Chapter 3: Mechanics

WAVES AND OSCILLATIONS

Periodic Motion

If a motion repeats itself in equal interval of time, then the motion is said to be periodic motion.

Since displacement of a particle in periodic motion can be expressed in terms of sine or coseine, it is often called harmonic motion.

If a particle in periodic motion moves back and forth over the same path, the motion is said to be oscillatory or vibratory.

Simple Harmonic Motion (SHM)

Simple Harmonic motion is the motion of a body when the force acting on it is proportional to the body's displacement but in opposite direction.

Here,

$$F = -kx.$$

In simple harmonic motion, the acceleration 'a' of a body is proportional to its displacement in opposite direction and they are related by w_0^2.

ie, $a(t) = -w_0^2 x(t)$

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In SHM, displacement $\{x(t)\}$ is represented by
$x = A \sin(\omega_0 t + \phi)$ or $x = A \cos(\omega_0 t + \phi)$

Here,

x = displacement $\qquad \omega_0 t + \phi$ = phase $\qquad \omega_0$ = angular frequency

A = amplitude $\qquad \phi$ = phase angle $\qquad t$ = time.

Harmonic Oscillator

Body undergoing simple harmonic motion is called harmonic oscillator.

→ In SHM, displacement, velocity, acceleration are sinusoidal functions of time.

→ The angular frequency, frequency and time period of SHM depends upon the mass 'm' and force constant 'k'.

$$\omega = \sqrt{\frac{k}{m}} \qquad \therefore T = 2\pi \sqrt{\frac{m}{k}}$$

Linear Harmonic Oscillator:

A body with mass 'm' that moves under the influence of a Hooke's law restoring force exhibiting SHM is called linear simple harmonic oscillator.

Eg: block-spring oscillator.

Simple Hamonic Motion in Loaded spring:

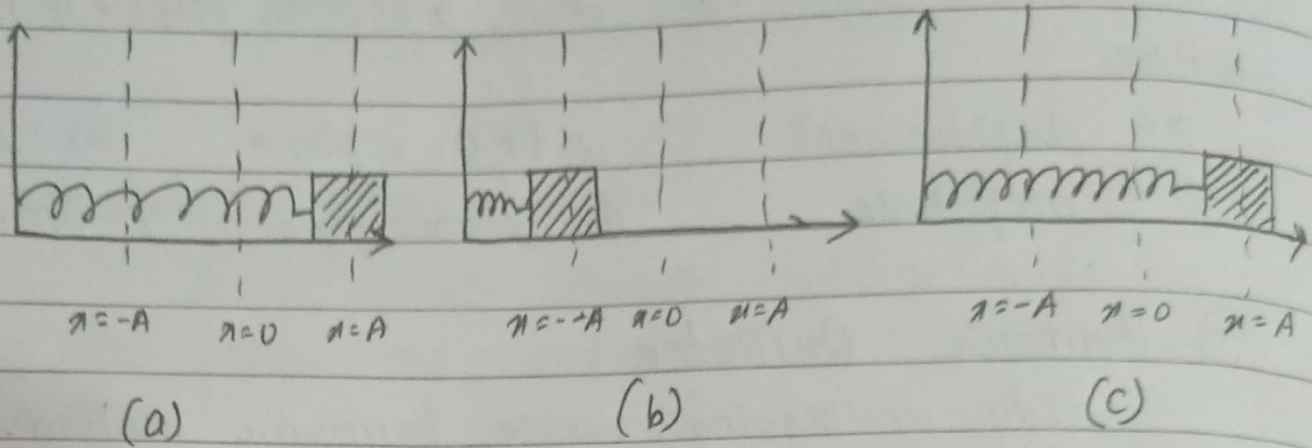

Fig: Oscillation in loaded spring.

Consider a horizontal spring of force constant 'k' whose one end is attached to a rigid support and the next end is attached to a body of mass 'm' rest on a frictionless surface.

Consider the spring is stretched to a distance A from its mean position $x=0$ and then released. The force exerted by the spring on the body at any position x from mean position is given by Hooke's law as.

$$F_{spring} = -kx \quad \text{———— (i)}$$

This of force is the resultant force exerted on the body which is also called restoring or elastic force.
But,
From Newton's second law of motion,

$$F_{resultant} = m\frac{d^2x}{dt^2} \quad \text{———— (ii)}$$

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So, equating (i) and (ii);

$$m\frac{d^2x}{dt^2} = -kx$$

or, $$\frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{——— (iii).}$$

We know, $\omega_0 = \sqrt{\frac{k}{m}}$ $\therefore \omega_0^2 = \frac{k}{m}$.

Putting in eq^n(iii), we get.

$$\frac{d^2x}{dt^2} = -\omega_0^2 x \quad \text{——— (iv)}$$

This represents the differential equation of the motion of the body attached to spring.

Also, we know,

$$v = \frac{dx}{dt}$$

Here, v = velocity at time t.

So, $$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = v \times \frac{dv}{dx}$$

Putting in eq^n(iv), we get.

$$v \cdot \frac{dv}{dx} = -\omega_0^2 x$$ or, $$v \cdot dv = -\omega_0^2 x\, dx \quad \text{—— (v)}$$

Integrating eq^n (v) on both sides, we get.

$$\int v \cdot dv = -\int \omega_0^2 x \, dx$$

or $\frac{1}{2}v^2 = -\omega_0^2 \times \frac{1}{2}x^2 + C$

or, $\frac{v^2}{2} = -\frac{\omega_0^2 x^2}{2} + C$ — (vi)

Here, C = constant of integration.

Now, when displacement is maximum, the velocity is equal to 0.

The maximum displacement is called amplitude (A).

So, at maximum displacement i.e, extreme position.

$x = \pm A$, $v = 0$.

Using in eq^n (vi),

$$0 = -\frac{1}{2}\omega_0^2 A^2 + C$$

$$\therefore C = \frac{1}{2}\omega_0^2 A^2$$

Hence, eq^n (vi) becomes.

$$\frac{1}{2}v^2 = \frac{-1}{2}\omega_0^2 x^2 + \frac{1}{2}\omega_0^2 A^2 \quad \text{—— (vii)}$$

or $v^2 = -\omega_0^2 x^2 + \omega_0^2 A^2$

or ~~$v = \omega_0\sqrt{A^2 - x^2}$~~ or, $v = \omega_0\sqrt{A^2 - x^2}$ —(viii)

or $\frac{dx}{dt} = \omega_0\sqrt{A^2 - x^2}$

or, $$\frac{dx}{\sqrt{A^2 - x^2}} = \omega_0\, dt \quad \text{—— (ix)}$$

Integrating eqn (ix) on both sides, we get.

$$\int \frac{dx}{\sqrt{A^2 - x^2}} = \int \omega_0\, dt$$

or, $\sin^{-1}\left(\frac{x}{A}\right) = \omega_0 t + \phi$

or, $x = A\sin(\omega_0 t + \phi)$ —— (x)

Here, ϕ = phase constant ie, another integration constant

Eqn (x) is the solution of eqn (iv).

Replacing $t = t + \frac{2\pi}{\omega_0}$ in eqn (x).

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So, new displacement,

$$x' = A\sin\left(\omega_0\left(t+\frac{2\pi}{\omega_0}\right)+\phi\right)$$

$$= A\sin\left(\frac{\cancel{\omega_0}(\omega_0 t+2\pi)}{\cancel{\omega_0}}+\phi\right)$$

$$\text{or } x' = A\sin(\omega_0 t+2\pi+\phi)$$

$$= A\sin(\omega_0 t+\phi) = x$$

Here, the body's position is ~~react~~ repeated after every $(2\pi/\omega_0)$ time.

So, this is called time period of oscillation and given by,

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \quad \text{--- (xi)}$$

The velocity in terms of time,
differentiating eqn (x) w.r.t t,

$$\frac{dx}{dt} = \frac{d}{dx}\left(A\sin(\omega_0 t+\phi)\right)$$

$$\text{or, } v = A\omega_0\cos(\omega_0 t+\phi) \quad \text{--- (xii)}$$

Now,
the kinetic energy and potential energy of the body in term of position,

$$K(x) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2(A^2-x^2)$$

and

$$U(x) = \frac{1}{2}kx^2$$

The kinetic energy and potential energy of the body in terms of time,

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t+\phi)$$

$$= \frac{1}{2}KA^2 \cos^2(\omega_0 t + \phi)$$

and

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \sin^2(\omega_0 t + \phi)$$

~~So, the total energy~~ $\cancel{(E_0) = K + U}$

$$\cancel{= \frac{1}{2}m\omega_0^2 A^2 - \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{2}kx^2}$$

So, the total energy $(E_0) = K(t) + U(t)$

$$= \frac{1}{2}m\omega_0^2 A^2 \cos^2(\omega_0 t + \phi) + \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}m\omega_0^2 A^2 \quad \therefore E_0 = \frac{1}{2}KA^2$$

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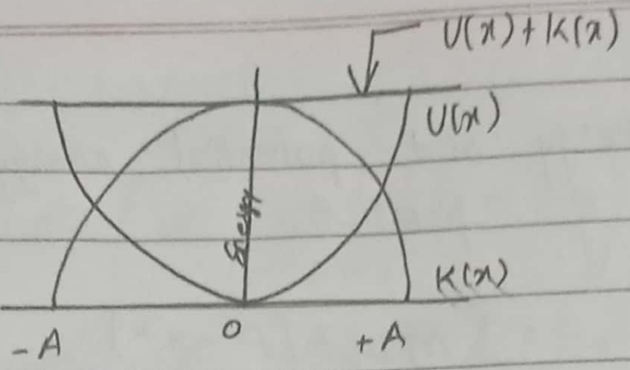


~~Fig.~~ Fig: Energy vs position graph

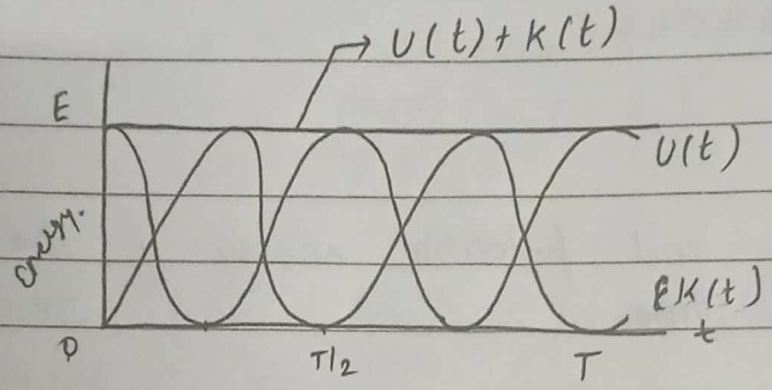


Fig: Energy vs time graph

The kinetic energy is maximum at mean position and minimum at extreme position.
Hence,

$KE_{max} = \frac{1}{2} kA^2$ $\qquad$ $KE_{min} = 0$.

The average KE

$$KE_{avg} = \frac{KE_{max} + KE_{min}}{2} = \frac{\frac{1}{2} kA^2 + 0}{2}$$

$$\therefore KE_{avg} = \frac{1}{4} kA^2$$

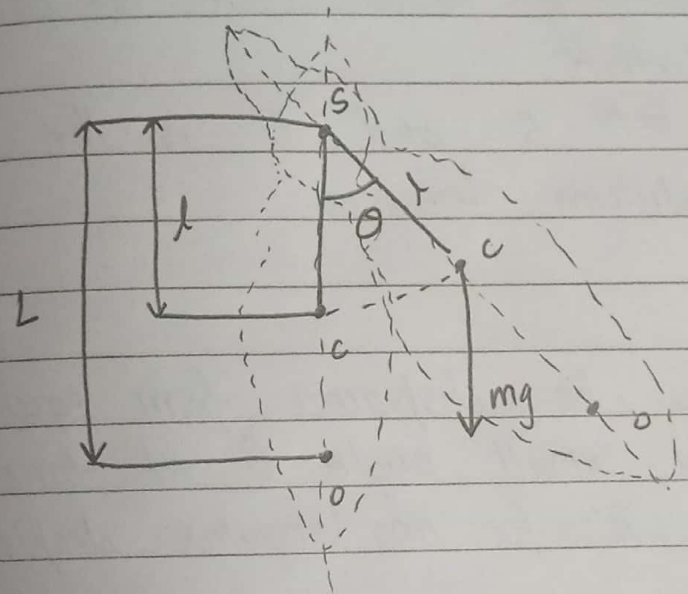
The potential energy is maximum at extreme position and minimum at mean position. The maximum potential energy (U_{max}) = $\frac{1}{2} kA^2$ and minimum potential energy (U_{min}) = 0.

The average PE,

$$U_{E\,avg} = \frac{U_{min} + U_{max}}{2} = \frac{0 + \frac{1}{2}kA^2}{2}$$

$$\therefore U_{E\,avg} = \frac{1}{4}kA^2.$$

Compound (Physical) Pendulum:



The physical (compound) pendulum is a rigid body of any shape capable to oscillate in a vertical plane about a horizontal axis passing through it.

The point of intersection of vertical plane and horizontal axis is called point of suspension S.

C is the centre of mass of the pendulum. The distance between S and C is the length of the pendulum l.

The compound pendulum can be converted into a simple pendulum by concentrating the whole mass of the pendulum at a point.

When the mass of compound pendulum is concentrated at a point to form a simple pendulum such that the time period of the resulting simple pendulum is equal to that of the compound pendulum, the point of concentration is called the point of oscillation.

The distance betn S and O is the length of equivalent simple pendulum L.

Now, the rigid body be displaced from equilibrium position ~~posit~~ by a small angle θ at time 't'. The restoring torque τ for an angular displacement θ,

$$\tau = -mgl\sin\theta \quad \text{— (i)}$$

Now, if I be the moment of inertia of a body about an axis of rotation and $\left(\frac{d^2\theta}{dt^2}\right)$ is its angular acceleration.

From Newton's second law of motion, the restoring torque is

$$\tau_{res} = I\frac{d^2\theta}{dt^2} \quad \text{— (ii)}$$

Therefore, equating (i) and (ii).

$$I \cdot \frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

or, $$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I}\sin\theta \quad \text{—— (iii)}$$

For sufficiently small angular displacement, $\sin\theta \approx \theta$ (radian).

So, eq^n (iii) becomes,

$$\frac{d^2\theta}{dt^2} \approx -\frac{mgl}{I}\theta \quad \text{—— (iv)}$$

We know,

$$a = -\omega_0^2 x \quad \text{—— (v)} \qquad \text{where, } \omega_0 = \sqrt{\frac{mgl}{I}}$$

Comparing (iv) and (v),

$$\omega_0 = \sqrt{\frac{mgl}{I}}$$

Hence, the time period of the compound pendulum is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I}{mgl}} \quad \text{—— (v).}$$

Now,

If I_g = moment of inertia.

K = radius of gyration of the compound pendulum about the horizontal axis passing through the centre of mass are parallel to the axis passing through center of suspension.

Using theorem of parallel axes,

$$I = I_g + ml^2$$

or $$I = mK^2 + ml^2 = m(K^2 + l^2)$$

Hence, the time period of oscillation.

$$T = 2\pi\sqrt{\frac{m(K^2+l^2)}{mgl}}$$

$$= 2\pi\sqrt{\frac{K^2/l + l}{g}} \quad \text{— (vi)}$$

or, $$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{— (vii)}.$$

Thus, the time period of physical pendulum is same as that of simple pendulum of the length $L = \frac{K^2}{l} + l$.

This length $\left(L = \frac{K^2}{l} + l\right)$ is the length of equivalent simple pendulum and reduced length.

Now, consider a point O on the other side C on a line with SC produced such that
$SO = l + \frac{K^2}{l}$ or, $CO = \frac{K^2}{l}$.

Here, O = point of oscillation.
S = point of suspension

Now, if the pendulum is inverted and made to oscillate about O,
the new time period (T') is obtained by substituting K^2/l in place of l in eqn (vi).

$$T' = 2\pi \sqrt{\frac{\frac{K^2}{K^2/l} + K^2 l}{g}}$$

$$T' = 2\pi \sqrt{\frac{K^2/l + l}{g}} = T \quad \text{— (viii)}$$

Thus, the time period about the center of oscillation is same as the time period about center of suspension.

Hence, O and S are interchangeable or reciprocal to each other.

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Rearranging eq^n (vi);

$$l^2 - \frac{T^2 g}{4\pi^2} l + k^2 = 0 \quad \text{———} \quad (ix)$$

Eq^n(ix) is quadratic in nature for l.
Let l_1 and l_2 be two roots of l.

So

$$l_1 + l_2 = \frac{T^2 g}{4\pi^2} \qquad l_1 l_2 = k^2$$

That means;

$$T = 2\pi \sqrt{\frac{l_1 + l_2}{g}}$$

Here, sum and product of l_1 and l_2 are ~~proo~~ positive.

Therefore, for any value of 'T', there are two points at l_1 and l_2 ~~from~~ center of gravity and same side of it.

So, there must be two other points on the other side of the center of gravity for which the time period will be same.

Hence, there are four points collinear with center of gravity about which the time period is the same.

The graph of T versus l is shown in figure.

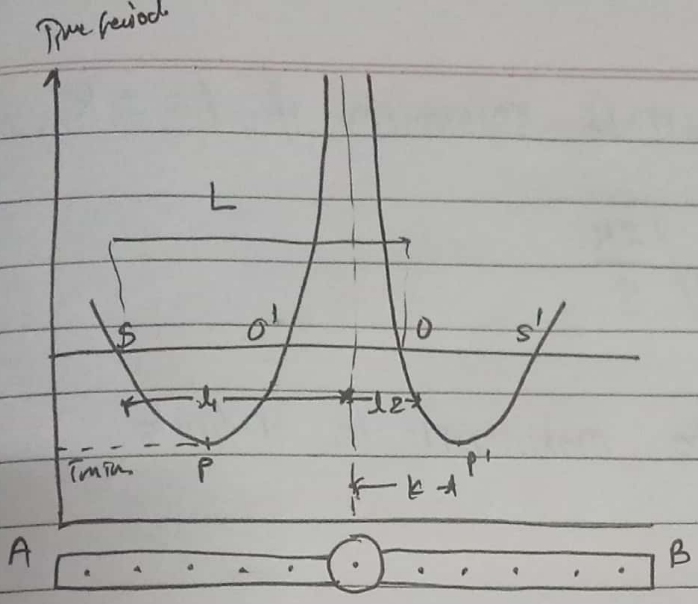

Fig: T.v.l graph.

* Condition for minimum time period:

From eq^n (ix);

$$T^2 = \frac{4\pi^2}{g}\left(\frac{K^2}{l} + l\right)$$

Differenting w.r.t. l,

$$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left(-\frac{K^2}{l^2} + 1\right)$$

For T to be minimum; $\frac{dT}{dl} = 0$

we have,

$$-\frac{K^2}{l^2} + 1 = 0$$

or, $l^2 = K^2$

$\therefore l = \pm K$

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ie, the time period will be minimum if $l = \pm K$.
Then,

$$T_{min} = 2\pi \sqrt{\frac{2K}{g}}$$

The time period will be maximum ie, infinite when $l = 0$.

Damped Harmonic Oscillation:

Damped Harmonic Oscillation are vibrating systems for which the amplitude of vibration decreases over time ~~to zero~~. In damped harmonic motion, the amplitude of oscillation gradually decreases to zero due to friction. The resistive force is called damping force.

Let us consider a horizontal spring of force constant 'k' loaded with mass 'm' and set to oscillate freely.
The angular frequency of oscillation $(\omega_0) = 2\pi \sqrt{\frac{k}{m}}$.

The ~~load~~ when resistive force acts on spring, the load of mass 'm' experiences two forces.

$F_{spring} = -kx$ — (i)

and

$F_{damp} = -bv = -b\frac{dx}{dt}$ — (ii)

The resultant force exerted on the load is

$$F = F_{spring} + F_{damp}$$
$$= -kx - b \cdot \frac{dx}{dt} \quad \text{--- (iii)}$$

From Newton's second law of motion,

$$F = ma$$

or, $F = m \times \frac{d^2x}{dt^2}$.

So, eq^n (iii) becomes,

$$m \frac{d^2x}{dt^2} = \cancel{F} - kx - b\frac{dx}{dt}$$

or, $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$ —— (iv)

Let $\frac{b}{m} = 2\gamma$ i.e, $\gamma = \frac{b}{2m}$ called damping constant.

The reciprocal of damping constant $\left(\tau = \frac{1}{2\gamma} = \frac{m}{b}\right)$ is called relaxation time.

We know,

$$\omega_0^2 = \frac{k}{m}$$

So, eq^n (iv) becomes,

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{—— (v)}$$

Eqⁿ (v) is the differential equation of damped harmonic oscillation.

Let us suppose $x = Ae^{\alpha t}$ and substiting in eqⁿ (v),

$$\alpha^2 Ae^{\alpha t} + 2\gamma\alpha Ae^{\alpha t} + \omega_0^2 Ae^{\alpha t} = 0$$

or, $\alpha^2 + 2\gamma\alpha + \omega_0^2 = 0$ because $Ae^{\alpha t} \neq 0$

So,

$$\alpha = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Let $\beta = \sqrt{\gamma^2 - \omega_0^2}$

So,

$$\alpha = -\gamma \pm \beta \quad \text{——— (vi)}$$

So, $\beta = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{\dfrac{1}{4\tau^2} - \omega_0^2}$ ——— (vii)

So, from eqⁿ (vi) : $\alpha = -\gamma + \beta$ and $\alpha = -\gamma - \beta$

So, eqⁿ(v) has two solutions:

$x_1 = A_1 e^{(-\gamma+\beta)t}$ and $x_2 = A_2 e^{(-\gamma-\beta)t}$

So, complete solⁿ $x = x_1 + x_2$

$$x = A_1 e^{(-\gamma+\beta)t} + A_2 e^{(-\gamma-\beta)t}$$

$$= A_1 e^{-\gamma t + \beta t} + A_2 e^{-\gamma t - \beta t}$$

$\cancel{= A_1 e}$

or $x = e^{-\gamma t}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right)$ ——— (viii)

Case 1: Overdamped motion

If the damping force is very high, then
$\gamma >> \omega_0$ i.e, $\frac{1}{4\tau^2} > \omega_0^2$ and β is a real.

[∵ From eq^n (viii)] but A_1 & A_2 are constants from the initial condition that has to be determined.

In this case,
the damping is so large due to which oscillation doesn't take place and particle returns to its equilibrium position gradually.

Here, amplitude decreases without any oscillation.
This is called over-damping
aperiodic
inharmonic.

Case 2: Critically Damped Motion

In this case,
$\beta = 0$ ie, $\frac{1}{4\tau^2} = \omega_0^2$

To prevent solution breakdown, if damping force is normal,
γ is slightly greater than ω_0
β is very small but real.

So eq^n(viii) becomes:

$$x = e^{-\gamma t}\left(A_1 e^{\beta t} + A_2 e^{-\beta t}\right)$$

$$= e^{-\gamma t}\left[A_1\left(1+\beta t + \frac{(\beta t)^2}{2}\dots\right) + A_2\left(1-\beta t + \frac{(\beta t)^2}{2} - \dots\right)\right]$$

$$\left[\because e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \dots, \quad e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{2} - \dots\right]$$

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Here. if $\lim_{\beta \to 0}$, terms with β^2 and highest order of β can be neglected.

$$x = e^{-\gamma t}(A_1 + A_1\beta t + A_2 - A_2\beta t)$$
$$= e^{-\gamma t}[(A_1 + A_2) + (A_1 - A_2)\beta t]$$

Let $A_1 + A_2 = M$ and $\beta(A_1 - A_2) = N$.

$$\therefore x = e^{-\gamma t}(M + Nt)$$

Here, the amplitude decreases faster than in case I. This situation is called critically damping.

(*) Case III : Under damped motion ie, Damped Oscillating Motion

If damping force is very low,

$\gamma << \omega_0$ ie, $\frac{1}{4\tau^2} < \omega_0^2$.

Hence, β will be complex.

$$\beta = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{-(\omega_0^2 - \gamma^2)} = i\sqrt{\omega_0^2 - \gamma^2} = i\omega$$

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \quad \text{—— (ix)}$$

So, eq^n (viii) becomes:

$$x = e^{-\gamma t}(A_1 e^{i\omega t} + A_2 e^{-i\omega t})$$
$$= e^{-\gamma t}[A_1(\cos\omega t + i\sin\omega t) + A_2(\cos\omega t - i\sin\omega t)]$$
$$= e^{-\gamma t}[(A_1 + A_2)\cos\omega t + (A_1 - A_2)i\sin\omega t]$$

let $C = A_1 + A_2$ and $D = (A_1 - A_2)i$

$$x = e^{-\gamma t}\left[C\cos\omega t + D\sin\omega t\right]$$

$$= e^{-\gamma t} \times \sqrt{C^2+D^2}\left(\frac{C}{\sqrt{C^2+D^2}}\cos\omega t + \frac{D}{\sqrt{C^2+D^2}}\sin\omega t\right)$$

$$= ae^{-\gamma t}(\sin\phi\cos\omega t + \cos\phi\sin\omega t)$$

ie, $x = ae^{-\gamma t}(\sin \omega t+\phi)$ ——— (x)

Here, $a = \sqrt{C^2+D^2}$, $\sin\phi = \frac{C}{\sqrt{C^2+D^2}}$, $\cos\phi = \frac{D}{\sqrt{C^2+D^2}}$

According to eq^n (x),
amplitude part $ae^{-\gamma t}$ decreases exponentially with time and finally it becomes zero.
but
motion is still periodic with angular frequency 'ω' and phase angle ϕ.
This is called underdamping.

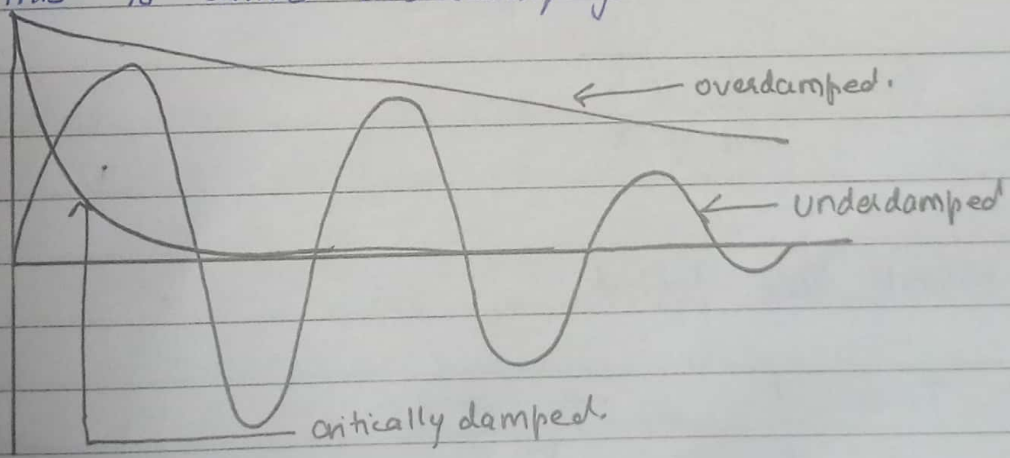

Fig: different types of damping.

Time period of the damped harmonic oscillatory motion is the time interval between two successive maxima and minima in the damped oscillator.

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \frac{1}{4\tau^2}}} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

The frequency of damped oscillator is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

From eq^n (X), the amplitude of damped oscillating motion at instant t is $ae^{-\gamma t}$.

If $b = 0$ or damping is negligibly small, the motion of the osillator is undamped and the frequency is called the natural frequency and is

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

So, natural time period.

$$T = \frac{1}{f_0} = 2\pi\sqrt{\frac{m}{k}}$$

Forced or Driven Harmonic Oscillation

When a body is displaced and then released, the body oscillates with its own natural frequency is given by

$$\omega_0 = \sqrt{\frac{k}{m}}$$ (in absence of frictional force)

and,

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$ (in presence of small friction force)

The damping can be overcome by applying some external oscillatory force on the body.

As a result, its own frequency immediately dies out and the body starts to oscillate with frequency of external oscillating force.

This type of oscillation is called forced/driven harmonic oscillation.

Now, when frequency of external force is the same as that of natural frequency of the oscillator, the amplitude of oscillation will be maximum.

This condition of obtaining maximum amplitude of oscillation is called resonance and the frequency at which resonance occurs is called resonant frequency.

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By Newton's second law of motion, the equation of motion of driven oscillator is

$$F = ma \quad \text{——— (i)}$$

In eq^n (i), F = sum of restoring force and damping force.

$$F_{rest} = -kx \quad \text{— (ii)} \qquad \text{and} \qquad F_{damp} = -b\frac{dx}{dt} \quad \text{— (iii)}$$

Let the applied external oscillating force is

$$F_{ext} = F_0 e^{i\omega' t}$$

Here,

F_0 = amplitude of external force.

ω' = angular frequency of external force.

The resultant force experienced by the body is.

$$F = F_{spring} + F_{damp} + F_{ext}$$

$$= -kx - b\frac{dx}{dt} + F_0 e^{i\omega' t} \quad \text{— (iv)}$$

From ~~of~~ Newton's second ~~of~~ law of motion, we have, $F = m\frac{d^2x}{dt^2}$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 e^{i\omega' t}$$

or, $$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = F_0 e^{i\omega' t} \quad \text{——— (v).}$$

Let $\frac{b}{m} = \frac{1}{\tau} = 2\gamma$, $\frac{k}{m} = \omega_0^2$ and $\frac{F_0}{m} = f_0$.

So, eqⁿ (v) becomes;

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = f_0 e^{i\omega' t} \quad \text{—(vi)}.$$

Here,
b = positive constant
k = force constant of the spring
γ = damping constant
m = mass of the load.

Eqⁿ(vi) is the differential eqⁿ of forced oscillation.
Let us assume,

$$x = Ae^{i(\omega' t+\phi)} \quad \text{—— (vii)}$$

Here,
A = amplitude of the oscillation
ϕ = phase angle

Substituting in eqⁿ (vi), we get.

$$\frac{d^2\left(Ae^{i(\omega' t+\phi)}\right)}{dt^2} + 2\gamma \frac{d\left(Ae^{i(\omega' t+\phi)}\right)}{dt} + \omega_0^2\left(Ae^{i(\omega' t+\phi)}\right) = f_0 e^{i\omega' t}$$

or, $-\omega'^2 Ae^{i(\omega' t+\phi)} + i2\gamma\omega' Ae^{i(\omega' t+\phi)} + \omega_0^2 Ae^{i(\omega' t+\phi)} = f_0 e^{i\omega' t}$.

or, $-\omega'^2 Ae^{i\phi} + i2\gamma\omega' Ae^{i\phi} + \omega_0^2 Ae^{i\phi} = f_0$

or, $-\omega'^2 A + i2\gamma\omega' A + \omega_0^2 A = f_0 e^{-i\phi} = f_0(\cos\phi - i\sin\phi)$

or, $(\omega_0^2 - \omega'^2)A + i2\gamma\omega' A = f_0\cos\phi - if_0\sin\phi \quad \text{—— (viii)}$

Equating real and imaginary parts of eqⁿ (viii), we get.

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$$(\omega_0^2 - \omega'^2)A = f_0 \cos\phi \quad \text{—(ix)}$$
$$2\gamma\omega' A = -f_0 \sin\phi \quad \text{—(x)}$$

Squaring and adding (ix) and (x), we get

$$(\omega_0^2 - \omega'^2)^2 A^2 + 4\gamma^2\omega'^2 A^2 = f_0^2\cos^2\phi + f_0^2\sin^2\phi$$

or, $$\left[(\omega_0^2 - \omega'^2)^2 + 4\gamma^2\omega'^2\right] A^2 = f_0^2$$

or, $$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega'^2)^2 + 4\gamma^2\omega'^2}} \quad \text{—— (xi)}$$

Dividing eq^n (x) by (ix), we get.

$$\frac{2\gamma\omega' A}{(\omega_0^2 - \omega'^2)A} = \frac{-f_0 \sin\phi}{f_0 \cos\phi} \quad \text{or,} \quad \frac{\sin\phi}{\cos\phi} = \frac{2\gamma\omega'}{\omega_0^2 - \omega'^2}$$

or, $$\tan\phi = \frac{2\gamma\omega'}{\omega'^2 - \omega_0^2}$$

$$\therefore \phi = \tan^{-1}\left(\frac{2\gamma\omega'}{\omega'^2 - \omega_0^2}\right) \quad \text{—(xii)}$$

Hence, eq^n(7) becomes.

$$x = \frac{f_0}{\sqrt{(\omega_0^2 - \omega'^2)^2 + 4\gamma^2\omega'^2}} \exp\left[i\left(\omega' t + \tan^{-1}\left(\frac{2\gamma\omega'}{\omega'^2 - \omega_0^2}\right)\right)\right]$$

—— (xiii)

Eq^n (xiii) gives the displacement of the load at any time in the forced oscillation.

(*) Case I:

For no damping ie, forced free oscillation.

$\gamma = 0$.

So

$$A = \frac{f_0}{\omega_0^2 - \omega'^2} \quad \text{—— (xiv)}$$

The amplitude goes to infinity as $\omega' \approx \omega_0$

(*) Case II:

The amplitude has maximum value at a frequency of external oscillating force.

To get maximum amplitude, we have.

$$\frac{dA}{d\omega'} = 0$$

or, $$\frac{d}{d\omega'}\left(\frac{f_0}{\sqrt{(\omega_0^2-\omega'^2)^2+4\gamma^2\omega'^2}}\right) = 0$$

or, $$f_0\left(\frac{-1}{2}\right)\left[(\omega_0^2-\omega'^2)^2+4\gamma^2\omega'^2\right]^{\frac{-3}{2}}\left[2(\omega_0^2-\omega'^2)(-2\omega')+8\gamma^2\omega'\right] = 0$$

or, $$-4\omega'(\omega_0^2-\omega'^2-2\gamma^2) = 0$$

ie, $$(\omega_0^2-\omega'^2-2\gamma^2) = 0$$

$$\therefore \omega' = \sqrt{\omega_0^2-2\gamma^2} \quad \text{—— (xv)}$$

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Hence, the amplitude is maximum when the frequency of external oscillating force is equal to

$$\omega' = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$$

This situation of maximum amplitude is called resonance.

Resonance is the phenomenon of making the amplitude maximum by matching the frequency of external oscillating force with the frequency of free or natural oscillation.

The maximum amplitude is;

$$A(max) = \frac{f_0}{\sqrt{(\omega_0^2 - (\omega_0^2 - 2\gamma^2))^2 + 4\gamma^2(\omega_0^2 - 2\gamma^2)}}$$

$$= \frac{f_0}{\sqrt{4\gamma^{4} + 4\gamma^2\omega_0^2 - 8\gamma^{4}}}$$

$$= \frac{f_0}{\sqrt{4\gamma^2\omega_0^2 - 4\gamma^4}} \qquad \therefore A_{max} = \frac{f_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$$

This means, the maximum amplitude is achieved as damping decreases.

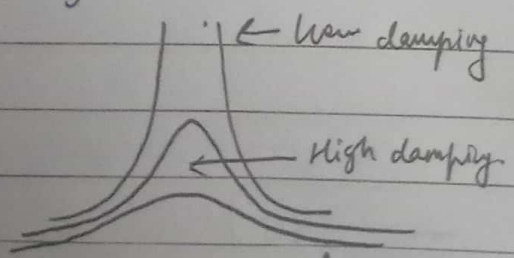

Fig: Amplitude vs external frequency.