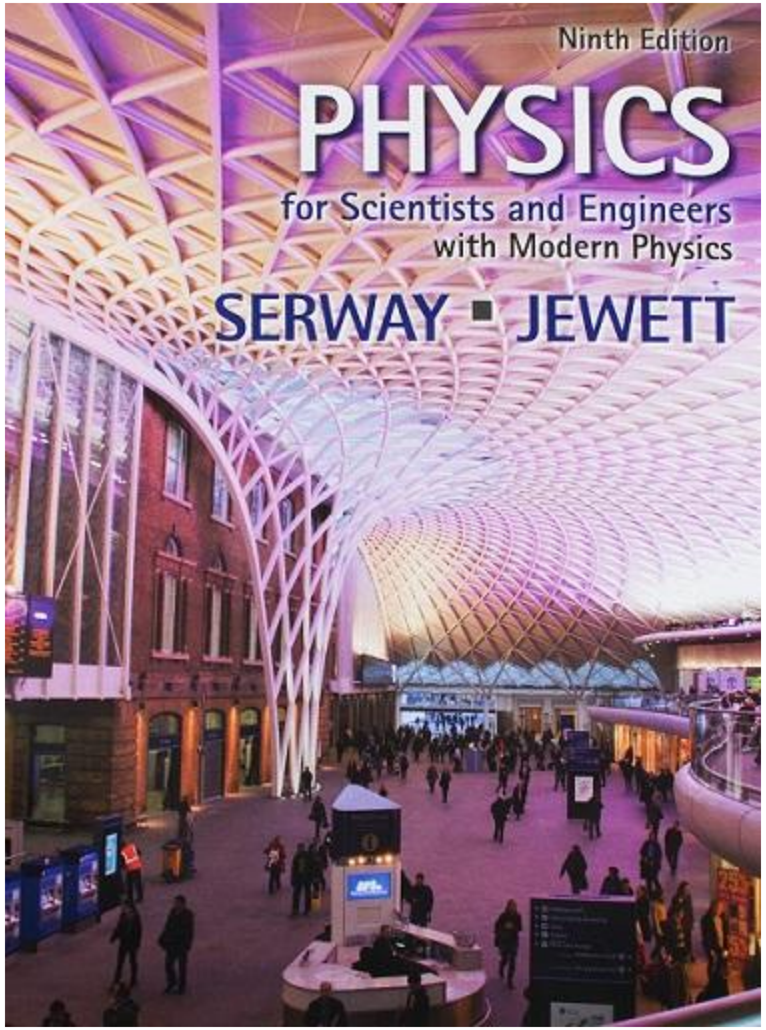
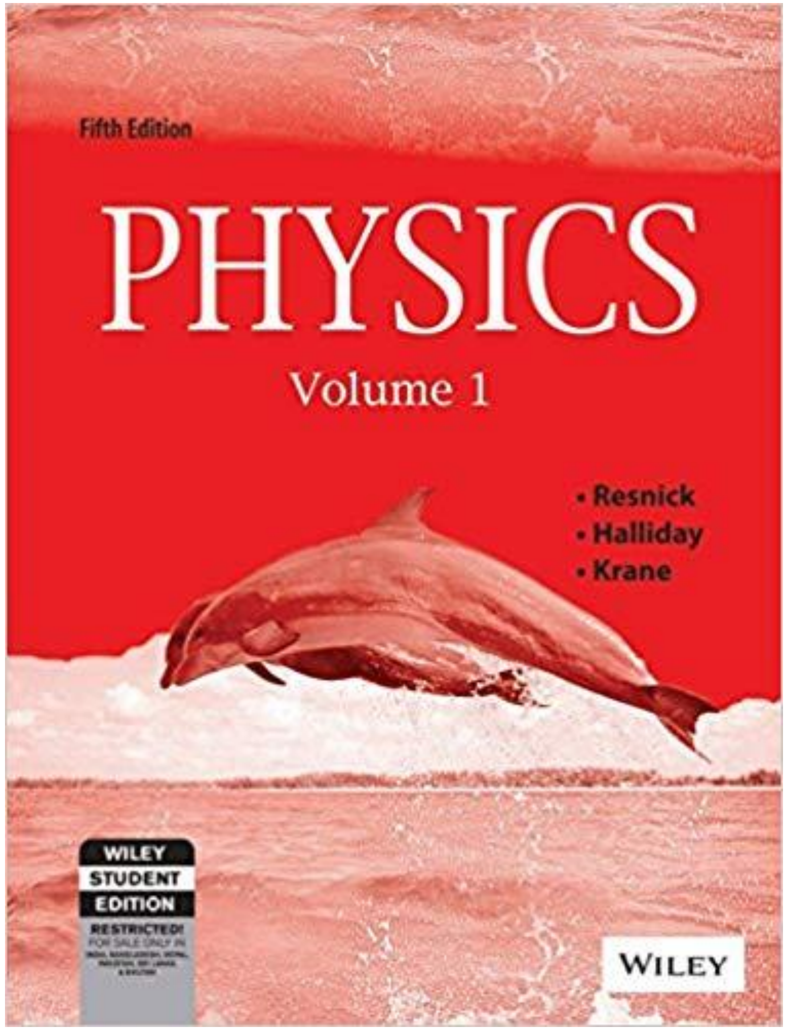
PHYSICS

General Physics I (PHYS 101)





Ganesh Kuwar Chhetri

Dept. of Physics, School of Science, Kathmandu University

WAVE AND OSCILLATION

Course Outline

- Simple Harmonic Oscillator
- Compound Pendulum
- Damped Harmonic Oscillator
- Forced Harmonic Oscillation

Simple Harmonic Motion (SHM)

Periodic Motion

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, and the back-and-forth motion of the pistons in a car engine. This kind of motion is called **periodic motion** or **oscillation.**

Simple Harmonic Motion (SHM)

- **Simple harmonic motion** is the motion of a body when the force acting on it is proportional to the body's displacement but in the opposite direction.

$$\boxed{F = -kx}$$

- In **simple harmonic motion**, the acceleration *a* of a body is proportional to the displacement *x* but opposite in sign, and the two quantities are always related by a constant (ω_0^2):

$$\boxed{a(t) = -\omega_0^{\,2} x(t)}$$

In many cases this condition is satisfied if the displacement from equilibrium is small.

- In **simple harmonic motion (SHM)**, the displacement *x(t)* of a particle from its equilibrium position is described by the equation

$$\boxed{x = A\sin(\omega_0 t + \phi) \quad \text{or} \quad x = A\cos(\omega_0 t + \phi)}$$

where A is the **amplitude** of the displacement $\omega_0 t + \phi$ is the **phase** of the motion, ω_0 is the **angular frequency**, and ϕ is the **phase constant or phase angle**.

- A body that undergoes simple harmonic motion is called a **harmonic oscillator.**
- The displacement, velocity, and acceleration in **SHM** are sinusoidal functions of time.
- The angular frequency, frequency, and period in **SHM** do not depend on the amplitude, but only on the mass *m* and force constant *k*.

The Linear Simple Harmonic Oscillator

The Linear Harmonic Oscillator

- A object with mass m that moves under the influence of a Hooke's law restoring force given by $F = -k\,x$ exhibits simple harmonic motion with

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency)}$$

and $$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period)}$$

Such a system is called a **linear simple harmonic oscillator.**

- The block–spring system of Figure Lo-1 is a **linear simple harmonic oscillator.**

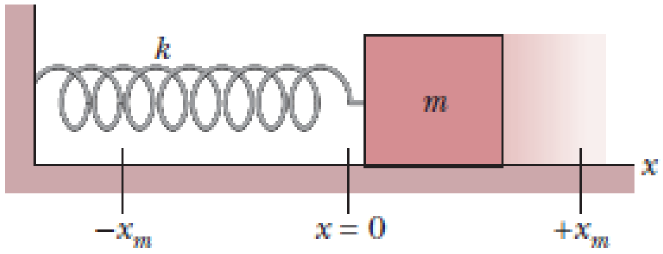


Figure Lo-1

Simple Harmonic Oscillation

- Consider a physical system that consists of a block of mass *m* attached to the end of a spring, with the block free to move on a horizontal, frictionless surface Figure H_O-1.

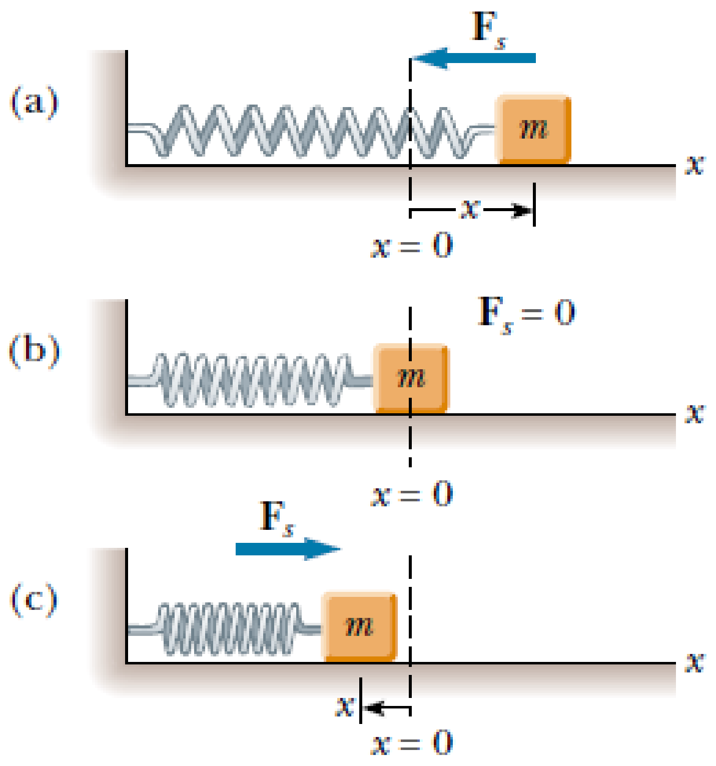


Figure H_O-1: A block attached to a spring moving on a frictionless surface. (Harmonic Oscillator)

- When the block is displaced a small distance *x* from equilibrium, the spring exerts on the block a force that is proportional to the displacement and given by Hooke's law:

$$\underbrace{F_s}_{\text{restoring force}} = -kx \qquad \text{............} \qquad (1)$$

- Applying Newton's second law to the motion of the block, together with Equation (1), we obtain,

$$F_s = -kx = ma_x$$

$$\text{or,} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\therefore \quad \boxed{\frac{d^2x}{dt^2} = -\omega_0^2 x} \qquad \text{...........} \qquad (2)$$

$$\text{where } \underbrace{\omega_0}_{\text{anglular frequency}} = \sqrt{\frac{k}{m}}$$

Simple Harmonic Oscillation

- A differential equation representing simple harmonic motion:

$$\boxed{\frac{d^2x}{dt^2} = -\omega_0^2 x} \qquad \text{...........} \qquad (2)$$

The minus sign means the acceleration and displacement always have opposite signs.

In SHM, the acceleration a_x is proportional to the displacement x but opposite in sign and the two quantities are related by the square of the angular frequency ω_0.

- Eq. (2) can be written as

$$\mathrm{v}\frac{d\mathrm{v}}{dx} = -\omega_0^2 x$$

$$\text{or,} \qquad \mathrm{v}d\mathrm{v} = -\omega_0^2 x dx \qquad \text{...........} \qquad (3)$$

- Integrating both sides of Eq. (3), we get

$$\int \mathrm{v}d\mathrm{v} = -\omega_0^2 \int x dx$$

$$\text{or,} \qquad \frac{\mathrm{v}^2}{2} = -\omega_0^2\left(\frac{x^2}{2}\right) + C \qquad \text{....................} \qquad (4)$$

where C is a constant of integration.

- At extreme position,

$$x = \pm A, \ \mathrm{v} = 0$$

$$\therefore \quad C = \frac{1}{2}\omega_0^2 A^2$$

Hence

$$\mathrm{v} = \omega_0\sqrt{A^2 - x^2}$$

$$\text{or,} \qquad \frac{dx}{dt} = \omega_0\sqrt{A^2 - x^2}$$

$$\text{or,} \qquad \frac{dx}{\sqrt{A^2 - x^2}} = \omega_0 \, dt \qquad \text{....................} \qquad (5)$$

Integrating both sides of Eq. (5), we get

$$\sin^{-1}\left(\frac{x}{A}\right) = \omega_0 t + \phi$$

$$\therefore \qquad \boxed{x = A\sin(\omega_0 t + \phi)} \qquad \text{....................} \qquad (6)$$

This is the solution of Equation (2).

Simple Harmonic Oscillation

Simple Harmonic Oscillation

For simple harmonic motion :

Displacement at time t,

Phase

$$x(t) = A\sin\left(\omega_0 t + \phi\right)$$

Amplitude

Time

Phase constant or phase angle

Angular frequency

Period in SHM

- The period T is the time required for one complete oscillation, or **cycle.**
- When time t of the equation (6) is replaced by , $t' = t + \frac{2\pi}{\omega_0}$ then the new displacement is

$$x' = A\sin\left[\omega_0\left(t + \frac{2\pi}{\omega_0}\right) + \phi\right] = A\sin\left[\left(\omega_0 t + 2\pi\right) + \phi\right] = A\sin\left[2\pi + \left(\omega_0 t + \phi\right)\right]$$

$$= A\sin\left(\omega_0 t + \phi\right) \quad = x$$

- That means the body's position is repeated after every $\frac{2\pi}{\omega_0}$ time.

Therefore, period in simple harmonic oscillation is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

Simple Harmonic Oscillation

Frequency in Simple Harmonic Motion

- The inverse of the period is called the **frequency** f of the motion.
- The frequency represents the number of oscillations or cycles per unit time.
- The frequency in SHM is

$$f = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Angular Frequency in SHM

- The **angular frequency** ω_0 is related to the period and frequency of the motion by

$$\omega_0 = \frac{2\pi}{T} = 2\pi f$$

- The frequency and period depend only on the mass of the block and on the force constant of the spring.
- In the SI system, it is measured in hertz: $1 \text{ hertz} = 1\text{Hz} = 1 \text{ oscillation per second} = 1 \text{ s}^{-1}$

Displacement, Velocity, and Acceleration in SHM

- In simple harmonic motion (SHM), the displacement of a body from its equilibrium position is given by

$$x = A\sin(\omega_0 t + \phi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(S–1)}$$

- Differentiating equation (S-1) leads to equations for the particle's SHM **velocity** and **acceleration** as functions of time:

$$\text{v} = \omega_0 A\cos(\omega_0 t + \phi) \qquad (\text{velocity})$$

and $$a = -\omega_0^{\ 2} A\sin(\omega_0 t + \phi) \qquad (\text{acceleration})$$

- The displacement from the equilibrium position, velocity, and acceleration all vary sinusoidally with time .

Energy in Simple Harmonic Motion

- The force exerted by an ideal spring is a conservative force, and the vertical forces do no work, so the total mechanical energy of the system is *conserved*.
- The kinetic energy in simple harmonic motion vary with time and is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0{}^2A^2\cos^2(\omega_0 t + \phi) \qquad \left[K = \frac{1}{2}m\omega_0{}^2\left(A^2 - x^2\right)\right]$$

- The potential energy in simple harmonic motion vary with time and is given by

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0{}^2A^2\sin^2(\omega_0 t + \phi) \qquad \left[U = \frac{1}{2}m\omega_0{}^2x^2\right]$$

- The total mechanical energy in simple harmonic motion is given by

$$E = K + U = \frac{1}{2}m\omega_0{}^2A^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}m\omega_0{}^2A^2\sin^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}m\omega_0{}^2A^2 = \frac{1}{2}kA^2$$

$$\therefore \boxed{E = \frac{1}{2}kA^2}$$

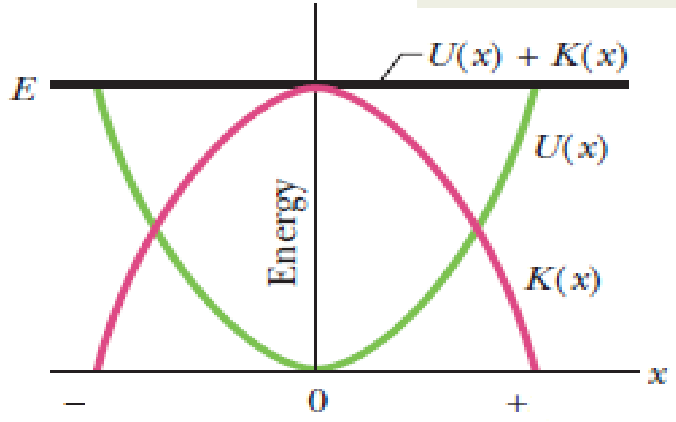


Figure Ho-2

As *position* changes, the energy shifts between the two types, but the total is constant

That is, **the total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude.**

The Physical Pendulum (Compound Pendulum)

The Physical Pendulum

- Any rigid body mounted so that it can swing in a vertical plane about some axis passing through it is called a **physical pendulum** or **a compound pendulum**.
- Actually all real pendulums are physical pendulums.
- Figure P-1shows an arbitrary physical pendulum pivoted about a horizontal frictionless axis through O and displaced from the equilibrium position by an angle θ.

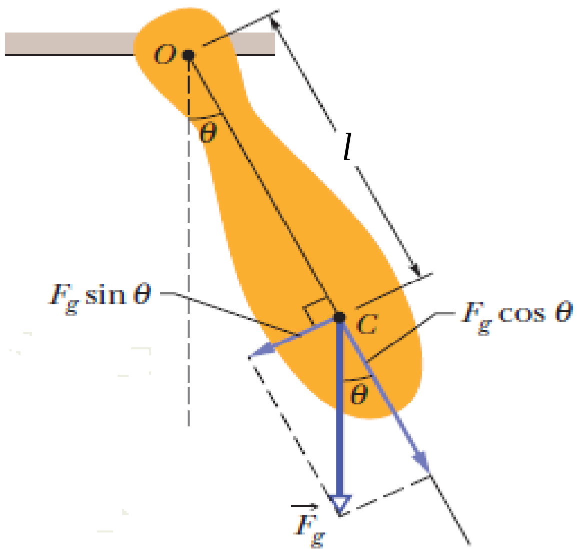


Figure P-1 A physical pendulum

The oscillation is in the *xy* plane. The z-axis is out of the page.

- The distance from pivot O to the centre of mass C is l , the moment of inertia of the body about an axis of rotation through O is I , and the mass of the body is m.
- Using the law of motion $\sum \tau_z = m\alpha_z$, we obtain

$$-mgl\sin\theta = I\frac{d^2\theta}{dt^2} \quad \dots\dots\dots\dots\dots\dots \quad (1)$$

The minus sign indicates that the torque about O tends to decrease θ. That is, the force of gravity produces a restoring torque.

- For a sufficiently small angular displacement, $\sin\theta \approx \theta$ (in radian) and the equation of motion reduces to

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgl}{I}\right)\theta \quad \dots\dots\dots\dots\dots\dots \quad (2)$$

This equation is of the same form as equation $\frac{d^2x}{dt^2} = -\omega_0^2 x$

Therefore, the motion is simple harmonic motion and the angular frequency is $\omega_0 = \sqrt{\frac{mgl}{I}}$.

Hence, the time period of physical pendulum is $\boxed{T = 2\pi\sqrt{\frac{I}{mgl}}}$

The Physical Pendulum (Compound Pendulum)

Time Period of Compound Pendulum

$$T = 2\pi\sqrt{\frac{I_{CM} + ml^2}{mgl}} = 2\pi\sqrt{\frac{mK^2 + ml^2}{mgl}} = 2\pi\sqrt{\frac{\left(\frac{K^2}{l} + l\right)}{g}}$$

Using parallel axes theorem, $I = I_{CM} + ml^2 = mK^2 + ml^2$

where I_{CM} is the moment of inertia of physical pendulum about an axis passing through its centre of mass

K is the radius of gyration of physical pendulum about an axis passing through its centre of mass

$$\therefore \boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

$$L = \frac{K^2}{l} + l$$

$L \rightarrow$ length of the equivalent simple pendulum (the distance between the centres of suspension and oscillation)

The Physical Pendulum (Compound Pendulum)

Minimum Time Period of Compound Pendulum

The time period of compound pendulum is

$$T = 2\pi\sqrt{\frac{\left(\frac{K^2}{l}+l\right)}{g}}$$

$$\Rightarrow \quad T^2 = \frac{4\pi^2}{g}\left(\frac{K^2}{l}+l\right)$$

Differentiating with respect to l , we have

$$2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left(-\frac{K^2}{l^2}+1\right)$$

Clearly, T will be maximum or a minimum when $\frac{dT}{dl}=0$ i.e., when $l = K$.

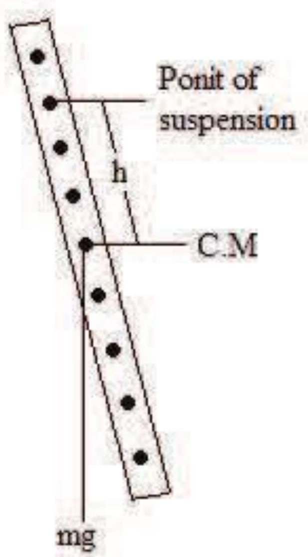
Since $\frac{d^2T}{dl^2}$ comes out to be *positive*, it is clear that T is a minimum when $l = K$.

- Therefore, Minimum Time-Period:

$$T_{\min} = 2\pi\sqrt{\frac{\left(\frac{K^2}{K}+K\right)}{g}}$$

$$\therefore \boxed{T_{\min} = 2\pi\sqrt{\frac{2K}{g}}}$$

Compound Pendulum



The compound pendulum can be converted into a simple pendulum by concentrating the whole mass of the pendulum at a point. When the mass of the compound pendulum is concentrated at a point to form a simple pendulum such that the time period of the resulting simple pendulum is equal to that of the compound pendulum, then the point of concentration is called the **point of oscillation**.

MCQ & FILL IN THE BLANKS

WAVE AND OSCILLATION

- **The minimum time period of compound pendulum is**

[a] $T_{\min} = \sqrt{2\pi \frac{2k}{g}}$ **[b]** $T_{\min} = \pi\sqrt{\frac{2k}{g}}$ **[c]** $T_{\min} = 2\pi\sqrt{\frac{k}{g}}$ **[d]** $T_{\min} = \pi\sqrt{\frac{8k}{g}}$ $\Re$

- **A 0.500-kg cart connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless air track. If the amplitude of the motion is 3.00 cm, then the total energy of the system is ……….**

$E = 9.00\times10^{-3}\,J$

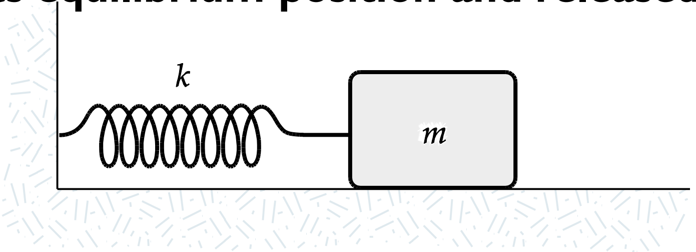
- **A compound pendulum of length *l* and mass *M* swings back and forth with period *T*. If the mass is doubled, what is the new period?**

[a] $\sqrt{2}T$ **[b]** $\sqrt{T}$ **[c]** T **[d]** $\frac{1}{\sqrt{2}}T$

- **In an engine, a piston oscillates with simple harmonic motion so that its position varies according to the expression** $x = (5.00\text{ cm})\cos\left(2t + \frac{\pi}{6}\right)$ **where *x* is in centimeters and *t* is in seconds. The velocity (in cm/s) of the body at *t* = 1.0 s is ………..**

$\mathrm{v} = -5.00\ cm/s$

- **A mass *m* = 2.0 kg is attached to a spring having a force constant *k* = 290 N/m as in the figure. The mass is displaced from its equilibrium position and released. Its frequency of oscillation (in Hz) is approximately ……**



***f* = 1.9**

Damped Harmonic Oscillator

Damped Harmonic Oscillator

- In many real systems, dissipative forces, such as friction, retard the motion.

  Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be ***damped***.
- The decrease in amplitude caused by dissipative forces is called **damping.**
- When some resistive force acts on a free oscillation, then the amplitude of oscillation gradually decreases and becomes zero. This type of oscillation is called damped harmonic oscillation. .
- Figure D_H-1 shows a simple model of damped oscillator.
- We consider the oscillating body (of mass *m*) to be attached to a (massless) vane immersed in a fluid, in which it experience a viscous damping force $-bv$.

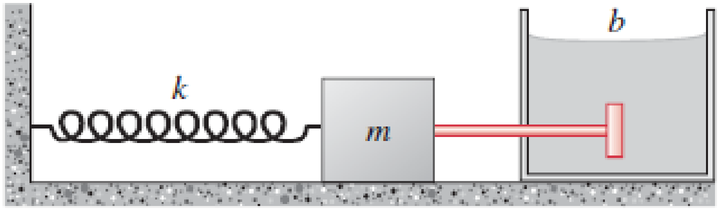


Figure D_H1:
A representation of a damped harmonic oscillator.
A vane immersed in a fluid exerts a damping force on the block as the block oscillates parallel to the *x* axis.

Damped Harmonic Oscillator

Damped Harmonic Oscillator

- The force on the block from the spring is $F_s = -kx$.
- The **damping force** on the block is $\vec{F}_{\text{damp}} = -b\vec{v}$

  Where b is a **damping constant** that depends on properties of the fluid and the size and shape of the vane that is immersed in the fluid.

  and $\vec{v}$ is the velocity of the vain and block.
- Let us assume that the gravitational force on the block is negligible relative to $\vec{F}_{\text{damp}}$ and F_s .

The *net* force on the body is then

$$\sum F_x = -kx - bv$$

From Newton's Second Law:

$$\sum F_x = ma_x$$

or, $$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

$\therefore$ $$\boxed{\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0}$$

.... (1)

Let $\frac{b}{m} = 2\gamma$ so that $\gamma = \frac{b}{2m}$ (called damping ratio) and

$$\sqrt{\frac{k}{m}} = \omega_0 \quad (\text{natural angular frequency})$$

Equation (1) reduces to

$$\boxed{\frac{d^2x}{d^2t} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0} \quad .... \quad (2)$$

This is the differential equation of **damped harmonic oscillator.**

Damped Harmonic Oscillator

- Let a solution of differential equation is $x = Ae^{\alpha t}$.

  Substituting this solution in the Equation (2), we get

$$\alpha^2 Ae^{\alpha t} + 2\gamma\alpha Ae^{\alpha t} + \omega_0^{\,2} Ae^{\alpha t} = 0$$

$$i.e. \quad \alpha^2 + 2\gamma\alpha + \omega_0^{\,2} = 0$$

$$i.e. \quad \alpha = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^{\,2}}}{2}$$

$$= -\gamma \pm \sqrt{\gamma^2 - \omega_0^{\,2}}$$

$$\therefore \quad \boxed{\alpha = -\gamma \pm \beta} \qquad \text{where } \beta = \sqrt{\gamma^2 - \omega_0^{\,2}}$$

Therefore, α have two values: $-\gamma - \beta$ and $-\gamma + \beta$.

- So Equation (2) must have two solutions:

$$x_1 = A_1 e^{(-\gamma-\beta)t} \text{ and } x_2 = A_2 e^{(-\gamma+\beta)t}$$

The complete solution is

$$x = x_1 + x_2$$

$$= A_1 e^{(-\gamma+\beta)t} + A_2 e^{(-\gamma-\beta)t}$$

$$\therefore \quad \boxed{x = e^{-\gamma t}(A_1 e^{\beta t} + A_2 e^{-\beta t})} \qquad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

This is the displacement equation of mass m executing damped SHM.

Case I:

When $\gamma \gg \omega_0$ and β is a real (damping force is very high), the condition is called **overdamping.**

The system no longer oscillates but returns to its equilibrium position without oscillation when it is displaced and released.

Damped Harmonic Oscillator

Case II :

If the damping force is normal, such that γ is slightly greater than ω_0 , and is real and very small, the condition is called critical damping.

Eq. (3) reduces to

$$x = e^{-\gamma t}(A_1 e^{\beta t} + A_2 e^{-\beta t})$$
$$= e^{-\gamma t}\left[A_1(1+\beta t+\cdots) + A_2(1-\beta t+\cdots)\right]$$
$$= e^{-\gamma t}\left[A_1 + A_1\beta t + A_2 - A_2\beta t\right]$$
$$= e^{-\gamma t}\left[(A_1+A_2) + (A_1-A_2)\beta t\right]$$

$$\therefore \quad \boxed{x = e^{-\gamma t}(M + N\beta t)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

where $(A_1 + A_2) = M$ and $(A_1 - A_2) = N$

The system no longer oscillates but returns to equilibrium faster than with overdamping when it is displaced and released.

Case III :

If the damping force is very low, such that $\gamma \ll \omega_0$. As a result, β becomes a complex

$$i.e.\ \beta = \sqrt{\gamma^2 - \omega_0^{\ 2}} = \sqrt{-(\omega_0^{\ 2} - \gamma^2)} = i\sqrt{\omega_0^{\ 2} - \gamma^2} = i\omega$$

$$\text{where} \quad \omega = \sqrt{\omega_0^{\ 2} - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Eq. (3) becomes

$$x = e^{-\gamma t}(A_1 e^{i\omega t} + A_2 e^{-i\omega t})$$
$$= e^{-\gamma t}\left[A_1(\cos\omega t + i\sin\omega t) + A_2(\cos\omega t - i\sin\omega t)\right]$$
$$= e^{-\gamma t}\left[(A_1+A_2)\cos\omega t + \{(A_1-A_2)i\}\sin\omega t\right]$$
$$= e^{-\gamma t}\left[C\cos\omega t + D\sin\omega t\right] \qquad \text{where } C = (A_1+A_2),$$
$$D = \{(A_1-A_2)i\}$$
$$= e^{-\gamma t}\sqrt{C^2+D^2}\left[\frac{C}{\sqrt{C^2+D^2}}\cos\omega t + \frac{D}{\sqrt{C^2+D^2}}\sin\omega t\right]$$

$$\text{Let} \quad \frac{C}{\sqrt{C^2+D^2}} = \sin\phi, \ \text{and} \ \frac{D}{\sqrt{C^2+D^2}} = \cos\phi,$$

$$\text{then} \quad x = e^{-\gamma t}\sqrt{C^2+D^2}\left[\sin\phi\cos\omega t + \cos\phi\sin\omega t\right] \qquad \therefore \quad \boxed{x = Ee^{-\gamma t}\sin(\omega t + \phi)}$$

The system oscillates with steadily decreasing amplitude. This situation is called under damping.

Damped Harmonic Oscillator

- Figure D0 shows the Graphs of displacement versus time

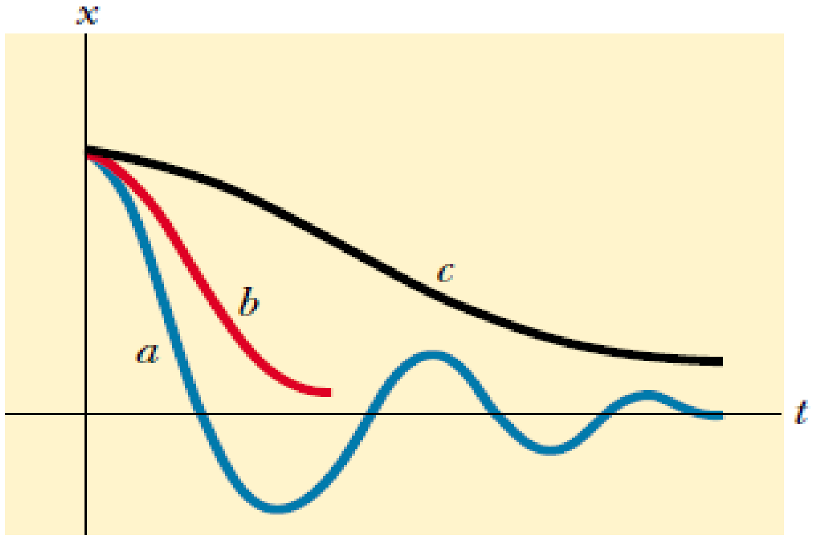


Figure D0
Graphs of displacement versus time for
(a) an underdamped oscillator,
(b) a critically damped oscillator,
(c) an overdamped oscillator.

Notes:

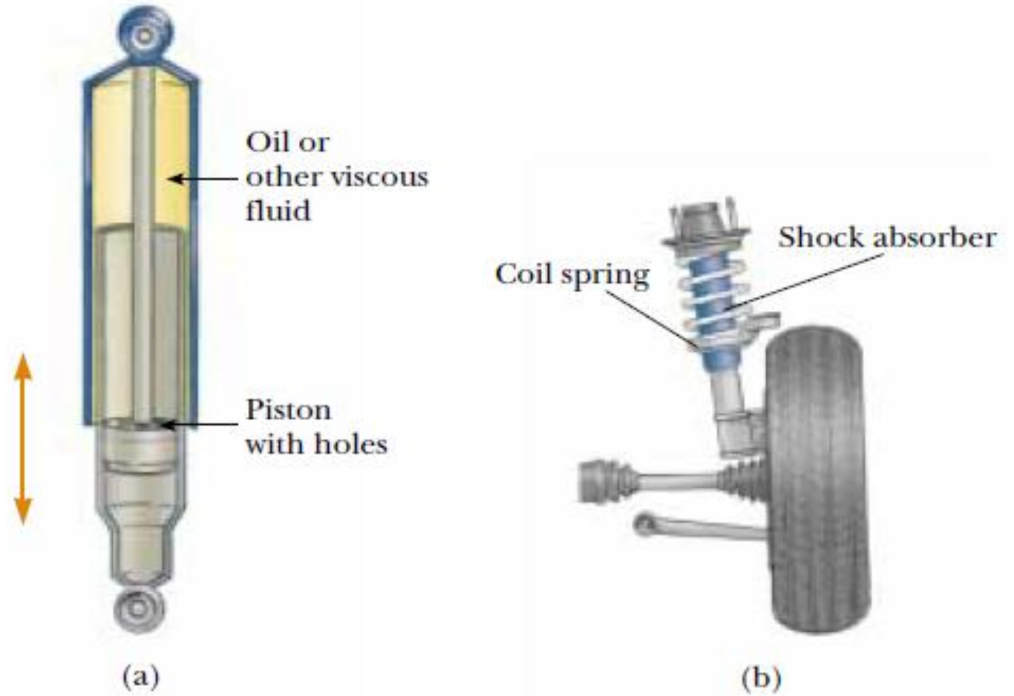


Figure D_H-5

a) A shock absorber consists of a piston oscillating in a chamber filled with oil.
As the piston oscillates, the oil is squeezed through holes between the piston and the chamber, causing a damping of the piston's oscillations.
b) One type of automotive suspension system, in which a shock absorber is placed inside a coil spring at each wheel.

Forced Harmonic Oscillation

- The damping can be overcome by applying some oscillating external force on the oscillating body. As a result, its own frequency immediately dies out and the body starts to oscillate with the frequency of external oscillating force. This type of oscillation is called **forced harmonic oscillation**.
- A common example of a forced oscillator is a damped oscillator driven by an external oscillating force, $F_{ext} = F_o \sin \omega' t$.

  where ω' is the angular frequency of the periodic force, and F_0 is a constant.

The resultant force experience by the body is

$$F = F_{\text{spring}} + F_{\text{damp}} + F_{\text{ext}}$$

$$= -kx - b\frac{dx}{dt} + F_o \sin \omega' t$$

- From Newton's second law of motion,

$$\sum \mathrm{F}_x = ma_x$$

$$\therefore \quad m\frac{d^2x}{d^2t} = -kx - b\frac{dx}{dt} + F_o \sin \omega' t$$

$$\text{or} \quad \boxed{\frac{d^2x}{d^2t} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_o}{m}\sin \omega' t} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Let $\frac{b}{m} = 2\gamma$, $\frac{k}{m} = \omega_0^{\ 2}$ and $\frac{F_0}{m} = \mathrm{f}_0$

Equation (1) reduces to

$$\frac{d^2x}{d^2t} + 2\gamma\frac{dx}{dt} + \omega_0^{\ 2}x = \mathrm{f}_0 \sin \omega' t \quad \ldots\ldots\ldots\ldots \quad (2)$$

This is the **differential equation of forced harmonic oscillator**.

Forced Harmonic Oscillation

- Let a solution of differential equation is $x = A\sin(\omega' t + \phi)$ .... (3)

  where A is the amplitude of the oscillation and ϕ is the phase angle.

- Substituting this solution in the Equation (2), we get

$$-\omega'^2 A\sin(\omega' t+\phi) + 2\gamma\ \omega' A\cos(\omega' t+\phi) + \omega_0^{\ 2} A\sin(\omega' t+\phi) = f_0 \sin\omega' t$$

or, $$\left(\omega_0^{\ 2} - \omega'^2\right) A\sin(\omega' t+\phi) + 2\gamma\omega' A\cos(\omega' t+\phi) = f_0 \sin\left[(\omega' t+\phi) - \phi\right]$$

or, $$\left(\omega_0^{\ 2} - \omega'^2\right) A\sin(\omega' t+\phi) + 2\gamma\omega' A\cos(\omega' t+\phi) = f_0 \sin(\omega' t+\phi)\cos\phi - f_0 \cos(\omega' t+\phi)\sin\phi$$

$\therefore$ $$\left\{\left(\omega_0^{\ 2} - \omega'^2\right) A\right\} \sin(\omega' t+\phi) + \left\{2\gamma\omega' A\right\} \cos(\omega' t+\phi) = \left\{f_0 \cos\phi\right\} \sin(\omega' t+\phi) + \left\{-f_0 \sin\phi\right\} \cos(\omega' t+\phi)$$

.... (4)

- Equating the coefficient of and from both sides of equation (4), we get

$$\left(\omega_0^{\ 2} - \omega'^2\right) A = f_0 \cos\phi \qquad \text{....................} \quad (5)$$

and $$2\gamma\omega' A = -f_0 \sin\phi \qquad \text{....................} \quad (6)$$

Forced Harmonic Oscillation

- Squaring and adding equations (5) and (6), we get

$$\left(\omega_0^{\,2}-\omega'^2\right)^2 A^2+\left(2\gamma\omega'\right)A^2=f_0^{\,2}$$

or, $$\left[\left(\omega_0^{\,2}-\omega'^2\right)^2+\left(2\gamma\omega'\right)\right]A^2=f_0^{\,2}$$

$\therefore$ $$\boxed{A=\frac{f_0}{\sqrt{\left(\omega_0^{\,2}-\omega'^2\right)^2+4\gamma^2\omega'^2}}}$$

.... (7)

Dividing Eq. (6) by Eq. (5), we get

$$\frac{2\gamma\omega' A}{\left(\omega_0^{\,2}-\omega'^2\right)A}=\frac{-f_0\sin\phi}{f_0\cos\phi}$$

or, $$\tan\phi=\frac{2\gamma\omega'}{\left(\omega'^2-\omega_0^{\,2}\right)}$$

$\therefore$ $$\boxed{\phi=\tan^{-1}\left(\frac{2\gamma\omega'}{\left(\omega'^2-\omega_0^{\,2}\right)}\right)}$$

.... (8)

- Therefore equation (3) becomes

$$x=\frac{f_0}{\sqrt{\left(\omega_0^{\,2}-\omega'^2\right)^2+4\gamma^2\omega'^2}}\sin\left[\omega' t+\tan^{-1}\left(\frac{2\gamma\omega'}{\left(\omega'^2-\omega_0^{\,2}\right)}\right)\right]$$

.... (9)

This is the displacement equation of mass m executing forced SHM.

Forced Harmonic Oscillation

Case I:

For no damping or free oscillation, $\gamma = 0$

So, Amplitude A becomes

$$\boxed{A = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega'^2\right)^2}} = \frac{F_0}{m\left(\omega_0^2 - \omega'^2\right)}}$$

The amplitude A goes to infinity as $\omega' \approx \omega_0$.

Case II:

The amplitude has maximum value at a frequency of external oscillating force.

To get the maximum value, we have

$$\frac{dA}{d\omega'} = 0 \quad \Rightarrow \frac{d}{d\omega'}\left[\frac{f_0}{\sqrt{\left(\omega_0^2 - \omega'^2\right)^2 + 4\gamma^2\omega'^2}}\right] = 0$$

$$\Rightarrow \quad f_0 \frac{d}{d\omega'}\left[\left\{\left(\omega_0^2 - \omega'^2\right)^2 + 4\gamma^2\omega'^2\right\}^{-\frac{1}{2}}\right] = 0$$

$$\Rightarrow \quad f_0\left[\left(-\frac{1}{2}\right)\left\{\left(\omega_0^2 - \omega'^2\right)^2 + 4\gamma^2\omega'^2\right\}^{-\frac{3}{2}}\right]$$

$$\left[2\left(\omega_0^2 - \omega'^2\right)\left(-2\omega'\right) + 8\gamma^2\omega'\right] = 0$$

$$\Rightarrow \quad \left[2\left(\omega_0^2 - \omega'^2\right)\left(-2\omega'\right) + 8\gamma^2\omega'\right] = 0$$

$$\Rightarrow \quad -4\omega'\left(\omega_0^2 - \omega'^2 - 2\gamma^2\right) = 0$$

$$\Rightarrow \quad \omega_0^2 - \omega'^2 - 2\gamma^2 = 0$$

$$\therefore \quad \boxed{\omega' = \sqrt{\omega_0^2 - 2\gamma^2}}$$

Hence the amplitude is maximum when the frequency of external oscillating force is equal to

$$\boxed{\omega' = \sqrt{\omega_0^2 - 2\gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}}$$

The amplitude is a function of the driving frequency ω' and reaches a peak at a driving frequency ω' close to the natural frequency ω_0 of the system. This behavior is called **resonance**.

, resonance occurs.

Forced Harmonic Oscillation

The maximum amplitude is

$$A_{max} = \frac{f_0}{\sqrt{\left\{\omega_0^2 - \left(\omega_0^2 - 2\gamma^2\right)\right\}^2 + 4\gamma^2\left(\omega_0^2 - 2\gamma^2\right)}}$$

$$= \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega_0^2 + 2\gamma^2\right)^2 + 4\gamma^2\omega_0^2 - 8\gamma^4}}$$

$$= \frac{f_0}{\sqrt{4\gamma^4 + 4\gamma^2\omega_0^2 - 8\gamma^4}}$$

$$= \frac{f_0}{\sqrt{4\gamma^2\omega_0^2 - 4\gamma^4}} = \frac{f_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$$

$$\therefore \boxed{A_{max} = \frac{F_0}{2m\gamma\sqrt{\omega_0^2 - \gamma^2}}}$$

That means the maximum amplitude (the amplitude at resonance) decreases as damping increases.

Figure D_F is a graph of amplitude as a function of frequency for a forced oscillator with and without damping.

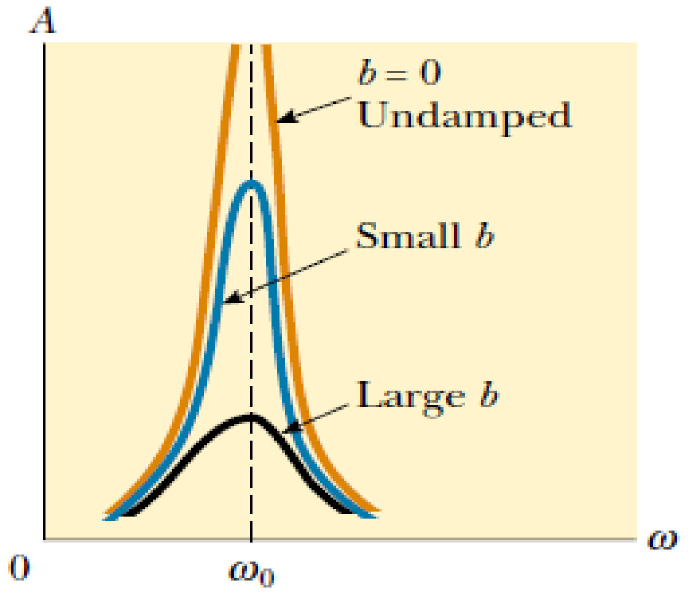


Figure D_F
Graph of amplitude versus frequency for a damped oscillator when a periodic driving force is present. When the frequency of the driving force equals the natural frequency resonance occurs.

Forced Harmonic Oscillation

Notes:

When the damping is small, the forced oscillations reach their maximum displacement amplitude when the driving frequency is equal to natural frequency. This condition is known as resonance and the corresponding frequency ω' is called the resonant angular frequency:

$$\boxed{\omega' = \omega_0} \quad \text{(resonance condition).}$$

A bridge has natural frequencies that can be set into resonance by an appropriate driving force. A dramatic example of such resonance occurred in 1940, when resonant vibrations destroyed the Tacoma Narrows Bridge in the state of Washington (four months and six days after it was opened for traffic.

In September 1985, buildings of intermediate height collapsed in Mexico City as a result of an earthquake far from the city. Taller and shorter buildings remained standing. One of the reason is resonance.

All mechanical structures have one or more natural angular frequencies, and if a structure is subjected to a strong external driving force that matches one of these angular frequencies, the resulting oscillations of the structure may rupture it. Thus, for example, aircraft designers must make sure that none of the natural angular frequencies at which a wing can oscillate matches the angular frequency of the engines in flight. A wing that flaps violently at certain engine speeds would obviously be dangerous.

SUMMARY

Notes:

Simple Harmonic Motion

- When the acceleration of an object is proportional to its displacement from some equilibrium position and is in the direction opposite the displacement, the object moves with simple harmonic motion.

- The hallmark of SHM is $\boxed{a(t) = -\omega_0^2 x(t)}$:

  (1) The particle's acceleration is always opposite its displacement and

  (2) The two quantities are always related by a constant (ω_0^2).

- The position x of a simple harmonic oscillator varies periodically in time according to the expression $x = A\sin(\omega_0 t + \phi)$.

- The total energy of a simple harmonic oscillator is a constant of the motion and is given

  $$E = \frac{1}{2}kA^2.$$

Notes:

Compound Pendulum:

- Time period of Compound Pendulum

$$T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{mK^2 + ml^2}{mgl}} = 2\pi\sqrt{\frac{\left(\frac{K^2}{l} + l\right)}{g}}$$

where I is the moment of inertia of physical pendulum about an axis passing through its pivot point

m is the mass of the physical pendulum

and l is the distance between the pivot point and centre of mass

- The time period of compound pendulum is minimum when its length is equal to its radius of gyration about the axis through its centre of mass $(l = K)$:

$$\boxed{T_{\min} = 2\pi\sqrt{\frac{2K}{g}}}$$

- The time period of compound pendulum is maximum when its length is zero, i.e., when the centre of gravity itself is the point of suspension.

SUMMARY

Notes:

Damped Oscillation

- When a damping force $(\vec{F}_{damp} = -b\vec{v})$ is added to a simple harmonic oscillator, the motion is called a damped oscillation.
- When the magnitude of the retarding force is small such that $\frac{b}{2m} < \omega_0$, the system is said to be **underdamped.**
  The resulting motion is described by
  $$\boxed{x = Ae^{-\frac{b}{2m}t}\sin(\omega t + \phi)} \qquad \text{(oscillator with little damping)}$$
  The angular frequency of oscillation is given by $\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$.
  The system oscillates with decaying amplitude.
- When b reaches a critical value b_c such that $\frac{b_c}{2m} = \omega_0$, the system is said to be **critically damped**.
  When the system is displaced, it returns to equilibrium without oscillating.
- If the medium is so viscous that the retarding force is large compared with the restoring force—that is, if $\frac{b}{2m} > \omega_0$—the system is **overdamped.**
  When the system is displaced, it returns to equilibrium without oscillating but more slower than with critical damping.

Notes:

Forced Oscillation

- When a sinusoidally varying driving force is added to a damped harmonic oscillator, the resulting motion is called **a forced oscillation or a driven oscillation**.

- If an external driving force with angular frequency ω' acts on an oscillating system with natural angular frequency ω_0, the system oscillates with angular frequency ω'.

- The amplitude of the oscillation is given by

$$A=\frac{F_0/m}{\sqrt{\left(\omega'^2-\omega_0^2\right)^2+\left(\frac{b\omega'}{m}\right)^2}} \qquad \text{.....................} \qquad (2)$$

where $\omega_0=\sqrt{\frac{k}{m}}$ is the natural frequency of the system

The amplitude is a function of the driving frequency ω' and reaches a peak at a driving frequency ω' close to the natural frequency ω_0 of the system. This behavior is called **resonance.**

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Thank you