# KATHMANDU UNIVERSITY

DHULIKHEL , KAVRE

SUBJECT: MATH104

ASSIGNMENT: 3

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<Q.1>: Find all the local maxima, local minima and saddle points of the functions.

(a): $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

Soln:

Given,

$f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

the above function is differentiable for all $x$ and $y$. Hence, the function has extreme points where $f_x$ and $f_y$ are simultaneous zero.

So,

$$f_x = 2x + y + 3 = 0 \quad \text{— (i)}$$

and $$f_y = x + 2y - 3 = 0 \quad \text{— (ii)}$$

Solving (i) and (ii), $\cancel{2x+y+3} = \cancel{x+2y-3}$

or, $(x,y) = (-3,3)$

Local extrema only exists at $(-3,3)$

So, $f_{xx} = 2$ $\qquad f_{yy} = 2$

$(f_x)_y = 1$

Now,

$f_{xx} f_{yy} - (f_x)_y^2 = 2 \times 2 - 1 = 3.$

Since $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$, $f$ has local minimum at $(-3,3)$

$\therefore f_{min} = -5$

(b): $f(x,y) = x^2+xy+3x+2y+5$

Soln:

Given,

$f(x,y) = x^2+xy+3x+2y+5$

the above function is differentiable for all $x$ and $y$. Hence, the function has extreme points where $f_x$ and $f_y$ are simultaneously zero.

So,

$f_x = 2x+y+3 = 0$ — (i)

$f_y = x+2 = 0$ — (ii)

Solving (i) and (ii), we get $\therefore (x,y) = (-2,1)$

Local extrema ~~the~~ only exists at $(-2,1)$

$f_{xx} = 2$ $\quad f_{yy} = 0$

$f_{xy} = 1$

Now,

$f_{xx}f_{yy} - f_{xy}^2 = 2\times 0 - 1^2$
$= -1 < 0$

Since $f_{xx}f_{yy} - f_{xy}^2 < 0$, saddle point exists at $(-2,1)$.

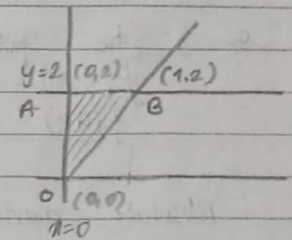
<Q.2>: Find the absolute maxima and minima of the functions on the given domains.

(a): $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x=0, y=2, y=2x$ in the first quadrant

Soln.

Given,

$f(x,y) = 2x^2-4x+y^2-4y+1$




For interior points,

$f_x = 4x-4=0$ — (i)

$f_y = 2y-4=0$ — (ii)

Solving (i) and (ii), we get. $(x,y) = (1,2)$

So, $f(1,2) = -5$

Along OA, $x=0$

$\therefore f(0,y) = y^2-4y+1$

So,

$f'(0,y) = 2y-4 = 0$ $\quad \therefore y=2$.

So points on OA, $(0,0)$, $(0,2)$,

So $f(0,0) = 1$

$f(0,2) = -4$

Along AB, $y=2$

$\therefore f(x,2) = 2x^2-4x-3$

So, $f'(x,2) = 4x-4=0$ $\quad \therefore x=1$

& points on AB, $(0,2)$, $(1,2)$

$\therefore f(1,2) = -5$ $\quad \therefore f(0,2) = -4$

Along OB, $y = 2x$

So,

$f(x, 2x) = 2x^2 - 4x + 4x^2 - 8x + 1 = 0$

$= 6x^2 - 12x + 1 = 0$

~~$\therefore x = 1.91, 0.08$~~ So,

$f'(x, 2x) = 12x - 12$

$\therefore x = 1$

when ~~$x = 1.91$~~, the points

~~$(1.91, 3.82)$~~, ~~$(0.08, 0.16)$~~

So, the points on OB, $(0,0)$, $(1,2)$

$f(1,2) = -5$

$f(0,0) = 1$

$\therefore$ Absolute maximum $= 1$ at $(0,0)$

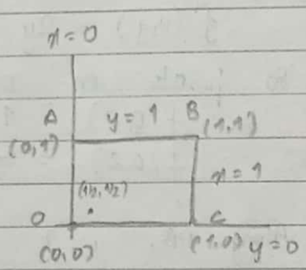
Absolute minimum $= -5$ at $(1,2)$

<b>: $f(x,y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \le x \le 1$, $0 \le y \le 1$.

Soln.

Given,

$f(x,y) = 48xy - 32x^3 - 24y^2$




For interior points,

$f_x = 48y - 96x^2 = 0$ — (i)

$f_y = 48x - 48y = 0$ — (ii)

From eqn(ii), $x = y$.

So, $x = 1/2, 0$.

interior points are $\left(\frac{1}{2}, \frac{1}{2}\right)$, $(0,0)$

$\therefore f(0,0) = 0$

$\therefore f(1/2, 1/2) = 2$

Along OA, ~~$y = 0$~~ $x = 0$

$\therefore f(0,y) = -24y^2$

So,

$f'(0,y) = -48y = 0$

$\therefore y = 0$

The points along OA are $(0,0)$ and $(0,1)$

$\therefore f(0,0) = 0$

$\therefore f(0,1) = -24$

Along OC, $y = 0$

$\therefore f(x,0) = -32x^3$

So, $f'(x,0) = -96x^2 = 0$

$\therefore x = 0$

The points along OC are $(0,0)$, $(1,0)$

$\therefore f(0,0) = 0$

$\therefore f(1,0) = -32$

Along AB, $y = 1$

$\therefore f(x,1) = 48x - 32x^3 - 24$

So,

$f'(x,1) = 48 - 96x^2 = 0$

$\therefore x = \sqrt{1/2}$

The points along AB, $(0,1)$; $(1,1)$, $(1/\sqrt{2}, 1)$, ~~$(-1/\sqrt{2}, 1)$~~

$\therefore f(0,1) = -24$ $\quad f(1,1) = -8$

$\therefore f(1/\sqrt{2}, 1) = -1.37$

Along BC, $x = 1$

$f(1, y) = 48y - 32 - 24y^2$

So,

$f'(1,y) = 48 - 48y = 0$

$\therefore y = 1$

points along BC, $(1,1), (1,0)$.

$f(1,1) = -8$

$f(1,0) = -32$

$\therefore$ Absolute maxima $= 2$ at $(1/2, 1/2)$

Absolute minima $= -32$ at $(1,0)$

<Q.3>: Maximum the function $f(x,y,z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

Soln:

Given,

$f(x,y,z) = x^2 + 2y - z^2$

the two constraints are,

$g_1(x,y,z) = 2x - y$

$g_2(x,y,z) = y + z$

Now,

$\nabla f = 2x\hat{i} + 2\hat{j} - 2z\hat{k}$

$\nabla g_1 = 2\hat{i} - \hat{j}$

$\nabla g_2 = \hat{j} + \hat{k}$

We know,

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

or, $2x\hat{i} + 2\hat{j} - 2z\hat{k} = \lambda(2\hat{i} - \hat{j}) + \mu(\hat{j} + \hat{k})$

or, $2x\hat{i} + 2\hat{j} - 2z\hat{k} = 2\lambda\hat{i} - \lambda\hat{j} + \mu\hat{j} + \mu\hat{k}$

or, $2x\hat{i} + 2\hat{j} - 2z\hat{k} = 2\lambda\hat{i} + (-\lambda + \mu)\hat{j} + \mu\hat{k}$

Equating corresponding components,

$2x = 2\lambda$ or $x = \lambda$

$2 = -\lambda + \mu$

$\lambda = -2z - 2$

$-2z = \mu$

$\therefore z = -\mu/2$

$x = -2z - 2$

$\therefore z = -\left(\frac{x+2}{2}\right)$

~~Substituting in g and $g_2$.~~

~~$2\lambda - 0 = 0$ $\therefore \lambda = 0$~~

Now,

$2x - y = 0$

or $y = 2x$

and $y + z = 0$

or $2x - \left(\frac{x+2}{2}\right) = 0$

$\therefore x = 2/3$

$y = 2 \times \frac{2}{3} = \frac{4}{3}$

$z = -\left(\frac{x+2}{2}\right) = -\frac{4}{3}$

$\therefore$ The maximum value is $f(x,y,z) = \frac{4}{3}$ at $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$

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<Q.4>: Find the extreme values of $f(x,y,z) = x-y+z$ on the unit sphere $x^2+y^2+z^2=1$.

Soln.

Given,

$$f(x,y,z) = x-y+z$$

given constraint,

$$x^2+y^2+z^2 = 1$$

$$g(x,y,z) = x^2+y^2+z^2-1$$

Now, $\nabla f = \hat{i} - \hat{j} + \hat{k}$

$\nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

We know,

$$\nabla f = \lambda \nabla g \quad \text{---(i)}$$

or $\hat{i} - \hat{j} + \hat{k} = \lambda 2x\hat{i} + \lambda 2y\hat{j} + \lambda 2z\hat{k}$

~~Equating corresponding components.~~

$\cancel{\lambda 2x = 1}$ $\cancel{\lambda 2y = -1}$ $\cancel{\lambda 2z = 1}$

$\cancel{\therefore x = 1/2}$ $\cancel{\therefore y = -1/2}$ $\cancel{z = 1/2}$

$\cancel{2x\lambda = 1}$ $\cancel{-2y\lambda = 1}$ $\cancel{2z\lambda = 4}$

$\cancel{\lambda = \frac{1}{2x}}$, $\cancel{\lambda = \frac{-1}{2y}}$, $\cancel{\lambda = \frac{1}{2z}}$

~~when $\lambda = 1$, $x = 1/2$~~

or $\hat{i} - \hat{j} + \hat{k} = \lambda(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$

$\therefore (x,y,z) = \lambda^{-1}(1/2, -1/2, 1/2)$

Here, we have,

$$x^2+y^2+z^2 = 1$$

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$$\frac{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}{\lambda^2} = 1$$

or $\frac{3}{4} = \lambda^2$ $\quad \therefore \lambda = \pm \frac{\sqrt{3}}{2}$

Now, from (i).

Equating corresponding components.

$2x\lambda = 1$, $2y\lambda = -1$, $2z\lambda = 1$

We have, $\lambda = \pm\frac{\sqrt{3}}{2}$.

So,

$x = \pm\frac{1}{\sqrt{3}}$, $y = \pm\frac{1}{\sqrt{3}}$, $z = \pm\frac{1}{\sqrt{3}}$

Now the points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$, $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, $(1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, $(1/\sqrt{3}, -1/\sqrt{3}, +1/\sqrt{3})$

$f(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = 1/\sqrt{3}$

$f(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}) = -1/\sqrt{3}$

$f(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}) = -1/\sqrt{3}$

$f(1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3}) = 1/\sqrt{3}$

$f(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = 1/\sqrt{3}$

$f(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}) = -\sqrt{3}$

$f(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) = -1/\sqrt{3}$

$f(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}) = 1/\sqrt{3}$

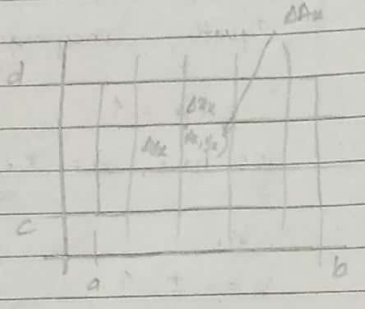
$f(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}) = \sqrt{3}$

$\therefore$ The local maximum $= \sqrt{3}$ at $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$

local minimum $= -\sqrt{3}$ at $(-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$

<Q-5>. Define the double integral of a function $f(x,y)$ over a rectangular region in $xy$ plane. state first form and Fubini's theorem in plane. Sketch the region of integration of the function $f(x,y) = x/y$ over the region in the first quadrant bounded by $y = x$, $y = 2x$, $x = 1$, $x = 2$ and then integrate it.

Soln:



$R: a \le x \le b, \quad c \le y \le d$

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

When a limit of the sums $S_n$ exists, giving the same limiting value no matter what choices are made, then the function $f$ is said to be integrable and the double integral of $f$ over $R$ written as.

$$\iint_R f(x,y)\, dA = \iint_R f(x,y)\, dx\, dy$$

Fubini's theorem in 1st form states that,
" If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b$, $c \le y \le d$ then.

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Fubini's theorem of stronger form states that,
" If $f(x,y)$ is continuous on a region $R$.

(i): If $R$ is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with $g_1(x)$ and $g_2(x)$ is continuous on $[a,b]$ then.

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

(ii) If $R$ is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with $h_1$ and $h_2$ are continuous on $[c,d]$

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\, dx\, dy$$

Now from Fubini's theorem in stronger form.

$$\iint_R f(x,y)\, dA = \int_1^2 \int_x^{2x} \frac{x}{y}\, dy\, dx$$

$$= \int_1^2 (x \ln y)_{x}^{2x}\, dx = \int_1^2 (x \ln 2)\, dx$$

$$= \left( \frac{x^2}{2} \ln 2 \right)_1^2$$

$$= 2 \ln 2 - \tfrac{1}{2} \ln 2 = \frac{3}{2} \ln 2$$

<Q.6>: Evaluate the following:

(a): $\int_0^3\int_0^2 (4-y^2)\,dy\,dx$

Sol$^n$:

$$= \int_0^3 \left[\int_0^2 4\,dy - \int_0^2 y^2\,dy\right] dx$$

$$= \int_0^3 \left[(4y)_0^2 - \left(\frac{y^3}{3}\right)_0^2\right] dx$$

$$= \int_0^3 \left\{(4\times2 - 4\times0) - \left(\frac{8}{3}-\frac{0}{3}\right)\right\} dx$$

$$= \int_0^3 \left(8-\frac{8}{3}\right) dx = \int_0^3 \frac{16}{3}\,dx = \frac{16}{3}\times 3 = 16$$

(b): $\int_{-1}^0\int_{-1}^1 (x+y+1)\,dx\,dy$

Sol$^n$:

$$= \int_{-1}^0 \left(\frac{x^2}{2}+xy+x\right)_{-1}^1 dy = \int_{-1}^0 \left(\left(\frac{1^2}{2}+y+1\right)-\left(\frac{1}{2}+(-1)y+(-1)\right)\right)dy$$

$$= \int_{-1}^0 \left(\cancel{\frac{1}{2}}+y+1-\cancel{\frac{1}{2}}+y+1\right)dy = \int_{-1}^0 2y+2$$

$$= \cancel{(y^2)_{-1}^0} = \cancel{0^2-(-1)^2 = -1}. \quad =(y^2+2y)_{-1}^0$$

$$=1$$

(c): $\int_0^\pi\int_0^x (x\sin y)\,dy\,dx$

Sol$^n$:

$$= \int_0^\pi \left(x(-\cos y)\right)_0^x dx = \int_0^\pi(-x\cos x + x)\,dx$$

$$= \int_0^\pi (x - x\cos x)\,dx = \int_0^\pi x\,dx - \int_0^\pi x\cos x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^\pi - \int_0^\pi x\cos x\,dx = \frac{\pi^2}{2} - \int_0^\pi x\cos x\,dx$$

$$= \frac{\pi^2}{2} - \left[x\int\cos x\,dx - \int\left[\frac{dx}{dx}\int\cos x\,dx\right]dx\right]_0^\pi$$

$$= \frac{\pi^2}{2} - \left[x\sin x - \int \sin x\,dx\right]_0^\pi = \frac{\pi^2}{2} - \left[x\sin x + \cos x\right]_0^\pi$$

$$= \frac{\pi^2}{2} - \left[(\pi\times\sin\pi + \cos\pi) - (0\cdot\sin 0 + \cos 0)\right]$$

$$= \frac{\pi^2}{2} - [-1-1] = \frac{\pi^2}{2}+2$$

$\{d\}: \int_0^{\ln 8} \int_0^{\ln y} (e^{x+y})\, dx\, dy$

$= \int_0^{\ln 8} \left( e^y \int_0^{\ln y} (e^x)\, dx \right) dy$

$= \int_0^{\ln 8} e^y \left( e^x \right)_0^{\ln y} dy$

$= \int_0^{\ln 8} e^y \left( e^{\ln y} - e^0 \right) dy \quad = \int_0^{\ln 8} e^y (y-1)\, dy$

$= \left\{ (y-1) \int e^y\, dy - \int \left( \frac{d(y-1)}{dy} \int e^y dy \right) dy \right\}_0^{\ln 8}$

$= \left[ (y-1) e^y - \int 1 \cdot e^y\, dy \right]_0^{\ln 8}$

$= \left[ (y-1) e^y - e^y \right]_0^{\ln 8} \quad = \left[ y e^y - 2e^y \right]_0^{\ln 8}$

$= \left( \ln 8 \cdot e^{\ln 8} - 2e^{\ln 8} \right) - \left( 0 \cdot e^0 - 2 \cdot e^0 \right).$

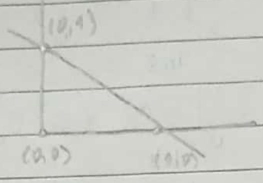
$= 8 \ln 8 - 16 + 2 \quad = 8 \ln 8 - 14$

(e): the integral of $f(x,y) = x^2+y^2$ over the triangular region with vertices $(0,0)$, $(1,0)$, $(0,1)$.

Soln:

Given, $f(x,y) = x^2+y^2$

If $x$ constant, $0 \le y \le 1-x$.




Now,

$$= \int_0^1\int_0^{1-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left(x^2y + \frac{y^3}{3}\right)_0^{1-x} dx \qquad = \int_0^1 \left(x^2 + \frac{1}{3}\right)dx$$

$$= \left(\frac{x^3}{3} + \frac{x}{3}\right)_0^1 = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

$$= \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3}\right)dx$$

$$= \int_0^1 \left(\frac{3x^2-3x^3+1-3x^2+3x-x^3}{3}\right)dx$$

$$= \frac{1}{3}\int_0^1 (1+3x-4x^3)\,dx$$

$$= \frac{1}{3}\left[x+\frac{3x^2}{2}-x^4\right]_0^1$$

$$= \frac{1}{3}\left[1+\frac{3}{2}-1 \right. \qquad \frac{1}{3}$$

$$= \int_0^1\int_0^{1-x}(x^2+y^2)\,dy\,dx \qquad = \int_0^1 \left(x^2y+\frac{y^3}{3}\right)_0^{1-x}dx$$

$$= \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3}\right)dx$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{1-3x^2+3x-x^3}{3}\right)dx$$

$$= \int_0^1 x^2\,dx - \int_0^1 x^3\,dx + \int_0^1 \frac{1}{3}\,dx - \int_0^1 \frac{3x^2}{3}\,dx + \int_0^1 \frac{3x}{3}\,dx - \int_0^1 \frac{x^3}{3}\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 + \left(\frac{x}{3}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^4}{12}\right)_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} - \frac{1}{3} + \frac{1}{2} - \frac{1}{12}$$

$$= \frac{1}{3} - \frac{1}{4} - \frac{1}{12}$$

Soln:

Now,

$$\int_0^1\int_0^{1-x}(x^2+y^2)\,dy\,dx$$

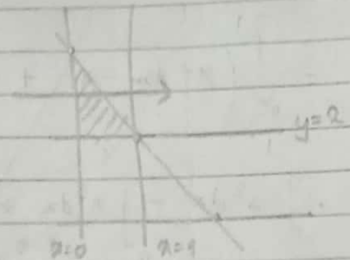
$$= \int_0^1 \left(x^2y+\frac{y^3}{3}\right)_0^{1-x}dx \qquad = \int_0^1 x^2-x^3+\frac{1-x^3+3x^2+3x}{3}$$

$$= \frac{1}{3}\int_0^1 6x^2-4x^3+3x-x^3+1 \qquad = \frac{1}{6}$$

<Q.7>: Reverse the order of integration and evaluate.

(a): $$\int_0^1 \int_2^{4-2x} dy\,dx$$

Sol$^n$:




R: $2 \le y \le 4-2x$

So, $y = 2$ — (i)

$y = 4-2x$ — (ii)

Solving (i) and (ii), $\therefore x = 1$

When $x=0,\ y=4$

when $x=1,\ y=2$

The horizontal line enters through $x=0$ and exist through $x = y-4/2$

So, equivalent integral.

$$\int_2^4 \int_0^{(y-4)/2} dx\,dy$$

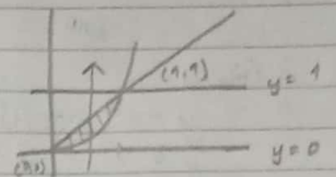
$$= \int_2^4 \frac{y-4}{2}\,dx \quad = \int_2^4 \frac{y}{2}\,dy - \int_2^4 2\,dy$$

$$= \left(y^2\right)_2^4 - \left(2y\right)_2^4$$

$$= (16-4) - (8-4) = 12-4 = 8$$

(b): $$\int_0^1 \int_y^{\sqrt{y}} dx\,dy$$

Sol$^n$:




R: $0 \le y \le 1$

$y \le x \le \sqrt{y}$

$x = y$ — (i)

$x = \sqrt{y}$ — (ii) Solving (i) and (ii),

when $y=0,\ x=0$

when $y=1,\ x=1.$

$y - \sqrt{y} = 0$

$\sqrt{y}(\sqrt{y}-1) = 0$ $\quad \therefore y = 0, 1.$

Also, arranging (i) and (ii), $x = y$ and $y = x^2$.

The vertical line enters through $y = x^2$ and exists through $y = x$

So,
$$\int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 (x - x^2)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 \quad \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(c): $\int_0^1 \int_1^{e^x} dy\,dx$

Soln:

Given,

$y = e^x$ — (i)

$y = 1$ — (ii)

when $x=0$, $y=1$

when $x=1$, $y=e$

Rearranging (i), $y = e^x$

$\therefore x = \ln y$

The integral is,

$$\int_1^e \int_0^{\ln y} dx\,dy$$

$$= \int_1^e \ln y \, dy$$

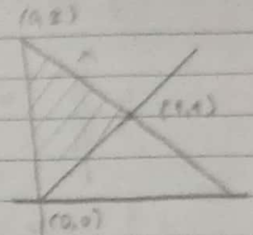
$$= \left[ y \ln y - y \right]_1^e$$

$$= (e - e) - (\ln 1 - 1)$$

$$= 1.$$

(8): Find the volume of the region that lies under the paraboloid $z = x^2+y^2$ and above the triangle enclosed by the line $y = x$ and $x = 0$ and $x+y = 2$ in the $xy$-plane.

Soln:




Here,
the range of integration,

$x \le y \le 2-x$

$0 \le x \le 1$

$$\text{the volume} = \int_0^1 \int_x^{2-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left( x^2 y + \frac{y^3}{3} \right)_x^{2-x} dx$$

$$= \int_0^1 \left[ x^2(2-x) + \frac{(2-x)^3}{3} \right] - \left[ x^3 + \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \left( 2x^2 - x^3 + \frac{8 - 12x + 6x^2 - x^3}{3} - x^3 - \frac{x^3}{3} \right) dx$$

$$= \int_0^1 \left( 2x^2 - 2x^3 + \frac{8 - 12x + 6x^2 - 2x^3}{3} \right) dx$$

$$= \int_0^1 \left( \frac{6x^2 - 6x^3 + 8 - 12x + 6x^2 - 2x^3}{3} \right) dx$$

$$= \int_0^1 \frac{12x^2 - 8x^3 - 12x + 8}{3}$$

$$= \frac{4}{3}\int_0^1 (3x^2 - 2x^3 - 3x + 2)\, dx$$

$$= \frac{4}{3}\left[ x^3 - \frac{x^4}{2} - \frac{3x^2}{2} + 2x \right]_0^1$$

$$= \frac{4}{3} \text{ cub units.}$$

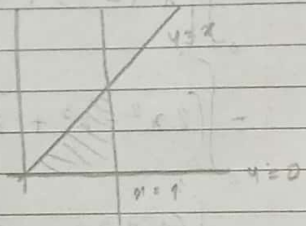
<Q.9>: Find the volume of the prism whose base is in the xy plane bounded by x-axis and the line $y = x$ and $x = 1$ whose top face is $z = f(x,y) = 3 - x - y$.

Soln:

Here,

R: $0 \le y \le x$

$0 \le x \le 1$




the ~~rate~~ volume

$$= \int_0^1 \int_0^x (3 - x - y)\, dy\, dx$$

$$= \int_0^1 \left(3y - xy - \frac{y^2}{2}\right)_0^x = \int_0^1 3x - x^2 - \frac{x^2}{2}$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \cancel{3x1} - \cancel{\frac{3x1^2}{2}} = ,$$

$$\left(\frac{3x^2}{2} - \frac{x^3}{2}\right)_0^1 = 1. \text{ cub units.}$$

<Q.10>: Change cartesian integrals into equivalent polar integrals and evaluate:

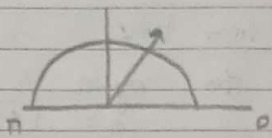
(a) $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} dy\, dx$

Soln:

Given,

R: $-1 \le x \le 1$ and

$0 \le y \le \sqrt{1-x^2}$




the region is bounded by x-axis and $y = \sqrt{1-x^2}$ or, $x^2 + y^2 = 1$ ie, $r = 1$.

$$\therefore \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} dy\, dx = \int_0^{\pi} \int_0^1 r\, dr\, d\theta$$

Evaluating,

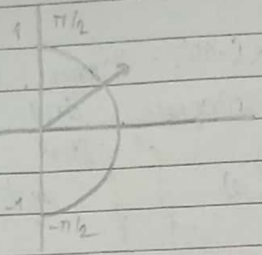
$$\int_0^{\pi} \left(\frac{r^2}{2}\right)_0^1 = \frac{\pi}{2}$$

(b): $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy$

Soln:

We know,

$x^2+y^2 = r^2$



R: $-1 \le y \le 1$

$0 \le x \le \sqrt{1-y^2}$

From here;

the region is bounded by y-axis and $x = \sqrt{1-y^2}$ on $x^2+y^2 = 1$

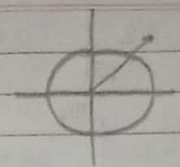
$\therefore r = 1.$

$\therefore \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2+y^2)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^2\, r\,dr\,d\theta$

$= \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4}\right)_0^1 d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{4}\,d\theta$

$= \frac{1}{4}\left[\theta\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{4}.$

(c): $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx.$



Soln.

Given,

R: $-a \le x \le a$ and $-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

The region is bounded within $y = -\sqrt{a^2-x^2}$ and $x^2+y^2 = a^2$ $\therefore r = a$ in all quadrant.

So $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = \int_{0}^{2\pi}\int_{0}^{a} r\,dr\,d\theta$

$= \int_{0}^{2\pi} \frac{a^2}{2}\,d\theta = \pi a^2$

(d): $\int_{0}^{6}\int_{2}^{y} x\,dx\,dy$

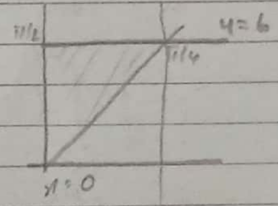
Soln.

We know,

R: $0 \le y \le 6$ R: $2 \le x \le y$

and

$x = r\cos\theta$



So,

$\int_{0}^{6}\int_{2}^{y} x\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_{0}^{6\operatorname{cosec}\theta} r^2\cos\theta\,dr\,d\theta$

$= \int_{\pi/4}^{\pi/2} \left(\frac{r^3}{3}\right)_0^{6\operatorname{cosec}\theta} \cos\theta\,d\theta$

$$= 72\int_{\pi/4}^{\pi/2} \operatorname{cosec}^3\theta \cos\theta \, d\theta$$

$$= 72\int_{\pi/4}^{\pi/2} \operatorname{cosec}^2\theta \cot\theta \, d\theta$$

Let $u = \cot\theta$ $\therefore du = -\operatorname{cosec}^2\theta \, d\theta$

when $\theta = \pi/4$, $u = 1$ and $\theta = \pi/2$, $u = 0$.

So,

$$= 72\int_{1}^{0} -u\,du = 72\left(\frac{u^2}{2}\right)_0^1 = 36$$

(iv): $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dy\,dx$

Soln,

$$= \int_{-\pi/2}^{\pi/2}\int_{0}^{1} \ln(r^2+1)\,r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_{1}^{2} \ln u\,du\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[u\ln u - u\right]_1^2 d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}(2\ln 2 - 2 - \ln 1 + 1)\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}(2\ln 2 - 1)\,d\theta = \int_{-\pi/2}^{\pi/2}\ln 2\,d\theta - \int_{-\pi/2}^{\pi/2}\frac{1}{2}\,d\theta = \pi\ln 2 - \frac{\pi}{2}$$

<Q.11>: Find the average value of $f(x,y) = x\cos xy$ over the rectangle $R: 0 \le x \le \pi$ and $0 \le y \le 1$.

Soln,

Given,

$f(x,y) = x\cos xy$

and

$R:$ $0 \le x \le \pi$ and $0 \le y \le 1$.

Now,

$$\iint_R dA = \int_0^{\pi}\int_0^1 dy\,dx = \int_0^{\pi} 1\cdot dx = \pi$$

and

$$\iint_R f(x,y)\,dA = \int_0^{\pi}\int_0^1 x\cos xy\;dy\,dx$$

$$= \int_0^{\pi}\left(\sin xy\right)_0^1 dx = \int_0^{\pi} \sin x\,dx$$

$$= \left|-\cos x\right|_0^{\pi} = 2$$

$\therefore$ Average value $= \dfrac{\iint_R f(x,y)\,dA}{\iint_R dA} = \dfrac{2}{\pi}$

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<Q.12>: Evaluate the integrals:

(i): $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\, dz\, dx\, dy$

Soln:

$= \int_0^1\int_0^1 \left(x^2z + y^2z + \frac{z^3}{3}\right)_0^1 dx\,dy$

$= \int_0^1\int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dx\,dy$

$= \int_0^1 \left(\frac{x^3}{3} + y^2x + \frac{x}{3}\right)_0^1 dy \quad = \int_0^1 \left(\frac{1}{3} + y^2 + \frac{1}{3}\right) dy$

$= 1.$

<b>: $\int_0^1\int_0^\pi\int_0^\pi (y \sin z)\, dx\, dy\, dz$

Soln,

$= \left(\int_0^1 1 \cdot dx\right)\left(\int_0^\pi y\, dy\right)\left(\int_0^\pi \sin z\, dz\right)$

$= 1 \times \frac{\pi^2}{2} \times (-\cos z)_0^\pi \quad = \pi^2$

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<Q.13>: Find the volume of the sphere $x^2+y^2 \cancel{=a^2} + z^2 = a^2$ by using triple integral

Soln.

~~Volume~~ $\int_0^a\int_0^a\int_0^a$

Volume of sphere in first octant

$= V = 8 \iiint dx\,dy\,dz.$

We know,

$z^2 = a^2 - x^2 - y^2$

$\therefore z = \pm\sqrt{a^2-x^2-y^2}$

$z = 0$

$y^2 = a^2 - x^2$

$\therefore y = \pm\sqrt{a^2-x^2}$

Thus,

z varies from $z=0$ to $z = \sqrt{a^2-x^2-y^2}$

y varies from $y=0$ to $y = \sqrt{a^2-x^2}$

x varies from $x=0$ to $x = a$.

Volume (V) $= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$

$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$

$= 8 \int_0^a \left(\frac{a^2-x^2}{2} \cdot \sin^{-1}\frac{y}{\sqrt{a^2-x^2}} + \frac{y}{2}\sqrt{a^2-x^2-y^2}\right)_0^{\sqrt{a^2-x^2}} dx$

$= 8 \int_0^a \left(\frac{a^2-x^2}{2} \sin^{-1}(1)\right) dx$

$$= 8\int_0^a \left(\frac{a^2-x^2}{2} \times \frac{\pi}{2}\right) dx$$

$$= 2\pi \int_0^a a^2 - x^2$$

$$= 2\pi \left[a^3 - a^3/3\right] = \frac{4\pi a^3}{3}$$

<Q.14>: Find the average value of $F(x,y,z)$ over the given region.

(a): $F(x,y,z) = x^2+9$ over the cube in the first octant bounded by $x=2, y=2, z=2$.

Soln.

Given,

$f(x,y,z) = x^2+9$

Now $\iiint_R dv = \int_0^2\int_0^2\int_0^2 (x^2+9)\, dz\,dy\,dx$

$$= \int_0^2\int_0^2 \left(x^2 z + 9z\right)_0^2 dy\,dx$$

$$= \int_0^2\int_0^2 (2x^2+18)\,dy\,dx$$

$$= \int_0^2 \left(2x^2 y + 18y\right)_0^2 dx$$

$$= \int_0^2 (4x^2+36)\,dx$$

$$= \left(\frac{4x^3}{3} + 36x\right)_0^2$$

$$= \frac{4\times 8}{3} + 36\times 2$$

$$= \frac{32}{3} + 72 = \frac{248}{3}$$

and $\int_0^2\int_0^2\int_0^2 dx\,dy\,dz = 8$

$\therefore$ Average value $= \frac{248}{3\times 8} = \frac{31}{3}$

<b>: $F(x,y,z) = x^2+y^2+z^2$ over the cube in the first octant bounded by planes $x=2, y=2, z=2$.

Soln:

$$\iiint_R dx\,dy\,dz = \int_0^2\int_0^2\int_0^2 dx\,dy\,dz$$

$$= 8$$

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$$= \int_0^2 \int_0^2 \int_0^2 (x^2+y^2+z^2)\, dx\, dy\, dz$$

$$= \int_0^2 \int_0^2 \left( \frac{x^3}{3} + y^2 x + z^2 x \right)_0^2 dy\, dz$$

$$= \int_0^2 \left( \frac{8}{3} + 2y^2 + 2z^2 \right) dy\, dz$$

$$= \int_0^2 \left( \frac{8}{3} y + \frac{2y^3}{3} + 2z^2 y \right)_0^2 dz$$

$$= \int_0^2 \left( \frac{16}{3} + \frac{16}{3} + 4z^2 \right) dz$$

$$= \int_0^2 \left( \frac{32}{3} + 4z^2 \right) dz = \left( \frac{32}{3} z + \frac{4z^3}{3} \right)_0^2$$

$$= \frac{64}{3} + \frac{32}{3} = \frac{96}{3}$$

$$\therefore \text{Average value} = \frac{\iiint_R f(x,y,z)\, dV}{\iiint_R dV}$$

$$= \frac{96}{3 \times 8} = 4$$

⟨16⟩: Evaluate the following

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{\rho^3}{3}\right)_0^{\frac{1-\cos\phi}{2}} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{1-\cos\phi}{2}\right)^3 \times \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{(1-\cos\phi)^3}{6} \sin\phi \, d\phi \, d\theta$$

$$\frac{d\cos\phi}{d\phi} = -\sin\phi \quad \text{or,} \quad d\cos\phi = -\sin\phi \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} (1-\cos\phi)^3 \, d\cos\phi \, d\theta$$

$$= \int_0^{2\pi} \left( -\frac{1}{24} \frac{(1-\cos\phi)^4}{4} \right)_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \frac{-1}{6} \, d\theta \quad = \frac{-\pi}{3}$$

⟨Q.no 18⟩ i) $\displaystyle\int_0^3\int_0^4\int_{x=y/2}^{x=y/2+1} \left(\frac{2x-y}{2}\right) + \frac{z}{3} \, dx\,dy\,dz$ by applying the transformation.

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}, \quad w = \frac{z}{3}$$

Sol$^n$:

Now,

$$u = \frac{2x-y}{2} \qquad y = 2v$$

$$\text{or,} \quad x = u+v \qquad z = 3w$$

Again,

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\text{or,} \ \frac{\partial x}{\partial u} = \frac{\partial(u+v)}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = \frac{\partial(2v)}{\partial u} = 0, \quad \frac{\partial z}{\partial u} = \frac{\partial(3w)}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = \frac{\partial(u+v)}{\partial v} = 1, \quad \frac{\partial y}{\partial v} = \frac{\partial(2v)}{\partial v} = 2, \quad \frac{\partial z}{\partial v} = \frac{\partial(3w)}{\partial v} = 0$$

$$\frac{\partial x}{\partial w} = \frac{\partial(u+v)}{\partial w} = 0, \quad \frac{\partial y}{\partial w} = \frac{\partial(2v)}{\partial w} = 0, \quad \frac{\partial z}{\partial w} = \frac{\partial(3w)}{\partial w} = 3$$

$$\text{So,} \ |J| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

Now

| $xyz$ eq$^n$ | $uvw$ eq$^n$ | simplied form |
|---|---|---|
| $x = y/2$ | $u+v = v$ | $u = 0$ |
| $x = y/2 + 1$ | $u+v = v+1$ | $u = 1$ |
| $y = 0$ | $2v = 0$ | $v = 0$ |
| $y = 4$ | $2v = 4$ | $v = 2$ |
| $z = 0$ | $3w = 0$ | $w = 0$ |
| $z = 3$ | $3w = 3$ | $w = 1$ |

So

$$\int_0^3\int_0^4\int_{y/2}^{y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz = 6\int_0^1\int_0^2\int_0^1 (u+w)\,du\,dv\,dw$$

$$= 6\int_0^1\int_0^2\left(\frac{u^2}{2}+uw\right)_0^1 dv\,dw$$

$$= 6\int_0^1\int_0^2\left(\frac{1}{2}+w\right)dv\,dw$$

$$= 6\int_0^1\left(\frac{v}{2}+vw\right)_0^2 dw$$

$$= 6\left(w+w^2\right)_0^1 = 12$$

<Q.no-17>: Define the Jacobian determinant or Jacobian of the coordinate transformation $x = g(u,v)$, $y = h(u,v)$.
Find the Jacobian $\partial(x,y)/\partial(u,v)$ for transformation

(a): $x = u\cos v$, $y = u\sin v$

Sol$^n$:

Jacobian of the transformation $x = g(u,v)$ $y = h(u,v)$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = \frac{\partial u\cos v}{\partial u} = \cos v \qquad \frac{\partial x}{\partial v} = \frac{\partial u\cos v}{\partial v} = -u\sin v$$

$$\frac{\partial y}{\partial u} = \frac{\partial u\sin v}{\partial u} = \sin v \qquad \frac{\partial y}{\partial v} = \frac{\partial u\sin v}{\partial v} = u\cos v$$

$$|J| = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix}$$

$$= \cos v \times u\cos v + u\sin^2 v$$

$$= u$$

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(b): ~~$x = 2u-1$, $y = 2$~~ $x = u\sin v$, $y = u\cos v$

Soln:

$$|J| = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

$\frac{\partial x}{\partial u} = \frac{\partial\, u\sin v}{\partial u} = \sin v$ $\qquad \frac{\partial x}{\partial v} = \frac{\partial\, u\sin v}{\partial v} = u\cos v$

$\frac{\partial y}{\partial u} = \frac{\partial\, u\cos v}{\partial u} = \cos v$ $\qquad \frac{\partial y}{\partial v} = \frac{\partial\, u\cos v}{\partial v} = -u\sin v$

$$|J| = \begin{vmatrix} \sin v & \cos v \\ u\cos v & -u\sin v \end{vmatrix}$$

$$= -u\sin^2 v + u\cos^2 v$$
$$= u(\cos^2 v - \sin^2 v)$$
$$= 2u\cos 2v$$

<Q.no 17> (ii).

(a): $x = u\cos v$, $y = u\sin v$, $z = w$.

Soln:

$$|J| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

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$\frac{\partial x}{\partial u} = \frac{\partial u\cos v}{\partial u} = \cos v$ $\qquad \frac{\partial x}{\partial v} = \frac{\partial u\cos v}{\partial v} = -u\sin v$ $\qquad \frac{\partial x}{\partial w} = \frac{\partial u\cos v}{\partial w} = 0$

$\frac{\partial y}{\partial u} = \frac{\partial u\sin v}{\partial u} = \sin v$ $\qquad \frac{\partial y}{\partial v} = \frac{\partial u\sin v}{\partial v} = u\cos v$ $\qquad \frac{\partial y}{\partial w} = \frac{\partial u\sin v}{\partial w} = 0$

$\frac{\partial z}{\partial u} = \frac{\partial w}{\partial u} = 0$ $\qquad \frac{\partial w}{\partial v} = 0$ $\qquad \frac{\partial w}{\partial w} = 1$

So,
$$|J| = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v$$
$$= u$$

<Q.no. 18> (iii) Soln:

$x = 2u-1$, $\quad y = 3v-4$, $\quad z = 1/2(w-4)$

$$|J| = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

$\frac{\partial x}{\partial u} = \frac{\partial(2u-1)}{\partial u} = 2$ $\qquad \frac{\partial y}{\partial u} = \frac{\partial(3v-4)}{\partial u} = 0$ $\qquad \frac{\partial z}{\partial u} = \frac{1}{2}(w-4) = 0$

$\frac{\partial x}{\partial v} = \frac{\partial(2u-1)}{\partial v} = 0$ $\qquad \frac{\partial y}{\partial v} = \frac{\partial(3v-4)}{\partial v} = 3$ $\qquad \frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\frac{1}{2}(w-4) = 0$

$\frac{\partial x}{\partial w} = \frac{\partial(2u-1)}{\partial w} = 0$ $\qquad \frac{\partial y}{\partial w} = \frac{\partial(3v-4)}{\partial w} = 0$ $\qquad \frac{\partial z}{\partial w} = \frac{\partial}{\partial w}\frac{1}{2}(w-4) = \frac{1}{2}$

$$|J| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{vmatrix} = 3$$