INFINITE SERIES:

Infinite series:

If $\{a_n\}$ is a sequence of numbers, then the sum $a_1 + a_2 + \cdots + a_n + \cdots$ is an infinite series.

Here, a_n is the n^{th} term of the series.

It is denoted as:

$$\sum_{n=1}^{\infty} a_n.$$

Partial Sums:

A partial sum of an infinite series is the sum of the finite numbers of consecutive terms beginning with the first term.

ie,

If $\{a_n\}$ is a sequence and $\sum a_n$ is a corresponding series, then the sequence $\{S_n\}$ is defined as,

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots \qquad \vdots$$
$$S_n = a_1 + a_2 + \cdots + a_n$$

Here,

S_n = the sequence of partial sum of the series $\sum a_n$.

We know,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

So,

if $\lim_{n\to\infty} S_n = L$ (any finite value), $\sum a_n$ converges

and

if $\lim_{n\to\infty} S_n = \pm\infty$ or DNE, $\sum a_n$ diverges.

ie,

If sequence of partial sum converges, the series $\sum a_n$ also converges.

and.

If sequence of partial sum diverges, the series $\sum a_n$ also diverges.

Divergence test:

Let $\sum a_n$ is the corresponding infinite series of sequence a_n.

If $\lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ diverges

but

if $\lim_{n\to\infty} a_n = 0$, $\sum a_n$ may converge or diverge

ie, divergence test is not conclusive.

Questions:

(i): $\sum_{n=1}^{\infty} 2n.$

Sol,

Here,

$a_n = 2n.$

The series corresponds to,

$2+4+6+8+ \ldots\ldots$

This is arithmetic series.

So,

$S_n = \left(\frac{a_1+a_n}{2}\right) \times n$

$= \left(\frac{2+2n}{2}\right) \times n$

$\therefore S_n = (n+1)n$

So,

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (n+1)n$

$= \infty.$

The infinite series diverges.

Using divergence test,

Here,

$a_n = 2n$

So,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2n = \infty$

Since, $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

$\because$ Here, both sequence and series diverges.

Since,

$\lim_{n\to\infty} a_n = \infty$

sequence is also diverge.

(ii): $\sum_{n=1}^{\infty} \frac{5n+3}{7n-4}$

Sol^n:

Here,

$a_n = \frac{5n+3}{7n-4}$

~~The~~ So,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n+3}{7n-4}$

$= \lim_{n\to\infty} \frac{5}{7}$ [$\because$ Using L-Hopital]

$= \frac{5}{7}$

Here, $\lim_{n\to\infty} a_n = \frac{5}{7} \neq 0$ so.

the sequence converges but the series diverges.

(iii): $\sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+2)}\right)$

Sol^n:

Here,

$a_n = \frac{1}{(n+1)(n+2)}$

So,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(n+1)(n+2)}$

$= 0$

Since, $\lim_{n\to\infty} a_n = 0$, the divergente test ~~change~~ is inconclusive

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So,

$$a_n = \frac{1}{(n+1)(n+2)}$$

$$= \frac{(n+2)-(n+1)}{(n+1)(n+2)}$$

$$\therefore a_n = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

So,

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$$

$$\dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-2}\right)$$

$$\therefore S_n = \frac{1}{2} - \frac{1}{n-2}$$

Hence,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{n-2}$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$

Since,

$\lim_{n\to\infty} S_n = \frac{1}{2}$ is finite value.

Thus, $\sum a_n$ converges.

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Infinite Geometric Series:

Let a = first term

r = common ratio

n = no. of terms then, the series

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

is an infinite geometric series.

It is denoted as $\sum_{n=1}^{\infty} ar^{n-1}$.

We know, for geometric series,

$S_n = \frac{a(1-r^n)}{1-r}$ for all $r \neq 1$

*) Case 1:

If $|r| = 1$,

The corresponding series becomes,

$S_n = a + a + \cdots + a$

$\therefore S_n = na$

So, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty$

The series diverges.

*) Case 2:

If $|r| > 1$

$\lim_{n\to\infty} r^n = \infty$

So,

$\lim_{n\to\infty} S_n = \lim_{r^n\to\infty} \frac{a(r^n-1)}{r-1} = \infty$

The series diverges.

*) Case 3:

If $|r| < 1$

$\lim_{n\to\infty} r^n = 0$ and $S_n = \frac{a}{1-r}$

So,

$\lim_{n\to\infty} S_n = \lim_{r^n\to\infty} \frac{a}{1-r} = 0$

The series converges.

So, # for a infinite geometric ~~sequence~~ series

if

$|r| \geq 1$, series diverges

$|r| < 1$, series converges.

Questions

(i): $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5}{4^{n-1}}$

Soln.

The corresponding series is,

$5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \cdots$

$= 5\left(1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \cdots\right)$

Here, $|r| = \left|-\frac{1}{4}\right| = \frac{1}{4} < 1$

Since $|r| < 1$, the series converges.

(ii): # $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

Soln.

The corresponding series is.

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdots$$

Here,

$$|r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$$

Since $|r| < 1$, the series converges.