Lecture 10

Magnetostatic

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Outline

Introduction

Stationary charges (Charges at rest) produce constant electric field **Electrostatics**.

Current at a point is defined as the flow of charge per unit time and give by $I = \dfrac{dq}{dt}$

Steady current (current remains constant with respect to time) produces constant magnetic field **Magnetostatics**.

The magnetic force

The magnetic force on charge Q moving with velocity $\vec{v}$ in magnetic field $\vec{B}$ is

$$\vec{F}_{mag} = Q\left(\vec{v} \times \vec{B}\right) \tag{1}$$

The direction of force $\vec{F}_{mag}$ is that of vectors $\vec{v} \times \vec{B}$, i.e. $\perp$ to the plane containing $\vec{v}$ and $\vec{B}$ and thus a force $\vec{F}$ will always be a side-way force.

Special Cases:

1. Magnetic forces acting on a charged particle will be **maximum** if $\vec{v} \perp \vec{B}$ i.e. $\theta = 90^0$. The maximum force is $F = QvB$.
2. Magnetic force will be **minimum** (i.e. $F_{\text{mag}} = 0$) when
   a. $Q = 0$ i.e. particle is uncharged
   b. $\vec{v} = 0$ i.e. particle is at rest
   c. $\theta = 0^\circ$ or 180° $\vec{v}$ parallel or anti-parallel to $\vec{B}$.

The magnetic force (contd.)

If $Q = 1$ C, $\vec{v} = 1$ m/s and $\sin\theta = 1$ then $\vec{F} = \vec{B}$.

The force experienced by a unit test charge projected with a unit velocity is called the **strength of magnetic field or magnetic induction**. It is a vector quantity denoted by $\vec{B}$.

The S.I. unit of $\vec{B}$ is $\frac{\mathrm{N}}{C(m/s)}$ $=\mathrm{N}\cdot\mathrm{A}^{-1}\cdot\mathrm{m}^{-1}$.

This is given a special name wb$\cdot$m^{-2} or Tesla (T).

C.G.S. unit 1 gauss (G)=10^{-4} Tesla (T)

Magnetic field is defined as the space around a magnet or a current-carrying conductor. More precisely, the space around a magnet or a current-carrying conductor in which needle of a compass gets influenced is called magnetic field.

The magnetic force (contd.)

If a charge particle moves through a region in which both an electric and magnetic field are present then the resultant force is given by

$$\begin{aligned}\vec{F} &= \vec{F}_e + \vec{F}_{mag} \\ &= Q\vec{E} + Q\left(\vec{v}\times\vec{B}\right) = Q\left[\vec{E} + \left(\vec{v}\times\vec{B}\right)\right]\end{aligned}$$

$$\vec{F} = Q\left[\vec{E} + \left(\vec{v}\times\vec{B}\right)\right] \tag{2}$$

This force $\vec{F}$ is known as **Lorentz force** and relation is **Lorentz relation**.

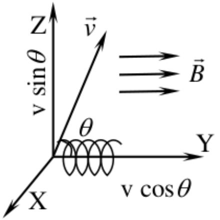
Magnetic force does no work

$$W_{mag} = \int dW_{mag} = \int \vec{F}_{mag}\cdot d\vec{l} = \int Q\left(\vec{v}\times\vec{B}\right)\cdot\vec{v}dt = 0$$

Since $\vec{v}\cdot\left(\vec{v}\times\vec{B}\right) = 0$

Cyclotron Motion (The first modern particle accelerator)

Let a charge Q of mass m and velocity $\vec{v}$. Suppose θ be the angle between $\vec{v}$ and $\vec{B}$. We resolve $\vec{v}$ in to $v_1 = v\cos\theta$ and $v_2 = v\sin\theta$ along and perpendicular to $\vec{B}$. Due to component $v\sin\theta$, Q moves in circular path and due to component $v\cos\theta$ it travels along Y-direction. So the resultant motion will be a helix along $\vec{B}$.



For circular motion, with radius r and velocity v_2, the force due to magnetic field provides the centripetal force, i.e.

$$Qv_2B = \frac{mv_2^2}{r} \implies r = \frac{mv_2}{QB} \tag{3}$$

Cyclotron Motion (The first modern particle accelerator) (contd.)

Therefore, the momentum of the charge particle is $p = mv_2 = QBr$

This Equation (3) shows that radius of circular path is proportional to the velocity of charge particle. This equation (3) also known as the **cyclotron formula** because it describes the motion of particle in cyclotron.

When $\vec{v}$ and $\vec{B}$ are perpendicular to each other (i.e. $\theta = 90^0$)

$$\frac{v_2}{r} = \omega = 2\pi f = \frac{QB}{m} \implies \boxed{f = \frac{QB}{2\pi m}}$$

This relation shows that frequency of revolution is independent of velocity of charge particle.

The magnetic force:-Cycloid Motion

Suppose that $\vec{B}$ points in the X-direction and $\vec{E}$ in the Z-direction as shown in the figure. A particle at rest released from the origin. Initially, the particle is at rest so the magnetic force is zero and the electric field accelerates the charge in Z-direction. As it picks up speed, a magnetic force develops which pulls the charge around to the right.The faster it goes, the stronger $\vec{F}_{\text{mag}}$ becomes. As a result it curves the particle back around towards the Y-axis.

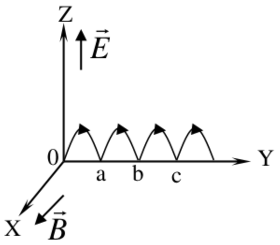


Figure 1: Cycloid

At this point the charge is moving against the electric force so it begins to slow down the magnetic force then decreases and electrical force take over bringing the charge to rest at point ‘a’ in figure 1. The process is repeated and entire path looks like as shown called **cycloid**.

Magnetic flux

The elemental magnetic flux $d\Phi_m$ across an elemental surface area $d\vec{a}$ is defined as

$$d\Phi_m = \vec{B} \cdot d\vec{a}$$

Therefore, the total magnetic flux Φ_m across a surface S is given by

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a}$$

If $\vec{B}$ is uniform and normal to surface S then

$$\Phi_m = \int_S B da = BS$$

Thus magnetic flux is defined as the number of lines of force perpendicularly throughout from the surface.

Line, Surface and Volume current density

Consider the charges are flowing along the line. The line current density is simply the current along the line, i.e.

$$I = \frac{dq}{dt} = \frac{dq}{dl_{\parallel}}\frac{dl_{\parallel}}{dt} = \lambda v$$

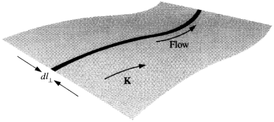


Figure 2

Consider the charges are flowing over the surface with velocity $\vec{v}$ as shown in figure 2.

The surface current density is defined as the *current per unit width-perpendicular-to-flow*. If the current in the ribbon of figure 2 is dI within the elemental width $dl_{\perp}$, then the surface current density is

$$K = \frac{dI}{dl_{\perp}}$$

Line, Surface and Volume current density (contd.)

But $\frac{dI}{dl_\perp} = \frac{dt dI}{dl_\perp dl_\parallel}\frac{dl_\parallel}{dt} = \frac{dq}{da}v = \sigma v$. The surface current density can also be written as

$$\vec{K} = \sigma\vec{v}$$

When the charge is following in the three-dimensional region, then the volume current density is defined as the *current per unit area-perpendicular-to-flow*.

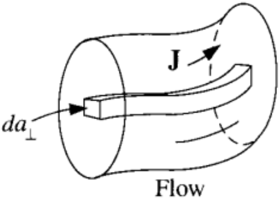


Figure 3: volume current

Line, Surface and Volume current density (contd.)

Consider a tube of elemental cross section area $da_\perp$, running parallel to the flow as shown in figure 3. If the current in this tube is dI, the volume current density is

$$J = \frac{dI}{da_\perp}$$

But $\dfrac{dI}{dl_\perp} = \dfrac{dtdI}{da_\perp dl_\parallel}\dfrac{dl_\parallel}{dt} = \dfrac{dq}{d\tau}v = \rho v$. The volume current density can also be written as

$$\vec{J} = \rho\vec{v}$$

Note:

$$\boxed{I = \lambda v, \qquad \vec{K} = \sigma\vec{v}, \qquad \vec{J} = \rho\vec{v}}$$

Continuity Equation

Suppose a tube of volume V and elemental cross-section $da_\perp$ running parallel to flow as shown in figure 3. The small amount of current when charges pass through small area is then $dI = \vec{J} \cdot da_\perp = \vec{J} \cdot \hat{n}\, da$. Here $\hat{n}$ is unit vector normal to cross-section. So the total current may be obtained by integrating dI over total surface.

$$I = \int_S \vec{J} \cdot \hat{n}\, da$$

Also we know,

$$I = \frac{dQ}{dt} = \frac{d}{dt} \int_V \rho\, d\tau = \int \frac{\partial \rho}{\partial t} d\tau$$

Continuity Equation (contd.)

Here we have taken the partial derivative with respect to time.

From law of conservation of charge the rate at which the charge leaving the volume at the cost of the rate of decrease of charge inside the volume

$$\int_S \left(\vec{J}\cdot\hat{n}\right) da = -\frac{d}{dt}\int_V \rho \, d\tau$$

$$\int_V \left(\nabla\cdot\vec{J}\right) d\tau = -\int_V \frac{\partial\rho}{\partial t} d\tau$$

$$\int_V \left(\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t}\right) d\tau = 0$$

$$\therefore \boxed{\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0} \tag{4}$$

Continuity Equation (contd.)

This equation is gives the relation between charge density and current density. This is the precise mathematical statement of local charge conservation; it is called continuity equation. If the charge density is independent of time, then we get the steady current. In this case the continuity equation becomes

$$\boxed{\nabla \cdot \vec{J} \ = 0}$$

Force on current carrying conductor

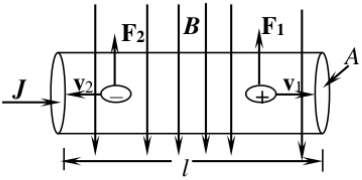


Figure 4

Suppose a current carrying conductor of length l, cross-sectional area A is carrying current I as shown in figure 4. A magnetic field $\vec{B}$ is perpendicular to the length of wire. Let $\vec{v}_1$ be the velocity of positive charges q_1 then force on them due to magnetic field is $q_1(\vec{v}_1 \times \vec{B})$. If there are n_1 number of positive charges per unit volume, then total

Force on current carrying conductor (contd.)

force on them due to magnetic field is $\vec{F}_1 = (n_1 A l) q_1 (\vec{v}_1 \times \vec{B})$. Similarly for n_2 number of negative charges per unit volume with velocity $\vec{v}_2$, the force on them is $\vec{F}_2 = (n_2 A l) q_2 (\vec{v}_2 \times \vec{B})$. Since q_1 and q_2 have opposite signs and $\vec{v}_1$ and $\vec{v}_2$ have opposite direction, $\vec{F}_2$ has the same direction as $\vec{F}_1$. Thus, the total force on both the charges (i.e. on the conductor) is

$$\begin{aligned}
\vec{F} &= \vec{F}_1 + \vec{F}_2 \\
&= (n_1 A l) q_1 (\vec{v}_1 \times \vec{B}) + (n_2 A l) q_2 (\vec{v}_2 \times \vec{B}) \\
&= (A l)(n_1 q_1 \vec{v}_1 + n_2 q_2 \vec{v}_2) \times \vec{B} \\
&= A l (\vec{J} \times \vec{B}) \\
\therefore \vec{F} &= I (\vec{l} \times \vec{B}) \qquad \text{This is the required expression for force.}
\end{aligned}$$

Force on current carrying conductor (contd.)

- Magnetic force on a system of moving point charges

$$\boxed{\vec{F}_m = \sum_{i=1}^{n} q_i \left(\vec{v}_i \times \vec{B}\right)}$$

- For a continuous system of charges

$$\boxed{\vec{F}_m = \int dq \left(\vec{v} \times \vec{B}\right)}$$

Where $\vec{v}$ is the velocity of elemental charge dq in magnetic field $\vec{B}$.

Force on current carrying conductor (contd.)

- The magnetic force on line-current is

$$\boxed{\vec{F}_m = \int \lambda dl \left(\vec{v} \times \vec{B}\right) = \int \lambda v \left(d\vec{l} \times \vec{B}\right) = \int I \left(d\vec{l} \times \vec{B}\right)}$$

- The magnetic force on surface-current is

$$\boxed{\vec{F}_m = \int \sigma da \left(\vec{v} \times \vec{B}\right) = \int \left(\sigma \vec{v} \times \vec{B}\right) da = \int \left(\vec{K} \times \vec{B}\right) da}$$

- The magnetic force on volume-current is

$$\boxed{\vec{F}_m = \int \rho d\tau \left(\vec{v} \times \vec{B}\right) = \int \left(\rho \vec{v} \times \vec{B}\right) d\tau = \int \left(\vec{J} \times \vec{B}\right) d\tau}$$

The magnetic field of a steady current (Biot-Savart law)

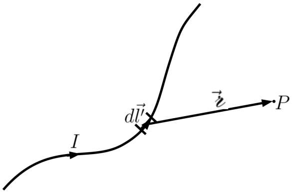


Figure 5

The magnetic field of a steady line current at point P with position vector $\vec{r}$ is given by the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^3} \tag{5}$$

The magnetic field of a steady current (Biot-Savart law) (contd.)

The integration is taken along the current path, in the direction of flow; $d\vec{l'}$ is an elemental length along the wire, and $\vec{\mathscr{r}}$ is the separation vector from the source to the point P as shown in figure 5. The constant μ_0 is called the permeability of free space; it value is is

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N \cdot A^{-2}}$$

For an elemental segment $d\vec{l'}$ of the wire, the elemental magnetic field at P is

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l'} \times \vec{\mathscr{r}}}{\mathscr{r}^3} \tag{6}$$

The magnetic field of a steady current (Biot-Savart law) (contd.)

Since $Idl = \frac{I}{dl_\perp} dl_\perp dl = Kda$ and $Idl = \frac{I}{da_\perp} da_\perp dl = Jd\tau$. In vector form

$$I d\vec{l} \equiv \vec{K} da \equiv \vec{J} d\tau \tag{7}$$

The Biot-Savart law i.e. Equation (5) can be written in term of surface and volume current density as

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \hat{\mathscr{r}}}{\mathscr{r}^2} da' = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau' \tag{8}$$

1. **Find the magnetic field a distance r from a long straight wire carrying a steady current I.**

Solution:

Suppose a straight conductor AB carrying a current I is placed along the X-axis as shown in figure 6.

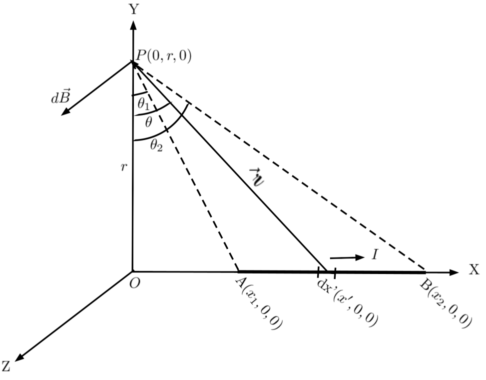


Figure 6

Applications of Biot-Savart law (contd.)

The current is flowing along +ve X-direction. Let dx' be an elemental length with coordinate $(x', 0, 0)$. The coordinates of ends A and B are respectively $(x_1, 0, 0)$ and $(x_2, 0, 0)$. Let $P(0, r, 0)$ be a point on Y-axis at a distance r from origin.The separation vector of P from dx' is $\vec{\mathscr{r}} = -x'\hat{i} + r\hat{j}$ and $\mathscr{r} = \sqrt{x'^2 + r^2}$. Now, $\quad d\vec{l'} \times \vec{\mathscr{r}} = (dx'\hat{i}) \times (-x'\hat{i} + r\hat{j}) = rdx'\hat{k}$

The magnetic field at P due to the current on dx' is

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l'} \times \vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I}{4\pi} \frac{rdx'}{(r^2 + x'^2)^{3/2}} \hat{k}$$

From figure 6, $x' = r\tan\theta$ and $dx' = r\sec^2\theta d\theta$ and hence

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{r^2 \sec^2\theta d\theta}{r^3 \sec^3\theta} \hat{k} = \frac{\mu_0 I}{4\pi r} \cos\theta d\theta \hat{k} \tag{9}$$

The net magnetic field at P can be calculated by integrating equation (9) from the limit $\theta \to \theta_1$ to $\theta \to \theta_2$ as shown in figure 6, i.e.

$$\vec{B} = \frac{\mu_0 I}{4\pi r}\int_{\theta_1}^{\theta_2} \cos\theta d\theta \hat{k}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi r}\left(\sin\theta_2 - \sin\theta_1\right)\hat{k} \tag{10}$$

This shows that the magnetic field points along the direction of Z-axis. Similarly, if P lies at a distance r on Z-axis, then the magnetic field points along -ve Y-axis. In general, the magnetic field is circulating the current carrying conductor according the right-hand thumb rule. Considering $\hat{\phi}$ as the unit vector tangential to the field line, the equaiton (10) conveniently be written as

$$\vec{B} = \frac{\mu_0 I}{4\pi r}\left(\sin\theta_2 - \sin\theta_1\right)\hat{\phi} \tag{11}$$

If the straight conductor is symmetric about YZ-plane, then $\theta_1 = -\theta_0$ and $\theta_2 = \theta_0$ and equation (11) reduces to

$$\vec{B} = \frac{\mu_0 I}{4\pi r}\left[\sin\theta_0 - \sin(-\theta_0)\right]\hat{k} = \frac{\mu_0 I}{4\pi r} 2\sin\theta_0 \hat{k} = \frac{\mu_0 I}{2\pi r}\sin\theta_0 \hat{\phi}$$

If the straight conductor is infinitely long along both sides with center at origin, then $\theta_2 = +\frac{\pi}{2}$ and $\theta_1 = -\frac{\pi}{2}$ and equation (11) gives

$$\vec{B} = \frac{\mu_0 I}{4\pi r}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right]\hat{k} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

2. **Find the magnetic field a distance z above the centre of a circular loop of radius R which carries a steady current I.**

Solution:-

Consider a circular loop of radius R carrying a steaday current I on XY-plane with center at origin O as shown in figure 7.

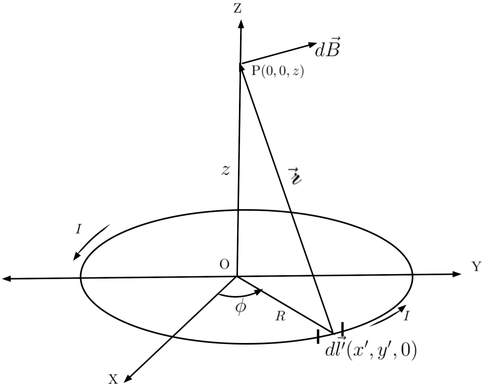


Figure 7

$P(0,0,z)$ is a point on Z-axis at a distance z from origin. Let's take an elemental vector length $d\vec{l'}$ on the ring with coordinate $(x',y',0)$ such that

$$x' = R\cos\phi, \quad \text{and} \quad y' = R\sin\phi$$

The elemental vector length $d\vec{l'}$ can be written as

$$d\vec{l'} = dx'\hat{i} + y'\hat{j} = -R\sin\phi d\phi\hat{i} + R\cos\phi d\phi\hat{i} = (-\sin\phi\hat{i} + \cos\phi\hat{j})Rd\phi$$

The separation vector of P from $d\vec{l'}$ is

$$\vec{\mathscr{r}} = -x'\hat{i} - y'\hat{j} + z\hat{k} = -R\cos\phi\hat{i} - R\sin\phi\hat{j} + z\hat{k}$$

and

$$\mathscr{r} = \sqrt{R^2 + z^2}$$

Applications of Biot-Savart law (contd.)

Again,

$$
\begin{aligned}
d\vec{l}' \times \vec{\mathscr{r}} &= \left[\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right) R\,d\phi\right] \times \left(-R\cos\phi\,\hat{i} - R\sin\phi\,\hat{j} + z\hat{k}\right) \\
&= R\,d\phi\left(R\sin^2\phi\,\hat{k} + z\sin\phi\,\hat{j} + R\cos^2\phi\,\hat{k} + z\cos\phi\,\hat{i}\right) \\
&= R\left(z\cos\phi\,d\phi\,\hat{i} + z\sin\phi\,d\phi\,\hat{j} + R\,d\phi\,\hat{k}\right)
\end{aligned}
$$

The magnetic field at P due to the current on the elemental length $d\vec{l}'$ is

$$
\begin{aligned}
d\vec{B} &= \frac{\mu_0 I}{4\pi}\,\frac{d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^3} \\
&= \frac{\mu_0 I}{4\pi}\,\frac{R\left(z\cos\phi\,d\phi\,\hat{i} + z\sin\phi\,d\phi\,\hat{j} + R\,d\phi\,\hat{k}\right)}{\left(R^2+z^2\right)^{3/2}}
\end{aligned}
$$

Applications of Biot-Savart law (contd.)

$$\therefore d\vec{B} = \frac{\mu_0 IR}{4\pi\left(R^2+z^2\right)^{3/2}}\left[z\cos\phi\, d\phi\,\hat{i} + z\sin\phi\, d\phi\,\hat{j} + R\, d\phi\,\hat{k}\right]$$

The magnetic field at P due to the current through whole loop is

$$\vec{B} = \frac{\mu_0 IR}{4\pi\left(R^2+z^2\right)^{3/2}}\left[\cancelto{0}{z\int_0^{2\pi}\cos\phi\, d\phi}\,\hat{i} + \cancelto{0}{z\int_0^{2\pi}\sin\phi\, d\phi}\,\hat{j} + R\int_0^{2\pi} d\phi\,\hat{k}\right]$$

$$= \frac{\mu_0 IR}{4\pi\left(R^2+z^2\right)^{3/2}} R\, 2\pi\hat{k}$$

Hence,

$$\boxed{\vec{B} = \frac{\mu_0 IR^2}{2\left(R^2+z^2\right)^{3/2}}\hat{k}} \tag{12}$$

Applications of Biot-Savart law (contd.)

If there are N number of circular current loops of same radii, the magnetic field at P is

$$\vec{B} = \frac{\mu_0 I R^2 N}{2\left(R^2 + z^2\right)^{3/2}}\hat{k} \tag{13}$$

At the center of the circular loop, $z = 0$ and

$$\boxed{\vec{B} = \begin{cases} \dfrac{\mu_0 I}{2R}\hat{k}, & \text{for single loop} \\ \dfrac{\mu_0 I N}{2R}\hat{k}, & \text{for } N \text{ turns} \end{cases}} \tag{14}$$

End of Lecture 10

Thank you