Span:

If $\vec{v_1}, \vec{v_2}, \dots, \vec{v_p} \in \mathbb{R}^n$, then the set of all linear combinations of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_p}$ is denoted by span $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_p}\}$.

ie,

Span $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_p}\}$ is the collection of all vectors that can be expressed in the form $c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_p\vec{v_p}$

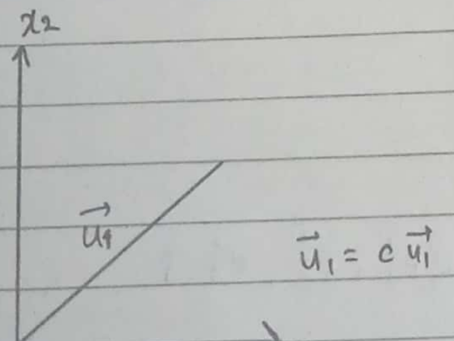
where, $c_1, c_2, \dots, c_p$ are scalars.

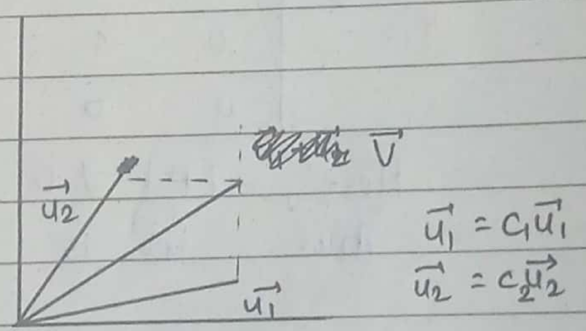
Note:

(i): b is span $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_p}\}$, then the system of linear equations repraented has solution.

(ii) If pivot positions of all the row exists, then the span existe

(iii) If the deteminant of the coefficient matrix is not equal to zero, then the span exists.





Span $(\vec{u})$

The span of $\vec{u}$ ie, all the vectors $c\vec{u_1}$ where c is scalar is about lines.

It exists through a line only.

span $(\vec{u}, \vec{v})$

$\vec{v} = \vec{u_1} + \vec{u_2}$

The span of $\vec{v}$ ie, span $(\vec{u_1}, \vec{u_2})$ ~~exists~~ or, the set of all vectors $c_1\vec{u_1} + c_2\vec{u_2}$ where, c_1 & c_2 are scalar is about plane.

It exists through a plane only.

<Q>: Prove that $\mathbb{R}^3$ is the span of vectors $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

ie, Is $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ span in $\mathbb{R}^3$?

Soln:

Writing $\vec{e_1}, \vec{e_2}, \vec{e_3}$ in matrix form,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here,
All the pivot elements of rows exists, thus
$\mathbb{R}^3$ is span of $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$.

OR,

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Here, $|A| \neq 0$.
Thus, $\mathbb{R}^3$ is span of $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$

OR,

Let $\vec{u} = c_1\vec{e_1} + c_2\vec{e_2} + c_3\vec{e_3}$ be an linear combination of $\vec{e_1}, \vec{e_2}, \vec{e_3}$

$$\vec{u} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or, $\vec{u} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \in \mathbb{R}^3$

Thus, $\mathbb{R}^3$ is span of $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$.

<Q>: Let $\vec{u_1} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$

Does span $\{\vec{u_1}, \vec{u_2}\} \in \mathbb{R}^3$ exists ?

Soln.

The coefficient matrix is.

$$\begin{bmatrix} 1 & 5 \\ -2 & -13 \\ 3 & -3 \end{bmatrix}$$

Here, the determinant doesn't exist.
Thus,
span $\{\vec{u_1}, \vec{u_2}\} \in \mathbb{R}^3$ doesn't exist.

Also, the pivot positions of all the rows doesn't exists.

[Q] For what value of n will $\vec{y}$ be in span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, if $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

and $\vec{y} = \begin{bmatrix} -4 \\ 3 \\ n \end{bmatrix}$

Soln.

Here,

$$\vec{y} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

Here, c_1, c_2, c_3 are scalars.

So,

$$\begin{bmatrix} -4 \\ 3 \\ n \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

which implies that,

$$c_1 + 5c_2 - 3c_3 = -4 \quad \text{— (i)}$$
$$-c_1 - 4c_2 + c_3 = 3 \quad \text{— (ii)}$$
$$2c_1 + 3c_2 + 0 = n \quad \text{— (iii)}$$

The augmented matrix is,

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ 2 & 3 & 0 & n \end{array}\right]$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & -7 & 6 & n+8 \end{array}\right]$$

Applying $R_3 \to R_3 + 7R_2$,

$$\sim \left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -8 & n+1 \end{array}\right]$$

From R_3; $-8c_3 = n+1$

$$\therefore c_3 = \frac{n+1}{-8}$$

From R_2, $c_2 - 2 \times c_3 = -1$

$$\therefore c_2 = -1 - \frac{(2n+2)}{4} = \frac{-6-2n}{4} = \frac{-(6+2n)}{4}$$

from R_1, $c_1 + 5c_2 - 3c_3 = -4$

$$\therefore c_1 = \frac{17n+25}{8}$$

From eqn (iii),

$$2 \times \left(\frac{17n+25}{8}\right) + 3 \times -1\left(\frac{6+2n}{4}\right) = n$$

or, $\dfrac{17n+25}{4} - \dfrac{18+6n}{4} = n$

or, $17n + 25 - 18 + 6n = 4n$

or, $7n = 7$ $\quad \therefore n = 1$.