Column Space of a Matrix:

Let $A_{m \times n} = [\vec{v_1} \ \vec{v_2} \ \dots \ \vec{v_n}]$ where, $\vec{v_1}, \vec{v_2}, \dots \vec{v_n} \in \mathbb{R}^m$

The column space of matrix A is defined as the set of all linear combinations of column vectors of A.

It is denoted as col (A) or C(A).

$col(A) / C(A) = \{ c_1\vec{v_1} + c_2\vec{v_2} + \dots c_n \vec{v_n} \}$.

where, $c_1, c_2, \dots, c_n$ are arbitrary constants.

It can also be defined as the subspace spanned by column vectors of A.

ie,

$col(A)$ or $C(A) = span \{ \vec{v_1}, \vec{v_2}, \dots \vec{v_n} \}$.

Note: For $A\vec{x} = \vec{b}$ is consistent ie, has one solution if $\vec{b}$ is column space of $\vec{A}$.

<Q>: Find the column space of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Soln:

Date
Page

The column vectors of A are

$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Let $\vec{v}$ be the column space.
Then,

$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$

as column space is the set of linear combinations of $\vec{v_1}, \vec{v_2}, \vec{v_3}$.
So,

$\vec{v} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Here, c_1, c_2, c_3 are arbitrary constant $\in \mathbb{R}$.

So,

$col(A) = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

OR,

Let $\vec{b} = (x, y, z)$
such that,

$\vec{b} = c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3}$

Date

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

which implies that

$c_1 + c_2 + c_3 = x$ — (i)

$2c_1 + c_3 = y$ — (ii)

$3c_1 + c_2 + 2c_3 = z$ — (iii)

Writing in augmented matrix form,

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 2 & 0 & 1 & y \\ 3 & 1 & 2 & z \end{array}\right]$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 3R_1$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -2 & -1 & y-2x \\ 0 & -2 & -1 & z-3x \end{array}\right]$

Applying $R_3 \to R_3 - R_2$.

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -2 & -1 & y-2x \\ 0 & 0 & 0 & z-x-y \end{array}\right]$

For $\vec{b}$ to be ~~linear~~ span / column space of a_1, a_2, a_3, the system of linear equations must be consistent. ie, $z - x - y = 0$.

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - 2R_2$.

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & x-y \\ 0 & 1 & 3 & y \\ 0 & 0 & -1 & -2y+z-x \end{array}\right]$$

Applying $R_3 \to (-1) \times R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & x-y \\ 0 & 1 & 3 & y \\ 0 & 0 & 1 & x+2y-z \end{array}\right]$$

Appling $R_1 \to R_1 + R_3$ and $R_2 \to R_2 - 3R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2x+y-z \\ 0 & 1 & 0 & -3x-5y+3z \\ 0 & 0 & 1 & x+2y-z \end{array}\right]$$

From R_3; $c_3 = (x+2y-z)$
From R_2, $c_2 = (-3x-5y+3z)$
From R_1, $c_1 = (2x+y-z)$

Thus, the column space or span is

$col(A) = \vec{b} = (2x+y-z)\vec{a_1} + (-3x-5y+3z)\vec{a_2} + (x+2y-z)\vec{a_3}$.

<Q>: Describe span of $\vec{a_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{a_2} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\vec{a_3} = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$

Sol^n:

Let $\vec{b} = (x, y, z)$
such that

$\vec{b} = c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3}$

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$

which implies that

$c_1 + c_2 + 2c_3 = x$ — (i)
$c_2 + 3c_3 = y$ — (ii)
$c_1 + 3c_2 + 8c_3 = z$ — (iii)

The augmented matrix is,

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & x \\ 0 & 1 & 3 & y \\ 1 & 3 & 8 & z \end{array}\right]$$

Applying $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & x \\ 0 & 1 & 3 & y \\ 0 & 2 & 6 & z-x \end{array}\right]$$

Date
Page

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & x-y \\ 0 & 1 & 3 & y \\ 0 & 0 & 0 & z-x-2y \end{array}\right]$$

For $\vec{b}$ to be span or column space of $\vec{a_1}, \vec{a_2}, \vec{a_3}$, the system of linear equations must be consistent.

So, $z - x - 2y = 0$.

So, From R_3, $0 \times c_3 = 0$

ie c_3 is free variable.

Let $c_3 = k$.

From R_2, $c_2 + 3c_3 = y$

$\therefore c_2 = \cancel{8}y - 3\cancel{c}k$

From R_1, $c_1 - c_3 = x - y$

or $c_1 = x - y + t$

When $t = 0$,

$c_3 = 0$ $\quad c_2 = y$ $\quad c_1 = x - y$

Thus, the span or column ~~ve~~ space

$col(\cancel{A}\, a_1, a_2, a_3) = span\{\vec{a_1}, \vec{a_2}, \vec{a_3}\} = \vec{b} =$
$(x-y)\vec{a_1} + y\,\cancel{b}\,\vec{a_2} + 0 \cdot \vec{a_3}$
$= (x-y)\vec{a_1} + y\,\vec{a_2}$