General Physics I (PHYS 101)

Lecture 11

Harmonic Oscillation

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Outline

Damped Harmonic Oscillation

If no frictional force acts on the oscillation, it would oscillate indefinitely. But in actual practice, the oscillator almost always lies in resting medium (air, oil etc) and the amplitude of oscillation gradually decreases to zero as a result of friction. The motion is said to be damped by friction and is called the damped harmonic motion. The resistive force is called damping force.

Consider horizontal spring of force constant k loaded with mass m and is set to oscillate freely. The angular frequency of oscillation is given by $\omega_0 = 2\pi\sqrt{\frac{k}{m}}$. When some resistive (damping) force acts on the spring then the load experiences two forces: one is the restoring force $F_{\text{spring}} = -kx$ and the other is the damping force F_{damp}. In

Damped Harmonic Oscillation (contd.)

general the damping force is proportional to the velocity of the load and directed opposite to the motion, i.e. $F_{\text{damp}} = -bv = -b\frac{dx}{dt}$, where b is some positive constant. The resultant force exerted on the load is

$$F = F_{\text{spring}} + F_{\text{damp}} = -kx - b\frac{dx}{dt}$$

But from Newton's second law of motion, $F = ma = m\frac{d^2x}{dt^2}$, therefore

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \qquad (1)$$

Let $\dfrac{b}{m} = 2\gamma$, so that $\gamma = \dfrac{b}{2m}$ is called the damping constant. The reciprocal of damping constant i.e. $\tau = \dfrac{1}{2\gamma} = \dfrac{m}{b}$ is the relaxation time. We have, $\dfrac{k}{m} = \omega_0^2$. Now equation (1) reduces to

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{2}$$

Equation (2) is the differential equation of damped harmonic oscillation. To solve the equation, let's assume a trivial solution, $x = Ae^{\alpha t}$. Substituting this solution in the above equation, we get

$$\alpha^2 Ae^{\alpha t} + 2\gamma\alpha Ae^{\alpha t} + \omega_0^2 Ae^{\alpha t} = 0$$

Damped Harmonic Oscillation (contd.)

$$\implies \alpha^2 + 2\gamma\alpha + \omega_0^2 = 0$$

$$\implies \alpha = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$\implies \alpha = -\gamma \pm \beta \tag{3}$$

with

$$\beta = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{\frac{1}{4\tau^2} - \omega_0^2} \tag{4}$$

According to equation (3) α has two values; $-\gamma + \beta$ and $-\gamma - \beta$. So the equation (2) must have two solutions; $x_1 = A_1 e^{(-\gamma+\beta)t}$ and

Damped Harmonic Oscillation (contd.)

$x_2 = A_2 e^{(-\gamma-\beta)t}$. The complete solution is the linear combination of these two solutions i.e.

$$x = x_1 + x_2 = A_1 e^{(-\gamma+\beta)t} + A_2 e^{(-\gamma-\beta)t}$$

$$= A_1 e^{-\gamma t+\beta t} + A_2 e^{-\gamma t-\beta t} = A_1 e^{-\gamma t} e^{\beta t} + A_2 e^{-\gamma t} e^{-\beta t}$$

$$x = e^{-\gamma t}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right) \tag{5}$$

Damped Harmonic Oscillation (contd.)

Case I: Over damped motion If the damping force is very high, then $\gamma \gg \omega_0$ i.e. $\frac{1}{4\tau^2} > \omega_0^2$ and β is a real. The solution is given by equation (5). But A_1 and A_2 are constants to be determine from the initial condition.

In this case the frictional force or damping is so large such that oscillation do not take place and particle returns to its equilibrium position gradually.The amplitude decreases without any oscillation. The situation is called over damping or aperiodic or inharmonic.

Case II: Critically damped motion In this case $\beta = 0$ i.e. $\dfrac{1}{4\tau^2} = \omega_0^2$

To avoid the breakdown of the soluton, if the damping force is normal, γ is slightly greater than ω_0 , and β is still real but very small.

Now equation (5) reduces to

$$
\begin{aligned}
x &= e^{-\gamma t}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right) \\
&= e^{-\gamma t}\left[A_1\left(1 + \beta t + \frac{(\beta t)^2}{2} + \cdots\right) + A_2\left(1 - \beta t + \frac{(\beta t)^2}{2} - \cdot\right.\right.
\end{aligned}
$$

Damped Harmonic Oscillation (contd.)

In this limit when $\beta \to 0$ terms containing β^2 and higher order power of β may be neglected. Hence

$$\begin{aligned} x &= e^{-\gamma t}\left(A_1 + A_1\beta t + A_2 - A_2\beta t\right) \\ &= e^{-\gamma t}\left[\left(A_1 + A_2\right) + \left(A_1 - A_2\right)\beta t\right] \\ &= e^{-\gamma t}\left(M + Nt\right) \end{aligned}$$

where $M = A_1 + A_2$ and $N = \left(A_1 - A_2\right)\beta$. The amplitude decreases even faster than that in Case I. This situation is called critically damping.

Case III: Under damped motion (Damped oscillating motion) If the damping force is very low, such that $\gamma \ll \omega_0$ i.e $\frac{1}{4\tau^2} < \omega_0^2$. As a result, β becomes a complex, i.e.

$$\beta = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{-\left(\omega_0^2 - \gamma^2\right)} = i\sqrt{\omega_0^2 - \gamma^2} = i\omega$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{6}$$

Now the equation (5) becomes

$$
\begin{aligned}
x &= e^{-\gamma t}\left(A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right) \\
&= e^{-\gamma t}\left[A_1(\cos\omega t + i\sin\omega t) + A_2(\cos\omega t - i\sin\omega t)\right] \\
&= e^{-\gamma t}\left[(A_1 + A_2)\cos\omega t + (A_1 - A_2)\, i\sin\omega t\right] \\
&= e^{-\gamma t}\left(C\cos\omega t + D\sin\omega t\right) \\
&= e^{-\gamma t}\sqrt{C^2+D^2}\left(\frac{C}{\sqrt{C^2+D^2}}\cos\omega t + \frac{D}{\sqrt{C^2+D^2}}\sin\omega t\right) \\
&= ae^{-\gamma t}(\sin\phi\cos\omega t + \cos\phi\sin\omega t)
\end{aligned}
$$

$$\text{i.e. } x = ae^{-\gamma t}\sin(\omega t + \phi) \tag{7}$$

Damped Harmonic Oscillation (contd.)

where, $a = \sqrt{C^2 + D^2}$, $\cos\phi = \dfrac{C}{\sqrt{C^2 + D^2}}$,

$\sin\phi = \dfrac{D}{\sqrt{C^2 + D^2}}$, $C = A_1 + A_2$, and $D = i(A_1 - A_2)$

According to equation (7) the amplitude part $ae^{-\gamma t}$ decreases exponentially with time and finally it becomes zero but motion is still periodic with angular frequency ω and phase angle ϕ. This situation is called under-damping. The displacement versus time graph is as shown in figure 1.

Damped Harmonic Oscillation (contd.)

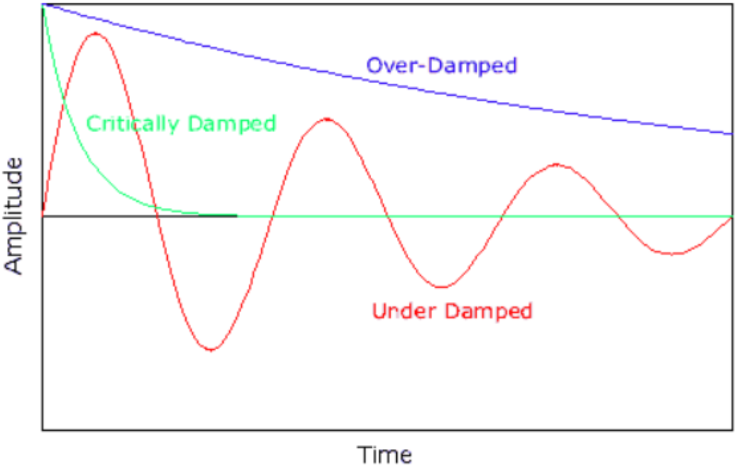

Figure 1: Different type of damping situation

Damped Harmonic Oscillation (contd.)

The time interval between two successive maxima and minima in the damped oscillator is known as period of the motion and is given by

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \frac{1}{4\tau^2}}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

The frequency of damped oscillator is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

From equation (7) the amplitude of damped oscillating motion at any instant t is $ae^{-\gamma t}$

Damped Harmonic Oscillation (contd.)

If $b = 0$ or damping is negligibly small, the motion of the oscillator is undamped and the frequency is called the natural frequency and is given by

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Also the natural time period is given by

$$T = \frac{1}{f_0} = 2\pi\sqrt{\frac{m}{k}}$$

Forced or Driven Harmonic Oscillation

When a body is displaced and then released, the body oscillates with its own natural frequency given by

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \text{(in absence of frictionl force)}$$

and, $$\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} \qquad \text{(in presence of small frictionl force)}$$

The damping can be overcome by applying some oscillating external force on the oscillating body. As a result, its own frequency immediately dies out and the body starts to oscillate with the frequency of external oscillating force. This type of oscillation is called forced or driven harmonic oscillation.

Forced or Driven Harmonic Oscillation (contd.)

Now when the frequency of external force is the same as that of natural frequency of the oscillator, the amplitude of oscillation will be maximum. This condition is called the resonance and the frequency at which the resonance occurs is called the resonant frequency.

The equation of motion of forced (driven) oscillator is given by Newton's second law $F = ma$ in which F is the sum of restoring force $-kx$, the damping force $-b\dfrac{dx}{dt}$ and the applied external oscillating force is $F_{\text{ext}} = F_0 e^{i\omega' t}$ (for simplicity). Where F_0 and ω' are the amplitude and angular frequancy of the external force.

Forced or Driven Harmonic Oscillation (contd.)

The resultant force experienced by the body is

$$F = F_{\text{spring}} + F_{\text{damp}} + F_{\text{ext}} = -kx - b\frac{dx}{dt} + F_0 e^{i\omega' t}$$

But from Newton's second law of motion, we have, $F = m\dfrac{d^2x}{dt^2}$, therefore

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 e^{i\omega' t}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}e^{i\omega' t} \tag{8}$$

Let $\dfrac{b}{m} = \dfrac{1}{\tau} = 2\gamma, \quad \dfrac{k}{m} = \omega_0^2$ and $\dfrac{F_0}{m} = f_0$.

Forced or Driven Harmonic Oscillation (contd.)

Now the equation (8) reduces to

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f_0 e^{i\omega' t} \tag{9}$$

where b is some positive constant, k is the force constant of the spring, m is the mass of the load, γ is the damping constant. The equation (9) represents the differential equation of forced oscillation. To solve this equation, let's assume a trivial solution

$$x = Ae^{i(\omega' t + \phi)} \tag{10}$$

where A is the amplitude of the oscillation and ϕ is the phase angle to be determined. Substituting the value of x in equation (9).

We get

$$\frac{d^2\left(Ae^{i(\omega' t+\phi)}\right)}{dt^2}+2\gamma\frac{d\left(Ae^{i(\omega' t+\phi)}\right)}{dt}+\omega_0^2\left(Ae^{i(\omega' t+\phi)}\right)=f_0e^{i\omega' t}$$
$$\Longrightarrow\ -\omega'^2Ae^{i(\omega' t+\phi)}+i2\gamma\omega' Ae^{i(\omega' t+\phi)}+\omega_0^2Ae^{i(\omega' t+\phi)}=f_0e^{i\omega' t}$$
$$\Longrightarrow\ -\omega'^2Ae^{i\omega' t}e^{i\phi}+i2\gamma\omega' Ae^{i\omega' t}e^{i\phi}+\omega_0^2Ae^{i\omega' t}e^{i\phi}=f_0e^{i\omega' t}$$
$$\Longrightarrow\ -\omega'^2Ae^{i\phi}+i2\gamma\omega' Ae^{i\phi}+\omega_0^2Ae^{i\phi}=f_0$$
$$\Longrightarrow\ -\omega'^2A+i2\gamma\omega' A+\omega_0^2A=f_0e^{-i\phi}=f_0(\cos\phi-i\sin\phi)$$
$$\Longrightarrow\ \left(\omega_0^2-\omega'^2\right)A+i2\gamma\omega' A=f_0\cos\phi-if_0\sin\phi \tag{11}$$

Equating the real and imaginary part of equation (11), we get

$$\left(\omega_0^2 - \omega'^2\right) A = f_0 \cos\phi \tag{12}$$

$$2\gamma\omega' A = -f_0 \sin\phi \tag{13}$$

Squaring and adding equation (12) and (13), we get

$$\left(\omega_0^2 - \omega'^2\right)^2 A^2 + 4\gamma^2\omega'^2 A^2 = f_0^2 \cos^2\phi + f_0^2 \sin^2\phi$$

$$\implies \left[\left(\omega_0^2 - \omega'^2\right)^2 + 4\gamma^2\omega'^2\right] A^2 = f_0^2$$

$$\implies A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega'^2\right)^2 + 4\gamma^2\omega'^2}} \tag{14}$$

Dividing equation (13) by equation (12), we get

$$\frac{2\gamma\omega' A}{\left(\omega_0^2 - \omega'^2\right) A} = \frac{-f_0 \sin\phi}{f_0 \cos\phi} \implies \frac{\sin\phi}{\cos\phi} = -\frac{2\gamma\omega'}{\omega_0^2 - \omega'^2} \implies \tan\phi = \frac{2\gamma\omega'}{\omega'^2 - \omega_0^2}$$

$$\implies \phi = \tan^{-1}\left(\frac{2\gamma\omega'}{\omega'^2 - \omega_0^2}\right) \tag{15}$$

Therefore,equation (10) becomes

$$x = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega'^2\right)^2 + 4\gamma^2\omega'^2}} \exp\left[i\left(\omega' t + \tan^{-1}\left(\frac{2\gamma\omega'}{\omega'^2 - \omega_0^2}\right)\right)\right] \tag{16}$$

Equation (16) gives the displacement of the load at any time in the forced oscillation.

Forced or Driven Harmonic Oscillation (contd.)

Case I:- For no damping i.e. free oscillation, $\gamma = 0$, and the amplitude A becomes

$$A = \frac{f_0}{\omega_0^2 - \omega'^2} \tag{17}$$

The amplitude goes to infinity as $\omega' \approx \omega_0$

Forced or Driven Harmonic Oscillation (contd.)

Case II:- The amplitude has maximum value at a frequency of external oscillating force. To get the maximum amplitude, we have

$$\frac{dA}{d\omega'} = 0$$

$$\implies \frac{d}{d\omega'}\left(\frac{f_0}{\sqrt{\left(\omega_0^2-\omega'^2\right)^2+4\gamma^2\omega'^2}}\right) = 0$$

$$\implies f_0\left(-\frac{1}{2}\right)\left[\left(\omega_0^2-\omega'^2\right)^2+4\gamma^2\omega'^2\right]^{-\frac{3}{2}}\left[2\left(\omega_0^2-\omega'^2\right)\left(-2\omega'\right)+8\gamma^2\omega'\right] = 0$$

$$\implies -4\omega'\left(\omega_0^2-\omega'^2-2\gamma^2\right) = 0$$

$$\implies \omega_0^2-\omega'^2-2\gamma^2 = 0$$

$$\implies \omega' = \sqrt{\omega_0^2-2\gamma^2} \qquad (18)$$

Forced or Driven Harmonic Oscillation (contd.)

Hence the amplitude is maximum when the frequency of external oscillating force is equal to

$$\omega' = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$$

This situation of maximum amplitude is called resonance. That means, the resonance is the phenomenon of making the amplitude maximum by matching the frequency of external oscillating force with the frequency of free or natural oscillation.

Forced or Driven Harmonic Oscillation (contd.)

The maximum amplitude is

$$
\begin{aligned}
A_(max) &= \frac{f_0}{\sqrt{\left(\omega_0^2 - \left(\omega_0^2 - 2\gamma^2\right)\right)^2 + 4\gamma^2\left(\omega_0^2 - 2\gamma^2\right)}} \\
&= \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega_0^2 + 2\gamma^2\right)^2 + 4\gamma^2\omega_0^2 - 8\gamma^4}} \\
&= \frac{f_0}{\sqrt{4\gamma^4 + 4\gamma^2\omega_0^2 - 8\gamma^2}} \\
&= \frac{f_0}{\sqrt{4\gamma^2\omega_0^2 - 4\gamma^4}} \\
\implies A_{\max} &= \frac{f_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}
\end{aligned}
$$

That means the maximum amplitude (the amplitude at resonance) decreases as damping increases. The amplitude A versus frequency of external oscillating force ω' is as shown in figure 2

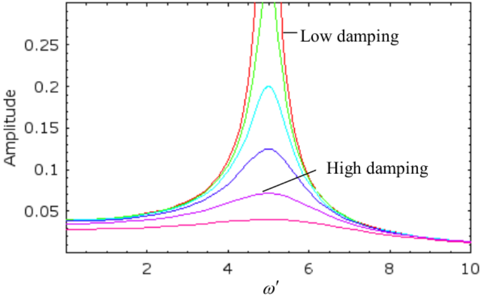

Figure 2: Amplitude versus external frequency of oscillating force