# Advanced Calculus
# Functions of Several Variables

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Absolute - Constrained Extreme Values

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# Absolute Extreme Values

## Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region $R$ into three steps.

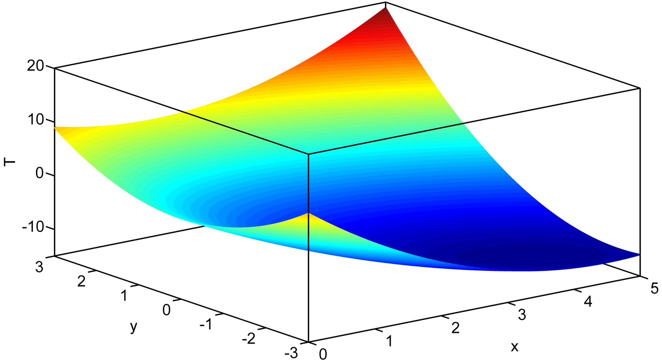
1. *List the interior points of R* where $f$ may have local maxima and minima and evaluate $f$ at these points. These are the critical points of $f$.
2. *List the boundary points of R* where $f$ has local maxima and minima and evaluate $f$ at these points. We show how to do this shortly.
3. *Look through the lists* for the maximum and minimum values of $f$. These will be the absolute maximum and minimum values of $f$ on $R$. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of $f$ appear somewhere in the lists made in Steps 1 and 2.

# Absolute Extreme Values

Find absolute maximum value and absolute minimum value of

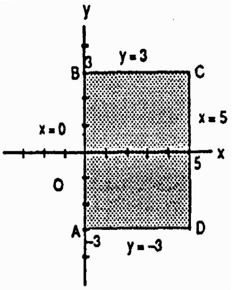
$$T(x, y) = x^2 + xy + y^2 - 6x$$

on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 3$.



# Solution

Given function: $T(x, y) = x^2 + xy + y^2 - 6x$, $0 \leq x \leq 5$, $-3 \leq y \leq 3$.




(i) On AB, $T(x, y) = T(0, y) = y^2$ on $-3 \leq y \leq 3$;
$T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $T(0, 0) = 0$,
$T(0, -3) = 9$, and $T(0, 3) = 9$

(ii) On BC, $T(x, y) = T(x, 3) = x^2 - 3x + 9$ on $0 \leq x \leq 5$;
$T'(x, 3) = 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$ and $y = 3$;
$T\left(\frac{3}{2}, 3\right) = \frac{27}{4}$ and $T(5, 3) = 19$

(iii) On CD, $T(x, y) = T(5, y) = y^2 + 5y - 5$ on
$-3 \leq y \leq 3$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and
$x = 5$; $T\left(5, -\frac{5}{2}\right) = -\frac{45}{4}$, $T(5, -3) = -11$ and $T(5, 3) = 19$

(iv) On AD, $T(x, y) = T(x, -3) = x^2 - 9x + 9$ on $0 \leq x \leq 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$;
$T\left(\frac{9}{2}, -3\right) = -\frac{45}{4}$, $T(0, -3) = 9$ and $T(5, -3) = -11$

(v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2 \Rightarrow (4, -2)$ is an interior critical point with $T(4, -2) = -12$. Therefore the absolute maximum is 19 at $(5, 3)$ and the absolute minimum is $-12$ at $(4, -2)$.

# Finding Absolute Extreme Values

Find absolute maximum value and absolute minimum value of

$$f(x, y) = x^2 + y^2$$

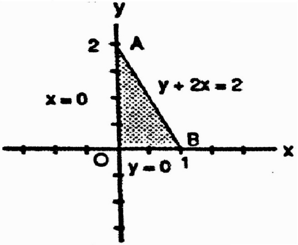
on the closed triangular plate $x = 0$, $y = 0$, $y + 2x = 2$ in the first quadrant.

Ans: Next Slide

# Solution

Given function: $f(x, y) = x^2 + y^2$, $x = 0$, $y = 0$, $y + 2x = 2$.

(i) On OA, $f(x, y) = f(0, y) = y^2$ on $0 \le y \le 2$;
$f'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $f(0, 0) = 0$ and
$f(0, 2) = 4$




(ii) On OB, $f(x, y) = f(x, 0) = x^2$ on $0 \le x \le 1$;
$f'(x, 0) = 2x = 0 \Rightarrow x = 0$ and $y = 0$; $f(0, 0) = 0$ and
$f(1, 0) = 1$

(iii) On AB, $f(x, y) = f(x, -2x + 2) = 5x^2 - 8x + 4$ on
$0 \le x \le 1$; $f'(x, -2x + 2) = 10x - 8 = 0 \Rightarrow x = \frac{4}{5}$
and $y = \frac{2}{5}$; $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$; endpoint values have been found above.

(iv) For interior points of the triangular region, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0 \Rightarrow x = 0$ and $y = 0$,
but $(0, 0)$ is not an interior point of the region.

Therefore the absolute maximum is 4 at $(0, 2)$ and the absolute minimum is 0 at $(0, 0)$.

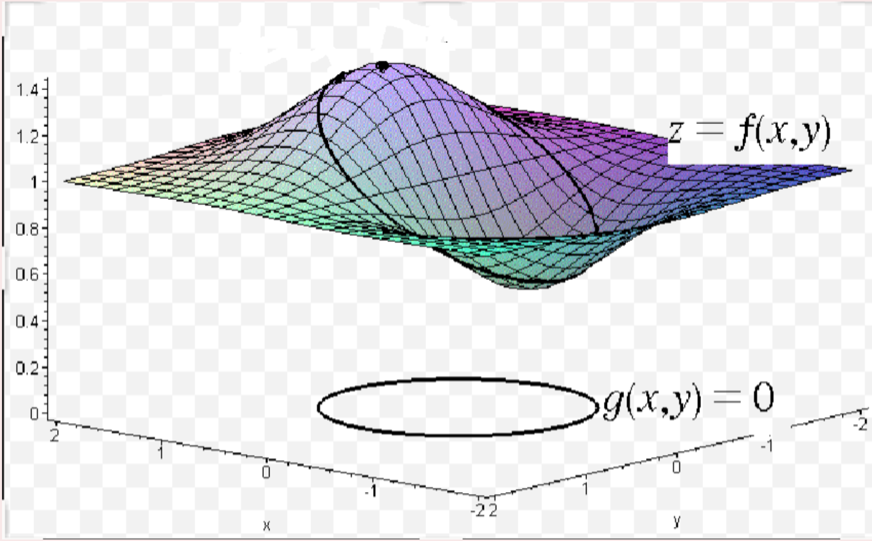
# Constrained Maxima and Minima

**Example**

Find the point $(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Closest Point: (5/3,5/6,-5/6)     Distance: 2.04 (Approx.)

## Lagrange Multiplier



### Lagrange Multiplier

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of $x$, $y$, $z$, and that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0.$$

For functions of two independent variables, the condition is similar, but without the variable $z$.

**EXAMPLE** Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

# Finding Extreme Values

**EXAMPLE** Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1.$

**Solution** We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of $x$, $y$, and $\lambda$ for which $\nabla f = \lambda \nabla g$ and $g(x, y) = 0.$

The gradient equation gives $y\mathbf{i} + x\mathbf{j} = \dfrac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$

from which we find $y = \dfrac{\lambda}{4}x, \quad x = \lambda y, \quad$ and $\quad y = \dfrac{\lambda}{4}(\lambda y) = \dfrac{\lambda^2}{4}y,$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

**Case 1:** If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

**Case 2:** If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation

$g(x, y) = 0$ gives $\dfrac{(\pm 2y)^2}{8} + \dfrac{y^2}{2} = 1, \quad 4y^2 + 4y^2 = 8 \quad$ and $\quad y = \pm 1.$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

1. Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.

Ans: max. 1 at $(0, \pm 1)$ and $(1, 0)$. min. $-1$ at $(-1, 0)$.

## Lagrange Multipliers with Two Constraints

Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0 \qquad \text{and} \qquad g_2(x, y, z) = 0$$

and $g_1$ and $g_2$ are differentiable, with $\nabla g_1$ not parallel to $\nabla g_2$, we find the constrained local maxima and minima of $f$ by introducing two Lagrange multipliers $\lambda$ and $\mu$ (mu, pronounced "mew"). That is, we locate the points $P(x, y, z)$ where $f$ takes on its constrained extreme values by finding the values of $x, y, z, \lambda$, and $\mu$ that simultaneously satisfy the equations

$$\boxed{\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0}$$

# Problem

- Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

- Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

  **Solution:** Here, $\nabla f = \nabla \lambda g_1 + \mu \nabla g_2$ results in

  [where, $g_1(x, y, z) = x + 2y + 3z - 6$ and $g_2(x, y, z) = x + 3y + 9z - 9$]

$$2x = \lambda + \mu, \quad 2y = 2\lambda + 3\mu, \quad 2z = 3\lambda + 9\mu$$

$$\implies x = \frac{\lambda + \mu}{2}, \quad y = \frac{2\lambda + 3\mu}{2}, \quad z = \frac{3\lambda + 9\mu}{2} \qquad (*)$$

- Now, with these substitutions in the following equations $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$

$$\implies \quad 7\lambda + 17\mu = 0 \text{ and } 34\lambda + 91\mu = 0$$

$$\implies \quad \lambda = 240/59, \ \mu = -78/59$$

So, $x = 81/59$, $y = 123/59$, $z = 9/59$ [from $(*)$].

Hence, $\boxed{\min(f) = 369/59}$ at $(81/59, 123/59, 9/59)$.

End of Unit 2