SEQUENCE:

→ Arrangement of any object or set of numbes in a particular order followed by some rule.

Types:
i) Arithmetic: constant difference.
ii) Geometric: constant ratio
iii) # Fibonacci: $a_n = a_{n-1} + a_{n-2}$
iv) Harmonic: reciprocal of all terms of arithemetic sequence.
v) Finite: finite numbers of terms
vi) infinite: infinite number of terms.

Monotonic Sequence:

- The sequence that is always moving in one direction only. It is of two types: increasing monotonic and decreasing monotonic.

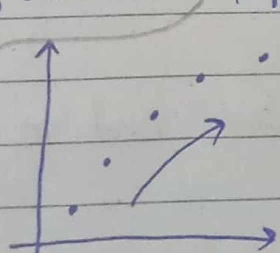
i) Increasing Monotonic Sequence

Here,
preceeding term is less than or equal to given term.

$a_1 \leq a_2 \leq a_3$ ......

$\frac{a_1}{a_2} \leq 1$

Graphically,



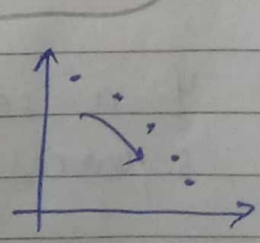
ii) Decreasing Monotonic Sequence.

Here,
preceeding term is greater than or equal to that given term.

$a_1 \geq a_2 \geq a_3$ ....

$\frac{a_1}{a_2} \geq 1$

Graphically,



Bounded sequence:

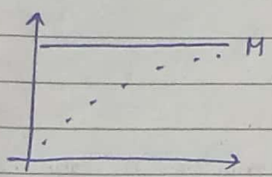
Bounded sequence are of two types:
Bounded above and bounded below.

i) Bounded above

→ A sequence $\{a_n\}$ is bounded above if there ~~is~~ exists a numbes M such that $a_n \le M$ for all n.

→ Sequence has a ceiling.

→ Here, M is upper bound of a_n.

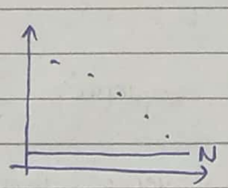


ii) Bounded below

→ A sequence $\{a_n\}$ is bounded below if there exists a numbes N such that $a_n \ge N$ for all n

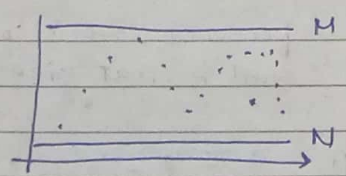
→ Sequence has a floor.

Here, N is lower bound of a_n.



A sequence is said to be simply bounded if

$$N \le a_n \le M.$$



Note: If a sequence $\{a_n\}$ is bounded and monotonic, sequence converges.

Eg: $a_n = \dfrac{1}{n^2}$

Soln:

Here,
$a_1 = 1 \quad a_2 = 1/4, \quad a_3 = 1/9, \quad a_4 = 1/16 \dots$

~~Hea~~ The sequence shows decreasing trend.

Checking for monotonic sequence:

$$a_n \ge a_{n+1}$$

or, $$\frac{1}{n^2} \ge \frac{1}{(n+1)^2}$$

Cross-multiplying, we get,

$$(n+1)^2 \ge n^2$$

or, $n^2 + 2n + 1 \ge n^2$

or, $2n + 1 \ge 0$ is true for all $n \ge 1$.

Hence, this is a decreasing monotonic sequence.

Checking for bound:

Since, the sequence is decreasing monotonic for $n \ge 1$, ~~$a_1 = 1$ is the lower~~, the first term is highest value.
Thus, a_n is bounded above $a_1 = 1$.

and
$1/n^2$ is always positive. So, a_n is bounded below 0.

Since sequence is bounded and monotonic, it converges.

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Eg: $a_n = \frac{3^n}{n!}$

Soln.

Here,

$a_1 = 3$ $a_2 = 4.5$ $a_3 = 4.5$ $a_4 = 27/8$ (3.375) $a_5 = 2.025$.

The sequence shows decreasing trend from a_2.

Checking for monotonic sequence:

$a_n \geq a_{n+1}$

or, $\frac{a_n}{a_{n+1}} \geq 1$

or, $\frac{3^n}{n!} \times \frac{(n+1)!}{3^{n+1}} \geq 1$

or, $\frac{\cancel{3^n} \times (n+1) \times \cancel{n!}}{\cancel{n!} \times \cancel{3^n} \times 3} \geq 1$

or, $(n+1) \geq 3$ or $n \geq 2$.

Here,
sequence is decreasing monotonic from $n \geq 2$.

Checking for bounded sequence:

Since this is decreasing monotonic sequence, it is bounded above $a_2 = 4.5$

and.
a_n is positive for $n \geq 2$, the sequence is bounded below 0.

Since the sequence is bounded and convergent, it converges.

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Convergence and Divergence of a Sequence

A sequence (a_n) converges to the number L if for every $\epsilon > 0$, there exists a corresponding integer N such that

$n > N \Rightarrow |a_n - L| < \epsilon$

If no such numbers L exist, we can say the series is divergent.

Here, if $\lim_{n \to \infty} a_n = L$ then, L = limit of sequence.

Note:

i) if $\lim_{n \to \infty} a_n = L$ (any finite value), it converges

ii) if $\lim_{n \to \infty} a_n = -\infty$ or ∞ or doesn't exist (infinite or undefined), it diverges.

Note: Generally, if the graph of sequence follows horizontal asymptote, it has a limit. So the sequence converges.

Sandwich Theorem of Sequence / Squeeze Theorem.

Let $\{a_n, b_n, c_n\}$ be a sequence of real numbers.

If $a_n \le b_n \le c_n$ holds for all n

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = L.$$

Questions: (Checking convergence or divergence)

i) $a_n = \dfrac{1}{3^n}$

Soln:

Here,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3^n}$$

$= 0$ is finite value.

Thus, the sequence converges.

(ii): $a_n = \dfrac{8n}{3n-5}$

Soln:

Here,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{8n}{3n-5}$$

$= \lim_{n\to\infty} \dfrac{8}{3}$ [Using L-Hopital]

$= \dfrac{8}{3}$ is finite value.

The sequence converges.

(iii) $a_n = \sin(n)$

Soln.

Here.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin(n)$$

it doesn't exist. because it oscillates between -1 and 1.

The sequence diverges.

(iv): $a_n = \cos\left(\dfrac{1}{n}\right)$

Soln.

Here,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right)$$

$= \cos 0 = 1$ is finite value.

The sequence converges.

(v): $a_n = \frac{\ln(n^4)}{5n}$

Soln:

Here,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n^4)}{5n}$

$= \lim_{n\to\infty} \frac{4 \ln n}{5n}$

$= \lim_{n\to\infty} \frac{4}{5} \times \frac{1}{n}$ [∵ Using L-Hopital rule]

$= 0$ ie, finite value

The sequence converges.

(vi): $a_n = \frac{(n+1)!}{n!}$

Soln:

Here,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)!}{n!}$

$= \lim_{n\to\infty} \frac{(n+1) \times n!}{n!}$

$= \infty$ ie, infinite value.

The sequence diverges.

(vii) $a_n = \frac{(n+1)!}{(n+2)!}$

Soln:

Here,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!}$

$= \lim_{n\to\infty} \frac{(n+1)!}{(n+2)(n+1)!}$

$= 0$ ie, finite value.

The sequence converges.

(viii): $a_n = \frac{4n}{\sqrt{n^2+5}}$

Soln:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n}{\sqrt{n^2+5}}$

$= \lim_{n\to\infty} \frac{4\cancel{n} \times 1/\cancel{n}}{\sqrt{n^2/n^2 + 5/n^2}}$

$= \lim_{n\to\infty} \frac{4}{\sqrt{1 + 5/n^2}}$

$= 4$ ie, finite value.

The sequence converges.

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(ix): $a_n = \left(1+\frac{1}{n}\right)^n$

Soln:

Here,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

Let $y = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

on $\ln y = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n$

$$= \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n}$$

$$= \lim_{n\to\infty} \frac{-1/n^2 \,/\, 1+1/n}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n}$$

on $\ln y = 1$

or, $y = e^1$

So,

$\lim_{n\to\infty} a_n = e$ ie, finite value.

The sequence converges.

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(x): $a_n = \frac{n}{2^n}$

Soln.

Here,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2^n}$$

$$= \lim_{n\to\infty} \frac{1}{2^n \times \ln(2)}$$

$= 0$ ie, finite value.

The sequence converges.

(xi): $a_n = \frac{1}{n^2}\sin(n)$.

Soln:

We know,

$$-1 \le \sin(n) \le 1$$

or, $$\frac{-1}{n^2} \le \frac{\sin(n)}{n^2} \le \frac{1}{n^2}$$

So,

$\lim_{n\to\infty} \frac{-1}{n^2} = \infty$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, then,

$\lim_{n\to\infty} \frac{\sin(n)}{n^2} = 0$ ie, finite value.

The sequence converges.

(xii) $a_n = \frac{1}{n}\sin(n)$.

Sol^n:

We know,

$-1 \le \sin(n) \le 1$

or $\frac{-1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$

Here,

$\lim_{n\to\infty} \frac{-1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$

So,

$\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$ i.e, finite value.

The sequence converges

(xiii): $a_n = \frac{\sin^2 n}{2n}$

Sol^n:

We know,

$-1 \le \sin(n) \le 1$

or, $0 \le \sin^2 n \le 1$

or $\frac{0}{2n} \le \frac{\sin^2 n}{n} \le \frac{1}{2n}$

Here,

$\lim_{n\to\infty} \frac{0}{2n} = 0$ and $\lim_{n\to\infty} \frac{1}{2n} = 0$

So,

$\lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$ i.e, finite value.

The sequence converges.

(xiv): If $\lim_{n\to\infty} \frac{1}{n} = 0$ converges, prove that $\lim_{n\to\infty} \frac{\cos n}{n}$ is convergent.

Sol^n:

We know,

$-1 \le \cos n \le 1$

or $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$

Here,

$\lim_{n\to\infty} \frac{-1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$

So,

$\lim_{n\to\infty} \frac{\cos n}{n} = 0$ i.e, finite value.

The sequence converges.