Date   No.

CHAPTER 2

ELECTROSTATIC FIELD

Electric Charge (q):

Charge is the fundamental and characteristics property of the elementary particles which make up matter.

It is a scalar quantity.
SI unit: Coulomb (C)

(*): Kind of Charge: i) Positive charge   ii) Negative charge.

(*) Properties of charge:

(i) Like charges repel each other and unlike charge attract each other.
(ii) Electric charge is quantized ie, $q = \pm ne$
(iii) Electric charge is conserved.
(iv) Electric charge is additive in nature
(v). The charge on a body is not affected by the speed of the body.

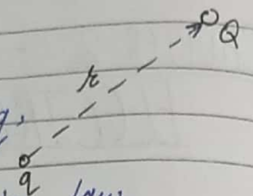
(*) Elementary charge

The magnitude of charge on a proton or an electron

$$e = 1.6 \times 10^{-19}\ C$$

Coulomb's law

The force on a test charge Q due to a single point charge q, which is at rest a distance 'r' away is given by Coulomb's law.



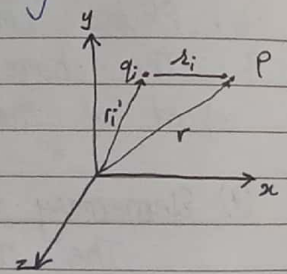
$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{r^2} \hat{r}$$

i.e., the magnitude of the electric force between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

The constant ε_0 is called permittivity of free space.

$$\varepsilon_0 = 8.85 \times 10^{-12} \; C^2 N^{-1} m^{-2}$$

If there are several point charges $q_1, q_2, ..., q_n$ at distances $r_1, r_2, ..., r_n$ from Q, the total force on Q is given by.



$$\vec{F} = \vec{F_1} + \vec{F_2} + \cdots$$

$$= \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1 Q}{r_1^2} \hat{r}_1 + \frac{q_2 Q}{r_2^2} \hat{r}_2 + \cdots \right)$$

$$= \frac{Q}{4\pi\varepsilon_0} \left(\frac{q_1}{r_1^2} \hat{r}_1 + \frac{q_2}{r_2^2} \hat{r}_2 + \cdots \right)$$

$$\therefore \vec{F} = Q\vec{E}$$

where,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \left(\sum_{i=1}^{n} \frac{q_i}{r_i^2} \hat{r}_i \right)$$

Using principle of superposition,

$$\vec{E}_{net} = \vec{E_1} + \vec{E_2} + \cdots$$

Electric Field

The electric field $\vec{E}$ at point in space is defined as the electric force $\vec{F}$ acting on a positive test charge Q placed at that point divided by the magnitude of the test charge.

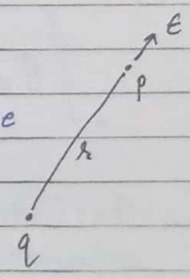
$$\vec{E} = \frac{\vec{F}}{Q}$$

SI unit = NC^{-1} (Newton per coulomb)

The electric field is vector quantity that varies from point to point.

(*): E.F. of a Point Charge

The electric field $\vec{E}$ produced at field point P by an isolated point charge q at the source point S is given below:

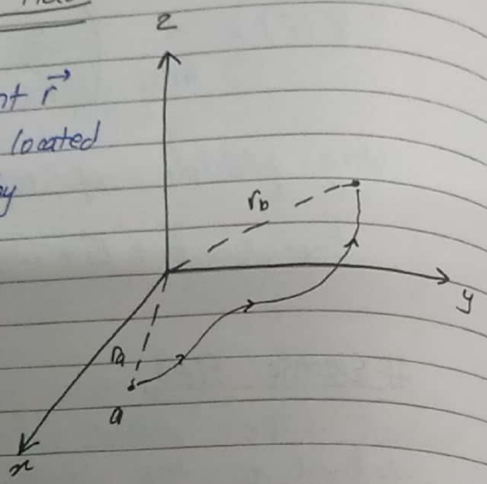


$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \vec{r}$$

Line Integral of Electric Field

The electric field at a point $\vec{r}$ due to a point charge q located at the origin is given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$



Now, the line integral of electric field.

$$\int_a^b \vec{E}\cdot d\vec{l} = \int_a^b \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\right)\left(dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}\,dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-1}{r}\right]_{r_a}^{r_b}$$

$$\therefore \int_a^b \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right]$$

The electric field due to stationary charges is conservative field. i.e, $\oint \vec{E}\cdot d\vec{l} = 0$.

The amount of workdone by the electric field $\vec{E}$ when a unit positive charge moves from point a and point b.

$$W_E = \int_a^b \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right]$$

Curl of Electric Field

The line integral of electric field around a closed path is zero

i.e, $\oint_c \vec{E}\cdot d\vec{l} = 0$

Using stokes theorem,

$$\oint_s (\nabla\times\vec{E})\cdot d\vec{a} = 0$$

$$\therefore \nabla\times\vec{E} = 0$$

The electric field at point $\vec{r}$ due to a point charge q located at the origin is.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

Now,

$$\nabla\times\vec{E} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}(x\hat{i} + y\hat{j} + z\hat{k})\right)$$

Now,

$$\nabla \times \vec{E} = \frac{q}{4\pi\varepsilon_0 r^3} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= \frac{q}{4\pi\varepsilon_0 r^3}\left[\hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\right]$$

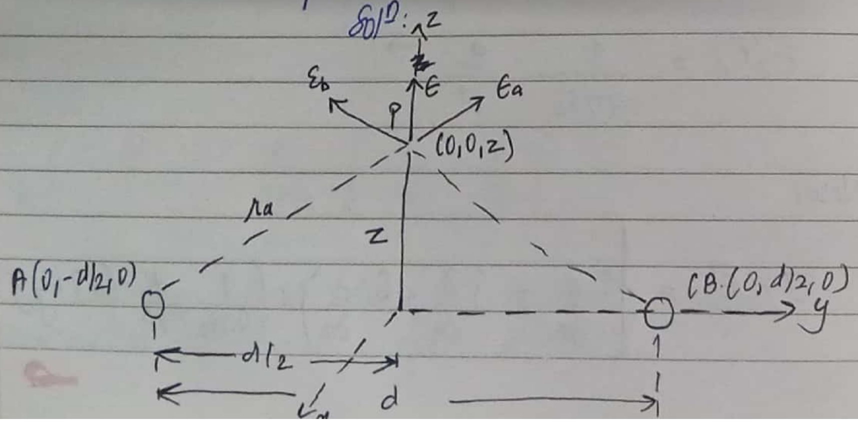
$$= \frac{q}{4\pi\varepsilon_0 r^3} \times 0$$

$$\therefore \nabla \times \vec{E} = 0$$

This shows that electric field is conservative field for stationary charges ie, conservative field has zero curl.

Example: Find the electric field or distance z above the midpoint between two equal charges q, d distance apart.

Sol^n:



Here, from figure,

$$\vec{r_a} = d/2\,\hat{j} + z\,\hat{k}$$

$$\vec{r_b} = -d/2\,\hat{j} + z\,\hat{k}$$

So,

$$r_a = \sqrt{\left(\frac{d}{2}\right)^2 + z^2} = \left(\frac{d^2}{4} + z^2\right)^{1/2}$$

$$r_b = \sqrt{\left(\frac{-d}{2}\right)^2 + z^2} = \left(\frac{d^2}{4} + z^2\right)^{1/2}$$

Now, the electric field at P due to charge at A,

$$\vec{E_A} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_A^3}\vec{r_A} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\left(\frac{d}{2}\hat{j} + z\hat{k}\right)$$

Similarly,

$$\vec{E_b} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\vec{r_B}}\vec{r_B} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\left(\frac{-d}{2}\hat{j} + z\hat{k}\right)$$

We know, E_T at P is equal to

$$E_T = E_A + E_B$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\left(\frac{d}{2}\hat{j}+z\hat{k}\right)+\frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\left(-d/2\,\hat{j}+z\hat{k}\right)$$

$$\therefore E_T = \frac{1}{4\pi\varepsilon_0}\, q\, \frac{2z}{\left(\frac{d^2}{4}+z^2\right)^{3/2}}\hat{k}$$

When $z >> d$,

$$E_T = \frac{1}{4\pi\varepsilon_0}\, q\, \frac{2z}{z^3}\hat{k} = \frac{1}{4\pi\varepsilon_0}\, q\, \frac{2}{z^2}\hat{k}$$

Continuous Charge Distribution

* Linear Charge Density

λ (charge per unit length) $= \frac{dq}{dl'}$ $\quad dq = \lambda \times dl'$

(*) Surface charge Density

σ (charge per unit area) $= \frac{dq}{da'}$ $\quad dq = \lambda \times da'$

(*) Volume charge Density:

ρ (charge per unit area) $= \frac{dq}{d\tau'}$ $\quad dq = \rho \times d\tau'$

$$\therefore q = \int \rho\, d\tau'$$

We know,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{r}.$$

The electric field of a line charge.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\cdot dl'}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\cdot dl'}{r^3}\vec{r}$$

The electric field of a surface charge.

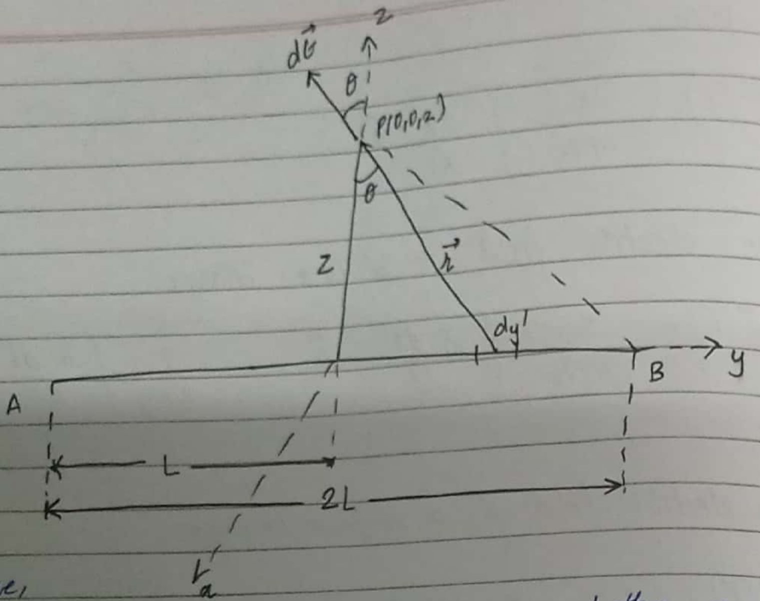
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\cdot da'}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\cdot da'}{r^3}\vec{r}$$

The electric field of a volume charge.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\cdot d\tau'}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\cdot d\tau'}{r^3}\hat{r}$$

Example: Find the electric field at a distance z above the midpoint of a straight line segment of length 2l which carries a uniform linear charge 'λ'.

Soln:



Here,
figure illustrate the coordinates and the geometry to be used.
We know,

$$\lambda = \frac{dq}{dl'}$$

Let us take an elemental line segment dy'.
So, $dl' = dy'$
So,

$$dq = \lambda \cdot dl'$$
$$= \lambda \cdot dy'$$

From the figure, $\vec{r} = -y'\hat{j} + z\hat{k}$

$$\therefore r = (y'^2 + z^2)^{1/2}$$

The charge on elemental length dy' at C along the line is

$$dq = \lambda \cdot dy'$$

The electric field at P due to charge dq is given by

$$d\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{dq}{r^3}\vec{r}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, dy'}{(y'^2+z'^2)^{3/2}} \times (-y\hat{j} + z\hat{k}) \quad \text{—— (i)}$$

The total electric field at P due to charge on the whole line segment AB is.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{\lambda\, dy'}{(y'^2+z'^2)^{3/2}} \cdot (-y'\hat{j} + z\hat{k})$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[-\int_{-L}^{L} \frac{y'\, dy'}{(y'^2+z^2)^{3/2}}\hat{j} + \int_{-L}^{L} \frac{z \cdot dy'\, \vec{k}}{(y'^2+z^2)^{3/2}}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[2\int_{0}^{L} \frac{z\, dy'\, \hat{k}}{(y'^2+z^2)^{3/2}}\right]$$

$$= \frac{2\lambda z}{4\pi\varepsilon_0}\left[\int_{0}^{L} \frac{dy'\, \hat{k}}{(y'^2+z^2)^{3/2}}\right]$$

Let $y' = z\tan\theta$

$\tan\theta = \frac{y'}{z}$ So, $dy' = z\sec^2\theta\, d\theta$

when $y' = 0$, $\theta = 0$
when $y' = L$, $\theta = \tan^{-1}(L/z) = \phi_0$

Then,
$$y'^2 + z^2 = z^2\tan^2\theta + z^2$$
$$= z^2(\tan^2\theta + 1)$$
$$= z^2\sec^2\theta$$

Then,
$$\vec{E} = \frac{2\lambda z}{4\pi\varepsilon_0}\int_0^{\tan^{-1}(L/z)} \frac{z^2\sec^2\theta\, d\theta}{z^3\sec^3\theta}\,\hat{k}$$
$$= \frac{2\lambda}{4\pi\varepsilon_0 z}\int_0^{\tan^{-1}(L/z)} \cos\theta\, d\theta\,\hat{k}$$
$$= \frac{2\lambda}{4\pi\varepsilon_0 z}\int_0^{\phi_0} \cos\theta\, d\theta\,\hat{k}$$
$$= \frac{2\lambda}{4\pi\varepsilon_0 z}\sin\theta\Big|_0^{\phi_0}$$
$$= \frac{2\lambda}{4\pi\varepsilon_0 z}\sin\phi_0\,\hat{k}$$
$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0 z}\frac{\lambda(2L)}{\sqrt{L^2+z^2}}\,\hat{k}$$
$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z\sqrt{L^2+z^2}}\,\hat{k} \quad \text{—(ii)}$$

(x) Special cases

i) For points far from the line $(z \gg L)$.
$$\vec{E} = \frac{2\lambda}{4\pi\varepsilon_0 z} \times \frac{L}{z}\,\hat{k}$$
$$= \frac{2\lambda L}{4\pi\varepsilon_0 z^2}\,\hat{k}$$
$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda(2L)}{z^2}\,\hat{k} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2}\,\hat{k}$$

ii) As line tends to infinity $(L \to \infty)$.
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z\sqrt{\frac{z^2}{L^2}+1}}\,\hat{k}$$
$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}\,\hat{k}$$ gives field of infinite straight wire.

If 3 marks Q, upto eq^n (ii)
If 5 marks Q, upto ~~eq^n~~ special case.