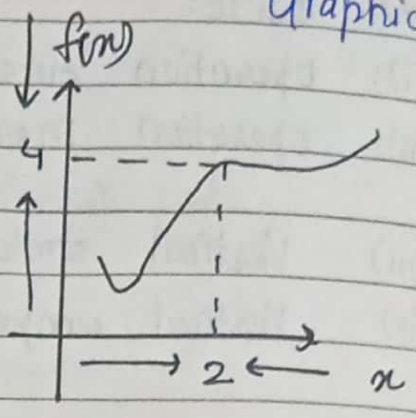
Limit of a Function:

$$\lim_{x \to a} f(x) = L$$

Graphically,

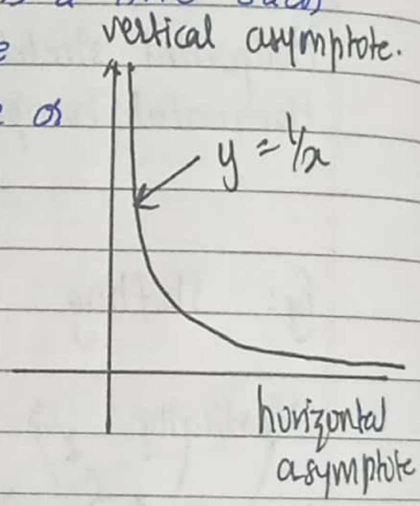


Eg: $$\lim_{x \to 2} x^2 - x + 4 = 4$$

* Indeterminant form: Meaningless expressions

Eg: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, ∞^{∞}, 0^{∞}

* Asymptote: ~~The line~~ of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the 'x' or 'y' coordinates tends to infinity.



Eg: $y = \frac{1}{x}$ is an infinity curve.

Sandwich theorem:

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some interval containing c, except possible at $x = c$ itself, then,

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = 1$$

We ~~k~~ get,

$$\lim_{x \to c} f(x) = 1.$$

Eg: Find $\lim_{x \to 0} f(x)$, where, $1 - \frac{x^2}{4} \leq f(x) \leq 1 + \frac{x^2}{4}$, $x \neq 0$

Sol^n:

Here,

$$\lim_{x \to 0} 1 - \frac{x^2}{4} = 1$$

$$\lim_{x \to 0} 1 + \frac{x^2}{4} = 1.$$

By Sandwich theorem;

$$\lim_{x \to 0} f(x) = 1.$$

Q) Find $\lim_{x \to 0} g(x)$, where, $2 - \cancel{2}x^2 \leq g(x) \leq 2\cos x$

Sol^n:

Here,

$$\lim_{x \to 0} 2 - x^2 = 2 - 0^2 = 2$$

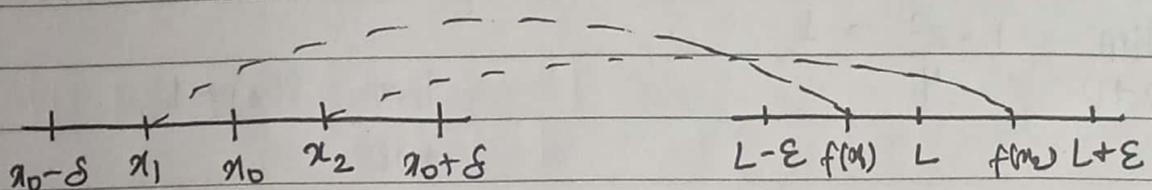
$$\lim_{x \to 0} 2\cos x = 2 \times \cos 0 = 2$$

By Sandwich theorem;

$$\lim_{x \to 0} g(x) = 2$$

$\varepsilon - \delta$ definition of limit

Let δ be the small positive ~~change~~ number that causes change in x and ε be the small positive change in $f(x)$ value induced by change in x by δ.



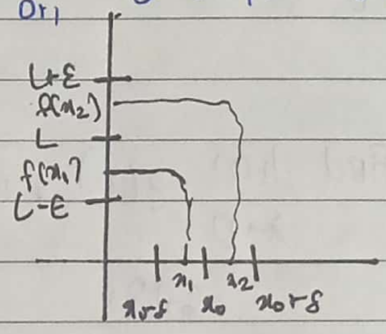
Here, $x_0 - \delta < x < x_0 + \delta$ or, $0 < |x - x_0| < \delta$

Then,

$|f(x) - L| < \varepsilon$

Then

$$\lim_{x \to x_0} f(x) = L$$



Def^n: The limit of the function $f(x)$ is L if x approaches to x_0 is

$$\lim_{x \to x_0} f(x) = L$$

if for every $\varepsilon > 0$, $\delta > 0$ such that if change in distance ~~from x_0~~ of x from x_0 is δ, then the distance of $f(x)$ from L is less than ε.

ie,

$0 < |x - x_0| < \delta$ then

$|f(x) - L| < \varepsilon$.

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Q: Prove that: $\lim_{x \to 3} (4x-5) = 7$

Soln:

We have to show,
if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$

Here,
$x_0 = 3$ $L = 7$ $f(x) = 4x - 5$

Case (I):
Here,
$|f(x) - L| = |(4x-5) - 7|$
$= |4x - 12| = 4|x-3|$

If $0 < |x - \cancel{3}^{x_0}| < \delta$ then. $|f(x) - L| < \varepsilon$
ie, $4|x-3| < \varepsilon$
or $|x-3| < \frac{\varepsilon}{4}$

Thus, we can say: $\delta = \frac{\varepsilon}{4}$

Case (II):
We put $\varepsilon > 0$, $\delta = \frac{\varepsilon}{4}$
Here,
$|(4x-5) - 7| = 4|x-3| < 4\delta$
$= 4 \times \frac{\varepsilon}{4} = \varepsilon$

So, if $0 < |x-3| < \delta$, then $|(4x-5) - 7| < \varepsilon$.
From the defn of $\varepsilon - \delta$, we ~~to~~ can say
$\lim_{x \to 3} 4x - 5 = 7$ exists.

OR,

We need to show,

if $|x - x_0| < \delta$ then, $|f(x) - L| < \varepsilon$

Here, given is

$$|x-3| < \delta \quad \text{--- (i)}$$

Now,

$$|f(x) - L| = |4x - 5 - 7|$$
$$= |4x - 12|$$
$$= 4|x-3|$$
$$= 4\,|x-3| < 4\delta \quad [\because \text{From (i)}]$$

Now,

$$|f(x) - L| < 4\delta$$

Now, for $4\delta = \varepsilon$, $\delta = \frac{\varepsilon}{4}$

So,

$$|f(x) - L| < 4 \times \frac{\varepsilon}{4}$$

$$\therefore |f(x) - L| < \varepsilon$$

Here, since $|x - x_0| < \delta \rightarrow |f(x) - L| < \varepsilon$. ie,

Thus, $\lim\limits_{x \to 3} 4x - 5 = 7$ is proved.

& We can establish relationship betn δ & ε.

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Q: Prove that $\lim_{x \to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2 & \neq 2 \\ 1 & = 2 \end{cases}$

Soln:

We need to show,

if $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$

Here,

$x_0 = 2 \qquad L = 4 \qquad f(x) = x^2$

Given;

$|x - 2| < \delta$ —— (i)

Now,

$|f(x) - L| = |x^2 - 4|$

$= |x + 2| \cdot |x - 2|$ —— (ii)

Let $\delta = 1$ then,

$|x - 2| < 1$

or, $\pm (x - 2) < 1$

or $x - 2 < 1$

$\therefore x < 3$

Again,

$-(x-2) < 1$

$\therefore x > 1$

So,

$1 < x < 3$

if $x < 3$, then $x + 2 < 5$

Now, eqn (ii) becomes.

$|f(x) - L| < 5\delta$

Now, for $5\delta = \varepsilon$, $\delta = \frac{\varepsilon}{5}$

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So,

$\cancel{f(x) =} \; |f(x) - L| < \cancel{5} \cdot \frac{\varepsilon}{\cancel{5}}$

$\therefore \; |f(x) - L| < \varepsilon$

Here, if $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$.
ie, relationship betn δ and ε is established
Then,

$$\lim_{x \to 2} f(x) = 4 \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases} \text{ is proved.}$$

OR,

Here,
$x_0 = 2$, $L = 4$, $f(x) = x^2$
So, We know
$|f(x) - L| < \varepsilon$

or, $|x^2 - 4| < \varepsilon$
or $-\varepsilon < x^2 - 4 < \varepsilon$
or $4 - \varepsilon < x^2 < 4 + \varepsilon$
or $\sqrt{4 - \varepsilon} < x \; \sqrt{4 + \varepsilon}$ — (i)

Again,
$|x - 2| < \delta$ $[\because |x - x_0| < \delta]$
or, $-\delta < x - 2 < \delta$
or $2 - \delta < x < 2 + \delta$ — (ii)

$2 - \delta = \sqrt{4 - \varepsilon}$ and $2 + \delta = \sqrt{9 - \varepsilon}$
$\therefore \delta = 2 - \sqrt{4 - \varepsilon}$ $\quad \delta = -2 + \sqrt{4 - \varepsilon}$

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Thus, minimum $(2-\sqrt{4-\varepsilon}, -2+\sqrt{4+\varepsilon})$

which means.

$0<|x-2|<\delta$ lies on $(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$

Here, we have established relationship betn δ and ε.

Thus, from $\delta-\varepsilon$ definition, we can say

$$\lim_{x\to 2} f(x) = 4 \quad \begin{cases} x^2, & x\neq 2 \\ 1 & x=2 \end{cases}$$ is proved.

(Q): Find δ if $f(x)=x+1$, $L=5$, $x_0=4$, $\varepsilon=0.01$.

soln:

We know,

$|f(x)-L|<\varepsilon$

or, $|x+1-5|<0.01$

or $|x-4|<0.01$

or $-0.01<x-4<0.01$

or $3.99<x<4.01$ — (i)

Also,

$|x-x_0|<\delta$

or $|x-4|<\delta$

or, $\cancel{(x-4)<\delta<(x-4)}$ $-\delta<(x-4)<\delta$

or $4-\delta<x<4+\delta$ — (ii)

Using (i) and (ii);

$4-\delta=3.99$ $\qquad$ $4+\delta=4.01$

$\therefore \delta=0.01$ $\qquad$ $\therefore \delta=0.01$

$\therefore \delta=0.01$.

Q: If $f(x) = \frac{1}{x}$, $L = \frac{1}{4}$, $x_0 = 4$, $\varepsilon = 0.05$

Find δ.

Soln:

We know,

$|f(x) - L| < \varepsilon$

or $\left|\frac{1}{x} - \frac{1}{4}\right| < \varepsilon$

or $\left|\frac{4-x}{4x}\right| < \varepsilon$

or $-\varepsilon < \frac{4-x}{4x} < \varepsilon$

or, $-4\varepsilon < \frac{4-x}{x} < 4\varepsilon$

or $-4x\varepsilon < 4-x < 4x\varepsilon$

or $4x\varepsilon > x-4 > -4x\varepsilon$

or $4+4x\varepsilon > x > 4-4x\varepsilon$

or, $4+0.2x > x > 4-0.2x$

or $4-0.2x < x < 4+0.2x$ — (i)

Also; $|x - x_0| < \delta$

or $-\delta < \{x-4\} < \delta$

or $4-\delta < x < \delta+4$. — (ii)

From (i) and (ii),

$4-\delta = 4-0.2x$ $\qquad$ $\delta+4 = 4+0.2x$

$\therefore \delta = 0.2x$ $\qquad$ $\therefore \delta = 0.2x$

$\therefore \delta = 0.2x$

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or, $-0.05 < \frac{1}{x} - \frac{1}{4} < 0.05$

or, $\frac{1}{5} < \frac{1}{x} < \frac{3}{10}$

or, $5 > x > 10/3$ —— (i) or, $10/3 < x < 5$

Again,

$|x - x_0| \leq \delta$

or, $|x-4| < \delta$

or, $-\delta < x-4 < \delta$

or, $4-\delta < x < 4+\delta$ —— (ii)

From ~~eq~~ (i) and (ii);

$\cancel{5 = 4 - \delta}$ $\frac{10}{3} = 4 - \delta$ $5 = 4 + \delta$

or, $\therefore \delta = 1$

$\therefore \delta = 0.67$

$\boxed{\therefore \delta = 0.67, 1}$

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<Q7: Prove that: $\lim_{x \to 1} f(x) = 1$ if $f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$

Soln:

Here,

$x_0 = 1$, $L = 1$, $f(x) = 2$

Given;

$$|x-1| < \delta \quad \text{— (i)}$$

Now,

$$|f(x) - L| = |x^2 - 1|$$

$$= |x+1| \cdot |x-1| \quad \text{— (ii)}$$

Let $\delta = 0.5$ then,

$$|x-1| < 0.5$$

or, $\pm(x-1) < 0.5$

either,

$x - 1 < 0.5$

$\therefore x < 1.5$

or,

$-(x-1) < 0.5$

or $x - 1 > -0.5$

$\therefore x \geq 0.5$

$$0.5 < x < 1.5$$

if $x < 1.5$, $|x+1| = 2.5 = 5/2$

Now, eqn (ii) becomes.

$$|f(x) - L| < \frac{5}{2}\delta$$

for $\frac{5}{2}\delta = \varepsilon$, $\delta = \frac{2}{5}\varepsilon$

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So,
$$|f(x) - L| < \frac{5}{2} \times \frac{2}{5} \epsilon$$
$$\therefore |f(x) - L| < \varepsilon$$

Here, if $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.
i.e. relationship betn δ and ε is established.

Thus, $\lim_{x \to 1} f(x) = 1$ $\begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$ is proved.

OR,

We have,
$x_0 = 1$, $L = 1$, $f(x) = x^2$

And; $|f(x) - L| < \varepsilon$
or $|x^2 - 1| < \varepsilon$
or $-\varepsilon < x^2 - 1 < \varepsilon$
or $1 - \varepsilon < x^2 < 1 + \varepsilon$
or $\sqrt{1-\varepsilon} < x < \sqrt{1+\varepsilon}$ — (i)

Also,
$|x - L| < \delta$
or $-\delta < x - 1 < \delta$
or $1 - \delta < x < 1 + \delta$ — (ii)

From (i) and (ii),

$1 - \delta = \sqrt{1-\varepsilon}$ $\qquad$ $1 + \delta = \sqrt{1+\varepsilon}$
$\therefore \delta = 1 - \sqrt{1-\varepsilon}$ $\qquad$ $\therefore \delta = -1 + \sqrt{1+\varepsilon}$

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Thus, $\delta = \min\left(1-\sqrt{1-\varepsilon}\ ,\ -1+\sqrt{1+\varepsilon}\right)$

which implies
$0 < |x - x_0| < \delta$ a lies on $\left(\sqrt{1-\delta}\ ,\ \sqrt{1+\delta}\right)$

Thus, from the defn of limit,
we can say
$\lim_{x \to L} f(x) = L$ if $f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$ is proved.