Oscillatory Motion and Chaos

One example of simple harmonic motion is a massless string that is connected to a particle of mass m. We let θ be the angle the string makes with the vertical and assuming the string is always straight and that only gravity and tension of the string is acting on it. The force perpendicular to the string is given by $F_\theta = -mgsin(\theta)$ where g is the acceleration due to gravity and the minus to remind that it is a restoring force in the pendulum. Using Newton's Second Law of Motion, substituting the values, we get

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

which has the general solution

$$\theta = \theta_0 \sin(\omega t + \phi)$$

where ω and ϕ are based on the initial displacement. Now to consider the numerical approach to this solution, using the Euler Method would be

$$\omega_{i+1} = \omega_i - \frac{g}{l}\Delta t$$

$$\theta_{i+1} = \theta_i + \omega_i \Delta t$$

Looking at previous equations it might seem the job is already done. But we'll see some erroneous results in our model that cannot just be solved by taking a very small timestep. We'll see the amplitude i.e. maximum height that the pendulum will keep increasing as long there is a non zero value of the timestep. This is the first time we've encountered a computational model where the method is inherently unstable in this condition. When we take the energy of the pendulum which is

$$E = \frac{1}{2}ml^2(\omega^2 + \frac{g}{l}\theta^2)$$

Using the Euler Method, we'll get

$$E_{i+1} = E_i + \frac{1}{2}mgl(\omega_i * 2 + \frac{g}{l}\theta_i^2)(\Delta t)^2$$

Therefore, no matter how small the timestep, the model used in the system means that the energy will increase while in the real world energy is conserved and therefore does not have additional energy. In earlier problems, we got away with it because these errors were negligible enough that we were able to ignore it. But in this case, we are supposed to use other numerical methods.

The modification of Euler system called Euler-Cromer method, is an effective way to tackle this challenge. This method is very similar to the original Euler method, where you use the previous values of θ and ω to calculate the new value of ω but in this method, the current value of ω is used to calculate the new value of θ . While this may seem a very small change, this method conserves energy and therefore helps us model oscillatory methods better. The Euler-Cromer Method is very stable and yield accurate results for oscillatory problems.

3.2 Adding Dissipation, Nonlinearity and a Driving Force

Now we complicate the above differential equation by adding certain forces to make it more realistic.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}$$

where the second term on the right, models the friction with q being the coefficient of friction. Therefore, since the modified equation is still linear, therefore an analytic solution still exists. This differential equation describes three regions of distinct physical behavior. One regime is an **underdamped system** for small friction. while the other end of the extreme is **overdamped system** which is a monotonic, exponential decay of θ. And between these two extremes, there is the **critically damped system** which is the system that reaches to rest the fastest.

Another way to make the model more complicated and more realistic is to keep the driving force. A convenient choice for the type of driving force is a sinusoidal with respect to time with an amplitude of F_D and the angular frequency of Ω_D . This leads to the differential equation for the motion of the pendulum as:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$$

with the last term representing the external driving force. The driving force will pump energy in and out of the system and the external frequency of Ω_D will compete with the natural frequency of Ω. A unique case for this motion is when the external driving frequency is equal or very close to the natural frequency of the pendulum, causing **resonance,** which can increase amplitude by a significant amount if the friction is small. In the next section we'll see the 'chaotic' nature of the above differential system.

3.3 Chaos in the Driven Nonlinear Pendulum

Looking at the much more complicated and realistic complication for the motion of pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$$

If we keep constraints to this system, we can still see some order and some periodicity like the simple harmonic model. But if we don't restrain certain variables, especially the driving force over the value of 1.2, the periodicity and elegance disappears. There is no behavior to discern and will not be like the simple harmonic model we modeled earlier. This is an example of chaotic behavior. While the term *'chaotic' behavior* might evoke a sense of unpredictability and randomness, we were earlier to simulate this through the differential equation. The differential equation above accounts for all the possible movements for a pendulum with friction with an external force. While it seems like an apparent contradiction between determinism of the analytic and the chaotic nature of our models. It's much more beneficial to think of chaos as volatility rather than unpredictability. When the driving force is under 1.2, if you take two pendulums and everything being kept equal, tweak the initial angle very slightly, therefore the behavior of the two system might be very similar. But if the driving force is greater than 1.2, the two systems in a very small amount of time will have behaviors that are completely different from each other. While the irregular variation of $\Delta\theta$ cannot be described by any simple function but is corresponds to the relation $\Delta\theta \approx e^{\lambda t}$ where λ is known as a Lyapunov exponent. There are other ways to get information about chaotic system as well through phase-space plots, Poincare sections and stroboscopic plots.