$$= \frac{q^2}{4\pi\varepsilon_0 a}\left[-4 + \frac{2}{\sqrt{2}}\right]$$

$$\therefore U = \cancel{\frac{2q^2}{4\pi\varepsilon_0}} \quad 2 \times \frac{1}{4\pi\varepsilon_0} \times \frac{q^2}{a}\left[-2 + \frac{1}{\sqrt{2}}\right]$$

Energy of Continuous Charge Distribution

The total work necessary to assemble ~~then~~ n-point charges is given by.

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(V(\vec{r_i})\right) \quad \text{—— (i)}$$

where,
$V(\vec{r_i})$ is the potential of $\vec{r_i}$ due to all charges.

For volume charge density (ρ), eq^n (i) becomes.

$$W = \int \frac{1}{2} \rho V \, d\tau$$

From Gauss's law,

$$W = \frac{1}{2}\int \varepsilon_0 (\nabla \vec{E}) V \, d\tau \qquad \left[\because \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}\right]$$

$$\text{or,} \quad W = \frac{\varepsilon_0}{2}\int V(\nabla\cdot\vec{E})\, d\tau \qquad \left[\because \nabla(V\vec{E}) = V(\nabla\cdot\vec{E}) + \nabla V\cdot\vec{E}\right]$$

$$W = \frac{\varepsilon_0}{2}\int \left[-\nabla V\vec{E} + \nabla(V\vec{E})\right] d\tau$$

or, $$W = \frac{\varepsilon_0}{2}\left[\int \vec{E}\cdot\vec{E}\, d\tau + \int \nabla(V\vec{E})\, d\tau\right] \quad [\because -\nabla V = \vec{E}]$$

Using divergence theorem,

$$= \frac{\varepsilon_0}{2}\left[\int E^2\, d\tau + \oint V\vec{E}\cdot d\vec{a}\right]$$

When the integral is taken over all space, the surface integral goes to 0.

$$\therefore W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\, d\tau$$

$$= \int u_E\, d\tau \qquad \text{where, } u_E = \frac{\varepsilon_0 E^2}{2}$$

Here, the term u_E is energy density.
The unit is J/m^3.

<Q>: Find the energy of a uniformly charged spherical shell of total charge q and radius R.

Soln:

For ~~all~~ uniformly charged spherical shell, the electric field inside $(E = 0)$
and
outside $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$

We know,

$$W_{total} = \int_{\text{all space}} \frac{\varepsilon_0}{2} E^2\, d\tau$$

$$= \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\, d\tau$$

We know, $d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$= \frac{\varepsilon_0}{2}\left[\left(\int_0^\infty E^2 r^2\, dr\right)\left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)\right]$$

$$= \frac{\varepsilon_0}{2}\left[\int_R^\infty E_{out}^2 r^2\, dr\right] 2\times 2\pi$$

$$= \frac{4\pi\varepsilon_0}{2}\left[\int_R^\infty \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)^2 r^2\, dr\right]$$

$$= \frac{(4\pi\varepsilon_0)}{2} \times \left[\int_R^\infty \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \times \frac{q^2}{r^4} \times r^2\, dr\right]$$

~~= q~~

$$= \frac{q^2}{(4\pi\varepsilon_0)^2} \times \frac{(4\pi\varepsilon_0)}{2} \times \varepsilon_0 \left[\int_R^\infty \frac{1}{r^2}\, dr \right]$$

$$= \frac{1}{2} \times \frac{1}{4\pi\varepsilon_0} \times q^2 \times \frac{1}{R}$$

$$\therefore W_{total} = \frac{q^2}{8\pi\varepsilon_0 R}$$

[Q] Find the energy stored in a uniformly charged solid sphere of radius R and charge q.

Soln:

For a uniformly charged solid sphere of radius R, the electric field $\left(E_{inside} = \frac{1}{4\pi\varepsilon_0} \frac{qr}{R^3}\right)$

and

outside the electric field. $\left(E_{outside} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}\right)$

Therefore,

$$W_{total} = \frac{\varepsilon_0}{2} \int_{all\ space} E^2\, d\tau$$

We know, $d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

So,

$$= \frac{\varepsilon_0}{2} \int_{all\ space} E^2 (r^2 \sin\theta\, dr\, d\theta\, d\phi)$$

$$= \frac{\varepsilon_0}{2} \left[\left(\int_0^\infty E^2 r^2\, dr \right) \left(\int_0^\pi \sin\theta\, d\theta \right) \left(\int_0^{2\pi} d\phi \right) \right]$$

$$= \frac{2 \times 2\pi \times \varepsilon_0}{2} \left[\left(\int_0^R (E_{in})^2 r^2\, dr + \int_R^\infty (E_{out})^2 r^2\, dr \right) \right]$$

$$= 2\pi\varepsilon_0 \left[\int_0^R \left(\frac{1}{4\pi\varepsilon_0} \frac{qr}{R^3} \right)^2 r^2\, dr + \int_R^\infty \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \right)^2 r^2\, dr \right]$$

$$= 2\pi\varepsilon_0 \times \left(\frac{q}{4\pi\varepsilon_0} \right)^2 \left[\frac{1}{R^6} \int_0^R r^4\, dr + \int_R^\infty \frac{1}{r^2}\, dr \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q^2}{2} \left[\frac{1}{R^6} \frac{R^5}{5} + \frac{1}{R} \right]$$

$$\therefore W_{total} = \frac{6}{5} \left[\frac{1}{4\pi\varepsilon_0} \frac{q^2}{2R} \right]$$

Conductors:

Conductors are substances that contains large numbers of essentially free charge carriers.

The charge carriers are free to wander throughout the conducting material.

They respond to almost infinitesimal electric fields and they continue to move as long as they experience a field.

Insulators

Insulators / dielectric are substances in which all charged particles are bound rather strongly to constituent molecules.

The charged particles may shift their positions slightly in response to an electric field, but they don't leave the vicinity of their molecules.

Perfect Conductor:

A perfect conductor is a material containing an unlimited supply of completely free charges.

In real life, there are no perfect conductors.

*) Basic Electrostatic Properties

(i): Electric field $E=0$ inside conductor.

(ii) Volume charge density ($\rho=0$) inside conductor.

From Gauss's law,

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \rho=0 \text{ inside conductor}$$

$$\therefore \vec{E}=0.$$

(iii) Any net charge resides on the surface.

(iv) $\vec{E}$ is directly perpendicular to the surface just outside a conductor.

(v): A conductor is an equipotential.

For any two points within a given conductor,

$$V(a)-V(b) = -\int_a^b \vec{E}\cdot d\vec{l} = 0$$

$$\cancel{\therefore V(a) = 0}$$

$$\therefore V(a) = V(b).$$

(*): Note:

(i): If E and V are electric fields and electric potential at the midpoint of two equal and opposing point charges,

$$E \neq 0, \quad V=0.$$

(ii): The workdone in displacing a charge 2C through 0·5m on an equipotential surface is

(iii) A thin spherical conducting shell of radius R has charge q. Another charge Q is placed at a ~~distance~~ center of the shell. The electrostatic potential at point P at distance R/2 from the center of shell is.

$$V = V_1 + V_2$$

$$= \frac{1}{4\pi\varepsilon_0} \times \frac{Q}{R/2} + \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

$$\therefore V = \frac{1}{4\pi\varepsilon_0 R}(2Q+q)$$

(iv) The electrostatic potential energy of configuration of three charges $+2e$, $-e$, and $-2e$ placed at three corners A, B, C of an equilateral Δ of side l is.

$$U = -\frac{e^2}{\pi\varepsilon_0 l}$$

(v) The electrostatic potential energy of configuration of four charges $+q, -2q, -q, +2q$ placed at four corners A, B, C, D of a square of side 'a' is.

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{5q^2}{a\sqrt{2}}\right]$$

<Q>: A [(2,3)] and B [(5,7)] are in a region where the electric field is uniform. and is given by

$$\vec{E} = (4\hat{i} + 3\hat{j})\ N/C$$

Sol^n:

We know,

$$V_A - V_B = \int_A^B \vec{E}\cdot\vec{dl}$$

$$= \int_{(2,3)}^{(5,7)} (4\hat{i} + 5\hat{j})(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_{(2,3)}^{(5,7)} 4\cdot dx + 3\cdot dy$$

$$= 4\int_2^5 dx + 3\int_3^7 dy$$

$$= 12 + 12 = 24 \text{ Volt.}$$