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Unit: 2

FUNCTIONS OF SEVERAL VARIABLES

Functions of Several Variables

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A real-valued function f on D is a rule that assigns a unique (single) element

$$w = f(x_1, x_2, \ldots, x_n)$$

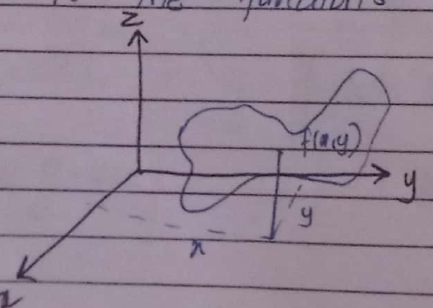
to each element in D.

The set D is the function's domain.

The set of w-values taken on by f is the function's range.

The symbol 'w' is the dependent variable of f, and f is said to be a function of the n independent variables x_1 to x_n.

The x_j's is the function's input variables.
The w is the function's output variables.



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Q7: Find the domain and the ranges for the following functions.

Functions	Domain	Range.
$z = \sqrt{y - x^2}$	$y - x^2 \geq 0$ $\therefore y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$.	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	$\mathbb{R} \times \mathbb{R}$ {entire plane}	$[-1, 1]$
$w = \sqrt{x^2 + y^2 + z^2}$	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ {entire space}.	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half space, $z > 0$	$(-\infty, \infty)$

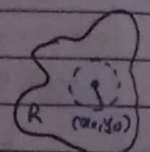
Interior Point:

A point (x_0, y_0) in a region (set) R in the xy-plane is an interior point of R if it is the center of a disk of positive radius that lies entirely in R.

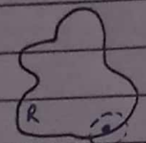
Boundary Point

A point (x_0, y_0) is a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie at R.

The boundary point need not belong to R.



Interior point



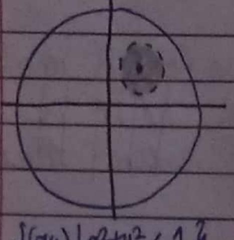
Boundary point

Open Sets (Region)

A region is said to be open if it consists entirely of interior points.

Closed Region

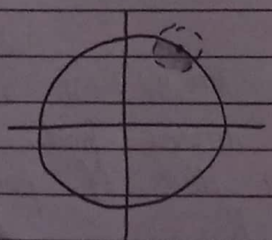
A region is said to be closed if it consists of all boundary points.



$\{(x,y) \mid x^2+y^2<1\}$

Open region

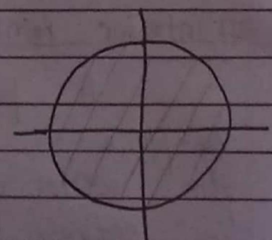
→ Every point is an interior point.



$\{(x,y) \mid x^2+y^2=1\}$

Boundary.

→ Boundary of disc



$\{(x,y) \mid x^2+y^2 \le 1\}$

Closed region

→ Contains all boundary points.

Bounded Regions:

A region in plane is bounded if it lies inside a disk of fixed radius.

Eg: Line segments, triangles, rectangles, disks, etc.

Unbounded Regions:

A region in plane is said to be unbounded if it is not bounded.

Eg: Line, Coordinate Axes, Half Planes, Planes.

In 3-d,

→ Open sets: Space, open balls, open half space $(z>0)$

→ Closed sets: Line, Planes, closed half space $(z \ge 0)$.

→ Neither open nor close: Cube with missing face.

Level Curves

The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y)=c$ is called a level curve of f.

The set of all points $(x, y, f(x,y))$ in space for (x,y) in the domain of f, is called the graph of f.

The graph of f is also called surface $z = f(x,y)$.

The curve in which the plane $z=c$ cuts a surface $z=f(x,$ is made up of the point representing $f(x,y)=c$, called contour lin

Level Surface of Function of three variables

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x,y,z) = c$ is called level surface of f.

<Q>: Describe the level surface of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$.

Sol^n:

Given,

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$

To find the level surface, let $f(x,y,z) = c$

So,

$\sqrt{x^2+y^2+z^2} = c$

$x^2+y^2+z^2 = c^2$ and $c \geq 0$.

<Q>: Find:

a) Domain
b) Range
c) Level Curves
d) Boundary of function's domain.
e) Determine if domain is open, closed or neither
f) Decide if domain is bounded or unbounded.

(1): $f(x,y) = \sqrt{y-x}$.

Sol^n:

a) Domain: $y - x \geq 0$ or, $y \geq x$.

b) Range: $z \geq 0$. i.e, $[0, \infty)$

c) Level curves;

$\sqrt{y-x} = c$

or, $y - x = c^2$ and $c \geq 0$.

d) Boundary of function's domain: [$\because$ presence of equality sign] $y = x$.

e) The domain is closed.

f) Since domain $y \geq x$, the function is unbounded.

(2): $f(x,y) = \ln(x^2+y^2)$.

Sol^n:

a) Domain: $(x,y) \neq (0,0)$

b) Range: $(-\infty, \infty)$

c) Level curves:

$\ln(x^2+y^2) = c$

or $x^2+y^2 = e^c$.

d) Boundary:

It has a single point $(0,0)$.

e) The domain is open.

f) The domain $(x,y) \neq (0,0)$, it is unbounded.

<3>: $f(x,y) = \dfrac{1}{\sqrt{16-x^2-y^2}}$

Soln.

Given,

$$f(x,y) = \frac{1}{\sqrt{16-(x^2+y^2)}}$$

a) Domain: $x^2+y^2 < 16$

b) Range: $z \geq \frac{1}{4}$

c) Level curves:

$$c = \frac{1}{\sqrt{16-x^2-y^2}}$$

$$\text{or } \cancel{c^2 = \frac{1}{16-x^2-y^2}}$$

$$\text{or } \cancel{c^2 16 - x^2c^2 - y^2c^2 = 1}$$

$$\text{or } \cancel{c^2 16 = (x^2+y^2)c^2}$$

$$\cancel{\text{or } x^2+y^2 = 16}$$

$$k^2 = 16 - x^2 - y^2$$

$$\text{or } x^2+y^2 = 16-k^2.$$

circles with radius < 4.

(d) The boundary:

If $k=0$,

$x^2+y^2=16$ is the boundary but it doesn't belong to domain.

(e) The domain is open

(f): The domain is bounded.

<Q>: Find an equation for the level curve/surface of the given function which passes through the given point.

<a>: $f(x,y) = 16-x^2-y^2$ at $(2\sqrt{2}, \sqrt{2})$

Soln:

Given, $f(x,y) = 16-x^2-y^2$

and

$$f(2\sqrt{2}, \sqrt{2}) = 16-(2\sqrt{2})^2-(\sqrt{2})^2 = 6$$

So,

$$f(x,y) = 6$$

$$\text{or } 16-x^2-y^2 = 6$$

$$\therefore x^2+y^2 = 10.$$

(b): $f(x,y,z) = \sqrt{x-y} - \ln z$ at $(3,-1,1)$

Soln:

Given, $f(x,y,z) = \sqrt{x-y} - \ln z$

At $(3,-1,1)$,

$f(3,-1,1) = \sqrt{3+1} - \ln 1$

$= 2$

Now,

$f(x,y,z) = 2$

or $\sqrt{x-y} - \ln z = 2$.

(c): $f(x,y,z) = \ln(x^2+y^2+z^2)$ at $(-1,2,1)$

Soln:

Given,

$f(x,y,z) = \ln(x^2+y^2+z^2)$

Let $f(x,y,z) = c$

So,

$x^2+y^2+z^2 = e^c$

Now

At $(-1,2,1)$,

$\ln(6) = c$

or, $6 = e^c$

$\therefore x^2+y^2+z^2 = 6$.

Limits in Two Dimensions

We say that limit of function (x,y) approaches the limit L as (x,y) approaches (x_0,y_0) i.e,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for every $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

for all domains of f

$|f(x,y) - L| < \epsilon$ whenever ~~$0<\sqrt{}$~~

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

(x) theorem:

Let k = any real numbers.

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$.

Then,

1) Sum rule: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L+M$

2) Difference rule: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L-M$

(3) Constant multiple rule. $\lim_{(x,y)\to(x_0,y_0)} k f(x,y) = kL$

(4) Product rule. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$

(5) Quotient rule: $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$.

(6) Power rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, $n \geq 0$

(7) Root rule: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$

Q7: Evaluate the following limits:

(a): $\lim_{(x,y)\to(0,1)} \frac{x+y}{2xy+x+y}$

Sol^n.

Given

$= \lim_{(x,y)\to(0,1)} \frac{x+y}{2xy+x+y}$

$= \frac{0+1}{2\times0\times1+0+1} = 1.$

(b): $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Sol^n:

$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)\times(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}$

$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}}$

$= 0$

(c) $\lim_{\substack{(x,y)\to(2,2)\\ x-y\neq0}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

Sol^n.

$\lim_{(x,y)\to(2,2)} \frac{x+2\sqrt{x}-y-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

$= \lim_{(x,y)\to(2,2)} \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})+2(\sqrt{x}-\sqrt{y})}{(\sqrt{x}-\sqrt{y})}$

$= \lim_{(x,y)\to(2,2)} \frac{\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y}+2)}{\cancel{(\sqrt{x}-\sqrt{y})}}$

$= 2+2\sqrt{2}.$

<(d)>: $$\lim_{(x,y)\to(\pi/2,\,0)} \frac{\cos y + 1}{y - \sin x}$$

Soln:

Given,

$$\lim_{(x,y)\to(\pi/2,\,0)} \frac{\cos y + 1}{y - \sin x}$$

$$= \frac{\cos 0 + 1}{0 - \sin \pi/2} = -2$$

<(e)>: $$\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2 - xy + 4x^2 - 4x}$$

Soln.

$$\lim_{(x,y)\to(2,-4)} \frac{y+4}{(x-2)^2 - xy}$$

$$= \frac{-4+4}{(2-2)^2 - 2\times(-4)} = \frac{0}{8} = 0$$

<(f)>: $$\lim_{\cancel{P} \to (\pi/2,\,\pi/2,\,0)} \sin^2 x + \sin y \cos z$$

Soln.

$$\lim_{P\to(\pi/2,\,\pi/2,\,0)} \sin^2\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \times \cos 0$$

$$= 1 + 1\times 1$$

$$= 2$$

Continuity

A function $f(x,y)$ is continuous at the point (x_0, y_0) if

(i): f is defined as (x_0, y_0)

(ii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

(iii) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

A function is continuous if it is continuous at every point of the domain.

x) Two-path Test for Non-existence of a Limit

If a function $f(x,y)$ has different limits along two different paths in the domain of f as (x,y) approaches (x_0, y_0) then, $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ ~~exists~~ doesn't exist.

<Q>: show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous at origin.

Show that limit of the function doesn't exist at origin. {... is not continuous at origin but continuous at other points.}

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Soln.

Given,

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

At origin, $y = mx$ i.e. path approaching origin.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \frac{2x^2 m}{x^2 + m^2x^2}$$

$$= \frac{2m \cancel{x^2}}{\cancel{x^2}(1+m^2)}$$

$$\therefore L = \frac{2m}{(1+m^2)}$$

Here, L depends on 'm' and 'm' depends on path. So, limiting value changes with path

Note: If no 'm' in limiting ~~pa~~ value, it is path independent.

For $m=0$, $L=0$

For $m=1$, $L=1$

By two path test, we can confirm that limit of the function doesn't exist at (0,0)

So, the function is not continuous at (0,0).

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(*) Note:

If (x_0, y_0) is other than origin,

$$f(x_0, y_0) = \frac{2x_0 y_0}{x_0^2+y_0^2} \text{ is defined } (x_0^2+y_0^2 \neq 0)$$

$\therefore$ The limit exists at (x_0, y_0)

Thus, function is continuous except origin.

<A>: Find the limits of the given functions at origin if they exist.

1): $f(x,y) = \dfrac{x^4-y^2}{x^4+y^2}$

Soln.

~~At~~ Along $y = mx^2$,

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx^2}} \frac{x^4 - (mx^2)^2}{x^4 + (mx^2)^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4 - m^2x^4}{x^4+m^2x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{x^4}(1-m^2)}{\cancel{x^4}(1+m^2)}$$

$$\therefore L = \frac{1-m^2}{1+m^2}$$

Since the existence of 'm' in limiting value, it is path dependent

$\therefore$ It is discontinuous at (0,0).

<2>: $f(x,y) = \dfrac{x^2-y^6}{x^2+y^6}$

Sol^n:

Given,

$\lim f(x,y) = \dfrac{x^2-y^6}{x^2+y^6}$

Along $y = mx^{1/3}$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=mx^{1/3}}} \frac{x^2 - m^6x^2}{x^2+m^6x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{x^2}(1-m^6)}{\cancel{x^2}(1+m^6)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{1-m^6}{1+m^6}$$

$$\therefore L = \frac{1-m^6}{1+m^6}$$

Due to the existence of 'm' in limiting value, it is path dependent.

Thus, it is not continuous at (0,0).

<3>: $f(x,y) = \dfrac{x^2}{x^2+y^2}$

Sol^n:

We know,

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

Given,

$f(x,y) = \dfrac{x^2}{x^2+y^2} = \dfrac{r^2\cos^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= \cos^2\theta$

Since the existence of 'θ' indicates the function to being path dependent, we can conclude that limit of function doesn't exist at origin

When $\theta = 0$, $\cos^2\theta = 1$

when $\theta = \pi/2$, $\cos^2\theta = 0$

Thus, $f(x,y)$ is not continuous at (0,0).

<B>: At what points, given function are continuous?

(1): $f(x,y) = \sin(x+y)$ $\Rightarrow$ all (x,y).

(2) $f(x,y) = \dfrac{x^2+y^2}{x^2-3x+2}$ $\Rightarrow$ $(-\infty,\infty) - \{1,2\}$

Now, $x^2-3x+2 \neq 0$

So, $x = 1, 2$.

<3>. $f(x,y,z) = \frac{1}{|y|+|z|}$ ⇒ all points except $(x,0,0)$

Partial ~~Diff~~ Derivatives

The partial derivative of $f(x,y)$ with respect to x at point (x_0, y_0) is

$$f_x = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

The partial derivative of $f(x,y)$ with respect to y at point (x_0,y_0) is

$$f_y = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

<A>: From first principle, find the derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

<a>: $f(x,y) = 1 - x^2 - y^2 - 2xy$ at $(1,1)$

Soln.

Now,

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

or, $$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = \lim_{h\to 0} \frac{f(1+h, 1) - f(1,1)}{h}$$

$$= \lim_{h\to 0} \frac{\{1-(1+h)^2 - 1^2 - 2(1+h)\times 1\} - \{1 - 1^2 - 1^2 - 2\times 1\times 1\}}{h}$$

$$= \lim_{h\to 0} \frac{\{\cancel{1} - (1+2h+h^2) - \cancel{1} - (2+2h)\} - \{\cancel{1} - \cancel{1} - 1 - 2\}}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{1} - 2h - h^2 - \cancel{2} - 2h + \cancel{3}}{h}$$

$$= \lim_{h\to 0} \frac{-h^{\cancel{2}}}{\cancel{h}} - \frac{4\cancel{h}}{\cancel{h}}$$

$$= -4.$$

Again,

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

or, $$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \lim_{h\to 0} \frac{f(1, 1+h) - f(1,1)}{h}$$

$$= \lim_{h\to 0} \frac{\{1 - (1)^2 - (1+h)^2 - 2\cdot 1(1+h)\} - \{\cancel{1} - 1^2 - 1^2 - 2\times 1\times 1\}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1} - \cancel{1} - \cancel{1} - 2h - h^2 - \cancel{2} - 2h + \cancel{3}}{h} = \lim_{h\to 0} \frac{-h^{\cancel{2}}}{\cancel{h}} - \frac{4\cancel{h}}{\cancel{h}}$$

$$= -4.$$

<b>: $f(x,y,z) = x^2y^2z^2$ at $(1,2,3)$

Soln:

Given,

$$f(x,y,z) = x^2y^2z^2$$

Now,

i) $$\left.\frac{\partial f}{\partial x}\right|_{(1,2,3)} = \lim_{h\to 0} \frac{f(x_0+h, y_0, z_0) - f(x_0,y_0,z_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h,2,3) - f(1,2,3)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^2 2^2 3^2 - 1^2 2^2 3^2}{h}$$

$$= \lim_{h\to 0} \frac{(1+2h+h^2)36 - 36}{h} = \lim_{h\to 0} \frac{\cancel{36} + 72h + 36h^2 - \cancel{36}}{h}$$

$$= \lim_{h\to 0} \frac{36h^{\cancel{2}}}{\cancel{h}} + \frac{72\cancel{h}}{\cancel{h}}$$

$$= 72$$

ii) $$\left.\frac{\partial f}{\partial y}\right|_{(1,2,3)} = \lim_{h\to 0} \frac{f(x_0, y_0+h, z_0) - f(x_0,y_0,z_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(1,2+h,3) - f(1,2,3)}{h}$$

$$= \lim_{h\to 0} \frac{1^2 \times (2+h)^2 \times 3^2 - 1^2 2^2 3^2}{h}$$

$$= \lim_{h\to 0} \frac{9(4+4h+h^2) - 36}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{36} + 36h + 9h^2 - \cancel{36}}{h} = \lim_{h\to 0} \frac{36h}{h} + \frac{9h^{\cancel{2}}}{\cancel{h}}$$

$$= 36.$$

(iii) $$\left.\frac{\partial f}{\partial z}\right|_{(1,2,3)} = \lim_{h\to 0} \frac{f(1,2,3+h) - f(1,2,3)}{h}$$

$$= \lim_{h\to 0} \frac{1^2 2^2 (3+h)^2 - 1^2 2^2 3^2}{h}$$

$$= \lim_{h\to 0} \frac{4(9+6h+h^2) - 36}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{36} + 24h + 4h^2 - \cancel{36}}{h}$$

$$= \lim_{h\to 0} \frac{24\cancel{h}}{\cancel{h}} + \frac{4h^{\cancel{2}}}{\cancel{h}}$$

$$= 24$$

<B>: Find f_x, f_y if

(1) $f(x,y) = xy^2$

Sol^n:

(i) $f_x = \frac{\partial f}{\partial x} = \frac{\partial(xy^2)}{\partial x}$

$= y^2 \frac{\partial x}{\partial x} = y^2$

(ii) $f_y = \frac{\partial f}{\partial y} = \frac{\partial(xy^2)}{\partial y} = x \frac{\partial y^2}{\partial y}$

$= 2xy$

(2) $f(x,y) = e^{-x} \sin(x+y)$

Sol^n:

(i): $f_x = \frac{\partial f}{\partial x} = \frac{\partial(e^{-x} \cdot \sin(x+y))}{\partial x}$

$= e^{-x} \cdot \frac{\partial(\sin(x+y))}{\partial(x+y)} \times \frac{\partial(x+y)}{\partial x} + \sin(x+y) \cdot \frac{\partial e^{-x}}{\partial x}$

$= e^{-x} \cdot \cos(x+y) + \sin(x+y)(-e^{-x})$

$\therefore f_x = e^{-x}(\cos(x+y) - \sin(x+y))$

(ii) $f_y = \frac{\partial f}{\partial y} = \frac{\partial(e^{-x} \cdot \sin(x+y))}{\partial y}$

$= e^{-x} \left[\frac{\partial \sin(x+y)}{\partial(x+y)} \times \frac{\partial(x+y)}{\partial y} \right]$

$\therefore f_y = e^{-x} \cdot \cos(x+y)$

<C>: Find f_x and f_y at $(1,2)$ if $f(x,y) = 1-x+y-3x^2y$

Sol^n:

Given,

$f(x,y) = 1 - x + y - 3x^2y$.

Now,

(i): $f_{x\,(1,2)} = \frac{\partial f}{\partial x} = \frac{\partial(1-x+y-3x^2y)}{\partial x}$

$= 0 - 1 + 0 - \cancel{6y}\ 6xy$

$= -1 - 6xy$

$= -1 - 6 \times 1 \times 2$ $\therefore f_x = -13$

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(ii): $f_{y(1,2)} = \frac{\partial f}{\partial y} = \frac{\partial(1-x+y-3x^2y)}{\partial y}$

$= 0 - 0 + 1 - 3x^2$

$= 1 - 3\times1^2$

$= 1 - 3 = -2$

$\therefore f_{y(1,2)} = -2$

Second Order Partial Derivatives

$\frac{\partial^2 f}{\partial x^2}$ or f_{xx} $\qquad$ $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

$\frac{\partial^2 f}{\partial x \partial y}$ or f_{yx} $\qquad$ $\frac{\partial^2 f}{\partial y \partial x}$ or f_{xy}

In fact,

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y$

$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x$

(X): Laplace's Equation: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

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<Q>: Show that $f(x,y,z) = e^{3x+4y}.\cos 5z$ satisfies Laplace's equation.

Soln:

Given,

$f(x,y,z) = e^{3x+4y}.\cos 5z$

(i): $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

$= \frac{\partial}{\partial x}\left(\frac{\partial e^{(3x+4y)}}{\partial e^{(3x+4y)}} . \frac{\partial(3x+4y)}{\partial x} . \cos 5z\right)$

$= \frac{\partial}{\partial x}(e^{3x+4y} . 3 . \cos 5z)$

$= 3\cos 5z \left[\frac{\partial}{\partial e^{3x+4y}}\left(e^{3x+4y}\right) \times \frac{\partial(3x+4y)}{\partial x}\right]$

$\therefore \frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y}\cos 5z$

(ii): $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial(e^{3x+4y}.\cos 5z)}{\partial y}\right)$

$= \frac{\partial}{\partial y}\left[\frac{\partial e^{(3x+4y)}}{\partial(3x+4y)} . \frac{\partial(3x+4y)}{\partial y} . \cos 5z\right]$

$= \cos 5z \frac{\partial}{\partial y}\left(e^{3x+4y} . 4\right)$

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$$= 4\cos 5z \cdot \left(\frac{\partial(e^{3x+4y})}{\partial(3x+4y)} \times \frac{\partial(3x+4y)}{\partial y} \right)$$

$$\therefore \frac{\partial^2 f}{\partial y^2} = 16\cos 5z \, e^{3x+4y}$$

(iii): $\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial(e^{(3x+4y)} \cdot \cos 5z)}{\partial z}\right)$

$$= e^{(3x+4y)} \cdot \frac{\partial}{\partial z}\left(\frac{\partial \cos 5z}{\partial z}\right)$$

$$= e^{3x+4y} \frac{\partial}{\partial z}(-5 \sin 5z)$$

$$\frac{\partial^2 f}{\partial z^2} = -25 \, e^{3x+4y} \cdot \cos 5z$$

Now,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 9e^{3x+4y}\cos 5z + 16e^{3x+4y}\cos 5z - 25e^{3x+4y}\cos 5z$$

$$= 0$$

$f(x,y,z)$ satisfies Laplace's equation.

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Mixed Derivative Theorem: (Euler's or Clairaut's)

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (a,b) and are all continuous at (a,b) then,

$$f_{xy}(a,b) = f_{yx}(a,b)$$

<Q>: Verify mixed derivative theorem.

(a): $f(x,y) = y + \frac{x}{y}$

Soln:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial(y + x/y)}{\partial x} = 0 + \frac{1}{y} = \frac{1}{y}$$

$$f_y = \frac{\partial f}{\partial x} = \frac{\partial(y + x/y)}{\partial y} = 1 - \frac{x}{y^2}$$

Now,

$$(f_x)_y = \frac{\partial(1/y)}{\partial y} = \frac{-1}{y^2}$$

$$(f_y)_x = \frac{\partial(1 - x/y^2)}{\partial x} = -\frac{1}{y^2}$$

$$\therefore (f_x)_y = (f_y)_x.$$

(b) $w = x\sin y + y\sin x + xy$

Sol^n:

Now,

$$\frac{\partial w}{\partial x} = \frac{\partial\, x\sin y}{\partial x} + \frac{\partial\, y\cdot\sin x}{\partial x} + \frac{\partial\, xy}{\partial x}$$

$$\therefore f_x = \sin y + y\cos x + y$$

Again,

$$\frac{\partial w}{\partial y} = \frac{\partial\, x\sin y}{\partial y} + \frac{\partial\, y\sin x}{\partial y} + \frac{\partial\, xy}{\partial y}$$

$$\therefore f_y = x\cos y + \sin x + x$$

Now,

$$(f_x)_y = \frac{\partial f_x}{\partial y} = \frac{\partial \sin y}{\partial y} + \frac{\partial\, y\cos x}{\partial y} + \frac{\partial\, y}{\partial y}$$

$$= \cancel{\cos y = y\sin x + 1}$$

$$\therefore (f_x)_y = \cos y + \cos x + 1$$

$$(f_y)_x = \frac{\partial f_y}{\partial x} = \frac{\partial\, x\cos y}{\partial x} + \frac{\partial \sin x}{\partial x} + \frac{\partial x}{\partial x}$$

$$= \cos y + \cos x + 1.$$

$\therefore (f_x)_y = (f_y)_x$ proved.

Linearization:

The ~~definition~~ linearization of a function $f(x,y)$ at a point (x_0, y_0) when f is differentiable is the function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

(x): Standard Linear Approximation:

The approxiamation $f(x,y) \approx L(x,y)$ is the standard linear approximation of f at (x_0, y_0).

<Q7: Find the linearization $L(x,y)$ of the function $f(x,y) = x^2+y^2+1$ at $(1,1)$.

Sol^n:

Given,

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial (x^2+y^2+1)}{\partial x} = 2x + y^2$$

At $(1,1)$

$$f_x(1,1) = 3$$

$$f(1,1) = 1^2+1^2+1 = 3$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial (x^2+y^2+1)}{\partial y} = x^2 + 2y$$

At $(1,1)$

$$f_y(1,1) = 1 + 2\times 1 = 3.$$

We know,

$$L(1,1) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$= 3 + 3(x-1) + 3(y-1)$$
$$= \cancel{3} + 3x - \cancel{3} + 3y - 3$$

$$\therefore L(1,1) = 3x + 3y - 3 \quad = 3(x+y-1)$$

Total Differentiation:

If we move from (x_0, y_0) to a point (x_0+dx, y_0+dy) nearby, the resulting differential in f is.

$$df = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

This change in linearization of f is called the total differential of f.

(X): Note:

For more than two variables: At (x_0, y_0, z_0)

$$L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$$

$$\cancel{df = f_x(P_0) + f_y(P_0)}$$

$$df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$

Absolute, Relative and % Change

	True/ Actual	Estimate/ Approxiamation
Absolute	Δf	df
Relative	$\dfrac{\Delta f}{f(x_0,y_0)}$	$\dfrac{df}{f(x_0,y_0)}$
%	$\dfrac{\Delta f}{f(x_0,y_0)} \times 100\%$	$\dfrac{df}{f(x_0,y_0)} \times 100\%$