KATHMANDU UNI.

DHULIKHEL, KAVRE

Subject: Maths (101)

Assignment No: 1

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<Q.1>: Find domain and range of:

a): $f(x) = 1+x^2$.

Sol^n:

Given,

$f(x) = 1+x^2$

Here, $f(x)$ is defined for every value of x.

So, Domain $= (-\infty, \infty)$

For Range:

$y = 1+x^2$

or $x^2 = y-1$

or $x = \sqrt{y-1}$

ie, $y-1 \geq 0$

or $y \geq 1$

$\therefore$ Range $= [1, \infty)$

<b>. $f(x) = 1-\sqrt{x}$

Given,

$f(x) = 1-\sqrt{x}$

Here, $f(x)$ is defined for all values of $x \geq 0$ as for negative values of x, $f(x)$ becomes complex

Domain $= [0, \infty)$

For range:

$$y = 1 - \sqrt{x}$$

or $\sqrt{x} = 1 - y$

or $x = (1-y)^2$

We know,

$(1-y)^2 \geq 0$

or $1 - y \geq 0$

$\therefore y \leq 1$

Range $= (-\infty, 1]$

(c): $f(z) = \sqrt{4-z^2}$

Soln:

Given,

$f(z) = \sqrt{4-z^2}$

Here,

$4 - z^2 = (2-z)(2+z)$

either, $z = 2$ or, $z = -2$

Domain: $[-2, 2]$

For range:

$y = \sqrt{4-z^2}$

or $y^2 = 4 - z^2$

or $z^2 = 4 - y^2$ or $z = \sqrt{4-y^2}$

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We know,

$(4-y^2) \geq 0$

or $4 \geq y^2$

or $y^2 \leq 2^2$

$\therefore y \leq 2$

Here, y must be positive.

~~$\therefore$ Range = $(-\infty, 2]$~~ $\therefore$ Range = $[0, 2]$

(d): $f(t) = \dfrac{1}{1+\sqrt{t}}$

Soln:

Given,

$f(t) = \dfrac{1}{1+\sqrt{t}}$

Here, $f(t)$ is defined for all values of $t \geq 0$ as any ~~valu~~ negative value of t, the soln will be a complex number.

Domain: $\in [0, \infty)$

For range: $y = \dfrac{1}{1+\sqrt{t}}$

or $\dfrac{1}{y} = 1+\sqrt{t}$

or $\sqrt{t} = \dfrac{1}{y} - 1$

or $t = \left(\dfrac{1}{y} - 1\right)$

Here, t is defined for all values of y except $y=0$ and y ~~$\therefore$ Range~~ must be positive.

$\therefore$ Range: $(0, \infty)$

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Q.2): What symmetries, if any, do the graphs have.

(a): $y = -x^2$

Soln.

→ Checking for symmetry on ~~x-axis~~ y-axis
Replacing x by $-x$.

$$y = -(-x)^2$$
or, $y = -x^2$ = original function.

This function has symmetry on y-axis.

(b): $y = -1/x$

Soln:

→ Checking symmetry on y-axis:
Replacing x by $-x$.

$$y = -\frac{1}{(-x)}$$

or $y = \frac{1}{x}$ ≠ original function

This function has no symmetry on y-axis.

→ Checking symmetry on x-axis:
Replacing y by $-y$

$-y = \frac{-1}{x}$ or, $y = \frac{1}{x}$ ≠ original function

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This function has no symmetry on x-axis

→ Checking symmetry about origin:

Replacing x by $-x$.

$y = -\left(\frac{1}{-x}\right)$ or $y = \frac{1}{x}$

Here,

$f(-x) = -f(x)$

So, this function has symmetry about origin.

(c). $y = \sqrt{|x|}$

Soln.

→ Checking for symmetry on y-axis:

Replacing x by $-x$.

$y = \sqrt{|-x|}$

or $y = \sqrt{x}$

Here,

$f(-x) = f(x)$.

So, the function has symmetry on y-axis.

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(d): $y = (-x)^{3/2}$
Soln:
→ checking for symmetry on y-axis.
Replacing x by -x.
$y = \{-(-x)\}^{3/2}$
$y = x^{3/2}$

Here, $f(-x) \neq f(x)$ so, it is not symmetrical along y-axis
but
$f(-x) = f(x)$. So, the function has symmetry about origin.

(Q.no.3): Determine if the functions is odd, even or neither.

(a): $f(x) = 3$
Soln
Given,
$f(x) = 3$
or $f(x) = 3 + 0 \cdot x$

Here putting $x = -x$.
$f(-x) = 3 + 0 \cdot (-x)$
$\therefore f(x) = 3$

ie, $f(-x) = f(x)$
The $f(x)$ is even function.

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(b): $f(x) = x^3 + x$

Soln:

Given,

$f(x) = x^3 + x$

Replacing $x = -x$,

$$f(-x) = (-x)^3 + (-x)$$
$$= -x^3 - x$$
$$= -(x^3 + x)$$

$\therefore f(-x) = -f(x)$.

This is odd function.

(c): $f(x) = \dfrac{1}{x^2 + x + 1}$

Soln,

Given,

$$f(x) = \frac{1}{x^2 + x + 1}$$

Replacing $x = -x$.

$$f(-x) = \frac{1}{(-x)^2 - x + 1}$$
$$= \frac{1}{x^2 - x + 1}$$

Here, $f(x) \neq f(x)$ and $f(-x) \neq -f(x)$. So, this function is neither odd nor even.

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<Q.no.4>: find domain and ranges of f, g, f+g, f·g.

(a): $f(x) = x$, $g(x) = \sqrt{x-1}$

Soln:

For ~~(a)~~ $f(x)$:

Given,

$f(x) = x$

Here, x is defined for all $x \in \mathbb{R}$ $(-\infty, \infty)$

Domain = $(-\infty, \infty)$

Here,

$y = x$

∴ Range = $(-\infty, \infty)$

For $g(x)$:

Given,

$g(x) = \sqrt{x-1}$

Here, $g(x)$ is defined for all values of $x \geq 1$

Domain = $[1, \infty)$

Here,

$y = \sqrt{x-1}$

or $y^2 = x-1$

∴ $x = y^2 + 1$

Here, for all values of domain, $f(x) \geq 1$

So,

Range = $[1, \infty)$

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For $(f+g)(x)$:

$f(x)+g(x) = x+\sqrt{x-1}$

Here, $f(x)$ is defined for all values of $x \geq 1$.
$\therefore$ Domain $= \epsilon \ [1,\infty)$

~~for~~: Range:

$y = x+\sqrt{x-1}$

or, Here, putting all values of domain on $(f+g)(x)$.
~~Range =~~ the value of $y \geq 1$.
So,
Range $= [1,\infty)$

(b): For $(f\cdot g)(x)$.

$f(x)\cdot g(x) = x\sqrt{x-1}$

Here, $f(x)$ is defined for all values of $x \geq 1$.
Domain $= \epsilon- [1,\infty)$

Here, ~~Range.~~

$y = x\sqrt{x-1}$

~~on~~ Putting all values of domain of $f(x)$, y will have value ≥ 0.

$\therefore$ Range $= [0,\infty)$

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(b): $f(x) = \sqrt{x+1}$, $g(x) = \sqrt{x-1}$

Soln:

For $f(x)$:

$f(x) = \sqrt{x+1}$

Here, $f(x)$ is defined for all values of $x \geq 0$.

$\therefore$ Domain $= [0, \infty)$

Also, for all values of domain, the value of y is always $y \geq 1$. So,

Range of function $= [1, \infty)$

For $g(x)$:

$g(x) = \sqrt{x-1}$

Here, $g(x)$ is defined for all value of $x \geq 1$.

$\therefore$ Domain $= [1, \infty)$

Also, for all values of $x \in$ Domain, the value of $f(x)$ is always $y \geq 0$.

So,

Range $= [0, \infty)$

For $f(x) + g(x)$.

$f(x) + g(x) = \sqrt{x+1} + \sqrt{x-1}$

Here,

Domain of $f(x) = [0, \infty)$

Domain of $g(x) = [1, \infty)$

$\therefore$ Domain of $f(x) + g(x) = [1, \infty)$

$\{[0, \infty) \cup [1, \infty)\}$

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~~Range~~: Also, for all value of $x \in [0, \infty) \cup [1, \infty)$
the value is $f(x)$ = is $y \geq \sqrt{2}$
$\therefore$ Range = $[\sqrt{2}, \infty)$

for $(f \cdot g)(x)$

$f(x) \times g(x) = \sqrt{(x+1)} \cdot \sqrt{(x-1)}$
$= \sqrt{x^2 - 1}$

~~Here We know,~~
~~$x^2 - 1 \geq 0$~~ $\quad (x+1) \geq 0 \quad (x-1) \geq 0$
or ~~$x^2 \geq 1$~~ $\quad$ or $x \geq -1 \quad x \geq 1$
~~$\therefore x \geq 1$~~ $\quad \therefore x \geq -1, 1$
~~$\therefore$ Domain = $[1, \infty)$~~

Domain = $(-\infty$, smallest root$] \cup [$largest root$, \infty)$
$= (-\infty, -1] \cup [1, \infty)$

on $y = \sqrt{x^2 - 1}$
or $y^2 = x^2 - 1$
on $x^2 = y^2 + 1$
or $x = \sqrt{y^2 + 1}$

Here ~~f(x) is~~ x is defined for all values of ~~$y \geq 0$~~ y.
& but (y^2+1) overally is positive.

So,
Range = $[0, \infty)$

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<Q.no.5> : Find domain and ranges of f/g and g/f if $f(x) = 2$, $g(x) = x^2+1$

Soln.

For $f(x)/g(x)$.

$$\frac{f(x)}{g(x)} = \frac{2}{x^2+1}$$

Here, f(x) is defined for all values of x

for

Domain ~~x^2+1~~ $= (-\infty, \infty)$

For range:

$$y = \frac{2}{x^2+1}$$

or $x^2+1 = \frac{2}{y}$

or $x^2 = \frac{2}{y} - 1$

or, $x = \sqrt{\frac{2}{y} - 1}$

or $x = \sqrt{\frac{2-y}{y}}$

~~Her~~ We know,

$2 - y \geq 0$

$\therefore y \leq 2$ but $y \neq 0$. as x is not defined when $y = 0$.

Range: $(-\infty, 2] - \{0\}$

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For $g(x) / f(x)$,

$$\frac{g(x)}{f(x)} = \frac{x^2+1}{2}$$

Here, $(g/f)(x)$ is defined for all values of x.

So, Domain = $(-\infty, \infty)$

Also,

$$y = \frac{x^2+1}{2}$$

$$\Rightarrow 2y - 1 = x^2$$

$$\Rightarrow \sqrt{2y-1} = x$$

Here,

$$2y - 1 \geq 0$$

$$\Rightarrow 2y \geq 1$$

$$\Rightarrow y \geq \frac{1}{2}$$

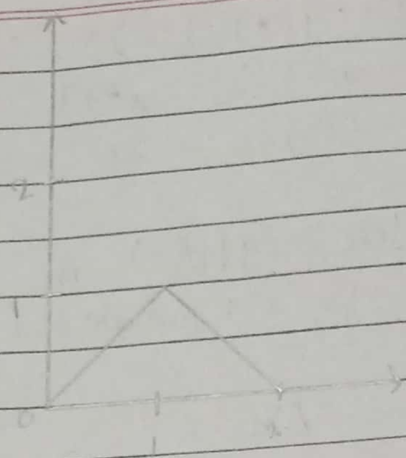
Range : $[1/2, \infty)$

Q.no.67: Graph the functions:

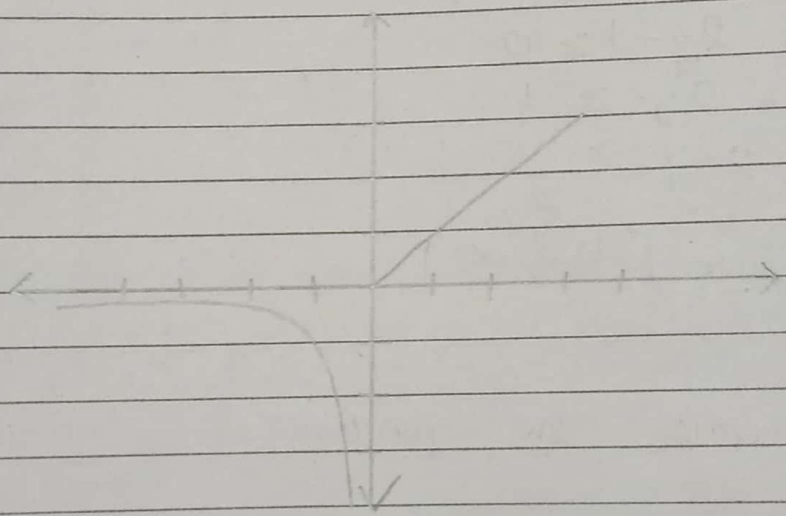
a): $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \end{cases}$

Soln:

x	0	1	2
y	0	1	0



(b) : $g(x) = \begin{cases} 1/x & x < 0 \\ x & 0 \le x \end{cases}$



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Q.no 7: Evaluate the limits:

(a): $\lim\limits_{x\to 2} \dfrac{x-\sqrt{8-x^2}}{\sqrt{x^2+12}-4}$

Sol^n

$= \lim\limits_{x\to 2} \dfrac{x-\sqrt{8-x^2}}{\{\sqrt{x^2+12}-4\}}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$= \lim\limits_{x\to 2} \dfrac{(x-\sqrt{8-x^2})(\sqrt{x^2+12}+4)}{(\sqrt{x^2+12}-4)(\sqrt{x^2+12}+4)} \times \dfrac{(x+\sqrt{8-x^2})}{(x+\sqrt{8-x^2})}$

$= \lim\limits_{x\to 2} \dfrac{x\sqrt{x^2+12}+4x-\sqrt{(8-x^2)(x^2+12)}-4\sqrt{8-x^2}}{x^2+12-16}$

$= \lim\limits_{x\to 2}$

$= \lim\limits_{x\to 2} \dfrac{(x^2-8+x^2)(\sqrt{x^2+12}+4)}{(x^2-4)(x+\sqrt{8-x^2})}$

$= \lim\limits_{x\to 2} \dfrac{2\cancel{(x^2-4)}}{\cancel{(x^2-4)}} \cdot \dfrac{(\sqrt{x^2+12}+4)}{(x+\sqrt{8-x^2})}$

$= \lim\limits_{x\to 2} \dfrac{2(\sqrt{x^2+12}+4)}{x+\sqrt{8-x^2}}$

$= \dfrac{2(\sqrt{4+12}+4)}{2+\sqrt{8-4}} = \dfrac{2\times 8}{4} = 4$

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(b): $\lim_{x \to 2} \frac{4x - x^2}{2 - \sqrt{x}}$

Sol^n:

$$= \lim_{x \to 2} \frac{4x - x^2}{2 - \sqrt{x}} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 4} \frac{(4x - x^2)(2 + \sqrt{x})(4x + x^2)}{(2 - \sqrt{x})(2 + \sqrt{x})\ (4x + x^2)}$$

$$= \lim_{x \to 4} \frac{(16x^2 - x^4)(2 + \sqrt{x})}{(4 - x)(4x + x^2)}$$

$$= \lim_{x \to 4} \frac{x^2(16 - x^2)(2 + \sqrt{x})}{x(4 - x)(4\cancel{x} + x)}$$

$$= \lim_{x \to 4} \frac{\cancel{x^2}\cancel{(16 - x^2)}(2 + \sqrt{x})}{\cancel{x}\cancel{(16 - x^2)}}$$

$$= 4 \times (2 + \sqrt{4})$$

$$= 16$$

(c): $\lim_{x \to -2^-} (x - 3) \frac{|x + 2|}{x + 2}$

Sol^n:

$$\lim_{x \to -2^-} (x - 3) \frac{|x + 2|}{x + 2}$$

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For, $x < -2$, $|x+2| = -(x+2)$

Then,

$$\lim_{x \to -2^-} (x-3) . \frac{-\cancel{(x+2)}}{\cancel{(x+2)}}$$

$$= \lim_{x \to -2^-} -(x-3)$$

$$= -(2-3)$$

$$= 5$$

(d): $\lim_{t \to 4^+} \frac{|t|}{t}$

Soln:

For, $x > 4$, $|t| = t$

Then,

$$\lim_{t \to 4^+} \frac{t}{t}$$

$$= 1$$

(e): $\lim_{x \to 4^-} \frac{|x|}{x}$

Soln

For $x < 4$, $|x| = -x$

Then,

$$\lim_{x \to 4^-} \frac{-x}{x} = -1$$

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(f): $\lim_{x \to \infty} \sqrt{x} - \sqrt{x-3}$

Soln

$= \lim_{x \to \infty} \sqrt{x} - \sqrt{x-3}$ $[\infty - \infty \text{ form}]$

$= \lim_{x \to \infty} \frac{(\sqrt{x} - \sqrt{x-3})(\sqrt{x} + \sqrt{x-3})}{(\sqrt{x} + \sqrt{x-3})}$

$= \lim_{x \to \infty} \frac{\cancel{x} - \cancel{x} + 3}{\sqrt{x} + \sqrt{x-3}} = \frac{3}{\infty} = 0$

(g): $\lim_{x \to \infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

Soln:

$= \lim_{x \to \infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{x \to \infty} \frac{\frac{\sqrt[3]{x}}{\sqrt[3]{x}} - \frac{\sqrt[5]{x}}{\sqrt[3]{x}}}{\frac{\sqrt[3]{x}}{\sqrt[3]{x}} + \frac{\sqrt[5]{x}}{\sqrt[3]{x}}}$

$= \lim_{x \to \infty} \frac{1 - \frac{1}{x^{2/15}}}{1 + \frac{1}{x^{2/15}}}$

$= \frac{1 - 0}{1 + 0} = 1$

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(h): $\lim_{\theta \to 3^-} \frac{|\theta|}{\theta}$

Soln:

$\lim_{\theta \to 3^-} \frac{[\theta]}{\{\theta\}}$

Here, $\theta \to 3^-$, $[\theta] = 3$

So,

$= \frac{3}{3} = 1$

Q.no(8): Find the limits:

(a): $\lim_{x \to 0^+} \frac{2}{x^{2/3}}$

Soln.

$\lim_{x \to 0^+} \frac{2}{x^{2/3}}$

$= \frac{2}{(0)^{2/3}} = \infty$

(b): $\lim_{x \to \pi/2^-} \tan x$

Soln.

$= \lim_{x \to \pi/2} \tan x$

$= \tan \pi/2 = \infty$

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(c): $\lim_{t \to \infty} \frac{2-t+\sin t}{t+\cos t}$

Soln:

$= \lim_{t \to \infty} \frac{2-t+\sin t}{t+\cos t}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

$= \lim_{t \to \infty} \frac{\frac{2}{t} - \frac{t}{t} + \frac{\sin t}{t}}{\frac{t}{t} + \frac{\cos t}{t}}$

$= \frac{0-1+0}{1+0} = -1$

(d): $\lim_{x \to \pi/2^+} \sec x$

Soln:

$= \lim_{x \to \pi/2^+} \sec x$

$= \sec \pi/2 = \infty$

Q.No(9): State Sandwich theorem and use it to find following limits.

Statement: Suppose that $g(x) \le f(x) \le h(x)$ for all x in some interval containing c except possible at $x=c$ itself then,

$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = 1.$ Hence, we get $\lim_{x \to c} f(x) = 1$

(a): Show that if $\lim_{x \to c} |f(x)| = 0$ the $\lim_{x \to c} f(x) = 0$.

Soln:

We have,

$$\lim_{x \to c} |f(x)| = 0$$

We know,

$$-|f(x)| \le f(x) \le |f(x)|$$

Here,

$$\lim_{x \to c} -|f(x)| = 0$$

$$\lim_{x \to c} |f(x)| = 0$$

Therefore, by the defn of Sandwich theorem,

$$\lim_{x \to c} f(x) = 0$$

Hence, proved.

(b): If $\sqrt{2-x^2} \le f(x) \le 2\cos x$ for all x, find $\lim_{x \to 0} f(x)$.

Soln:

Here,

$$\lim_{x \to 0} \cancel{\sqrt{2}}\ 2 - x^2 = 2 - 0^2 = 2$$

$$\lim_{x \to 0} 2\cos x = 2 \times \cos 0 = 2$$

By Sandwich theorem,

$$\lim_{x \to 0} f(x) = 2$$

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(Q): If $1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$, find $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$

Soln:

Given,

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

Now,

$$\lim_{x \to 0} 1 - \frac{x^2}{6} = 1 - \frac{0^2}{6} = 1$$

$$\lim_{x \to 0} 1 = 1$$

By Sandwich theorem,

$$\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = 1.$$

(Q.10): State "$\epsilon - \delta$" definition of the limit for function $f(x)$ at $x = c$. Find $\delta > 0$.

Soln:

The limit of the function $f(x)$ is L if x approaches to x_0 is

$$\lim_{x \to x_0} f(x) = L$$

if for every $\epsilon > 0$, $\delta > 0$ such that if change in distance of x from x_0 is δ, then the distance of $f(x)$ from L is less than ϵ.

i.e, $0 < |x - x_0| < \delta$ then, $|f(x) - L| < \epsilon$.

(a): $f(x) = x+1$, $x_0 = 4$, $\epsilon = 0.01$.

Soln.

~~We know,~~

$\cancel{|f(x) - L| < \epsilon}$

or, $\cancel{|x+1|}$

Given,

$f(x) = x+1$, $x_0 = 4$, $\epsilon = 0.01$.

Now,

$\lim_{x \to x_0(4)} = 4+1 = 5$

$\therefore L = 5$

Now,

$|f(x) - L| < \epsilon$

or, $|x+1-5| < \epsilon$

or, $|x-4| < \epsilon$

or, $\cancel{-(x-4) < \epsilon < (x-4) \text{ — (i)}}$

or, $|x-4| < 0.01$

or, $-0.01 < x-4 < 0.01$

or, $3.99 < x < 4.01$ — (i).

Also,

$|x - x_0| < \delta$

or, $|x-4| < \delta$

or, $-\delta < x-4 < \delta$

or, $4-\delta < x < 4+\delta$ — (ii)

Comparing (i) and (ii);

$4-\delta = 3.99$ $\qquad$ $4+\delta = 4.01$

$\therefore \delta = 0.01$ $\qquad$ $\therefore \delta = 0.01$

$\therefore \delta = 0.01$.

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(b): $\lim_{x\to 1} f(x)=1$ if $f(x)=\begin{cases} x^2 & \text{if } x\neq 1 \\ 2 & \text{if } x=1 \end{cases}$

Soln.

Here,

$f(x)=x^2$, $L=1$, $x_0=1$,

Now,

$0<|x-1|<\delta \Rightarrow |f x^2-1|<\varepsilon$

$(\because |x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon)$

Then,

Here,

$|x-1|<\delta$

or, $-\delta<x-1<+\delta$

or, $1-\delta<x<1+\delta$ — (i)

Also,

$|x^2-1|<\varepsilon$

or, $-\varepsilon<\{x^2-1\}<\varepsilon$

or $1-\varepsilon<x^2<1+\varepsilon$

or $\sqrt{1-\varepsilon}<x<\sqrt{1+\varepsilon}$ — (ii)

Comparing (i) and (ii);

$1-\delta=\sqrt{1-\varepsilon}$

or $\therefore \delta=1-\sqrt{1-\varepsilon}$

Also,

$1+\delta=\sqrt{1+\varepsilon}$

or $\delta=\sqrt{1+\varepsilon}-1$

$\therefore \delta=\text{minimum}\{(1-\sqrt{1-\varepsilon}, \sqrt{1+\varepsilon}-1)$

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(c): $f(x) = \sqrt{1-5x}$, $x_0 = -3$, $\varepsilon = 0.5$

Soln:

$$\lim_{x \to x_0} f(x)$$

$$= \lim_{x \to -3} \sqrt{1 - 5\times(-3)}$$

$$= 4 \qquad \therefore L = 4.$$

Here,

$f(x) = \sqrt{1-5x}$, $x_0 = -3$, $\varepsilon = 0.5$, $L = 4$.

We know,

$$0 < |x+3| < \delta \implies |\sqrt{1-5x} - 4| < \varepsilon.$$

Here,

$$|x+3| < \delta$$

or $-\delta < x+3 < \delta$

or $-\delta - 3 < x < \delta - 3$ — (i)

Also,

$$|\sqrt{1-5x} - 4| < \varepsilon$$

or $-\varepsilon < \sqrt{1-5x} - 4 < \varepsilon$

or, $-0.5 < \sqrt{1-5x} - 4 < 0.5$

or, $3.5 < \sqrt{1-5x} < 4.5$

or, $12.25 < 1 - 5x < 20.25$

or, $11.25 < -5x < 19.25$

or, $-2.25 > x > -3.85$ ~~(ii)~~ or, $-3.85 < x < -2.25$ — (ii)

Comparing (i) and (ii);

$-\delta - 3 = -3.85$

or $\delta = 0.85$

$\delta - 3 = \cancel{-3.85}\ -2.25$

$\therefore \delta = \cancel{0.85}\ 0.75$

$\therefore \delta = (0.85, 0.75)$ minimum

$= 0.75$

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(11): Evaluate:

(a): $\lim_{x \to \pi/2} (\tan x)^{\pi/2 - x}$

Let $y = (\tan x)^{\pi/2 - x}$

Taking log on both sides,

$$\log y = \left(\frac{\pi}{2} - x\right) \log \tan x$$

Taking limits on both sides

$$\lim_{x \to \pi/2} \log y = \lim_{x \to \pi/2} \frac{\log \tan x}{(\pi/2 - x)^{-1}} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\tan x} \times \sec^2 x}{(-1) \times (\frac{\pi}{2} - x)^{-2} \times (-1)} \quad [\text{Using L-Hopital}]$$

$$= \lim_{x \to \pi/2} \frac{\cancel{\cos x}}{\sin x} \times \frac{1}{\cancel{\cos x}} \times \sec x \times \left(\frac{\pi}{2} - x\right)^2$$

$$= \lim_{x \to \pi/2} \frac{2}{\sin 2x} \times \left(\frac{\pi}{2} - x\right)^2 \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \pi/2} 2 \cdot \left(\frac{\pi}{2} - x\right) \times (-1) \times \frac{2}{2\cos 2x}$$

$$= 0.$$

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or, $\lim_{x \to \pi/2} \log y = 0$

$\therefore \lim_{x \to \pi/2} y = e^0 = 1$

$\therefore \lim_{x \to \pi/2} (\tan x)^{\pi/2 - x} = 1.$

(b): $\lim_{x \to 1^+} (x-1)^{1-x}$

Soln.

$\lim_{x \to 1^+} (x-1)^{1-x}$ $[0^0 \text{ form}]$

Let $y = (x-1)^{1-x}$

Putting log on both sides,

$\log y = (1-x) \log(x-1)$

Taking limit on both sides

$\lim_{x \to 1^+} \log y = \lim_{x \to 1^+} (\overset{1-x}{\cancel{x-1}}) \log(x-1)$

Using L-Hopital.

$= \lim_{x \to 1^+} \frac{\log(x-1)}{(1-x)^{-1}}$

$= \lim_{x \to 1^+} \frac{1}{(x-1)} \times \frac{1}{(-1) \times (1-x)^{-2} \times (-1)}$

$= \lim_{x \to 1^+} \frac{(1-x)^2}{(x-1)} = \lim_{x \to 1^+} -(1-x) = 0$

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or $\lim_{x \to 1^+} \log y = 0$

or $\lim_{x \to 1^+} y = e^0 = 1.$

$\therefore \lim_{x \to 1^+} (x-1)^{1-x} = 1.$

(c): $\lim_{x \to 0^+} (1+x)^{\frac{1}{x}}$

Soln:

$\lim_{x \to 0^+} (1+x)^{1/x}$

Let $y = (1+x)^{1/x}$

Putting log on both sides,

$$\log y = \frac{1}{x} \log(1+x)$$

Taking limit on both sides,

$$\lim_{x \to 0^+} \log y = \lim_{x \to 0^+} \frac{1}{x} \log(1+x)$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1+x} \cdot x}{1} \quad \text{[L-Hopital Rule]}$$

$$= \lim_{x \to 0^+} \frac{1}{1+x} \times x$$

$$= \frac{1}{+1} = +1$$

$\lim_{x \to 0^+} y = e^1$ $\therefore \lim_{x \to 0^+} (\cancel{x}1+x)^{\frac{1}{x}} = e$

(d): $\lim\limits_{x \to 0^-} \dfrac{x - \sin^{-1} x}{x^3}$

Sol^n:

$$= \lim_{x \to 0^-} \frac{x - \sin^{-1} x}{x^3} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0^-} \frac{1 - \dfrac{1}{\sqrt{1-x^2}}}{3x^2}$$

$$= \lim_{x \to 0^-} \cancel{1 - (\sqrt{1-x^2})^{-1}} \quad \frac{1 - (\sqrt{1-x^2})^{-1}}{3x^2}$$

$$= \lim_{x \to 0^-} - \frac{(-1) \times (1-x^2)^{-3/2} \times -x}{6x}$$

$$= \lim_{x \to 0^-} \frac{-\cancel{x} \, (1-x^2)^{3/2}}{6\cancel{x}}$$

$$= \frac{-1}{6}$$

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(e): $\lim_{x\to\pi/2} (\sin x)^{\tan x}$

Sol^n:

$\lim_{x\to\pi/2} (\sin x)^{\tan x}$ ~~0^∞ form~~

Let $y = (\sin x)^{\tan x}$

Taking log on both sides,

$$\log y = \tan x \log \sin x$$

Taking limit on both sides,

$$\lim_{x\to\pi/2} \log y = \lim_{x\to\pi/2} \tan x \log \sin x$$

$$= \lim_{x\to\pi/2} \frac{\log \sin x}{(\tan x)^{-1}}$$

$$= \lim_{x\to\pi/2} \frac{\frac{1}{\sin x} \times \cos x}{(-1)(\tan x)^{-2} \times \sec^2 x}$$

$$= \lim_{x\to\pi/2} \frac{\cot x \times \tan^2 x}{-\sec^2 x}$$

$$= \lim_{x\to\pi/2} \frac{\tan x}{-\sec^2 x} = \frac{\sin x}{\cos x} \times -\cos^2 x$$

$$= \lim_{x\to\pi/2} \frac{\log \sin x}{\cot x}$$

$$= \lim_{x\to\pi/2} \frac{\cot x}{-\operatorname{cosec}^2 x} = -\frac{\cos x}{\sin x} \times \sin^2 x$$

$$\lim_{x\to\pi/2} \log y = 0 \qquad = \lim_{x\to\pi/2} y = e^0 = 1$$

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$$\therefore \lim_{x \to \pi/2} (\sin x)^{\tan x} = 1$$

(f): $\lim_{\theta \to \pi/2} \dfrac{1-\cos\theta}{1+\cos 2\theta}$

Soln.:

$$\lim_{\theta \to \pi/2} \frac{1-\cos\theta}{1+\cos 2\theta}$$

$$= \lim_{\theta \to \pi/2} \frac{-(-\sin\theta)}{2\sin 2\theta} \quad [\text{L-Hopital rule}]$$

$$= \lim_{\theta \to \pi/2} \frac{\cos\theta}{-4\cos 2\theta} = \frac{0}{-4} = 0$$

(g): $\lim_{x \to 0} \dfrac{\cos x - 1}{x}$

Soln.

$$\lim_{x \to 0} \frac{-\sin x}{1} = 0$$

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(h) $\lim_{x\to 0} \frac{\ln x - 1}{x - e}$

Soln.

$= \lim_{x\to 0} \frac{\ln x - 1}{x - e}$

$= \lim_{x\to 0} \frac{\frac{1}{x} - \cancel{1}}{1 - e}$ $\quad = \lim_{x\to 0} \frac{\frac{1}{x}}{1 - 0}$

$= \lim_{x\to 0} \frac{x^{-1} - 1}{1 - \cancel{e}}$ $\quad = \lim_{x\to 0} \frac{1}{x}$

$= \lim_{x\to 0} \frac{-1 x^{-2}}{e}$ $\quad = \lim_{x\to 0} \frac{0}{1} = \infty$ (limit don't exist)

$= 0$ $\quad = \infty$.

(Q.no. 12) : For what value of a is the function.

(a): $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 3 \\ 2ax & \text{if } x \geq 3 \end{cases}$ continuous at every x.

Soln:

Here,

$\lim_{x\to 3^-} x^2 - 1$

$= \lim_{x\to 3^-} 9 - 1 = 8$

For the function to be continuous,

$\lim_{x\to 3^+} 2ax = 8$

or, $26a = 8$ $\therefore a = 4/3$

(b): $f(x) = \begin{cases} x^2 & \text{if } x < 3 \\ 4ax^3 - 1 & \text{if } x \geq 3 \end{cases}$ continuous at every x.

Soln:

$\lim_{x \to 3^-} x^2$

$= \lim_{x \to 3} = 3^2 = 9$

For the function to be continuous,

$\lim_{x \to 3^+} 4ax^3 - 1 = 9$

or, $4 \times a \times 27 - 1 = 9$

or, $108a = 9$ $\therefore a = \frac{1}{12}$

(Q.no 13): Find slope, the equation of tangent and normal to the following curves at given point:

(a): $x^2 + xy - y^2 = 1$ at $(2,3)$

Soln:

Given,

$x^2 + xy - y^2 = 1$

Differentiating both sides w.r.t x,

$\frac{dx^2}{dx} + \frac{dxy}{dx} - \frac{dy^2}{dx} = \frac{d \cdot 1}{dx}$

or, $2x + x \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} = 0$

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$$6 + 2\frac{dy}{dx} - 6\frac{dy}{dx} = 0$$

or $6 = 4\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{3}{2} = m$ (slope)

~~At~~ We have,

$m = 3/2$ $\qquad (x_1, y_1) = (2, 3)$.

Let The eqn of tangent be.

$y - y_1 = m(x - x_1)$

or, $y - 3 = \frac{3}{2}(x - 2)$

or, $2y - \cancel{6} = 3x - \cancel{6}$

or $3x - 2y = 0$. which is reqd. eqn of tangent

Let the eqn of normal be.

$y - y_1 = \frac{-1}{m}(x - x_1)$

or, $y - 3 = \frac{-1}{(3/2)}(x - 2)$

or $y - 3 = \frac{-2}{3}(x - 2)$

or, $3y - 9 = -2x + 4$

or $2x + 3y - 13 = 0$

which is the reqd. eqn of normal.

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Q(b): $x\sin 2y = y\cos 2x$. at $(\pi/4, \pi/2)$.

Soln:

$$x\sin 2y = y\cos 2x$$

Differentiating both sides w.r.t x,

$$\frac{d}{dx}(x\sin 2y) = \frac{d}{dx}(y\cos 2x)$$

or, $x\cdot\frac{d\sin 2y}{d2y}\times\frac{dy}{dx}\times 2 + \sin 2y\cdot 1 = y\cdot\frac{d\cos 2x}{d2x}\times 2 + \cos 2x\cdot\frac{dy}{dx}$

or, $2x\cdot\cos 2y\times\frac{dy}{dx} + \sin 2y = -2y\sin 2x + \cos 2x\cdot\frac{dy}{dx}$

or, $2x\cos 2y\frac{dy}{dx} - \cos 2x\cdot\frac{dy}{dx} = -2y\sin 2x - \sin 2y$

or, $\frac{dy}{dx}\left(2x\cos 2y - \cos 2x\right) = -2y\sin 2x - \sin 2y$

or, $\frac{dy}{dx}\left(2\cdot\frac{\pi}{4}\cdot\cos 2\cdot\frac{\pi}{2} - \cos 2\times\frac{\pi}{4}\right) = -2\times\frac{\pi}{2}\cdot\sin 2\frac{\pi}{4} - \sin 2\cdot\frac{\pi}{2}$

or, $\frac{dy}{dx}\left(-\frac{\pi}{2} - 0\right) = -\pi - 0$

or, $\frac{dy}{dx} = \frac{-\pi}{\left(-\frac{\pi}{2}\right)}$ $\therefore \frac{dy}{dx} = 2 = m$

We have,

$m = 2$ $(x_1, y_1) = (\pi/4, \pi/2)$

Let the eqn of tangent be

$$y - y_1 = m(x - x_1)$$

or, $y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right)$

or, $y - \pi/2 = 2x - \pi/2$ or, $2x - y = 0$

which is the reqd. eqn

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Let the eq^n of the normal be.

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

or, $y - \frac{\pi}{2} = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$

or $2y - \pi = -x + \frac{\pi}{4}$

or $x + 2y - \frac{5\pi}{4} = 0$

or, $x + 2y - 225 = 0$ which is the reqd eq^n.

(c): $x^2 \cos 2x - \sin y = 0$ at $(0, \pi)$

Soln:

or, $x^2 \cos 2x - \sin y = 0$

or, $x^2 \cos 2x = \sin y$

Diff. both sides w.r.t. x,

$x^2 \cdot \frac{d\cos 2x}{d2x} \times 2 + \cos 2x \cdot \frac{dx^2}{dx} = \frac{d\sin y}{dy} \times \frac{dy}{dx}$

or $-2x^2 \sin 2x + 2x \cos 2x = \cos y \times \frac{dy}{dx}$

or $-2 \times 0^2 \sin 2 \times 0 + 2 \times 0 \cdot \cos 2 \times 0 = -1 \times \frac{dy}{dx}$

or $0 + 0 = -1 \times \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = 0. \Rightarrow m = 0.$

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Let the tangent be

$y - y_1 = m(x - x_1)$

or, $y - \pi = \cancel{m}\, 0 \times (x - 0)$

$\therefore y = \pi$

Let the normal be.

$y - y_1 = \frac{-1}{m}(x - x_1)$

or $0 = \cancel{\pi x} - x + 0$

$\therefore x = 0$ which is the eqn of normal.

(d): $y = e^x + 2$ at $(0, 2)$

Soln:

$y = e^x + 2$

Diff both sides w.r.t x,

$\frac{dy}{dx} = \frac{de^x}{dx} + \frac{d.2}{dx}$

$\therefore \frac{dy}{dx} = e^0 = 1.$ $m = 1$

We have,

$m = 1,$ $(x_1, y_1) = 0, 2.$

Let the eqn of tangent be.

$y - y_1 = m(x - x_1)$

or $y - 2 = 1 \times x$

or $x - y + 2 = 0$ which is the reqd eqn.

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Let the eq^n of the normal be:

$$y-y_1 = \frac{-1}{m}(x-x_1)$$

or, $$y-2 = \frac{-1}{1}(x)$$

$\therefore$ $x+y-2=0$ which is the reqd eq^n.

<Q.no14>: Does the curve $y=x^4-2x^2+2$ have any horizontal tangent? If yes, where?

Sol^n:

Given,

$$y = x^4 - 2x^2 + 2$$

Diff. both sides w.r.t x,

$$\frac{dy}{dx} = \frac{dx^4}{dx} - 2\cdot\frac{dx^2}{dx} + \frac{d.2}{dx}$$

or, $$\frac{dy}{dx} = 4x^3 - 4x$$

For horizontal tangent,

$$\frac{dy}{dx} = 0$$

So, $4x^3 - 4x = 0$

or, $~~4x~~$ $4x(x^2-1) = 0$

either, $x = 0$ or, $x = \pm 1$

When,

$x=0$, $y=2$

$x=1$, $y=1$

$x=-1$, $y=1$

$\therefore$ The curve has horizontal tangents at $(0,2)$, $(1,1)$, $(-1,1)$

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(Q.no15): The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent to the line at point $(1, 0)$. Find a, b, c.

Soln:

Given,

$y = x^2 + ax + b$.

Since the curve passes through $(1, 0)$.

$0 = 1 + a + b$

or $a + b = -1$ —(i)

Also,

$y = cx - x^2$

Since the curve passes through $(1, 0)$.

$0 = c - 1$

$\therefore c = 1$ —(ii)

As two curves have common tangent,

$\frac{dy}{dx} = \frac{d(x^2 + ax + b)}{dx} = 2x + a$

At $(1, 0)$, $\frac{dy}{dx} = 2 + a$ —(iii)

Also,

$\frac{dy}{dx} = \frac{d(cx - x^2)}{dx}$

$= c - 2x$

At $(1, 0)$, $\frac{dy}{dx} = c - 2$ —(iv)

From (iii) & (iv),

$2 + a = c - 2$ or, $-3 + a = 0$ $\therefore a = -3$

Putting value of a in (ii),

$$-3+b=-1$$
$$\therefore b=2$$

$\therefore$ The values of a, b, c are $-3, 2, 1$.