Chapter : 2: Mechanics:

ROTATIONAL DYNAMICS

Rigid body:

A rigid body is defined as a solid body in which the particle are completely arranged so that the inter-particle distance is small and fixed and their positions are not disturbed by any external force applied on it.

A rigid body undergoes both translational and rotational motion.

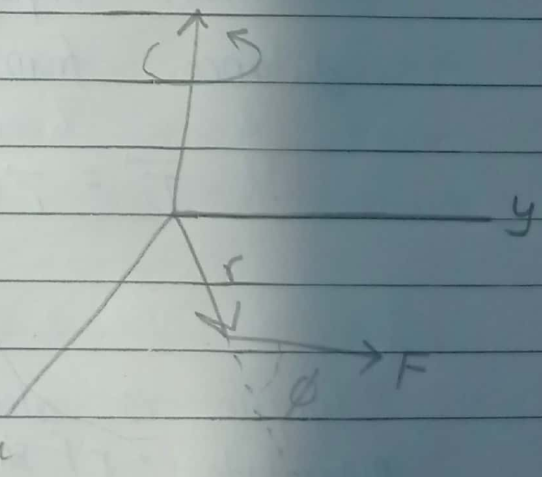
Torque and Angular Momentum:

Consider a particle of mass 'm' and linear velocity $\vec{v}$ rotating ~~to~~ about fixed point O.

The particle is located at position vector $\vec{r}$ relative to its axis of rotation. If the particle's linear momentum is $\vec{p}$, then angular momentum $\vec{L}$ of particle with respect to point O is defined as.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

$$\therefore \vec{L} = rp \sin\phi \hat{n} \quad -(i).$$



Here, angular vector is vector quantity
magnitude = $rp \sin\phi \, \hat{n}$
direction = $\perp^r$ to plane of $\vec{r}$ and $\vec{p}$ and specified by right hand rule.

If $\phi = 90°$, $L = pr$
$\Rightarrow L$ = linear momentum $\times$ $\perp^r$ distance from axis.
= moment of linear momentum.

If $\vec{F}$ force is applied to particle,

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$$

or, $$\vec{F} = \frac{d\vec{p}}{dt}$$

Taking cross-product with $\vec{r}$ on both sides,

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$$

We know, torque is the product of force and the distance from the axis of rotation,

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} \quad -(ii)$$

Torque is also a vector quantity.
magnitude = $rF \sin\phi$
direction = $\perp^r$ to plane of $\vec{r}$ and $\vec{F}$
ϕ = angle betn $\vec{F}$ and $\vec{r}$.

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Diff. (i) w.r.t time, we get.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times p + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$$

We know, $\vec{v} \times \vec{v} = 0$.

so,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \tau.$$

Hence, the rate of change of angular momentum is equal to the torque.

This is analogy for Newton's law ~~of~~ for translational motion.

Torque and Angular Momentum of System of Particles

Consider the system of n-particles of point masses $m_1, m_2, \ldots m_n$ with position vectors $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}$ respectively and angular momenta $L_1 = \vec{r_1} \times \vec{p_1}$, $L_2 = \vec{r_2} \times \vec{p_2}$ and so on.

The total angular momentum of system is equal to sum of all angular momenta of particles of the system ie, Date. No.

$$\vec{L} = \vec{L_1} + \vec{L_2} + \dots + \vec{L_n} = \sum_{i=1}^{n} L_i \quad \text{— (i)}$$

Differentiating eq^n(i) w.r.t. t,

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L_1}}{dt} + \frac{d\vec{L_2}}{dt} + \dots + \frac{d\vec{L_n}}{dt}$$

$$\text{or, } \frac{d\vec{L}}{dt} = \vec{\tau_1} + \vec{\tau_2} + \dots + \vec{\tau_n} \quad \text{— (ii).}$$

We know, two types of ~~extee~~ torques acts on the system i.e, due to internal forces and due to external forces. So, eq^n(ii) becomes.

$$\frac{d\vec{L}}{dt} = (\vec{\tau}_{1int} + \vec{\tau}_{2int} + \dots + \vec{\tau_n}) + (\vec{\tau}_{1ext} + \vec{\tau}_{2ext} \dots + \vec{\tau}_{n\,ext}) \quad \text{— (iii)}$$

All the internal forces are in pair, equal in magnitude and opposite direction. So, sum of ~~all~~ torque due to internal forces is zero.
So, eq^n(iii) becomes.

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{1ext} + \vec{\tau}_{2ext} + \dots + \vec{\tau}_{n\,ext} = \vec{\tau}_{ext} \quad \text{— (iv).}$$

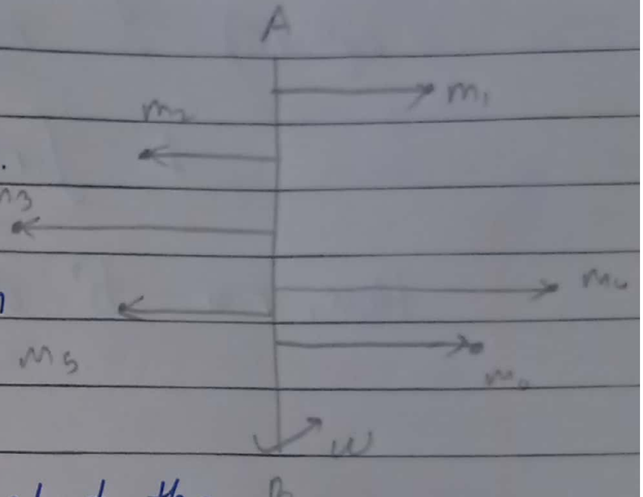
$\vec{\tau}_{ext}$ = resultant of all ext. torques acting on the particles.

If $\vec{\tau}_{ext} = 0$, then, $\frac{d\vec{L}}{dt} = 0$ ie, $\vec{L}$ = constant.

If resultant of all torques is zero, the total angular momentum remains conserved although the individual particle may experience external torques.

This is principle of conservation of total angular momentum.

Rotational K.E. and Moment of inertia (Rotational inertia)



Consider a system of point mass particles $m_1, m_2, \ldots, m_n$ and their $\perp^r$ distance from axis of rotation AB is $r_1, r_2, \ldots, r_n$.

Consider all particles rotate about the axis with same angular velocity ω.

So, velocities, $v_1 = \omega r_1$, $v_2 = \omega r_2$, ..., $v_n = \omega r_n$.

Hence, total kinetic energy of system of rotating particles is.

$$K_{rot} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots + \frac{1}{2} m_n v_n^2$$

$$= \frac{1}{2} m_1 \omega^2 r_1^2 + \frac{1}{2} m_2 \omega^2 r_2^2 + \ldots + \frac{1}{2} m_n \omega^2 r_n^2$$

$$= \frac{1}{2} \left(\sum_{i=1}^{n} m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2$$

$\therefore K_{rot} = \frac{1}{2} I \omega^2$ — (i).

and here,
$$I = \sum_{i=1}^{n} m_i r_i^2 \quad \text{— (ii).}$$

Eqn (ii) is moment of inertia or rotational inertia of system of particle.

For a rigid body system, the m_i of eqn (ii) is replaced by dm. So,

$$I = \int r^2 dm \quad \text{— (iii).}$$

Here, r = ⊥r distance of small mass dm from axis of rotation.

Radius of Gyration

The distance from the axis of rotation to the point where total mass of the body is supposed to be concentrated ~~it~~ such that the moment of inertia about the axis remains same.

So, moment of inertia based on radius of gyration of mass M,

$$I = MK^2 \Rightarrow K = \sqrt{\frac{I}{M}} \quad \text{— (i).}$$

The radius of gyration can also be defined as a distance whose squared value multiplied with the total mass of system / body gives the moments of inertia about the given axis.

It depends upon,

- Shape of body
- size of body
- axis of rotation.

⊛ Physical significance of M.o.I

Moment of inertia plays the same role in rotational motion as mass does in translation motion. This is physical significance of moment of inertia.