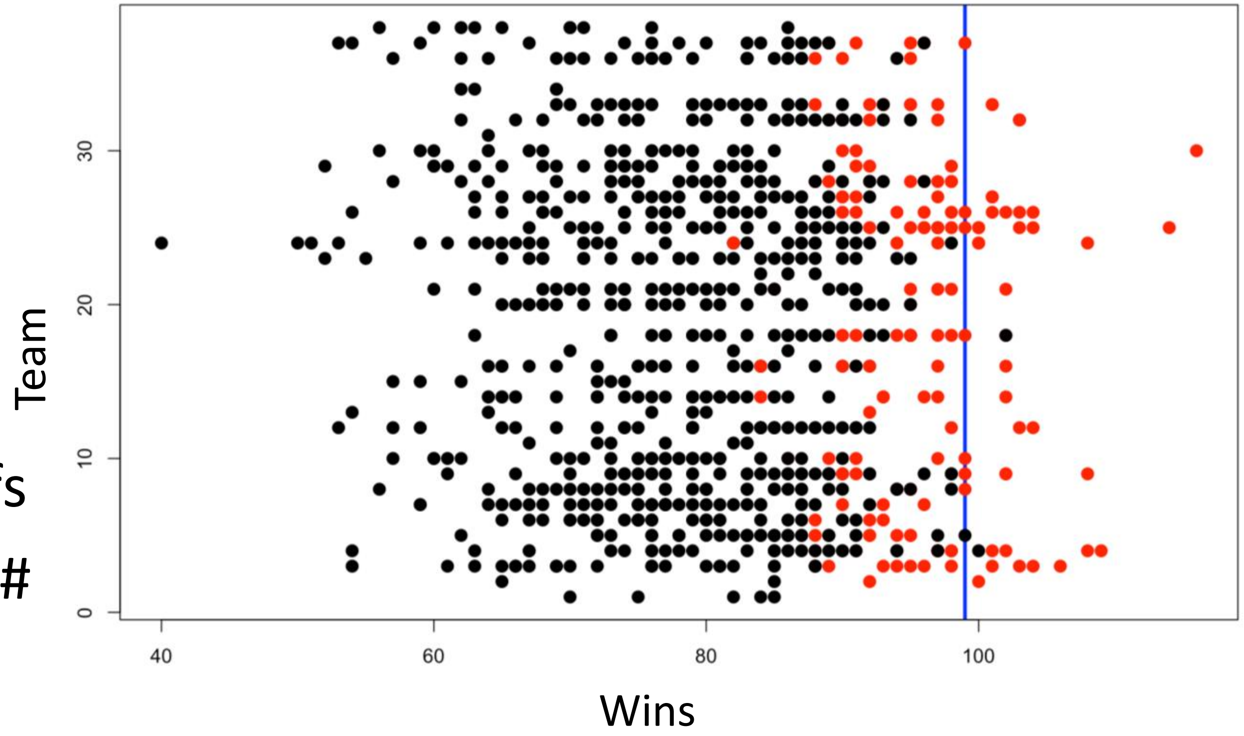
Case Study – Data (baseball-reference.com and MITx)

- 1232 rows, 15 variables
- Statistics for 40 teams from 1962 to 2012
- Oakland A was trying to make playoffs in 2002 and so, 902 rows of data from pre-2002 dates used.

Team	League	Year	RS	RA	W	OBP	SLG	BA	Playoffs	RankSeason	RankPlayoffs	G	OOBP	OSLG
ANA	AL	2001	691	730	75	0.327	0.405	0.261	0			162	0.331	0.412
ARI	NL	2001	818	677	92	0.341	0.442	0.267	1	5	1	162	0.311	0.404
ATL	NL	2001	729	643	88	0.324	0.412	0.26	1	7	3	162	0.314	0.384
BAL	AL	2001	687	829	63	0.319	0.38	0.248	0			162	0.337	0.439
BOS	AL	2001	772	745	82	0.334	0.439	0.266	0			161	0.329	0.393
CHC	NL	2001	777	701	88	0.336	0.43	0.261	0			162	0.321	0.398
CHW	AL	2001	798	795	83	0.334	0.451	0.268	0			162	0.334	0.427
CIN	NL	2001	735	850	66	0.324	0.419	0.262	0			162	0.341	0.455
CLE	AL	2001	897	821	91	0.35	0.458	0.278	1	6	4	162	0.341	0.417
COL	NL	2001	923	906	73	0.354	0.483	0.292	0			162	0.35	0.48
DET	AL	2001	724	876	66	0.32	0.409	0.26	0			162	0.357	0.461

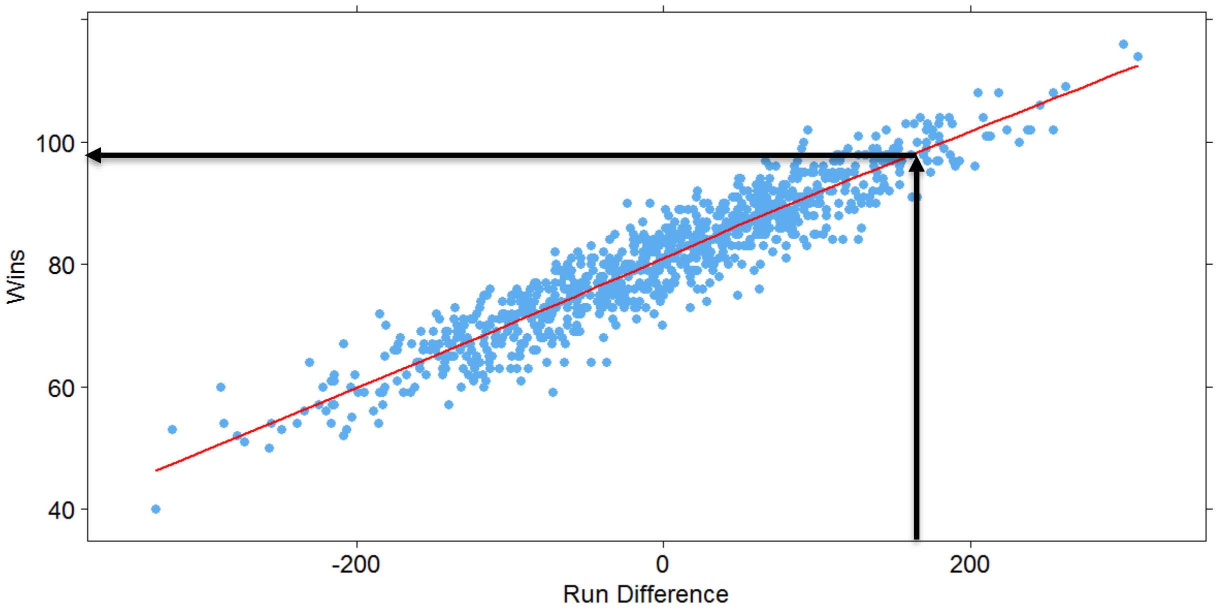
Case Study – Scatter plot

- No. of wins for each team
- Red – Case when team went to playoffs
- Black – Case when team did not go to playoffs
- Vertical blue line – DePodesta's estimate for # of wins required (99)



Case Study – Scatter plot

- DePodesta also estimated that a team on an average needed to score 169 runs more (814-645) per game than their opponent to make the 99 wins
- Strong correlation = 0.94
- Model also predicted 99 wins for a 169-run difference



$$W = 80.881375 + 0.105766 * RD$$
$$W = 80.881375 + 0.105766 * 169 = 98.8$$

CSE 7302c

Case Study – Regression for RS

- Run difference = Runs Scored (RS) – Runs Allowed (RA)
- RS is a function of OBP (On Base Percentage), SLG (Slugging Percentage) and BA (Batting Average)
- Adj. $R^2 = 0.93$

```
Call:
lm(formula = RS ~ OBP + SLG + BA, data = moneyball)

Residuals:
    Min      1Q  Median      3Q     Max 
-70.941 -17.247  -0.621  16.754  90.998 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -788.46      19.70 -40.029  < 2e-16 ***
OBP          2917.42     110.47  26.410  < 2e-16 ***
SLG          1637.93      45.99  35.612  < 2e-16 ***
BA           -368.97     130.58  -2.826  0.00482 ** 
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 24.69 on 898 degrees of freedom
Multiple R-squared:  0.9302, Adjusted R-squared:   0.93 
F-statistic:  3989 on 3 and 898 DF,  p-value: < 2.2e-16
```

- However, coefficient of BA is negative, which is non-intuitive (higher batting average leading to lower chance of winning!). This indicates multi-collinearity.
- Removing BA gives a model with Adj. $R^2 = 0.9294$

```
Call:
lm(formula = RS ~ OBP + SLG, data = moneyball)

Residuals:
    Min      1Q  Median      3Q     Max 
-70.838 -17.174  -1.108  16.770  90.036 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -804.63      18.92  -42.53   <2e-16 ***
OBP          2737.77      90.68   30.19   <2e-16 ***
SLG          1584.91      42.16   37.60   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 24.79 on 899 degrees of freedom
Multiple R-squared:  0.9296, Adjusted R-squared:  0.9294 
F-statistic:  5934 on 2 and 899 DF,  p-value: < 2.2e-16
```

$$\boxed{RS = -804.96 + 2737.77 * OBP + 1584.91 * SLG}$$

Case Study – Regression for RA

- RA is a function of OOBP (Opponent On Base Percentage) and OSLG (Opponent Slugging Percentage)
- Missing values removed. 902 values got dropped to 90.
- Adj. $R^2 = 0.9052$

```
Call:
lm(formula = RA ~ OOBP + OSLG, data = moneyball)

Residuals:
    Min      1Q  Median      3Q     Max 
-82.397 -15.178  -0.129  17.679  60.955 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -837.38      60.26 -13.897  < 2e-16 ***
OOBP         2913.60     291.97   9.979 4.46e-16 ***
OSLG         1514.29     175.43   8.632 2.55e-13 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 25.67 on 87 degrees of freedom
  (812 observations deleted due to missingness)
Multiple R-squared:  0.9073, Adjusted R-squared:  0.9052 
F-statistic: 425.8 on 2 and 87 DF,  p-value: < 2.2e-16
```

$$\boxed{RA = -837.38 + 2913.60 * OOBP + 1514.29 * OSLG}$$

Case Study – Prediction

- Predict how many runs A's will score and allow in 2002 indicating whether they will make the playoffs or not.
- Inputs to RS and RA models are average team OBP, SLG, OOBP and OSLG values in 2001, assuming team quality remains the same in 2002.
- Values in 2001 (data file has for the entire season including playoffs; the values below are for the regular season as predictions are for that part only)
  - OBP: 0.339
  - SLG: 0.430
  - OOBP: 0.307
  - OSLG: 0.373

Case Study – Prediction

- Equations

$$RS = -804.96 + 2737.77 * OBP + 1584.91 * SLG$$

$$RA = -837.38 + 2913.60 * OOBP + 1514.29 * OSLG$$

$$W = 80.881375 + 0.105766 * RD$$

- Calculations

$$RS = -804.96 + 2737.77 * 0.339 + 1584.91 * 0.430 = 804.66 \sim 805$$

$$RA = -837.38 + 2913.60 * 0.307 + 1514.29 * 0.373 = 621.93 \sim 622$$

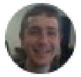
$$W = 80.881375 + 0.105766 * 183 = 100.2 \sim 100$$

- Results

Metric	Model Prediction	DePodesta's Estimate	Actual
RS	805	810	800
RA	622	660	654
Wins	100	95	103

Theoretical World vs the Practical World - Advice

Is it true that the majority of business problems can be solved with linear and logistic regression models?



Ryan Barnes, Data Scientist at Mountain America Credit Union (2015-present)

Answered Jun 23 · Upvoted by Edward Williams, M.A. Statistics, University of Wisconsin - Madison (1968) and Martin Lukac, Ph.D. Sociology & Statistics, KU Leuven (2020) · 2 min read

Let me let you in on a secret about the difference between school (data science competitions too) and the real world. In the real world things break. Data shifts because the guy entering it into the system leaves the job and the new guy does it a little bit differently. The world changes around your model, like the NBA 3-point line gets moved back, and so your data distribution on 3 point attempts made shifts. Other things out of your control happen.

In the real world you are constantly balancing between getting the "right answer", getting an answer quickly, and getting a solution that isn't fragile, and that is easy to debug when it does break (because it will break). In my professional life, time and again I thought that a linear model wasn't powerful enough, and started with something more complicated. Then I was forced to come back to a linear model. Why?

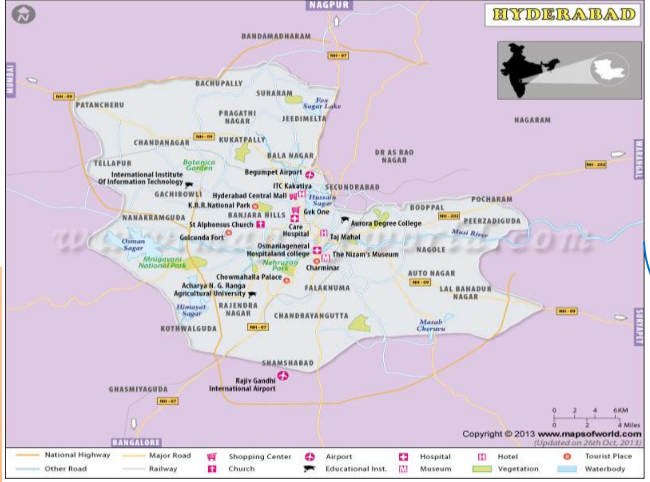
Once I had an optimization problem, I had developed a cool genetic algorithm to solve the problem for setting the optimal cutoff values for a fraud model. It would spin for a day or two up to a week depending on how complex the rule it was trying to optimize, but it would get fantastic results. It worked like a charm every time we needed to set these thresholds. Turns out nobody used it. When asked why, the humans weren't patient enough to wait for the machine to think.

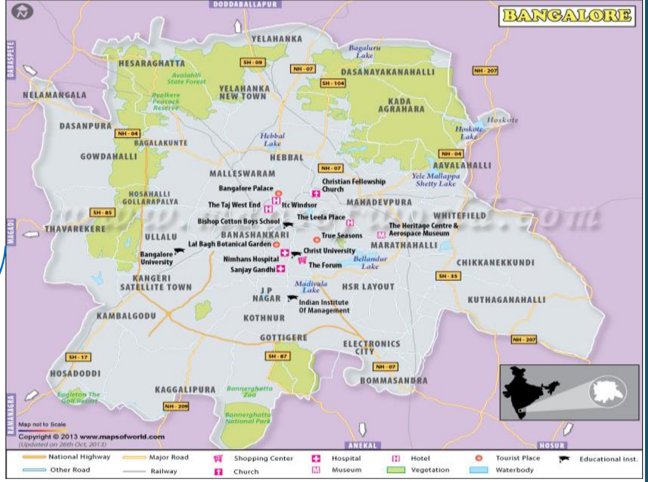
So I threw together a linear regression to set the thresholds. The result wasn't nearly as optimal. But it ran in a couple of seconds. Everyone uses that system. It gets them better results than just using a gut feeling, and it is fast. Are we leaving money on the table? Maybe, depends on your perspective, if no one uses it, we are leaving way more money on the table than by doing a linear regression.

How about if that thing broke. It was nearly impossible to debug, and the results were stochastic to boot. So you never knew if you had the best possible result. With a linear model, I can write a unit test. I can figure out why it gave the answer that it did, and it is just a more solid algorithm that is nearly impossible to break.

So to answer your question, can you solve any business problem with linear regression and probability models? Probably not, I'm looking at PR or HR problems for example, but in terms of data science, they are rock solid models and should be your go to models. Only when they won't work, and you are 100% sure that they aren't working should you move onto anything else.

Source: https://www.quora.com/Is-it-true-that-the-majority-of-business-problems-can-be-solved-with-linear-and-logistic-regression-models
Last accessed: July 12, 2018





Inspire...Educate...Transform.

HYDERABAD
2nd Floor, Jyothi Imperial, Vamsiram Builders, Old Mumbai Highway, Gachibowli, Hyderabad - 500 032
+91-9701685511 (Individuals)
+91-9618483483 (Corporates)

BENGALURU
Floors 1-3, L77, 15th Cross Road, 3A Main Road, Sector 6, HSR Layout, Bengaluru – 560 102
+91-9502334561 (Individuals)
+91-9502799088 (Corporates)

Social Media

Web: http://www.insofe.edu.in

Facebook: https://www.facebook.com/insofe

Twitter: https://twitter.com/Insofeedu

YouTube: http://www.youtube.com/InsofeVideos

SlideShare: http://www.slideshare.net/INSOFE

LinkedIn: http://www.linkedin.com/company/international-school-of-engineering

This presentation may contain references to findings of various reports available in the public domain. INSOFE makes no representation as to their accuracy or that the organization subscribes to those findings.