KATHMANDU UNIVERSITY

DHULIKHEL , KAVRE

Subject: MATH104

Assignment No: 2

Submitted by:

Ashraya Kadel
CE Batch 2022
Roll No: 25

Submitted to:

Ganga Ram Phaijoo

Department of Mathematics.

SUBMISSION DATE: 29 | 09 | 2023

Date ______
Page ______

Q.1: Find the function's domain, range, level curve, boundary of the function's domain, determine if the domain is open or closed and decide if the domain is bounded or unbounded if the function is defined by the equation

$$f(x,y) = \ln(x^2+y^2).$$

Soln:

Given,

$$f(x,y) = \ln(x^2+y^2)$$

(i): Domain: $(x,y) \neq (0,0)$

(ii) Range: $(-\infty, \infty)$

(iii) Level curves:

Let $\ln(x^2+y^2) = c$

or, $x^2+y^2 = e^c$

(iv) Boundary point: $(x,y) = (0,0)$

(v) The domain is open

(vi) Since Domain $(x,y) \neq (0,0)$, it is unbounded.

Q.2: Evaluate the following limits (if exists)

(a): Soln:

Given,

$$\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \cancel{\frac{\sqrt{x}(\sqrt{x}+2) - \sqrt{y}(\sqrt{y}+2)}{\sqrt{x}-\sqrt{y}}} = \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})+2(\sqrt{x}}{(\sqrt{x}-\sqrt{y})}$$

$$= \lim_{(x,y)\to(4,0)} \frac{\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+2)(\sqrt{y}+2)(\sqrt{x}+\sqrt{y}+2)}{\cancel{(\sqrt{x}-\sqrt{y})}}$$

$= 2$

(b): $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

Soln:

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(\sqrt{x}+\sqrt{y})\cancel{(\sqrt{x}-\sqrt{y})}}{\cancel{(\sqrt{x}-\sqrt{y})}}$$

$= 0$

(c): $\lim_{(x,y)\to(\pi/2,0)} \frac{\cos y+2}{y-\sin x}$

Soln.

$$= \lim_{(x,y)\to(\pi/2,0)} \frac{\cos 0+2}{0-\sin \pi/2} = -3$$

(d): $\lim_{P_0(\pi,0,3)} z e^{-2y} \cos 2x$

Soln:

$$= \lim_{P_0\to(\pi,0,3)} 3\times e^{-2\times 0}\cdot\cos 2\times\pi$$

$= 3\times1\times1 = 3$

(e): $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ along $y^2=x$.

Soln:

Given,

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$$

Along $y^2=x$,

$$= \lim_{(x,y)\to(0,0)} \frac{2x\times x}{2x^2} = 1$$

(Q.3): Show that limits of given functions don't exist at origin.

(a): $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$.

Soln:

At $(0,0)$,

$f(0,0) = \frac{0}{\sqrt{0+0}} = \frac{0}{0}$ (indeteeminant form).

So, limit does exist at origin.

<Q.5>: Define $f(0,0)$ in a way that extends $f(x,y) = xy\left(\frac{x^2-y^2}{x^2+y^2}\right)$ to be continuous at the origin.

Soln:

Given,

$f(x,y) = xy\left(\frac{x^2-y^2}{x^2+y^2}\right)$

Along $y = mx$

$f(x,mx) = x \cdot mx \left(\frac{x^2 - m^2x^2}{x^2 + m^2x^2}\right)$

$= mx^2 \cdot \frac{\cancel{x^2}}{\cancel{x^2}}\left(\frac{1-m^2}{1+m^2}\right)$

$= mx^2 \left(\frac{1-m^2}{1+m^2}\right)$

At $f(0,0)$

$f(0,0) = 0$

Then, redefining the function ~~fto~~

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

<b>: $f(x,y) = \frac{x^4-y^2}{x^4+y^2}$

Soln.

At $(0,0)$

$f(0,0) = \frac{0^4-0^2}{0^4+0^2} = \frac{0}{0}$ (indeteeminant form)

So, limit doesn't exist at origin.

<Q.4>: Prove that the function defined by

$$f(x,y) = \begin{cases} x^2/x^2+y^2 & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$ is

discontinuous at $(0,0)$.

Soln:

Given,

$$f(x,y) = \begin{cases} x^2/x^2+y^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $y = mx$ at origin,

$f(x, mx) = \frac{x^2}{x^2 + m^2x^2} = \frac{\cancel{x^2}\ 1}{\cancel{x^2}(1+m^2)}$

$= \frac{1}{1+m^2}$

When $m = 1$, $f(x,mx) = 1/2$

when $m = 2$, $f(x,mx) = 1/3$

The existence of 'm' shows path dependency.

Hence, it is not continuous at $(0,0)$.

<Q.6>: Define first partial derivative of the function $f(x,y)$ with respect to x and y and give its geometrical meaninings.

Ans:

The first partial derivative of the function $f(x,y)$ w.r.t x is denoted by $\partial f/\partial x$ and is defined as the rate of change of $f(x,y)$ w.r.t x while keeping y constant.

$$f_x = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

The first partial derivative of the function $f(x,y)$ w.r.t y is denoted by $\partial f/\partial y$ and is defined as the rate of change of $f(x,y)$ w.r.t y while keeping x constant.

$$f_y = \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}.$$

Geometrically, $\partial f/\partial x$ represents the slope of the tangent line to the surface defined by $f(x,y)$ in the x-direction.

Geometrically, $\partial f/\partial y$ represents the slope of the tangent line to the surface defined by $f(x,y)$ in the y-direction.

<Q.7>: Calculate the first order partial derivatives f_x, f_y, f_z of the following functions.

<a>: $f(x,y) = (x^2-1)(y+2)$

Soln:

Given,

$$f(x,y) = (x^2-1)(y+2)$$

Now,

i) $$\frac{\partial f}{\partial x} = \frac{\partial[(x^2-1)(y+2)]}{\partial x}$$

$$= (x^2-1)\frac{\partial(y+2)}{\partial x} + (y+2)\frac{\partial(x^2-1)}{\partial x}$$

$$= 0 + (y+2)2x$$

$$\therefore \frac{\partial f}{\partial x} = 2x(y+2)$$

(ii) $$\frac{\partial f}{\partial y} = \frac{\partial[(x^2-1)(y+2)]}{\partial y}$$

$$= (x^2-1)\frac{\partial(y+2)}{\partial y} + (y+2)\frac{\partial(x^2-1)}{\partial y}$$

$$= (x^2-1)1 + 0$$

$$\therefore \frac{\partial f}{\partial y} = x^2-1$$

Date
Page

<b>: $f(x,y) = \dfrac{x+y}{xy-1}$

Soln.

Given, $f(x,y) = \dfrac{x+y}{xy-1}$

(i): $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{x+y}{xy-1}\right)$

$= \dfrac{(xy-1)\dfrac{\partial(x+y)}{\partial x} - (x+y)\dfrac{\partial(xy-1)}{\partial x}}{(xy-1)^2}$

$= \dfrac{(xy-1) - y(x+y)}{(xy-1)^2}$

$\therefore \dfrac{\partial f}{\partial x} = \dfrac{(xy-1) - y(x+y)}{(xy-1)^2}$

<c>: $f(x,y,z) = e^{x+y+z}$

Soln:

Given, $f(x,y,z) = e^{x+y+z}$

(i): $\dfrac{\partial f}{\partial x} = \dfrac{\partial e^{(x+y+z)}}{\partial(x+y+z)} \times \dfrac{\partial(x+y+z)}{\partial x}$

$\therefore \dfrac{\partial f}{\partial x} = e^{x+y+z}$

Date
Page

(ii): $\dfrac{\partial f}{\partial y} = \dfrac{\partial(e^{x+y+z})}{\partial(x+y+z)} \times \dfrac{\partial(x+y+z)}{\partial y}$

$\therefore \partial f/\partial y = e^{x+y+z}$

(iii) $\dfrac{\partial f}{\partial z} = \dfrac{\partial e^{x+y+z}}{\partial(x+y+z)} \times \dfrac{\partial(x+y+z)}{\partial z}$

$\therefore \dfrac{\partial f}{\partial z} = e^{(x+y+z)}$

<d>: $f(x,y) = \ln(x+y)$

Soln:

Given. $f(x,y) = \ln(x+y)$

(i): $\dfrac{\partial f}{\partial x} = \dfrac{\partial \ln(x+y)}{\partial(x+y)} \times \dfrac{\partial(x+y)}{\partial x}$

$\therefore \dfrac{\partial f}{\partial x} = \dfrac{1}{x+y}$

(ii) $\dfrac{\partial f}{\partial y} = \dfrac{\partial \ln(x+y)}{\partial(x+y)} \times \dfrac{\partial(x+y)}{\partial y}$

$\therefore \dfrac{\partial f}{\partial y} = \dfrac{1}{x+y}$

~~(iii)~~ <e>: $f(x,y,z) = \sin^{-1}(xyz)$

Sol^n:

Given,

$f(x,y,z) = \sin^{-1}(xyz)$

Now,

(i): $\dfrac{\partial f}{\partial x} = \dfrac{\partial \sin^{-1}(xyz)}{\partial xyz} \times \dfrac{\partial (xyz)}{\partial x}$

$\therefore \partial f/\partial x = \dfrac{1}{\sqrt{1-x^2y^2z^2}} \times yz$

(ii) $\dfrac{\partial f}{\partial y} = \dfrac{\partial \sin^{-1}(xyz)}{\partial xyz} \times \dfrac{\partial (xyz)}{\partial y}$

$\therefore \partial f/\partial y = \dfrac{1}{\sqrt{1-x^2y^2z^2}} \times xz$

(iii) $\dfrac{\partial f}{\partial z} = \dfrac{\partial \sin^{-1}(xyz)}{\partial (xyz)} \times \dfrac{\partial (xyz)}{\partial z}$

$\therefore \partial f/\partial z = \dfrac{1}{\sqrt{1-x^2y^2z^2}} \times xy$

<f>: $f(x,y,z) = yz\ln(xy)$

Sol^n:

Given,

$f(x,y,z) = yz\ln(xy)$

(i): $\dfrac{\partial f}{\partial x} = \dfrac{\partial\, yz(\ln xy)}{\partial xy} \times \dfrac{\partial xy}{\partial x}$

$= yz \times \dfrac{1}{xy} \times y \qquad \therefore \dfrac{\partial f}{\partial x} = \dfrac{yz}{x}$

(ii): $\dfrac{\partial f}{\partial y} = \dfrac{\partial\, yz(\ln xy)}{\partial xy} \times \dfrac{\partial xy}{\partial y}$

$= yz \times \dfrac{1}{xy} \times x \qquad \therefore \dfrac{\partial f}{\partial y} = z$

(iii) $\dfrac{\partial f}{\partial z} = \dfrac{\partial\, yz(\ln xy)}{\partial xy} \times \dfrac{\partial xy}{\partial z}$

$= yz \cdot \dfrac{1}{xy} \times 0$

$\therefore \dfrac{\partial f}{\partial z} = 0$

Date ____ Page ____

<Q.8>: Calculate the second order partial derivatives.

<a>: $f(x,y) = \sin(xy)$

Soln:

Given,

$f(x,y) = \sin(xy)$

Now

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial(\sin(xy))}{\partial xy} \times \frac{\partial xy}{\partial x}$$

$$\therefore f_x = y\cos(xy)$$

$$f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial y\cos(xy)}{\partial(xy)} \times \frac{\partial(xy)}{\partial x}$$

$$= -y^2 \sin xy$$

Again,

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial(\sin xy)}{\partial xy} \times \frac{\partial xy}{\partial y}$$

$$\therefore f_y = x\cancel{y}\cos xy$$

$$f_{yy} = \frac{\partial(y\cos xy)}{\cancel{\partial y}\,\partial xy} \times \frac{\partial xy}{\partial y}$$

$$= \cancel{y} - x^2 \sin xy$$

Date ____ Page ____

<b>: $h(x,y) = xe^y + y + 1$

Soln.

$$h_x = \frac{\partial h}{\partial x} = \frac{\partial(xe^y + y + 1)}{\partial x}$$

$$= \frac{\partial xe^y}{\partial x} + \frac{\partial y}{\partial x} + \frac{\partial \cdot 1}{\partial x}$$

$$h_x = \cancel{x}\, e^y$$

$$h_{xx} = \frac{\partial h_x}{\partial x} = \frac{\partial e^y}{\partial x} = 0$$

Again,

$$h_y = \frac{\partial h}{\partial y} = \frac{\partial x\cancel{e}^y}{\partial y} + \frac{\partial y}{\partial y} + \frac{\partial \cdot 1}{\partial x}$$

$$\therefore h_y = \cancel{\partial x e}\; xe^y + 1$$

$$h_{yy} = \frac{\partial h_y}{\partial y} = \frac{\partial xe^y}{\partial y} + \frac{\partial \cdot 1}{\partial y}$$

$$\therefore h_{yy} = xe^y$$

<Q.9>: Use the limit definition of derivative to compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of the following functions.

(a): $f(x,y) = 1 - x + y - 3x^2y$ at $(1,2)$

Soln:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = \lim_{h\to 0} \frac{f(1+h,2) - f(1,2)}{h}$$

$$= \lim_{h\to 0} \frac{1-(1+h)+2-3(1+h)^2 2 - (1-1+2-3\times 1^2\times 2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1}-\cancel{1}-h+2-6(1+2h+h^2)-(-4)}{h}$$

$$= \lim_{h\to 0} \frac{-h+\cancel{2}-\cancel{6}-12h-6h^2+\cancel{4}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-h}-13h-6h^2}{h}$$

$$= \lim_{h\to 0} \frac{-13\cancel{h}}{\cancel{h}} - \frac{6h^{\cancel{2}}}{\cancel{h}}$$

$\therefore f_{x(1,2)} = -13-6\times 0 = -13$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = \lim_{h\to 0} \frac{f(1,2+h) - f(1,2)}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{1}-\cancel{1}+(2+h)-3\times 1^2\times(2+h)) - (\cancel{1}-\cancel{1}+2-3\times 1^2\times 2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2}+h-\cancel{6}-3h+\cancel{4}}{h} = \lim_{h\to 0} \frac{-2h}{h}$$

$\therefore f_{y(1,2)} = -2$

(b): $f(x,y) = 4+2x-3y-xy^2$ at $(-2,1)$

Soln:

Given,

$f(x,y) = 4+2x-3y-xy^2$

Now,

(i): $$\left.\frac{\partial f}{\partial x}\right|_{(-2,1)} = \lim_{h\to 0} \frac{f(h-2,1) - f(-2,1)}{h}$$

$$= \lim_{h\to 0} \frac{4+2(h-2)-3\times 1-(h-2)1^2 - (4+2\times(-2)-3\times 1-(-2)\times 1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}+2h-\cancel{4}-3-h+2-(\cancel{4}-\cancel{4}-3+2)}{h}$$

$$= \lim_{h\to 0} \frac{h-\cancel{1}+\cancel{1}}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

(ii): $\frac{\partial f}{\partial y}\Big|_{(-2,1)} = \lim_{h\to 0} \frac{f(-2,h+1) - f(-2,1)}{h}$

$= \lim_{h\to 0} \frac{(4+2\times(-2)-3\times(h+1)-(-2)(h+1)^2) - (4+2\times(-2)-3\times1-(-2)\times1^2)}{h}$

$= \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}-3h-3+2(h^2+2h+1) - (4-4-3+2)}{h}$

$= \lim_{h\to 0} \frac{-3h-\cancel{3}+2h^2+4h+\cancel{2}+\cancel{1}}{h}$

$= \lim_{h\to 0} \frac{2h^2}{\cancel{h}} + \frac{\cancel{h}}{\cancel{h}}1 = 1$

$\therefore f_y|_{(-2,1)} = 1$

= ti

Q.10: Find dy/dx if $x^2+\sin y-2y=0$ and find $\partial z/\partial x$ and $\partial z/\partial y$ if $z^3-xy+yz+y^3-2=0$

Soln:

Given,

$x^2+\sin y-2y=0$

Now,

$\frac{\partial f}{\partial x} = \frac{\partial x^2}{\partial x} + \frac{\partial \sin y}{\partial x} - \frac{\partial 2y}{\partial y}$

$\therefore \frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = \frac{\partial x^2}{\partial y} + \frac{\partial \sin y}{\partial y} - \frac{\partial 2y}{\partial y}$

$\therefore \frac{\partial f}{\partial y} = f_y = \cos y - 2$

So,

$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x}{\cos y - 2}$

$\therefore \frac{dy}{dx} = \frac{2x}{2-\cos y}$

Again,

~~$z^3 - xy + yz + y^3 - 2 = 0$~~

~~Partially differentiating both sides w.r.t. x,~~

~~Now,~~ ~~ffa~~

$z^3 - xy + yz + y^3 - 2 = 0$

Partially differentiating both sides w.r.t. x,

$\frac{\partial z^3}{\partial z}\times\frac{\partial z}{\partial x} - \frac{\partial xy}{\partial x} + \frac{\partial yz}{\partial x} + \frac{\partial y^3}{\partial x} - \frac{\partial 2}{\partial x} = 0$

or, $3z^2\times\frac{\partial z}{\partial x} - y + 0 + 0 - 0 = 0$

$\therefore \frac{\partial z}{\partial x} = \frac{y}{3z^2}$

Partially differentiating w.r.t y,

$$\frac{\partial z^3}{\partial z} \times \frac{\partial z}{\partial y} - \frac{\partial xy}{\partial y} + \frac{\partial yz}{\partial y} + \frac{\partial y^3}{\partial y} - \frac{\partial 2}{\partial y} = 0$$

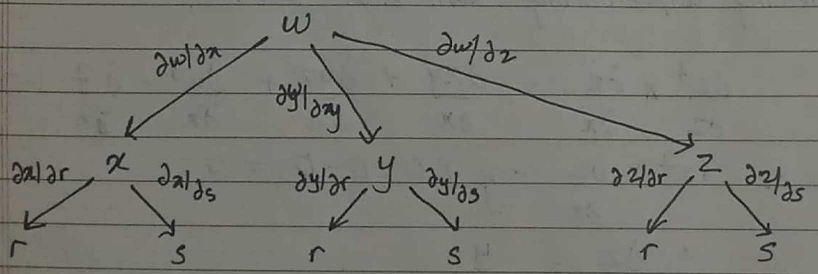
or $3z^2 \times \frac{\partial z}{\partial y} - x + z + 3y^2 - 0 = 0$

or $3z^2 \times \frac{\partial z}{\partial y} = \cancel{3y}\ x - 3y^2 - z$

$\therefore \frac{\partial z}{\partial y} = \frac{x - 3y^2 - z}{3z^2}$

<Q.11>: Draw a tree diagram and write a net chain rule formula for $\partial w/\partial r$ and $\partial w/\partial s$ for $w = f(x, y, z)$, $x = g(r,s)$, $y = h(r,s)$
$z = k(r,s)$.

Soln:



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial r}$$

and $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial s}$

<Q.12> Evaluate dw/dt at a given value of t.

<a>: $w = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$ at $t = 0$.

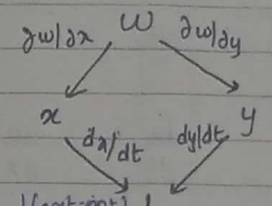
Given,

$w = x^2 + y^2$

$x = \cos t + \sin t$ Given value, $t = 0$.

$y = \cos t - \sin t$

Now,



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt}$$

$$= \frac{\partial (x^2+y^2)}{\partial x} \times \frac{d(\cos t + \sin t)}{dt} + \frac{\partial (x^2+y^2)}{\partial y} \times \frac{d(\cos t - \sin t)}{dt}$$

$= 2x(\cos t - \sin t) + 2y(-\sin t - \cos t)$

$= 2[x(\cos t - \sin t) - y(\sin t + \cos t)]$

At $t = 0$,

$\left.\frac{dw}{dt}\right|_{t=0} = 2[x(\cos 0 - \sin 0) - y(\sin 0 + \cos 0)]$

$= 2x - 2y = 2(x - y)$

<b>: $w = \ln(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, $z = 4t$
at $t = 3$.

Soln:

Given,

$w = \ln(x^2+y^2+z^2)$

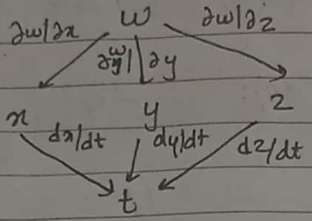
$x = \cos t$

$y = \sin t$

$z = 4t$

and

$t = 3$



Now,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt} + \frac{\partial w}{\partial z} \times \frac{dz}{dt}$$

$$= \left(\frac{\partial(\ln(x^2+y^2+z^2))}{\partial(x^2+y^2+z^2)} \times \frac{\partial(x^2+y^2+z^2)}{\partial x} \times \frac{d \cos t}{dt} \right) + \left(\frac{\partial \ln(x^2+y^2+z^2)}{\partial(x^2+y^2+z^2)} \times \frac{\partial(x^2+y^2+z^2)}{\partial y} \times \frac{d \sin t}{dt} \right) + \left(\frac{\partial \ln(x^2+y^2+z^2)}{\partial(x^2+y^2+z^2)} \times \frac{\partial(x^2+y^2+z^2)}{\partial z} \times \frac{d\, 4t}{dt} \right)$$

$$= \frac{1}{(x^2+y^2+z^2)} \times 2x \times -\sin t + \frac{1}{(x^2+y^2+z^2)} \times 2y \times \cos t + \frac{1}{(x^2+y^2+z^2)} \times 2z \times 4$$

$$\therefore \frac{dw}{dt} = \frac{1}{(x^2+y^2+z^2)} \{ 8z + 2y\cos t - 2x \sin t \}$$

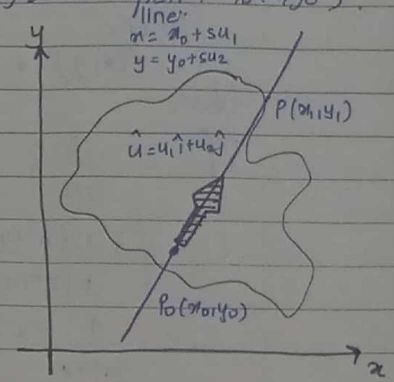
At $t = 3$

$$\left.\frac{dw}{dt}\right|_{t=3} = \frac{1}{(x^2+y^2+z^2)} \{ 8z + 2y \cos 3 - 2x \sin 3 \}$$

$$= \frac{1}{(x^2+y^2+z^2)} \{ 8z + 1.98y - 0.1x \}$$

(Q.13): Define directional derivative of a function $f(x,y)$ at the point $P_0(x_0,y_0)$ in the direction of unit vector $\hat{u}$. What is the difference between the partial derivatives and directional derivative of a function $f(x,y)$ at point $P_0(x_0,y_0)$.

Soln:



The derivative of f at $P_0(x_0,y_0)$ in direction of unit vector
$\hat{u} = u_1 \hat{i} + u_2 \hat{j}$

$$\left(\frac{df}{ds}\right)_{\hat{u},P_0} = \lim_{s \to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

provided that the limit exists.

If $\hat{u} = \vec{i}$, $(D_{\hat{u}} f)_{P_0} = \partial f/\partial x$ at P_0

If $\hat{u} = \vec{j}$, $(D_{\hat{u}} f)_{P_0} = \partial f/\partial y$ at P_0

Partial derivatives gives the change of the function with respect to individual variables. ~~Geo~~ Geometrically, it gives the tangent slopes in the x and y directions at P_0.

Directional derivatives analyzes the rate of change in a chosen direction. Geometrically, it gives tangent slope in a specific direction at P_0.

<Q.14>: Find the gradient of the given functions at given points.

<a>: $f(x,y) = \ln(x^2+y^2)$ at $(1,1)$

Soln:

Given,

$f(x,y) = \ln(x^2+y^2)$

Now,

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= \left\{\frac{\partial \ln(x^2+y^2)}{\partial(x^2+y^2)} \times \frac{\partial(x^2+y^2)}{\partial x}\right\}\hat{i} + \left\{\frac{\partial \ln(x^2+y^2)}{\partial(x^2+y^2)} \times \frac{\partial(x^2+y^2)}{\partial y}\right\}\hat{j}$$

$$= \frac{1}{x^2+y^2} \times 2x\,\hat{i} + \frac{1}{x^2+y^2} \times 2y\,\hat{j}$$

$$= \frac{2}{x^2+y^2}(x\hat{i} + y\hat{j})$$

At $(1,1)$,

$$\nabla f_{(1,1)} = \frac{2}{1^2+1^2}(1\hat{i} + 1\hat{j})$$

$$\therefore \nabla f_{(1,1)} = \hat{i} + \hat{j}$$

<b>: $f(x,y,z) = e^x + y\cos z + (y+1)\sin^{-1}x$ at $(0,0,\pi/6)$

Soln:

Given,

$f(x,y,z) = e^x + y\cos z + (y+1)\sin^{-1}x$

Now,

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\frac{\partial e^x}{\partial x} + \frac{\partial y\cos z}{\partial x} + (y+1)\frac{\partial \sin^{-1}x}{\partial x}\right)\hat{i} + \left(\frac{\partial e^x}{\partial y} + \frac{\partial y\cos z}{\partial y} + \frac{\partial(y+1)\sin^{-1}x}{\partial y}\right)\hat{j} + \left(\frac{\partial e^x}{\partial z} + \frac{\partial y\cos z}{\partial z} + \frac{\partial(y+1)\sin^{-1}x}{\partial z}\right)\hat{k}$$

$$= \left(e^x + 0 + \frac{1}{\sqrt{1-x^2}}\right)\hat{i} + \left(0 + \cos z + \sin^{-1}x\right)\hat{j} + \left(0 - y\sin z + 0\right)\hat{k}$$

$$= \left(e^x + \frac{1}{\sqrt{1-x^2}}\right)\hat{i} + \left(\cos z + \sin^{-1}x\right)\hat{j} - \left(y\sin z\right)\hat{k}$$

At $(0,0,\pi/6)$

$$= \left(e^0 + \frac{1}{1}\right)\hat{i} + \left(\cos\pi/6 + \sin^{-1}0\right)\hat{j} - \left(0 \times \sin\pi/6\right)\hat{k}$$

$$= 2\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

<Q.15>: Find the derivative of following functions at point P_0 in direction of vector $\vec{A}$.

<a>: $f(x,y) = 2xy - 3y^2$ at $P_0(5,5)$, $\vec{A} = 4\hat{i} + 3\hat{j}$

Soln:

Given,

$\vec{A} = 4\hat{i} + 3\hat{j}$

$|\vec{A}| = \sqrt{4^2+3^2} = 5$

$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$ — (i)

Comparing with $\hat{u} = u_1\hat{i} + u_2\hat{j}$,

$u_1 = 4/5 = 0.8$, $u_2 = 3/5 = 0.6$

We know,

$$\left(\frac{df}{ds}\right)_{\hat{u}, P_0(x_0,y_0)} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$$

So,

$$\left(\frac{df}{ds}\right)_{\hat{u},(5,5)} = \lim_{s\to 0} \frac{f(5+0.8s, 5+0.6s) - f(5,5)}{s}$$

$$= \lim_{s\to 0} \frac{\{2(5+0.8s)(5+0.6s) - 3(5+0.6)^2\} - \{2\times5\times5 - 3\times5^2\}}{s}$$

$$= \lim_{s\to 0} \frac{(10+1.6s)(5+0.6s) - 3(25+6s+0.36s^2) - (-25)}{s}$$

$$= \lim_{s\to 0} \frac{-0.12s^2}{s} - \frac{4s}{s}$$

$= -4.$

$$\therefore \left(\frac{df}{ds}\right)_{\hat{A},(5,5)} = -4.$$

<b>: $f(x,y,z) = xe^y + yz$ at $P_0(2,0,0)$, $\vec{A} = \hat{i} + 2\hat{j}$

Soln:

Given,

$f(x,y,z) = xe^y + yz$ at $P_0(2,0,0)$

$\vec{A} = \hat{i} + 2\hat{j}$

$|\vec{A}| = \sqrt{1^2+2^2} = \sqrt{5}$

$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}$ — (i)

Comparing (i) with $\hat{u} = u_1\hat{i} + u_2\hat{j}$

$u_1 = 1/\sqrt{5}$, $u_2 = 2/\sqrt{5}$, $u_3 = 0$

We know

$$\left(\frac{df}{ds}\right)_{\hat{A}, P_0(x_0,y_0,z_0)} = \lim_{s\to 0} \frac{f(x_0+su_1, y_0+su_2, z_0+su_3) - f(x_0,y_0,z_0)}{s}$$

$$\left(\frac{df}{ds}\right)_{\hat{A},(2,0,0)} = \lim_{s\to 0} \frac{f(2+\cancel{\cdot} 1/\sqrt{5}\, s, 0+2s/\sqrt{5}, 0+0) - f(2,0,0)}{s}$$

$$= \lim_{s\to 0} \frac{(2+s/\sqrt{5})e^0 + (2s/\sqrt{5}) - (2\times e^0 + 0\times 0)}{s}$$

$$= \lim_{s \to 0} \frac{\cancel{2} + s/\sqrt{5} + 2s/\sqrt{5} - \cancel{2}}{s}$$

$$= \lim_{s \to 0} \frac{3\cancel{s}}{\sqrt{5}\,\cancel{s}} = \frac{3}{\sqrt{5}}$$

$$\therefore \left(\frac{df}{ds}\right)_{\hat{A},\,(2,0,0)} = \frac{3}{\sqrt{5}}$$

<Q.167>: Find the directions in which the function increase and decrease most rapidly at P_0. Then, find the derivatives of the functions in there directions.

Q.7: $f(x,y) = x^2 + xy + y^2$ at $P_0(-1,1)$.

Soln:

Given,

$$f(x,y) = x^2 + xy + y^2$$

and

$$P_0(-1,1) = (x_0, y_0) \quad \text{or } x_0 = -1,\ y_0 = 1$$

Now,

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= \frac{\partial(x^2+xy+y^2)}{\partial x}\hat{i} + \frac{\partial(x^2+xy+y^2)}{\partial y}\hat{j}$$

$$= \left\{\frac{\partial x^2}{\partial x} + \frac{\partial xy}{\partial x} + \frac{\partial y^2}{\partial x}\right\}\hat{i} + \left\{\frac{\partial x^2}{\partial y} + \frac{\partial xy}{\partial y} + \frac{\partial y^2}{\partial y}\right\}\hat{j}$$

$$= (2x+y)\hat{i} + (2y+x)\hat{j}$$

At $(-1,1)$,

$$(\nabla f)_{(-1,1)} = \{2\times(-1)+1\}\hat{i} + \{2\times 1 + (-1)\}\hat{j}$$

$$= (-2+1)\hat{i} + (2-1)\hat{j}$$

$$\therefore (\nabla f)_{(-1,1)} = -\hat{i} + \hat{j}$$

Now,

$$|\nabla f| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

The direction of most rapid increase is given by

$$\hat{u} = \frac{\nabla f}{|\nabla f|} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

The derivative in this direction

$$(D_{\hat{u}} f)_{(-1,1)} = |\nabla f| = \sqrt{2}$$

The direction of most rapid decrease is given by

$$-\hat{u} = -\left(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right) = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

The derivative in this direction $(D_{-\hat{u}} f)_{(-1,1)} = -|\nabla f|$

$$= -\sqrt{2}$$

Date ____ Page ____

<Q.b>: $h(x,y,z) = \ln xy + \ln yz + \ln xz$, $P_0(1,1,1)$

Soln:

Given,

$$h(x,y,z) = \ln xy + \ln yz + \ln xz$$

and

$$P_0(x_0,y_0,z_0) = (1,1,1)$$

Now,

$$\nabla h = \frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j} + \frac{\partial h}{\partial z}\hat{k}$$

$$= \left(\frac{\partial \ln xy}{\partial xy} \times \frac{\partial xy}{\partial x}\right)\hat{i} + \left(\frac{\partial \ln yz}{\partial yz} \times \frac{\partial yz}{\partial y}\right)\hat{j} + \left(\frac{\partial \ln xz}{\partial xz} \times \frac{\partial xz}{\partial z}\right)\hat{k}$$

$$= \left(\frac{1}{xy} \times y\right)\hat{i} + \left(\frac{1}{yz} \times z\right)\hat{j} + \left(\frac{1}{xz} \times x\right)\hat{k}$$

$$= \frac{1}{x}\hat{i} + \frac{1}{y}\hat{j} + \frac{1}{z}\hat{k}$$

At $(1,1,1)$,

$$(\nabla h)_{(1,1,1)} = \hat{i} + \hat{j} + \hat{k}$$

$$|\nabla h| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

The direction of most rapid increase is given by

$$\hat{u} = \frac{\nabla h}{|\nabla h|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Date ____ Page ____

The derivative in this direction

$$(D_{\hat{u}} f)_{(1,1,1)} = |\nabla h| = \sqrt{3}$$

The direction of most rapid decrease is given by,

$$-\hat{u} = -\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$\therefore -\hat{u} = -\frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

The derivative in this direction

$$(D_{-\hat{u}} f)_{(1,1,1)} = -|\nabla h| = -\sqrt{3}.$$

<Q.17>: Find the equations for the

(a) tangent plane

(b) normal line at a point P_0 on the given surfaces.

<a>: $x^2+y^2+z^2 = 3$, $P_0(1,1,1)$

Soln.

Given,

$$x^2+y^2+z^2 = 3$$

and

$$P_0(x_0,y_0,z_0) = P_0(1,1,1)$$

So,

$f(x,y,z) = x^2+y^2+z^2$

Now,

$f_x = \frac{\partial f}{\partial x} = 2x$ At $(1,1,1)$, $f_x = 2$

$f_y = \frac{\partial f}{\partial y} = 2y$ At $(1,1,1)$, $f_y = 2$

$f_z = \frac{\partial f}{\partial z} = 2z$ At $(1,1,1)$, $f_z = 2$

~~We know~~ The required eqn of tangent line is,

$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$

or, $2(x-1) + 2(y-1) + 2(z-1) = 0$

or $2x-2+2y-2+2z-2=0$

or $2x+2y+2z-6=0$

which is the reqd eqn of tangent line.

Normal lines at (x_0,y_0,z_0),

$x = x_0 + f_x(P_0)t$ or, $x = 1+2t$

$y = y_0 + f_y(P_0)t$ or, $y = 1+2t$

$z = z_0 + f_z(P_0)t$ or $z = 1+2t$

which are the reqd eqn of normal lines.

(b): $x^2 - xy - y^2 - z = 0$, $P_0(1,1,-1)$

Soln.

Given,

$f(x,y,z) = x^2 - xy - y^2 - z$

and

$P_0(x_0,y_0,z_0) = (1,1,-1)$

Now,

$f_x = \frac{\partial f}{\partial x} = \frac{\partial(x^2-xy-y^2-z)}{\partial x} = 2x-y$

At $(1,1,-1)$, $f_x = 1$

$f_y = \frac{\partial f}{\partial y} = \frac{\partial(x^2-xy-y^2-z)}{\partial y} = -x-2y$

At $(1,1,-1)$ ~~-3~~, $f_y = -3$

$f_z = \frac{\partial f}{\partial z} = \frac{\partial(x^2-xy-y^2-z)}{\partial z} = -1$

At $(1,1,-1)$, $f_z = -1$

The reqd eqn of tangent line is

$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0$

or, $1(x-1) + (-3)(y-1) + (-1)(z+1) = 0$

or $x-1-3y+3-z-1=0$

or $x-3y-z+1=0$

which is the reqd eqn of tangent line.

Date

Page

Normal lines at (x_0, y_0, z_0)

$$x = x_0 + f_x(P_0)t \quad \text{or,} \quad x = 1 + t$$
$$y = y_0 + f_y(P_0)t \quad \text{or,} \quad y = 1 - 3t$$
$$z = z_0 + f_z(P_0)t \quad \text{or} \quad z = -1 - t$$

which is the required eq^n of normal lines.