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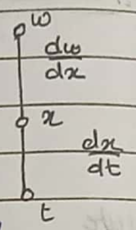
Chain Rule:

(a) for single variable:

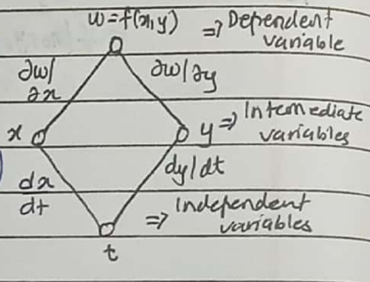
when $w=f(x)$ is differentiable function of x and $x=g(t)$ is a differential function of t, w is differentiable function of t and



$$\frac{dw}{dt} = \frac{dw}{dx} \times \frac{dx}{dt}$$

(b) Two Independent Variables:

If $w=f(x,y)$ is differentiable and $x=x(t)$, $y=y(t)$ are differentiable functions of t, then the composite $w=f(x(t),y(t))$ is differentiable function of t.



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt}$$

⟨Q⟩: Use chain rule to find the derivative of $w=xy$ with respect to t along the path $x=\cos t$; $y=\sin t$. What is derivative's value at $t=\pi/2$?

Soln:

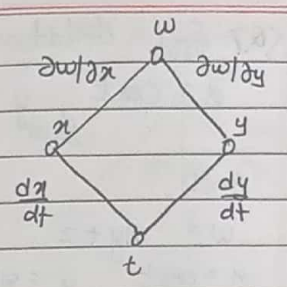
Given,

$w=xy$.

Date. No.

$x=\cos t$

$y=\sin t$



Now,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt}$$

$$= \frac{\partial xy}{\partial x} \times \frac{d\cos t}{dt} + \frac{\partial xy}{\partial y} \times \frac{d\sin t}{dt}$$

$$= -y\sin t + x\cos t$$

$$= -\sin^2 t + \cos^2 t$$

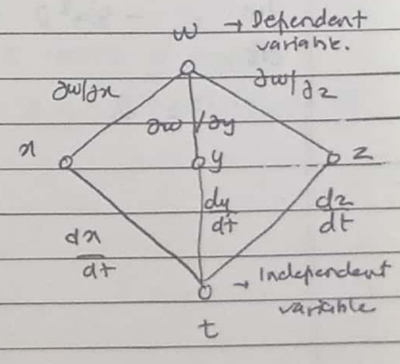
At $t=\pi/2$,

$$= -\sin^2 \pi/2 + \cos^2 \pi/2$$

$$\therefore \frac{dw}{dt} = -1.$$

(c): One independent, three intermediates variables.

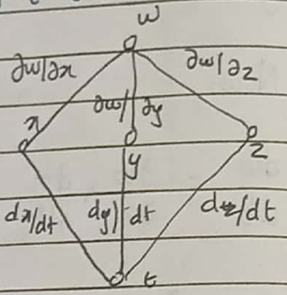
Suppose that $w=f(x,y,z)$ and x,y,z are differentiable functions of t and



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt} + \frac{\partial w}{\partial z} \times \frac{dz}{dt}$$

Q. find dw/dt if $w = xy + z$,
$x = \cos t$, $y = \sin t$, $z = t$. at $t = 0$

Sol^n:



Given,
$w = xy + z$
$x = \cos t$ $y = \sin t$ $z = t$

Now

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt} + \frac{\partial w}{\partial z} \times \frac{dz}{dt}$$

$$= \frac{\partial(xy+z)}{\partial x} \times \frac{d\cos t}{dt} + \frac{\partial(xy+z)}{\partial y} \times \frac{d\sin t}{dt} + \frac{\partial(xy+z)}{\partial z} \times \frac{dt}{dt}$$

$$= -y\sin t + x\cos t + (1 \times 1)$$

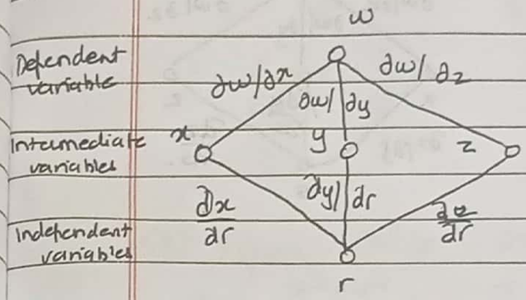
$$= -\sin^2 t + \cos^2 t + 1$$

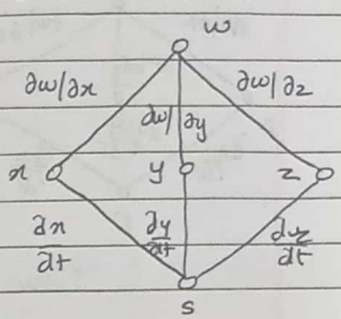
At $t = 0$,

$$\frac{dw}{dt} = -\sin^2 0 + \cos^2 0 + 1$$

$$= 2$$

(d) Two independent variables, three intermediate variables.





Suppose that $w = f(x,y,z)$ and $x = g(r,s)$, $y = h(r,s)$ and $z = k(r,s)$.
If all four functions are differentiable, then w has partial derivatives with respect to r and s given by formulae.

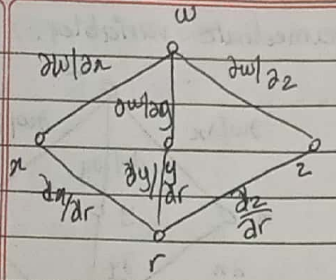
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial r}$$

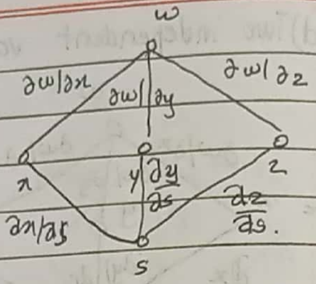
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial s}$$

Q. Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s.
$w = x^2 + 2y + z^2$, $x = r/s$, $y = r^2 + \ln s$, $z = 2r$.

Sol^n.

Given,
$w = x^2 + 2y + z^2$
$x = r/s$
$y = r^2 + \ln s$
$z = 2r$





Now,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial r} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial r}$$

$$= \frac{\partial(x+2y+z^2)}{\partial x} \times \frac{\partial(r/s)}{\partial r} + \frac{\partial(x+2y+z^2)}{\partial y} \times \frac{\partial(r^2+\ln s)}{\partial r} + \frac{\partial(x+2y+z^2)}{\partial z} \times \frac{\partial(2r)}{\partial r}$$

$$= \left(1 \times \frac{1}{s}\right) + \left(2 \times 2r\right) + \left(2z \times 2\right)$$

$$\therefore \frac{\partial w}{\partial r} = \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$$

Again,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial s}$$

$$= \frac{\partial(x+2y+z^2)}{\partial x} \times \frac{\partial(r/s)}{\partial s} + \frac{\partial(x+2y+z^2)}{\partial y} \times \frac{\partial(r^2+\ln s)}{\partial s} + \frac{\partial(x+2y+z^2)}{\partial z} \times \frac{\partial(2r)}{\partial s}$$

$$= 1 \cdot r \cdot \frac{-1}{s^2} + 2 \times \frac{1}{s} + 2z \times 0$$

$$= \frac{-r}{s^2} + \frac{2}{s} \qquad \therefore \frac{\partial w}{\partial s} = \frac{2}{s} - \frac{r}{s^2}$$

<Q>: Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if $w = x^2+y^2$, $x = r-s$, $y = r+s$.

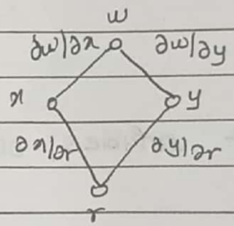
Soln:

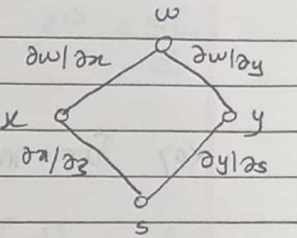
Given,

$w = x^2+y^2$

$x = r-s$

$y = r+s$.





Now,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial r}$$

$$= \frac{\partial(x^2+y^2)}{\partial x} \times \frac{\partial(r-s)}{\partial r} + \frac{\partial(x^2+y^2)}{\partial y} \times \frac{\partial(r+s)}{\partial r}$$

$$= 2x \times 1 + 2y \times 1$$

$$= 2r - \cancel{2s} + 2r + \cancel{2s}$$

$$\therefore \frac{\partial w}{\partial r} = 4r$$

Again,

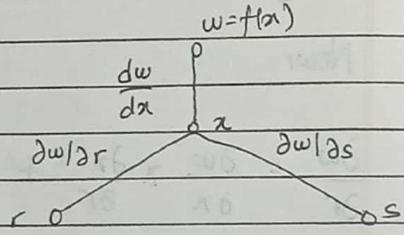
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial s}$$

$$= \frac{\partial (x^2+y^2)}{\partial x} \times \frac{\partial (r-s)}{\partial s} + \frac{\partial (x^2+y^2)}{\partial y} \times \frac{\partial (r+s)}{\partial s}$$

$$= 2x \times (-1) + 2y \times 1$$
$$= 2y - 2x$$
$$= 2(r+s) - 2(r+s)$$

$$\therefore \frac{\partial w}{\partial s} = 4s$$

<e> Two independent variables, one intermediate variables.

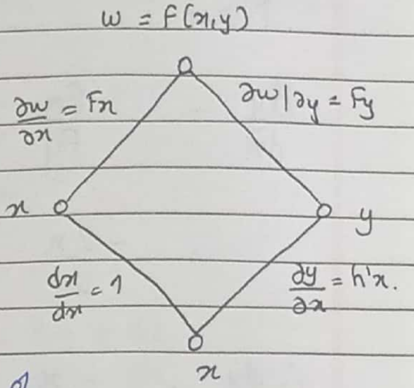


If $w = f(x)$ and
$x = g(r,s)$
then,

$$\frac{\partial w}{\partial r} = \frac{dw}{dx} \times \frac{\partial w}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx} \times \frac{\partial w}{\partial s}$$

<f>: Implicit Differentiation.



Suppose that

i) the function $F(x,y)$ is differentiable

(ii) The equation $F(x,y) = 0$ defines y implicitly as function of x, say $y = h(x)$.

Then, $\frac{dy}{dx} = -\frac{F_x}{F_y}$

$$0 = \frac{dw}{dx}$$
$$= F_x \times 1 + F_y \times \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{F_x}{F_y}$$

<Q>: Find dy/dx if $x^2+y^2 = a^2$.

Soln:

Now,

$$F_x = \frac{\partial x^2}{\partial x} \cancel{\times \frac{\partial x^2}{\partial x}} + \frac{\partial y^2}{\partial x} - \frac{\partial a^2}{\partial x}$$
$$= 2x$$

$$F_y = \frac{\partial x^2}{\partial y} + \frac{\partial y^2}{\partial y} - \frac{\partial a^2}{\partial y} = 2y$$

Now

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

$$= \frac{-2x}{2y}$$

$$\therefore \frac{dy}{dx} = \frac{-x}{y}$$

<Q>: Find the following derivatives:

(i): $\frac{dw}{dt}$ if $w = 2ye^x - \ln z$

$x = \ln(t^2+1)$

$y = \tan^{-1}(t)$ $\quad z = t$. $\quad$ at $t = 1$.

Soln:

Given,

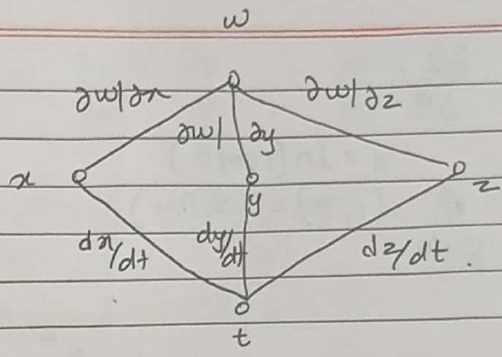
$w = 2ye^x - \ln z$

$x = \ln(t^2+1)$

$y = \tan^{-1}(t)$

$z = t$

Now,



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt} + \frac{\partial w}{\partial z} \times \frac{dz}{dt}$$

$$= \frac{\partial(2ye^x - \ln z)}{\partial x} \times \frac{d(\ln(t^2+1))}{d(t^2+1)} \times \frac{d(t^2+1)}{dt} + \frac{\partial(2ye^x - \ln z)}{\partial y} \times \frac{d\tan^{-1}t}{dt} + \frac{\partial(2ye^x - \ln z)}{\partial z} \times \frac{\cancel{\partial t}\, dt}{\partial\, dt}$$

$$= \left(2ye^x \times \frac{1}{1+t^2} \times 2t\right) + \left(2e^x \times \frac{1}{1+t^2}\right) + \left(\frac{-1}{z}\right)$$

$$= \frac{2e^x(2yt+1)}{1+t^2} - \frac{1}{z}$$

Replacing values of x, y, z.

$$= \frac{2e^{\ln(t^2+1)}\{2\times\tan^{-1}(t)\times t + 1\}}{(1+t^2)} - \frac{1}{t}$$

$$= \frac{2\cancel{(t^2+1)}(2t\cdot\tan^{-1}(t)+1)}{\cancel{(t^2+1)}} - \frac{1}{t}$$

At $t = 1$,

$$= 4\tan^{-1}(1) + 2 - 1$$

$$= \cancel{4} \times \frac{\pi}{\cancel{4}} + 2 - 1$$

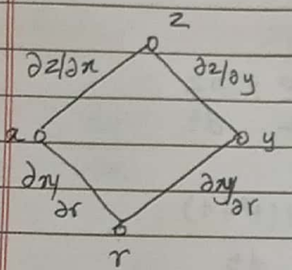
$$\therefore \frac{dw}{dt} = \pi + 1$$

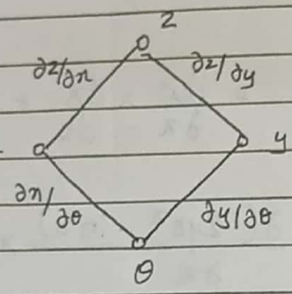
(ii) $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$

$z = 4e^{x} \ln y$ $\quad x = \ln(r\cos\theta)$

$y = r\sin\theta$ $\quad$ at $(r,\theta) = (2, \pi/4)$

Soln:





We know,

(i) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial r}$

$= \frac{\partial (4e^{x}\ln y)}{\partial x} \times \frac{\partial \ln(r\cos\theta)}{\partial r} + \frac{\partial (4e^{x}\ln y)}{\partial y} \times \frac{\partial r\sin\theta}{\partial r}$

$= 4e^{x}\ln y \times \left\{ \frac{\partial \ln(r\cos\theta)}{\partial(r\cos\theta)} \times \frac{\partial(r\cos\theta)}{\partial r} \right\} + \frac{4e^{x}}{y} \times \sin\theta$

$= 4e^{x}\ln y \times \frac{1}{r\cancel{\cos\theta}} \times \cancel{\cos\theta} + \frac{4e^{x}}{y} \sin\theta$

$= \ln(r\sin\theta) \times 4e^{\ln(r\cos\theta)} \times \frac{1}{r} + \frac{4e^{\ln(r\cos\theta)}}{r\sin\theta} \times \sin\theta$

$= \frac{4\cancel{r}\cos\theta \times \ln(r\sin\theta)}{\cancel{r}} + \frac{4\cancel{r}\cos\theta \times \cancel{\sin\theta}}{\cancel{r\sin\theta}}$

$= 4\cos \ln(r\sin\theta) + 4\cos\theta$

$= 4 \times \cos\left(\frac{\pi}{4}\right) \times \ln\left(2 \times \sin\left(\frac{\pi}{4}\right)\right) + 4 \times \cos\left(\frac{\pi}{4}\right) = 3.8086$

$\therefore \partial z/\partial r = 3.8086$

(ii) $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial \theta}$

$= \frac{\partial (4e^{x}\ln y)}{\partial x} \times \frac{\partial \ln(r\cos\theta)}{\partial \theta} + \frac{\partial (4e^{x}\ln y)}{\partial y} \times \frac{\partial (r\sin\theta)}{\partial \theta}$

$= \frac{\partial (4e^{x}\ln y)}{\partial x} \times \frac{\partial \ln(r\cos\theta)}{\partial (r\cos\theta)} \times \frac{\partial (r\cos\theta)}{\partial \theta} + \frac{\partial (4e^{x}\ln y)}{\partial y} \times \frac{\partial (r\sin\theta)}{\partial \theta}$

$= -4e^{x}\ln y \times \frac{1}{\cancel{r}\cos\theta} \times \cancel{r}\sin\theta + \frac{4e^{x}}{y} \times r\cos\theta$

$= -4e^{x} \ln y \times \tan\theta + \frac{4e^{x}}{y} r\cos\theta$

$= -4 \times e^{\ln(r\cos\theta)} \times \ln(r\sin\theta) \times \tan\theta + \frac{4e^{\ln(r\cos\theta)}}{r\sin\theta} \times \cancel{\frac{\partial r\sin\theta}{\partial\theta}}\, r\cos\theta$

$= -4r\cos\theta \times \ln(r\sin\theta) \tan\theta + \frac{4\cancel{r}\cos\theta \times r\cos\theta}{\cancel{r}\sin\theta}$

$= -4r\cos\theta \ln(r\sin\theta) \tan\theta + 4\cot\theta\cos\theta$

$= -4 \times 2 \times \cos\pi/4 \times \ln(2 \times \sin\pi/4) \tan\pi/4 + 4\cot\pi/2 \cos\pi/2$

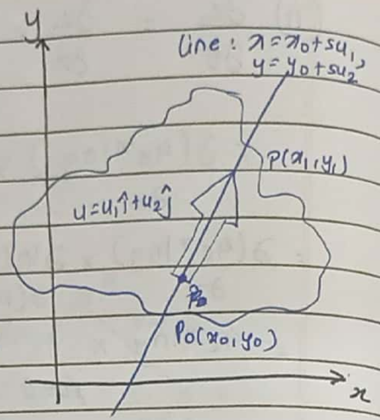
$= 2\sqrt{2} + 1.96$

$= 2.828 + 1.96 = 4.788$

$\therefore \frac{\partial z}{\partial \theta} = 4.788$

Directional Derivative

The derivative of f at $P_0(x_0, y_0)$ in direction of unit vector
$\hat{u} = u_1\hat{i} + u_2\hat{j}$ is the number.



$$\left(\frac{df}{ds}\right)_{\hat{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided that the limit exists.

$\vec{P_0P}$ = direction
$= (x_1 - x_0)\hat{i} + (y_1 - y_0)\hat{j}$

Now,
Along x-axis, $\hat{u} = u_1\vec{i} + u_2\vec{j} = \hat{i} = 1\cdot\vec{i} + 0\cdot\vec{j}$
$u_1 = 1$, $u_2 = 0$.

So,
$$\left(\frac{df}{ds}\right)_{\vec{i}} = \lim_{s \to 0} \frac{f(x_0 + s \times u_1, y_0 + su_2) - f(x_0, y_0)}{s}$$
$$= \lim_{s \to 0} \frac{f(x_0 + s, y_0) - f(x_0, y_0)}{s}$$
$$= \frac{\partial f}{\partial x}$$

$\therefore$ If $\hat{u} = \vec{i}$, $(D_{\hat{u}} f)_{P_0} = \frac{\partial f}{\partial x}$ at P_0.

Along y-axis, $\hat{u} = u_1\hat{i} + u_2\hat{j} = \hat{j} = 0\cdot\vec{i} + 1\cdot\hat{j}$
So, $u_1 = 0$, $u_2 = 1$

So,
$$\left(\frac{df}{ds}\right)_{\vec{j}} = \lim_{s \to 0} \frac{f(x_0 + su_1 + y_0 + su_2) - f(x_0, y_0)}{s}$$
$$= \lim_{s \to 0} \frac{f(x_0, y + s) - f(x_0, y_0)}{s}$$
$$= \frac{\partial f}{\partial y}$$

$\therefore$ If $\hat{u} = \hat{j}$, $(D_{\hat{u}} f)_{P_0} = \frac{\partial f}{\partial y}$ at P_0

(Q) Find the derivative of $f(x, y) = 2xy - 3y^2$ at $P_0(5, 5)$ in the direction of $\vec{u} = 4\vec{i} + 3\vec{j}$ from the definition.

Soln.

Given,
$f(x, y) = 2xy - 3y^2$

$P_0(\overset{x_0, y_0}{\cancel{5, 5}}) = (5, 5)$

$\vec{u} = 4\hat{i} + 3\hat{j}$

$\therefore |\vec{u}| = \sqrt{4^2 + 3^2} = 5$

$\therefore \hat{u} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$ — (i) Comparing with $\hat{u} = u_1\hat{i} + u_2\hat{j}$

$u_1 = \frac{4}{5}$, $u_2 = \frac{3}{5}$

Date. No.

Now,

$$\left(\frac{df}{ds}\right)_{\hat{u},(5,5)} = \lim_{s\to 0} \frac{f(5+0.8s,\ 5+0.6s) - f(5,5)}{s}$$

$$= \lim_{s\to 0} \frac{\{2(5+0.8s)(5+0.6s) - 3(5+0.6s)^2\} - \{2\times5\times5 - 3\times5^2\}}{s}$$

$$= \lim_{s\to 0} \frac{(10+1.6s)(5+0.6s) - 3(25+6s+0.36s^2)+25}{s}$$

$$= \lim_{s\to 0} \frac{(50+6s+8s+0.96) - (75+18s+1.08s^2)+25}{s}$$

$$= \lim_{s\to 0} \frac{-0.12s^2 - 4s}{s}$$

$$= \lim_{s\to 0} -0.12s - 4$$

$$= -4.$$

Q7: Find the derivative of $f(x,y) = x^2+xy$ at $P_0(1,2)$ in direction of $\vec{u} = \vec{i}+\vec{j}$ using definition.

Soln.

Given,

$f(x,y) = x^2+xy$

$P_0(x,y) = \cancel{(5,5)}\ (1,2)$

$\vec{u} = \vec{i}+\vec{j}$

$|\vec{u}| = \sqrt{2}$ $\quad\therefore \hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

So, $u_1 = 1/\sqrt{2}$, $u_2 = 1/\sqrt{2}$

Date. No.

Now,

$$\left(\frac{df}{ds}\right)_{\hat{u},(1,2)} = \lim_{s\to 0} \frac{f(1+s/\sqrt{2},\ 2+s/\sqrt{2}) - f(1,2)}{s}$$

$$= \lim_{s\to 0} \frac{\{(1+s/\sqrt{2})^2 + (1+s/\sqrt{2})(2+s/\sqrt{2})\} - \{1^2+1\times2\}}{s}$$

$$= \lim_{s\to 0} \frac{\{(1+\sqrt{2}s+s^2/2) + (2+s/\sqrt{2}+2s/\sqrt{2}+s^2/2)\} - (3)}{s}$$

$$= \lim_{s\to 0} \frac{\{1+\sqrt{2}s+s^2/2+2+3s/\sqrt{2}+s^2/2\} - 3}{s}$$

$$= \lim_{s\to 0} \frac{\cancel{3}+s^2+5/\sqrt{2}\,s - \cancel{3}}{s}$$

$$= \lim_{s\to 0} \frac{s^2+5/\sqrt{2}\,s}{s}$$

$$\therefore \left(\frac{df}{ds}\right)_{\hat{u},(1,2)} = 5/\sqrt{2}$$

Directional Derivative as Dot Product

(x) Gradient:

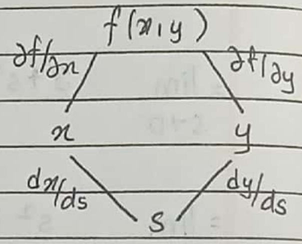
The gradient vector (gradient) of $f(x,y)$ at a point $P_0(x_0,y_0)$ is the vector.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

(x) Theorem:

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0,y_0)$ then,

$$\left(\frac{df}{ds}\right)_{\hat{u},\,P_0} = (\nabla f)_{P_0}\cdot\hat{u}$$



~~Now~~ We have,

$x = x_0 + su_1$ and $y = y_0 + su_2$

and $\hat{u} = u_1\hat{i} + u_2\hat{j}$

By chain rule,

$$(D_{\hat{u}} f)_{P_0} = \left(\frac{df}{ds}\right)_{\hat{u},\,P_0} = \left.\frac{\partial f}{\partial x}\right|_{P_0}\frac{dx}{ds} + \left.\frac{\partial f}{\partial y}\right|_{P_0}\frac{dy}{ds}$$

$$= \underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)_{P_0}}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\right)}_{\text{Direction of } u}$$

$$\therefore \left(\frac{df}{ds}\right)_{\hat{u},\,P_0} = (\nabla f)_{P_0}\ \hat{u}$$

Hence, proved.

Q7: Find the derivative of $f(x,y) = 2xy - 3y^2$ at $P_0(5,5)$ in direction of $\vec{u} = 4\hat{i} + 3\hat{j}$ using gradient.

Soln.

Given,

$f(x,y) = 2xy - 3y^2$

$P_0(x_0,y_0) = (5,5)$

$\vec{u} = 4\hat{i} + 3\hat{j}$ $\therefore |\vec{u}| = 5$

$\therefore \hat{u} = 4/5\,\hat{i} + 3/5\,\hat{j}$

We know,

$$\left(\frac{df}{ds}\right)_{\hat{u},\,P_0} = (\nabla f)_{P_0}\ \hat{u}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\left(\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}\right)$$

$$= \frac{4}{5}\times\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\times\frac{3}{5}$$

$$= \frac{4}{5}\times\frac{\partial(2xy-3y^2)}{\partial x} + \frac{3}{5}\times\frac{\partial(2xy-3y^2)}{\partial y}$$

$$= \frac{4}{5}(2y - 0) + \frac{3}{5}(2x - 6y)$$

At (5,5),

$$\cancel{= \frac{4}{5}\times 2\times 5 + \frac{3}{5}\times 2\times 5 - \frac{3}{5}\times 6\times 5}$$

$$= 4/5\times 2\times 5 - 3/5\times 2\times 5 + 3/5\times 6\times 5$$

$$\therefore \left(\frac{df}{ds}\right)_{\hat{u},\,P_0} = -4$$

Date. No.

Q7: Find ∇f if $f(x,y,z) = x^2+y^2 - 2z^2 + z\ln z$ at $(1,1,1)$.

Soln.

Given,

$$f(x,y,z) = x^2+y^2-2z^2+z\ln z$$

Now

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial(x^2+y^2-2z^2+z\ln z)}{\partial x}\hat{i} + \frac{\partial(x^2+y^2-2z^2+z\ln z)}{\partial y}\hat{j} + \frac{\partial(x^2+y^2-2z^2+z\ln z)}{\partial z}$$

$$\cancel{= 2x\hat{i} + 2y\hat{j} - 4z + \left(z\times\frac{1}{z} + \ln z\right)}$$

~~At (1,1,1)~~

$$= 2x\hat{i} + 2y\hat{j} - (4z+1)\hat{k}$$

At $(1,1,1)$.

$$\therefore \nabla f_{P_0} = 2\hat{i} + 2\hat{j} - 5\hat{k}$$

Date. No.

⇒ Algebra Rules for Gradients

(i) Sum rule: $\nabla(f+g) = \nabla f + \nabla g$

(ii) Difference rule: $\nabla(f-g) = \nabla f - \nabla g$

(iii) Constant multiple rule.

$$\nabla(kf) = k\nabla f$$

(iv) Product rule:

$$\nabla(fg) = f\nabla g + g\nabla f$$

(v) Quotient rule:

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$

Q7: Find the derivative of $f(x,y) = x^2\sin 2y$ at $(1, \pi/2)$ in the direction $\vec{v} = 3\hat{i} - 4\hat{j}$

Soln.

Given,

$$f(x,y) = x^2\sin 2y$$

$$P_0(x_0,y_0) = (1, \pi/2)$$

$$\vec{v} = 3\hat{i} - 4\hat{j}$$

$$|\vec{v}| = \sqrt{3^2+4^2} = 5$$

$$\therefore \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j} = \left(\frac{3}{5}\right)\hat{i} + \left(\frac{-4}{5}\right)\hat{j}$$

Date. No.

We know,

$$\left(\frac{df}{ds}\right)_{\hat{u},P_0} = (\nabla f)\cdot\hat{u}$$

$$= \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}\right)\left(\frac{3}{5}\hat{\imath} + \left(\frac{-4}{5}\right)\hat{\jmath}\right)$$

$$= \frac{3}{5}\times\frac{\partial f}{\partial x} - \frac{4}{5}\times\frac{\partial f}{\partial y}$$

$$= \frac{3}{5}\times\frac{\partial(x^2\sin 2y)}{\partial x} - \frac{4}{5}\times\frac{\partial(x^2\sin 2y)}{\partial y}$$

$$= \frac{3}{5}\times 2x\sin 2y - \frac{4}{5}\times x^2\, 2\cos 2y$$

At $(1, \pi/2)$

$$= \frac{3}{5}\times 2\times 1\times\sin 2\times\frac{\pi}{2} - \frac{4}{5}\times 1^2\times 2\times\cos 2\times\frac{\pi}{2}$$

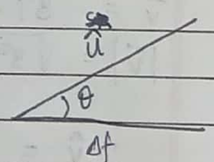
$$= 0 + \frac{8}{5}$$

$$\therefore \left(\frac{df}{ds}\right)_{\hat{u},P_0} = \frac{8}{5}$$

(x) Properties of Directional Derivatives

We have,

$$(D_{\hat{u}} f)_{P_0} = (\nabla f)_{P_0}\cdot\hat{u} = |\nabla f|_{P_0}\cos\theta$$



Date. No.

P.1: The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and u is in the direction of ∇f. At each point P in its domain, f increases most rapidly in direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_u f = |\nabla f|\cos(0) = |\nabla f|$$

P.2: The function f decreases most rapidly when $\cos\theta = -1$ or when $\theta = 0$ and the direction is $-\nabla f$. The derivative in this direction is

$$D_u f = |\nabla f|\cos(\pi) = -|\nabla f|$$

P.3: Any direction u orthogonal to gradient $\nabla f \neq 0$ is a direction of zero change in f because $\theta = \pi/2$.

$$D_u f = |\nabla f|\cos(\pi/2) = 0.$$

(Q7): Find the directions in which the following functions $f(x,y,z) = x/y - yz$ at $P_0(4,1,1)$ increases and decreases most rapidly. Also find the derivatives in this directions.

Soln:

Date. No.

Given,

$$f(x,y,z) = \frac{x}{y} - yz$$

~~at (4,1,1)~~ $P_0(x_0,y_0,z_0) = (4,1,1)$

Now,

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial(x/y - yz)}{\partial x}\hat{i} + \frac{\partial(x/y - yz)}{\partial y}\hat{j} + \frac{\partial(x/y - yz)}{\partial z}\hat{k}$$

$$= \frac{1}{y}\hat{i} + \left(-\frac{x}{y^2} - z\right)\hat{j} + \left(-y\right)\hat{k}$$

$$\therefore \nabla f = \hat{i} - 5\hat{j} - \vec{k}$$

$$\therefore |\nabla f| = \sqrt{1^2 + (-5)^2 + (-1)^2} = 3\sqrt{3}$$

(i): The direction of most rapid increase is given by

$$\hat{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{3\sqrt{3}}\left(\hat{i} - 5\hat{j} - \hat{k}\right)$$

$$\therefore \hat{u} = \frac{1}{3\sqrt{3}}\hat{i} - \frac{5}{3\sqrt{3}}\hat{j} - \frac{1}{3\sqrt{3}}\hat{k}$$

Date. No.

The derivative in this direction $(D_{\hat{u}}f)_{P_0} = |\nabla f| = 3\sqrt{3}$

(ii) The direction of most rapid decrease is given by

$$-\hat{u} = -\left(\frac{1}{3\sqrt{3}}\hat{i} - \frac{5}{3\sqrt{3}}\hat{j} - \frac{1}{3\sqrt{3}}\hat{k}\right)$$

$$\therefore -\hat{u} = -\frac{1}{3\sqrt{3}}\hat{i} + \frac{5}{3\sqrt{3}}\hat{j} + \frac{1}{3\sqrt{3}}\hat{k}$$

The derivative in this direction $(D_{-\hat{u}}f)_{P_0} = -|\nabla f| = -3\sqrt{3}$.

Tangent Planes and Normal Lines

(a) Tangent plane to the level surface $f(x,y,z) = c$ at $P_0(x_0,y_0,z_0)$ is

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$$

(b): Normal line to the level surface $f(x,y,z) = c$ at $P_0(x_0,y_0,z_0)$ is

$$x = x_0 + f_x(P_0)t,\quad y = y_0 + f_y(P_0)t,\quad z = z_0 + f_z(P_0)t$$

(c) Plane tangent to a surface $z = f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$ is

~~$f_x(x-x_0)$~~

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

Q. Find the equation of the tangent lines to the level curves $f(x,y) = c$ at given points.

(i) ~~f(x²~~ $x^2+y^2=4$ at $(\sqrt{2}, \sqrt{2})$

Soln:

Given,

$$f(x,y) = x^2+y^2$$

Now,

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial(x^2+y^2)}{\partial x} = 2x$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial(x^2+y^2)}{\partial y} = 2y$$

At $(\sqrt{2}, \sqrt{2})$,

$f_x = 2\sqrt{2}$ $\quad$ $f_y = 2\sqrt{2}$

We know, the equation of tangent line is.

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$$

or $2\sqrt{2}(x-\sqrt{2}) + 2\sqrt{2}(y-\sqrt{2}) = 0$

or $2\sqrt{2}x - 4 + 2\sqrt{2}y - 4 = 0$

or, $2\sqrt{2}x + 2\sqrt{2}y = 8$

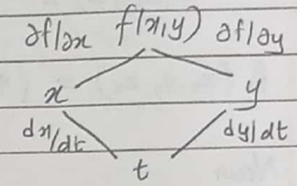
or, $y = -x + 2\sqrt{2}$

which is the required equation.

Q. If a differential function $f(x,y)$ has a constant value c along a smooth curve. $\vec{r}(t) = g(t)\hat{i} + h(t)\hat{j}$ then,

$$\nabla f \cdot \frac{d\vec{r}}{dt} = 0.$$

Soln.



We have,

$$f(x,y) = c$$

Now,

$$\frac{df}{dt} = 0 \quad \text{--- (i)}$$

From chain rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\partial y} \times \frac{dy}{dt}$$

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right)$$

$$= \nabla f \cdot \frac{1}{dt} \times (dx\hat{i} + dy\hat{j})$$

$$\therefore \frac{df}{dt} = \nabla f \cdot \frac{d\vec{r}}{dt} \quad \text{--- (ii)}$$

Equating (i) & (ii), we get.

$$\therefore \nabla f \cdot \frac{d\vec{r}}{dt} = 0$$

<Q7: Find the tangent plane and normal line of the surface $f(x,y,z) = x^2+y^2+z-9$ at $P_0(1,2,4)$.

Soln:

Given,

$$f(x,y,z) = x^2+y^2+z-9$$

$$P_0(x_0,y_0,z_0) = (1,2,4)$$

Now

$f_x = 2x$ $\quad f_y = 2y$

$f_z = 1$

At $(1,2,4)$, $f_x = 2$, $f_y = 4$, $f_z = 1$

We know, the equation of tangent is.

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$$

or $2(x-1) + 4(y-2) + 1(z-4) = 0$

or $2x-2+4y-8+z-4=0$

or $2x+4y+z = 14$

which is the reqd eqn

Again,

the normal lines to the level surface,

$x = x_0 + f_x(P_0)t$ $\quad \therefore x = 1+2t$

$y = y_0 + f_y(P_0)t$ $\quad \therefore y = 2+4t$

$z = z_0 + f_z(P_0)t$ $\quad \therefore z = 4+t$.

which are the reqd eqn of normal lines.

<Q7: Estimate how much the value of $f(x,y,z) = y\sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 units from $P_0(0,1,0)$ straight towards $P_1(2,2,-2)$.

Soln:

Given,

$P_1 = (2,2,-2)$ and $P_0 = (0,1,0)$.

$$f(x,y,z) = y\sin x + 2yz$$

The Vector $\overrightarrow{P_0P_1} = (2-0)\hat{i} + (2-1)\hat{j} + (-2-0)\hat{k}$

$= 2\hat{i} + \hat{j} - 2\hat{k}$

$\therefore |\overrightarrow{P_0P_1}| = \sqrt{2^2+1^2+(-2)^2} = \sqrt{4+4+1} = 3$

$\therefore$ The direction of the vector $\overrightarrow{P_0P_1} = \dfrac{\overrightarrow{P_0P_1}}{|P_0P_1|}$

$\therefore \hat{u} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$

Now,

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial(y\sin x+2yz)}{\partial x}\hat{i} + \frac{\partial(y\sin x+2yz)}{\partial y}\hat{j} + \frac{\partial(y\sin x+2yz)}{\partial z}\hat{k}$$

$$= y\cos x\,\hat{i} + (\sin x+2z)\hat{j} + 2y\,\hat{k}$$

At $(0,1,0)$,

$$\nabla f|_{(0,1,0)} = \hat{i} + 2\hat{k}$$

Therefore,

Date. No.

$$\nabla f|_{P_0} \cdot \hat{u} = (i + 2\hat{k}) \cdot \left(\frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{2}{3} - \frac{4}{3}$$

$$\therefore \nabla f|_{P_0} \cdot \hat{u} = -\frac{2}{3}$$

We know,

$$\frac{df}{ds} = \nabla f|_{P_0} \cdot \hat{u}$$

or, $$df = (\nabla f|_{P_0}\ \hat{u})\ ds$$

$$= -\frac{2}{3} \times 0.1 = -0.067 \text{ units}$$

The estimated change is -0.067 units.