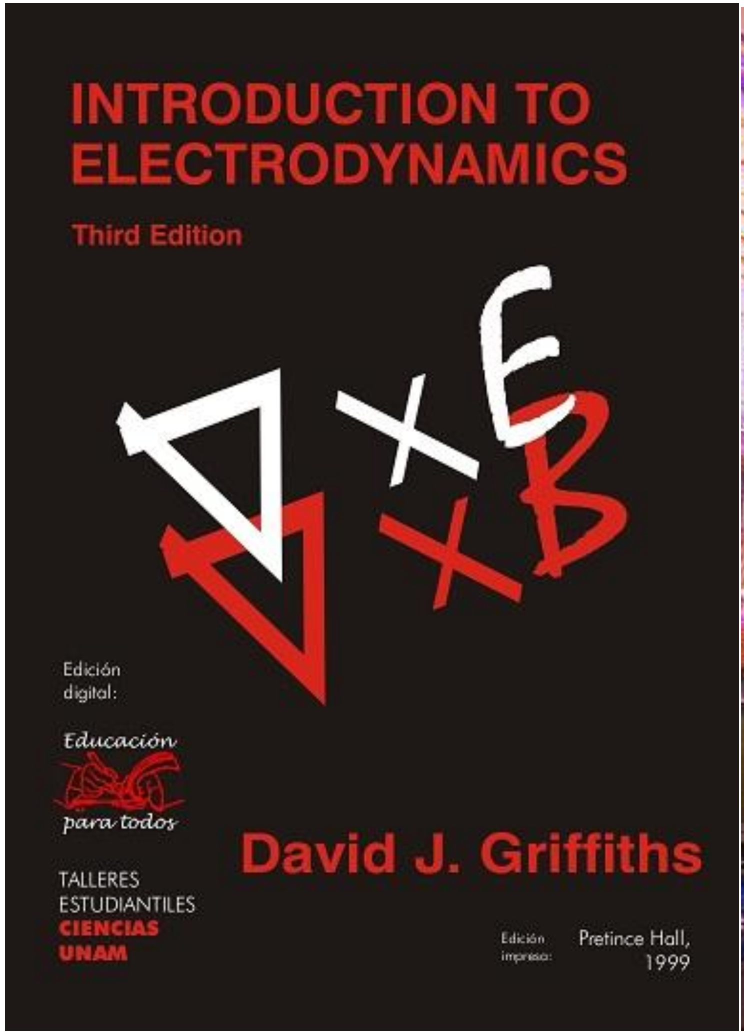
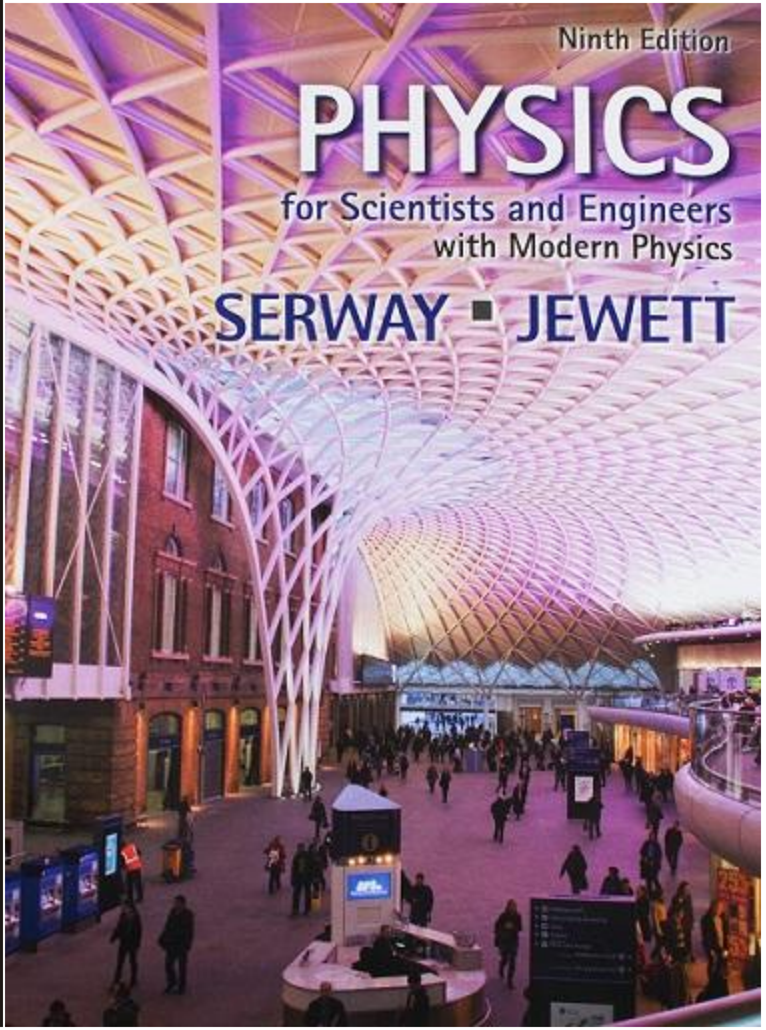
PHYSICS

General Physics II (PHYS 102)





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Course Outline

MAGNETOSTATICS

- Steady Current, Oersted's Experiment
- The Biot-Savart Law and its applications
- Divergence of Magnetic Field
- Ampere's Law
- Application of Ampere's Law
- Magnetic Vector Potential
- Sample Problems

Steady Current

MAGNETOSTATICS

Steady Current:

- Stationary charges produce electric fields that are constant in time; hence the term electrostatics.
- Steady currents produce magnetic fields that are constant in time; the theory of steady currents is called magnetostatics.

Stationary Charges	$\Rightarrow$	constant electric fields:	electrostatics
Steady Currents	$\Rightarrow$	constant magnetic fields:	magnetostatics

Steady current I mean a continuous flow that has been going on forever, without change and without charge piling up anywhere.

- **In magnetostatics,**

Continuity Equation :

$$\nabla \cdot \vec{J} = 0$$

when a Steady current flows in a wire, its magnitude I must be the same all along the line

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0 \text{ in magnetostatics}$$

Oersted's Experiment

MAGNETOSTATICS

Oersted's Experiment:

- The discovery that currents produce magnetic fields was made by **Hans Christian Oersted** in 1820.

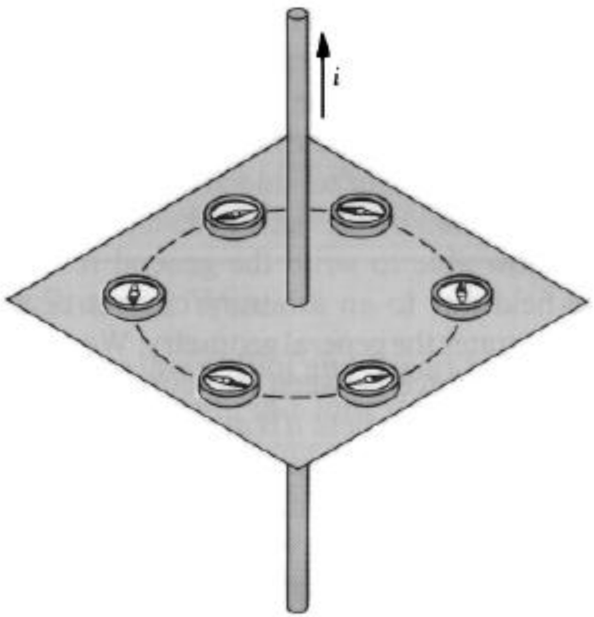

Figure O- 1
Oersted's experiment

- **Oersted** observed that, as illustrated in Figure O-1, when a compass is placed near a straight current-carrying wire, the needle always aligns perpendicular to the wire.

- This was the first experimental link between electricity and magnetism, and it provided the beginning of the development of a formal theory of electromagnetism.

- In modem terms,

  we analyze **Oersted's experiment** by saying that the current in the wire sets up a magnetic field, which exerts a torque on the compass needle and aligns it with the field.

The Biot –Savart Law

The Biot-Savart Law:

- Shortly after Oersted's discovery in 1820 that a compass needle is deflected by a current-carrying conductor, Jean-Baptiste Biot (1774–1862) and Félix Savart (1791– 1841) performed quantitative experiments on the force exerted by an electric current on a nearby magnet.
- From their experimental results, Biot and Savart arrived at a mathematical expression that gives the magnetic field at some point in space in terms of the current that produces the field.

- **The Biot-Savart Law:**

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

where μ_0 is a constant called Permeability of free space:

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T.m/A}$$

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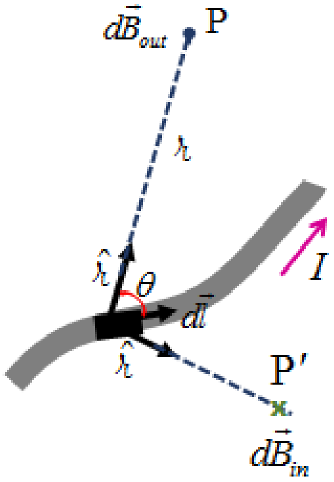


The direction of the filed is into the page at P′

The total magnetic field created at some point by a current of finite size:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

The magnetic field described by the Biot–Savart law is the field due to a given current-carrying conductor.

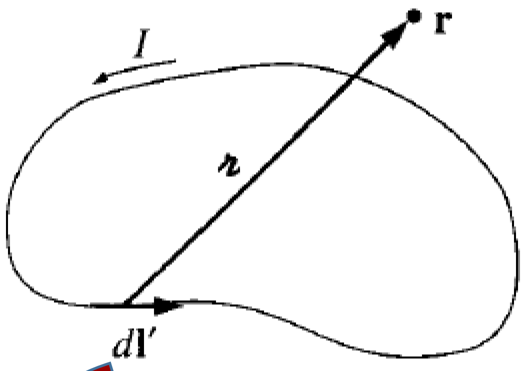
The Biot-Savart Law

MAGNETOSTATICS

The Magnetic Field of a Steady Current:

- The magnetic field of a steady line current is given by the **Biot-Savart law**:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{I\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'\times\vec{\mathscr{r}}}{\mathscr{r}^3}$$



$\vec{\mathscr{r}}$

the vector
from the source point
to the field point

an element
of length along the wire

Permeability of free space:

$$\mu_0 = 4\pi\times10^{-7}\ \text{N/A}^2$$

The Biot-Savart Law for surface current:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,da' = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\,da'$$

The Biot-Savart Law for volume current:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\vec{\mathscr{r}}}{\mathscr{r}^3}\,d\tau'$$

As the starting point for magnetostatics, the Biot-Savart law plays a role analogous to Coulomb's law in electrostatics. Indeed, the $\frac{1}{\mathscr{r}^2}$ dependence is common to both laws.

Applications of The Biot-Savart Law

The magnetic field a distance 's' form a long straight wire carrying a steady current *I* :

MAGNETOSTATICS

AB → a straight wire carrying a steady current I

$$y' = s\tan\theta$$
$$\therefore\ dy' = s\sec^2\theta\, d\theta$$

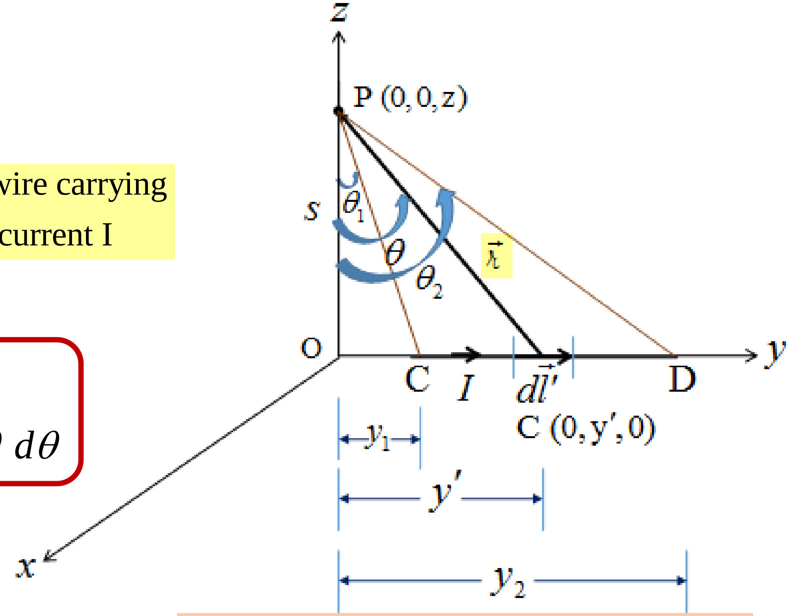


Figure B.S- I
illustrates the geometry and the coordinates to be used

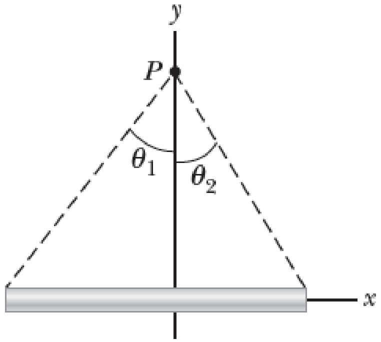
From Figure,

$$d\vec{l}' = dy'\,\hat{j}$$
$$\vec{\mathscr{r}} = -y'\,\hat{j} + s\,\hat{k}$$
$$\mathscr{r} = \sqrt{\left(y'^2 + s^2\right)}$$

$$\therefore\ d\vec{l}' \times \vec{\mathscr{r}} = dy'\,\hat{j} \times \left(-y'\,\hat{j} + s\,\hat{k}\right)$$
$$= s\,dy'\,\hat{i}$$

$d\vec{l}' \times \vec{\mathscr{r}} \rightarrow$ points out of the page

The magnetic field at P a distance S from a long straight wire carrying a steady current I



$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^3}}$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{s\,dy'\,\hat{i}}{\left(s^2 + y'^2\right)^{\frac{3}{2}}}$$

put $y' = s\tan\theta$

$dy' = s\sec^2\theta\, d\theta$

$$\text{So,}\ \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{s\left(s\sec^2\theta\, d\theta\right)}{s^3\sec^3\theta}\,\hat{i}$$

$$= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta\,\hat{i}$$

$$= \frac{\mu_0 I}{4\pi s}\left[\sin\theta\right]_{\theta_1}^{\theta_2}\,\hat{i}$$

$$\because\ \boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)\hat{i}}$$

For an infinitely long, straight wire

$$\theta_1 = -\frac{\pi}{2}\ \&\ \theta_2 = \frac{\pi}{2}$$

$$\boxed{B = \frac{\mu_0 I}{2\pi s}}$$

Applications of The Biot-Savart Law

MAGNETOSTATICS

Magnetic Field Due to a Current in a Long Straight Wire

$$B = \frac{\mu_0 i}{2\pi s}$$

The magnetic field vector at any point is tangent to a circle.

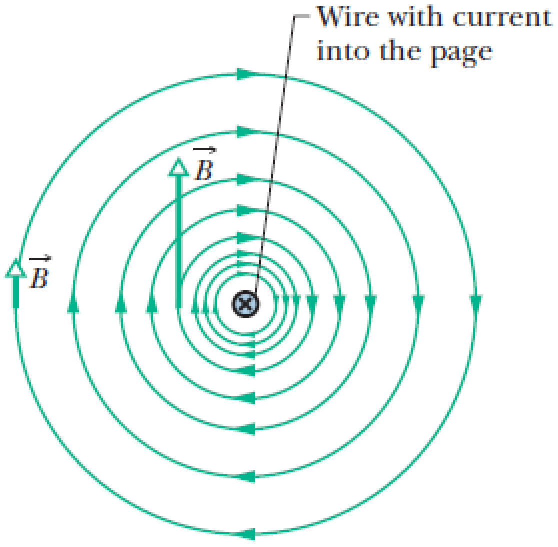


Figure-MF

The magnetic field lines produced by a current in a long straight wire form concentric circles around the wire.

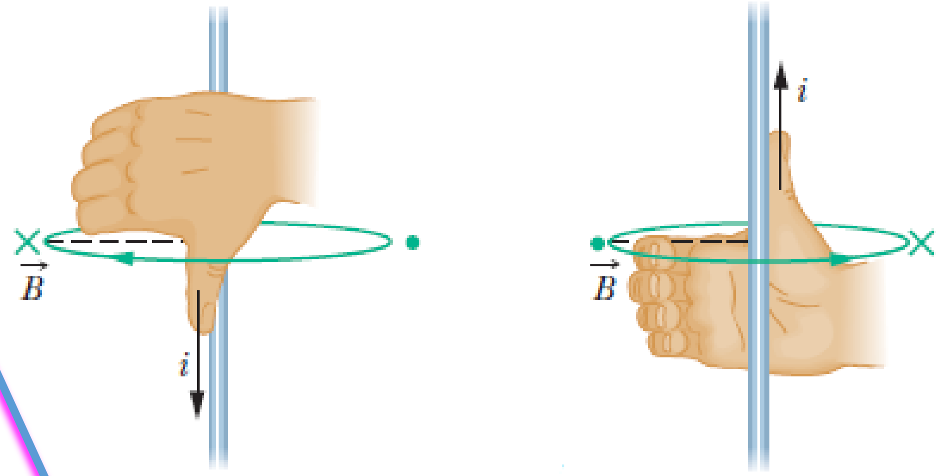


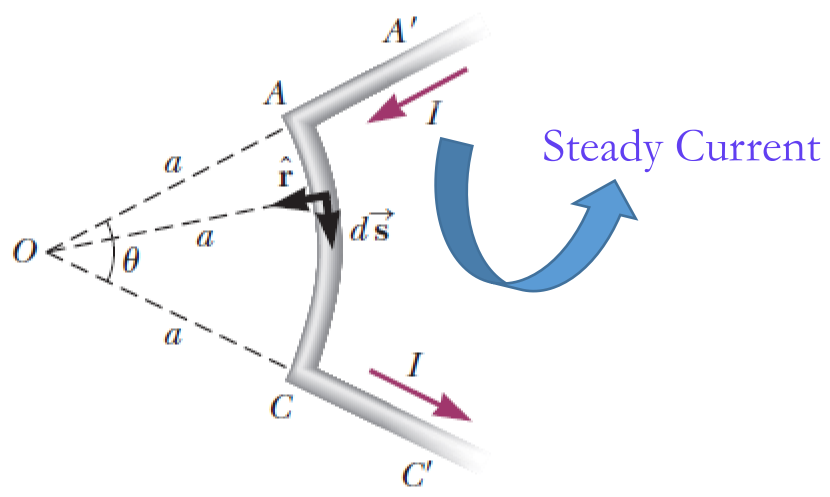
Figure-MC

A right-hand rule gives the direction of the magnetic field due to a current in a wire.

Applications of The Biot-Savart Law

MAGNETOSTATICS

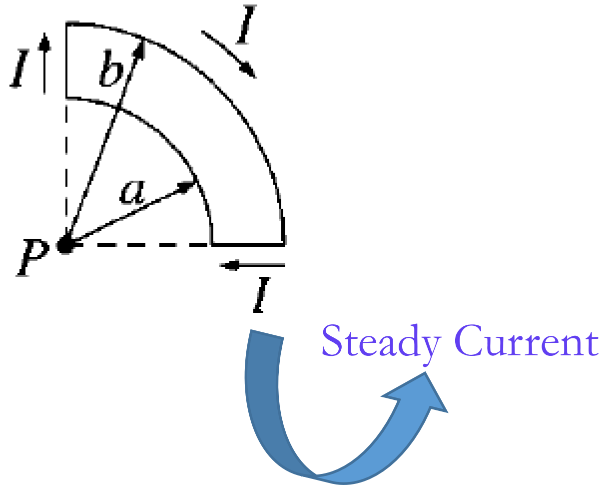
Magnetic Field Due to a Curved Wire Segment



- The magnetic field at O due to the current in the straight segments AA' and CC' is zero because $d\vec{s}$ is parallel to $\hat{r}$ along these paths, which means that $d\vec{s} \times \vec{r} = 0$ for these paths.
- The magnetic field at O due only to the current in the curved portion of the wire of length s is

$$\boxed{B = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \vec{r}}{r^3} = \frac{\mu_0 I}{4\pi a^2}\int ds = \frac{\mu_0 I}{4\pi a^2}s = \frac{\mu_0 I}{4\pi a^2}(a\theta) = \frac{\mu_0 I\theta}{4\pi a}}$$

Magnetic Field Due to a Curved Wire Segments



The magnetic field at point P:

$$B = \left[\frac{\mu_0 I\left(\frac{\pi}{2}\right)}{4\pi a} - \frac{\mu_0 I\left(\frac{\pi}{2}\right)}{4\pi b}\right](\text{out})$$

$$\boxed{\therefore B = \frac{\mu_0 I}{8}\left[\frac{1}{a} - \frac{1}{b}\right](\text{out})}$$

Applications of The Biot-Savart Law

MAGNETOSTATICS

The Magnetic Force Between Two Parallel Conductors

- Consider two long, straight, parallel wires separated by a distance a and carrying currents I_1 and I_2 in the same direction as in Figure MF-I
- The field at the wire I due to the current on the wire 2 is

$$\boxed{B_2 = \frac{\mu_0 I_2}{2\pi a}}$$

- The magnitude of force exerted on one wire 1 due to the magnetic field set up by the other wire 2 is

$$F_1 = I_1\, l\, B_2 = I_1\, l \left(\frac{\mu_0 I_2}{2\pi a}\right) = \frac{\mu_0 I_1 I_2}{2\pi a} l$$

- When the field set up at wire 2 by wire 1 is calculated, the force $\vec{F}_2$ acting on wire 2 is found to be equal in magnitude and opposite in direction to $\vec{F}_1$.

The direction of $\vec{F}_1$ is toward wire 2 because $\vec{l} \times \vec{B}_2$ is in that direction.

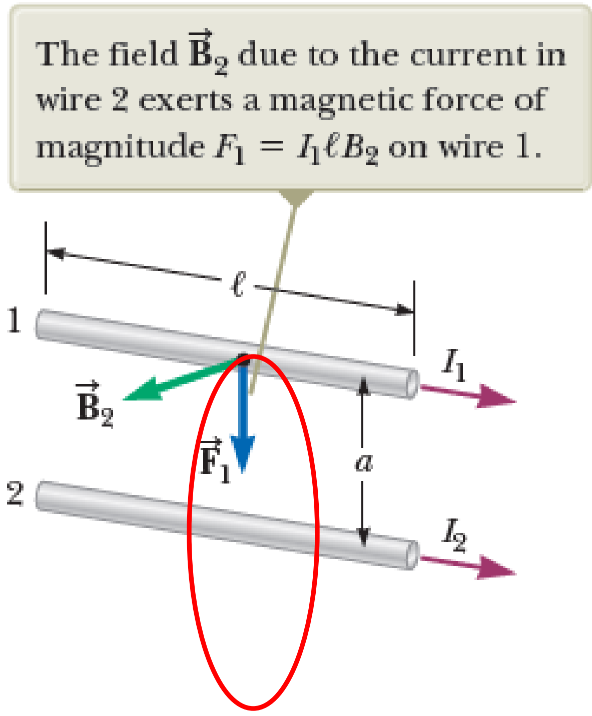


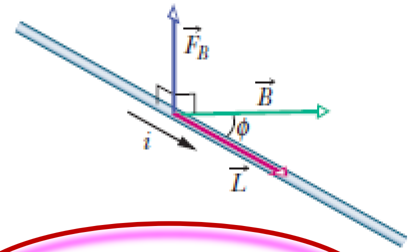
Figure MF-I

- Force per unit length:

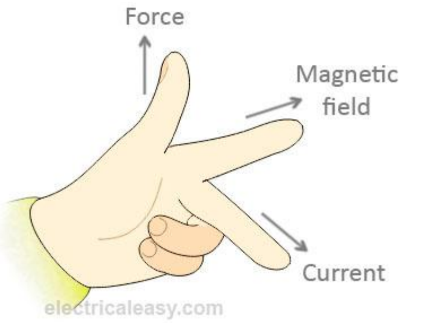
$$\boxed{\frac{F_B}{l} = \frac{\mu_0 I_1 I_2}{2\pi a}}$$

Parallel conductors carrying currents in the same direction attract each other.

Magnetic Force on a Current-Carrying Conductor:



$$\vec{F}_B = i\,\vec{L} \times \vec{B}$$



Fleming's Left-hand Rule:

MAGNETOSTATICS

The Magnetic Field at the Centre of a Square Loop

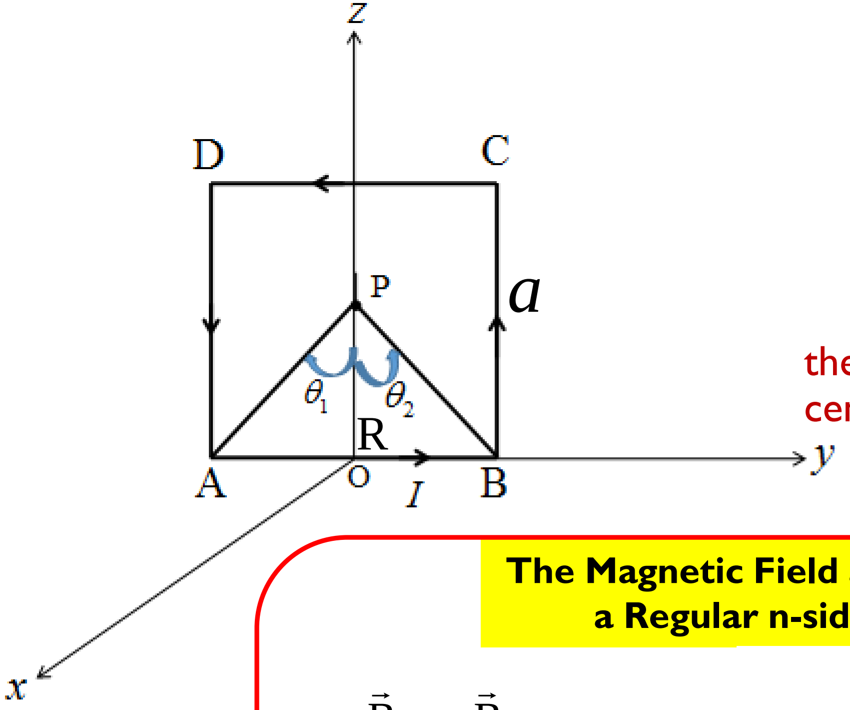
- Magnetic field at P due to the current I in the wire AB (one side of the square loop) is

$$\boxed{\vec{B}_1 = \frac{\mu_0 I}{4\pi R}\left(\sin\theta_2 - \sin\theta_1\right)\hat{i}}$$

$$= \frac{\mu_0 I}{4\pi R}\left[\sin\frac{\pi}{4} - \sin\left(-\frac{\pi}{4}\right)\right]\hat{i}$$

$$= \frac{\mu_0 I}{4\pi R}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\hat{i}$$

$$= \frac{\mu_0 I}{4\pi R}\sqrt{2}\ \hat{i}$$



the length of the side of a square

$$R = \frac{a}{2}$$

the distance from centre to the side

- Magnetic field at P due to the flowing through the square loop ABCD:

$$\vec{B} = 4\,\vec{B}_1 = 4\frac{\mu_0 I}{4\pi R}\sqrt{2}\ \hat{i} = \sqrt{2}\frac{\mu_0 I}{\pi R}\hat{i}$$

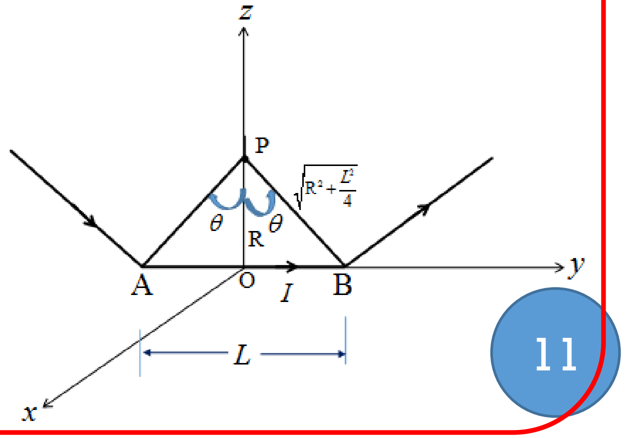
$$\boxed{\therefore\ \vec{B} = \sqrt{2}\frac{\mu_0 I}{\pi R}\hat{i} = 2\sqrt{2}\frac{\mu_0 I}{\pi a}\hat{i}}$$

The Magnetic Field at the Centre of a Regular n-sided Polygon

$$\vec{B} = n\,\vec{B}_1$$

$$= n\frac{\mu_0 I}{4\pi R}\left(2\sin\theta\right)\hat{i}$$

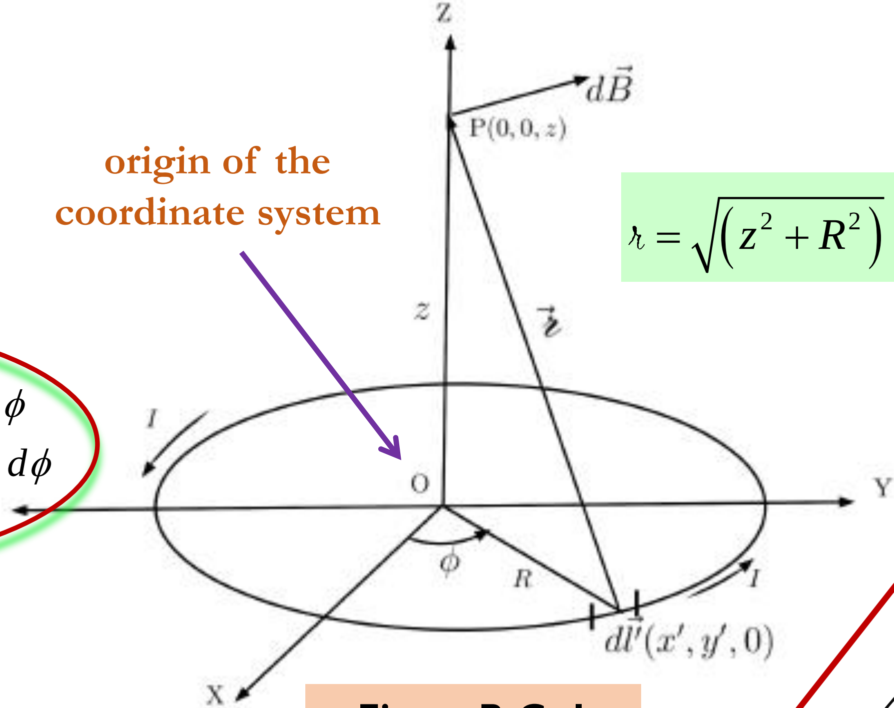
$$= \frac{\mu_0 I}{4\pi R}\frac{C}{\sqrt{R^2 + \frac{L^2}{4}}}\hat{i}$$



Applications of The Biot-Savart Law

The magnetic field a distance *z* above the center of a circular loop of radius *R*, which carries a steady current *I*

- A circular loop of radius *R* → lies on the xy –plane
- P → a point at a distance *z* above the centre O of circular loop



origin of the coordinate system

Figure B-C- 1

$$\mathscr{r} = \sqrt{\left(z^2+R^2\right)} = \left(z^2+R^2\right)^{\frac{1}{2}}$$

From Figure,

$$d\vec{l}' = dx'\,\hat{i} + dy'\,\hat{j}$$

$$\vec{\mathscr{r}} = -x'\hat{i} - y'\,\hat{j} + z\,\hat{k}$$

$$x' = R\cos\phi\,, \qquad y' = R\sin\phi$$

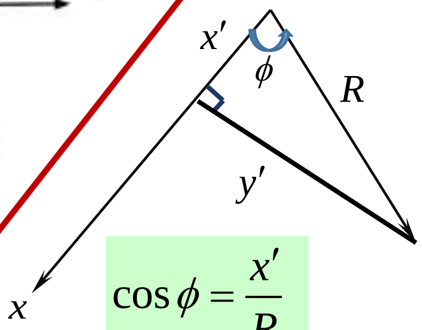
$$\therefore\ dx' = -R\sin\phi\, d\phi \qquad dy' = R\cos d\phi$$

$$\therefore\ d\vec{l}' \times \vec{\mathscr{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx' & dy' & 0 \\ -x' & -y' & z \end{vmatrix}$$

$$= z dy'\,\hat{i} - z dx'\,\hat{j} + \left(-y'dx' + x'dy'\right)\hat{k}$$

$$= zR\cos\phi d\phi\,\hat{i} - z\left(-R\sin\phi d\phi\right)\hat{j} + \left\{\left(-R\sin\phi\right)\left(-R\sin\phi d\phi\right) + \left(R\cos\phi\right)\left(R\cos\phi d\phi\right)\right\}\hat{k}$$

$$= zR\cos\phi d\phi\,\hat{i} + zR\sin\phi d\phi\,\hat{j} + R^2 d\phi\,\hat{k}$$



$$\cos\phi = \frac{x'}{R}$$

$$\sin\phi = \frac{y'}{R}$$

Applications of The Biot-Savart Law

MAGNETOSTATICS

The magnetic field a distance *z* above the center of a circular loop of radius *R*, which carries a steady current *I*

- The magnetic field at P due to a steady current I in a circular loop:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \left\{ zR\cos\phi d\phi\ \hat{i} + zR\sin\phi d\phi\ \hat{j} + R^2 d\phi\ \hat{k} \right\} \Big/ \left(z^2+R^2\right)^{\frac{3}{2}}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{R^2 d\phi\ \hat{k}}{\left(z^2+R^2\right)^{\frac{3}{2}}} \qquad \left[\because \int_0^{2\pi}\cos\phi d\phi = 0, \int_0^{2\pi}\sin\phi d\phi = 0\right]$$

$$= \frac{\mu_0 I}{4\pi}\frac{R^2}{\left(z^2+R^2\right)^{\frac{3}{2}}}(2\pi)\hat{k} \qquad \left[\because \int_0^{2\pi} d\phi = 2\pi\right]$$

$$\boxed{\therefore\ \vec{B}(\vec{r}) = \frac{\mu_0 I}{2}\frac{R^2}{\left(z^2+R^2\right)^{\frac{3}{2}}}\hat{k}}$$

Cases:

- If P lies at the centre of circular loop, *z* = 0

$$\boxed{\therefore\ \vec{B}(\vec{r}) = \frac{\mu_0 I}{2R}\hat{k}}$$

- If there are N numbers of loops tightly gathered together,

$$\boxed{\therefore\ \vec{B}(\vec{r}) = N\frac{\mu_0 I}{2}\frac{R^2}{\left(z^2+R^2\right)^{\frac{3}{2}}}\hat{k}}$$

Applications of The Biot-Savart Law

MAGNETOSTATICS

The magnetic field on the axis of a tightly wound solenoid (helical coil) consisting of n turns per unit length wrapped around a cylindrical tube of radius 'a' and carrying current I.

- Number of turns on an elemental length dy:

$$\mathrm{dN} = n\,dy$$

- The magnetic field at P due to the current I flowing through dN turns of solenoid:

$$d\vec{B} = \mathrm{dN}\,\frac{\mu_0 I}{2}\frac{a^2}{\left(a^2+y^2\right)^{\frac{3}{2}}}\,\hat{j}$$

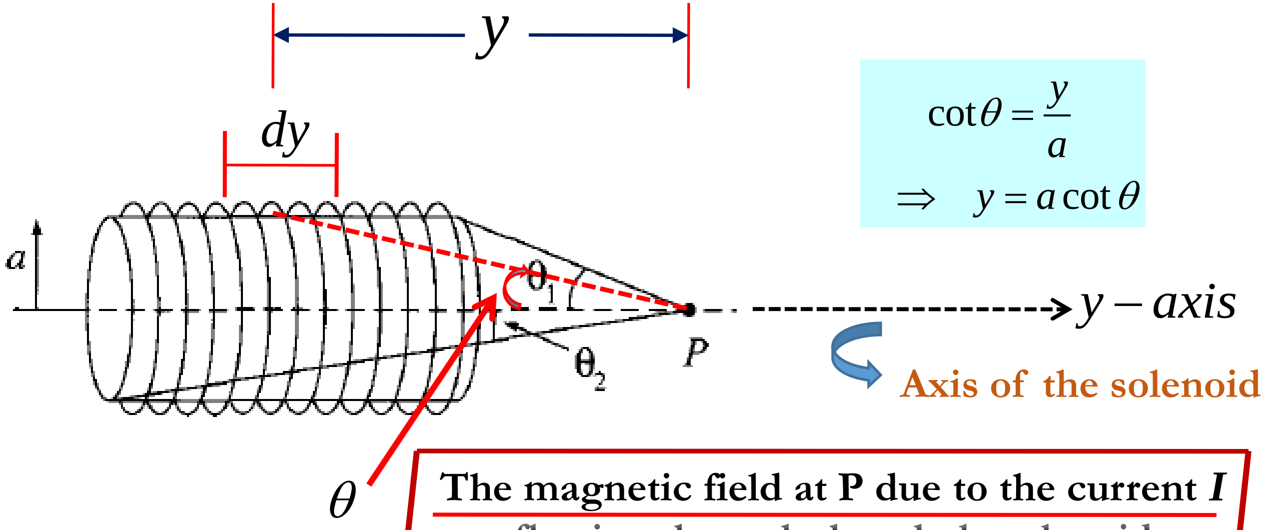
put $\quad y = a\cot\theta$

$$\Rightarrow \mathrm{dy} = -a\,\mathrm{cosec}^2\theta\, d\theta$$

$$\& \ \left(a^2+y^2\right)^{\frac{3}{2}} = \left(a^2+a^2\cot^2\theta\right)^{\frac{3}{2}} = \left[a^2\left(1+\cot^2\theta\right)\right]^{\frac{3}{2}} = a^3\,\mathrm{cosec}^3\theta$$

$$\therefore\ d\vec{B} = \mathrm{dN}\,\frac{\mu_0 I}{2}\frac{a^2}{\left(a^2+y^2\right)^{\frac{3}{2}}}\,\hat{j} \qquad = n\,dy\,\frac{\mu_0 I}{2}\frac{a^2}{\left(a^2+y^2\right)^{\frac{3}{2}}}\,\hat{j}$$

$$= n\left(-a\,\mathrm{cosec}^2\theta\, d\theta\right)\frac{\mu_0 I}{2}\frac{a^2}{a^3\,\mathrm{cosec}^3\theta}\,\hat{j} = n\frac{\mu_0 I}{2}\left(-\sin\theta\, d\theta\right)\hat{j}$$



$$\cot\theta = \frac{y}{a}$$

$$\Rightarrow \quad y = a\cot\theta$$

The magnetic field at P due to the current I flowing through the whole solenoid

$$\vec{B} = n\frac{\mu_0 I}{2}\left[\int_{\theta_1}^{\theta_1}\left(-\sin\theta\, d\theta\right)\right]\hat{j}$$

$$= n\frac{\mu_0 I}{2}\left[\cos\theta\right]_{\theta_1}^{\theta_2}\hat{j}$$

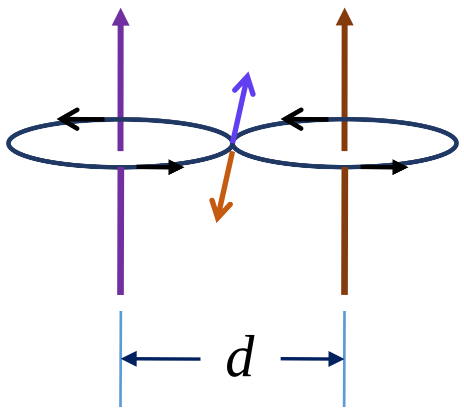
$$\therefore\ \vec{B} = n\frac{\mu_0 I}{2}\left(\cos\theta_2 - \cos\theta_1\right)\hat{j}$$

For an infinite solenoid $\left[\theta_1 = \pi, \theta_2 = 0\right]$

$$\vec{B} = \left(\mu_0 n I\right)\hat{j}$$

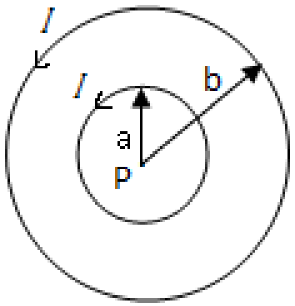
Sample Problems

- Two long and straight conducting wires separated with distance *d* and each of carrying current *I* in the same direction. The magnetic field at the point midway between the wires is zero .



$$\vec{B} = -\frac{\mu_0 I}{2\pi\left(\frac{d}{2}\right)}\hat{i} + \frac{\mu_0 I}{2\pi\left(\frac{d}{2}\right)}\hat{i} = 0$$

- The magnitude of the magnetic field at point P if *a* = *R* and *b* = 2*R* is

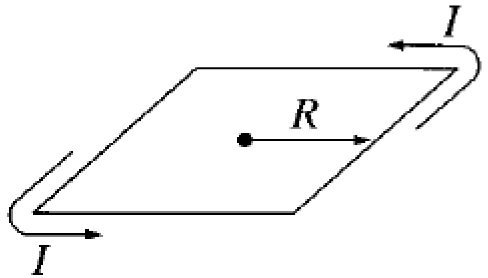


$$B = \frac{\mu_0 I}{2a} + \frac{\mu_0 I}{2b} = \frac{\mu_0 I}{2R} + \frac{\mu_0 I}{2(2R)} = \frac{3}{4}\frac{\mu_0 I}{R}$$

Sample Problems

MAGNETOSTATICS

- The magnetic field at the centre of a square loop of side '**a**', which carries a steady current I, is



$$B = 2\sqrt{2}\frac{\mu_0 I}{\pi a}$$

- Consider a solenoid that is very long compared with its radius. Of the following choices, what is the most effective way to increase the magnetic field in the interior of the solenoid?

  [a] double its length , keeping the number of turns per unit length constant.

  [b] reduce its radius by half, keeping the number of turns per unit length constant.

  [c] double its radius , keeping the number of turns per unit length constant.

  [d] overwrap the entire solenoid with an additional layer of current-carrying wire.

For an infinite solenoid

$$B = (\mu_0 n I)$$

MAGNETOSTATICS

The Divergence of Magnetic Field:

- **The Biot-Savart law** for the general case of a volume current reads

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau' \quad \text{..........} \quad (1)$$

This formula gives the magnetic field at a point $\vec{r} = (x, y, z)$ in terms of an integral over the current distribution $\vec{J}(x', y', z')$. [Figure B.D-1]

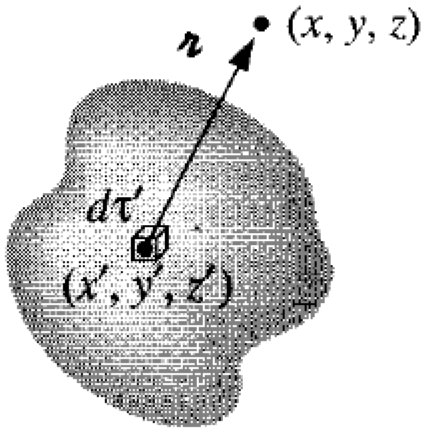


Figure B.D-1

It is best to be absolutely explicit at this stage:

$\vec{B}$ is a function of (x, y, z),

$\vec{J}$ is a function of (x', y', z'),

$\vec{\mathscr{r}} = (x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}$

$d\tau' = dx'dy'dz'$.

- The integration is over the primed coordinates; the divergence is to be taken with respect to the unprimed coordinates.

The divergence of magnetic field:

$$\nabla\cdot\vec{B} = \nabla\cdot\left[\frac{\mu_0}{4\pi}\int\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'\right]$$

$$= \frac{\mu_0}{4\pi}\int\left[\nabla\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\left(\nabla\times\vec{J}\right) - \vec{J}\cdot\left(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right]d\tau'$$

Using product rule

$$\nabla\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\left(\nabla\times\vec{A}\right) - \vec{A}\cdot\left(\nabla\times\vec{B}\right)$$

But,

$$\nabla\times\vec{J} = 0, \quad \left[\therefore \vec{J} \text{ doesn't depend on the unprimed variables } (x, y, z)\right]$$

$$\& \quad \nabla\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0$$

$$\therefore \nabla\cdot\vec{B} = 0$$

This conforms that magnetic monopole does not exist.

$$\boxed{\int_V \left(\nabla\cdot\vec{B}\right)d\tau = 0 \Rightarrow \oint_S \vec{B}\cdot d\vec{a} = 0}$$

Ampère's Law

MAGNETOSTATICS

Ampere's Law:

- **Figure A.L-1** is a perspective view of the magnetic field surrounding a long, straight, current-carrying wire.

- The line integral of magnetic field $\vec{B}$ around a circular path of radius a, centered at the wire, is

$$\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi a}\, dl = \frac{\mu_0 I}{2\pi a}\oint dl$$

$$= \frac{\mu_0 I}{2\pi a}(2\pi a) = \mu_0 I$$

- Although this result was calculated for the special case of a circular path surrounding a wire of infinite length, it holds for a closed path of any shape (an amperian loop) surrounding a current that exists in an unbroken circuit.

- The general case, known as **Ampère's law**, can be stated as follows:

The line integral of magnetic field $\vec{B}$ around any closed path is equal to μ_0 times the total steady current passing through any surface bounded by the closed path:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0\, I_{\text{enc}}$$

total current enclosed by the integration path

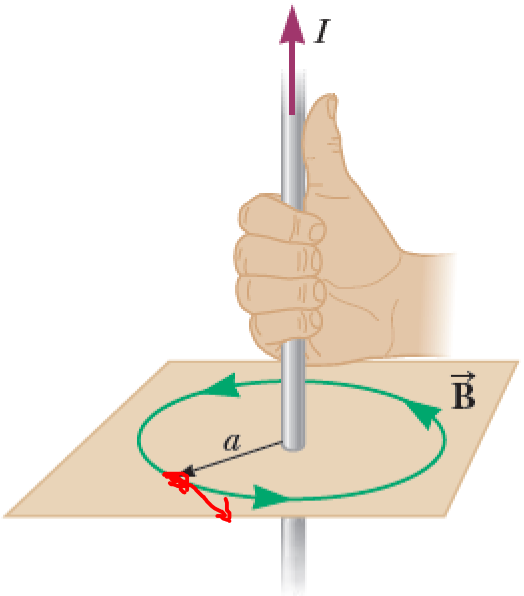
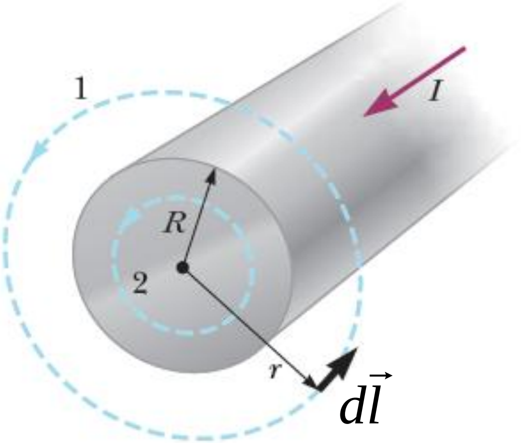


Figure A.L-1
The right-hand rule for determining the direction of the magnetic field surrounding a long, straight wire carrying a current.

Andre-Marie Ampère
French Physicist (1775–1836)
Ampère is credited with the discovery of electromagnetism, which is the relationship between electric currents and magnetic fields. Ampère died at the age of 61 of pneumonia.

The Magnetic Field Created by a Long Current-Carrying Wire:

- **A long, straight wire of radius R carrying a steady current I uniformly distributed across the cross section of the wire.**



The magnetic field a distance r from the center of the wire in the regions $r \geq R$:

Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\text{or, } B\oint dl = \mu_0 I$$

$$\text{or, } B(2\pi r) = \mu_0 I$$

From symmetry, $\vec{B}$ must be constant in magnitude and parallel to $d\vec{l}$ at every point on this circle.

$$\therefore \quad B = \frac{\mu_0 I}{2\pi r} \quad (\text{for } r \geq R)$$

The magnetic field a distance r from the center of the wire in the regions $r < R$:

Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

I_{enc} the current passing through the plane of the circle

$$\text{or, } B\oint dl = \mu_0 \left(\frac{\pi r^2}{\pi R^2} I\right)$$

$$\text{or, } B(2\pi r) = \mu_0 \left(\frac{\pi r^2}{\pi R^2} I\right)$$

$$\therefore \quad B = \left(\frac{\mu_0 I}{2\pi R^2}\right) r \quad (\text{for } r < R)$$

Application of Ampère's Law

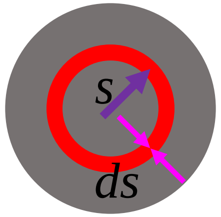
MAGNETOSTATICS

The Magnetic Field Created by a Long Current-Carrying Cylindrical Wire:

- **A steady current I flows down long, cylindrical wire of radius R.**
- **Current is distributed in such a way that $\vec{J}$ is proportional to S, the distance from the axis.**

The current in the shaded patch (red) is

$$J\ da_{\perp} = (ks)(2\pi s ds) = 2\pi k\left(s^2 ds\right)$$



Magnetic Field Outside the Wire

Ampere's Law

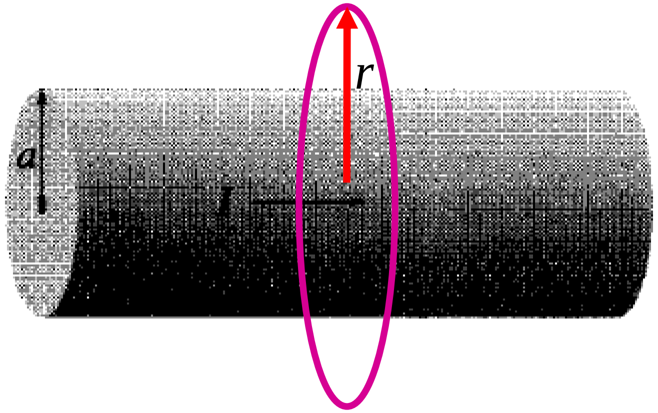
$$\oint \vec{\mathrm{B}}\cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

$$\text{or, } B\oint dl = \mu_0\left(\int_0^a 2\pi k\left(s^2 ds\right)\right)$$

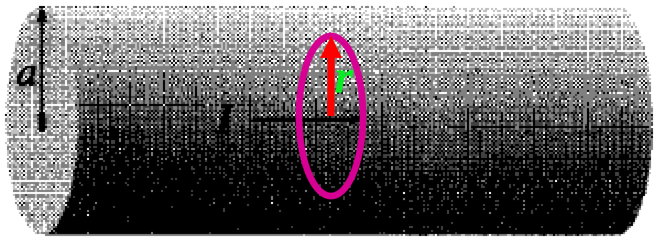
$$\text{or, } B(2\pi r) = \mu_0 2\pi k\left(\int_0^a s^2 ds\right)$$

$$\text{or, } B(2\pi r) = \mu_0 2\pi k\frac{a^3}{3}$$

$$\therefore \quad B = \frac{\mu_0 k a^3}{3r}\ (\text{for } r > a)$$



Magnetic Field Inside the Wire



Ampere's Law

$$\oint \vec{\mathrm{B}}\cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

$$\text{or, } B\oint dl = \mu_0\left(\int_0^r 2\pi k\left(s^2 ds\right)\right)$$

$$\text{or, } B(2\pi r) = \mu_0 2\pi k\left(\int_0^a s^2 ds\right)$$

$$\text{or, } B(2\pi r) = \mu_0 2\pi k\frac{r^3}{3}$$

$$\therefore \quad B = \frac{\mu_0 k r^2}{3}\ (\text{for } r < a)$$

Solenoid

MAGNETOSTATICS

Solenoid:

- A solenoid is a long wire wound in the form of a helix.

Figure S-1 shows the magnetic field lines surrounding a loosely wound solenoid. The field lines in the interior are nearly parallel to one another, are uniformly distributed, and are close together, indicating that the field in this space is strong and almost uniform.

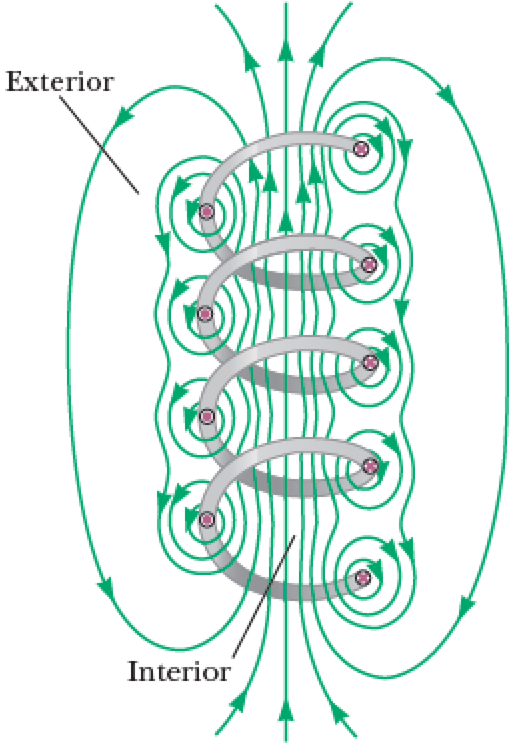


Figure S-1

An **ideal solenoid** is approached when the turns are closely spaced and the length is much greater than the radius of the turns. For such a solenoid carrying a current I, the external field is close to zero and the interior field is uniform over a great volume.

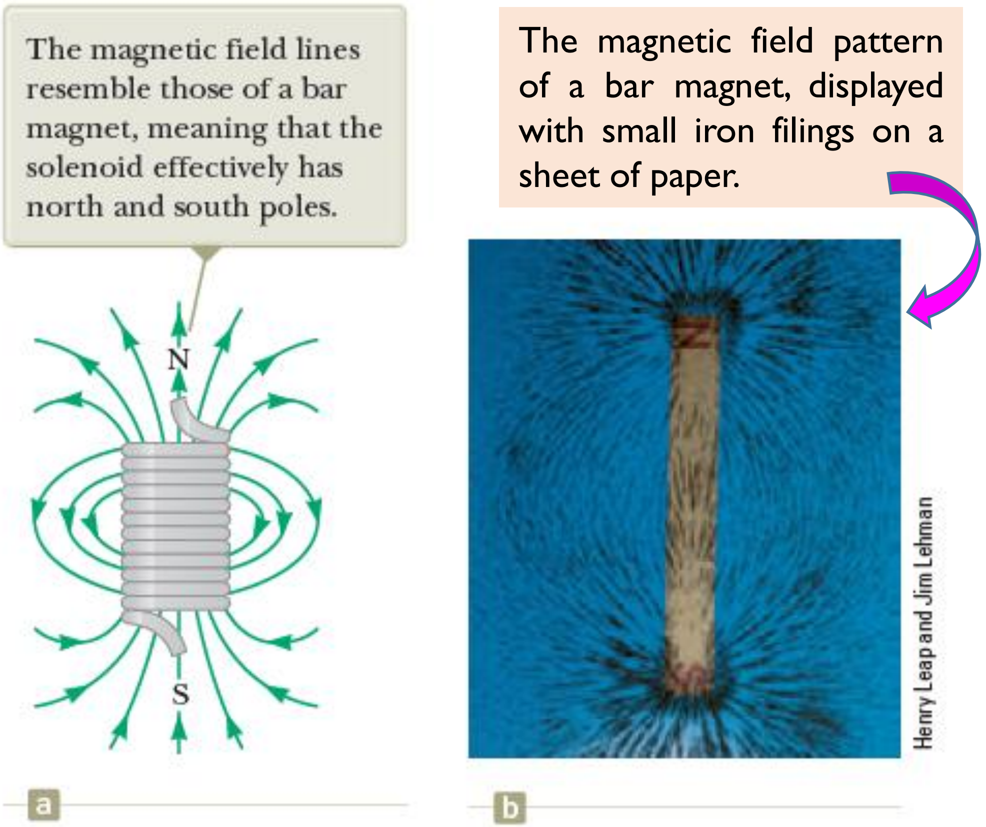


Figure S-2

Figure S-2 (a) Magnetic field lines for a tightly wound solenoid of finite length, carrying a steady current. The field in the interior space is strong and nearly uniform

Magnetic Field of a Solenoid

MAGNETOSTATICS

Magnetic Field of a Solenoid:

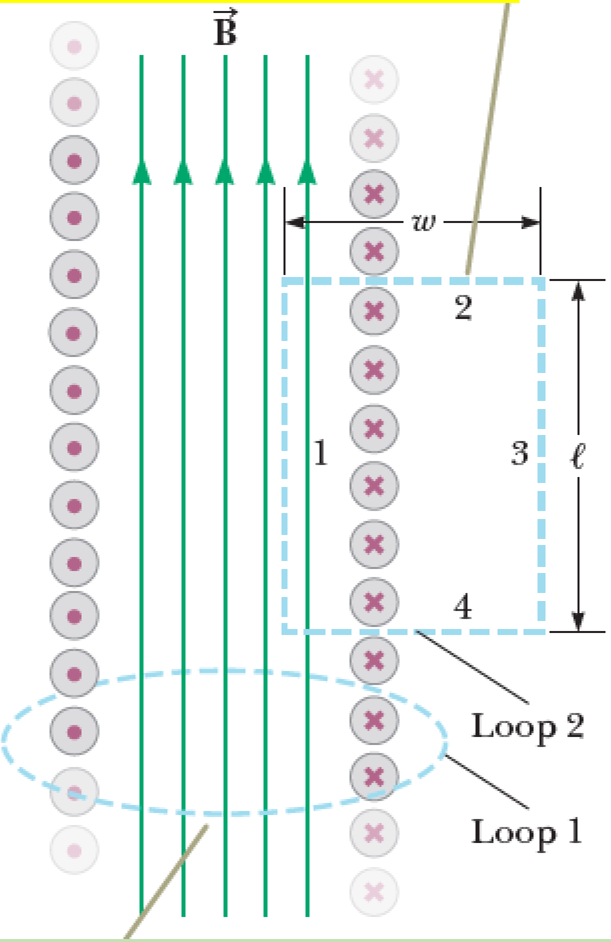


Figure S-3

Cross-sectional view of an ideal solenoid carrying a current I, where the interior magnetic field is uniform and the exterior field is close to zero

For the rectangular path (Amperian loop 2) of length ℓ, and width w

The line integral of magnetic Field over the closed rectangular path (Amperian loop 2)

$$\oint \vec{B}\cdot d\vec{l} = \int_{path1} \vec{B}\cdot d\vec{l} + \int_{path2} \vec{B}\cdot d\vec{l} + \int_{path3} \vec{B}\cdot d\vec{l} + \int_{path4} \vec{B}\cdot d\vec{l} \;.......... \quad (1)$$

$$= B\int_{path1} dl + 0 + 0 + 0$$

$$= Bl$$

The second and fourth integrals in Eq. 1 are zero because for every element of these paths B is either at right angles to the path (for points inside the solenoid) or is zero (for points outside). In either case, $\vec{B}\cdot d\vec{l}$ is zero, and the integrals vanish. The third integral, which includes the part of the rectangle that lies outside the solenoid, is zero because we have taken B as zero for all external points for an ideal solenoid.

Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \quad \Rightarrow \quad Bl = \mu_0\left[(nl)I\right]$$

$$\therefore \quad B = \mu_0 nI$$

n
$\downarrow$
number of turns per unit length

Magnetic Vector Potential

Magnetic Vector Potential:

- The divergence of magnetic filed is zero.

$$i.e.\quad \nabla\cdot\vec{B}=0 \quad \text{..........} \quad (1)$$

- The divergence of curl of a vector is always zero.

$$i.e.\quad \nabla\cdot\left(\nabla\times\vec{A}\right)=0 \quad \text{..........} \quad (2)$$

From Eq. (1) and (2), we get

$$\vec{B}=\nabla\times\vec{A} \quad \text{..........} \quad (3)$$

The vector $\vec{A}$ is called the magnetic vector potential and is defined as a vector field whose curl is equal to magnetic field.

- **If** $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ is magnetic vector potential,

then the y-component of magnetic field is $B_x=\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}$.

$$\vec{B}=\nabla\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

Magnetic Vector Potential

MAGNETOSTATICS

Magnetic Vector Potential:

- From Biot-Savart Law,

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\vec{\mathscr{r}}}{\mathscr{r}^3}\, d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \vec{J}\times\left(\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right) d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \vec{J}\times\left\{-\nabla\left(\frac{1}{\mathscr{r}}\right)\right\} d\tau' \qquad \left[\because \nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right]$$

$$= \frac{\mu_0}{4\pi}\int \left[\nabla\left(\frac{1}{\mathscr{r}}\right)\times\vec{J}\right] d\tau'$$

$$\left[\because \nabla\times\left(\frac{\vec{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\left(\nabla\times\vec{J}\right) + \nabla\left(\frac{1}{\mathscr{r}}\right)\times\vec{J} = \nabla\left(\frac{1}{\mathscr{r}}\right)\times\vec{J}\right]$$

$= 0$ $\left[\because \text{ J depends only on the source coordinates } (x', y', z')\right]$

$$= \frac{\mu_0}{4\pi}\int \left[\nabla\times\left(\frac{\vec{J}}{\mathscr{r}}\right)\right] d\tau'$$

$$\boxed{\therefore\ \vec{B} = \nabla\times\left[\frac{\mu_0}{4\pi}\int\left(\frac{\vec{J}}{\mathscr{r}}\right) d\tau'\right]}$$

comparing

$$\boxed{\vec{B} = \nabla\times\vec{A}} \Rightarrow \vec{A} = \frac{\mu_0}{4\pi}\int\left(\frac{\vec{J}}{\mathscr{r}}\right) d\tau'$$

For line and Surface Current,

Magnetic Vector Potential

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}}\, d\vec{l}'$$

&

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{\vec{K}}{\mathscr{r}}\, da'$$

Magnetic Flux

$$\phi_B = \int_s \vec{B}\cdot d\vec{a}$$

$$= \int_s \left(\nabla\times\vec{A}\right)\cdot d\vec{a}$$

$$= \oint \vec{A}\cdot d\vec{l}$$

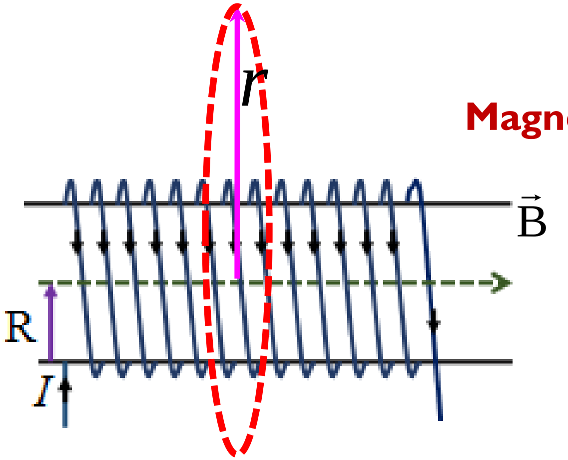
Magnetic Vector Potential

The vector potential of an infinite solenoid with n turns per unit length, radius R, and current I

- **Magnetic Field**

$$B = \begin{cases} \mu_0 n I, & \text{inside the solenoid} \\ 0, & \text{outside the solenoid} \end{cases}$$

Outside the Solenoid



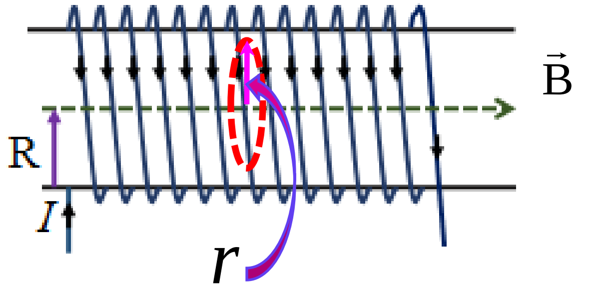
Magnetic Flux, $\phi_B = \int_s \vec{B} \cdot d\vec{a}$

$$\text{or, } \oint \vec{A} \cdot d\vec{l} = \int_s \vec{B} \cdot d\vec{a}$$

$$\text{or, } A(2\pi r) = (\mu_0 n I)\left[\pi R^2\right]$$

$$\therefore \quad A = \frac{1}{2}\frac{\mu_0 n I R^2}{r}$$

Inside the Solenoid



Magnetic Flux, $\phi_B = \int_s \vec{B} \cdot d\vec{a}$

$$\text{or, } \oint \vec{A} \cdot d\vec{l} = \int_s \vec{B} \cdot d\vec{a}$$

$$\text{or, } A(2\pi r) = (\mu_0 n I)\left[\pi r^2\right]$$

$$\therefore \quad A = \frac{1}{2}\mu_0 n I r$$

Magnetic Vector Potential

The magnetic vector potential of a finite segment of a straight wire carrying a current I and associated magnetic field

CD → a finite segment of a straight wire carrying a steady current I

From Figure,

$$d\vec{l}' = dy'\,\hat{j}$$

$$\vec{\mathscr{R}} = -y'\,\hat{j} + z\,\hat{k}$$

$$\mathscr{R} = \sqrt{\left(y'^2 + z^2\right)}$$

$$y' = z\tan\theta$$

$$\therefore\; dy' = z\sec^2\theta\, d\theta$$

$$\tan\theta_1 = \frac{y_1}{z} \quad \& \quad \sec\theta_1 = \frac{\sqrt{y_1^{\,2} + z^2}}{z}$$

$$\tan\theta_2 = \frac{y_2}{z} \quad \& \quad \sec\theta_2 = \frac{\sqrt{y_2^{\,2} + z^2}}{z}$$

Magnetic Vector Potential

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{R}}\, dl' = \frac{\mu_0 I}{4\pi}\int_{y_1}^{y_2} \frac{dy'\,\hat{j}}{\sqrt{y'^2 + z^2}}$$

$$= \frac{\mu_0 I}{4\pi}\left[\ln\left(y' + \sqrt{y'^2 + z^2}\right)\right]_{y_1}^{y_2}\hat{j}$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right)\right]$$

$$\boxed{\therefore\; \vec{A} = \frac{\mu_0 I}{4\pi}\left[\ln\left(y_2 + \sqrt{y_2^{\,2} + z^2}\right) - \ln\left(y_1 + \sqrt{y_1^{\,2} + z^2}\right)\right]\hat{j}}$$

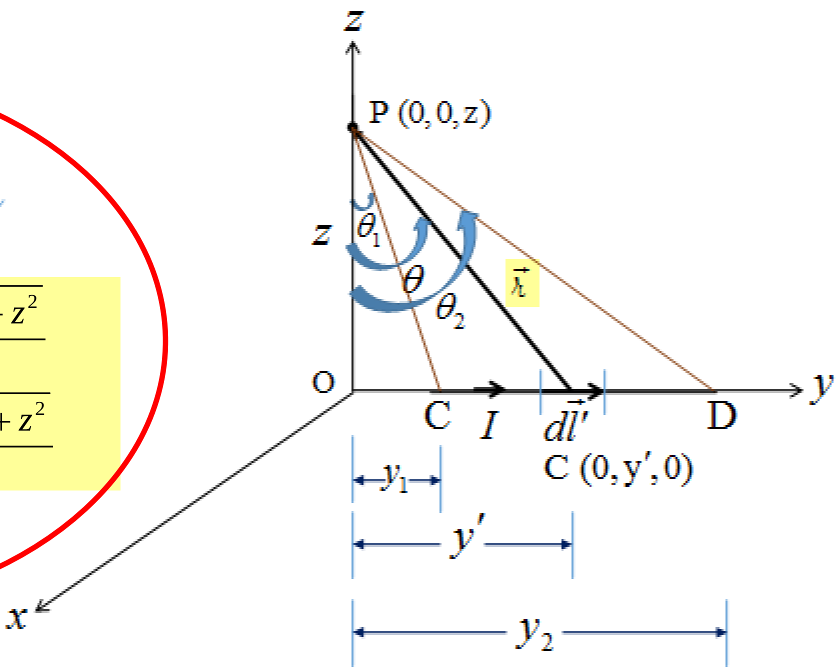


Figure V - I
illustrates the geometry and the coordinates to be used

Magnetic Vector Potential

Associated Magnetic Field

$$\boxed{\vec{B} = \nabla \times \vec{A}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & A_y & 0 \end{vmatrix}$$

$$= -\frac{\partial A_y}{\partial z}\hat{i} + \frac{\partial A_y}{\partial x}\hat{k}$$

$$= -\frac{\partial A_y}{\partial z}\hat{i}$$

$$= -\frac{\partial}{\partial z}\left[\frac{\mu_0 I}{4\pi}\left\{\ln\left(y_2 + \sqrt{y_2^{\,2} + z^2}\right) - \ln\left(y_1 + \sqrt{y_1^{\,2} + z^2}\right)\right\}\right]\hat{i}$$

$$= -\frac{\mu_0 I}{4\pi}\left[\frac{1}{\left(y_2 + \sqrt{y_2^{\,2} + z^2}\right)}\frac{1}{2}\frac{2z}{\sqrt{y_2^{\,2} + z^2}} - \frac{1}{\left(y_1 + \sqrt{y_1^{\,2} + z^2}\right)}\frac{1}{2}\frac{2z}{\sqrt{y_1^{\,2} + z^2}}\right]\hat{i}$$

$$= -\frac{\mu_0 I}{4\pi}\left[\frac{\cos\theta_2}{z\tan\theta_2 + z\sec\theta_2} - \frac{\cos\theta_1}{z\tan\theta_1 + z\sec\theta_1}\right]\hat{i}$$

$$= -\frac{\mu_0 I}{4\pi z}\left[\frac{\cos^2\theta_2}{1+\sin\theta_2} - \frac{\cos^2\theta_1}{1+\sin\theta_1}\right]\hat{i} \quad = \quad -\frac{\mu_0 I}{4\pi z}\left[\left(1-\sin\theta_2\right) - \left(1-\sin\theta_1\right)\right]\hat{i}$$

$$\tan\theta_1 = \frac{y_1}{z} \quad \& \quad \sec\theta_1 = \frac{\sqrt{y_1^{\,2} + z^2}}{z}$$

$$\tan\theta_2 = \frac{y_2}{z} \quad \& \quad \sec\theta_2 = \frac{\sqrt{y_2^{\,2} + z^2}}{z}$$

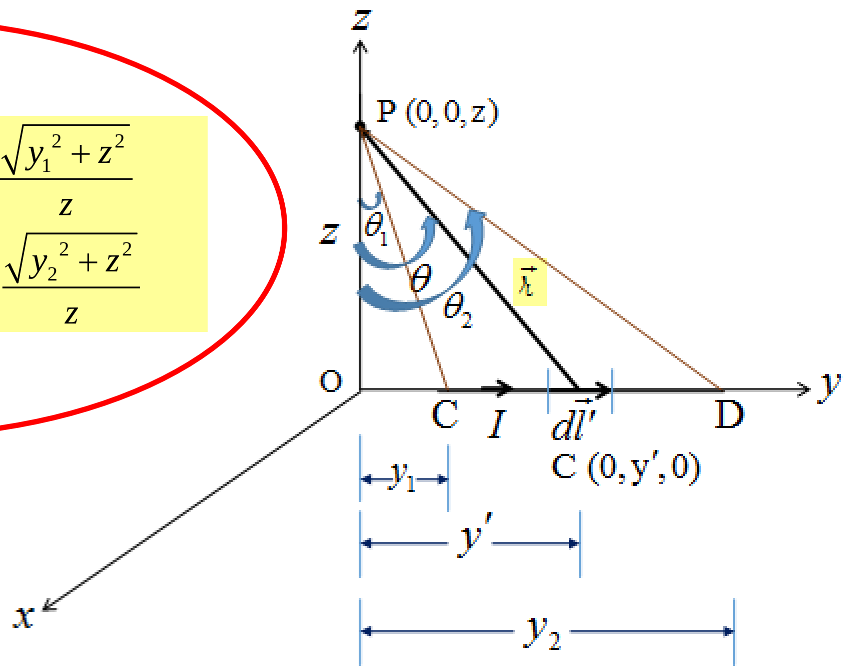


Figure V - I
illustrates the geometry and the coordinates to be used

$$\boxed{\therefore \quad \vec{B} = \frac{\mu_0 I}{4\pi z}\left[\sin\theta_2 - \sin\theta_1\right]\hat{i}}$$

Magnetic Vector Potential

If $\vec{B}$ is uniform, show that $\vec{A}(\vec{r}) = -\frac{1}{2}(\vec{r}\times\vec{B})$ works. That is check that $\nabla\cdot\vec{A} = 0$ and $\nabla\times\vec{A} = \vec{B}$.

MAGNETOSTATICS

$\overrightarrow{PQ} = d\vec{l} \longrightarrow$ an elemental vector length on the loop

- **The area of the triangle OPQ is**

$$d\vec{a} = \frac{1}{2}\left(\overrightarrow{OP}\times\overrightarrow{PQ}\right)$$
$$= \frac{1}{2}\left(\vec{r}\times d\vec{l}\right)$$

- **The magnetic flux crossing the triangle OPQ is**

$$d\phi = \vec{B}\cdot d\vec{a}$$
$$= \vec{B}\cdot\frac{1}{2}\left(\vec{r}\times d\vec{l}\right)$$
$$= \frac{1}{2}\left(\vec{B}\times\vec{r}\right)\cdot d\vec{l}$$
$$= -\frac{1}{2}\left(\vec{r}\times\vec{B}\right)\cdot d\vec{l}$$

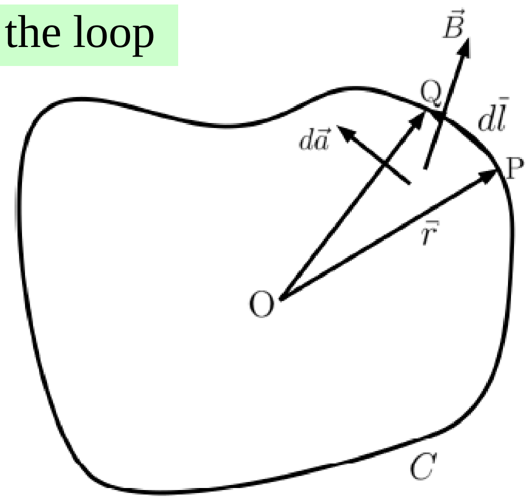


Figure V – I
A closed loop C

- **The magnetic flux crossing the whole loop is**

$$\phi_B = \oint_C \left\{-\frac{1}{2}\left(\vec{r}\times\vec{B}\right)\right\}\cdot d\vec{l} \quad \text{..........} \quad (1)$$

- **In term of vector potential, the flux through the whole loop is**

$$\phi_B = \oint_C \vec{A}\cdot d\vec{l} \quad \text{..........} \quad (2)$$

From Eq. (1) and Eq.(2), we get

$$\oint_C \vec{A}\cdot d\vec{l} = \oint_C \left\{-\frac{1}{2}\left(\vec{r}\times\vec{B}\right)\right\}\cdot d\vec{l}$$

$$\therefore \boxed{\vec{A} = -\frac{1}{2}\left(\vec{r}\times\vec{B}\right)}$$

Magnetic Vector Potential

If $\vec{B}$ is uniform, show that $\vec{A}(\vec{r}) = -\frac{1}{2}(\vec{r}\times\vec{B})$ works. That is check that $\nabla\cdot\vec{A} = 0$ and $\nabla\times\vec{A} = \vec{B}$.

MAGNETOSTATICS

- Now,

$$\nabla\cdot\vec{A} = \nabla\cdot\left[-\frac{1}{2}(\vec{r}\times\vec{B})\right]$$
$$= -\frac{1}{2}\left[\nabla\cdot(\vec{r}\times\vec{B})\right]$$
$$= -\frac{1}{2}\left[\vec{B}\cdot(\nabla\times\vec{r}) - \vec{r}\cdot(\nabla\times\vec{B})\right]$$
$$= 0 \qquad \left[\begin{array}{l}\because \ \nabla\times\vec{r} = 0 \\ \& \ \nabla\times\vec{B} = 0 \ \text{ for constant } \vec{B}\end{array}\right]$$

$$\therefore \quad \nabla\cdot\vec{A} = 0$$

$$\vec{r}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (yB_z - zB_y)\hat{i} - (xB_z - zB_x)\hat{j} + (xB_y - yB_x)\hat{k}$$

$$\therefore \nabla\times\vec{A} = \nabla\times\left[-\frac{1}{2}(\vec{r}\times\vec{B})\right] = -\frac{1}{2}\left[\nabla\times(\vec{r}\times\vec{B})\right]$$
$$= -\frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (yB_z - zB_y) & (zB_x - xB_z) & (xB_y - yB_x) \end{vmatrix}$$
$$= -\frac{1}{2}\left[(-B_x - B_x)\hat{i} - (B_y + B_y)\hat{j} + (-B_z - B_z)\hat{k}\right]$$
$$= B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
$$= \vec{B}$$

$$\therefore \quad \nabla\times\vec{A} = \vec{B}$$

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Thank you