Absolute Extreme Values

There are three steps to calculate the absolute extreme values on closed bounded region.

(i): List the interior points of R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.

(ii) List the boundary points of R where f has local maxima and minima and evaluate f at these points. We show how to do this shortly.

(iii) Look through the lists for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R.

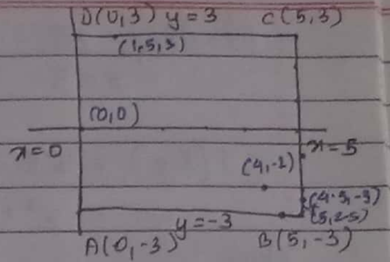
Since absolute extrema are also local extrema, we get value at step 1/2.

Q7: Find absolute maximum value and absolute minimum value of

$$T(x,y) = x^2 + xy + y^2 - 6x$$

on rectangular plate $0 \le x \le 5$, $-3 \le y \le 3$.

Soln:



Given,

$T(x,y) = x^2 + xy + y^2 - 6x$

For interior points,

$T_x = 2x + y - 6 = 0$ —— (i)

$T_y = x + 2y = 0$ —— (ii)

Solving (i) and (ii), we get.

$2x + y = 6$

$x + 2y = 0$ $\quad \therefore (x,y) = (4,-2)$

So, $T(4,-2) = 4^2 + 4\times(-2) + (-2)^2 - 6\times 4$
$= -12$

Now, along boundary,

(a) Along AB: $y = -3$

$\therefore T(x,-3) = x^2 + x(-3) + (-3)^2 - 6\times x$
$= x^2 - 9x + 9$

Now,

$T'(x,-3) = 0$

or, $2x - 9 = 0$

$\therefore x = 9/2$ $\quad \therefore y = -3$

So, points on AB are,

$(0,-3)$, $(5,-3)$, $(4.5,-3)$

Now,

$T(0,-3) = 9$

$T(5,-3) = 25 - 45 + 9 = -11$

$T(4.5,-3) = 81/4 - 81/2 + 9 = -11.25$

(b): Along BC,

$x = 5$

$\therefore T(5,y) = 25 + 5y + y^2 - 30$

$= y^2 + 5y - 5$

Now,

$T'(5,y) = 2y + 5 = 0$

or $2y + 5 = 0$

$\therefore y = -5/2$

So points on BC is $(5,-3)$, $(5,3)$, $(5,-2.5)$

Now,

$T(5,-3) = -11$

$T(5,3) = 19$

$T(5,-5/2) = -5$

(c): Along CD,

$y = 3$

$\therefore T(x,3) = \cancel{2x} + \cancel{3}\ x^2 + 3x + 9 - 6x$

$= x^2 - 3x + 9$

Now,

$T'(x,3) = 0$

or, $2x - 3 = 0$

$\therefore x = 3/2 \qquad y = 3$

So points along CD

$(0,3)$, $(5,3)$, $(1.5,3)$

Now,

$T(0,3) = 9$

$T(5,3) = 19$

$T(1.5,3) = 6.75$

(d): Along DA,

$x = 0$

$\therefore T(0,y) = y^2$

Now,

$T'(0,y) = 2y = 0$

$\therefore y = 0.$

So, points along DA,

$(0,0)$, $(0,3)$, $(0,-3)$

Now,

$T(0,0) = 0$

$T(0,3) = 9$

$T(0,-3) = 9$

So, the absolute maxima $= 19$ at $(5,3)$

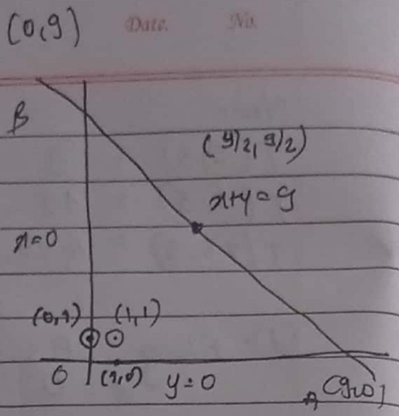
the absolute minima $= -12$ at $(4,-2)$

<Q7: Find the absolute maxima and minimum values of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on a triangular plane in 1st quadrant bounded by $x=0$, $y=0$ and $x+y=9$.

Soln,

Given,

$f(x,y) = 2+2x+2y-x^2-y^2$



for interior points,

$f_x = 2-2x = 0$ —(i)

$f_y = 2-2y = 0$ —(ii)

Solving (i) and (ii), we get

$x=1, \quad y=1$

$\therefore (x,y) = (1,1)$

So,

$f(1,1) = 2+2+2-1-1 = 4$

Now, along ~~AB~~ OB, $x=0$

$\therefore T(0,y) = 2+2y-y^2$

So,

$T'(0,y) = 2-2y = 0$

or $y=1$.

So points on OB $(0,0)$, $(0,9)$, $(0,1)$

So,

$T(0,0) = 2$ $\quad$ $T(0,1) = 3$

$T(0,9) = -61$

Now, along OA, $y=0$.

$\therefore T(x,0) = 2+2x-x^2$

So, $T'(x,0) = 2-2x = 0 \quad \therefore x = (1,0)$

So, points on OA $(0,0)$, $(1,0)$, $(9,0)$.

$\therefore f(0,0) = 2$

$f(9,0) = -61$

$f(1,0) = 3$

Now, along AB,

$x+y = 9$

or, $y = 9-x$.

So,

$f(x,y) = 2+2x+2(9-x)-(9-x)^2-x^2$

$= 2+2x+18-2x-(81-18x+x^2)-x^2$

$= 20-81+18x-x^2-x^2$

$= 18x-61-2x^2$.

So,

$f'(x,y) = f'(x,9-x) = 18-4x$

or $18-4x=0$

$\therefore x = 9/2$

So, $y = 9/2$

Points on AB, $(0,9)$, $(9,0)$, $(9/2, 9/2)$

$f(9/2, 9/2) = -20.5$.

Hence,

absolute maxima $= 4$ at $(1,1)$

absolute minima $= -61$ at $(0,9)$ and $(9,0)$

{Q7: find absolute maximum value and absolute minimum value of

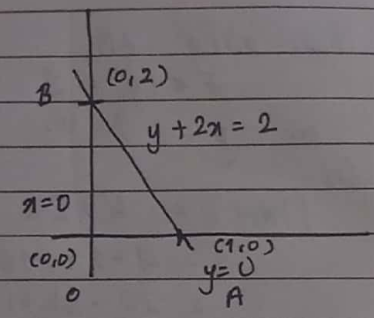
$$f(x,y) = x^2+y^2$$

on closed triangular plate $x=0$, $y=0$, $y+2x=2$ in the first quadrant.

Soln:

Given,

$f(x,y) = x^2+y^2$



For interior points,

$f_x = 2x = 0$ —(i)

$f_y = 2y = 0$ —(ii)

$\therefore (x,y) = (0,0)$

$f(0,0) = 0$

Along OB, $x=0$.

$T(0,y) = y^2$

$\therefore T'(0,y) = 2y = 0$

$\therefore y = 0$

$\therefore (x,y) = (0,0)$.

So, points on OB, $(0,0)$, $(0,2)$

$\therefore f(0,2) = 4$.

Along OA, $y=0$.

$T(x,0) = x^2$

$\therefore T'(x,0) = 2x = 0$ $\therefore x=0$ So, $(x,y) = (0,0)$

So, points on OA $(0,0)$, $(1,0)$

$f(1,0) = 1$

Along AB,

$y = 2-2x$

So,

$$f(x, 2-2x) = x^2+(2-2x)^2$$
$$= x^2+4-8x+4x^2$$
$$= 5x^2-8x+4$$

So,

$f'(x, 2-2x) = 10x-8 = 0$

$\therefore x = 4/5$

So, $y = 2/5$

So, points on AB $(4/5, 2/5)$, $(0,2)$, $(1,0)$

$\therefore f(4/5, 2/5) = 4/5 = 0.8$

Hence,

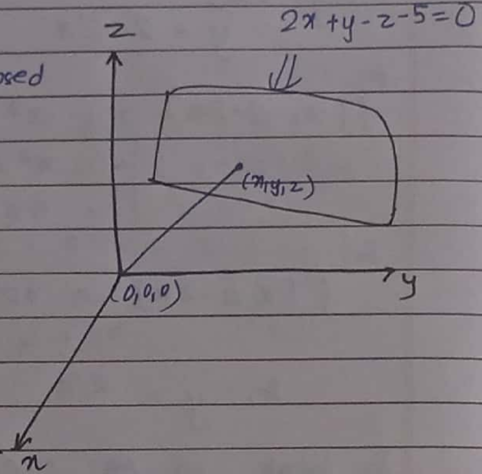
absolute minimum $= 0$ at $(0,0)$

absolute maxima $= 4$ at $(0,2)$

Constrained Maxima- Minima

Q7: Find the point (x,y,z) on the plane $2x+y-z-5=0$ that is closed to the origin.

Soln:



~~Now~~ Let $P(x,y,z)$ be the closed point.

$|OP| = d = \sqrt{x^2+y^2+z^2}$

We have to find x,y,z minimizes d. and x,y,z must be on the plane $2x+y-z-5=0$.

We need to minimize $f = x^2+y^2+z^2$ with constraint: $2x+y-z-5 = g$

Using Lagrange's multiplies,

$$\nabla f = \lambda(\nabla g)$$

So,

$$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda(2\hat{i} + \hat{j} - \hat{k})$$

or $2x\hat{i} + 2y\hat{j} + 2z\hat{k} = 2\lambda\hat{i} + \lambda\hat{j} - \lambda\hat{k}$

Equating corresponding components,

$2x = 2\lambda$ $\quad$ $2z = -\lambda$

$2y = \lambda$ $\quad$ and $\quad$ $2x+y-z = 5$

Now,

$x = \lambda$ $\quad$ $y = \frac{1}{2}\lambda$

$z = -\frac{1}{2}\lambda$

~~Now~~ Putting in constraint,

$$2 \times \lambda + \frac{1}{2} \times \lambda + \frac{1}{2}\lambda = 5$$

Multipling both sides by 2,

$$4\lambda + \lambda + \lambda = 10$$

$\therefore \lambda = 5/3$

$\therefore x = 5/3$ $\quad$ $y = 5/6$ $\quad$ $z = -5/6$

And.

distance $= \sqrt{(5/3)^2 + (5/6)^2 + (-5/6)^2}$

$= 2.04$ units.

Lagrange's Multiplies

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable and $\nabla g \neq 0$ when $g(x,y,z)=0$.

To find the local maximum and minimum values of subject to the constraint $g(x,y,z)=0$ (if there exists), find the values of x,y,z and that simultaneously ~~et~~ satisfy the equations $\nabla f = \lambda \nabla g$ and $g(x,y,z)=0$

For functions having ~~two~~ ~~ident~~ independent variables, the condition is similar, but without the variable z.

Q7: Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Soln:

Given,

$$f(x,y) = xy$$

and constraint,

$$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

So,

$$\nabla f = y\hat{i} + x\hat{j}$$

$$\nabla g = \frac{1}{4}x\hat{i} + y\hat{j}$$

Now,

$$\nabla f = \lambda(\nabla g)$$

or, $y\hat{i} + x\hat{j} = \frac{\lambda x}{4}\hat{i} + \lambda y\hat{j}$

Equating corresponding components,

$y = \frac{\lambda x}{4}$ ...(i) $\quad x = \lambda y$ ...(ii)

Putting (ii) in (i),

$y = \frac{\lambda^2 y}{4}$ or $y - \frac{\lambda^2}{4}y$

$y\left(1 - \frac{\lambda^2}{4}\right) = 0$

$\therefore y = 0, \quad \lambda = \pm 2$

Case 1: If $y = 0$, $x = 0$.

But $(0,0)$ doesn't exist on the ellipse. So, $y \neq 0$.

Case 2: If $y \neq 0$,

$\lambda = \pm 2$.

$x = \pm 2y$.

Now, putting in constraint,

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1$$

or $\frac{4y^2}{8} + \frac{y^2}{2}$ or, $2y^2 = 2$ $\quad \therefore y = \pm 1$

$\therefore x = \pm 2$

The extreme values exists at $(2,1), (-2,1)$, $(2,-1), (-2,-1)$

Now,

$f(2,1) = 2 \qquad f(2,-1) = -2$

$f(-2,1) = -2 \qquad f(-2,-1) = 2$

The greatest value $= 2$ at $(2,1)$ $(-2,-1)$

and

smallest value $= -2$ at $(-2,1)$, $(2,-1)$.

Q7: Find the extreme values of $f(x,y) = x^3+y^2$ on the circle $x^2+y^2 = 1$.

Soln:

Given function,

$$f(x,y) = x^3+y^2$$

given constraint;

$$g(x,y) = x^2+y^2-1 \quad \text{where } x^2+y^2 = 1 \quad \text{—(i)}.$$

Now,

~~$\nabla f = \lambda \nabla g$~~

$$\nabla f = 3x^2\hat{i} + 2y\hat{j}$$

$$\nabla g = 2x\hat{i} + 2y\hat{j}$$

We know,

$$\nabla f = \lambda \nabla g$$

or $3x^2\hat{i} + 2y\hat{j} = \lambda(2x\hat{i} + 2y\hat{j})$

or $3x^2\hat{i} + 2y\hat{j} = 2x\lambda\hat{i} + 2y\lambda\hat{j}$

Equating corresponding components,

$$3x^2 = 2\lambda x \quad \text{—— (ii)}$$

$$2y = 2\lambda y \quad \text{—— (iii)}$$

From (iii), $2y = 2\lambda y$

or $y(1-\lambda) = 0$

$\therefore y = 0, \quad \lambda = 1$

If $y = 0$, $x = \pm 1$

If $\lambda = 1$, $3x^2 = 2x$

or $x(3x-2) = 0$

$\therefore x = 0, \quad x = 2/3.$

If $x = 0$, $y = \pm 1$

If $x = 2/3$, $y = \pm\sqrt{5}/3$

So, the points are,

$(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$, $(2/3, +\sqrt{5}/3)$
$(2/3, -\sqrt{5}/3)$

Now,

$f(0,1) = 1$

$f(0,-1) = 1$

$f(1,0) = 1$

$f(-1,0) = -1$

$f(2/3, +\sqrt{5}/3) = 0.85$

$f(2/3, -\sqrt{5}/3) = 0.85$

$\therefore$ The maximum is 1 at $(0,\pm 1)$ & $(1,0)$
and
minimum is -1 at $(-1,0)$

(*) Lagrange's Multipliers with two constraints:

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

where, $g_1(x,y,z) = 0$. $g_2(x,y,z) = 0$

Q7: Minimize $f(x,y,z) = x^2+y^2+z^2$ subjected to the constraints $x+2y+3z=6$ and $x+3y+9z=9$

Soln:

Given,

$$f(x,y,z) = x^2+y^2+z^2$$

The two constraints are:

$$g_1(x,y,z) = x+2y+3z-6$$
$$g_2(x,y,z) = x+3y+9z-9$$

Now,

$$\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla g_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\nabla g_2 = \hat{i} + 3\hat{j} + 9\hat{k}$$

We know,

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

or, $2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda(\hat{i}+2\hat{j}+3\hat{k}) + \mu(\hat{i}+3\hat{j}+9\hat{k})$

or, $2x\hat{i} + 2y\hat{j} + 2z\hat{k} = (\lambda+\mu)\hat{i} + (2\lambda+3\mu)\hat{j} + (3\lambda+9\mu)\hat{k}$

Equating corresponding components, we get

$$x = \frac{\lambda+\mu}{2}, \quad y = \frac{2\lambda+3\mu}{2}, \quad z = \frac{3\lambda+9\mu}{2}$$

~~Substituting the values in g_1 and g_2,~~

$$\frac{\lambda+\mu}{2} + 2\left(\frac{2\lambda+3\mu}{2}\right) + 3\left(\frac{3\lambda+9\mu}{2}\right) = 6$$

or, $\lambda+\mu + 4\lambda+6\mu + 9\lambda + 27\mu = 12$

or, $14\lambda + 34\mu = 12$ — (a).

$$\frac{\lambda+\mu}{2} + 3\left(\frac{2\lambda+3\mu}{2}\right) + 9\left(\frac{3\lambda+9\mu}{2}\right) = 9$$

or, $\lambda+\mu + 12\lambda+18\mu + 54\lambda + 162\mu = 18$

or, $67\lambda + 181\mu = 18$ — (b).

Substituting the values in g_1 and g_2

$$\frac{\lambda+\mu}{2} + 2\left(\frac{2\lambda+3\mu}{2}\right) + 3\left(\frac{3\lambda+9\mu}{2}\right) = 0$$

or $7\lambda + 17\mu = 0$ — (a)

$$\frac{\lambda+\mu}{2} + 3\left(\frac{2\lambda+3\mu}{2}\right) + 9\left(\frac{3\lambda+9\mu}{2}\right) = 0$$

or $\frac{\lambda}{2} + \frac{\mu}{2} + \frac{6\lambda}{2} + \frac{9\mu}{2} + \frac{27\lambda}{2} + \frac{81\mu}{2} = 0$

$\lambda + \mu + 6\lambda + 9\mu + 27\lambda + 81\mu = 0$

or, $34\lambda + 91\mu = 0$ — (b)

Solving (a) and (b), we get. $\lambda = 240/59$ & $\mu = -78/59$

So, $x = 81/59$, $y = 123/59$, $z = 9/59$.

$\therefore$ Min(f) $= 369/59$ at $(81/59, 123/59, 9/59)$