Analyzing Knight Movement on a Chess Board Through Linear Algebra

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In this paper we attempt to answer the following question: given a knight placed randomly on a standard 8 x 8 chessboard is it possible for the knight to move to every square on the grid. The framework of linear algebra provides the tools to tackle this challenge. Using vectors, matrices, and their properties from linear algebra along with some clever modular arithmetic this result can be analyzed and proved.

(0, 7)	(1, 7)	(2, 7)	(3, 7)	(4, 7)	(5, 7)	(6, 7)	(7, 7)
(0, 6)	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)	(7, 6)
(0, 5)	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)	(7, 5)
(0, 4)	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)	(7, 4)
(0, 3)	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)	(7, 3)
(0, 2)	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)	(7, 2)
(0, 1)	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)	(7, 1)
(0, 0)	(1, 0)	(2, 0)	(3, 0)	(4, 0)	(5, 0)	(6, 0)	(7, 0)

Figure 1.

Problem Setup: Assign a grid-based numbering system to the chessboard squares with one corner labeled as (0,0) and the opposite as (7,7), as depicted in Figure 1. Place the knight on square (1,0). The knight has the ability to move vertically two spaces either up or down, combined with a single space movement horizontally either left or right, or horizontally two spaces either left or right, combined with a single space movement vertically either up or down. This results in a total of 8 potential movements, assuming the knight does not move off the board. These movements are captured by four primary vectors and their negative counterparts (achieved by multiplying the vector by (-1)). The top number in each vector specifies the horizontal shift from the knight's starting position, while the bottom number indicates the vertical shift.:

$$k_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad k_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$k_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad k_4 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

1. To address the original challenge, we'll begin by solving for specific example points and attempt to derive a general proof that applies to any point on the grid. Let's explore a specific example point to demonstrate the approach. Utilizing the site: https://www.random.org/integers/ we generate a pair of integers $(p, q) = (2{,}4)$.

We will now establish a vector equation that models the scenario, introducing free variables to account for the number of moves in each direction type:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + a_4 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

The vector $(a_1, a_2, a_3, a_4) \in z$ because the free variables are constrained to integer values, as only a whole number of moves in each direction is permissible (e.g., fractional moves are not feasible in this configuration).

$(p, q) = (2,4)$:

To address this, we establish a system of equations for each component. This results in the following two equations:

i. $1 + a_1 - a_2 + 2a_3 - 2a_4 = 2$
ii. $2a_1 + 2a_2 + a_3 + a_4 = 4$

Rearranging the first equation and multiplying by 2

$$\Rightarrow \quad a_1 - a_2 + 2a_3 - 2a_4 = 1$$

$$\Rightarrow \quad 2a_1 - 2a_2 + 4a_3 - 4a_4 = 2$$

Subtracting the modified first equation from the second equation we get:

$$\Rightarrow \quad 2a_1 + 2a_2 + a_3 + a_4 = 4$$

$$\Rightarrow \quad -(2a_1 - 2a_2 + 4a_3 - 4a_4) = 2$$

$$\Rightarrow \quad 4a_2 - 3a_3 + 5a_4 = 2$$

This situation describes a flat surface or a "plane" within a 4D space. Because the plane is 2D, we can choose two variables to vary freely and find solutions that fit the given equations. Let:

$$a_3 = s$$
$$a_4 = t$$

We then solve for a_1 and a_2 in terms of s and t

$$\Rightarrow \quad 4a_2 - 3s + 5t = 2$$

$$\Rightarrow \quad a_2 = \frac{2+3s-5t}{4}$$

Substituting back into equation i from above to solve for a_1:

$$\Rightarrow \quad a_1 - \frac{2 + 3s - 5t}{4} + 2s - 2t = 0$$

$$\Rightarrow \quad 4a_1 - 2 - 3s + 5t + 8s - 8t = 0$$

$$\Rightarrow \quad 4a_1 = 2 - 5s + 3t$$

$$\Rightarrow \quad a_1 = \frac{2 - 5s + 3t}{4}$$

Therefore, all quadruplets of the form $(\frac{2-5s+3t}{4}, \frac{2+3s-5t}{4}, s, t)$ are solutions.

But for this case we need integer solutions, and we need to ensure that all moves are valid in the sense the knight remains within the scope of the board.

Let's attempt to generate 3 pairs of integer solutions and plot them to ensure they are valid.

Assuming s and t are integers we need the following equations to hold:

From a_1:

$\frac{2-5s+3t}{4} \equiv 0(mod\ 4)1$

$$2-5s+3t \equiv 0(mod\ 4)$$

$$-5s+3t \equiv -2(mod\ 4)$$

$$-s-t \equiv -2(mod\ 4)$$

$$s+t \equiv 2(mod\ 4)$$

From a_2:

$$\frac{2+3s-5t}{4} \equiv 0(mod\ 4)$$

$$2+3s-5t \equiv 0(mod\ 4)$$

$$3s-5t \equiv -2(mod\ 4)$$

$$-s-t \equiv -2(mod\ 4)$$

$$s+t \equiv 2(mod\ 4)$$

Since all the above steps are reversible, the equation:

$s+t \equiv 2(mod\ 4)$ must hold for a valid integer solution.

Pairs (s,t) (mod 4) which satisfy the given equation above are:

$$(0,2),(2,0),(1,1)$$

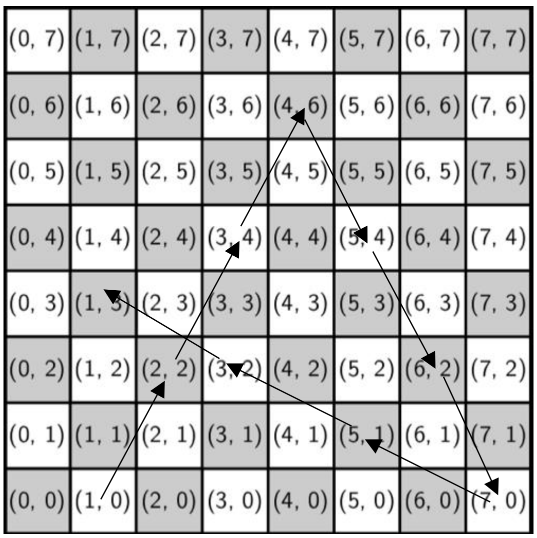
Test Case 1:

Let $s=0, t=2$ because $s+t \equiv 2(mod\ 4)$

After substitution in $(\frac{2-5s+3t}{4}, \frac{2+3s-5t}{4}, s, t)$ we get the quadruplet $(a_1, a_2, a_3, a_4) =$ $(2, -2, 0, 2)$

This can be represented on the grid as a valid sequence of moves as represented on the figure:

$(k_1, k_1, k_1, -k_2, -k_2, -k_2, k_4, k_4, k_4)$



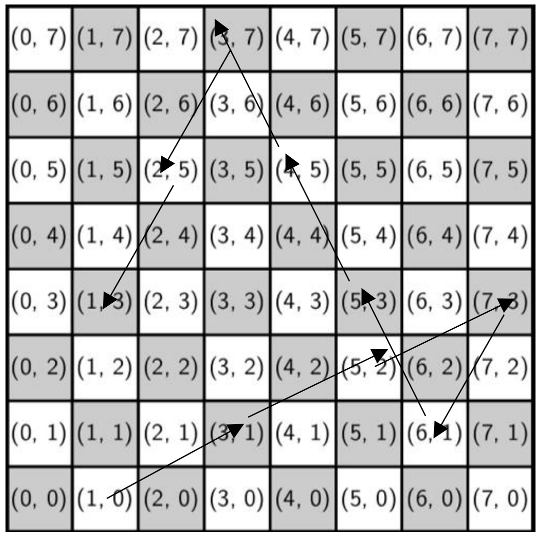
Test Case 2:

Let $s = 2, t = 0$ because $s + t \equiv 2(mod\ 4)$

After substitution in $(\frac{2-5s+3t}{4}, \frac{2+3s-5t}{4}, s, t)$ we get the quadruplet $(a_1, a_2, a_3, a_4) =$ $(2, -2, 2, 0)$

This can be represented on the grid as a valid sequence of moves as represented on the figure:

$(k_3, k_3, k_3, -k_1, k_2, k_2, k_2, -k_1, -k_1)$



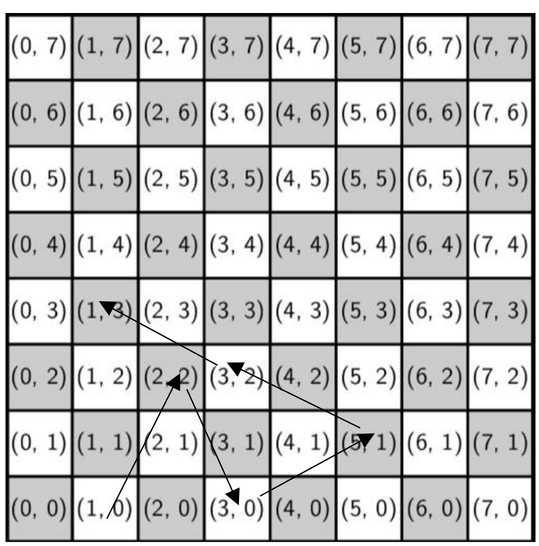
Test Case 3:

Let $s = 1, t = 2$ because $s + t \equiv 3(mod\ 4)$

After substitution in $(\frac{3-5s+3t}{4}, \frac{3+3s-5t}{4}, s, t)$ we get the quadruplet $(a_1, a_2, a_3, a_4) =$ $(1, -1, 1, 2)$

This can be represented on the grid as a valid sequence of moves as represented on the figure:

$(k_1, -k_2, k_3, k_4, k_4)$



General Case:

From this example, let's tackle the general case where $(p, q) = (a, b)$

As done earlier let's build two equations:

i. $1 + a_1 - a_2 + 2a_3 - 2a_4 = a$
ii. $2a_1 + 2a_2 + a_3 + a_4 = b$

From the first equation we obtain:

$$2a_1 - 2a_2 + 4a_3 - 4a_4 = 2a - 2$$

Subtracting this form the second equation we obtain:

$$2a_1 + 2a_2 + a_3 + a_4 = b$$
$$-(2a_1 - 2a_2 + 4a_3 - 4a_{4)} = -(2a - 2)$$

To get:

$$4a_2 - 3a_3 + 5a_4 = b - 2a + 2$$

As done before, assign two free variables $(s, t) = (a_3, a_4)$

After substitution we obtain:

$$4a_2 - 3s + 5t = b - 2a + 2$$

Solving for a_2:

$$4a_2 = b - 2a + 2 + 3s - 5t$$

$$a_2 = \frac{1}{4}(b - 2a + 2 + 3s - 5t)$$

Substituting these values back into the first equation we obtain:

$$1 + a_1 - \frac{1}{4}(b - 2a + 2 + 3s - 5t) + 2s - 2t = a$$

Solving for a_1:

$$4a_1 = 4a - 4 + (b - 2a + 2 + 3s - 5t) - 8s + 8t$$

$$4a_1 = b + 2a - 2 - 5s + 3t$$

$$a_1 = \frac{1}{4}(b + 2a - 2 - 5s + 3t)$$

So $(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}(b + 2a - 2 - 5s + 3t), \frac{1}{4}(b - 2a + 2 + 3s - 5t), s, t\right)$

For the solution set to be integers the following need to hold:

i. $\frac{1}{4}(b + 2a - 2 - 5s + 3t) \equiv 0(mod\ 4)$
ii. $\frac{1}{4}(b - 2a + 2 + 3s - 5t) \equiv 0(mod\ 4)$

From equation one:

$$\frac{1}{4}(b + 2a - 2 - 5s + 3t) \equiv 0(mod\ 4)$$

$$(b + 2a - 2 - 5s + 3t) \equiv 0(mod\ 4)$$

$$(b + 2a - 5s + 3t) \equiv 2(mod\ 4)$$
$$(b + 2a - s - t) \equiv 2(mod\ 4)$$

From equation two:

$$\frac{1}{4}(b - 2a + 2 + 3s - 5t) \equiv 0(mod\ 4)$$

$$(b - 2a + 2 + 3s - 5t) \equiv 0(mod\ 4)$$

$$(b - 2a + 3s - 5t) \equiv 2(mod\ 4)$$

$$(b - 2a - s - t) \equiv 2(mod\ 4)$$

Therefore, the following two equations must hold for integers s and t

i. $(b + 2a - s - t) \equiv 2(mod\ 4)$
ii. $(b - 2a - s - t) \equiv 2(mod\ 4)$

The constraints of s and t depend on the value of a and b $(mod\ 4)$

Let's do casework based on the parity of a and b

Case 1 (a$\equiv 1(mod\ 2)$ and b$\equiv 0(mod\ 2)$):

$$a = 2n + 1$$
$$b = 2m$$

Then:

$$(2m \pm 4n \pm 2 - s - t) \equiv 2(mod\ 4)$$

$$(2m - s - t) \equiv 0(mod\ 4)$$

Since the values of s and t can be chosen:

Let $s = m, t = m$:

Then:

$$(2m - m - m) \equiv 0(mod\ 4)$$

$$0 \equiv 0(mod\ 4)$$

All steps are reversible, so this is a valid substitution.

Let's use the example test case of $a = 5, b = 6$ to make sure this works:

Solve for m:

$$b = 2m$$

$$6 = 2m$$

$$m = 3$$

Then $s = 3, t = 3$:

Then $(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}(b + 2a - 2 - 5s + 3t), \frac{1}{4}(b - 2a + 2 + 3s - 5t), s, t\right)$ so $(a_1, a_2, a_3, a_4) = (2, -2, 3, 3)$

Notice this pair of numbers satisfies the general equation:

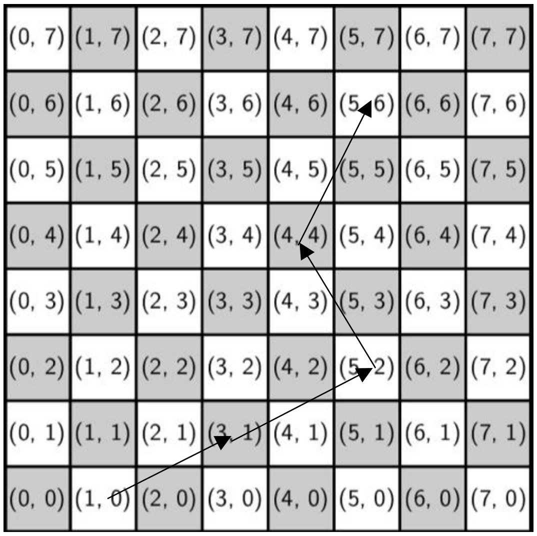
$$\begin{bmatrix}1\\0\end{bmatrix} + a_1\begin{bmatrix}1\\2\end{bmatrix} + a_2\begin{bmatrix}-1\\2\end{bmatrix} + a_3\begin{bmatrix}2\\1\end{bmatrix} + a_4\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}1\\2\end{bmatrix} - 2\begin{bmatrix}-1\\2\end{bmatrix} + 3\begin{bmatrix}2\\1\end{bmatrix} + 3\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}2\\4\end{bmatrix} + \begin{bmatrix}2\\-4\end{bmatrix} + \begin{bmatrix}6\\4\end{bmatrix} + \begin{bmatrix}-6\\3\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$$

$$\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$$

Let's look at the grid and generate a solution to $a = 5, b = 6$:



This sequence of moves (k_3, k_3, k_2, k_1) corresponds to the $(a_1, a_2, a_3, a_4) = (1,1,2,0)$.

Notice this solution is different from the one which is found through method above, even though both are valid, this is because the equation:

$$(2m - s - t) \equiv 0(mod\ 4)$$

Has an infinite number of general solutions to it depending on the values of s and t which can be constructed which be addressed later in this paper.

With a general solution (there will be multiple depending on the how s and t) in this case being:

$$n = \frac{a-1}{2}, m = \frac{b}{2}$$

So:

$$s = \frac{b}{2}, t = \frac{b}{2}$$

$$(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}\left(b + 2a - 2 - 5(\tfrac{b}{2}) + 3(\tfrac{b}{2})\right), \frac{1}{4}\left(b - 2a + 2 + 3(\tfrac{b}{2}) - 5(\tfrac{b}{2})\right), s, t\right)$$

Case 2 (a$\equiv 0(mod\ 2)$ and b$\equiv 1(mod\ 2)$):

$$a = 2n$$
$$b = 2m + 1$$

Then:

$$(2m + 1 \pm 4n - s - t) \equiv 2(mod\ 4)$$

$$(2m - s - t) \equiv 1(mod\ 4)$$

Let $s = m, t = m$-1:

Then:

$$(2m - m - (m - 1)) \equiv 1(mod\ 4)$$

$$1 \equiv 1(mod\ 4)$$

All steps are reversible, so this is a valid substitution.

In this case on general solution to this case is:

$$n = \frac{a}{2}, m = \frac{b-1}{2}$$

So:

$$s = \frac{b-1}{2}, t = \frac{b-1}{2} - 1$$

$(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}(b + 2a - 2 - 5s + 3t), \frac{1}{4}(b - 2a + 2 + 3s - 5t), s, t\right) = \left(\frac{1}{4}\left(b + 2a - 2 - 5(\frac{b-1}{2}) + 3(\frac{b-1}{2} - 1)\right), \frac{1}{4}\left(b - 2a + 2 + 3(\frac{b-1}{2}) - 5(\frac{b-1}{2} - 1)\right), s, t\right)$

Case 3 (a$\equiv 0(mod\ 2)$ and b$\equiv 0(mod\ 2)$):

$$a = 2n$$

$$b = 2m$$

Then:

$$(2m \pm 4n - s - t) \equiv 2(mod\ 4)$$

$$(2m - s - t) \equiv 2(mod\ 4)$$

Let $s = 2m, t = -2$

Then:

$$(2m - 2m - (-2)) \equiv 2(mod\ 4)$$

$$2 \equiv 2(mod\ 4)$$

All steps are reversible, so this is a valid substitution.

In this case on general solution to this case is:

$$n = \frac{a}{2}, m = \frac{b}{2}$$

So:

$$s = b, t = -2$$

$(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}(b + 2a - 2 - 5s + 3t), \frac{1}{4}(b - 2a + 2 + 3s - 5t), s, t\right) = \left(\frac{1}{4}(b + 2a - 2 - 5(b) + 3(-2)), \frac{1}{4}(b - 2a + 2 + 3(b) - 5(-2)), b, -2\right)$

Case 4 (a$\equiv 1(mod\ 2)$ and b$\equiv 1(mod\ 2)$):

$$a = 2n + 1$$

$$b = 2m + 1$$

Then:

$$(2m + 1 \pm 4n \pm 2 - s - t) \equiv 2(mod\ 4)$$

$$(2m + 1 - s - t) \equiv 0(mod\ 4)$$

$$(2m - s - t) \equiv 3(mod\ 4)$$

Let $s = 2m, t = -3$

Then:

$$(2m - 2m - (-3)) \equiv 3(mod\ 4)$$

$$3 \equiv 3(mod\ 4)$$

All steps are reversible, so this is a valid substitution.

In this case on general solution to this case is:

$$n = \frac{a-1}{2}, m = \frac{b-1}{2}$$

So:

$$s = 2(\tfrac{b-1}{2}), t = -3$$

$(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}(b + 2a - 2 - 5s + 3t), \frac{1}{4}(b - 2a + 2 + 3s - 5t), s, t\right) = \left(\frac{1}{4}\left(b + 2a - 2 - 5\left(2\left(\frac{b-1}{2}\right)\right) + 3(-3)\right), \frac{1}{4}\left(b - 2a + 2 + 3\left(2\left(\frac{b-1}{2}\right)\right) - 5(-3)\right), 2\left(\frac{b-1}{2}\right), -3\right)$

In this paper we analyzed and proved that starting at the point $(1,0)$ on the grid it is possible for the knight to move to any point (a, b) on the board given a fixed set of 4 moves which can be applied an integer number of times (including inverses) if they keep the knight on the board. We attempted to solve this problem by first generating a random test point and generating a vector equation, and noticing the solution span was a 2D plane in 4D space with two free variables. Using some modular arithmetic, we were able to generate an equation regarding the free variables for integer solutions, and all infinite solutions to the modular system would give a valid solution to the vector equation, of course the sequence would need to be determined in a such a way that the moves ensure the knight does not leave the board.

The following four general solutions were constructed based on the parity of (a, b):

If (a$\equiv 1(mod\ 2)$ and b$\equiv 0(mod\ 2)$):

- $(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}\left(b + 2a - 2 - 5(\frac{b}{2}) + 3(\frac{b}{2})\right), \frac{1}{4}\left(b - 2a + 2 + 3(\frac{b}{2}) - 5(\frac{b}{2})\right), s, t\right)$

If (a$\equiv 0(mod\ 2)$ and b$\equiv 1(mod\ 2)$):

- $(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}\left(b + 2a - 2 - 5(\frac{b-1}{2}) + 3(\frac{b-1}{2} - 1)\right), \frac{1}{4}\left(b - 2a + 2 + 3(\frac{b-1}{2}) - 5(\frac{b-1}{2} - 1)\right), s, t\right)$

If (a$\equiv 0(mod\ 2)$ and b$\equiv 0(mod\ 2)$):

- $(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}(b + 2a - 2 - 5(b) + 3(-2)), \frac{1}{4}(b - 2a + 2 + 3(b) - 5(-2)), b, -2\right)$

If (a$\equiv 1(mod\ 2)$ and b$\equiv 1(mod\ 2)$):

- $(a_1, a_2, a_3, a_4) = \left(\frac{1}{4}\left(b + 2a - 2 - 5\left(2\left(\frac{b-1}{2}\right)\right) + 3(-3)\right), \frac{1}{4}\left(b - 2a + 2 + 3\left(2\left(\frac{b-1}{2}\right)\right) - 5(-3)\right), 2\left(\frac{b-1}{2}\right), -3\right)$

A question may arise, does the result depend on the starting position of the knight. For instance, if the knight started at any position (l, m) would it be possible to move to any point (a, b) on the board. I claim this is true, here is a simple proof without much calculation:

Given the displacement vector $\begin{bmatrix} a - l \\ b - m \end{bmatrix}$ and find the sequence of moves given the above equations which lead the knight from $(1,0)$ to $(a - l + 1, b - m)$ applying this same sequence of moves to the point (l, m) would lead to (a, b) as the displacement is the same, there is just a shift in the starting and ending points on the board.

Here are some other interesting observations to notice:

- For each case done on the parity of (a, b) there are an infinite number of general solutions which can be constructed

  This is because the following equation $\frac{1}{4}(b \pm 2a \pm 2 - 5s + 3t) \equiv 0(mod\ 4)$ needed to be met, and there are infinite number of solutions for s and t based on the parity of a and b, and once a solution is found, s and t can be scaled by an integer multiple of 4 and the equation will still be valid

- Notice the modular system equations constructed had solutions formed a residual class (mod 4) for each of the four different parity combinations of (a, b)

  It could be noticed that based on the pairwise basis, each equation looped through all possible values (mod 4) (0,1,2,3)

  This is because the equation must be satisfied:

  $$(b \pm 2a - s - t) \equiv 2(mod\ 4)$$

  When the substitution of $2m + 1$ is made the left side may change by 1. If the right-side substitution is made to $2n + 1$ is the left side may change by up to 2, therefore, a change of 1,2,3 is possible on the left side, resulting in cycling of the entire residual class (mod 4)

Please Email Any Questions/Typos/Suggested Edits To: samyagj@outlook.com