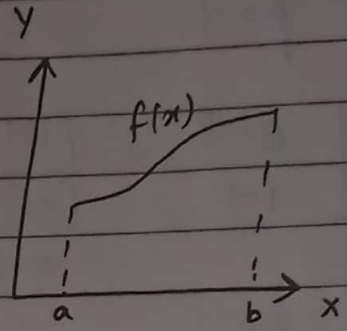
Unit: 5

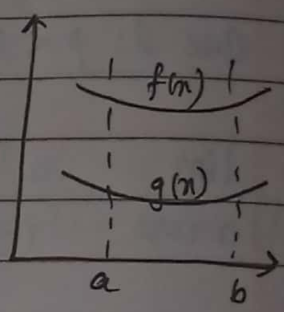
APPLICATIONS OF INTEGRATION

Area Between Two Curves

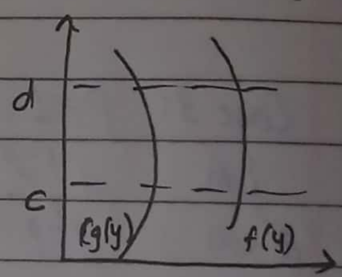
(i): Area $= \int_a^b f(x)\, dx$



(ii) Area of curve ~~about~~ using vertical rectangular base $= \int_a^b [\underset{\substack{\uparrow \\ \text{outer curve}}}{f(x)} - \underset{\substack{\uparrow \\ \text{inner curve.}}}{g(x)}]\, dx$



(iii) Area of curve using horizontal rectangular base $= \int_c^d [f(y) - g(y)]\, dy$



LQ7: Find the area between the curve $y = e^x$ and $y = x$ about x-axis from $x = 0$ and $x = 1$.

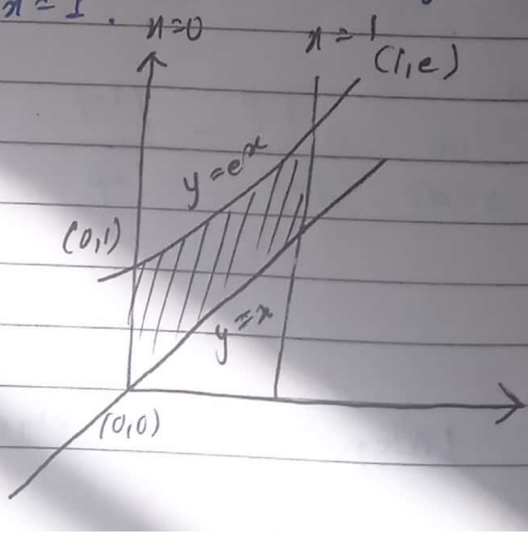
Soln:

Given,

$f(x) = e^x$

$g(x) = x$

Here, the region bounded by $(0,0)$, $(1,0)$, $(1,1)$, $(1,e)$.



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So, area with vertical rectangular base is

$$A = \int_0^1 [f(x) - g(x)]\, dx$$

$$= \int_0^1 (e^x - x)\, dx$$

$$= \int_0^1 e^x\, dx - \int_0^1 x\, dx$$

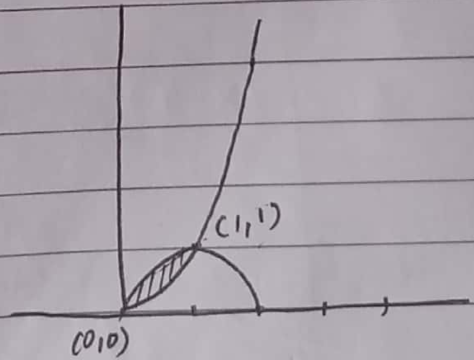
$$= e^x \Big|_0^1 - \frac{x^2}{2}\Big|_0^1$$

$$= [e^1 - e^0] - \left[\frac{1^2}{2} - \frac{0^2}{2}\right]$$

$$= (e-1) - \frac{1}{2} \qquad \therefore A = e - \frac{3}{2} \text{ square units.}$$

<Q7: Find the area of region enclosed by parabola $y = x^2$ and $y = 2x - x^2$.

Soln:



Given,

$f(x) = \cancel{x^2}\ 2x - x^2$ — (i)

$g(x) = \cancel{2x-}\ x^2$ — (ii)

Using eqn (ii) in eqn (i),

$x^2 = 2x - x^2$

or $2x^2 - 2x = 0$ or, $x(x-1)$ $\therefore x = 0$ or $x = 1$

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The intersecting points are (0,0) and (1,1).

Using vertical rectangular base.

Area between curves $= \int_0^1 [f(x) - g(x)]\, dx$

$$= \int_0^1 [(2x - x^2) - x^2]\, dx$$

$$= \int_0^1 2x\, dx - \int_0^1 2x^2\, dx$$

$$= 2\left[\int_0^1 x\, dx - \int_0^1 x^2\, dx\right]$$

$$= 2\left[\left.\frac{x^2}{2}\right|_0^1 - \left.\frac{x^3}{3}\right|_0^1\right]$$

$$= \frac{2}{3} \left. x^2 \right|_0^1 - \frac{1}{3} \left. x^3 \right|_0^1$$

$$= \frac{2}{3}\left[1^2 - 0^2\right] - \frac{1}{3}\left[1^3 - 0^3\right]$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units.}$$

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<Q>: $y^2 - 4x = 4$ and $4x - y = 16$

Soln:

Given,

$y^2 - 4x = 4$ $\qquad\qquad$ $4x - y = 16$

or, $x = \frac{y^2 - 4}{4}$ — (i) $\qquad\qquad$ or, $x = \frac{16 + y}{4}$ — (ii)

Putting eqn (i) in eqn (ii),

$$\frac{y^2 - 4}{4} = \frac{16 + y}{4}$$

or, $y^2 - y - 20 = 0$ $\qquad$ So,

$\therefore y = 5, -4.$

The area between curves:

$$A = \int_{-4}^{5} \left(\frac{y^2 - 4}{4} - \frac{16 + y}{4} \right) dy$$

$$= \int_{-4}^{5} \left(\frac{y^2 - y - 20}{4} \right) dy$$

$$= \frac{1}{4} \left[\int_{-4}^{5} y^2\, dy - \int_{-4}^{5} y\, dy - \int_{-4}^{5} 20\, dy \right]$$

$$= \frac{1}{4} \left[\frac{y^3}{3} \Big|_{-4}^{5} - \frac{y^2}{2} \Big|_{-4}^{5} - 20\, y \Big|_{-4}^{5} \right]$$

$$= \frac{1}{12} \left[5^3 - (-4)^3 \right] - \frac{1}{8} \left[5^2 - (-4)^2 \right] - 5 \left[5 - (-4) \right]$$

$$= \frac{63}{4} - \frac{9}{8} - 45 =$$

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$$A = \int_{-4}^{5} \left(\frac{16+y}{4} - \frac{y^2-4}{4} \right) dy$$

$$= \frac{1}{4} \left[\int_{-4}^{5} \cancel{16}(20+y-y^2)\ dy \right]$$

$$= \frac{1}{4} \int_{-4}^{5} 20 \cdot dy + \frac{1}{4} \int_{-4}^{5} y\ dy - \frac{1}{4} \int_{-4}^{5} y^3\ dy$$

$$= \frac{20}{4}\ y \Big|_{-4}^{5} + \frac{1}{8}\ y^2 \Big|_{-4}^{5} - \frac{1}{12}\ y^3 \Big|_{-4}^{5}$$

$$= 5(5-(-4)) + \frac{1}{8}\{5^2-(-4)^2\} - \frac{1}{12}\{5^3-(-4)^3\}$$

$$= 45 + \frac{9}{8} - \frac{63}{4}$$

$$= 30.375 \text{ sq. unit.}$$

<Q>: $y = x$ and $y = 2-(x-2)^2$

Soln:

Given,

$y = x$ — (i)

and

$y = 2-(x-2)^2$ —— (ii)

Putting eqn (i) in eqn (ii), we get.

$y = 2-(y-2)^2$

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or, $y = 2-(y-2)^2$

or $y = 2 - y^2 + 4y - 4$

or $y = -y^2 + 4y - 2$

or, $y^2 - 3y + 2 = 0$

$\therefore y = 2,\ y = 1.$

So the point of intersection is $(1,1)$ and $(2,2)$.
Hence,

Area of curves $= \displaystyle\int_1^2 2 - (x-2)^2 - x \, dx$

$$= \int_1^2 \cancel{2} - (x^2 - 4x + 4) - \cancel{2} \, dx$$

$$= \int_1^2 4x - x^2 - 4 \, dx$$

$$= 4\int_1^2 x \, dx - \int_1^2 x^2 \, dx - \int_1^2 4 \, dx$$

$$= \frac{4}{2} x^2 \Big|_1^2 - \frac{1}{3} x^3 \Big|_1^2 - 4x \Big|_1^2$$

$$= 2\{(2)^2 - (1)^2\} - \frac{1}{3}\{(2)^3 - (1)^3\} - 4(2-1)$$

$=$

Area of curves $= \int_1^2 \left(\left[2-(x-2)^2\right] - [x] \right) dx$

$$= \int_1^2 2\,dx - \int_1^2 (x-2)^2\,dx - \int_1^2 x\,dx$$

$$= 2x\Big|_1^2 - \frac{1}{2}x^2\Big|_1^2 - \int_1^2 x^2 - 4x + 4\,dx$$

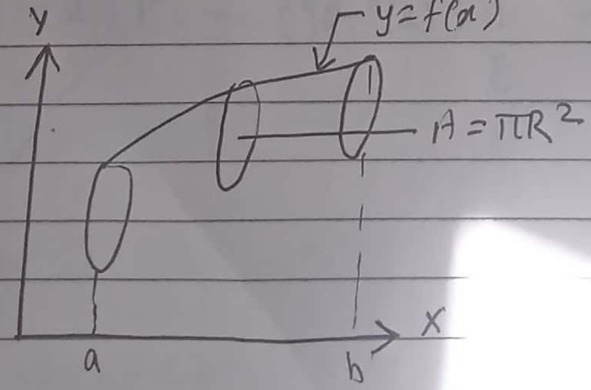
$$= 2x\Big|_1^2 - \frac{1}{2}x^2\Big|_1^2 - \left[\frac{1}{3}x^3\Big|_1^2 - \frac{4}{2}x^2\Big|_1^2 + 4x\Big|_1^2\right]$$

$$= 2(2-1) - \frac{1}{2}\{(2)^2-1^2\} - \left[\frac{1}{3}(2^3-1^3) - 2(2^2-1^2) + 4(2-1)\right]$$

$$= 2 - \frac{3}{2} - \frac{7}{3} + 6 - 4 = \frac{1}{6} \text{ sq. unit.}$$

Volume of Solid of Revolution

(*) Disk Method:



$$V = \int_a^b A(x)\, dx$$

About x-axis,

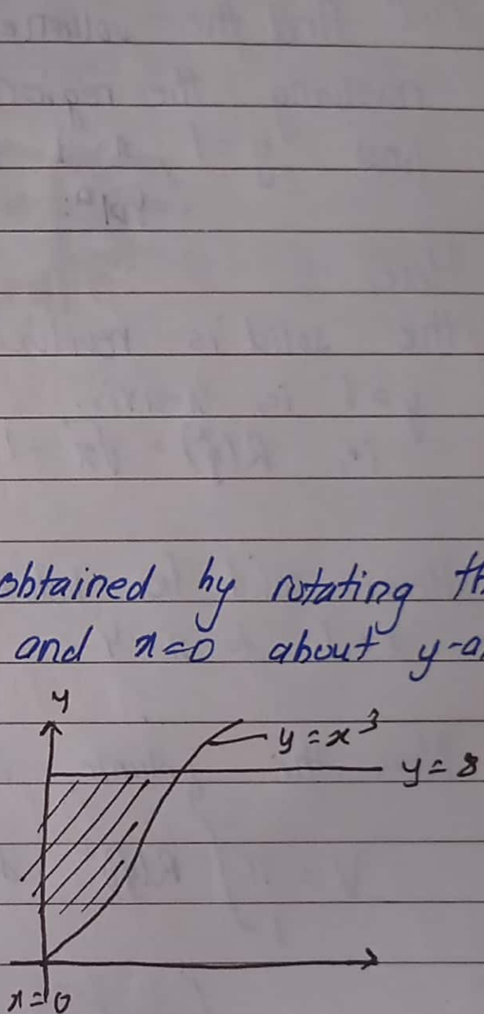
$$V = \int_a^b \pi \left(R(x)\right)^2 dx$$

About y-axis,

$$V = \int_c^d \pi \left(R(y)\right)^2 dy.$$

<Q>: find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about y-axis

Soln:



Given,

~~RE~~ $y = x^3$

$\therefore\ x = y^{1/3}$

$\therefore\ R(y) = y^{1/3}$

Using disk method,
volume of solid of revolution about y-axis,

$$V = \pi \int_0^8 \left(R(y)\right)^2 dx$$

$$= \pi\, y^{5/3} \times \frac{3}{5} \Big|_0^8 = \frac{3\pi}{5}\, y^{5/3} \Big|_0^8$$

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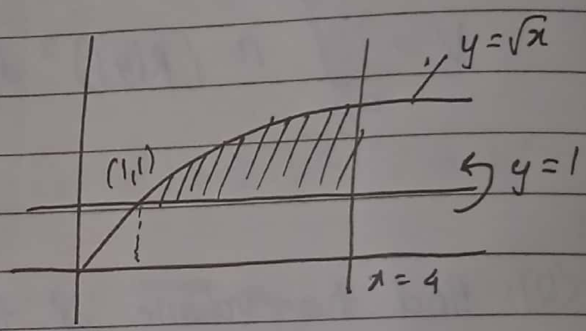
$$= \frac{3\pi}{5}\left(8^{5/3} - 0^{5/3}\right)$$

$$\therefore V = \frac{96\pi}{5} \text{ cub. unit.}$$

<Q>: Find the volume of ~~solution~~ solid generated by revolving the region bounded by $y = \sqrt{x}$ & lines $y = 1$, $x = 4$ about $y = 1$.

Soln:

Here,
the solid is revolved about $y = 1$ ie, x-axis.
i.e, $R(x) = \sqrt{x} - 1$



Here, solid lies between $x = 1$ to $x = 4$

Now, the volume of solid of revolution,

$$V = \pi \int_1^4 R(x)^2 \cdot dx$$

$$= \pi \int_1^4 (\sqrt{x} - 1)^2 \, dx$$

$$= \pi \int_1^4 (x - 2\sqrt{x} + 1) \, dx$$

$$= \pi\left(\int_1^4 x \, dx - 2\int_1^4 \sqrt{x} \, dx + \int_1^4 dx\right)$$

$$= \pi \left(\frac{x^2}{2} \Big|_1^4 - \frac{2}{3} x^{3/2} \Big|_1^4 + x \Big|_1^4 \right)$$

$$= \pi \left\{ \frac{1}{2} (4^2 - 1^2) - \frac{4}{3} (4^{3/2} - 1^{3/2}) + (4-1) \right\}$$

$$\therefore V = \cancel{\frac{29}{6}\pi} \quad \frac{7}{6}\pi \text{ cub. unit.}$$

<Q>: Find the volume of the solid obtained by rotating y-axis, the region under the curve $y = \sqrt{x}$ from 0 to 1.

Soln:

Given,

$$y = \sqrt{x}$$

$$\text{or } x = y^{1/2}$$

$$\therefore R(y) = y^{1/2}.$$

The ~~other~~ solides lies between $y = 0$ and $y = 1$.

Now, volume of solid of revolution,

$$V = \pi \int_0^1 (R(y))^2 \, dy$$

$$= \pi \int_0^1 (y^{1/2})^2 \, dy$$

$$= \pi \int_0^1 y \, dy = \frac{\pi}{2} y^2 \Big|_0^1$$

$$\therefore V = \frac{\pi}{2} \text{ cub. units.}$$

⟨Q⟩: Find the volume of solid generated by revolving the region between the ~~buss~~ parabola $x=y^2+1$ & the line $x=3$ about the line $x=3$.

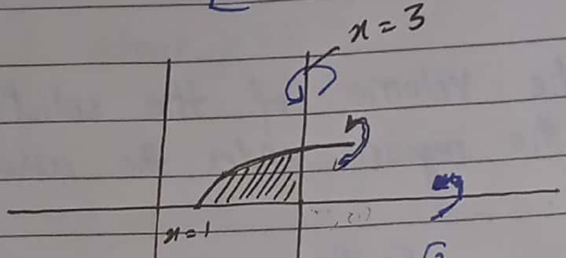
Sol^n:

Given,

The solid is revolved about $x=3$ ie, y-axis.

ie, $R(y) = y^2+1-3$ $\quad$ [(∵ shifting origin to $(3,0)$. [New $x = x-3$] (New $y = y-0$)]

∴ $R(y) = y^2-2$



Here, the solid lies between $x=1$ and $x=3$.

&, point of intersection $(1, 0)$ and $(3, \sqrt{2})$

Now,

volume of solid of revolution $(V) = \int_0^{\sqrt{2}} \pi (R(y))^2\, dy$

$$= \pi \int_0^{\sqrt{2}} (y^4 - 4y^2 + 4)\, dy$$

$$= \pi \left[\int_0^{\sqrt{2}} y^4\, dy - 4\int_0^{\sqrt{2}} y^2\, dy + 4\int_0^{\sqrt{2}} dy \right]$$

$$= \pi \left[\frac{1}{5} y^5 \Big|_0^{\sqrt{2}} - \frac{4}{3} y^3 \Big|_0^{\sqrt{2}} + 4 \cdot y \Big|_0^{\sqrt{2}} \right]$$

$$= \frac{\pi \sqrt{2}^5}{5} - \frac{4\pi}{3}(\sqrt{2})^3 + \pi 4\sqrt{2}$$

$$= 3.01\,\pi \text{ cub. un}$$

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⊛ Washer Method:

If $R(x)$ is outer radius and $r(x)$ is inner radius then,

volume of solid of revolution about x-axis,

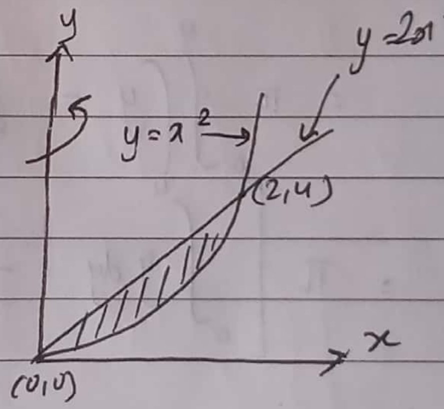
$$V = \int_a^b \pi \left([R(x)]^2 - [r(x)]^2\right) dx$$

Volume of solid of revolution about y-axis,

$$V = \int_c^d \pi \left([R(y)]^2 - [r(y)]^2\right) dy.$$

<Q>: The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about y-axis to generate the solid. Find the volume of the solid.

Soln.



Given,

$y = x^2$ —— (i)

and

$y = 2x$ —— (ii)

Solving (i) and (ii), we get.

$x^2 - 2x = 0$

or $x(x-2) = 0$

$\therefore x = 0, 2.$

The points of intersection is $(0,0)$ and $(2,4)$.

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Here,

$y = x^2$

$\therefore x = \sqrt{y}$

$\therefore R(y) = \sqrt{y}$ {outer radius}

and,

$y = 2x$

or $x = \frac{y}{2}$

$\therefore r(y) = \frac{y}{2}$ {inner radius}.

Now, volume of solid of revolution about y-axis,

$$V = \int_0^4 \pi \left[(R(y))^2 - (r(y))^2 \right] dy$$

$$= \pi \int_0^4 \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 \right] dy$$

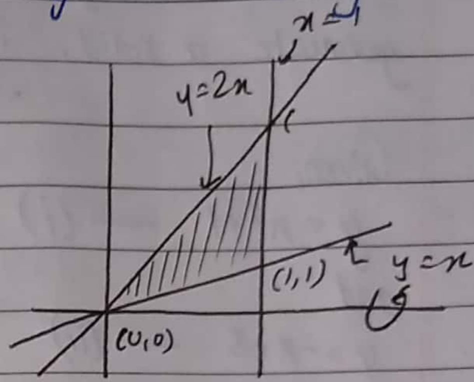
$$= \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy$$

$$= \pi \left[\int_0^4 y \, dy - \frac{1}{4} \int_0^4 y^2 \, dy \right]$$

$$= \pi \left[\frac{1}{2} y^2 \Big|_0^4 - \frac{1}{\cancel{4}12} y^3 \Big|_0^4 \right]$$

$$= \pi \times \frac{1}{2}(4^2) - \frac{1}{12}(4)^3 \qquad \therefore V = \frac{8\pi}{3}$$

<Q7: Find the volume of the solid generated by revolving the region bounded by $y = 2x$ and $y = x$ and $x = 1$ about x-axis.



Soln:

Given,

$y = 2x$ — (i)

$y = x$ — (ii)

Solving (i) and (ii), we get.

$\cancel{2x = x}$ $\quad \cancel{y = 2y}$

$\cancel{\text{or } 2x - x =}$

The points of intersection $(0,0)$ and $(1,1)$.

Here,

outer radius $(R(x)) = 2x$

inner radius $(r(x)) = x$

So the volume of solid of revolution about x-axis is.

$$V = \pi \int_0^1 \left[(R(x))^2 - (r(x))^2 \right] dx$$

$$= \pi \int_0^1 \left[(2x)^2 - (x)^2 \right] dx$$

$$= \pi \left[\int_0^1 4x^2 dx - \int_0^1 x^2 dx \right]$$

$$= \pi \left[\frac{4}{3} x^3 \Big|_0^1 - \frac{1}{3} x^3 \Big|_0^1 \right]$$

$$= \frac{4}{3}\pi - \frac{\pi}{3} = \pi \text{ cub. units}$$

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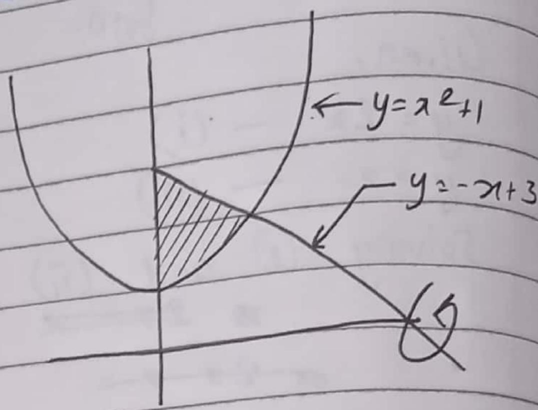
<Q>: The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about axis to generate a solid. Find the volume of solid of revolution.

Soln:



Given,

$y = x^2+1$ — (i)

and

$y = -x+3$ — (ii)

Solving (i) and (ii), we get.

$-x+3 = x^2+1$

or, $x^2+x-2 = 0$

$\therefore x = 1, -2.$

Here,

outer radius $(R(x)) = (-x+3)$

inner radius $(r(x)) = (x^2+1)$

Now, the volume of solid of revolution about x-axis,

$$V = \int_{-2}^{1} \pi \left[(R(x))^2 - (r(x))^2\right] dx$$

$$= \pi \left[\int_{-2}^{1} (-x+3)^2\, dx - \int_{-2}^{1} (x^2+1)^2\, dx\right]$$

$$= \pi \left[\int_{-2}^{1} (x^2-6x+9)\, dx - \int_{-2}^{1} (x^4+2x^2+1)\, dx\right]$$

$$= \pi \left[\int_{-2}^{1} x^2 dx - \int_{-2}^{1} 6x\, dx + \int_{-2}^{1} 9 \cdot dx - \int_{-2}^{1} x^4 dx - 2\int_{-2}^{1} x^2 dx - \int_{-2}^{1} 1 \cdot dx\right]$$

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$$= \pi \left[\frac{1}{3} x^3 \Big|_{-2}^{1} - \frac{6}{2} x^2 \Big|_{-2}^{1} + 9x \Big|_{-2}^{1} - \frac{1}{5} x^5 \Big|_{-2}^{1} - \frac{2}{3} x^3 \Big|_{-2}^{1} \mp x \Big|_{-2}^{1} \right]$$

$$= \pi \left[\frac{1^3 - (-2)^3}{3} - 3\{(1)^2 - (-2)^2\} + 9(1-(-2)) - \frac{1^5 - (-2)^5}{5} - \frac{2}{3}\{(1)^3 - (-2)^3\} - \{1-(-2)\} \right]$$

$$\therefore V = \frac{117\pi}{5}$$

Arc length (Length of curve)

If f' is continuous about $[a, b]$, then arc length of curve.

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

and

$$f' = \frac{df}{dx}$$

$$L = \int_c^d \sqrt{1 + f'(y)^2}\, dy$$

<Q7: Find the length of curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \quad \text{in interval} \quad 0 \le x \le 1.$$

Soln:

Given,

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$$f(x) = \frac{4\sqrt{2}}{3} x^{3/2} - 1$$

$$f'(x) = \frac{df(x)}{dx}$$

$$= \frac{d}{dx}\left(\frac{4\sqrt{2}}{3} x^{3/2} - 1\right)$$

$$= \frac{d}{dx}\left(\frac{4\sqrt{2}}{3} x^{3/2}\right) - \frac{d \cdot 1}{dx}$$

$$= \frac{4\sqrt{2}}{3} \frac{dx^{3/2}}{dx}$$

$$= \frac{\cancel{4}^{2}\sqrt{2}}{\cancel{3}} x^{1/2} \times \frac{\cancel{3}}{\cancel{2}}$$

$$= \cancel{\frac{4\sqrt{2} \times \sqrt{x}}{3}} \; 2\sqrt{2}\, x \qquad \therefore f'(x)^2 = \cancel{\frac{32}{9}x} \; 8x$$

Here, $f(x)$ is continuous on $[0,1]$.

Now,

length of arc $(L) = \int_0^1 \sqrt{1 + f'(x)} \, dx$

$$= \int_0^1 \sqrt{1 + \cancel{\frac{32}{9}}^{8} x} \, dx$$

$$= \int_0^1 \sqrt{8x+1} \, dx$$

$$= \int_0^1 (8x+1)^{1/2} \, dx$$

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$$= \frac{(8x+1)^{1+1/2}}{8(1+1/2)} \Bigg|_0^1$$

$$= \frac{2}{3\times 8} (8x+1)^{3/2} \Bigg|_0^1$$

$$= \frac{2}{3\times 8}(8\times 1+1)^{3/2} - \frac{2}{3\times 8}(8\times 0+1)^{3/2}$$

$\therefore L = \frac{13}{6}$ units.

<Q>: Find the length of the arc of the semicubical parabola $y^2 = x^3$ between $(1,1)$ and $(4,8)$.

Soln:

Given,

$y^2 = x^3$

or $y = x^{3/2}$ $\therefore f(x) = x^{3/2}$

Now,

$$df(x) = \frac{df(x)}{dx} = \frac{dx^{3/2}}{dx} = \frac{3\sqrt{x}}{2}$$

~~This~~ $f(x)$ is continuous on $[1,4]$.

$$\therefore f'(x)^2 = \frac{9x}{4}$$

Now,

Length of arc (L) = $\int_{1}^{4} \sqrt{1+f'(x)^2}\, dx$

$$L = \int_{1}^{4} \sqrt{1+\frac{9}{4}x}\, dx$$

$$= \frac{1}{2}\int_{1}^{4} \sqrt{4+9x}\, dx$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{9(1+1/2)}\Bigg|_{1}^{4}\right]$$

$$= \frac{1}{2}\, \frac{2\times(4+9x)^{3/2}}{9\times 3}\Bigg|_{1}^{4}$$

$$= \frac{1}{27}(4+9x)^{3/2}\Big|_{1}^{4}$$

$$= \frac{1}{27}\left[(4+9\times4)^{3/2} - (4+9\times1)^{3/2}\right]$$

= 7.63 units.

Surface Area of ~~Curve~~ Solid of Revolution.

Let f is continuous on $[a, b]$, then surface area of curve (SA) about x-axis,

$$S.A = \int_{a}^{b} 2\pi y\sqrt{1-(f'(x))^2}\, dx = \int_{a}^{b} 2\pi f(x)\sqrt{1-(f'(x))^2}\, dx$$

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Surface area of ~~curve~~ solid of revolution about y-axis.

$$S \cdot A = \int_c^d 2\pi x \sqrt{1+[f'(y)]^2}\, dy$$

$$= \int_c^d 2\pi f(y) \sqrt{1+[f'(y)]^2}\, dy.$$

<Q7: Find the area of surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$ about x-axis.

Sol^n:

Given,

$$y = 2\sqrt{x}$$

$$\therefore f(x) = 2\sqrt{x}$$

Now,

$$f'(x) = \frac{d f(x)}{dx} = \frac{d(2\sqrt{x})}{dx} = \frac{\cancel{2} \times 1}{\cancel{2}\sqrt{x}} = \frac{1}{\sqrt{x}}$$

~~$\therefore f$~~ Thus, $f(x)$ is continuous on $[1,2]$

$$f'(x)^2 = \frac{1}{x}$$

Now, the surface area of solid of revolution about x-axis,

$$SA = \int_a^b 2\pi f(x) \sqrt{1-(f'(x))^2}\, dx$$

$$= \int_1^2 2\pi \times 2\sqrt{x} \times \sqrt{1 - \frac{1}{x}}\, dx$$

$$= 4\pi \int_1^2 \sqrt{\cancel{x}} \cdot \sqrt{\frac{x-1}{\cancel{x}}} \, dx$$

$$= 4\pi \int_1^2 \sqrt{x-1} \, dx$$

$$= 4\pi \, 2 \times \frac{(x-1)^{3/2}}{3} \Big|_1^2$$

$$= \frac{8\pi}{3} \left[(2-1)^{3/2} - (1-1)^{3/2} \right]$$

$$SA = \frac{8\pi}{3} \text{ sq. units}$$

<Q7>: The arc of the parabola $y = x^2$ from $(1,1)$ ~~and~~ to $(2,4)$ is rotated about y-axis. Find the surface area of solid of revolution.

Soln:

Given,

$$y = x^2$$

or $x = y^{1/2}$

$\therefore f(y) = y^{1/2}$.

$$f'(y) = \frac{d f(y)}{dy} = \frac{d y^{1/2}}{dy} = \frac{1}{2\sqrt{y}}$$

Thus, $f(y)$ is continuous in $[1,4]$

$$\therefore f'(y)^2 = \frac{1}{4y}$$

Now, the surface area of solid of revolution about y-axis.

$$SA = \int_1^4 2\pi f(y) \sqrt{1+f'(y)}\, dy$$

$$= \int_1^4 2\pi \times \sqrt{y} \times \sqrt{1 + \frac{1}{4y}}\, dy$$

$$= \int_1^4 2\pi \times \sqrt{y} \times \sqrt{\frac{4y+1}{4y}}\, dy$$

$$= \int_1^4 \cancel{2}\pi \times \frac{1}{\cancel{2}} \times \cancel{\sqrt{y}} \times \frac{\sqrt{4y+1}}{\cancel{\sqrt{y}}}\, dy$$

$$= \pi \int_1^4 \sqrt{4y+1}\, dy$$

$$= \pi \left. \frac{(4y+1)^{3/2}}{2\cancel{4} \times \frac{3}{\cancel{2}}} \right|_1^4$$

$$= \frac{\pi}{6} \left. (4y+1)^{3/2} \right|_1^4$$

$$= \frac{\pi}{6} \left[(4\times4+1)^{3/2} - (4\times\overset{1}{\cancel{4}}+1)^{3/2} \right]$$

$$= 9.81\ \pi \text{ sq units.}$$