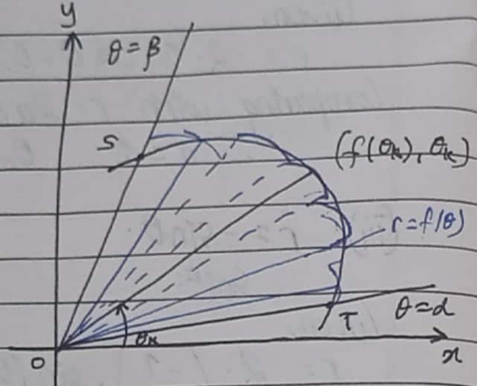
Area Bounded by Polar Curves



Here,
the region TOS is bounded by rays $\theta = \alpha$ and $\theta = \beta$.

the curve $r = f(\theta)$

Let us divide the curve into n numbers of non-overlapping fan-shaped circular sectors.

Let us consider a typical sector with radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$.

$$A_K = \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k.$$

Then, the area of the region is approximately,

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k.$$

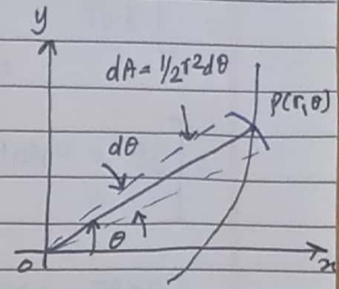
If there are continuously infinitesimally small sectors ie, $n \to \infty$

Then,

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$

$$= \int_{\alpha}^{\beta} \frac{1}{2} \cancel{f(\theta_k)} \left(f(\theta)\right)^2 d\theta$$

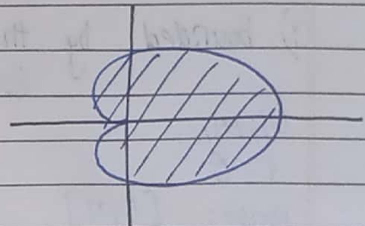
$$\therefore A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$



<Q>: Find the area of the region in the plane enclosed by the cardioid $r = 2(1+\cos\theta)$

Soln:

Given,
$r = 2(1+\cos\theta)$ gives cardioid.



The graph runs from 0 to 2π.
So, area of the region.

$$A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta \qquad \left(\because A = 2\int_0^{\pi} \tfrac{1}{2} r^2 d\theta\right)$$

$$= \frac{1}{2}\int_0^{2\pi} 4(1+\cos\theta)^2 d\theta$$

$$= \int_0^{2\pi} (2 + 4\cos\theta + 2\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (2 + 4\cos\theta + 1 + \cos 2\theta)\, d\theta = \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$

$$= \int_0^{2\pi} 3\cdot d\theta + \int_0^{2\pi} 4\cos\theta \, d\theta + \int_0^{2\pi} \cos 2\theta \, d\theta$$

$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

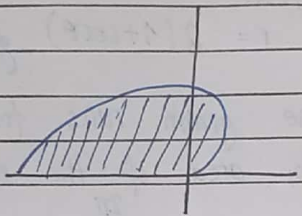
$$= (3\times 2\pi - 3\times 0) + 0 + 0 \qquad = 6\pi \text{ sq}\cdot\text{units.}$$

<Q.2>: find the area of the regions.

i) bounded by the spiral $r=\theta$ for $0\le\theta\le\pi$.

Soln:

Given,
$r=\theta$
range: $[0,\pi]$



Now,

$$\text{Area}(A) = \frac{1}{2}\int_0^{\pi} r^2 d\theta \qquad = \frac{1}{2}\int_0^{\pi} \theta^2 d\theta$$

$$= \frac{1}{2}\left[\frac{\theta^3}{3}\right]_0^{\pi}$$

$$= \frac{1}{6}\left[\pi^3 - 0^3\right] \qquad = \frac{\pi^3}{6} \text{ sq. units.}$$

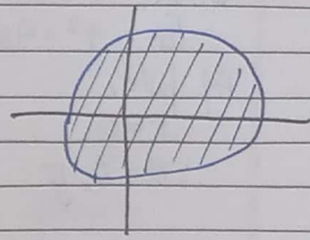
(ii) Inside the oval limacon, $r = 4+2\cos\theta$

Soln:

Given,
$r = 4+2\cos\theta$

The oval limacon graphs region from 0 to 2π.

Now,



$$\text{Area}(A) = \frac{1}{2}\int_0^{2\pi} r^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (4+2\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4(2+\cos\theta)^2 d\theta$$

$$= 2\int_0^{2\pi} (4+4\cos\theta+\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (8+8\cos\theta+2\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} 8\cdot d\theta + \int_0^{2\pi} 8\cos\theta\, d\theta + \int_0^{2\pi} 2\cos^2\theta \, d\theta$$

$$= 8\theta\Big|_0^{2\pi} + 8\sin\theta\Big|_0^{2\pi} + \int_0^{2\pi} 1\, d\theta + \int_0^{2\pi} \cos 2\theta\, d\theta$$

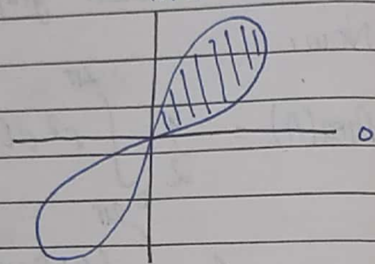
$$= 8\theta\Big|_0^{2\pi} + 8\sin\theta\Big|_0^{2\pi} + \theta\Big|_0^{2\pi} + \frac{\sin 2\theta}{2}\Big|_0^{2\pi}$$

Date. No.

$= (16\pi - 0) + 0 + (2\pi - 0) + 0 = 18$ sq. units.

(iii) Inside one loop of lemniscate $r^2 = 4\sin 2\theta$.

Sol^n.



Given,

For $r^2 = 4\sin 2\theta$,

At pole,

$r = 0$

$\therefore \theta = 0, \frac{\pi}{2}$

Now,

$$\text{Area (A)} = \frac{1}{2}\int_0^{\pi/2} 4\sin 2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= -\cancel{2}\left[\frac{\cos 2\theta}{\cancel{2}}\right]_0^{\pi/2}$$

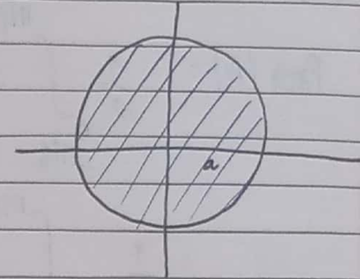
$$= -\cos 2\times\pi/2 - \cos 2\times 0$$

$= 2$ sq. units.

Date. No.

(iv): Inside circle $r = a$.

Sol^n:



Given,

$r = a$

Range: $[0, 2\pi]$.

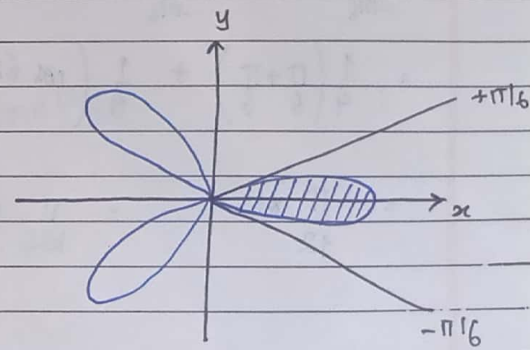
Now,

$$\text{Area (A)} = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} 1\cdot d\theta = \pi a^2 \text{ sq. units.}$$

(v) Inside one petal of $r = \cos 3\theta$

Sol^n:



Given,

$r = \cos 3\theta$

At pole, $r = 0$

$0 = \cos 3\theta$

or $3\theta = \pm\pi/2, \pm 3\pi/2$

$\therefore \theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}$

Range $[-\pi/6, \pi/6]$

Now,

$$\text{Area }(A) = \frac{1}{2}\int_{-\pi/6}^{\pi/6} r^2\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\,d\theta$$

$$= \frac{1}{4}\int_{-\pi/6}^{\pi/6} 2\cos^2 3\theta\,d\theta$$

$$= \frac{1}{4}\int_{-\pi/6}^{\pi/6} (1+\cos 6\theta)\,d\theta$$

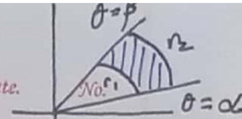
$$= \int_{-\pi/6}^{\pi/6} \frac{1}{4}\,d\theta + \int_{-\pi/6}^{\pi/6} \frac{1}{4}\cos 6\theta\,d\theta$$

$$= \frac{1}{4}\left(\frac{\pi}{6}+\frac{\pi}{6}\right) + \frac{1}{4}\left(\cos 6\times\frac{\pi}{6} - \cos 6\times\left(-\frac{\pi}{6}\right)\right)$$

$$= \frac{\pi}{12} + 0 \quad = \frac{\pi}{12} \text{ sq.units.}$$

Area betn two curves:

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right)d\theta$$



<Q.3>: Find the area shared by.

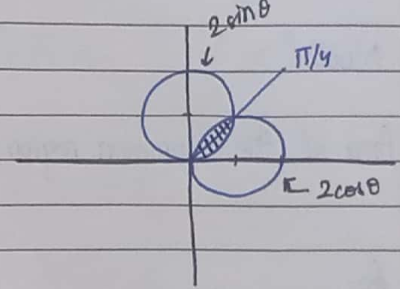
(i): $r = 2\cos\theta$ and $r = 2\sin\theta$.

Soln:

Given,

$r_1 = 2\cos\theta$

$r_2 = 2\sin\theta$



Now,

$2\cos\theta = 2\sin\theta$

or, $\sin(90^\circ - \theta) = \sin\theta$

$\therefore \theta = \pi/4$.

The common area is symmetrical above and below $\pi/4$.

Now,

$$\text{Area }(A) = \frac{1}{2}\times 2\int_0^{\pi/4} (2\sin\theta)^2\,d\theta$$

$$= 2\int_0^{\pi/4} 2\sin^2\theta\,d\theta$$

$$= 2\int_0^{\pi/4} (1-\cos 2\theta)\,d\theta$$

$$= \int_0^{\pi/4} 2\,d\theta - \int_0^{\pi/4} 2\cos 2\theta \quad = \left(2\cdot\frac{\pi}{4} - 2\cdot 0\right) - \left(\sin 2\times\pi/4 - \sin 2\times 0\right)$$

$$= \frac{\pi}{2} - 0 - 1 - 0 \quad = \pi/2 - 1 \text{ sq units.}$$

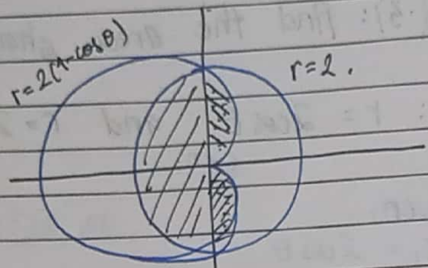
(ii): $r = 2$ and $r = 2(1-\cos\theta)$

Sol^n:

Given,

$r = 2$

$r = 2(1-\cos\theta)$



Now

Area of the common region = Area of semicircle in negative x-axis + Area of the dotted region. —(i)

So,

Area of dotted region $(A_1) = \frac{1}{2} \times 2 \int_0^{\pi/2} [2(1-\cos\theta)]^2 \, d\theta$

$$= 4 \int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta$$

$$= 4 \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= 2 \int_0^{\pi/2} (2 - 4\cos\theta + 2\cos^2\theta) \, d\theta$$

$$= 2 \int_0^{\pi/2} (3 - 4\cos\theta + \cos 2\theta) \, d\theta$$

$$= 2 \left[\int_0^{\pi/2} 3 \cdot d\theta - \int_0^{\pi/2} 4\cos\theta \, d\theta + \int_0^{\pi/2} \cos 2\theta \, d\theta \right]$$

$$= 2 \left[3\theta \Big|_0^{\pi/2} - 4\sin\theta \Big|_0^{\pi/2} + \frac{\sin 2\theta}{2} \Big|_0^{\pi/2} \right]$$

$$= 6\theta \Big|_0^{\pi/2} - 8\sin\theta \Big|_0^{\pi/2} + \sin 2\theta \Big|_0^{\pi/2}$$

$$= (3\pi - 0) - (8\sin\pi/2 - 8\sin 0) + 0$$

$$= 3\pi - 8 \text{ sq. units.}$$

And.

area of the common semicircle $(A_2) = \frac{1}{2} \int_{-\pi/2}^{\pi/2} r^2 \, d\theta$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} 4 \, d\theta$$

$$= 2 \int_{-\pi/2}^{\pi/2} d\theta = 2 \, [\theta]_{-\pi/2}^{\pi/2}$$

$$= 2 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 2\pi$$

$\therefore$ $A = A_1 + A_2 = 5\pi - 8$ sq. units

<Q.4>: Find the area inside of $r = 4\cos\theta$ and to the right of the vertical line $r = \sec\theta$.

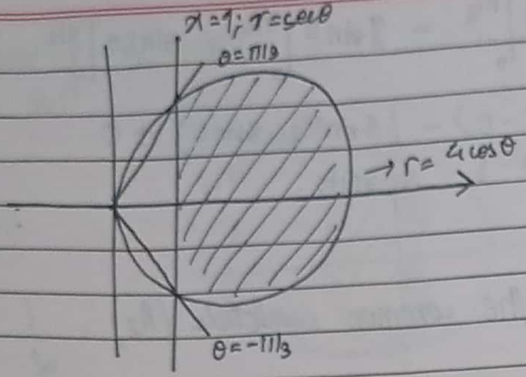
Sol^n:

Given,

$r = 4\cos\theta$

$r = \sec\theta$ or $r\cos\theta = 1$ $\therefore x = 1$.

Date. No.



Now solving the given equations;

$r = 4\cos\theta \qquad r = \sec\theta$

or, $4\cos\theta = \sec\theta$

or, $4\cos^2\theta = 1$

or $\cos^2\theta = \frac{1}{4}$ $\qquad$ or, $\cos\theta = \frac{1}{2}$

$\therefore \theta = \pm \pi/3$

~~The~~ The area of shaded region (A) = $\cancel{\frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos\theta)^2 d\theta}$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[(4\cos\theta)^2 - (\sec\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[16\cos^2\theta - \frac{1}{\cos^2\theta}\right] d\theta$$

Date. No.

Since the curve is symmetrical with x-axis.

$$A = 2 \times \frac{1}{2}\int_0^{\pi/3}\left[16\cos^2\theta - \sec^2\theta\right] d\theta$$

$$= \int_0^{\pi/3}\left[16\cos^2\theta - \sec^2\theta\right] d\theta$$

$$= \int_0^{\pi/3} 16\cos^2\theta\, d\theta - \int_0^{\pi/3}\sec^2\theta\, d\theta$$

$$= 8\int_0^{\pi/3} 2\cos^2\theta\, d\theta - \int_0^{\pi/3}\sec^2\theta\, d\theta$$

$$= 8\int_0^{\pi/3}(1+\cos 2\theta)\, d\theta - \int_0^{\pi/3}\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/3} 8\, d\theta + \int_0^{\pi/3} 8\cos 2\theta\, d\theta - \int_0^{\pi/3}\sec^2\theta\, d\theta$$

$$= 8\cdot[\theta]_0^{\pi/3} + \frac{8}{2}\left[\sin 2\theta\right]_0^{\pi/3} - [\tan\theta]_0^{\pi/3}$$

$$= 8\cdot\frac{\pi}{3} + 4\cdot\left(\sin 2\times\pi/3 - \sin 2\cdot 0/3\right) - \tan\pi/3$$

$$= \frac{8\pi}{3} - \sqrt{3} + 2\sqrt{3}$$

$$\therefore A = \frac{8\pi}{3} + \sqrt{3} \text{ sq. units.}$$

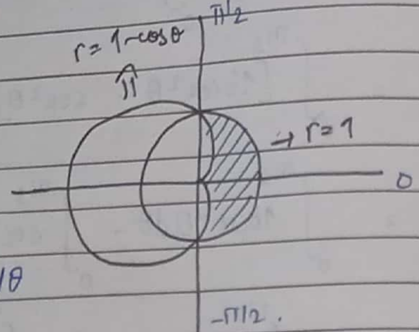
⟨Q.5⟩: Find the area of the region inside the circle $r = 1$ outside the cardioid $r = 1 - \cos\theta$.

Soln:

Given,

$r = 1$

$r = 1 - \cos\theta$



Here,

$$\text{Area}(A) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (r_2^2 - r_1^2)\, d\theta$$

Since the region is symmetrical with x-axis,

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} [1^2 - (1-\cos\theta)^2]\, d\theta$$

$$= \int_0^{\pi/2} [\cancel{1} - \cancel{1} + 2\cos\theta - \cos^2\theta]\, d\theta$$

$$= \int_0^{\pi/2} 2\cos\theta\, d\theta - \int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2[\sin\theta]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} 2\cos^2\theta\, d\theta$$

$$= 2\left[\sin\frac{\pi}{2} - \sin 0\right] - \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta$$

$$= 2 - \frac{1}{2}\left[\int_0^{\pi/2} 1\, d\theta - \int_0^{\pi/2} \cos 2\theta\, d\theta\right]$$

$$= 2 - \frac{1}{2}\left[\theta\Big|_0^{\pi/2} - \frac{\cancel{\cos}\ \sin 2\theta}{2}\Big|_0^{\pi/2}\right]$$

$$= 2 - \frac{1}{2}\left[\frac{\pi}{2} - 0 - \left(\frac{\sin 2\times\pi/2}{2} - \frac{\sin 2\times 0}{2}\right)\right]$$

$$= 2 - \frac{1}{2}\left[\frac{\pi}{2}\right]$$

$$= 2 - \frac{\pi}{4} \text{ sq. units.}$$