Lecture 07

Electrostatic Field in Matter

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Outline

1. Electric Dipole
   - Electric Dipole Moment $(\vec{p})$
   - Electric potential and field at a point due to a short dipole
   - Torque and Energy stored on a dipole placed in an electric field
     - Net Force on the dipole
     - Torque on the dipole:
     - Potential energy stored in the dipole

A pair of equal and opposite point charges separated by a small distance is called an electric dipole. The distance between two charges is called dipole length and its centre is called dipole centre.

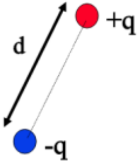

Figure 1: An electric dipole

Electric Dipole:-Electric Dipole Moment ($\vec{p}$)

- Electric dipole moment is defined as the product of magnitude of either charge and the vector distance separating the two charges.
- Electric Dipole Moment: $\vec{p} = q\vec{d}$
- The direction of electric dipole moment vector is from the negative charge toward the positive charge.
- It is a vector quantity.
- SI unit of Electric Dipole Moment is C m.

Electric Dipole:-

Electric potential and field at a point due to a short dipole

Consider a short dipole of dipole moment $\vec{p}$ and dipole length $\vec{d}$ with dipole center O at the origin. The dipole consist the charge $-q$ at the position $\left(0,0,-\frac{d}{2}\right)$ and another charge $+q$ at position $\left(0,0,+\frac{d}{2}\right)$ as shown in Figure 2. Let $P(x,y,z)$ be a point with position vector $\vec{r}$ at which the electric potential and field are to be determined.

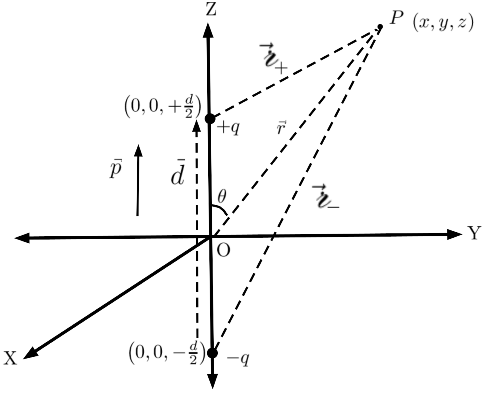

Figure 2

Electric Dipole:-

Electric potential and field at a point due to a short dipole (contd.)

From Figure 2, the separation vectors of P from $+q$ and $-q$ are respectively

$$\vec{\mathscr{r}}_{+} = x\hat{i} + y\hat{j} + \left(z - \frac{d}{2}\right)\hat{k}$$

$$\vec{\mathscr{r}}_{-} = x\hat{i} + y\hat{j} + \left(z + \frac{d}{2}\right)\hat{k}$$

The magnitude of them are

$$\mathscr{r}_{+} = \sqrt{x^2 + y^2 + \left(z - \frac{d}{2}\right)^2}$$

$$\mathscr{r}_{-} = \sqrt{x^2 + y^2 + \left(z + \frac{d}{2}\right)^2}$$

Electric Dipole:-

Electric potential and field at a point due to a short dipole (contd.)

Since, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$

$$\mathscr{r}_+ = \sqrt{x^2 + y^2 + z^2 - zd + \frac{d^2}{4}} = \sqrt{r^2 - zd + \frac{d^2}{4}}$$

and

$$\mathscr{r}_- = \sqrt{x^2 + y^2 + z^2 - zd + \frac{d^2}{4}} = \sqrt{r^2 + zd + \frac{d^2}{4}}$$

For a short dipole, either P is vary far from the dipole or the charges are closed to each other, i.e. $r \gg d$. Then,

$$\frac{1}{\mathscr{r}_+} = \left(r^2 - zd + \frac{d^2}{4}\right)^{-\frac{1}{2}} \approx \left(r^2 - zd\right)^{-\frac{1}{2}} = \frac{1}{r}\left(1 - \frac{zd}{r^2}\right)^{-\frac{1}{2}} \approx \frac{1}{r}\left(1 + \frac{zd}{2r^2}\right)$$

Electric Dipole:-

Electric potential and field at a point due to a short dipole (contd.)

Here, we have applied the binomial expansion and neglected the higher order terms. Similarly,

$$\frac{1}{\mathscr{r}_-} \approx \frac{1}{r}\left(1 - \frac{zd}{2r^2}\right)$$

The electric potential at P due to the short dipole is

$$\begin{aligned} V(x,y,z) &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\left(1 + \frac{zd}{2r^2}\right) - \frac{1}{r}\left(1 - \frac{zd}{2r^2}\right)\right] \\ &= \frac{1}{4\pi\varepsilon_0}\frac{zqd}{r^3} \end{aligned}$$

Electric potential and field at a point due to a short dipole (contd.)

Since $p = qd$, the electric potential at P is

$$\boxed{V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\frac{pz}{r^3}} \tag{1}$$

From Figure 2, $z = r\cos\theta$

$$\boxed{V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}} \tag{2}$$

Since $p\cos\theta = \vec{p}\cdot\hat{r}$, then

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}} \tag{3}$$

The electric field at P can be calculated by taking the negative gradient ($\because \vec{E} = -\nabla V$), of any expression of potential defined by Equations (1), (2) or (3)

Method-I (Cartesian Coordinate):- Using Equation (1)

$$\begin{aligned}\vec{E} &= -\nabla V \\ &= -\frac{p}{4\pi\varepsilon_0}\nabla\left(\frac{z}{r^3}\right) \\ &= -\frac{p}{4\pi\varepsilon_0}\left[z\nabla\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\nabla z\right]\end{aligned}$$

Electric Dipole:-

Electric potential and field at a point due to a short dipole (contd.)

Using $\nabla r^n = n r^{n-1}\hat{r}$ and $\nabla z = \hat{i}\dfrac{\partial z}{\partial x} + \hat{j}\dfrac{\partial z}{\partial y} + \hat{k}\dfrac{\partial z}{\partial z} = \hat{k}$

$$\vec{E} = -\frac{p}{4\pi\varepsilon_0}\left[-\frac{3z}{r^4}\hat{r} + \frac{1}{r^3}\hat{k}\right]$$

Therefore,

$$\vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(\frac{3z}{r}\hat{r} - \hat{k}\right) = \frac{p}{4\pi\varepsilon_0 r^3}\left(3\cos\theta\,\hat{r} - \hat{k}\right) \tag{4}$$

The magnitude is

$$E = \sqrt{E^2} = \sqrt{\vec{E}\cdot\vec{E}}$$

Electric Dipole:-

Electric potential and field at a point due to a short dipole (contd.)

$$= \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{\left(3\cos\theta\hat{r}-\hat{k}\right)\cdot\left(3\cos\theta\hat{r}-\hat{k}\right)}$$

$$= \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{9\cos^2\theta - 6\cos\theta(\hat{r}\cdot\hat{k})+1}$$

Since $\hat{r}\cdot\hat{k} = \cos\theta$, we can have

$$E = \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{3\cos^2\theta+1} \tag{5}$$

Method-II (Spherical Polar coordinate):- Using Equation (2)

The operator ∇ in spherical polar coordinate can be defined as

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{6}$$

Electric Dipole:-

Electric potential and field at a point due to a short dipole (contd.)

$$\text{So,} \quad \vec{E}(r,\theta) = -\nabla \mathrm{V}$$

$$= -\left[\frac{\partial \mathrm{V}}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \mathrm{V}}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \mathrm{V}}{\partial \phi}\hat{\phi}\right]$$

$$= -\left[\left\{\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right)\right\}\hat{r} + \frac{1}{r}\left\{\frac{\partial}{\partial \theta}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right)\right\}\hat{\theta}\right]$$

$$= -\left[\frac{p\cos\theta}{4\pi\varepsilon_0}\left\{\frac{\partial}{\partial r}\left(r^{-2}\right)\right\}\hat{r} + \frac{p}{4\pi\varepsilon_0 r^3}\left\{\frac{\partial}{\partial \theta}(\cos\theta)\right\}\hat{\theta}\right]$$

$$= -\left[\frac{p\cos\theta}{4\pi\varepsilon_0}\left(\frac{-2}{r^3}\right)\hat{r} + \frac{p}{4\pi\varepsilon_0 r^3}(-\sin\theta)\,\hat{\theta}\right]$$

$$= -\left[\frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}(-2\cos\theta)\,\hat{r} + \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}(-\sin\theta)\,\hat{\theta}\right]$$

Electric Dipole:-

Electric potential and field at a point due to a short dipole (contd.)

$$\therefore \boxed{\vec{E}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\left[2\cos\theta\;\hat{r} + \sin\theta\;\hat{\theta}\right]} \tag{7}$$

The magnitude of electric field:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\left[\sqrt{(2\cos\theta)^2 + (\sin\theta)^2}\right] = \frac{p}{4\pi\varepsilon_0 r^3}\left[\sqrt{4\cos^2\theta + \sin^2\theta}\right]$$

$$\therefore E = \frac{p}{4\pi\varepsilon_0 r^3}\left[\sqrt{3\cos^2\theta + 1}\right] \tag{8}$$

Cases:

Case I:- At a point on the axis of a dipole $\left(\theta = 0^0\right)$,

$$V(\vec{r}) \cong \frac{1}{4\pi\varepsilon_0}\frac{p}{r^2} \;\&\; E(\vec{r}) = \frac{2p}{4\pi\varepsilon_0 r^3}$$

Case II:- At a point on the perpendicular bisector of a dipole

$$\left(\theta = 90^0\right),\; V = 0 \;\&\; E = \frac{p}{4\pi\varepsilon_0 r^3}$$

Problem: Show that the electric field of a (perfect) dipole can be written in the coordinate free form $\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\left[3\left(\vec{p}\cdot\hat{r}\right)\hat{r} - \vec{p}\right]}$

Solution: Using equation (3)

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

$$= -\frac{1}{4\pi\varepsilon_0}\left[\nabla\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\left[\left(\vec{p}\cdot\vec{r}\right)\left\{\nabla\left(\frac{1}{r^3}\right)\right\} + \left(\frac{1}{r^3}\right)\left\{\nabla\left(\vec{p}\cdot\vec{r}\right)\right\}\right]$$

Electric Dipole:-

Electric potential and field at a point due to a short dipole (contd.)

Here, $\nabla(\vec{p}\cdot\vec{r}) = \nabla(xp_x + yp_y + zp_z) = p_x\hat{i} + p_y\hat{j} + p_z\hat{k} = \vec{p}$

$$\vec{E}(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\left[(\vec{p}\cdot\vec{r})\left(\frac{-3\vec{r}}{r^5}\right) + \left(\frac{1}{r^3}\right)(\vec{p})\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}\right]$$

Therefore,

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]} \tag{9}$$

Electric Dipole:-

Torque and Energy stored on a dipole placed in an electric field

Consider an electric dipole of dipole moment $\vec{p} = q\vec{d}$ with dipole length d and dipole center O in an external electric field $\vec{E}$ directed from left to right as shown in Figure 3. The dipole is capable to rotate about an axis through O and perpendicular to the field lines. Let the dipole makes an angle θ with electric field at an instant of time t.

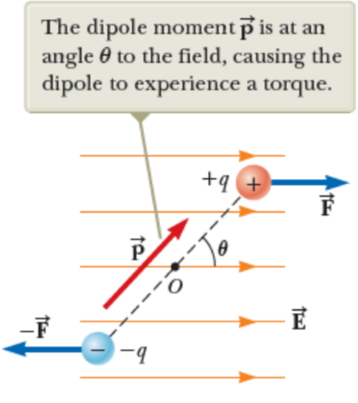

Figure 3

Electric Dipole:-

Torque and Energy stored on a dipole placed in an electric field

Net Force on the dipole

Each of the charges is modeled as a particle in an electric field.

The Force acting on the charge $+q$ is $\vec{F}_+ = q\vec{E}$.

The Force acting on the charge $-q$ is $\vec{F}_- = -q\vec{E}$.

Therefore the net force

$$\begin{aligned}\vec{F}_{net} &= \vec{F}_+ + \vec{F}_- \\ &= q\vec{E} + \left(-q\vec{E}\right)\end{aligned}$$

$$\therefore \vec{F}_{net} = 0$$

Hence, the net force on an electric dipole in a uniform external electric field is zero.

But these two equal and opposite forces are separated by a perpendicular distance $d_{\perp} = d\sin\theta$ and constitute a couple which rotates the dipole in clockwise direction, tending it in the direction of field.

Electric Dipole:-

Torque and Energy stored on a dipole placed in an electric field

Torque on the dipole:

$$\begin{aligned}\vec{\tau} &= \left(\vec{r}_+ \times \vec{F}_+\right) + \left(\vec{r}_- \times \vec{F}_-\right)\\ &= \left[\left(\frac{\vec{d}}{2}\right) \times \left(q\vec{E}\right)\right] + \left[\left(-\frac{\vec{d}}{2}\right) \times \left(-q\vec{E}\right)\right]\\ &= q\vec{d} \times \vec{E}\end{aligned}$$

$$\therefore \boxed{\vec{\tau} = \vec{p} \times \vec{E}} \tag{10}$$

Therefore, the torque, $\vec{\tau}$, on an electric dipole with dipole moment $\vec{p}$ in an uniform electric field $\vec{E}$ is $\vec{\tau} = \vec{p} \times \vec{E}$.

Consider the dipole is to rotate from angular position θ_1 to θ_2 against the torque due to electric field. The electric field exerts the torque to rotate the dipole clockwise, i.e. the direction of torque by it pointing into the plane of figure. The magnitude of the torque exerted by electric field is $\tau = pE\sin\theta$ pointing into the plane of figure. In order to rotate the dipole against this torque, we should exert same amount the torque at every moment in the direction out to the plane of figure.

Electric Dipole:-

Torque and Energy stored on a dipole placed in an electric field

Potential energy stored in the dipole (contd.)

Now, the amount of work done to rotate the dipole from angular position θ_1 to θ_2 against electric field is

$$W = \int_{\theta_1}^{\theta_2} \tau_{\text{app}} d\theta$$

$$= \int_{\theta_1}^{\theta_2} pE\sin\theta d\theta$$

$$= -pE\left(\cos\theta_2 - \cos\theta_1\right)$$

Electric Dipole:-

Torque and Energy stored on a dipole placed in an electric field

Potential energy stored in the dipole (contd.)

This amount of work done results the change in potential energy of the dipole between the angular position θ_1 and θ_2, i.e.

$$\Delta U = U(\theta_2) - U(\theta_1) = -pE(\cos\theta_2 - \cos\theta_1)$$

Let the initial angular position is chosen at $\theta_1 = \frac{\pi}{2} = 90^\circ$ and the final position as $\theta_2 = \theta$, then we can have

$$U(\theta) - U\left(\frac{\pi}{2}\right) = -pE\left(\cos\theta - \cos\frac{\pi}{2}\right) = -pE\cos\theta$$

Electric Dipole:-

Torque and Energy stored on a dipole placed in an electric field

Potential energy stored in the dipole (contd.)

Assuming the potential at the reference position $\frac{\pi}{2}$ is zero, i.e. $U\left(\frac{\pi}{2}\right) = 0$, we get

$$U(\theta) = -pE\cos\theta = -\vec{p}\cdot\vec{E} \tag{11}$$

Equation (11) gives the potential energy stored on the electric dipole placed in an electric field at any angular position with respect to the angular position $\frac{\pi}{2}$ where the potential energy is assumed to be zero.

The potential energy is

minimum when $\vec{p}$ becomes parallel to $\vec{E}$ i.e. $U_{\text{min}} = -pE$

and maximum when they are anti-parallel i.e. $U_{\text{max}} = +pE$

End of Lecture 07

Thank you