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KATHMANDU UNIVERSITY

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Subject: MATH104

Assignment No: 4

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<Q.1>: Define Beta function and establish the formula:
Ans:

Beta function or first Eulerian Integral denoted by $B(m,n)$.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \{m>0,\ n>0\}$$

(i): $B(m,n) = B(n,m)$.

Solⁿ:

We know,

$$B(m,n) = \int_0^\infty x^{m-1}(1-x)^{n-1}\,dx$$

Let $1-x=t$ then, $dx = -dt$. So.

$$B(m,n) = \int_1^0 -\{(1-t)^{m-1} t^{n-1}\,dt\}$$

$$= \int_0^1 (1-t)^{m-1} t^{n-1}\,dt = B(n,m)$$

$\therefore B(m,n) = B(n,m)$.

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$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Sol^n:

We know,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

Let $x = \sin^2\theta$

So, $dx = 2\sin\theta\cos\theta\, d\theta$.

when $x=0,\ \theta=0$

when $x=1,\ \theta=\pi/2$

So,

$$B(m,n) = \int_0^{\pi/2} \sin^{2(m-1)}\theta\,(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta\cdot\sin\theta\cdot\cos^{2n-2}\theta\cdot\cos\theta\cdot d\theta$$

$$= \int_0^{\pi/2} 2\sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\,d\theta.$$

$$\therefore B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cdot\cos^{2n-1}\theta\,d\theta$$

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(ii): $B(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Sol^n.

We know,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $x = \frac{y}{1-y}$. So, $1+x = 1+\frac{y}{1-y} = \frac{1}{1-y}$.

So, $dx = \frac{1}{(1-y)^2}\,dy$

When $x=0,\ y=0$.

when $x=\infty,\ y=1$

So,

$$\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{y^{m-1}}{(1-y)^{m-1}}\,\frac{(1-y)^{m+n}}{(1-y)}\,dy$$

$$= \int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

$$\therefore B(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}\,dy.$$

Now,

$$B(m,n) = B(n,m) = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \int_0^1 \frac{y^{n-1}}{(1-y)^{n-1}}\,\frac{(1-y)^{m+n}}{(1-y)}\,dy$$

$$= \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$ Hence, proved.

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(iii): $B(m,n) = \frac{n-1}{m+n-1} B(m,n-1) = \frac{m-1}{m+n-1} B(m-1,n)$

Soln:

Here,

$$\frac{n-1}{m+n-1} B(m,n-1) = \frac{(n-1)}{(m+n-1)} \cdot \frac{\Gamma(m)\Gamma(n-1)}{\Gamma(m+n-1)}$$

$$= \frac{\Gamma(m) \quad (n-1)\Gamma(n-1)}{(m+n-1)\Gamma(m+n-1)}$$

$$= \frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \frac{n-1}{m+n-1} B(m,n-1) = B(m,n).$$

Again,

$$\frac{m-1}{m+n-1} B(m-1,n) = \frac{(m-1)}{(m+n-1)} \frac{\Gamma(m-1)\Gamma(n)}{\Gamma(m+n-1)}$$

$$= \frac{(m-1)\ \Gamma(m-1)\ \Gamma(n)}{(m+n-1)\ \Gamma(m+n-1)}$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \frac{m-1}{m+n-1} B(m-1,n) = B(m,n)$$

Hence, proved.

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<Q.2>: Define Gamma Function and prove the following.

Ans:

The Gamma function or second Eulerian integral denoted by. $\Gamma(p)$.

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx \qquad (p>0)$$

(i): $\Gamma(1) = 1$

Soln:

We know,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx.$$

So,

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx \qquad = \int_0^\infty \frac{e^{-x}}{-1}$$

$$= \lim_{a\to\infty} \left(\frac{e^{-x}}{-1}\right)_0^a \qquad = \frac{e^{-a}}{-1} + \frac{e^0}{1}$$

$$\therefore \Gamma(1) = 1$$

(ii) $\Gamma(n+1) = n\Gamma(n)$.

Soln:

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1} dx$$

$$= \int_0^\infty e^{-x} x^n dx$$

$$= \lim_{a\to\infty} \left[x^n \int_0^a e^{-x} dx - \int_0^a \left(\frac{dx^n}{dx} \int_0^a e^{-x} dx \right) dx \right]$$

$$= \lim_{a\to\infty} x^n \left(\frac{e^{-x}}{-1} \right)_0^a + n \int_0^\infty x^{n-1} e^{-x} dx$$

So

$$\Gamma(n+1) = n\Gamma(n) + \lim_{a\to\infty} \left(-x^n e^{-x} \right)_0^a$$

Since, for $n>0$, $\lim_{a\to\infty} \frac{b^n}{e^b} = 0$.

$\therefore \Gamma(n+1) = n\Gamma(n)$.

(iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Soln:

We know,

$$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

If $m = 1/2$,

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin \pi/2}$$

or, $\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \pi$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Q.37: Relation betn Beta and Gamma functions.
Show that

(i): $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Soln:

We know,

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt \quad (m>0) \quad -(i).$$

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Let $t = z x$. then, $dt = z dx$.
when $t=0$, $x=0$.
when $t=\infty$, $x=\infty$

From (i), we get.

$$\Gamma(m) = \int_0^\infty e^{-zx} (zx)^{m-1} z\,dx$$

$$= \int_0^\infty z^m e^{-zx} x^{m-1} dx$$

$$\text{or, } \Gamma(m)\left[\int_0^\infty e^{-z} z^{n-1} dz\right] = \int_0^\infty z^m e^{-zx} x^{m-1} dx \left(\int_0^\infty e^{-z} z^{n-1} dz\right)$$

$$\text{or, } \Gamma(m)\,\Gamma(n) = \int_0^\infty\int_0^\infty z^{m+n-1} e^{-z(1+x)} x^{m-1} dx\,dz$$

Integrating z from 0 to ∞,

$$\Gamma(m)\,\Gamma(n) = \int_0^\infty\left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz\right] x^{m-1} dx$$

$$\text{or, } \Gamma(m)\,\Gamma(n) = \int_0^\infty \frac{\Gamma(m,n) \cdot x^{m-1}}{(1+x)^{m+n}} dx$$

$$\text{or } \Gamma(m)\,\Gamma(n) = \Gamma(m,n) \cdot \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\therefore B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m,n)}$$

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(ii): $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

Sol^n:

Given,

$$\int_0^\infty e^{-x^2} dx$$

Let $x^2 = t$

so, $2x\,dx = dt$ or $dx = dt/2\sqrt{t}$

When $x=0$, $t=0$
when $x=\infty$, $t=\infty$

$$= \int_0^\infty \frac{e^{-t}}{2t^{1/2}} dt$$

Let $p-1 = -1/2$. $\therefore p = 1/2$.

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{p-1} dt$$

$$= \frac{1}{2}\Gamma(p) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Hence, proved.

(iii): $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma(p+1/2)\,\Gamma(q+1/2)}{2\Gamma(p+q+2/2)}$

Sol^n:

We know,

$= \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$

and

$\frac{1}{2} B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$

Let $p = 2m-1$ $\quad \therefore m = \frac{p+1}{2}$

$q = 2n-1$ $\quad \therefore n = \frac{q+1}{2}$

$\therefore \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma(p+1/2)\,\Gamma(q+1/2)}{2\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$

$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

<Q.4>: Prove that:

(a): $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$

Sol^n.

$LHS = \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$

We know,

$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$

or, $\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin \pi/4}$

$\therefore \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$ proved.

(b): $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{\sqrt{3}}\pi$.

Sol^n:

$LHS = \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$

We know,

$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$

or $\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{\pi}{\sin \pi/3}$ $\quad \therefore \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{\sqrt{3}}\pi$

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(iii): $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\cdots\cdots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4.$

Solⁿ:

LHS =

$\cancel{\Gamma\left(\frac{1}{9}\right)\Gamma}\ \Gamma\left(1-\frac{8}{9}\right)\Gamma\left(\frac{8}{9}\right)\ \Gamma\left(1-\frac{7}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(1-\frac{6}{9}\right)\Gamma\left(\frac{6}{9}\right)$

$\Gamma\left(\frac{5}{9}\right)\Gamma\left(1-\frac{5}{9}\right)$

$= \frac{\pi}{\sin 8\pi/9} \times \frac{\pi}{\sin 7\pi/9} \times \frac{\pi}{\sin 5\pi/9} \times \frac{\pi}{\sin 6\pi/9}$

$= \frac{\pi^4}{\sin 160 \times \sin 140 \times \sin 120 \times \sin 100} = \frac{16}{3}\pi^4.$

Hence, proved

<Q.5>: Evaluate the following integrals:

(i): $\int_0^{\pi/2} \sin^6\theta \cos^4\theta \, d\theta$

Solⁿ.

$= \int_0^{\pi/2} \sin^{2\times 7/2 - 1}\theta \cdot \cos^{2\times 5/2 - 1}\theta \, d\theta$

Here, $2M - 1 = 6 \quad \therefore M = 7/2$

$2N - 1 = \cancel{6}\ 4 \quad \therefore N = 5/2$

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Now,

$\int_0^{\pi/2} \sin = \frac{1}{2} \frac{\Gamma(7/2)\,\Gamma(5/2)}{\Gamma(7/2 + 5/2)}$

$= \frac{1}{2} \times \Gamma\left(\frac{7}{2}\right) \times \Gamma\left(\frac{5}{2}\right) \times \frac{1}{\Gamma(\cancel{6})\ 5+1)}$

$= \frac{1}{2} \times \Gamma\left(\frac{5}{2}+1\right) \times \Gamma\left(\frac{3}{2}+1\right) \times \frac{1}{\cancel{\Gamma(6)}\ 5\,\cancel{\Gamma(5)}\ \Gamma(4+1)}$

$= \frac{1}{2} \times \frac{5}{2} \times \frac{3}{2} \times \Gamma\left(\frac{5}{2}\right) \times \Gamma\left(\frac{3}{2}\right) \times \frac{1}{\cancel{\Gamma(6)}\ 5\times 4 \times \Gamma(3+1)}$

$= \frac{1}{2} \times \frac{5}{2} \times \frac{3}{2} \times \Gamma\left(\frac{3}{2}+1\right) \times \Gamma\left(\frac{1}{2}+1\right) \times \frac{1}{\cancel{\Gamma(6)}\ 5\times 4\times 3 \times \Gamma(2+1)}$

$= \frac{1}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{3}{2} \times \Gamma\left(\frac{1}{2}+1\right) \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \times \frac{1}{5\times 4\times 3\times 2\times \Gamma(1+1)}$

$= \frac{1}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{5\times 4\times 3\times 2\times 1\times 1}$

$= \frac{1}{2} \times \frac{\cancel{5}}{2} \times \frac{3}{2} \times \frac{\cancel{3}}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\cancel{5}} \times \frac{1}{4} \times \frac{1}{\cancel{3}} \times \frac{1}{2}$

$= \frac{3\pi}{512}$

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(ii): $\int_0^a x^3 (a^2 - x^2)^{3/2} \, dx$ when $x = 0, \theta = 0$
when $x = a, \theta = \pi/2$

Soln.

Let $x = a\sin\theta$. $dx = a\cos\theta \, d\theta$

$= \int_0^{\pi/2} a^3 \sin^3\theta \, (a^2 - a^2\sin^2\theta)^{3/2} \, d\theta$

$= \int_0^{\pi/2} a^3 \sin^3\theta \times \{a^2(1-\sin^2\theta)\}^{3/2} \, d\theta$

$= \int_0^{\pi/2} a^6 \sin^3\theta \cdot \cos^3\theta \, d\theta$

$= a^6 \int_0^{\pi/2} \sin^3\theta \cos^3\theta \, d\theta$

~~$= a^6 \int_0^{\pi/2} \sin$~~ $= a^6 \int_0^{\pi/2} \sin^{2\times2-1}\theta \cdot \cos^{2\times2-1}\theta \, d\theta$

Here, $2M-1 = 3$ and $2N-1 = 3$
$\therefore M = 2$ $\therefore N = 2$

$= \frac{a^6}{2} \times \frac{\Gamma(M)\Gamma(N)}{\Gamma(M+N)} = \frac{a^6}{2} \times \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}$

$= \frac{a^6}{2} \times \frac{\Gamma(1+1) \times \Gamma(1+1)}{\Gamma(3+1)} = \frac{a^6}{2} \frac{1\times\Gamma(1) \times 1 \times \Gamma(1)}{3\,\Gamma(2+1)}$

$= \frac{a^6}{2} \times \frac{1\times1\times1\times1}{3\times2\times\Gamma(1+1)} = \frac{a^6}{2} \times \frac{1}{3\times2\times1\times1} = \frac{a^6}{12}$

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<Q.64: Define limit, continuity, and derivative of the vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{j}$. State the component test for continuity of the vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$.
Let $\vec{r}(t) = \sqrt{(1-t^2)}\,\vec{i} + 3t\,\vec{j} - 7\vec{k}$. At what value of t is the vector function $\vec{r}$ continuous?

Soln:

(pen changed)

Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D and $\vec{L}$ be a vector.
We say that r has limit $\vec{L}$ as t approaches t_0 ~~and~~ ~~teH~~

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$.

$$|\vec{r}(t) - \vec{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

If $\vec{L} = \vec{L_1}\,\vec{i} + \vec{L_2}\,\vec{j} + \vec{L_3}\,\vec{k}$ then, it can be shown as $\lim_{t\to t_0} \vec{r}(t) = \vec{L}$ precisely when,

$$\lim_{t\to t_0} f(t) = L_1, \quad \lim_{t\to t_0} g(t) = L_2, \quad \lim_{t\to t_0} h(t) = L_3.$$

A vector function $\vec{r}(t)$ is continuous at a point $t = t_0$ in its domain if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$

The function is continuous if it is continuous at every point in its domain.

The vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ has derivative at t if f, g, h have derivatives at t. The derivative is the vector function

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} =$$

$$= \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k}$$

Given,

$$\vec{r}(t) = \sqrt{1-t^2}\,\vec{i} + 3t\vec{j} - 7\vec{k}$$

At $t = t_0$,

$$\vec{r}(t_0) = \sqrt{1-t_0^2}\,\vec{i} + 3t_0\vec{j} - 7\vec{k}$$

Now,

$$\lim_{t \to t_0} \vec{r}(t) = \lim_{t \to t_0} \sqrt{1-t^2}\,\vec{i} + \lim_{t \to t_0} 3t\,\vec{j} + \lim_{t \to t_0} 7\vec{k}$$

$$= \cancel{\lim_{t \to t_0}} \sqrt{1-t_0^2}\,\vec{i} + 3t_0\vec{j} + 7\vec{k}$$

For $\vec{r}$ to be continuous,

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$

or $\sqrt{1-t_0^2}\,\vec{i} + 3t_0\vec{j} - 7\vec{k} = \sqrt{1-t_0^2}\,\vec{i} + 3t_0\vec{j} - 7\vec{k}$

Here, $\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$ for all $t_0 \in [-1, 1]$

continuous in interval $[-1, 1]$

<Q.7>: Define the smooth curve. Is the vector function $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$ smooth in interval $[-\pi, \pi]$.

Soln:

A vector function $\vec{r}$ is differentiable if it is differentiable at every point in its domain. The curve traced by r is smooth if dr/dt is continuous and never zero. that is, if f, g, h have continuous first derivatives that are not simultaneously zero.

Given,

$$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k} \quad ; \quad [-\pi, \pi]$$

$$\vec{r}'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$$

When $t = -\pi$, $-\sin t = 0$, $\cos t = -1$.

$t = \pi$, $-\sin t = 0$, $\cos t = -1$

When $t = -\pi$,

$$\vec{r}'(-\pi) = -\sin\pi\,\vec{i} + \cos\pi\,\vec{j} + \vec{k}$$
$$= -\vec{j} + \vec{k}$$

$$\lim_{t_0 \to -\pi} \vec{r}(t) = \lim_{t_0 \to \pi} -\sin\pi\,\vec{i} + \lim_{t_0 \to \pi} \cos\pi\,\vec{j} + \lim_{t_0 \to \pi} \vec{k}$$

$$= -\vec{j} + \vec{k}$$

$$\therefore \lim_{t_0 \to \pi} \vec{r}(t) = \vec{r}(-\pi)$$

Also,
when $t = \pi$,

$$\vec{r}'(\pi) = -\sin\pi\,\vec{i} + \cos\pi\,\vec{j} + \vec{k}$$
$$= -\vec{j} + \vec{k}$$

$$\lim_{t_0 \to \pi} \vec{r}(t) = \lim_{t_0 \to \pi} -\sin\pi\,\vec{i} + \lim_{t_0 \to \pi} \cos\pi\,\vec{j} + \lim_{t_0 \to \pi} \vec{k}$$
$$= -\vec{j} + \vec{k}$$

$$\therefore \lim_{t_0 \to \pi} \vec{r}(t) = \vec{r}(\pi)$$

Hence, $\vec{r}(t)$ is smooth in interval $[-\pi, \pi]$.

<Q.8>: Find $d\vec{r}/dt$ if

(a): $\vec{r}(t) = \ln\sqrt{1-t^2}\,\vec{i} + \sqrt{(1-t^2)}\,\vec{j}$

Sol^n:

Given,
$$\vec{r}(t) = \ln\sqrt{1-t}\,\vec{i} + \sqrt{1-t^2}\,\vec{j}$$

$$\therefore \vec{r}'(t) = \frac{d\vec{r}(t)}{dt} = \left\{\frac{d\ln\sqrt{1-t}}{d\sqrt{1-t}} \times \frac{d\sqrt{1-t}}{d(1-t)} \times \frac{d(1-t)}{dt}\right\}\vec{i} + \left\{\frac{d\sqrt{1-t^2}}{d(1-t^2)} \times \frac{d(1-t^2)}{dt}\right\}\vec{j}$$

$$= \left\{\frac{1}{\sqrt{1-t}} \times \frac{1}{2\sqrt{1-t}} \times (-1)\right\}\vec{i} + \left\{\frac{1}{2\sqrt{1-t^2}} \times -2t\right\}\vec{j}$$

$$= \frac{-1}{2(1-t)}\vec{i} - \frac{t}{\sqrt{1-t^2}}\vec{j}$$

(b): $\vec{r}(t) = (\sin^{-1}2t)\vec{i} + (\tan^{-1}3t)\vec{j} + 1/t\,\vec{k}$

Sol^n:

Given,
$$\vec{r}(t) = (\sin^{-1}2t)\vec{i} + (\tan^{-1}3t)\vec{j} + 1/t\,\vec{k}$$

$$\therefore \vec{r}'(t) = \frac{d\vec{r}(t)}{dt} = \left\{\frac{d(\sin^{-1}2t)}{d2t} \times 2\right\}\vec{i} + \left\{\frac{d\tan^{-1}3t}{d3t} \times 3\right\}\vec{j} + \left\{\frac{d(1/t)}{dt}\right\}\vec{k}$$

$$= \frac{2}{\sqrt{1-4t^2}}\vec{i} + \frac{1}{(1-4t^2)}\vec{j} - \frac{1}{t^2}\hat{k}$$

(c): $\vec{r}(t) = \frac{(2t-1)}{(2t+1)}\vec{i} + \ln(1-4t^2)\vec{j} + (\sec t)\vec{k}$

Sol^n:

Given,

$$\vec{r}(t) = \left(\frac{2t-1}{2t+1}\right)\vec{i} + \ln(1-4t^2)\vec{j} + (\sec t)\vec{k}$$

$$\vec{r}'(t) = \frac{d\vec{r}(t)}{dt} = \frac{d}{dt}\left(\frac{2t-1}{2t+1}\right)\vec{i} + \left\{\frac{d\ln(1-4t^2)}{d(1-4t^2)} \times \frac{d(1-4t^2)}{dt}\right\}\vec{j} + \frac{d}{dt}\sec t\,\vec{k}$$

$$= \frac{4}{(2t+1)^2}\vec{i} + \frac{1}{(1-4t^2)} \times (-8t)\vec{j} + \sec t\tan t\,\vec{k}$$

$$= \frac{4}{(2t+1)^2}\vec{i} - \frac{8t}{(1-4t^2)}\vec{j} + \sec t\tan t\,\vec{k}$$

(Q.no.9): Prove that:

$\frac{d}{dt}(\vec{u}\times\vec{v}) = \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$ for two vectors u and v.

Sol^n.

$$\frac{d}{dt}(\vec{u}\times\vec{v}) = \lim_{h\to 0}\frac{\vec{u}(t+h)\times\vec{v}(t+h) - \vec{u}(t)\times\vec{v}(t)}{h}$$

$$= \lim_{h\to 0} \cancel{u(t+h)\times v(t+h) - u(t)\times v(t+h)}$$

$$= \lim_{h\to 0}\frac{\vec{u}(t+h)\times\vec{v}(t+h) - \vec{u}(t)\times\vec{v}(t+h) + \vec{u}(t)\times\vec{v}(t+h) - \vec{u}(t)\times\vec{v}(t)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\vec{u}(t+h)-\vec{u}(t)}{h}\times\vec{v}(t+h) + \vec{u}(t)\times\frac{\vec{v}(t+h)-\vec{v}(t)}{h}\right]$$

$$= \lim_{h\to 0}\frac{\vec{u}(t+h)-\vec{u}(t)}{h}\times\lim_{h\to 0}\vec{v}(t+h) + \lim_{h\to 0}\vec{u}(t)\times\lim_{h\to 0}\frac{\vec{v}(t+h)-\vec{v}(t)}{h}$$

$$= \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$$

$$\therefore \frac{d}{dt}(\vec{u}\times\vec{v}) = \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$$

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Q.no. 10: The vector $\vec{r}(t)$ defines the position of a particle moving in the plane / space in time t. Find the particle's velocity, acceleration, speed and direction of motion at time specified.

(a): $\vec{r}(t) = (t^2+1)\vec{i} + (2t-1)\vec{j}$, $t = 1/2$.

Soln:

Given,

$\vec{r}(t) = (t^2+1)\vec{i} + (2t-1)\vec{j}$ $\quad t = 1/2$.

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d(t^2+1)}{dt}\vec{i} + \frac{d(2t-1)}{dt}\vec{j}$$

$$= 2t\,\vec{i} + 2\vec{j}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d\,2t}{dt}\vec{i} + \frac{d\,2}{dt}\vec{j}$$

$$= 2\vec{i}$$

At $t = 1/2$,

$\vec{v}(1/2) = 2 \times 1/2\,\vec{i} + 2\vec{j} = \vec{i} + 2\vec{j}$

$|\vec{v}(t)| = \sqrt{4t^2+4}$, $|\vec{v}(1/2)| = \sqrt{1+4} = \sqrt{5}$

$\vec{a}(1/2) = 2\vec{i}$

Direction at $t = 1/2$: $\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{1}{\sqrt{5}}\vec{i} + \dfrac{2}{\sqrt{5}}\vec{j}$

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(b): $\vec{r}(t) = (\cos 2t)\vec{i} + (3\sin 2t)\vec{j}$, $t = 0$.

Soln.

Given,

$\vec{r}(t) = (\cos 2t)\vec{i} + (3\sin 2t)\vec{j}$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \left\{\frac{d\cos 2t}{d2t} \times 2\right\}\vec{i} + \left\{3 \cdot \frac{d\sin 2t}{d2t} \times 2\right\}\vec{j}$$

$$= -2\sin 2t\,\vec{i} + 6\cos 2t\,\vec{j}$$

$$|\vec{v}(t)| = \sqrt{\{-2\sin 2t\}^2 + \{6\cos 2t\}^2}$$

$$= \sqrt{4\sin^2 2t + 36\cos^2 2t}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \left\{\frac{d(-2\sin 2t)}{d2t} \times 2\right\}\vec{i} + \left\{6 \times \frac{d\cos 2t}{d2t} \times 2\right\}\vec{j}$$

$$= -4\cos 2t\,\vec{i} - 12\sin 2t\,\vec{j}$$

When $t = 0$,

$\vec{v}(0) = 6\vec{j}$

$|\vec{v}(0)| = \sqrt{4\sin^2 2\times0 + 36 \cdot \cos^2 2\times0} = 6$

$\vec{a}(0) = -4\vec{i}$

Direction: $\dfrac{\vec{v}(0)}{|\vec{v}(0)|} = \vec{j}$

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(c): $\vec{r}(t) = (1+t)\vec{i} + \frac{t^2}{\sqrt{2}}\vec{j} + \frac{t^3}{3}\vec{k}$, $t = 1$

Soln:

Given,

$$\vec{r}(t) = (1+t)\vec{i} + \frac{t^2}{\sqrt{2}}\vec{j} + \frac{t^3}{3}\vec{k}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \vec{i} + \sqrt{2}t\,\vec{j} + t^2\vec{k}$$

$$|\vec{v}(t)| = \sqrt{1^2 + (\sqrt{2})^2 + (t^2)^2} = \sqrt{3+t^4}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \sqrt{2}\vec{j} + 2t\vec{k}$$

When $t = 1$,

$$\vec{v}(1) = \vec{i} + \sqrt{2}\vec{j} + \vec{k}$$

$$|\vec{v}(1)| = \sqrt{3+1^4} = 2$$

$$\vec{a}(t) = \sqrt{2}\vec{j} + 2\vec{k}$$

$$\text{Direction} = \frac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{1}{2}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{2}\vec{k}$$

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(11): Solve the initial value problem:

$$\frac{d^2\vec{r}}{dt^2} = -32\vec{k}$$

with the initial conditions: $\vec{r}(0) = 100\vec{k}$ and $d\vec{r}/dt\,\big|_{t=0} = 8\vec{i} + 8\vec{j}$

Soln:

Given,

$$\frac{d^2\vec{r}}{dt^2} = -32\vec{k}$$

On integrating w.r.t t,

$$\frac{d\vec{r}}{dt} = -32t\,\vec{k} + \vec{c}$$

At $t = 0$,

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\vec{i} + 8\vec{j}$$

or $0 + \vec{c} = 8\vec{i} + 8\vec{j}$ $\therefore \vec{c} = 8\vec{i} + 8\vec{j}$

So, $\frac{d\vec{r}}{dt} = 8\vec{i} + 8\vec{j} - 32t\vec{k}$

On integrating w.r.t t,

$$\vec{r}(t) = 8t\,\vec{i} + 8t\,\vec{j} - 16t^2\,\vec{k} + \vec{c}$$

We know,

$\vec{r}(0) = 100\vec{k}$

$\Rightarrow 0 + 0 + 0 + \vec{c} = 100\vec{k}$ $\therefore \vec{c} = 100\vec{k}$

$$\therefore \vec{r}(t) = 8t\,\vec{i} + 8t\,\vec{j} + (-16t^2 + 100)\vec{k}$$

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<Q.12>: Find the arc length parameter along the curve $\vec{r}(t) = (e^t \cos t)\vec{i} + (e^t \sin t)\vec{j} + (e^t)\vec{k}$ from the point where $t=0$ by evaluating the integral $s = \int_0^t |v(\tau)|\, d\tau$ and then find the length of the curve for $-\ln 4 \le t \le 0$.

Sol^n:

Given,

$\vec{r}(t) = (e^t \cos t)\vec{i} + (e^t \sin t)\vec{j} + (e^t)\vec{k}$

$\therefore \frac{df}{dt} = \frac{d\, e^t \cos t}{dt} = e^t \cos t - e^t \sin t.$

$\frac{dg}{dt} = \frac{d\, e^t \sin t}{dt} = e^t \sin t + e^t \cos t$

$\frac{dh}{dt} = \frac{de^t}{dt} = e^t$

So,

Arc length parameter $= \int_0^t \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$

$= \int_0^t \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t) + (e^t)^2}$

$= \int_0^t \sqrt{2e^{2t}(\cos^2 t + \sin^2 t) + e^{2t}} = \int_0^t \sqrt{3e^{2t}}$

$= \sqrt{3} \int_0^t e^t = \sqrt{3}\, e^t.$

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So,

arc length $= \sqrt{3} \int_{-\ln 4}^{0} e^t$

$= \sqrt{3}\, \left| e^t \right|_{-\ln 4}^{0} = \sqrt{3}\left(e^0 - e^{\ln 4^{-1}}\right)$

$= \sqrt{3}\left(1 - \frac{1}{4}\right) = \frac{3\sqrt{3}}{4}$ units.

~~<Q.no.13>: Prove the relations:~~

~~(a) $K = \dfrac{\left|\frac{d^2 y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$ (b): $K = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$ where~~

~~symbols have usual meaning.~~

~~Sol^n.~~

(a):

{P.T.O.}

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<Q.14>: Find $\vec{T}, \vec{N}, \vec{B}, K, \tau$.

(a) $\vec{r}(t) = e^t\cos t\,\vec{i} + e^t\sin t\,\vec{j} + 2\vec{k}$

Soln:

Given,

$\vec{r}(t) = e^t\cos t\,\vec{i} + e^t\sin t\,\vec{j} + 2\vec{k}$

$\vec{v}(t) = \dfrac{d\vec{r}(t)}{dt} = (e^t\cos t - e^t\sin t)\vec{i} + (e^t\cos t + e^t\sin t)\vec{j}$

$|\vec{v}(t)| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\cos t + e^t\sin t)}$

$= \sqrt{e^{2t}\cos^2 t - \cancel{2e^{2t}\cos t\sin t} + e^{2t}\sin^2 t + e^{2t}\cos^2 t \cancel{+2e^{2t}\cos t\sin t} + e^{2t}\sin^2 t}$

$= \sqrt{2e^{2t}(\cos^2 t + \sin^2 t)}$

$= \sqrt{2}\, e^t$

$\vec{T} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{(e^t\cos t - e^t\sin t)\vec{i} + (e^t\cos t + e^t\sin t)\vec{j}}{\sqrt{2}e^t}$

$= \dfrac{\cancel{e^t}(\cos t - \sin t)}{\sqrt{2}\cancel{e^t}}\vec{i} + \dfrac{\cancel{e^t}(\cos t + \sin t)}{\sqrt{2}\cancel{e^t}}\vec{j}$

$\therefore \vec{T} = (\cos t - \sin t)\vec{i} + (\cos t + \sin t)\vec{j}$

$\dfrac{d\vec{T}}{dt} = \dfrac{d\{(\cos t - \sin t)\vec{i} + (\cos t + \sin t)\vec{j}\}}{dt}$

$= \dfrac{d(\cos t - \sin t)}{dt}\vec{i} + \dfrac{d(\cos t + \sin t)}{dt}\vec{j}$

$= (-\sin t - \cos t)\vec{i} + (-\sin t + \cos t)\vec{j}$

$\left|\dfrac{d\vec{T}}{dt}\right| = \sqrt{\{-(\sin t + \cos t)\}^2 + (\cos t - \sin t)^2} = \sqrt{\sin^2 t + \cos^2 t \cancel{+2\sin t\cos t} + \cos^2 t \cancel{-2\cos t\sin t} + \sin^2 t} = \sqrt{2(\sin^2 t + \cos^2 t)}$

$\therefore |d\vec{T}/dt| = \sqrt{2}$

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$\vec{N} = \dfrac{d\vec{T}/dt}{|d\vec{T}/dt|} = \dfrac{(-\sin t - \cos t)}{\sqrt{2}}\vec{i} + \dfrac{(\cos t - \sin t)}{\sqrt{2}}\vec{j}$

$K = \dfrac{1}{|v|}\left|\dfrac{d\vec{T}}{dt}\right| = \dfrac{1}{\cancel{\sqrt{2}}e^t} \times \cancel{\sqrt{2}} = \dfrac{1}{e^t}$

(b): $\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j} + 3\vec{k}$

Soln.

Given,

$\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j} + 3\vec{k}$

$\vec{v}(t) = \dfrac{d\vec{r}(t)}{dt} = -\sin t\,\vec{i} + \sin t\,\vec{i} + t\cos t\,\vec{i} + \cos t\,\vec{j} - \cos t\,\vec{j} - t\sin t\,\vec{j}$

$\therefore \vec{v}(t) = t\cos t\,\vec{i} - t\sin t\,\vec{j}$

$|\vec{v}(t)| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = t$

$\vec{T} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{t\cos t\,\vec{i} - t\sin t\,\vec{j}}{t} = \cos t\,\hat{i} - \sin t\,\vec{j}$

$\dfrac{d\vec{T}}{dt} = \dfrac{d(\cos t\,\vec{i} - \sin t\,\vec{j})}{dt} = -\sin t\,\vec{i} - \cos t\,\vec{j}$

$\left|\dfrac{d\vec{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{-\sin t\ \hat{i} + -\cos t\ \hat{j}}{1} = -\sin t\ \hat{i} - \cos t\ \hat{j}$$

$$K = \frac{1}{|v|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{t} \times 1 = \frac{1}{t}$$

<c>: $\vec{r}(t) = (\cosh t)\vec{i} + (\sinh t)\vec{j} + t\vec{k}$

Soln.

Given, $\vec{r}(t) = (\cosh t)\vec{i} + (\sinh t)\vec{j} + t\vec{k}$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \sinh t\ \vec{i} + \cosh t\ \vec{j} + \vec{k}$$

$$|\vec{v}(t)| = \sqrt{(\sinh t)^2 + (\cosh t)^2 + 1^2}$$
$$= \sqrt{\sinh^2 t + \cosh^2 t + \cosh^2 t - \sinh^2 t}$$
$$= \cosh t\ \sqrt{2}$$

$$\vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\sinh t}{\sqrt{2}\cdot\cosh t}\vec{i} + \frac{\cosh t}{\cosh t\cdot\sqrt{2}}\vec{j} + \frac{1}{\cosh t\sqrt{2}}\vec{k}$$

$$= \frac{1}{\sqrt{2}}\tanh t\ \vec{i} + \frac{1}{\sqrt{2}}\vec{j} + \frac{\operatorname{sech} t}{\sqrt{2}}\vec{k}$$

$$\frac{d\vec{T}}{dt} = \frac{d}{dt}\frac{1}{\sqrt{2}}\tanh t\ \vec{i} + \frac{d}{dt}\frac{1}{\sqrt{2}}\vec{j} + \frac{d}{dt}\frac{1}{\sqrt{2}}\operatorname{sech} t\ \vec{k}$$
$$= \frac{1}{\sqrt{2}}\operatorname{sec}^2 h t\ \vec{i} + \frac{1}{\sqrt{2}}(\operatorname{sech} t\cdot\tanh t)\vec{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sec}^2 h t\ \tanh^2 t}$$
$$= \sqrt{\frac{1}{2}\operatorname{sech}^2 t(\operatorname{sech}^2 t + \tanh^2 t)}$$
$$\therefore \left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sqrt{2}}\operatorname{sech} t$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\ \vec{i} + \frac{1}{\sqrt{2}}(-\operatorname{sech} t\cdot\tanh t)\vec{k}}{\frac{1}{\sqrt{2}}\operatorname{sech} t}$$
$$= \operatorname{sech} t\ \vec{i} - \tanh t\ \vec{k}$$

$$K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$
$$= \frac{1}{\cosh t\sqrt{2}} \times \frac{1}{\sqrt{2}}\operatorname{sec} h t$$
$$= \frac{1}{\sqrt{2}}\operatorname{sec}^2 h t.$$