Lecture 12

Magnetostatics Field in Matter

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Keshav Raj Sigdel

Assistant Professor

Department of Physics

School of Science

Kathmandu University

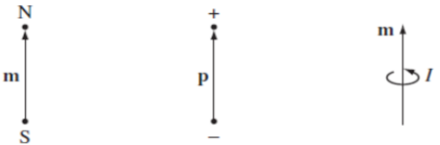
Outline

1. Magnetic Dipole
   - Torques and Energy on a Magnetic Dipoles placed in a magnetic field

2. Magnetization
   - Bound current and its physical interpretation

Magnetic Dipole

A magnetic dipole is a closed circulation of electric current. e.g. a single loop of wire with some constant current through it. The revolution of an electron in clockwise direction is equivalent to a conventional current in the anticlockwise direction and electronic orbit behaves like a magnetic dipole.



(a) Magnetic dipole (Gilbert model) (b) Electric dipole (c) Magnetic dipole (Ampère model)

Magnetic Dipole (contd.)

The magnetic dipole moment of a current loop is defined as the product of current and area enclosed by the loop, i.e.

$$\vec{m} = IA\hat{n} \tag{1}$$

where I is the current, A is the area of the loop and $\hat{n}$ is the unit vector normal to the surface enclosed by the loop. The magnetic dipole moment is the vector quantity. The direction of $\vec{m}$ is perpendicular to the plane of current loop, determined by the right hand rule in which as four fingers point along the direction of current, the thumb gives the direction of magnetic dipole moment. SI unit of magnetic dipole moment $\vec{m}$ is $A{\cdot}m^2$.

Torques and Energy on a Magnetic Dipoles placed in a magnetic field

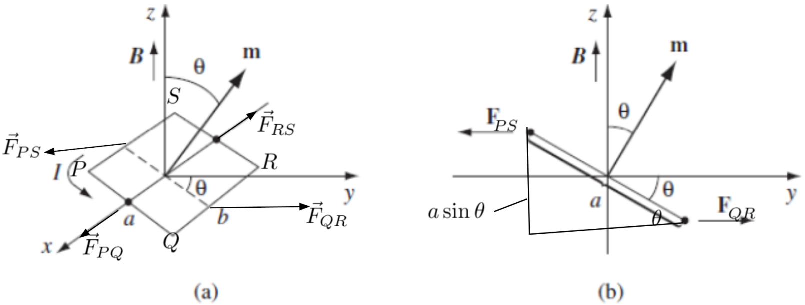


Figure 1

Magnetic Dipole

Torques and Energy on a Magnetic Dipoles placed in a magnetic field (contd.)

A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field. Let's calculate the torque on a rectangular current loop $PQRSP$ in a uniform field B. Center the loop at the origin, and tilt it an angle θ from the z-axis towards the y-axis. Let $\vec{B}$ point in the z-direction.

The force on side PQ is $\vec{F}_{PQ} = BIa\sin(90^\circ + \theta)\hat{i} = BIa\cos\theta\hat{i}$

The force on side QR is $\vec{F}_{QR} = BIb\sin 90^\circ\hat{j} = BIb\hat{j}$

The force on side RS is $\vec{F}_{RS} = -BIa\sin(90^\circ - \theta)\hat{i} = -BIa\cos\theta\hat{i}$

The force on side SP is $\vec{F}_{SP} = -BIb\sin 90^\circ\hat{j} = -BIb\hat{j}$

Torques and Energy on a Magnetic Dipoles placed in a magnetic field

(contd.)

Here, forces $\vec{F}_{PQ}$ and $\vec{F}_{RS}$ are equal in magnitude, opposite in direction and lie on the same line of action. So they cancel each other and just tend to stretch the loop along +ve and -ve x-direction.
But the forces $\vec{F}_{QR}$ and $\vec{F}_{SP}$ are equal in magnitude, opposite in direction and separated by a perpendicular distance $a\sin\theta$ as shown in figure 1. So they constitute the couple. The moment of couple i.e. torque is given by

$$\tau = \text{magnitude of a force} \times \text{perpendicular distance}$$

Torques and Energy on a Magnetic Dipoles placed in a magnetic field (contd.)

$$
\begin{aligned}
&= F_{QR} a \sin\theta \\
&= BIba \sin\theta \\
\implies \tau &= Bm \sin\theta \qquad (2)
\end{aligned}
$$

since $m = Iab$.

In vector form it is written as

$$\vec{\tau} = \vec{m} \times \vec{B} \qquad (3)$$

Magnetic Dipole

Torques and Energy on a Magnetic Dipoles placed in a magnetic field

(contd.)

Consider the magnetic dipole is to rotate from angular position θ_1 to θ_2 against the torque due to magnetic field. The magnetic field exerts the torque to rotate the dipole anti-clockwise with angular acceleration, i.e. the direction of torque by it pointing to +ve x-axis. The magnitude of the torque exerted by magnetic field is $\tau = mB\sin\theta$ pointing +ve x-direction. In order to rotate the dipole against this torque, we should exert same amount of the torque at every moment

Magnetic Dipole

Torques and Energy on a Magnetic Dipoles placed in a magnetic field (contd.)

in the -ve x-direction. Now the amount of work done to rotate the dipole from angular position θ_1 to θ_2 against magnetic field is

$$W = \int_{\theta_1}^{\theta_2} \tau_{\text{app}} d\theta = \int_{\theta_1}^{\theta_2} mB \sin\theta d\theta = -mB\left(\cos\theta_2 - \cos\theta_1\right)$$

This amount of work done results the change in potential energy of the dipole between the angular position θ_1 and θ_2, i.e.

$$\Delta U = U\left(\theta_2\right) - U\left(\theta_1\right) = -mB\left(\cos\theta_2 - \cos\theta_1\right)$$

Magnetic Dipole

Torques and Energy on a Magnetic Dipoles placed in a magnetic field (contd.)

Let the initial angular position is chosen at $\theta_1 = \dfrac{\pi}{2} = 90^\circ$ and the final position as $\theta_2 = \theta$, then we can have

$$U(\theta) - U\left(\frac{\pi}{2}\right) = -mB\left(\cos\theta - \cos\frac{\pi}{2}\right) = -mB\cos\theta$$

Assuming the potential at the reference position $\frac{\pi}{2}$ is zero, i.e. $U\left(\frac{\pi}{2}\right) = 0$, we get

$$U(\theta) = -mB\cos\theta = -\vec{m}\cdot\vec{B} \tag{4}$$

Torques and Energy on a Magnetic Dipoles placed in a magnetic field (contd.)

Equation (4) gives the potential energy stored on the magnetic dipole placed in a magnetic field at any angular position with respect to the angular position $\frac{\pi}{2}$ where the potential energy is assumed to be zero. The potential energy is minimum when $\vec{m}$ becomes parallel to $\vec{B}$ i.e. $U_{\text{min}} = -mB$ and maximum when they are anti-parallel i.e. $U_{\text{max}} = +mB$

Magnetization

All the substances consist of many tiny dipoles. In the absence of external magnetic field, those dipoles orient randomly and hence the total dipole moment is zero. If the substance is placed in a uniform magnetic field, the dipoles moment align or tend to align along some directions and the substance is said to be magnetized.

Magnetic dipole moment per unit volume of a magnetized material is called magnetization ($\vec{M}$). If $d\vec{m}$ be the magnetic dipole moment of an elemental volume $d\tau$ of magnetized material, then the magnetization is defined as

$$\vec{M} = \frac{d\vec{m}}{d\tau} \implies d\vec{m} = \vec{M} d\tau$$

It is a vector quantity. Its direction is same as the direction of net magnetic dipole moment.

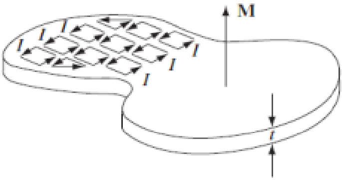


Figure 2

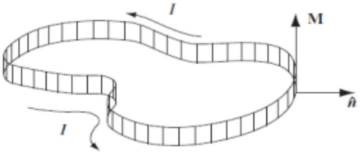


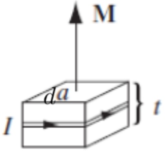
Figure 3

Consider a thin slab of thickness t with uniform magnetization M as shown in figure 2. The magnetic dipoles are indicated by tinny current loop. It can be noted that from figure 2, all the internal current cancel to each other but at the edge there is no adjacent loop to do the cancellation. The current left at the edge without cancellation is called

Magnetization :-Bound current and its physical interpretation (contd.)

bound current. As a result, whole slab is equivalent to a single ribbon of current I flowing around the boundary as shown in figure 3.

Let us take a tiny loop of area da and thickness t as shown in figure 4. In terms of magnetization, the dipole moment



$$dm = M\,d\tau = M\,t\,da \qquad (5)$$

Figure 4

In terms of circulatory current

$$dm = I\,da \qquad (6)$$

Magnetization :-Bound current and its physical interpretation (contd.)

$$\therefore I = Mt \tag{7}$$

The surface current density is defined as the current per unit length perpendicular-to-flow (i.e. the length is taken perpendicular to the direction of flow of current).

The surface bound current density K_b is

$$K_b = \frac{I}{t} = \frac{Mt}{t} = M \tag{8}$$

In vector form

$$\vec{K}_b = \vec{M} \times \hat{n} \tag{9}$$

where $\hat{n}$ is the unit normal vector drawn outward to the surface.
Equation (9) shows that the surface bound current over the top or the bottom of the slab is zero. Since $\vec{M}$ and $\hat{n}$ are parallel or anti-parallel to each other.
When the magnetization is non-uniform then the internal current will no longer cancel each other. Consider a magnetized material whose magnetization varies from point to point, i.e. magnetization is a function of position co-ordinate of tiny dipole.

Magnetization :-Bound current and its physical interpretation (contd.)

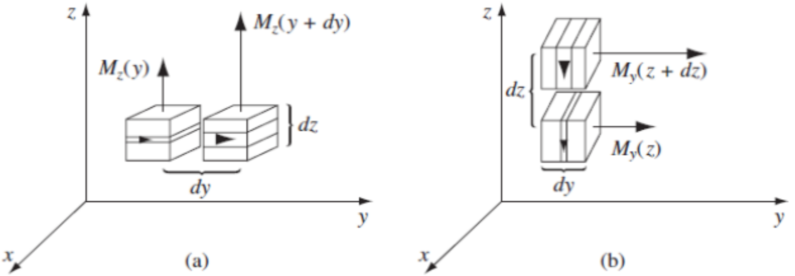


Figure 5

Magnetization :-Bound current and its physical interpretation (contd.)

Let us take two adjacent chunks whose magnetizations are along the positive z-axis as shown in figure 5(a). The magnetization increases as increase in y (greater arrow indicates the greater magnetization). At the interface where they join, there is a net current in the +ve x-direction and is given by

$$I_{+x} = [M_Z(y+dy) - M_Z(y)]\,dz = \left[M_Z(y) + \frac{\partial M_Z(y)}{\partial y}dy - M_Z(y)\right] dz$$

$$\Rightarrow I_{+x} = \frac{\partial M_Z}{\partial y} dydz \tag{10}$$

Similarly, if we consider two adjacent chunks with magnetizations are along the y-axis as in figure 5(b). The magnetization is increasing as

Magnetization :-Bound current and its physical interpretation (contd.)

increase of z. At the interface, there is net current along negative x-axis and is given by

$$I_{-x} = [M_y(z+dz) - M_y(z)]\,dy = \left[M_y(z) + \frac{\partial M_y(z)}{\partial z}dz - M_y(z)\right]dy$$

$$\Rightarrow I_{-x} = \frac{\partial M_y}{\partial z}dydz \qquad (11)$$

Therefore, the net current along x-axis is

$$I_x = I_{+x} - I_{-x} = \frac{\partial M_z}{\partial y}dydz - \frac{\partial M_y}{\partial z}dydz = \left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right)dydz \qquad (12)$$

Magnetization :-Bound current and its physical interpretation (contd.)

The volume bound current density is the current per unit area perpendicular-to-flow, (i.e. area is taken perpendicular to the flow of current).

So, x-component of volume bound current density is

$$(J_b)_x = \frac{I_x}{dydz} = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} \tag{13}$$

Similarly, the y- and z-component of volume bound current densities can be calculating by considering the chunks on xy- and zx-planes as

$$(J_b)_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} \tag{14}$$

Magnetization :-Bound current and its physical interpretation (contd.)

and

$$(J_b)_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \tag{15}$$

Therefore, the volume bound current density

$$\begin{aligned}
\vec{J}_b &= (J_b)_x \hat{i} + (J_b)_y \hat{j} + (J_b)_z \hat{k} \\
&= \left(\frac{\partial M_y}{\partial z} - \frac{\partial M_z}{\partial y} \right) \hat{i} + \left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x} \right) \hat{j} + \left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \right) \hat{k} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M_x & M_y & M_z \end{vmatrix}
\end{aligned}$$

Magnetization :-Bound current and its physical interpretation (contd.)

$$\therefore \vec{J}_b = \nabla \times \vec{M} \tag{16}$$

Also divergence of curl of a vector function is always zero
$\nabla \cdot (\nabla \times \vec{M}) = 0 \implies \nabla \cdot \vec{J}_b = 0$ This is the conservation law of charges.
i.e. The effect of magnetization is to establish bound current $\vec{J}_b = \nabla \times \vec{M}$ within the material and $\vec{K}_b = \vec{M} \times \hat{n}$ on the surface.

End of Lecture 12

Thank you