The radius of gyration can also be defined as a distance whose squared value multiplied with the total mass of system / body gives the moments of inertia about the given axis.

It depends upon,
- Shape of body
- size of body
- axis of rotation.

⊛ Physical significance of M.o.I

Moment of inertia plays the same role in rotational motion as mass does in translation motion. This is physical significance of moment of inertia.

$\perp^r$ and $\parallel$ Axes Theorem of Moment of Inertia

(a): $\perp^r$ axes theorem:

- This theorem is only applicable for plane lamina.

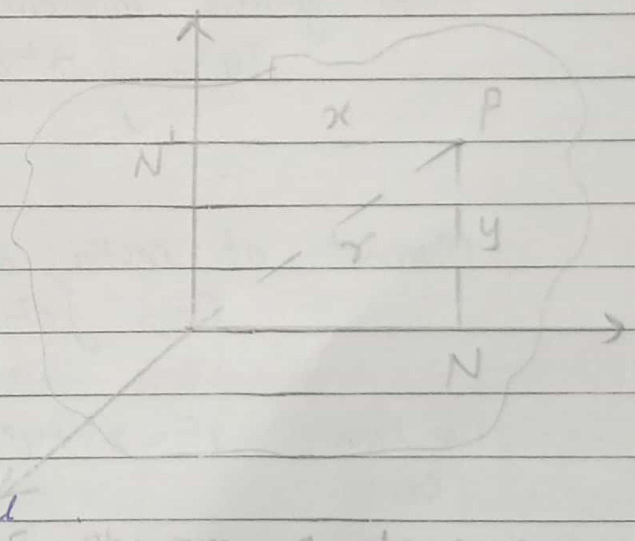


Fig: Theorem of $\perp$ axis

⊛ Statement:

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moment of inertia of the lamina about any two mutually $\perp$ axes, passing through its own plane, intersecting each other at the point through which the perpendicular axis passes.

⊛ Proof:

Let us consider plane lamina on XOY plane. the lamina is made up of a large number of particles.

Consider a small mass element dm at P. From P, PN and PN' are done $\perp^r$ to x and y-axis respectively.

Now,

$PN' = x$, $PN = y$.

About x-axis, moment of inertia of whole lamina.

$$I_x = \int y^2\, dm \quad \text{—— (i)}$$

About y-axis, moment of inertia of whole lamina.

$$I_y = \int x^2\, dm \quad \text{—— (ii)}$$

Moment of inertia of the whole lamina about z-axis is.

$$I_z = \int r^2\, dm \quad \text{—— (iii)}$$

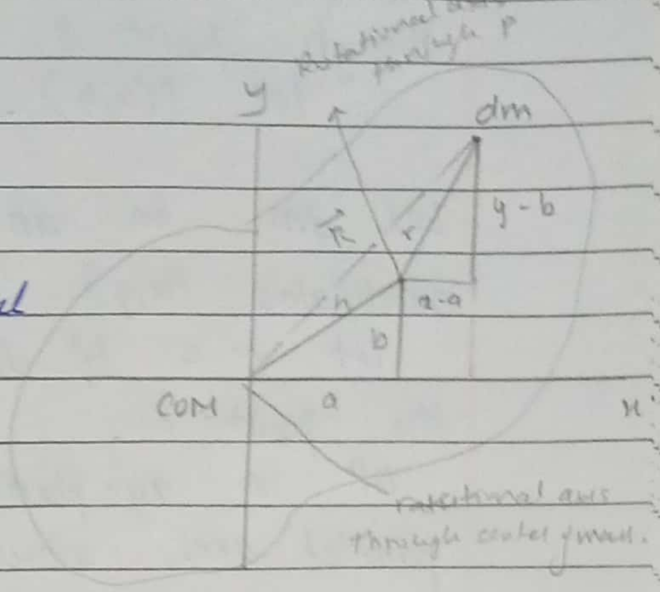
We know, $r^2 = x^2 + y^2$.

So,

$$I_z = \int (x^2+y^2)\, dm = \int x^2\, dm + \int y^2\, dm = I_x + I_y.$$

Hence, $I_z = I_x + I_y$. — (iv).

(b): Parallel Axes theorem:



(*) Statement:

The moment of inertia of a body about an axis is equal to the moment of inertia of the body about the axis through center of mass and parallel to the given axis plus the product of total mass of the body and square of ⊥r distance betn the axes.

Let I = moment of inertia of a body about an axis.

I_{com} = moment of inertia of the body about the axis passes through center of mass and parallel to given axis.

M = total mass.

h = ⊥r distance betn axes.

So,

$$I = I_{com} + Mh^2 \quad \text{— (i)}$$

(*) Proof:

Let O = center of mass of any arbitrarity shaped rigid body

M = ~~mo~~ total mass.

origin is put at point O.

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Let us consider an axis passing through $\perp^r$ to the plane of the body and another axis passing through point P parallel to first axis.
let P(a,b).

Let dm be an element of mass with general coordinates (x,y).
let 'r' = $\perp^r$ distance of elementary mass dm from the point P.
OP in xy-plane is $\perp^r$ distance betn two parallel axes, which is equal to h.

The moment of inertia of the rigid body about the axis passing through point P is given by.

$$I = \int r^2\, dm = \int \{(x-a)^2 + (y-b)^2\}\, dm$$

$$= \int (x^2 - 2xa + a^2 + y^2 - 2yb + b^2)\, dm$$

Rearranging,

$$I = \int (x^2+y^2)\, dm + \int (a^2+b^2)\, dm - 2a\int x\, dm - 2b\int y\, dm \quad \text{(ii)}$$

However, the coordinates of the center of mass by definition are given as.

$$x_{cm} = \frac{1}{M}\int x\, dm \quad \text{and} \quad y_{cm} = \frac{1}{M}\int y\, dm.$$

since, COM lies on z-axis, so, $x_{cm} = y_{cm} = 0$.

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Hence, eq^n (ii) becomes.

$$I = \int (x^2+y^2)\,dm + \int (a^2+b^2)\,dm \quad \text{— (iii)}$$

So, $x^2+y^2 = R^2$
and, $a^2+b^2 = h^2$.

Substituting in eq^n (iii), we get.

$$I = \int R^2\,dm + \int h^2\,dm. \quad \text{— (iv)}.$$

So,

$$I = \int R^2\,dm + h^2 \int dm \quad \text{— (v)}$$

Now,
$\int R^2\,dm$ = moment of inertia of the rigid body about the axis passing through center of mass.
Hence,

$$I = I_{com} + Mh^2 \quad \text{— (vi)}.$$

I_{com} = moment of inertia of the rigid body about an axis through its center of mass.

Moment of inertia of a Slender Rod

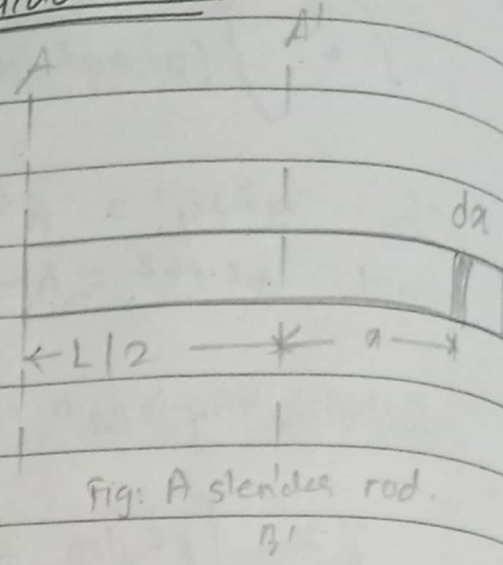


Fig: A slender rod.

Consider,

L = length of rod.

M = mass of rod.

AB = axis passing through CM and $\perp^r$ to rod.

$A'B'$ = axis through one end and $\perp^r$ to rod.

dx = elemental length to which rod is divided.

x = distance of dx from CM.

dm = mass of dx.

Now,

The mass of the elemental length dx is $dm = \frac{M}{L} dx$

The moment of inertia of slender rod about AB is,

$$I_{cm} = \int_{-L/2}^{+L/2} x^2 \, dm = \frac{M}{L} \int_{-L/2}^{+L/2} x^2 \, dx = \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L} \left(\frac{L^3}{8} + \frac{L^3}{8} \right) = \frac{ML^2}{12}$$

Using ∥ axes theorem, the moment of inertia about axis $A'B'$,

$$I_{A'B'} = I_{cm} + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2$$

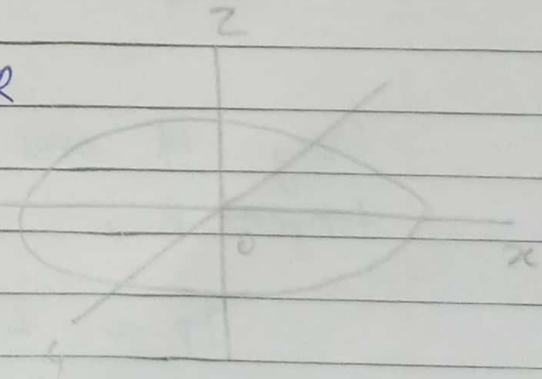
$$\therefore I_{A'B'} = \frac{1}{3}ML^2$$

Here, radius of gyration about AB is $K_{AB} = \sqrt{\frac{I_{com}}{M}}$

$$\therefore K_{AB} = \frac{L}{2\sqrt{3}}$$

radius of gyration about $A'B'$ = $K_{A'B'} = \frac{L}{\sqrt{3}}$

Moment of Inertia about the Circular Ring



Consider a circular ring of radius = R
& M = mass of circular ring
on xy plane with center of origin O.

Fig: M.I of ring.

Here,
z-axis is ⊥r to plane of ring and passing through center of mass of ring.

Let dm be elementary mass on ring lying at 'R' distance from z-axis.

The ~~amou~~ M.g.I about z-axis is.

$$I_z = \int R^2 \, dm. \quad \text{— (i)}$$

Since R is constant,

$$I_z = R^2 \int dm$$

$$\therefore I_z = MR^2 \quad \text{—— (ii)}.$$

Since the ring is symmetrical about x- and y-axis,
$I_x = I_y$.

Using ⊥r axes theorem,

$$I_x + I_y = I_z$$

or $$2 I_x = MR^2$$

$$\therefore I_x = \frac{1}{2} MR^2 = I_y.$$

Hence, the moment of inertia ~~of~~ of ring about x- or y-axis ie, $\frac{1}{2} MR^2$.

Using ∥ axes theorem, the moment about a tangent is,

$$I_T = I_x + MR^2 = \frac{3}{2} MR^2$$

$$\therefore I_T = \frac{3}{2} MR^2.$$