Advanced Calculus

Functions of Several Variables

GR Phaijoo, PhD

Department of Mathematics

School of Science, Kathmandu University

Kavre, Dhulikhel

Partial Derivatives - Chain Rule

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The Chain Rule

Chain Rule For Function of Single Variable

When $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is differentiable function of t and

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

Theorem: Chain Rule for Function of Two Independent Variable

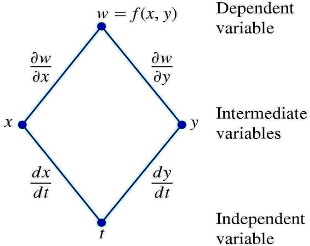
If $w = f(x, y)$ is a differentiable and $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t)$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Tree (Branch) Diagram

Chain Rule - Two independent variables

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

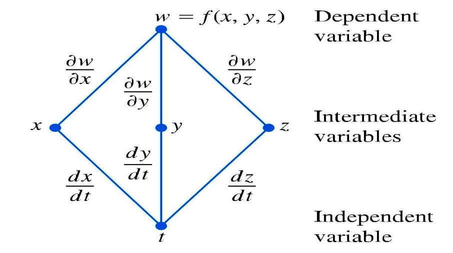
$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.}$$

Example: Use chain rule to find derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi/2$? Ans: -1.

Theorem; Chain Rule

Suppose that $w = f(x, y, z)$ and x, y, z are differentiable functions of t, then w is differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$



Example

Find $\dfrac{dw}{dt}$ if

$w = xy + z, x = \cos t, y = \sin t, z = t$

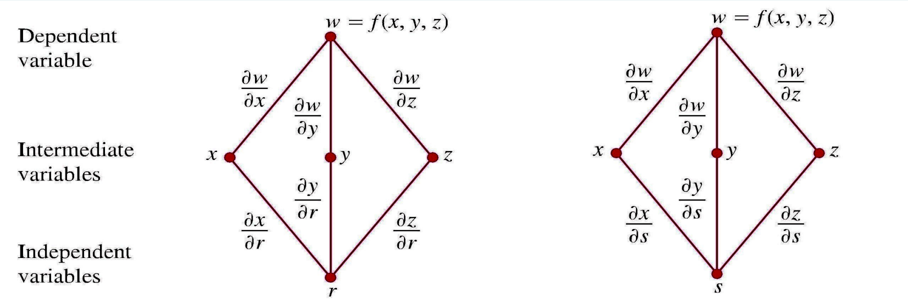
at $t = 0$. Ans. 2

Two Independent Variables, Three Intermediate Variables

Theorem; Chain Rule

Suppose that $w = f(x, y, z)$ and $x = g(r, s)$, $y = h(r, s)$ and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}, \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$



EXAMPLE Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution Using the formulas in Theorem, we find

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\
&= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r
\end{aligned}$$

Substitute for intermediate variable z.

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}
\end{aligned}$$

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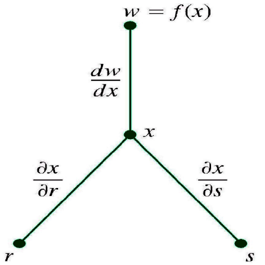
EXAMPLE Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, x = r - s, y = r + s \qquad \text{Ans:} \quad 44r, 4s$$

Two Independent Variables, One Intermediate Variables

If $w = f(x)$ and $x = g(r, s)$, then

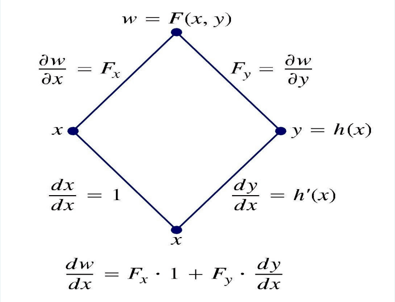


If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$

Implicit Differentiation

Suppose that
1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, y) = 0$ defines y implicitly as a function of x, say $y = h(x)$. Then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.



$$\begin{aligned} 0 &= \frac{dw}{dx} \\ &= F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx} \end{aligned}$$

Therefore, $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$

Example: Find $\dfrac{dy}{dx}$ if $x^2 + y^2 = a^2$.

Directional Derivative

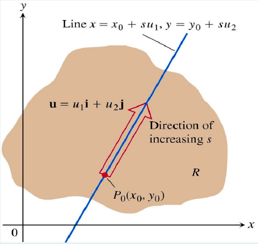
Definition

The derivative of f at $P_0(x_0, y_0)$ in the direction of unit vector $\hat{u} = u_1\vec{i} + u_2\vec{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\hat{u},P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided that the limit exists.

Directional derivative defined by $(*)$ is also denoted by $(D_{\hat{u}}f)P_0$ "The derivative of f at P_0 in the direction of $\hat{u}$"



- If $\hat{u} = \vec{i}$,
  $(D_{\hat{u}}f)_{P_0} = \dfrac{\partial f}{\partial x}$ at P_0.
- If $\hat{u} = \vec{j}$,
  $(D_{\hat{u}}f)_{P_0} = \dfrac{\partial f}{\partial y}$ at P_0.

Example

Find the derivative of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction of $\vec{u} = \vec{i} + \vec{j}$ using definition. Ans: $5/\sqrt{2}$.

Calculation of Directional Derivative as a Dot Product

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0. ∇f is read "grad f" and "del f."

We begin with the line $x = x_0 + su_1, \quad y = y_0 + su_2,$ through $P_0(x_0, y_0)$, parametrized with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then by the Chain Rule we find

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds} \qquad \text{Chain Rule for differentiable } f$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0}u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0}u_2 \qquad dx/ds = u_1 \text{ and } dy/ds = u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\mathbf{i} + u_2\mathbf{j}\right]}_{\text{Direction } \mathbf{u}}.$$

THEOREM — The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}.$$

Examples

1. Find ∇f if $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln z$ at (1,1,1).
   $Ans : 2\vec{i} + 2\vec{j} - 4\vec{k}$.

2. Find the derivative of $f(x, y) = x^2 \sin 2y$ at $(1, \pi/2)$ in the direction of $\vec{v} = 3\vec{i} - 4\vec{j}$. Ans. $8/5$.

Algebra of Gradients

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

Properties of Directional Derivatives

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Example

Find the directions inwhich the function $f(x, y, z) = x/y - yz$ increases and decreases most rapidly at $P_0(4, 1, 1)$. Then, find the derivatives of the function in these directions.

1. We have, $\nabla f|_{P_0} = \vec{i} - 5\vec{j} - \vec{k}$.

   The function f increases most rapidly in the direction of ∇f at P_0 given by $\hat{u} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{1}{3\sqrt{3}}(\vec{i} - 5\vec{j} - \vec{k}) = \dfrac{1}{3\sqrt{3}}\vec{i} - \dfrac{5}{3\sqrt{3}}\vec{j} - \dfrac{1}{3\sqrt{3}}\vec{k}$.

   The derivative in this direction is $(D_{\hat{u}}f)_{P_0} = |\nabla f| = 3\sqrt{3}$.

2. We have, $\nabla f|_{P_0} = \vec{i} - 5\vec{j} - \vec{k}$.

   The function f decreases most rapidly in the direction of $-\nabla f$ at P_0 given by

   $$-\hat{u} = -\frac{\nabla f}{|\nabla f|} = -\frac{1}{3\sqrt{3}}(\vec{i} - 5\vec{j} - \vec{k}) = -\frac{1}{3\sqrt{3}}\vec{i} + \frac{5}{3\sqrt{3}}\vec{j} + \frac{1}{3\sqrt{3}}\vec{k}.$$

   The derivative in this direction is $(D_{-\hat{u}}f)_{P_0} = -|\nabla f| = -3\sqrt{3}$.

Directional Derivatives

Estimating the Change in f in a Direction u

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\text{Directional derivative}} \underbrace{ds}_{\text{Distance increment}}$$

EXAMPLE Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Directional Derivatives

EXAMPLE Estimate how much the value of

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will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Solution We first find the derivative of f at P_0 in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of f at P_0 is

$$\nabla f|_{(0,1,0)} = ((y\cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$

Tangent Planes and Normal Lines

Tangent Plane

Tangent Plane to the level surface $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Normal Line

Normal line to the level surface $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$x = x_0 + f_x(P_0)t,\ y = y_0 + f_y(P_0)t,\ z = z_0 + f_z(P_0)t$$

Tangent Plane to a Surface

Plane tangent to a surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Example: Find the tangent plane and normal line of the surface $f(x, y, z) = x^2 + y^2 + z - 9 = 0$ at $(1, 2, 4)$ and find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.