KB Manandhar

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Sartha

Name: Shisir Kharel **Subject:** Calculus and linear algebra

Std: Geomatics 1st year **Div:** **Roll No.:**

School/College:

S.No.	Date	Title	Page No.	Remarks
		100 marks		
		25 marks (Internal) ⇓ 100 20 marks exam + 5 (Assignment) (Avg being final mark)		
		75 marks (final exam)		
		(1) functions, limits & continuity		
		(2) Differentiation		
		(3) Application of derivative		
		(4) Integration		
		(5) Application of definite integral		
		(6) Sequence and series		
		(7) System of linear equation		
		(8) vector space		
		(9) Eigen value and eigen vector.		

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Note

Greatest integer function.

The ~~greatest~~ function whose value at any number 'x' is the greatest integer less than or equal to 'x' is called greatest integer function or integer floor function.

eg:- $\lfloor 2.1 \rfloor = 2$

$\lfloor 5 \rfloor = 5$

$\lfloor 4.1 \rfloor = 4$

$\lfloor -2.3 \rfloor = -3$

$\lfloor -0.1 \rfloor = -1$

Least integer function

The function whose value at any number 'x' is the least integer greater than or equal to 'x' is called least integer function or integer ceiling function.

eg:- $\lceil 2.1 \rceil = 3$

$\lceil -2.3 \rceil = -2$

$\lceil -4.5 \rceil = -4$

$\lceil 0.1 \rceil = 1$

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Note

find Domain and Range of $f: \mathbb{R} \to \mathbb{R}$.

(i) $f(x) = \sqrt{x^2 - 5}$

for f(x) to exist, $x^2 - 5 \geq 0$

$x^2 \geq 5$

$x \geq \pm\sqrt{5}$

So, $D = \mathbb{R} - x^2 < 5$

$= \mathbb{R} - (-\sqrt{5}, \sqrt{5})$

Range $= [0, \infty)$

(ii) $f(x) = \sin^2 x$

$D = \mathbb{R}$

Range $= [0, 1]$

(iii) $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$

D exist, $1 - x^2 > 0$

$1 > x^2$

Domain $= (-1, 1)$

Range $= (0, \infty)$

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Note

Q. $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

such that,

	Do.	Ra
$f(x) = x^2 - 1$	$(-\infty, \infty)$	$[-1, \infty)$
$g(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$

find the domain and range of $fog(x)$, $gof(x)$, $fof(x)$, $gog(x)$.

Soln

$$fog(x)$$

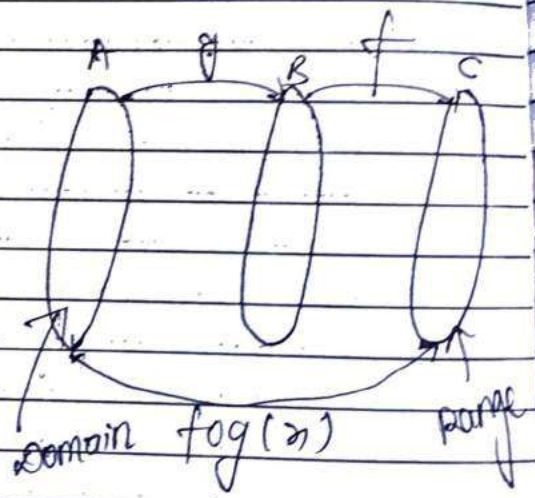
$$= f(\sqrt{x})$$

$$= (\sqrt{x})^2 - 1$$

$$fog(x) = x - 1$$

Domain is $[0, \infty)$

Range is $[-1, \infty)$



$$gof(x)$$

$$= g(x^2 - 1)$$

$$gof(x) = \sqrt{x^2 - 1}$$

Domain is $(-\infty, \infty)$

Range is $[0, \infty)$

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Note

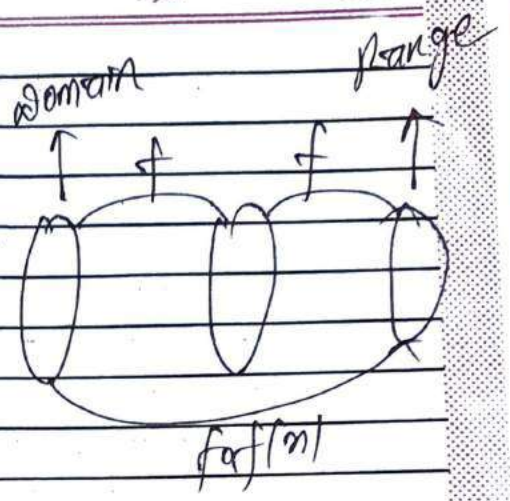
$fof(x)$

$= f(x^2-1)$

$= (x^2-1)^2-1$

$= x^4+1-2x^2-1$

$= x^4-2x^2$



Domain is $(-\infty, \infty)$

Range is $[-1, \infty)$

$gog(x)$

$= g(\sqrt{x})$

$= \sqrt{\sqrt{x}}$

$= x^{1/4}$

Domain is $[0, \infty)$

Range is $[0, \infty)$

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Note

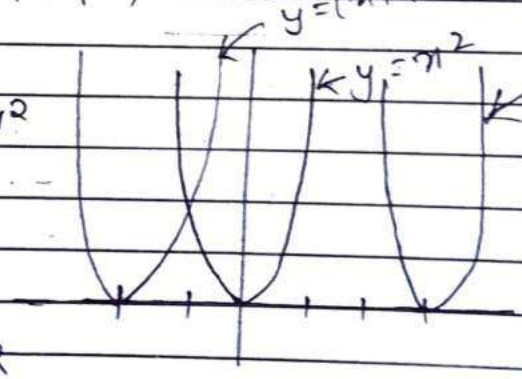
Graph and shifting of graph

① To shift the graph of function $y = f(x)$ straight up, add a positive constant to the right hand side of the formula/function $y = f(x)$.

② To shift the graph of function $y = f(x)$ straight down, add a negative constant to right hand side of function $y = f(x)$

③ To shift the graph of $y = f(x)$ to the left, add a positive constant to x.

④ To shift the graph of $y = f(x)$ to the right, add a negative constant to x.



$y = (x-h)^2 + k$

(h, k) vertex

$y = (x+2)^2$

or $y = \{x - (-2)\}^2 + 0$

$\therefore$ vertex $= (-2, 0)$

$y = (x-3)^2$

$y = \{x - (+3)\}^2 + 0$

$\therefore$ vertex $= (3, 0)$

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Scaling a Graph

Note (1) vertical and horizontal scaling:

If $c > 1$, $y = f(x)$ then (1) $y = c \cdot f(x)$

streches the graph of f vertically by a factor of c.

Cont vertical scaling

(2) If $y = \frac{1}{c} f(x)$, compress the graph of f vertically by a factor of c.
$(c > 1)$

(3) If $y = f(cx)$, compress the graph of f horizontally by a factor of c.

(4) If $y = f\left(\frac{x}{c}\right)$, streches the graph of f horizontally by a factor of c.

plot $y^2 = x$, $c = 2$

$y = \sqrt{x}$

(1) $y = 2\sqrt{x}$

(2) $y = \frac{\sqrt{x}}{2}$

(3) $y = \sqrt{2x}$

(4) $y = \sqrt{\frac{x}{2}}$

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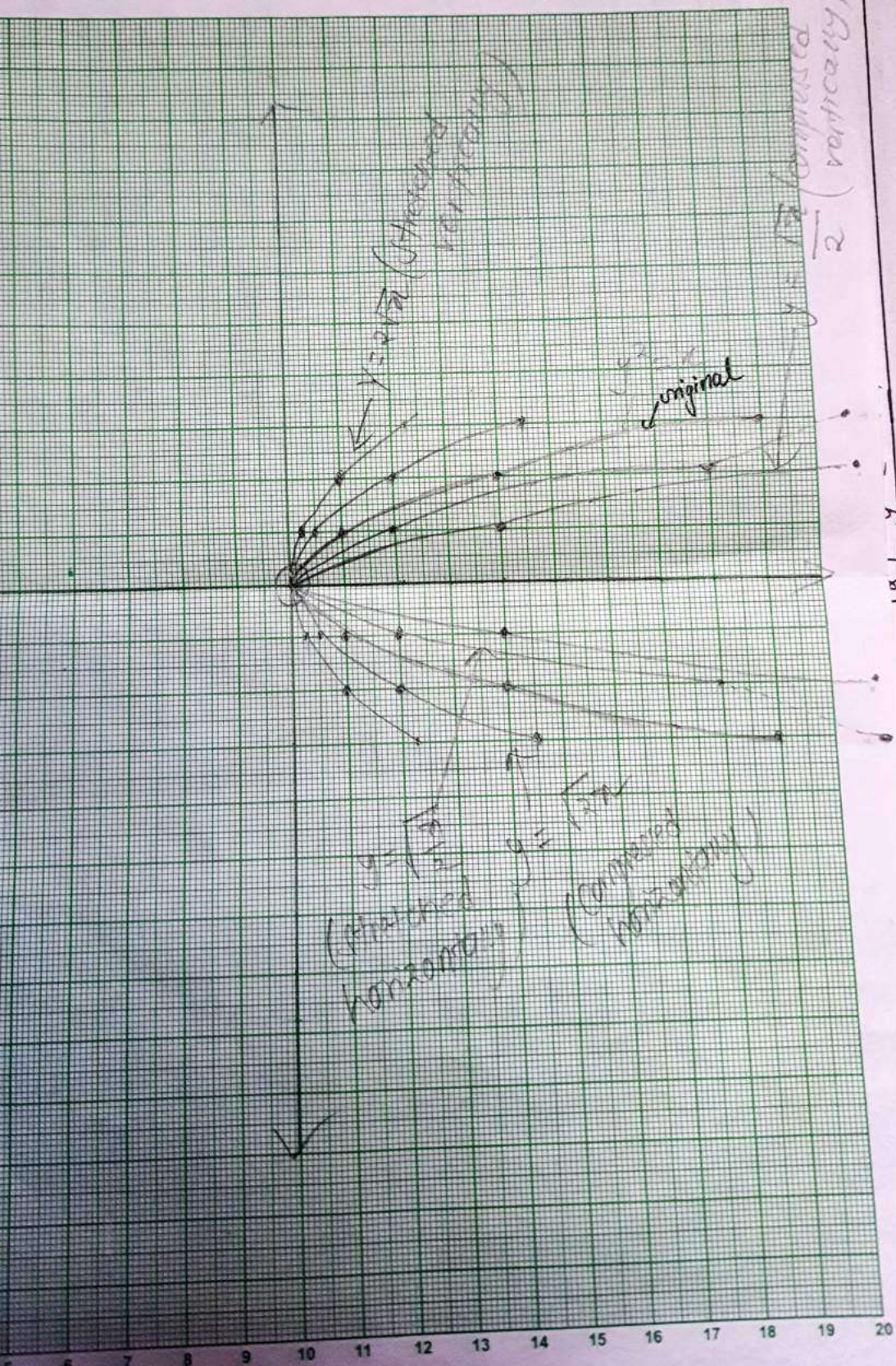

Note

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① $y = \sqrt{x}$

x	1	4	9
y	1	2	3

② $y = 2\sqrt{x}$

x	1	4	9
y	2	4	6

③ $y = \frac{\sqrt{x}}{2}$

x	1	4	9
y	0.5	1	1.5

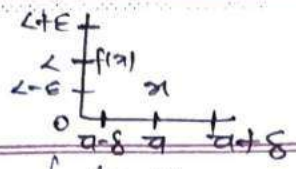
④ $y = \sqrt{2x}$

x	2	8	18
y	2	4	6

⑤ $y = \sqrt{\frac{x}{2}}$

x	2	8	18
y	1	2	3

$-|f(x)| \le f(x) \le |f(x)|$



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formal definition of limit

Note

Let $f(x)$ be defined on an open interval about 'a' except possibly at 'a' itself. We say that $f(x)$ has limit 'L' as x approaches 'a'

i.e.

$$\lim_{x \to a} f(x) = L$$

for

If every positive number, $\varepsilon > 0$, there exists a corresponding ~~ve~~ number, $\delta > 0$, such that

$$|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Sandwich theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x, in some open interval containing 'a' except at $x = a$ itself. Also, suppose that $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} h(x) = L$

then, $\lim_{x \to a} f(x) = L$

Q. If $1 - \frac{x^2}{4} \le f(x) \le 1 + \frac{x^2}{2}$. for all $x \ne 0$.

find $\lim_{x \to 0} f(x)$.

Solution

$$= \lim_{x \to 0} 1 - \frac{x^2}{4} = 1 - \frac{0}{4} = 1$$

again,

$$= \lim_{x \to 0} 1 + \frac{x^2}{2} = 1 + \frac{0}{2} = 1$$

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so given in formal limit definition question, then,

$L = f(x_0)$, → multiple functions $x \to$ 0 have $x \neq a$ type of function be $f(x)$ in $|f(x)-L| < \epsilon$

Note

Now, by sandwich theorem,

we get,

$$\lim_{x \to 0} f(x) = 1$$

Properties of limit

if $\lim_{x \to a} f(x) = L$

and $\lim_{x \to a} g(x) = M$

then,

1. $\lim_{x \to a} (f(x) + g(x)) = L + M$
2. $\lim_{x \to a} (f(x) - g(x)) = L - M$
3. $\lim_{x \to a} (f(x) \cdot g(x)) = LM$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, if $g(x) \neq 0$
5. $\lim_{x \to a} (f(x))^{\frac{m}{n}} = L^{\frac{m}{n}}$

Q. find the limit of $\lim_{x \to 3} (3x-4)$ and verify the definition.

soln

$$= \lim_{x \to 3} (3x-4)$$

$$= 3 \cdot 3 - 4$$

$$= 5$$

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$c < x - a < b$

then, $\delta = b$.

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Note

To verify,

Let $|f(x) - L| < \varepsilon$

$\Rightarrow |3x - 4 - 5| < \varepsilon$

$\Rightarrow |3x - 9| < \varepsilon$

$\Rightarrow |3(x-3)| < \varepsilon$

$\Rightarrow |3|\,|x-3| < \varepsilon$

$\Rightarrow |x-3| < \frac{\varepsilon}{3}$

$\therefore |x-3| < \delta$, where $\frac{\varepsilon}{3} = \delta$

verified

Q. Find the limit of $\lim_{x \to 1} \sqrt{5-x}$ and δ if $\varepsilon = 1$.

Soln

$\lim_{x \to 1} \sqrt{5-x}$

$= \sqrt{5-1}$

$= 2$

To verify,

Let $|f(x) - L| < \varepsilon$

or, $|\sqrt{5-x} - 2| < \varepsilon$

or, $|\sqrt{5-x} - 2| < 1$

or, ~~$5 - x + 4 - 2 \times 2 \times \sqrt{5-x} < 1$~~

or, $-1 < (\sqrt{5-x} - 2) < 1$

or, $1 < \sqrt{5-x} < 3$

or, $1 < 5 - x < 9$

or, $-4 < -x < 4$

or, $4 > x > -4$

$\Rightarrow -4 < x < 4$.

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Note

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$|x-1| < \delta$

or, $-\delta < x-1 < \delta$

or, $1-\delta < x < 1+\delta$

Comparing,

$1-\delta = -4$

$\therefore \delta = 5$

and

$1+\delta = 4$

$\delta = 3$

Least value is required value i.e. $\delta = 3$.

Right hand limit

We say that $f(x)$ has right hand limit L at x_0 and we write

$$\lim_{x \to x_0^+} f(x) = L$$

If for every positive number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$, such that for all x, $x_0 < x < x_0 + \delta$

$$\Downarrow$$

$$|f(x) - L| < \varepsilon.$$

Left hand limit

We say that $f(x)$ has left hand limit L at x_0 and we write.

$$\lim_{x \to x_0^-} f(x) = L$$

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Note

If for every number $\varepsilon > 0$, there exist a corresponding number $\delta > 0$, such that for all x,

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$$

Infinite limits

① We say that $f(x)$ approaches infinity as x approaches x_0 and we write $\lim_{x \to x_0} f(x) = +\infty$

If for every positive real number B, there exist a corresponding $\delta > 0$, such that for all x,

$$|x - x_0| < \delta \Rightarrow f(x) > B$$

② We say that $f(x)$ approaches $-\infty$ as x approaches x_0 and we write $\lim_{x \to x_0} f(x) = -\infty$.

If every negative real number $-B$, there exist a corresponding number $\delta > 0$, such that for all x,

$$|x - x_0| < \delta \Rightarrow f(x) < -B$$

Continuity

A function $f(x)$ is continuous at an interior point at $x = c$ of its domain if $\lim_{x \to c} f(x) = f(c)$

A function $f(x)$ is continuous at $x = c$ if and only if (iff) it satisfy the following three condition.

(i) $f(c)$ must be defined.

(ii) $\lim_{x \to c} f(x)$ exists.

(iii) $\lim_{x \to c} f(x) = f(c)$

Continuous extension of f(x) khanyo bhane, f(x) loi simplify garne ani certain point huda k hunxa na huda k hunchha yesari lekhne $f(x) = \begin{cases} --- & x \neq --- \\ --- & x = --- \end{cases}$ L.H.L

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Continuous at left end point

Note A function f(x) is continuous at left endpoint $x = a$ of its domain if $\lim_{x \to a^+} f(x) = f(a)$

Continuous at right end point → L.H.L

A function f(x) is continuous at right end point $x = b$ of its domain if $\lim_{x \to b^-} f(x) = f(b)$.

Properties of continuity

(1) If functions f(x) and g(x) are continuous at $x = c$, then the following functions are also continuous at $x = c$.

(i) $f(x) + g(x)$
(ii) $f(x) - g(x)$
(iii) $f(x) \cdot g(x)$
(iv) $c \cdot f(x)$ where c is any constant
(v) $\frac{f(x)}{g(x)}$ if $g(c) \neq 0$
(vi) $(f(x))^{m/n}$

(2) Every polynomial function is continuous at every point of a real line (all real numbers).

(3) Every rational function is continuous at every point where denominator is different from zero.

$\frac{2x^2+4}{x-2}$ continuous if $x \neq 2$.

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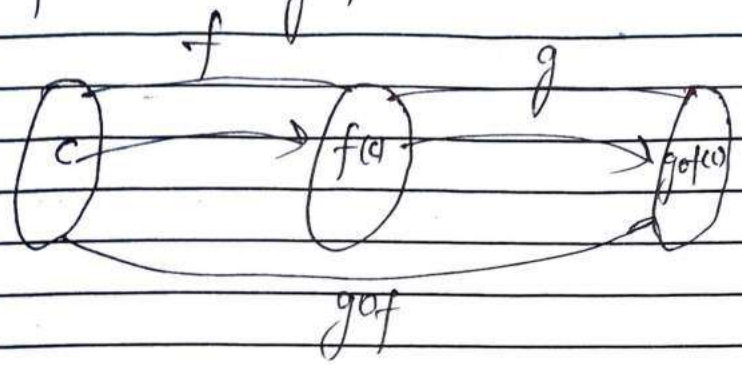
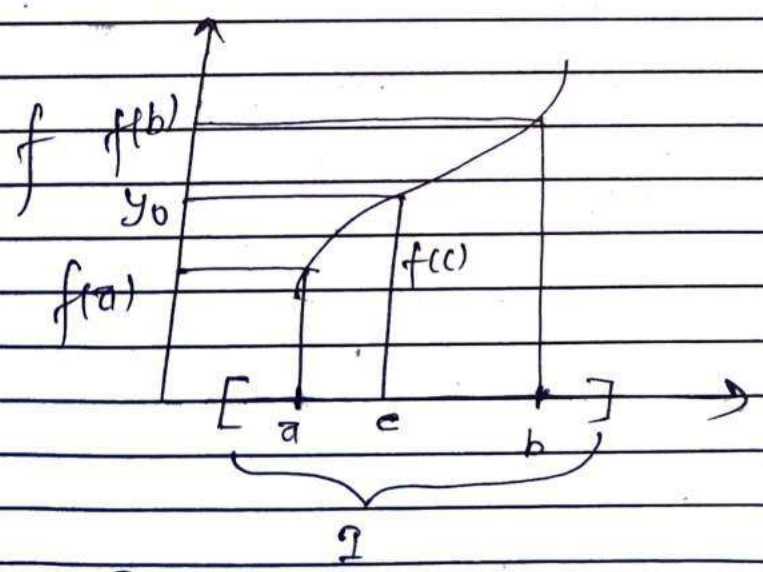
Note

(4) If 'f' is continuous at c and g is continuous at f(c) then gof is continuous at $x = c$.



(5) Suppose $f(x)$ is continuous at an interval 'I'. a and b are any two points on I. Then, if y_0 be any number between $f(a)$ and $f(b)$ then there exists a number 'c' between a and b such that $f(c) = y_0$.



It is also called intermediate value theorem.

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Types of discontinuity

Note

① Ordinary or Jump discontinuity

If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then $f(x)$ is said to be an ordinary or jump discontinuity at $x = a$.

eg: (i) $f x = \begin{cases} 3x+2 & x \geq 2 \\ 4x-9 & x < 2 \end{cases}$ at $x = 2$

(ii) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ at $x = 0$

(i) soln

for left hand limit,

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x-9) = 4 \times 2 - 9 = -1$

for right hand limit,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x+2) = 3 \times 2 + 2 = 8$

$\therefore$ L.H.L $\neq$ R.H.L, Hence discontinuous

(ii) soln

$$L.H.L = \lim_{x \to 0^-} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim_{x \to 0^-} \frac{e^{1/x}\left(1 - \frac{1}{e^{1/x}}\right)}{e^{1/x}\left(1 + \frac{1}{e^{1/x}}\right)}$$

$e^{\infty} = \infty$

$e = 2.72$

$\frac{1}{\infty} = 0$ $\quad \frac{1}{0} = \infty$

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$= \frac{1-0}{1+0} = 1$

Note

$R.H.L = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{1/x}-1}{e^{1/x}+1} = \lim_{x \to 0^+} \frac{e^{1/x}(1-\frac{1}{e^{1/x}})}{e^{1/x}(1+\frac{1}{e^{1/x}})}$

$= \frac{1-0}{1+0}$

$= 1$

$L.H.L = \lim_{x \to 0^-} \frac{e^{1/x}-1}{e^{1/x}+1}$

$= \lim_{x \to 0^-} \frac{\frac{1}{e^{-1/x}} - 1}{\frac{1}{e^{-1/x}} + 1}$

$= \frac{0-1}{0+1} = -1$

$\therefore L.H.L \neq R.H.L$, Hence discontinuous

(2) Infinite discontinuity

If one or both $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ tends to $+\infty$ or $-\infty$, then $f(x)$ is said to be infinite discontinuity

eg:- $f(x) = \frac{1}{x}$ at $x = 0$

$f(x) = e^{\frac{1}{x-a}}$ at $x = a$.

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Note

(3) Removable discontinuity

If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, but limiting value does not equals to $f(a)$ or $f(a)$ is not defined then $f(x)$ is said to be removal discontinuity at $x = a$.

eg:- $f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$.

$$f(x) = \begin{cases} 2x + 5 & < 1 \\ 5 & = 1 \\ 3x + 4 & > 1 \end{cases}$$

Soln

$L.H.L = \lim_{x \to 1^-} 2x + 5 = 7$

$R.H.L = \lim_{x \to 1^+} 3x + 4 = 7$

$f(1) = 5$

so,

not continuous.

To make continuous

$$f(x) = \begin{cases} 2x + 5 & < 1 \\ 7 & = 1 \\ 3x + 4 & > 1 \end{cases}$$

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Note

(c) Oscillating discontinuity

The discontinuity which is neither ordinary nor infinite discontinuity is called oscillating discontinuity. In such discontinuity, function may oscillate finitely or infinitely.

eg:- $\sin\frac{1}{x}$ at $x = 0$

$\tan\frac{1}{x}$ at $x = 0$

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$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

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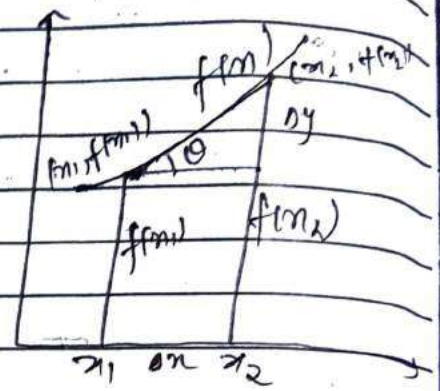
Average rate of change

Note

The average rate of change $y = f(x)$ w.r.t x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Q. find the average rate of change of following functions on given interval.

⇒ $f(x) = x^3 + 1$ $[-1, 1]$

⇒ $f(x) = \cot x$ $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

$f(1) = 1 + 1 = 2$

$f(-1) = -1 + 1 = 0$

So,

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{2 - 0}{1 - (-1)}$$

$$= 1$$

$$\frac{\Delta y}{\Delta x} = \frac{\cot\left(\frac{\pi}{2}\right) - \cot\left(\frac{\pi}{6}\right)}{\frac{\pi}{2} - \frac{\pi}{6}}$$

$$= \frac{0 - \sqrt{3}}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{-\sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$$

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$\lim_{x \to 0} \frac{\sin x}{x} = 1$, $\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

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Note

The average rate of change of f over closed interval $[x_1, x_2]$ is the slope of line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$. We know that the line passing through two points of the curve is secant to the curve, therefore the average rate of change of f from x_1 to x_2 identical with slope of secant.

Increment

An increment in a variable is net change in the variable. If x changes from x_1 to x_2, then increment in x is denoted by Δx i.e. $\Delta x = x_2 - x_1$.

Q. When a point A(1,2) moves to point B(3,5). Find the increment in x and y.

Soln

$\Delta x = x_2 - x_1 = 3 - 1 = 2$

$\Delta y = y_2 - y_1 = 5 - 2 = 3$

Derivative

The derivative of the function f(x) with respect to the variable 'x' is the function f'(x) whose value at x is

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$ if the limit exist.

Right hand derivative.

The right hand derivative of a function f(x) at x = a is

$$\boxed{f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}}$$

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$\frac{dx^n}{dx} = nx^{n-1}$

$\frac{dax^n}{dy} = anx^{n-1}$

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Note

Left hand limit

The left hand limit of a function $f(x)$ at $x = b$ is

$$\boxed{f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}}$$

Derivative exist only when L.H.D = R.H.Deriv.

Q. find $\frac{dy}{dx}$.

(1) $y = x^2 + 3x + 5$

$\therefore \frac{dy}{dx} = 2x + 3$

(2) $y = a^x$

$\therefore \frac{dy}{dx} = a^x \log a$

(3) $y = \log_a x$

$y = \log_a e \log_e x$

$\frac{dy}{dx} = \frac{\log_a e}{x}$

(4) $y = \sin^2 x$

$\frac{dy}{dx} = 2\sin x \cos x = \sin 2x$

(5) $y = \frac{\sec x + \tan x}{\sec x - \tan x}$

$y = (\sec x + \tan x)$

(6) $y = \log(x + \sqrt{x^2 + a^2})$

(7) $y = e^x \log x$

(8) $(ax^2 + bx + c)^{3/4} = y$

If $x = 2t + 5$ and $y = \sin t^2$. find $\frac{dy}{dx}$.

$\frac{dy}{dx} = dy/dt \,/\, dx/dt$

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Note

Implicit differentiation.

(i) $x^2+y^2+2xy=0$

differentiating on both sides,

or, $2x+\frac{dy^2}{dy}\frac{dy}{dx}+2y=0$

or, $2x+2y\frac{dy}{dx}+2y=0$

or, $\frac{dy}{dx}=\frac{-(2x+2y)}{2y}$

or, $2x+2y\frac{dy}{dx}+2\left[x\frac{dy}{dx}+y\frac{dx}{dx}\right]=0$

or, $2x+2y\frac{dy}{dx}+2\left[x\frac{dy}{dx}+y\right]=0$

or, $2x+2y\frac{dy}{dx}+2x\frac{dy}{dx}+2y=0$

or, $2x+2y+\frac{dy}{dx}(2x+2y)=0$

$\therefore \frac{dy}{dx}=\frac{-(2x+2y)}{2x+2y}=-1$

(ii) $x^y=y^x$

Taking log on both sides,

or, $y\log x=x\log y$

diff. on both sides,

or, $\frac{y}{x}+\log x\frac{dy}{dx}=\log y+\frac{x}{y}\frac{dy}{dx}$

or, $\frac{dy}{dx}\left(\log x-\frac{x}{y}\right)=\log y-\frac{y}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{\frac{x\log y-y}{x}}{\frac{y\log x-x}{y}}$

$\frac{dy}{dx}=\frac{xy\log y-y^2}{xy\log x-x^2}$

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If $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ = ∞ ($f(x)$ grows faster)
= 0 ($g(x)$ grows faster)
= some finite value (grows at equal rate)

(iii) $x^3 y^6 = (x+y)^4$

Note

Taking log on bothsides,

or $\log(x^3 y^6) = \log(x+y)^4$

or $3\log x + 6\log y = 4\log(x+y)$

or, $\frac{3}{x} + \frac{6}{y}\frac{dy}{dx} = 4$

Average rate of change

The average rate of change of function $f(x)$ w.r.t. x over the interval from x_0 to (x_0+h) is,

$$\text{Avg. r.o.c} = \frac{f(x_0+h) - f(x_0)}{h}$$

Instantaneous rate of change

The instantaneous rate of change of 'f' or $f(x)$ w.r.t. 'x' at $x = x_0$ is the derivative at x_0 i.e.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

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In numerical, displacement = $s(t_2) - s(t_1)$

avg. velocity = $\dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$

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Velocity

Note The velocity is the derivative of the position function $s = f(t)$ w.r.t. time 't'. At time 't', the velocity

$$v = \frac{ds}{dt} = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

Acceleration

It is the derivative of velocity w.r.t time 't'. If a body's position at time 't' is $s = f(t)$ then the body's accel^n at time 't' is $a = \dfrac{dv}{dt} = \dfrac{d^2 s}{dt^2}$

Q. Find the velocity and accel^n of following functions at given point/time.

(1) $s(t) = 2t^2 + 7$ at $t = 5$

$v = s'(t) = 4t$

$= 4 \times 5$ (at $t = 5$)

$= 20$ m/s

$a = s''(t) = 4$ m/s^2

(2) $s(t) = \sin^2 t$ at $t = \frac{\pi}{2}$

velocity $(v) = \dfrac{d\,s(t)}{dt} = 2\sin t \cos t = \sin 2t$

$= \sin 2 \times \frac{\pi}{2}$

$= 0$ m/s

$a = s''(t) = 2\cos 2t = 2\cos 2 \cdot \frac{\pi}{2} = -2$ m/s^2

$= 0$ (no velocity)

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Derivative of inverse function

Note To define the derivative of inverse functional a function $f(x)$, firstly, we find inverse function of given function $f(x)$ and then we find its derivative.

Theorem

If 'f' is differentiable at every point of an interval 'I' and $\frac{df}{d(x)}$ or f' is never zero on 'I'.

Then, f^{-1} is differentiable at every point of the interval $f(I)$. Then the value of $\frac{df^{-1}}{dx}$ at a particular point $f(a)$ is the reciprocal at the value of $\frac{df}{dx}$ at $x=a$

i.e.

$$\left(\frac{df^{-1}}{dx}\right)_{at\ x=f(a)} = \frac{1}{\left(\frac{df}{dx}\right)_{at\ x=a}}$$

Q. Let $f(x) = x^3-2$. find the value of $\frac{df^{-1}}{dx}$ at $x=2$ using theorem and general method.

Soln

$$f(x) = x^3-2$$

Let $f(x) = y$

$$y = x^3-2$$

$$x^3 = y+2$$

$$x = \sqrt[3]{y+2}$$

Hence, $f^{-1}(x) = (x+2)^{1/3}$

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Note

now,

$$\frac{df^{-1}(x)}{dx} = \frac{1}{3}(x+2)^{-2/3}$$

$$= \frac{1}{3(x+2)^{2/3}}$$

At $x = f(2)$

$= 2^3 - 2$

$= 6$

now,

$$= \frac{1}{3(6+2)^{2/3}}$$

$$= \frac{1}{3} \cdot 2^{3 \cdot \frac{2}{3}}$$

$$= \frac{1}{12}$$

By theorem,

$$\frac{df^{-1}}{dx} = \frac{1}{\frac{df}{dx} \text{ at } x=2}$$

$$\frac{df}{dx} = 3x^2 \text{ , at } x = 2$$

$$\frac{df}{dx} = 3 \times 2^2 = 12$$

$$\therefore \frac{df^{-1}}{dx} = \frac{1}{12}$$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

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~~Derivative of Inverse circular function~~

Note Theorem: If function $f(x)$ has derivative at $x = c$ then $f(x)$ is continuous at $x = c$ but converse may not be true

Proof:

Let $f'(x)$ exist at $x = c$ i.e. $f'(c)$ exist.
Now, we have to prove $f(x)$ is continuous at $x = c$.

i.e. $\lim_{x \to c} f(x) = f(c)$

or, $\lim_{h \to 0} f(x+h) = f(c)$

Now,

$$f(c+h) = f(c+h)$$
$$= f(c) + f(c+h) - f(c)$$
$$= f(c) + \frac{f(c+h) - f(c)}{h} \times h$$

Taking limit as $h \xrightarrow{\lim} 0$,

or $\lim_{h \to 0} f(c+h) = f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \times h$

or, $\lim_{h \to 0} f(c+h) = f(c) + f'(c) \times 0$

$\therefore \lim_{h \to 0} f(c+h) = f(c)$

now,

let $f(x) = |x|$

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

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Note

now, at $x = 0$.

$$R.H.L = \lim_{x \to 0^+} f(x)$$
$$= x$$
$$= 0$$

$$L.H.L = \lim_{x \to 0^-} f(x)$$
$$= -x$$
$$= -0$$
$$= 0$$

$$R.H.L = L.H.L$$

at $x = 0$, $f(0) = 0$

$\therefore$ The function $f(x)$ is continuous at $x = 0$.

Again, (at $x = 0$)

$$R.H.D = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0^+} \frac{0 + h - 0}{h}$$
$$= 1$$

$$L.H.D = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0^-} \frac{-(0+h) - 0}{h}$$
$$= \lim_{h \to 0^-} \frac{-h}{h}$$
$$= -1$$

$f(x)$ is continuous but $R.H.D \neq L.H.D$, hence Derivative doesn't exist.

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Horizontal tangent $= \frac{dy}{dx} = 0$

vertical tangent $= \frac{dx}{dy} = 0$

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Increasing function

Note

A function $y = f(x)$ is said to be increasing in the interval (a,b), if for every x_1, x_2 belongs to (a,b), then $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

$$\text{i.e. } \frac{dy}{dx} > 0$$

Decreasing function

A function $y = f(x)$ is said to be decreasing in the interval (a,b), if for every $x_1, x_2 \in (a,b)$, then $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

$$\text{i.e. } \frac{dy}{dx} < 0$$

Q. Examine whether the following functions are increasing or decreasing at given point.

① $f(x) = x^3 + 3x^2 - 100$ at $x = 1$ and $x = -2$.

Soln

$$f'(x) = \frac{df(x)}{dx}$$
$$= 3x^2 + 6x$$

at $x = 1$, $f'(x) = 3 + 6 = 9$ (Increasing)

at $x = -2$, $f'(x) = 3 \times 4 - 6 \times 2$
$= 0$ (Stationary)

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Note

Q. find the interval in which the following functions are increasing or decreasing.

(1) $f(x) = x^2 - 6x + 5$

(2) $f(x) = 5x^3 - 135x + 22$

Soln

(1) $f'(x) = 2x - 6$
$= 2(x-3)$

$\therefore$ $f(x)$ is increasing if $x > 3$ i.e $(3, \infty)$

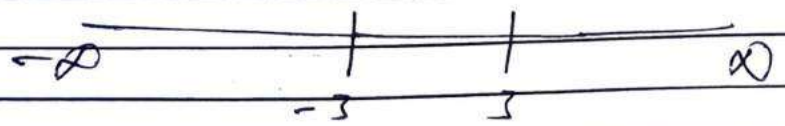
$\therefore$ $f(x)$ is decreasing if $x < 3$ i.e $(-\infty, 3)$.

(2) $f'(x) = 15x^2 - 135$
$= 15(x^2 - 9)$
$= 15(x-3)(x+3)$

~~$\therefore$ $f(x)$ is decreasing if $x > 3$ i.e. $(3, \infty)$~~

~~$\therefore$ $f(x)$ is decreasing if $x < 3$ i.e. $(-\infty, 3)$~~
~~$0 < x < 3$~~

for critical point, $f'(x) = 0 \Rightarrow x = 3$ or -3



$\therefore$ $f(x)$ is increasing on $(-\infty, -3) \cup (3, \infty)$

$\therefore$ $f(x)$ is decreasing on $(-3, 3)$

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Critical or stationary point

Note An interior point of the domain of a function, where f' is zero or undefined is called critical point of f.

Q. find critical point of $f(x) = x^3 + 2x^2 + 1$

$f(x) = \frac{1}{x}$

Soln

$f'(x) = 3x^2 + 4x$

for critical point, $3x^2 + 4x = 0$

or $x(3x + 4) = 0$

Either

$x = 0$

or $x = -\frac{4}{3}$

$f'(x) = \frac{-1}{x^2}$

for critical point, or $\frac{-1}{x^2} = \infty \left(\frac{1}{0}\right)$

or $\frac{1}{x^2} = \infty \left(\frac{1}{0}\right)$

or $x^2 = 0$

$\therefore x = 0$

Concavity

The graph of differentiable function $y = f(x)$ is concave up on an interval where $f'(x)$ is increasing and concave down on an interval where $f'(x)$ is decreasing.

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Note

The second derivative test for concavity.

If $y = f(x)$ be twice differentiable on interval 'I' then, $f''(x) > 0 \Rightarrow$ graph is concave up.

$f''(x) < 0 \Rightarrow$ graph is concave down.

Q. Determine where the graph is concave upward and concave downward of the following functions.

(1) $f(x) = x^4 - 2x^3 + 5$

soln

$f'(x) = 4x^3 - 6x^2$

$f''(x) = 12x^2 - 12x$

$= 12x(x-1)$

$(-\infty, 0)$ $(0, 1)$ $(1, \infty)$

$-\infty$ 0 1 ∞

for point of inflection,

$12x(x-1) = 0$

Either $x = 0$ or $x = 1$

$\therefore f(x)$ is increasing on, $(-\infty, 0) \cup (1, \infty)$

Concave up on $(-\infty, 0) \cup (1, \infty)$

Concave down on $(0, 1)$.

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$\log_a x = \frac{\log_b x}{\log_b a}$; b is desired base of log.

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L-hospital's rule → for $\frac{0}{0}, \frac{\infty}{\infty}$

Note Indeterminate form

(i) $\infty - \infty$
(ii) $0 \times \infty$
(iii) $0^{\infty}, 0^{0}, 1^{\infty}$

for $\infty - \infty$,

If $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = \infty$, $\lim_{x \to 0} f(x) - g(x)$ takes form $\infty - \infty$.

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x) \cdot g(x)}}$$

Example:

$$\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x \to 0} \frac{\sin x - x}{x \sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

= using L-Hospital's rule,

$$= \lim_{x \to 0} \frac{\cos x - 1}{x \cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{-\sin x}{x(-\sin x) + \cos x + \sin x}$$

$$= \frac{-0}{1}$$

$$= 0$$

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Note

for $0 \times \infty$,

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$

then $\lim_{x \to a} f(x) \cdot g(x)$

$$= \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}}$$

example, $\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \tan x$

$$= \lim_{x \to \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cot x} \left(\frac{0}{0} \text{ form}\right)$$

= Using L-Hospital's rule,

$$= \lim_{x \to \frac{\pi}{2}} \frac{1 - 0}{-\operatorname{cosec}^2 x}$$

$$= \frac{1}{-1} = -1$$

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for (iii) $0^{\infty}, 0^{0}, 1^{\infty}$

Note

for 0^0, If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

then, $\lim_{x \to a} (f(x))^{g(x)}$ takes form 0^0.

example:

$$= \lim_{x \to 0} (\sin x)^{2\tan x}$$

Let $y = (\sin x)^{2\tan x}$

or $\log y = 2\tan x \log \sin x$

or $\log y = \dfrac{2\log \sin x}{\cot x}$

Taking limit as $x \to 0$,

$$\lim_{x \to 0} \log y = \lim_{x \to 0} \frac{2\log \sin x}{\cot x}$$

$$= \lim_{x \to 0} \frac{2 \cdot \frac{1}{\sin x} \cos x}{-\operatorname{cosec}^2 x}$$

$$= -2 \times \frac{\cos x}{\cancel{\sin x}} \times \sin^2 x$$

$$= -2\cos x \sin x$$

$$= \lim_{x \to 0} -\sin 2x$$

$$\lim_{x \to 0} \log y = 0$$

$\lim_{x \to 0} y = e^0 = 1$, Hence, $\lim_{x \to 0} (\sin x)^{2\tan x} = 1$

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∞ can never be taken as extreme value because it is not defined.

for a right angled triangle, if hypotenuse is fix, area will be max^m if base = perpendicular

Extreme value of functions

Note

Global

Absolute extreme Value of function

Let f be a function with domain 'D' then, f has an absolute maximum value on 'D' at a point c if $f(x) \le f(c)$ for all $x \in D$, and an absolute minimum value at c' if $f(x) \ge f(c') \; \forall x \in D$.

Examples

(i) If $y = f(x) = x^2$, Domain $= (-\infty, \infty)$

Absolute min $= 0$

There is no absolute maxima. (∞ isn't defined)

(ii) $y = f(x) = x^2$, $D = (0, 5]$

Absolute min doesn't exist because we dont know what is near to zero.

Absolute max $= 25$

(iii) $y = x^2$, $D = [0, 5]$

A. Min $= 0$

A. Max $= 25$

(iv) $y = f(x) = x^2$, $D = (0, \infty)$

no absolute minima and maxima.

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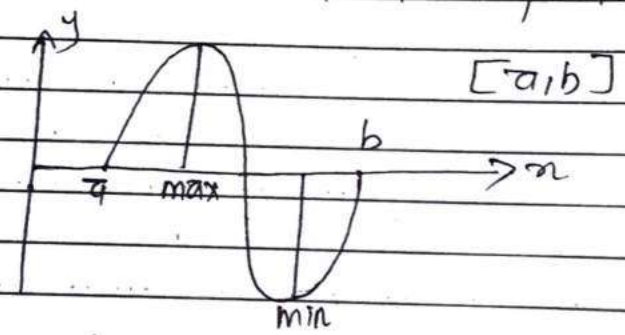
Max - Min theorem

Note

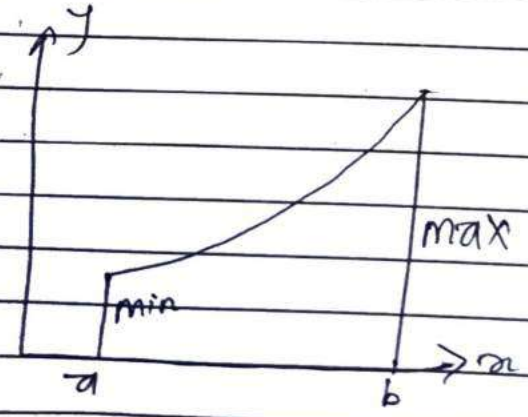
If function f is continuous at every point of a closed interval 'I', then there exist an absolute maximum value M and an absolute minimum value m, i.e. there are two points x_1 and x_2 in 'I' such that $f(x_1) = M$ and $f(x_2) = m$, then we can say,

~~$f(x) \leq M$, $f(x) \geq$~~ $m \leq f(x) \leq M$ for $x \in I$.

eg: Max and min at interior points.

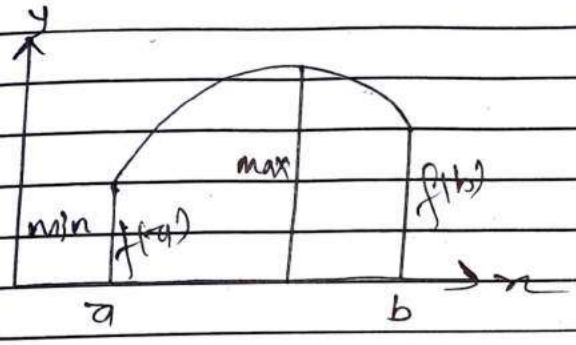


eg: Max and min at end points

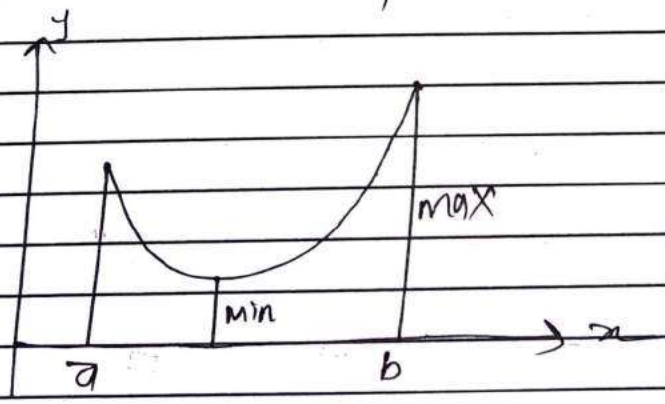


Note

eg: Max at interior point and min at end point.



eg: Min. at interior point and max at end point.



Q. find the absolute max. and absolute min. of $y = x^2+2x+5$ on $[-2,3]$.

Soln

$$f(x) = x^2+2x+5$$
$$f'(x) = 2x+2$$

for critical or stationary point,

$$2x+2 = 0$$
$$x = -1$$

now, $f(-1) = 1-2+5 = 4$

$f(-2) = 4-4+5 = 5$

$f(3) = 9+6+5 = 20$

Ab max $= 20$ at $x = 3$

Ab min $= 4$ at $x = -1$

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Note

Q. $f(x) = \cos x$ at $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Soln

$f'(x) = -\sin x$

for critical points

$f'(x) = 0$

$-\sin x = 0$

$x = 0,$

$f(0) = \cos 0 = 1$

$f\left(-\frac{\pi}{2}\right) = \cos 90 = 0$

$f\left(\frac{\pi}{2}\right) = \cos 90 = 0$

Ab. min $= 0$ at $x = -\frac{\pi}{2}, \frac{\pi}{2}$

Ab. max $= 1$ at $x = 0$

Q. $f(x) = \sin x$ at $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Soln

$f'(x) = \cos x$

$\cos x = 0$

$\therefore x = \frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$

$f\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

ab. min $= -1$ at $x = -\frac{\pi}{2}$

ab. min $= 1$ at $x = \frac{\pi}{2}$

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Local extreme values of function

Note

A function f has a local maximum value at an interior point 'c' of its domain if $f(x) \le f(c)$ for all x in some open interval containing 'c'. A function f has local minima at an interior point 'c' of its domain if $f(x) \ge f(c')$ for all x in some open interval containing 'c'.

Derivative test for local extreme values

→ If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has local max at $x = c$.

→ If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has local min at $x = c$.

Q. find the local maxima and minima of following functions

(i) $f(x) = x^4 - 4x^3 + 4x^2$

Soln

$f'(x) = 4x^3 - 12x^2 + 8x$

$f''(x) = 12x^2 - 24x + 8$

for critical point,

$f'(x) = 0$

or, $4x(x^2 - 3x + 2) = 0$

or, $4x\{(x-2)(x-1)\} = 0$

Either, $4x = 0$

$\Rightarrow x = 0$

OR $(x-1) = 0$ OR, $(x-2) = 0$

$\Rightarrow x = 1$ $\Rightarrow x = 2$

Note

Now, At $x = 0$,

$f''(0) = 0 - 0 + 8 = 8 = +ve$

$f(x)$ has local min. at $x = 0$

$f_{min} = 0^4 - 4.0^3 + 4.0^2$
$= 0$

At $x = 1$,

$f''(1) = 12 - 24 + 8$
$= -4$

$f(x)$ has local max. at $x = 1$,

$f_{max} = 1 - 4.1 + 4.1$
$= 1$

At $x = 2$,

$f''(2) = 48 - 48 + 8$
$= 8$

$\therefore f(x)$ has min at $x = 2$.

$f_{min} = 2^4 - 4.2^3 + 4.2^2$
$= 16 - 32 + 16$
$= -32$

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The nth derivative test for extreme value

Note

Let 'c' be a point in the interval in which the function $f(x)$ is defined and if $f'(c) = f''(c) = f'''(c) \ldots = f^{n-1}(c) = 0$ but $f^{n}(c) \neq 0$, then,

(i) $f(x)$ has neither max nor min at $x = c$ if n is odd.
(ii) $f(x)$ has max at $x = c$ if n is even and $f^{n}(c) < 0$.
(iii) $f(x)$ has min at $x = c$, if n is even and $f^{n}(c) > 0$.

eg: $f(x) = x^6$

soln

$f'(x) = 6x^5$

$f''(x) = 30x^4$

for critical point, $6x^5 = 0$

$\therefore x = 0$

$f''(0) = 80 \times 0^4 = 0$

now, $f'''(x) = 120x^3$

$f'''(0) = 0$

4th $\rightarrow f''''(x) = 360x^2$

$f''''(0) = 0$

5th $\rightarrow f'''''(x) = 720x$

$f'''''(0) = 0$

6th $\rightarrow f''''''(x) = 720$

$f''''''(0) = 720 \neq 0$

720 > 0 & 6 is even, so, $f(x)$ has minimum at $x = 0$,

$f_{min} = 0^6 = 0$

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Note

eg: $f(x) = -2x^5$

Soln

$f'(x) = -10x^4$

$f''(x) = -40x^3$

for C.P, $-10x^4 = 0$

$\therefore x = 0$

$f''(0) = 0$

$f'''(x) = -120x^2$

$f'''(0) = 0$

$f^{iv}(x) = -240x$

$f^{iv}(0) = 0$

$f^{v}(x) = -240$

$f^{v}(0) = -240 \neq 0$

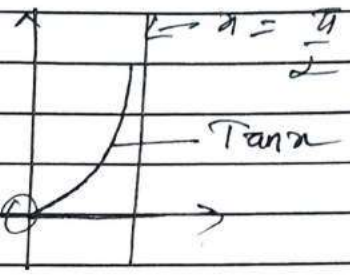
Here v is odd and $f^{v}(0) < 0$.

So, $f(x)$ has neither maximum nor minimum value.

Asymptotes

Note A straight line is said to be an asymptote to a given curve if the perpendicular distance from a point $p(x, y)$ in the curve to the straight line tends to 0 when x or y or both tends to infinity.

line $x = \frac{\pi}{2}$ is asymptote for $\tan x$.

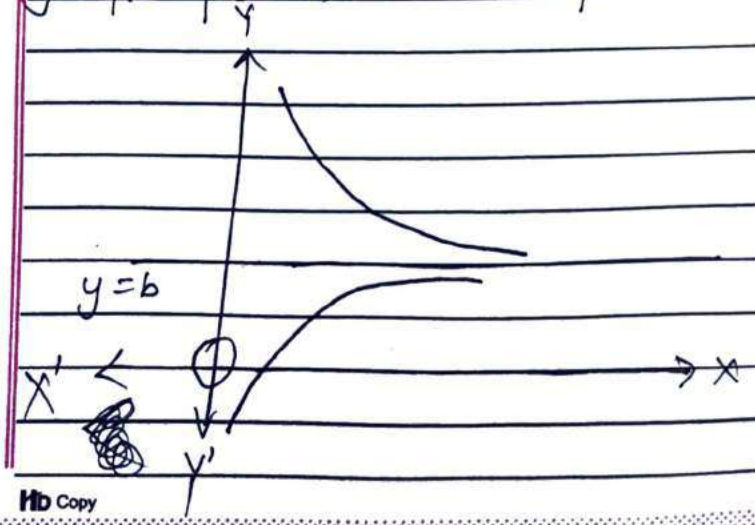


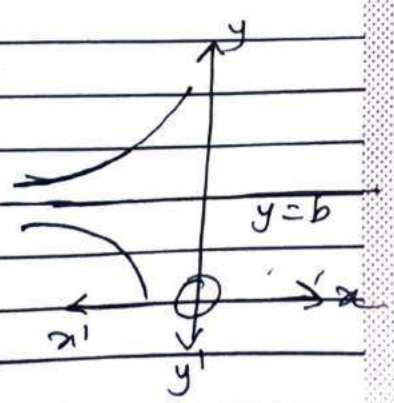
> Types of asymptotes of algebric curve.

(a) Parallel to x-axis (horizontal asymptotes)
(b) Parallel to y-axis (vertical asymptotes)
(c) Parallel to neither axis (oblique asymptotes)

Horizontal asymptotes

A line $y = b$ is a horizontal asymptote of the function $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$





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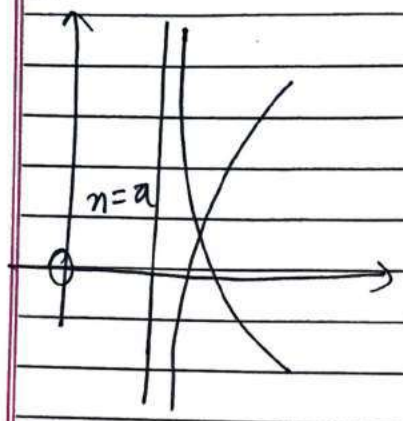
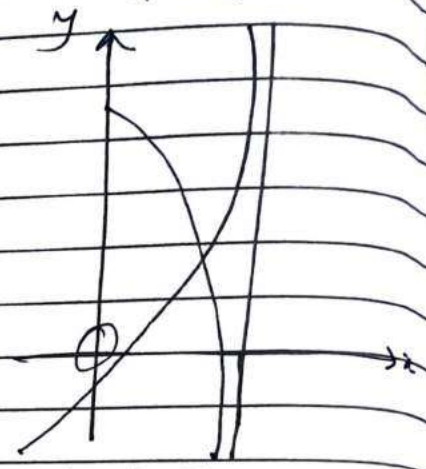
Note

Vertical asymptotes

A line $x = a$ is a vertical asymptote of the function $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$



$x \to a^+$



$x \to a^-$

Oblique asymptotes

A line $y = mx + c$ is said to be oblique asymptote if it is neither parallel to x-axis nor parallel to y-axis.

Equation of asymptotes, $f(x) = \frac{h(x)}{g(x)}$

① In the general, function $f(x) = \frac{h(x)}{g(x)}$ has asymptotes at $g(x) = 0$

eg:-

If $f(x) = \frac{x^2 + 5x + 10}{x - 1}$, then $x - 1 = 0$ is an asymptote

② If degree of $h(x)$ = degree of $g(x)$, then $f(x)$ can be expressed as $f(x) = K_1 + \frac{K_2}{g(x)}$.

Then, $f(x) = K_1 \Rightarrow y = K_1$ is an asymptote

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degree = heighest power of variable.

Page:

Note

example :-

If $f(x) = \frac{x+1}{x-1}$

$= \frac{x-1+2}{x-1}$

$= \frac{x-1}{x-1} + \frac{2}{x-1}$

$= 1 + \frac{2}{x-1}$

$\therefore f(x) = 1$ or $y = 1$ is an asymptote

(3) If degree of $h(x)$ = degree of $g(x)$ + 1, then $f(x)$ can be expressed as $f(x) = mx + c + \frac{k}{g(x)}$ then,

$f(x) = mx + c$ or, $y = mx + c$ is an asymptote

example :

$f(x) = \frac{x^2+2x+5}{x+1}$

$= \frac{x^2+x+x+1+4}{x+1}$

$= \frac{x(x+1)+1(x+1)+4}{x+1}$

$= \frac{(x+1)(x+1)+4}{(x+1)}$

$$\begin{array}{r|l} x+1 & x^2+2x+5 \quad (x+1 \\ & x^2+x \\ \hline & x+5 \\ & x+1 \\ \hline & 4. \end{array}$$

So, $x^2+2x+5 = (x+1)(x+1)+4$

$f(x) = x+1 + \frac{4}{x+1}$

$\therefore f(x) = x+1 \Rightarrow y = x+1$ is an asymptote

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Note

(4) If degree of $h(x)$ is greater than degree of $g(x)+1$ i.e.

degree of $h(x)$ > degree of $g(x)+1$,

then there is no asymptote.

Example: find the asymptotes and draw the graph.

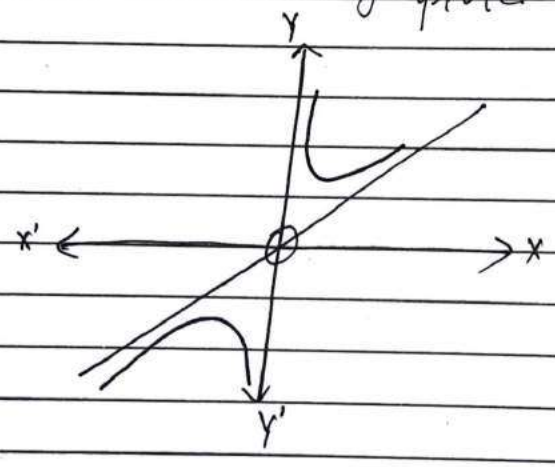
$$f(x) = \frac{x^2+1}{x}$$

dom

Here, $x=0$ is an asymptote.

Also,

$f(x) = x + \frac{1}{x}$, then $f(x) = x$ or $y = x$ is another asymptote.



Graphing the function (curve tracing)

Check list for graphing the function.

1) Check for symmetry (even or odd)
2) Check the asymptotes
3) Compute $f'(x)$ and solve for $f'(x) = 0$ or $f'(x) = \frac{1}{0}$ and find the interval where $f(x)$ is increasing or decreasing

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Note

4) Compute $f''(x)$ and concavity.
5) Evaluate $f(x)$ or y at special points (end points, critical points, point of inflection)
6) Graph $f(x)$ using above information.

example:

Q. $f(x) = x^{2/3}\left(\frac{5}{2} - x\right)$, Draw the graph.

Soln

Here,
$$f(-x) = (-x)^{2/3}\left(\frac{5}{2} - (-x)\right)$$
$$= x^{2/3}\left(\frac{5}{2} + x\right)$$
The given function is not symmetric.

Also, there is no asymptote.

$$f(x) = \frac{5}{2}x^{2/3} - x^{5/3}$$
$$f'(x) = \frac{5}{2} \cdot \frac{2}{3} \cdot x^{-1/3} - \frac{5}{3}x^{2/3}$$
$$= \frac{5}{3}x^{-1/3} - \frac{5}{3}x^{2/3}$$
$$= \frac{5}{3}\left(\frac{1}{x^{1/3}} - x^{2/3}\right)$$
$$= \frac{5}{3}\left(\frac{1-x}{x^{1/3}}\right)$$
for critical points,

$f'(x) = 0$,
$x = 1$ or, $f'(x) = \frac{1}{0}$, so, $x = 0$.

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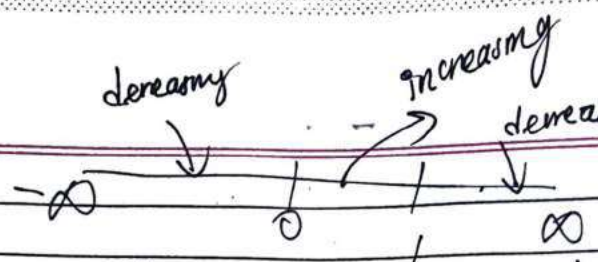


Note

Again, $f''(x) = \frac{5}{3}(x^{-1/3} - x^{2/3})$

$$f''(x) = \frac{5}{3}\left(-\frac{1}{3}x^{-4/3} - \frac{2}{3}x^{-1/3}\right)$$

$$= -\frac{5}{9}\left(x^{-4/3} + 2x^{-1/3}\right)$$

$$= \frac{-5}{9}\left(\frac{1}{x^{4/3}} + \frac{2}{x^{1/3}}\right)$$

for point of inflection,

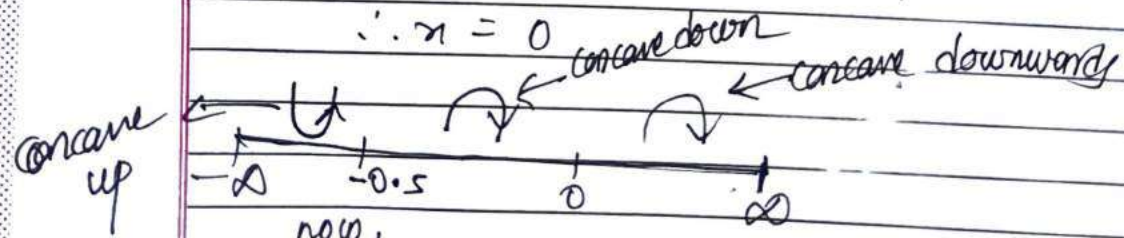
$f''(x) = 0$

or $\frac{-5}{9}\left(\frac{1+2x}{x^{4/3}}\right) = 0$

or $x = \frac{-1}{2} = -0.5$

also, $f''(x) = \frac{1}{0}$

$\therefore x = 0$



now,

x	-2	-0.5	0	1	+2.5	3
$y = f(x)$	7.14	1.88	0	1.5	0	-1.04

Note

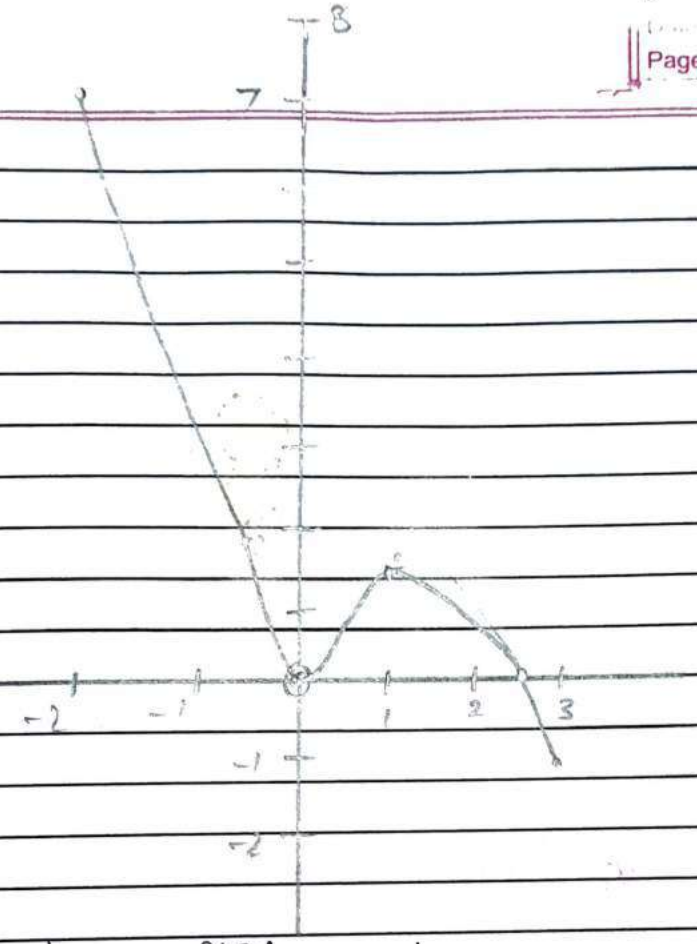


fig: $f(x) = x^{2/5}\left(\frac{5}{2} - x\right)$

Graphing $y = \frac{x^2-1}{x}$

Soln

$$f(-x) = \frac{(-x)^2-1}{-x} = \frac{-x^2-1}{x} = -f(x)$$

Hence, $f(x)$ is odd function, i.e. symmetric about origin.

for asymptotes

Here, $x = 0$ is an asymptote.

also,

$$y = x - \frac{1}{x}$$

Hence, $y = x$ is another asymptote.

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Note

for critical point

$$\frac{dy}{dx} = 1 + \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = 0$$

$$\text{or } \frac{x^2+1}{x^2} = 0$$

$$\text{or } x^2+1 = 0$$

$$\text{or, } x^2 = -1$$

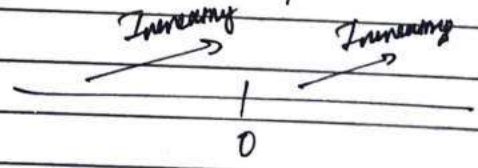
$$\text{or } x = \sqrt{-1}$$

There is no real solution.

$$\frac{dy}{dx} = \frac{1}{0}$$

$$\text{or } \frac{x^2+1}{x^2} = \frac{1}{0}$$

$\therefore x = 0$ (It is false because $x=0$ is already an asymptote, so, $x=0$ doesn't lie in curve)



for point of inflection

$$\frac{dy}{dx} = 1 + \frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = 0 - \frac{2}{x^3} = \frac{-2}{x^3}$$

$$\frac{d^2y}{dx^2} = \frac{1}{0}$$

$$\text{or, } \frac{-2}{x^3} = \frac{1}{0}$$

$\Rightarrow x = 0$ (not a point of curve)

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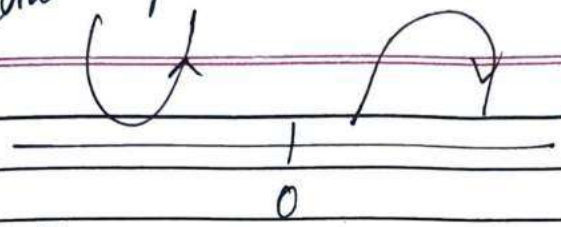
Concave up Concave down.

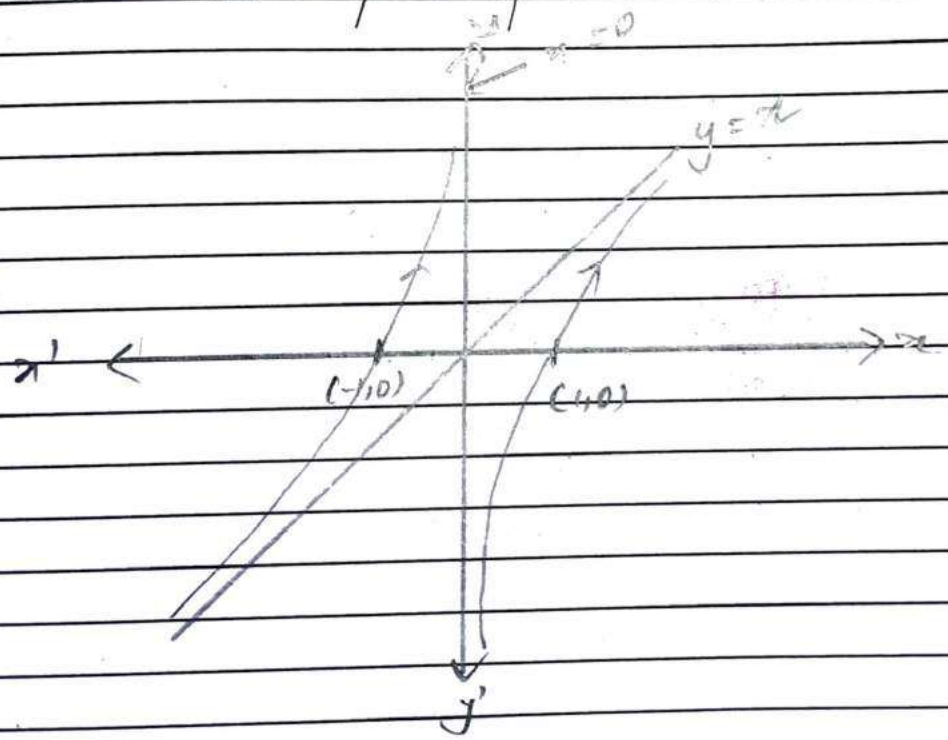


Note

Evaluating y at special points.

x	1	-1
y	0	0

$(1,0)$ and $(-1,0)$ are points of curve.



only $y' = \ldots$, given for general shape.

Then,

find critical point, point of inflection, decreasing, increasing, local maxima and minima as $(c, f(c))$ and then give some general shape to curve.

The Mean Value Theorems

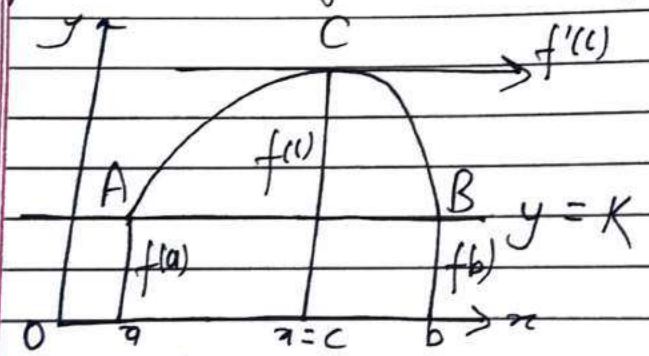
Rolle's theorem

Note

If a function $y = f(x)$ is continuous at every point of closed interval $[a,b]$ and differentiable in open interval (a,b), also $f(a) = f(b)$, then there exist $c \in (a,b)$ such that $f'(c) = 0$

> Geometrical meaning

If a differentiable curve intersect at two points of a horizontal line, then there exist at least one interior point where tangent drawn is parallel to X-axis.



$f(a) = f(b) = k$

Q. Verify Rolle's theorem and find the value of c for $f(x) = (x-2)^3 \cdot (x-4)^4$ on $[2,4]$.

Soln

1) $f(x) = (x-2)^3 \cdot (x-4)^4$ is continuous on $[2,4]$ because every polynomials are continuous at real numbers.

2) $f(x)$ is also differentiable on $(2,4)$.

3) $f(a) = f(b) = 0$

$$f'(x) = \left((x-2)^3 \cdot 4(x-4)^3\right) + \left((x-4)^4 \cdot 3 \cdot (x-2)^2\right)$$

$$= (x-2)^2 (x-4)^3 \left(4(x-2) + 3(x-4)\right)$$

$$f'(x) = (x-2)^2 (x-4)^3 (7x - 20)$$

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Here, $f''(x) = 0$

either, $(x-2)^2 = 0$

$\Rightarrow x = 2$ ✗

OR, $(x-4)^3 = 0$

$\Rightarrow x = 4$ ✗

OR, $7x - 20 = 0$

$\Rightarrow x = \frac{20}{7}$ ✓

(must lie between 2 and 4.10,

$c = \frac{20}{7} \in (2,4)$

$f'\left(\frac{20}{7}\right) = 0$

Hence, Rolle's theorem is verified.

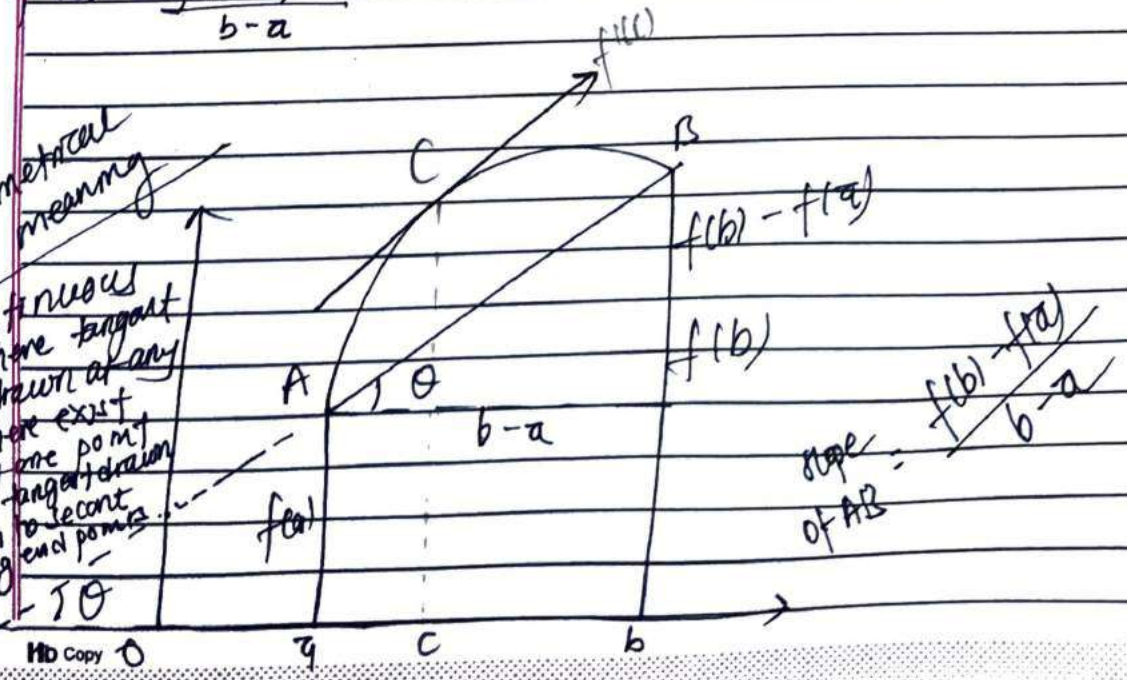
Mean Value Theorem (Lagrange's)

If a function $y = f(x)$ is continuous at every point of closed interval $[a,b]$ and differentiable at open interval (a,b), ~~also, $f'(c) = \frac{f(b)-f(a)}{b-a}$~~, then there exist at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Geometrical meaning

on a continuous curve where tangent can be drawn at any point, there exist at least one point c where tangent drawn is parallel to secant forming end points.



slope of AB $= \frac{f(b)-f(a)}{b-a}$

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Q. Verify mean value theorem for $f(x)$.

Note

$f(x) = x(x-1)(x-2)$ on $[0, \frac{1}{2}]$

$= (x^2 - x)(x-2)$

$= x^3 - 2x^2 - x^2 + 2x$

$= x^3 - 3x^2 + 2x$

Soln

1) $f(x)$ is a polynomial function so it is always continuous on $[0, \frac{1}{2}]$

2) $f(x)$ is differentiable on $(0, \frac{1}{2})$.

$f'(x) = 3x^2 - 6x + 2$

$\cancel{= 3x^2 - 5x - x + 2}$

Let the point $c \in (0, \frac{1}{2})$

then,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

or $3c^2 - 6c + 2 = \dfrac{\frac{1}{8} - \frac{3}{4} + 1 - 0}{\frac{1}{2}}$

or $3c^2 - 6c + 2 = \frac{3}{4}$

or $3c^2 - 6c + 2 - \frac{3}{4} = 0$

or $3c^2 - 6c + \frac{5}{4} = 0$

solving for c,

Here, $c = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{6 \pm \sqrt{36 - 4 \times 3 \times \frac{5}{4}}}{6} = \dfrac{6 \pm \sqrt{21}}{6}$

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Note

$\therefore c = 1.76 \notin (0, \frac{1}{2})$

$= 0.236 \in (0, \frac{1}{2})$

There exists a point $c \in (0, \frac{1}{2})$. so MVT is verified.

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Linearization

eqn of tangent is
$y = f(a) + f'(a)(x-a)$

Note If f is differentiable at $x = a$ then the approximating function $L(x) = f(a) + f'(a)\cdot(x-a)$ is the linearization of f at $x = a$.

The approximation $f(x) \approx L(x)$ of f is the standard linear approximation of f at $x = a$. The point $x = a$ is called centre of approximation.

example:

Q. find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$

soln

Here, $x = a$, $x = 0$

so, $a = 0$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f'(0) = \frac{1}{2}$

$f(0) = 1$

$L(x) = f(0) + f'(0)(x-0) = 1 + \frac{x}{2}$

Q. find the linearization of $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

soln

Here, $a = \frac{\pi}{2}$

$f'(x) = -\sin x$

$f'(\frac{\pi}{2}) = -1$

$f(\frac{\pi}{2}) = 0$

so, linearization is $L(x) = f(a) + f'(a)(x-a)$

$= 0 + (-1)(x - \frac{\pi}{2})$

$= -x + \frac{\pi}{2}$

Differential

Note Let $y = f(x)$ be a differentiable function. The differential dx is an independent differential and dy is dependent differential, where,

$$dy = f'(x)dx$$

example: ① find dy if $y = x^4 + 3x^2 + 7$, at $x = 1$ and $dx = 0.2$

Soln

$$\frac{dy}{dx} = 4x^3 + 6x$$

$$dy = (4x^3 + 6x)\,dx$$

$$= (4+6) \times 0.2$$

$$= 10 \times 0.2$$

$$= 2$$

② $y = \sin 2x$ at $x = \frac{\pi}{2}$, $dx = 0.1$

$$\frac{dy}{dx} = 2\cos 2x$$

$$dy = 2 \times \cos 2 \times \frac{\pi}{2} \times 0.1$$

$$= 2 \times (-1) \times 0.1$$

$$= -0.2$$

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Anti-derivative (Integration)

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Note

⇒ A function $F(x)$ is an antiderivative of a function $f(x)$ if

$$F'(x) = f(x), \text{ for all } x \text{ in the domain of } f(x).$$

⇒ The set of all anti-derivatives of f is the indefinite integral.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad \int dx = x + c$$

$$\int \sin x dx = -\cos x + c, \quad \int \tan x dx = -\log(\cos x) + c$$

or $\log(\sec x) + c$

$$\int \cot x dx = \log(\sin x) + c.$$

$$\int \sec x dx = \log(\sec x + \tan x) + c$$

$$\int \operatorname{cosec} x dx = \log(\operatorname{cosec} x - \cot x) + c$$

$$\int \sec^2 x dx = \tan x + c \qquad \boxed{\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a}}$$

$$\int \operatorname{cosec} x \cdot \cot x dx = -\cancel{\cot x} \operatorname{cosec} x + c$$

$$\int \operatorname{cosec}^2 x dx = -\cot x + c$$

$$\int \frac{1}{x} dx = \log x + c, \quad \int a^x dx = \frac{a^x}{\log a}$$

$$\int e^x dx = e^x + c.$$

Q. $\int \sin^4 x \, dx = \int (\sin^2 x)^2 dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \frac{1 - 2\cos 2x + \cos^2 2x}{4} dx$$

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$$\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a} + c$$

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Note

$$\frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{8}\int (2 - 4\cos 2x + 1 + \cos 4x)\, dx$$

$$= \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)\, dx$$

$$= \frac{1}{8}\left(3x - \frac{4\sin 2x}{2} + \frac{\sin 4x}{4}\right) + c$$

$$= \frac{1}{8}\left(3x - 2\sin 2x + \frac{\sin 4x}{4}\right) + c$$

Q. $\int \frac{2x+3}{x+7}\,dx$

$$= \int \frac{2x+14-11}{x+7}\,dx$$

$$= \int \frac{2x+14}{x+7}\,dx - \int \frac{11}{x+7}\,dx$$

yo x ko power mai kei xa bhane log lekhna parcha.

$$= \int \frac{2\cancel{(x+7)}}{\cancel{(x+7)}}\,dx - 11\log(x+7)$$

$$= 2x - 11\log(x+7) + c.$$

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Note

Q. $\int (e^{ax} + e^{-ax})^3 dx$

$= \int (e^{3ax} + 3 \cdot e^{2ax} \cdot e^{-ax} + 3e^{ax} e^{-2ax} + e^{-3ax}) dx$

$= \frac{e^{3ax}}{3a} + \frac{3 \cdot e^{ax}}{a} + \frac{3e^{-ax}}{-a} + \frac{e^{-3ax}}{-3a} + c$

Q. $\int \frac{1-\cos x}{1+\cos x} dx$

$= \int \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} dx$

$= \int \tan^2 \frac{x}{2} dx$

$= \int \left(\sec^2 \frac{x}{2} - 1\right) dx$

$= \frac{\tan \frac{x}{2} - x + c}{\frac{1}{2}}$

Q. $\int \frac{1}{x} \sin(\log x) dx$

Sum

Let $t = \log x$

$\frac{dt}{dx} = \frac{1}{x}$

$dx = \frac{dt}{\frac{1}{x}}$

putting values

$= \int \frac{1}{x} \sin t \cdot \frac{dt}{\frac{1}{x}}$

$= \cancel{\sin x} \int \sin t \, dt$

$= \cancel{\sin x} \frac{t^2}{2} + c$

$= \cancel{\sin x (\log x)^2} + c$

$= -\cos t + c$

$= -\cos(\log x) + c$

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* $\int e^x[f(x)+f'(x)]\,dx = \int e^x f(x)\,dx + c$

* $\int \frac{dz}{z} = \log z + c$

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By parts integration

Note $\boxed{\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx}$, LATE answer ki bahira Thikne.

$$I = \int e^{ax}\sin bx\,dx$$

OTHER Integration formulas

(1) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$

(2) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$

(3) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(4) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2}) + c$

(5) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(6) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log(x+\sqrt{x^2+a^2}) + c$

(7) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + c$

(8) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}) + c$

(9) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

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Note

eg 1. $\int \frac{dx}{9+x^2}$

Let $3\tan\theta = x$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$\therefore dx = d\theta\, 3\sec^2\theta$

now,

$\int \frac{3\sec^2\theta\, d\theta}{9+9\tan^2\theta} = \int \frac{3\cancel{\sec^2\theta}\, d\theta}{9\cancel{\sec^2\theta}}$

$= \frac{1}{3}\int d\theta$

$= \frac{1}{3}\theta$

$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

eg 1. $\int \frac{dx}{x^2\sqrt{x^2+1}}$, Let $x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}$

$= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\, \sec\theta}$

$= \int \frac{\sec\theta\, d\theta}{\tan^2\theta}$

$= \int \frac{1}{\cancel{\cos\theta}} \times \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, d\theta$

$= \int \cot\theta \cdot \operatorname{cosec}\theta\, d\theta$

$= -\operatorname{cosec}\theta + c$

$= -\cancel{\operatorname{cosec}(\tan}$

$= -\frac{\sqrt{1+x^2}}{x} + c$

$\frac{h}{p}$

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$$\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a} + C$$

$\cancel{2x \frac{1}{2} + 1}$

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eg:- $\int \frac{9}{(2x+1)(x+5)}\,dx$

Note

Let $\frac{9}{(2x+1)(x+5)} = \frac{A}{2x+1} + \frac{B}{x+5}$

or $9 = A(x+5) + B(2x+1)$

or $\cancel{9 = Ax + 5A + 10Bx + B}$

Let $x = -5$,

or $9 = B(-10+1)$

or $9 = B(-9)$

$\therefore B = -1$

Let $x = -\frac{1}{2}$

or $9 = A\left(-\frac{1}{2} + 5\right)$

or $9 = A\left(\frac{9}{2}\right)$

$\therefore A = 2$

now,

$$\frac{9}{(2x+1)(x+5)} = \frac{2}{2x+1} - \frac{1}{x+5}$$

$$I = \int \frac{2}{2x+1}\,dx - \int \frac{1}{x+5}\,dx$$

Let $2x+1 = z$

$\frac{dz}{dx} = 2$

$$= \int \frac{2}{z} \cdot \frac{dz}{2} - \int \frac{1}{x+5}\,dx$$

$$= \log(2x+1) - \log(x+5) + C$$

$$= \log\left(\frac{2x+1}{x+5}\right) + C \text{ ans}$$

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eg:- $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$

Note

$= \int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} \times \frac{\sec^2 x}{\sec^2 x}$

$= \int \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2}$

Let $a\tan x = t$

$\frac{dt}{dx} = a\sec^2 x$

$= \int \frac{\cancel{\sec^2 x}\, dt}{(t^2+b^2)\, a\cancel{\sec^2 x}}$

$= \frac{1}{a}\int \frac{dt}{t^2+b^2}$

$= \frac{1}{a}\cdot\frac{1}{b}\tan^{-1}\left(\frac{t}{b}\right)$

$= \frac{1}{ab}\tan^{-1}\left(\frac{t}{b}\right)$

$= \frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right)$

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$\sec^2\theta = 1 + \tan^2\theta$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

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Note

eg 1. $\displaystyle\int \frac{dx}{2+\cos x}$

$$= \int \frac{dx}{2 + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \times \frac{\sec^2 x/2}{\sec^2 x/2}$$

$$= \int \frac{\sec^2\frac{x}{2}\, dx}{2\sec^2\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\, dx}{2 + 2\tan^2\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\, dx}{3 + \tan^2\frac{x}{2}}$$

let $\tan\frac{x}{2} = t$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$dx = \frac{2\,dt}{\sec^2\frac{x}{2}}$$

$$= \int \frac{\cancel{\sec^2\frac{x}{2}}}{3 + t^2} \cdot \frac{2\,dt}{\cancel{\sec^2\frac{x}{2}}}$$

$$= 2\int \frac{dt}{t^2 + (\sqrt{3})^2}$$

$$= 2\,\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)$$

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Definite integral $\int_a^b$

Note $\int f(x)\,dx = \phi(x) + c$

$\boxed{\int_a^b f(x)\,dx = \phi(b) - \phi(a)}$ +c ra -c cancel hunxa bhanera yesma C narakheko

Let integration of $f(x)\,dx$ be $\phi(x) + c$, then we define the definite integral of $f(x)$ w.r.t. x from a to b is denoted by $\int_a^b f(x)\,dx$ and defined as;

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

(1) $\int_1^9 \sqrt{x}\,dx$

$= \int_1^9 x^{1/2}\,dx$

$= \left[\dfrac{x^{1/2+1}}{\frac{1}{2}+1}\right]_1^9$

$= \dfrac{2}{3}\left[x^{3/2}\right]_1^9$

$= \dfrac{2}{3}\left[9^{3/2} - 1^{3/2}\right]$

$= 17.33$

(2) $\int \dfrac{2x}{1+x^2}\,dx$

Let $1+x^2 = z$

$\dfrac{dz}{dx} = 2x$

$dx = \dfrac{dz}{2x}$

$= \int \dfrac{\cancel{2x} \cdot \frac{dz}{\cancel{2x}}}{z}$

$= \int_0^1 \dfrac{dz}{z}$

$= [\log z]_0^1$

$= [\log(1+x^2)]_0^1$

$= \log(1+1) - \log(1)$

$= 0.693$

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(iii) $\int_0^{\pi/2} \cos 3x \cos 2x \, dx$

Note

$= \frac{1}{2}\int_0^{\pi/2} 2\cos 3x \cos 2x \, dx$

$= \frac{1}{2}\int_0^{\pi/2} \cos(3x+2x) + \cos(3x-2x) \, dx$

$= \frac{1}{2}\int_0^{\pi/2} (\cos 5x + \cos x) \, dx$

$= \frac{1}{2}\left[\frac{\sin 5x}{5} + \sin x\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\frac{\sin 5\times\frac{\pi}{2}}{5} + \sin\frac{\pi}{2} - \sin 0 + \sin 0\right]$

$= \frac{1}{2}\left[\frac{1}{5} + 1\right]$

$= \frac{1}{\cancel{2}} \times \frac{\cancel{6}}{5} = \frac{3}{5}$

(iv) $\int \frac{1}{x^3+1} \, dx$

$= \int \frac{1 \, dx}{(x+1)(x^2-x+1)}$

Let $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

or $1 = A(x^2-x+1) + (Bx+C)(x+1)$

Let $x = -1$,

or $1 = A(1+1+1)$

$\therefore A = 1/3$

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Note

Let $x=0$,

or, $1 = A + C$

or, $1 = \frac{1}{3} + C$

$\therefore C = \frac{2}{3}$

Let $x = 1$,

or, $1 = A + (B+C)2$

or, $1 = A + 2B + 2C$

or, $1 = \frac{1}{3} + 2\times B + 2\times\frac{2}{3}$

or, $2B = 1 - \frac{1}{3} - \frac{4}{3}$

or, $2B = -\frac{2}{3}$

$\therefore B = \frac{-1}{3}$

Hence, $A = \frac{1}{3}, B = \frac{-1}{3}, C = \frac{2}{3}$

So, $\frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3(x+1)} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}$

$= \frac{1}{3(x+1)} + \frac{2-x}{3(x^2-x+1)}$

$I = \int \frac{1}{3(x+1)}\,dx + \int \frac{(2-x)}{3(x^2-x+1)}\,dx$

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$\frac{1}{6} \cdot \frac{2\sqrt{3}}{\frac{\sqrt{3}}{3}}$

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Note

$$= \frac{1}{3}\log(x+1) + \frac{1}{3}\int \frac{2-x}{x^2-x+1}\,dx$$

Let $x^2 - x + 1 = t$

$$\frac{dt}{dx} = 2x - 1$$

$$dx = \frac{dt}{2x-1}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\int \frac{2x-1-3}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\left[\int \frac{\cancel{2x-1}}{t}\cdot\frac{dt}{\cancel{2x-1}} - 3\int \frac{dx}{(x-\frac{1}{2})^2+1-\frac{1}{4}}\right]$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\left[\log t - 3\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right]$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\left[\log t - 3\cdot\frac{2}{\sqrt{3}}\cdot\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\left[\log t - 2\sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

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Fundamental theorem of calculus (Part-I)

Note If f is continuous on $[a,b]$, the $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and has derivative at every point of (a,b) and its derivative is $f(x)$.

i.e. $$\frac{dF(x)}{dx} = \frac{d\int_a^x f(t)\,dt}{dx}$$
$$= f(x)$$

The theorem says that;

1. The differential equation $\frac{dF(x)}{dx} = f(x)$ has a solution for every continuous function $f(x)$.
2. Every continuous function $f(x)$ is the derivative of some other function namely $\int_a^x f(t)\,dt$
3. Every continuous function has antiderivative.
4. The process of integration and differentiation are inverse of each other.

Example

1. $y = \int_0^x \cos t\,dt$

Soln $\frac{dy}{dx} = \frac{d\int_0^x \cos t\,dt}{dx} = \cos x$

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③ $y = \int_x^5 3t\sin t\,dt$ ...

④

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② $y = \int_0^x \frac{1}{1+t^2}\,dt$

Note

Soln

$$\frac{dy}{dx} = \frac{d\int_0^x \frac{1}{1+t^2}\,dt}{dx} = \frac{1}{1+x^2}$$

③ $y = \int_x^5 3t\sin t\,dt$

Soln

$$y = -\int_5^x 3t\sin t\,dt$$

$$\frac{dy}{dx} = -\frac{d\int_5^x 3t\sin t\,dx}{dx} = -3x\sin x$$

④ $y = \int_1^{x^2} \cos t\,dt$

Soln

Let $u = x^2$

$\frac{du}{dx} = 2x$

now,

$y = \int_1^u \cos t\,dt$

Hence, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= \frac{d\int_1^u \cos t\,dt}{du}\cdot 2x$

$= \cos u\cdot 2x$

$= 2x\cos x^2$ ans

$$= \frac{d\int_1^{x^2}\cos t\,dt}{dx^2}\cdot\frac{dx^2}{dx}$$

$= 2x\cos x^2$

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The fundamental theorem of calculus (part - II)

Note If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

eg:- $\int_0^{\pi} \cos x\,dx$

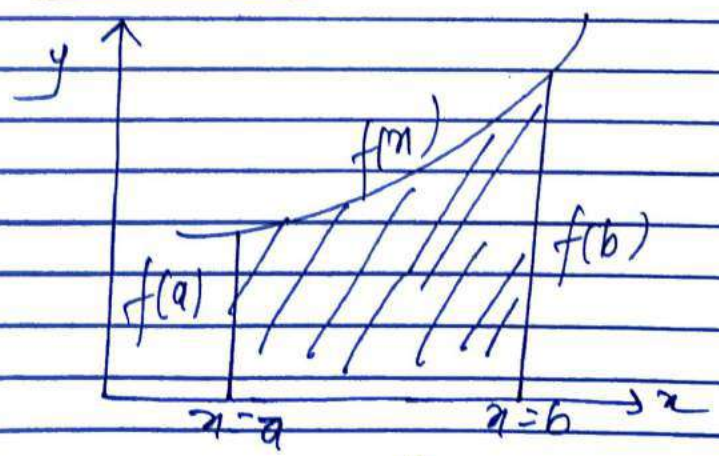
$= [\sin x]_0^{\pi}$

$= \sin\pi - \sin 0$

$= 0$

Note: The definition of definite integral is seen as the statement of fundamental theorem of calculus (part-II)

Geometrically



Definite integral $\int_a^b f(x)\,dx$ means the area bounded by the curve $f(x)$, the x-axis and two ordinates $f(a)$ and $f(b)$.

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Note

eg: find the area between $f(x)$ and x-axis with two ordinates $f(a)$ and $f(b)$.

(1) $y = f(x) = x^3 - 3x^2 + 2x, \; 0 \le x \le 2$

soln

$$\text{Area} = \int_0^2 y \, dx$$

$$= \int_0^2 (x^3 - 3x^2 + 2x) \, dx$$

$$= \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} \right]_0^2$$

$$= \left[\frac{x^4}{3} - x^3 + x^2 \right]_0^2$$

$$= \frac{16}{3} - 8 + 4$$

$$= \frac{16}{3} - 4 = \frac{4}{3} \text{ units.}$$

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Improper integral

Note ① Improper integral (Type - I)

Integral with infinite limits of integration are called improper integrals of type - I.

- If $f(x)$ is continuous on $[a, \infty)$, then,

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

- If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

- If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

$$= \lim_{b \to -\infty} \int_b^a f(x)\,dx + \lim_{c \to \infty} \int_a^c f(x)\,dx$$

In each case, if the limit is finite, we say that the improper integral is converged and the limit is value of improper integral.

If the limit is infinite, we say that the improper integral is diverged. ~~and the limit is value of improper integral.~~

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upper infinity

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Note

example: $\int_1^\infty \frac{\log x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\log x}{x^2}\,dx$

$= \log x \int_1^\infty \frac{1}{x^2}\,dx - \int_1^\infty \left(\frac{d\log x}{dx}\int_1^\infty \frac{1}{x^2}\,dx\right)dx$

$= \log x \left[\frac{-1}{x}\right]_1^\infty - \left[\frac{1}{x}\cdot\left[\frac{-1}{x}\right]_1^\infty\right]_1^\infty$

$= \log x - \left[\frac{1}{x}\right]_1^\infty$

$= \log x + 1$

$= \lim_{b\to\infty}\left[\log x \int \frac{1}{x^2}\,dx - \int\left(\frac{d\log x}{dx}\int \frac{1}{x^2}\,dx\right)dx\right]_1^b$

$= \lim_{b\to\infty}\left[\frac{-\log x}{x} - \int\left(\frac{1}{x}\cdot\frac{-1}{x}\right)dx\right]_1^b$

$= \lim_{b\to\infty}\left[\frac{-\log x}{x} + \int \frac{1}{x^2}\,dx\right]_1^b$

$= \lim_{b\to\infty}\left[\frac{-\log x}{x} - \frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[\frac{-\log b}{b} - \frac{1}{b} + \frac{0}{1} + \frac{1}{1}\right]$

using L-Hospital's rule,

$= \lim_{b\to\infty}\left[\frac{\frac{-1}{b}}{1} - \frac{1}{b} + 1\right]$

$= \frac{-1}{\infty} - 0 + 1 = 1$ ans

$= \frac{\log\infty}{\infty} - \frac{1}{\infty} - \left[-\log 1 + 1\right]$

$= 0 - 0 + 0 + 1$

$= 1$ ans

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both infinity

Note

example: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$\Rightarrow \cancel{\tan^{-1}x} \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$= \left[\tan^{-1}x\right]_{-\infty}^{0} + \left[\tan^{-1}x\right]_{0}^{\infty}$

$= \tan^{-1}0 - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \tan^{-1}(0)$

$= 0 + \frac{\pi}{2} + \frac{\pi}{2} - 0$

$= \pi$

lower limit infinity

Q. $\int_{-\infty}^{-1} \frac{2\,dx}{x^2-1} = 2\int_{-\infty}^{-1} \frac{1}{x^2-(1)^2}\,dx$

~~Let $x^2-1=t$~~ $= \lim_{a\to-\infty} 2\int_{a}^{-2} \frac{dx}{x^2-1^2}$

~~$\frac{dt}{dx} = 2x$~~

$= \lim_{a\to-\infty} \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\left[\log\left(\frac{x-1}{x+1}\right)\right]_{a}^{-2}$

$= \lim_{a\to-\infty} \log\left(\frac{-2-1}{-2+1}\right) - \log\left(\frac{a-1}{a+1}\right)$

$= \lim_{a\to-\infty} \log 3 - \log\left(\frac{1-\frac{1}{a}}{1+\frac{1}{a}}\right)$

$= \log 3 - \log 1$

$= \log 3$

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Improper integral (Type - II)

Note eg. $\int_1^5 \frac{1}{x-1}\,dx$, $\int_1^5 \frac{1}{x-5}\,dx$, $\int_1^5 \frac{1}{x-3}$

undefined at 1 | undefined at 5 | 1 - 5 ma 3 pani parxa. so, undefined at 3.

Integrals of functions that become infinite at a point within the interval of the integral are called improper integrals of type - II.

① If $f(x)$ is continuous on $(a,b]$ and is discontinuous at $x=a$, then

$$\int_a^b f(x)\,dx = \int_{a^+}^b f(x)\,dx \text{ or } \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

② If $f(x)$ is continuous at $[a,b)$ and is discontinuous at $x=b$, then,

$$\int_a^b f(x)\,dx = \int_a^{b^-} f(x)\,dx \text{ or } \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

③ If $f(x)$ is discontinuous at $x=c$ where, $a<c<b$ and continuous on $[a,c)$ and $(c,b]$

then, $\int_a^b f(x)\,dx = \int_a^{c^-} f(x)\,dx + \int_{c^+}^b f(x)\,dx$

$$= \lim_{d\to c^-}\int_a^d f(x)\,dx + \lim_{e\to c^+}\int_e^b f(x)\,dx$$

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$\Rightarrow$ example: $\int_0^1 \frac{1}{1-x}\,dx$

Note:

Let,

$$\lim_{a \to 1^-} \int_0^a \frac{1}{1-x}\,dx$$

$$= \lim_{a \to 1^-} \left[-\log(1-x)\right]_0^a$$

$$= \lim_{a \to 1^-} \left[-\log(1-a) + \cancel{\log 1}^{\,0}\right]$$

$$= \lim_{a \to 1^-} -\log(1-a)$$

$$= \lim_{a \to 1^-} \log\left(\frac{1}{1-a}\right)$$

$$= \lim_{a-1 \to 0} \log\left(\frac{1}{1-a}\right) = \log \infty = \infty \quad \left(\text{no value, Diverged}\right)$$

The improper integral is diverged.

$\Rightarrow$ $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$$= \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{a \to 0^+} \left[2\sqrt{x}\right]_a^1$$

$$= \lim_{a \to 0^+} \left[2 - 2\sqrt{a}\right]$$

$$= 2 - 2 \cdot 0$$

$$= 2.$$

Definite integral as limit of summation (Sums)

Note: Let $f(x)$ be a function continuous in the interval $[a,b]$ where $b > a$. Let the interval $[a,b]$ is divided into n-equal sub-intervals having each sub interval length 'h', such that, $nh = b-a$. then, the area between curve $f(x)$, x-axis and ordinates at $x=a$ & $x=b$ is defined as

$$S = \lim_{h \to 0} h \sum_{r=1}^{n} f(a+rh)$$

mostly used.

or,

$$S = \lim_{h \to 0} \sum_{r=0}^{n-1} f(a+rh)$$

example: Evaluate $\int_0^1 x^2 dx$ by summation method.

Soln

$a = 0, b = 1$

$b - a = nh = 1 - 0 = 1$

$f(x) = x^2$

Now,

$$\int_0^1 x^2 dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a+rh)$$

$$= \lim_{h \to 0} h \sum_{r=1}^{n} f(0+rh)$$

$$= \lim_{h \to 0} h \sum_{r=1}^{n} f(rh)$$

$$= \lim_{h \to 0} h \sum_{r=1}^{n} (rh)^2$$

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Note

$$= \lim_{h \to 0} h^3 \sum_{r=1}^{n} r^2$$

$$= \lim_{h \to 0} h^3 \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{h \to 0} h^3 \cdot \frac{n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$$

$$= \lim_{h \to 0} \frac{(nh)^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$$

As $\lim_{h \to 0}$, $\lim_{n \to \infty}$

$$= \frac{(1)^3(1+0)(2+0)}{6}$$

$$= \frac{1 \times 1 \times 2}{6} = \frac{1}{3} \text{ sq. unit.}$$

Initial value problem

The problem of finding a function of y of x when its derivative and its value y_0 at a particular point x_0 is given, called initial value problem.

Initial value problem may be higher order.

eg:- solve,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad y(0) = 0$$

or, $f(0) = 0$

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Note

Soln

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Integrating on both sides w.r.t. x,

$$\text{or } \int dy = \int \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\text{or } y = \sin^{-1}x + c \longrightarrow (1)$$

we have,

$$y(0) = 0$$

$$\text{or } 0 = \sin^{-1}(0) + c$$

$$\text{or } 0 = 0 + c$$

$$\therefore c = 0$$

Hence $c = 0$,

Keeping the value of c in (1),

we get $y = \sin^{-1}x$

<u>example</u>

$$\frac{d^2y}{dx^2} = 2-6x, \quad y'(0) = 4, \quad y(0) = 1$$

Soln

Integrating,

$$\text{or } \frac{dy}{dx} = \left(2x - \frac{6x^2}{2} + c_1\right) \longrightarrow (1)$$

$$\text{or } y' = 2x - \frac{6x^2}{2} + c_1$$

$$\text{or } 4 = 2\cdot 0 - \frac{6\times 0^2}{2} + c_1$$

$$\therefore c_1 = 4$$

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Note

$\therefore \frac{dy}{dx} = (2x - 3x^2 + 4) \longrightarrow (ii)$

again, integrating (ii),

we get,

or $y = \frac{2x^2}{2} - \frac{3x^3}{3} + 4x + C_2$

or $y = x^2 - x^3 + 4x + C_2 \longrightarrow (iii)$

we have,

$y(0) = 1,$

or, $1 = 0^2 - 0^3 + 4\times 0 + C_2$

$\therefore C_2 = 1$

putting value in (iii)

we get $\boxed{y = x^3 - x^3 + 4x + 1.}$

Average/Mean value of a function.

If f is integrable on $[a,b]$, then its average/mean value of $f(x)$ on $[a,b]$ is,

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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Theorem: Mean/Average value theorem.

Note If f is continuous on $[a,b]$, then there exist a point $c \in [a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

example:

find the average value of $f(x) = 4 - x^2$ on $[0,3]$, and verify the ~~rea~~ result.

Soln

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{3-0}\int_0^3 4 - x^2\,dx$$

$$= \frac{1}{3}\left[4x - \frac{x^3}{3}\right]_0^3$$

$$= \frac{1}{3}\left[4\times 3 - 9\right]$$

$$= 1$$

$\therefore$ ~~av~~ $(f) = 1$

Let $c \in [0,3]$, such that $f(c) = 1$

or $4 - c^2 = 1$

or $3 = c^2$

$\therefore c = \pm\sqrt{3}$

only $\sqrt{3} \in [0,3]$ $\Leftarrow$ Hence it is required point.

$-\sqrt{3} \notin [0,3]$

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Area between the curves

Note If f and g are continuous and $f(x) \geq g(x)$ on $[a, b]$ then the area between the curve $f(x)$ and $g(x)$ from a to b is given by

$$A = \int_a^b \{f(x) - g(x)\}\, dx$$

Similarly, the area between the curves $x = g(y)$ and $x = f(y)$ from $y = c$ to $y = d$, then

$$A = \int_c^d (f(y) - g(y))\, dy$$

example: find the area between the following two curves.
i.e. $y = x^4$ and $y = 8x$.

Soln

$y = x^4 \rightarrow (1)$ and $y = 8x \rightarrow (2)$

Solving (1) and (2),

or $x^4 = 8x$

or $x(x^3 - 8) = 0$

Either $x = 0$

or $x^3 - 8 = 0$

or $x = \sqrt[3]{8} = 2$

$$\text{Area} = \int_0^2 (x^4 - 8x)\, dx = \left(\frac{x^5}{5} - \frac{8x^2}{2}\right)\Big|_0^2 = 9.6 \text{ units}$$

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Length of plane curve

Note

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

If f is smooth on $[a,b]$, then, the length of the curve $y = f(x)$ from ~~$[a,b]$~~ is a to b is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

OR

If f is smooth on $[c,d]$, then the length of the curve $x = f(y)$ from c to d is,

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

example:

find the length of the curve $y = 5x^2 - 2$ $-1 \le x \le 2$.

Soln

for $y = f(x)$, we have,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

now, $\frac{dy}{dx} = 10x$

$$L = \int_{-1}^{2} \sqrt{1 + (10x)^2}\, dx$$

$$= \frac{1}{10}\left[\frac{10x}{2}\sqrt{1+(10x)^2} + \frac{1}{2}\log\left(10x + \sqrt{1+(10x)^2}\right)\right]_{-1}^{2}$$

$$= \frac{1}{10}\left(202.1 - (-51.708)\right) = 25.3808 \text{ units}$$

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Note

Q. find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$.

Soln

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$\frac{dy}{dx} = \frac{d\left(\frac{x}{2}\right)^{2/3}}{d\left(\frac{x}{2}\right)} \frac{d\frac{x}{2}}{dx}$$

$$= \frac{\cancel{2}}{3}\left(\frac{x}{2}\right)^{\frac{2}{3}-1} \frac{1}{\cancel{2}}$$

$$= \frac{1}{3\left(\frac{x}{2}\right)^{1/3}}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

It is not defined at 0

So, we cannot find the length of curve by using formula

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Therefore, we express x is the function of y.

We have, $y = \left(\frac{x}{2}\right)^{2/3}$

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$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a.(n+1)}$

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Note

or, $y^{3/2} = \frac{x}{2}$

or $x = 2y^{3/2}$

now,

$\frac{dx}{dy} = 2.\frac{dy^{3/2}}{dy} = \cancel{2} \times \frac{3}{\cancel{2}} \times y^{1/2} = 3y^{1/2}$

when $x = 0, y = 0$ $\left(\text{from } y = \left(\frac{x}{2}\right)^{2/3}\right)$

$x = 2, y = 1$

now,

length of curve is $L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$

$= \int_0^1 \sqrt{1 + 9y} \, dy$

$= \left[\frac{(1+9y)^{3/2}}{\frac{3}{2} \times 9}\right]_0^1$

$= 2.3424 - 0.074$

$= 2.2684$

Note:
Length निकालते, $\frac{dy}{dx}$ undefined at either extreme points भयो भने $\frac{dx}{dy}$ को use गर्ने y को value निकालेर and vice versa.

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Area of surface of revolution.

Note When the graph of non-negative function $y = f(x)$ is revolved around any axis, we get the surface and that surface is called area of surface of revolution.

(1) Revolution about x-axis

If the function $y = f(x) \geq 0$ is revolved about x-axis, then area of surface of revolution from $x = a$ to $x = b$ is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(2) Revolution about y-axis.

If the function $x = f(y) \geq 0$ is revolved about y-axis, then area of surface of revolution from $y = c$ to $y = d$ is,

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Q. Find the area of surface generated by revolving the curve $y = \sqrt{2x - x^2}$, $0.5 \leq x \leq 1.5$ about x axis.

Soln

$$y = \sqrt{2x - x^2}$$

$$\frac{dy}{dx} = \frac{1}{2(2x - x^2)^{1/2}} (2 - 2x) = \frac{1 - x}{\sqrt{2x - x^2}}$$

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Note

Here,

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1+\frac{(1-x)^2}{2x-x^2}}$$

$$= \sqrt{1+\frac{x^2+1-2x}{2x-x^2}}$$

$$= \sqrt{\frac{\cancel{2x}-\cancel{x^2}+\cancel{x^2}+1-\cancel{2x}}{2x-x^2}}$$

$$= \sqrt{\frac{1}{2x-x^2}}$$

now,

$$\text{Area} = S = \int_{0.5}^{1.5} 2\pi \cancel{\sqrt{2x-x^2}} \cdot \frac{1}{\cancel{\sqrt{2x-x^2}}}\, dx$$

$$= 2\pi \left[x\right]_{0.5}^{1.5}$$

$$= 2\pi \text{ square units } \#$$

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solids of revolution.

Volume of ~~Revolution of solids~~

Note ① About x-axis

The volume of solid generated by revolving about the x-axis, the region between the x-axis and the graph of continuous function $y=f(x)$ where $a \le x \le b$ is

$$V = \int_a^b \pi y^2 dx = \int_a^b \pi [f(x)]^2 dx$$

② About y-axis.

The volume of solid generated by revolving about the y-axis, the region between the y-axis and the graph of continuous function $x = f(y)$ where $c \le y \le d$ is

$$V = \int_c^d \pi x^2 dy = \int_c^d \pi [f(y)]^2 dy$$

Example 1: The area between the curve $y = \sqrt{x}$ and $0 \le x \le 4$ and x-axis is revolved about the x-axis to generate the solid. find the volume.

Soln =

$$V = \int_0^4 \pi y^2 dx$$

$$= \int_0^4 \pi (\sqrt{x})^2 dx = \pi \int_0^4 x dx = \pi \left[\frac{x^2}{2}\right]_0^4$$

$$= 8\pi \text{ cubic units.}$$

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Note

Volume of solid of revolution about a line parallel to x-axis

If the axis of revolution is a line parallel to x-axis (say $y = c$), then the volume of the solid generated by the region, revolving the line $y = c$ is;

$y = f(x)$.

$$V = \int_a^b \pi (y-c)^2 \, dx$$

Volume of solid of revolution about a line ~~of revolution~~ parallel to y-axis

$x = f(y)$

Say line be $x = a$, so, volume of solid generated by the region, revolving the line $x = a$ is;

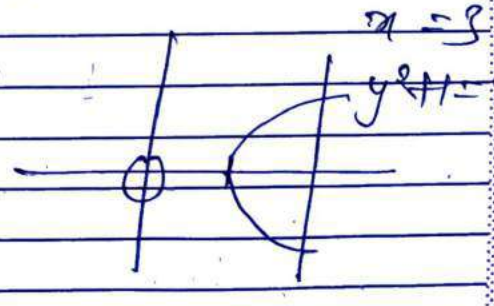
$$V = \int_c^d \pi (x-a)^2 \, dy$$

eg. Find the volume of the solid generated by revolving the region between the parabola $x = y^2+1$ and the line $x = 3$, about the line $x = 3$.

Soln

$x = y^2 + 1 \rightarrow (1)$

$x = 3 \rightarrow (2)$



solving (1) and (2),

$y^2 + 1 = 3$

$y^2 = 2$

$y = \pm\sqrt{2}$

so, $V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (x-3)^2 \, dy$

$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (y^2+1-3)^2 \, dy$

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Note

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (y^2-2)^2 \, dy$$

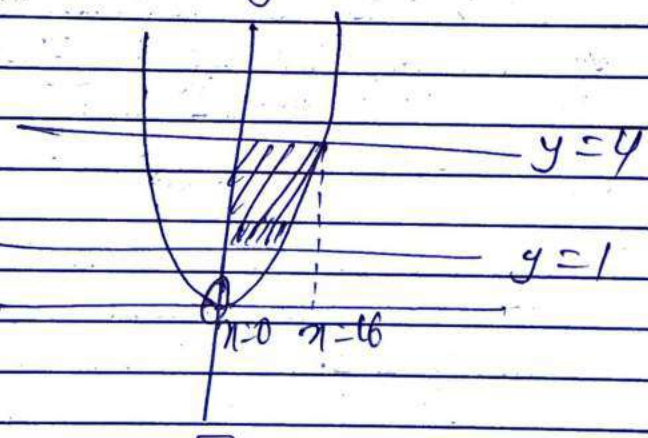
$$= \pi \left[\frac{y^5}{5} - \frac{4y^3}{3} + 4y \right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \pi \left[3.0169 - (-3.0169) \right]$$

$$= 6.0338\pi \text{ cubic units.}$$

eg: Find the volume of solid generated by revolving the region between the curve $y = \sqrt{x}$ and the line $y = 4$ about the line $y = 1$.

soln



$y = \sqrt{x}$ and $y = 4$

or $4 = \sqrt{x} \Rightarrow x = 16$

now, volume $= \int_0^{16} \pi (y-c)^2 \, dx$

$$= \int_0^{16} \pi (\sqrt{x} - 1)^2 \, dx$$

$$= \frac{116}{3}\pi \text{ cubic units.}$$

Volume of solid of revolution between two curves

Note ① About x-axis from $x=a$ to $x=b$.

$$V = \pi \int_a^b \{(f(x))^2 - (g(x))^2\} dx$$

② About y-axis from $y=c$ to $y=d$.

$$V = \pi \int_c^d \{(f(y))^2 - (g(y))^2\} dy$$

eg: The region bounded by the curve $y = x^2+1$ and the line $y = -x+3$ is revolved about x-axis to generate a solid, find the volume of solid.

Soln

$y = x^2+1 \rightarrow ①$

$y = -x+3 \rightarrow ②$

solving ① and ②,

$-x+3 = x^2+1$

or $x^2+x-2=0$

or $(x+2)(x-1)=0$

$\therefore$ Either $x=-2$ or 1.

volume $= \pi \int_{-2}^{1} \{(x^2+1)^2 - (3-x)^2\} dx$

$= \pi \int_{-2}^{1} (x^4+x^2-8+6x) dx$

$= -\frac{117}{5}\pi$ cubic units $= \frac{117}{5}\pi$ cu. units.

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eg: The region bounded by the parabola $y = x^2$ and the line $y = 2x$ is revolved about y-axis to generate a solid. find the volume of solid.

Note

soln

$$y = x^2 \rightarrow (1)$$
$$y = 2x \rightarrow (2)$$

or $x = \sqrt{y} \rightarrow (3)$

and

$$x = \frac{y}{2} \rightarrow (4)$$

solving (3) and (4),

$$\sqrt{y} = \frac{y}{2}$$

or, $2\sqrt{y} = y$

or, $y^2 = 4y$

or $y(y-4) = 0$

either $y = 0$ or $y = 4$.

so,

$$V = \pi \int_0^4 (\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 dy$$

$$= \pi \int_0^4 y - \frac{y^2}{4} \, dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4$$

HD Copy: $\therefore V = \frac{8}{3}\pi$ ~~units~~ cubic units

Sequence and Series

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Note

Defn :-

A infinite sequence (or sequence) of numbers is a function whose domain is the set of integers greater than or equal to some integer n.

A sequence is denoted by $\{a_n\}$.

for example :-

$a_n = 3n$ i.e. 3, 6, 9, 12 - - - - -

$a_n = \sqrt{n}$ i.e. $\sqrt{1}, \sqrt{2}, \sqrt{3}$ - - - -

A sequence $\{a_n\}$ is bounded above if there exist a number M such that $a_n \leq M$ for all n. The number M is an upper bound for sequence $\{a_n\}$. If M is an upper-bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$ then M is called the least upper bound for $\{a_n\}$.

example :- The sequence $\{a_n\} = \frac{1}{n}$

i.e. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

This sequence has least upper bound in 1.

example :- $a_n = n$

i.e. 1, 2, 3 - - - - -

Has no upper bound.

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Note: A sequence $\{a_n\}$ is called bounded below if there exist a real number m such that $a_n \geq m$ for all n.

example:-

$2, 2^2, 2^3, 2^4 \ldots$

$a_n = 2^n$

Here, $a_n \geq 2$

2 is greatest lower bound.

Convergence and divergence of sequence.

The sequence $\{a_n\}$ converges to the number L if for every positive integer $\varepsilon > 0$, there exist a corresponding integer N such that for ~~$n \geq N$~~

$n > N \ni |a_n - L| < \varepsilon$.

If no such number L exists, we say that $\{a_n\}$ is diverged.

if $\{a_n\}$ converges to 'L', we write

$\lim_{n \to \infty} a_n = L$ and L is called limit of sequence

Note: A sequence $\{a_n\}$ is said to be converge if $\lim_{n \to \infty} a_n =$ finite value

for eg:- $a_n = \frac{1}{n}$

$\lim_{n \to \infty} a_n = \frac{1}{n} = \frac{1}{\infty} = 0$ (finite value).

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Note

eg 1: $a_n = \frac{1}{2^n}$

$\lim_{n \to \infty} a_n = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$ (finite value)

$\therefore \{a_n\}$ is converged.

A sequence $\{a_n\}$ is said to be diverged if

$\lim_{n \to \infty} a_n \neq$ finite value.

eg 1: $a_n = 2n$

$\lim_{n \to \infty} 2n = 2 \times \infty = \infty$ (undefined)

$\therefore \{a_n\}$ is diverged.

Sub-sequence.

$a_n = n$ i.e. 1, 2, 3, 4, 5, 6 ....

$a_n = 2n$ i.e. 2, 4, 6, 8, 10 ... $\in$ sub-sequence.

If a_n converges to L, then its all subsequences also converge to L.

If the terms of one sequence appear in another sequence in their given order, we say that the first sequence is a subsequence of the second.

eg 1: if $a_n = \frac{1}{n}$, then $a_n = \frac{1}{2n}$ is a subsequence.

If $a_n = n$, then $a_n =$ prime number is a subsequence.

Note 1: If the sequence a_n converge to L, then all of its subsequences also converge to L.

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Note

Non-decreasing sequence

A sequence $\{a_n\}$ with the properties that $a_n \leq a_{(n+1)}$ for all n is called non decreasing sequence.

eg :- $\{a_n\} = 1, 2, 3, 3, 4, 5, 6, 6, 7 \ldots$

Increasing sequence

A sequence $\{a_n\}$ with the properties that $a_{n+1} > a_n$ for all n is called ~~non decre.~~ increasing sequence.

eg :- $\{a_n\} = 1, 3, 5, 7, 9 \ldots$

THEOREM : The non-decreasing sequence theorem.

A non-decreasing sequence of real numbers converges if and only if it is bounded from above.

If a non-decreasing sequence converges, it converges to its least upper bound

example :-

$$a_n = 2 - \frac{1}{n}$$

$$a_1 = 2 - \frac{1}{1} = 1$$

$$a_2 = 2 - \frac{1}{2} = 1.5$$

$$a_3 = 2 - \frac{1}{3} = 2 - 0.33 = 1.67$$

a_n converges to $\lim_{n \to \infty} a_n = 2 - \frac{1}{\infty} = 2$

Note

Theorem 2 :

Let $\{a_n\}$ and $\{b_n\}$ be sequence of real numbers and let A and B be real numbers such that

$$\lim_{n \to \infty} a_n = A \quad \& \quad \lim_{n \to \infty} b_n = B$$

then,

(i) $\lim_{n \to \infty} (a_n + b_n) = A + B$

(ii) $\lim_{n \to \infty} (a_n - b_n) = A - B$

(iii) $\lim_{n \to \infty} a_n \times b_n = AB$

(iv) $\lim_{n \to \infty} K \cdot a_n = KA$

(v) $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

The Sandwich Theorem

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some ~~integer~~ index (or integer) N and

$$\lim_{n \to \infty} a_n = L \text{ and } \lim_{n \to \infty} c_n = L \text{ then,}$$

$$\lim_{n \to \infty} b_n = L.$$

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Note

eg:- if $|b_n| \leq c_n$ and $\lim_{n\to\infty} c_n = 0$ then, prove that $\lim_{n\to\infty} b_n = 0$.

proof:-

given, $|b_n| \leq c_n$ for all $n \in \mathbb{N}$

$\Rightarrow -c_n \leq b_n \leq c_n$

now, $\lim_{n\to\infty} -c_n = -0 = 0$

$\lim_{n\to\infty} c_n = 0$

Hence, from Sandwich theorem,

$\lim_{n\to\infty} b_n = 0.$

Theorem 4:- Continuous function Theorem

Let $\{a_n\}$ be a sequence of real numbers if $a_n \to L$ $\left(\lim_{n\to\infty} a_n = L\right)$ and if f is a function that is continuous at 'L' and defined on all a_n then,

$$f(a_n) \to f(L)$$

i.e. $\lim_{n\to\infty} f(a_n) = f(L)$

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eg:- Prove that $\sqrt{\frac{n+1}{n}} \longrightarrow 1$.

Note

Soln

Let $a_n = \frac{n+1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = 1 + \frac{1}{n} = 1$$

let $f(x) = \sqrt{x}$, here $f(x)$ is continuous at $x = 1$ and defined at every terms of sequence.

Here, $f(a_n) \to f(L)$

or, $\sqrt{\frac{n+1}{1}} \to \sqrt{1}$

$\therefore \sqrt{\frac{n+1}{n}} \to 1$ #

Theorem - 5:- (L-Hospital's Rule).

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and $\{a_n\}$ is a sequence of real number such that $f(n) = a_n$ for $n \geq n_0$.

Then,

$$\lim_{x\to\infty} f(x) = L = \lim_{n\to\infty} a_n$$

eg:- $\lim_{n\to\infty} \frac{\log n}{n}$, prove that, $\lim_{n\to\infty} \frac{\log n}{n} = 0$.

Soln

$$= \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{1}{\infty} = 0$$

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Some formulas;

Note

1. $\lim_{n \to \infty} \frac{\log n}{n} = 0$

2. $\lim_{n \to \infty} \sqrt[n]{n} = 1$

3. $\lim_{n \to \infty} x^{1/n} = 1$ if $x > 0$

4. $\lim_{n \to \infty} x^n = 0$ if $|x| < 1$ or $-1 < x < 1$

5. $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

6. $\lim_{n \to \infty} \frac{x^n}{n!} = 0$

eg: $\lim_{n \to \infty} \frac{\log(n^2)}{n}$

$= \lim_{n \to \infty} \frac{2 \log n}{n}$

$= 2 \lim_{n \to \infty} \frac{\log n}{n}$

$= 2 \times 0 = 0$

eg: $\lim_{n \to \infty} \sqrt[n]{n^2}$

$= \lim_{n \to \infty} n^{2/n}$

$= \lim_{n \to \infty} \left(n^{\frac{1}{n}}\right)^2$

$= \lim_{n \to \infty} \left(\sqrt[n]{n}\right)^2$

$= 1^2 = 1$

eg: $\sqrt[n]{3n} = 3^{1/n} \cdot n^{1/n}$

$= 1 \times 1 = 1$

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Note

eg 1. $\lim_{n \to \infty} \left(\frac{-1}{2}\right)^n = 0$

eg 1. $\lim_{n \to \infty} \left(\frac{n-2}{n}\right)^n$

$= \lim_{n \to \infty} \left(1 + \frac{(-2)}{n}\right)^n$

$= e^{-2}$

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Note

Series

If $\{a_n\}$ is a sequence of numbers, then, the sum $a_1 + a_2 + a_3 + \dots a_n + \dots$ is an infinite series. The number a_n is the nth term of series. It is denoted by

$$\sum a_n \text{ or } \sum_{n=1}^{\infty} a_n.$$

Partial sum

Let $\{a_n\}$ is a sequence and $\sum a_n$ is a corresponding series then the sequence $\{S_n\}$ defined as;

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

It is the sequence of partial sum of the series $\sum a_n$.

If the sequence of partial sum converges then the series $\sum a_n$ is also converged.

If $S_n \to L$ then,

$$\sum a_n \to L.$$

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we write,

Note $a_1 + a_2 + a_3 + \cdots a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$

If the sequence of partial sum of the series does not converge, we say that, the series is diverged.

Geometrical series

The geometrical series $a + ar + ar^2 + \ldots$ converge to $\frac{a}{1-r}$ if $|r| < 1$.

and diverges if $|r| \geq 1$

example:

$$1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} \cdots$$

$$r = \frac{1}{\alpha} < 1$$

Converges to $\frac{a}{1-r}$

$$= \frac{1}{1 - \frac{1}{\alpha}}$$

$$= \alpha \quad \text{(converges)}$$

example (2) $2 + 4 + 8 + 16 + \cdots$

$r = 2 > 1$ (diverges)

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The nth term test for divergence.

Note

(1) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

but converse may not be true.

(2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} a_n$ fails to exist (become undefined) or become different from zero (0).

eg 1: $\sum n^2$

$= \lim_{n\to\infty} a_n$

$= \lim_{n\to\infty} n^2$

$= \infty^2$

$= \infty \neq 0$

i.e. Series diverged.

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eg: $\sum_{n=1}^{\infty} (-1)^{n+1}$

$= \lim_{n \to \infty} a_n$

$= \lim_{n \to \infty} (-1)^{n+1}$

$= (-1)^{\infty+1}$

$= (-1)^{\infty}$ (doesn't exist).

i.e. series is diverged.

eg:

$\sum a_n = 1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2}\right) + \cdots$

$a_n = \frac{1}{2^{n-1}}$

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2^{n-1}} = \frac{1}{2^{\infty}} = 0$

Here, $\lim_{n \to \infty} a_n = 0$ but the series diverges.

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The integral test for convergence & divergence

Note if $\sum_{n=a}^{\infty} a_n$ is a series, let $f(x)$ be defined for all $x \geq a$ and $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq a$, then $\sum_{n=a}^{\infty} a_n$ converges whenever $\int_a^{\infty} f(x)\,dx$ converges.

eg: $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ converges if $\int_2^{\infty} \frac{\ln x}{\sqrt{x}}\,dx$ converges.

Note: similar principal with divergence.

Theorem (p-test)

The infinite series $\sum \frac{1}{n^p}$ converge if $p > 1$ and diverge if $p \leq 1$.

eg: $\frac{1}{n^5}$, Here $p = 5 > 1$. (converged).

eg: $\frac{1}{n^{0.5}}$, Here $p = 0.5 < 1$ (diverged

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Limit comparision test

Note Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$.

(1) if $\lim_{n \to \infty} \frac{a_n}{b_n} = c > 0$

then, $\sum a_n$ and $\sum b_n$ both convergent or both divergent.

(2) if $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and if $\sum b_n$ is convergent then $\sum a_n$ also converges.

(3) if $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and if $\sum b_n$ is divergent then $\sum a_n$ is also divergent.

#. examine the following series converge or diverge.

$$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots$$

Here, $a_n = \frac{2n+1}{(n+1)^2} = \frac{n(2+\frac{1}{n})}{\{n(1+\frac{1}{n})\}^2} = \frac{n(2+\frac{1}{n})}{n^2(1+\frac{1}{n})^2}$

let $b_n = \frac{1}{n}$

Here, $\sum b_n = \sum \frac{1}{n}$ is diverge as $p = 1$.

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Note

now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2+\frac{1}{n}}{\cancel{n^2}(1+\frac{1}{n})^2}}{\frac{1}{\cancel{n^2}}}$$

$$= \frac{2+0}{(1+0)^2}$$

$$= 2 > 0$$

As, $\sum b_n$ is diverged, $\sum a_n$ is also diverged.

Hence, given series $\sum a_n$ is diverged.

The ratio test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho,$$

then,

(i) The series is converged if $\rho < 1$.
(ii) The series is diverged if $\rho > 1$
(iii) The series is not ~~the~~ conclusive if $\rho = 1$.
(inconclusive)

Example:- Check the convergence ~~of~~ divergence of the given series.

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Note

(1) $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$ (2) $\sum \frac{2^n}{n!}$ (3) $\frac{\sum 4^n n! n!}{(2n)!}$

(1)

Here, $a_n = \frac{2^n+5}{3^n}$

$a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$

Now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}+5}{3^{n+1}} \cdot \frac{3^n}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{2^n.2+5}{3^n.3} \times \frac{3^n}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{2^n\left(2+\frac{5}{2^n}\right)}{3} \times \frac{1}{2^n\left(1+\frac{5}{2^n}\right)}$$

$$= \frac{2+0}{3} \times \frac{1}{1+0}$$

$$= \frac{2}{3} < 1.$$

Hence the given series is converged.

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Note

(2)

Here,

$$a_n = \frac{2^n}{n!}$$

$$a_n = \frac{2^{n+1}}{(n+1)!}$$

now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}.2}{(n+1)\cancel{n!}} \times \frac{\cancel{n!}}{\cancel{2^n}}$$

$$= \frac{2}{\infty}$$

$$= 0 < 1$$

Hence the given series is converged.

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(3) Here,

$$a_n = \frac{4^n n! n!}{(2n)!}$$

$$a_{n+1} = \frac{4^{n+1} (n+1)!(n+1)!}{(2(n+1))!}$$

now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4^{n+1}(n+1)!(n+1)!}{(2(n+1))!} \times \frac{(2n)!}{4^n n! n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{4^n}.4.(n+1)\cancel{n!}(n+1)\cancel{n!}}{(2(n+1))!} \times \frac{(2n)!}{\cancel{4^n}.\cancel{n!}\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{4(n+1)(n+1)}{(2n+1)(2n+1)\cancel{(2n)!}} \times \cancel{(2n)!}$$

$$= \lim_{n\to\infty} \frac{4(n+1)\cancel{(n+1)}}{2\cancel{(n+1)}(2n+1)}$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{2n+1}$$

$$= \lim_{n\to\infty} \frac{2n+2}{2n+1}$$

$$= \lim_{n\to\infty} \frac{2+\frac{2}{n}}{2+\frac{1}{n}}$$

$$= \frac{2}{2} = 1$$

Hence, the series is inconclusive.

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The nth root test

Note

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n > N$ and suppose that $\lim_{n \to \infty} \sqrt[n]{a_n} = \rho$, then

(i) The series converges if $\rho < 1$.
(ii) The series diverges if $\rho > 1$
(iii) The series is inconclusive if $\rho = 1$.

eg:- Check them.

(a) $\sum \frac{n^2}{2^n}$ (b) $\sum \frac{2^n}{n^2}$

Soln

(a) Here, $a_n = \frac{n^2}{2^n}$

now,

$$\lim_{n \to \infty} \sqrt[n]{a_n}$$

$$= \lim_{n \to \infty} \left(\frac{n^2}{2^n}\right)^{1/n}$$

$$= \lim_{n \to \infty} \frac{(n^2)^{1/n}}{2}$$

$$= \lim_{n \to \infty} \frac{\left(n^{\frac{1}{n}}\right)^2}{2}$$

$$= \frac{1^2}{2} = \frac{1}{2} < 1$$

So, the series $\sum a_n$ is converged.

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Note

(6) Soln

Here, $a_n = \frac{2^n}{n^2}$

now, $\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}}$

$= \lim_{n\to\infty} \frac{(2^n)^{1/n}}{(n^2)^{1/n}}$

$= \lim_{n\to\infty} \frac{2}{(n^{1/n})^2}$

$= \frac{2}{(1)^2}$

$= 2 > 1$

Hence, the series $\sum a_n$ is diverged.

The alternating series

A series in which the terms are alternatively positive and negative is called an alternating series.

eg: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \frac{(-1)^{n+1}}{n} + \cdots$

Theorem: The alternating series test (Leibniz's test)

The series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - \cdots$

converge if all three of the following condition are satisfied.

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Note

(i) The a_n's are ~~all~~ all positive.

(ii) $a_n > a_{n+1}$ for some $n > N$

(iii) $\lim_{n \to \infty} a_n = 0$

eg:- $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

soln

P_1 : $\frac{1}{n} > 0$ for all $n > 0$.

P_2 : $a_n > a_{n+1}$ because $\frac{1}{n} > \frac{1}{n+1}$

P_3 : $\lim_{n \to \infty} a_n$

$= \lim_{n \to \infty} \frac{1}{n}$

$= 0$

Hence, $\sum (-1)^{n+1} \frac{1}{n}$ is convergent

Absolute convergence

A series $\sum a_n$ converges absolutely if corresponding series of absolute value i.e. $\sum |a_n|$ converges.

> Theorem: The absolute convergent test

If $\sum |a_n|$ converge, then $\sum_{n=1}^{\infty} a_n$ converges.

Note

eg: $\sum (-1)^{n+1} \frac{1}{n^3}$

Here, absolute series of above series is $\sum \frac{1}{n^3}$

By p-test $\sum \frac{1}{n^3}$ is converged.

Hence, $\sum (-1)^{n+1} \frac{1}{n^3}$ is convergent.

Conditionally convergent

A series that is converged but its absolute series is not converged is called conditionally convergent series.

eg: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$, this is converged.

absolute $= \sum_{n=1}^{\infty} \frac{1}{n}$, as $p=1$ (p-test), it is diverged.

Algebra

System of linear equation

Note A set of linear equations is called a system of linear equation.

eg:- $\left.\begin{aligned} 2x+3y+5z &= 100 \\ x-y+z &= 20 \\ 3x+y &= 25 \end{aligned}\right\}$

$\left.\begin{aligned} x+y &= 10 \\ 2x+2y &= 7 \end{aligned}\right\}$ are parallel lines, hence no solution.

for equations,

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Matrix of coefficient $= \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Augmented matrix $= \left(\begin{array}{ccc:c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$

Row-Echelon method (row of all zeros at bottom)

↳ french meaning = Ladder

first non zero entry in each row

eg:- $\begin{bmatrix} 2 & * & * & * \\ 0 & 5 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2, 5, 7, 10 are pivots.

* may be zero or non-zero

$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \leftarrow$ REF , $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftarrow$ not REF

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Note

eg:. $\begin{pmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow$ Row Echelon form.

2, 3, 1 are pivots.

eg:. $\begin{pmatrix} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 & -1 \\ 0 & 0 & 2 & 5 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow$ R.E.F.

Reduced Echelon form

eg:. $\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

i.e → make pivots one
→ make elements below and above pivots zero

eg:. $\begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow$ RREF

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \leftarrow$ neither REF nor RREF

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Note

eg: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ⇐ both

$\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ ⇐ REF
⇐ not RREF

$\begin{bmatrix} -1 \end{bmatrix}$ ⇐ REF, not RREF

$\begin{bmatrix} 1 \end{bmatrix}$ ⇐ REF, RREF (Both).

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ⇐ Neither

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ⇐ Both

eg: Solve the following system using Gaussian elimination method

$$4x - 11y + 13z = 3$$
$$4x - 12y + 10z = 0$$
$$x - 3y + 2z = -1$$

Soln The augmented matrix is;

$$\left(\begin{array}{ccc|c} 4 & -11 & 13 & 3 \\ 4 & -12 & 10 & 0 \\ 1 & -3 & 2 & -1 \end{array}\right)$$

We will convert this into Reduced Row Echelon Form. ~~toc~~

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Note

$R_1 \leftrightarrow R_3$

$$\left(\begin{array}{ccc:c} 1 & -3 & 2 & -1 \\ 4 & -12 & 10 & 0 \\ 4 & -11 & 13 & 3 \end{array}\right)$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 4R_1$

$$\left(\begin{array}{ccc:c} 1 & -3 & 2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 1 & 5 & 7 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{ccc:c} 1 & -3 & 2 & -1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

$R_3 \to \frac{R_3}{2}$

$$\left(\begin{array}{ccc:c} 1 & -3 & 2 & -1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$R_2 \leftrightarrow R_2 - 5R_3$

$$\left(\begin{array}{ccc:c} 1 & -3 & 2 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$R_1 \leftrightarrow R_1 + 3R_2$

$$\left(\begin{array}{ccc:c} 1 & 0 & 2 & -10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

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Note

$R_1 \leftrightarrow R_1 - 2R_3$

$$\begin{pmatrix} 1 & 0 & 0 & : & -4 \\ 0 & 1 & 0 & : & -3 \\ 0 & 0 & 1 & : & 2 \end{pmatrix}$$

Comparing, we get

$z = 2$

$y = -3$

$x = -4$

Can also be done by Row Echelon Form as;

$$\begin{pmatrix} 4 & -11 & 13 & : & 3 \\ 4 & -12 & 10 & : & 0 \\ 1 & -3 & 2 & : & -1 \end{pmatrix}$$

$R_2 \leftrightarrow R_2 - R_1$

$R_3 \leftrightarrow R_3 - \frac{R_1}{4}$

$$\begin{pmatrix} 4 & -11 & 13 & : & 3 \\ 0 & -1 & -3 & : & -3 \\ 0 & \frac{-1}{4} & \frac{-5}{4} & : & \frac{-7}{4} \end{pmatrix}$$

$R_3 \leftrightarrow R_3 - \frac{R_2}{4}$

$$\begin{pmatrix} 4 & -11 & 13 & : & 3 \\ 0 & -1 & -3 & : & -3 \\ 0 & 0 & \frac{-1}{2} & : & -1 \end{pmatrix}$$

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we get,

$$\boxed{-\frac{1}{2}z = -1}$$

$\therefore z = 2$

$$\boxed{-y - 3z = -3}$$

or, $-y - 6 = -3$

$\therefore y = -3$

$$\boxed{4x - 11y + 13z = 3}$$

or, $4x + 33 + 26 = 3$

$\therefore x = -14$

Solve the following two system.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -3x_1 + 7x_2 - 2x_3 &= 1 \\ -2x_1 + 3x_2 + 3x_3 &= -1 \end{aligned} \quad \text{and} \quad \begin{aligned} x_1 + 2x_2 - x_3 &= -2 \\ 3x_1 + 7x_2 - 2x_3 &= 5 \\ -2x_1 + 3x_2 + 3x_3 &= 3 \end{aligned}$$

Soln

Let's use RREF Method,

The augmented matrix is;

$$\left(\begin{array}{ccc:cc} 1 & 2 & -1 & 4 & -2 \\ 3 & 7 & -2 & 1 & 5 \\ -2 & 3 & 3 & -1 & 3 \end{array}\right)$$

$R_2 \leftrightarrow R_2 - 3R_1$, $R_3 \leftrightarrow R_3 + 2R_1$

$$\left(\begin{array}{ccc:cc} 1 & 2 & -1 & 4 & -2 \\ 0 & 1 & 1 & -11 & 11 \\ 0 & 7 & 1 & 7 & -1 \end{array}\right)$$

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$R_1 \to R_1 - 2R_2, \quad R_3 \to R_3 - 7R_2$

$$\left(\begin{array}{ccc:cc} 1 & 0 & -3 & 26 & -20 \\ 0 & 1 & 1 & -11 & 11 \\ 0 & 0 & -6 & 80 & -78 \end{array}\right)$$

$R_3 \leftrightarrow \frac{1}{-6} R_3$

$$\left(\begin{array}{ccc:cc} 1 & 0 & -3 & 26 & -20 \\ 0 & 1 & 1 & -11 & 11 \\ 0 & 0 & 1 & -10 & 13 \end{array}\right)$$

$R_1 \leftrightarrow R_1 + 3R_3, \quad R_2 \leftrightarrow R_2 - R_3$

$$\left(\begin{array}{ccc:cc} 1 & 0 & 0 & -16 & 15 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & -10 & 13 \end{array}\right)$$

solution of 1st system

solution of second system.

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Note

If the system of linear equations have solution, we can say that the system is consistent.

If the system of linear equations have no solution, the system is considered as inconsistent.

Row Rank

The row rank of a matrix is the number of non-zero rows in a row echelon form of matrix.

eg:- $\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Row Rank = 3

eg:- $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Row Rank = 2

Theorem

: A linear system is consistent if and only if the row rank of the coefficient matrix is equal to that of the augmented matrix.

Homogeneous system

A linear system is called homogeneous if all of the constant terms are zero.

eg:- $2x + 3y + 7z = 0$
$3x + 7y + 4z = 0$
$x - y + z = 0$

Note:- A Homogeneous system is always consistent.

Parametrically represented solution

When the augmented matrix of a system has been put into reduced echelon form, if number of rows (non zero) is less than no. of variables, then we should use parametrically represented solution method.

Also, there will be two types of variables;

⇒ A variable that corresponding to a leading 1 is called leading variable. (basic variable) and other variable are called non-leading variable. (free variable).

If no of eqn is less than no. of variable, it will surely give parametric solution.

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Q. find the solution of following system.

$$x_1 + 2x_2 - 3x_3 = 3$$
$$-2x_1 - 5x_1 + 4x_3 = 5$$
$$-5x_1 - 13x_2 + 9x_3 = 18$$

soln The augmented matrix is;

$$\left(\begin{array}{ccc:c} 1 & 2 & -3 & 3 \\ -2 & -5 & 4 & 5 \\ -5 & -13 & 9 & 18 \end{array}\right)$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + 5R_1$

$$\left(\begin{array}{ccc:c} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & -3 & -6 & 33 \end{array}\right)$$

$R_2 \to -R_2$

$$\left(\begin{array}{ccc:c} 1 & 2 & -3 & 3 \\ 0 & 1 & 2 & -11 \\ 0 & -3 & -6 & 33 \end{array}\right)$$

$R_1 \to R_1 - 2R_2$
$R_3 \to R_3 + 3R_2$

$$\left(\begin{array}{ccc:c} 1 & 0 & -7 & 25 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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now,

$$x_1 + 0x_2 - 7x_3 = 25 \longrightarrow (1)$$
$$x_2 + 2x_3 = -11 \longrightarrow (2)$$

x_1 and x_2 are leading variable.
x_3 is non-leading variable.

Let $x_3 = t$

from (2), $x_2 + 2t = -11$
$x_2 = -11 - 2t$

from (1), $x_1 - 7t = 25$
$\therefore x_1 = 7t + 25$

Required solution is,

$$x_1 = 7t + 25$$
$$x_2 = -11 - 2t$$
$$x_3 = t$$

when $t = 0$; $x_1 = 25$
$x_2 = -11$
$x_3 = 0$

If finite value is asked, we can put any value of t

Vector Space

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Linear combinations

If $\vec{u_1}, \vec{u_2} \ldots \vec{u_k}$ are vectors on $\mathbb{R}^n$ and if $c_1, c_2, c_3 \ldots c_k$ are scalars, the vector

$\vec{V} = c_1\vec{u_1} + c_2\vec{u_2} + \ldots + c_k\vec{u_k}$ is called linear combination of vectors.

eg:- Let $\vec{V} = (2,9)$, $\vec{u_1} = (2,-3)$
& $\vec{u_2} = (2,1)$

then, $\vec{V} = -2\vec{u_1} + 3\vec{u_2}$

$= -2(2,-3) + 3(2,1)$
$= (-4,6) + (6,3)$
$= (2,9)$

Q. Express the vector $\vec{V} = (7,7)$ as a linear combination of $\vec{u_1} = (-1,2)$
& $\vec{u_2} = (3,1)$

soln

Let $\vec{V} = c_1\vec{u_1} + c_2\vec{u_2}$

or, $(7,7) = c_1(-1,2) + c_2(3,1)$
or, $(7,7) = (-c_1, 2c_1) + (3c_2, c_2)$
or, $(7,7) = (-c_1 + 3c_2, 2c_1 + c_2)$

we can write,

$$-c_1 + 3c_2 = 7 \rightarrow (1)$$
$$2c_1 + c_2 = 7 \rightarrow (2)$$

solving eqn (1) and (2) by $(2(1) + (2))$

$$-2c_1 + 6c_2 = 14$$
$$2c_1 + c_2 = 7$$
$$\overline{7c_2 = 21}$$
$$c_2 = 3$$

Hence, $c_1 = 2$

Hence, we can express as;

$$\vec{v} = 2\vec{u_1} + 3\vec{u_2}$$

Note:- If we have the augmented matrix of a consistent system, then, the linear combination is possible and the scalars are found by solving the system.

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Q. Determine if $V = (1, -3, 2)$ can be written as the linear combination of $(1, 2, -1)$, $(3, 5, 2)$, $(4, 7, 1)$.

Soln

Let $\vec{V} = c_1 \vec{u_1} + c_2 \vec{u_2} + c_3 \vec{u_3}$

$$(1, -3, 2) = c_1(1, 2, -1) + c_2(3, 5, 2) + c_3(4, 7, 1)$$

or, $(1, -3, 2) = (c_1 + 3c_2 + 4c_3,\ 2c_1 + 5c_2 + 7c_3,\ -c_1 + 2c_2 + c_3)$

Equating, we get,

$$c_1 + 3c_2 + 4c_3 = 1$$

$$2c_1 + 5c_2 + 7c_3 = -3$$

$$-c_1 + 2c_2 + c_3 = 2$$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 2 & 5 & 7 & -3 \\ -1 & 2 & 1 & 2 \end{array}\right)$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$,

$$\begin{pmatrix} 1 & 3 & 4 & : & 1 \\ 0 & -1 & -1 & : & -5 \\ 0 & 5 & 5 & : & 3 \end{pmatrix}$$

$R_2 \to (-1) R_2$

$$\begin{pmatrix} 1 & 3 & 4 & : & 1 \\ 0 & 1 & 1 & : & 5 \\ 0 & 5 & 5 & : & 3 \end{pmatrix}$$

$R_3 \to R_3 - 5R_2$

$$\underbrace{\begin{pmatrix} 1 & 3 & 4 & : & 1 \\ 0 & 1 & 1 & : & 5 \\ 0 & 0 & 0 & : & -22 \end{pmatrix}}_{\text{Augmented matrix}}$$

$\underbrace{\qquad\qquad}_{\text{coefficient matrix}}$

Rank of coefficient matrix $= 2$
Rank of augment matrix $= 3$

$A\vec{x} = \vec{b} \longleftarrow$ vector equation

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Hence, the system of equation ~~iff~~ is inconsistent.

i.e. c_1, c_2, c_3 cannot be found so the linear combination is not possible.

Theorem:-

The linear system $A\vec{x} = \vec{b}$ has a solution if and only if (iff), $\vec{b}$ can be written/expressed as a linear combination of the column vectors of A.

eg:

$$\begin{pmatrix} 1 & 6 & 0 \\ 2 & 7 & 1 \\ 3 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

i.e. $\overset{\rightarrow x}{c_1}(1,2,3) + \overset{\rightarrow y}{c_2}(6,7,-2) + \overset{\rightarrow z}{c_3}(0,1,2) = (1,2,3)$

then only x, y, z can be found.

example:-

find the solution of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 10 & 5 \\ 2 & 7 & 1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

Soln

The augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & 3 & 4 & 6 \\ 3 & 10 & 5 & 4 \\ 2 & 7 & 1 & -2 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 3 & 4 & 6 \\ 0 & 1 & -7 & -14 \\ 0 & 1 & -7 & -14 \end{array}\right]$$

$R_1 \to R_1 - 3R_2$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & 25 & 48 \\ 0 & 1 & -7 & -14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $\vec{x} = (x, y, z)$.

then, $x + 25z = 48$

$y - 7z = -14$

x and y are leading variable;

z is non-leading variable.

Let $z = t$

or $x + 25t = 48$

$\therefore x = 48 - 25t$

$\therefore y = 7t - 14$

put $t = 1$,

$x = 23$

$y = -7$

$z = 1$

now,

$(6, 4, -2) = 23(1, 3, 2) - 7(3, 10, 7) + (4, 5, 1)$

$= (6, 4, -2)$

Hence, $(6, 4, -2) = 23(1, 3, 2) - 7(3, 10, 7) + (4, 5, 1)$

#

Span of vectors

The span of a set of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3 \ldots \vec{u}_k\}$ in $\mathbb{R}^n$ is the set of all linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3 \ldots \vec{u}_k$

i.e. $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + \ldots c_k\vec{u}_k$ is a span of $\vec{u}_1, \vec{u}_2, \vec{u}_3 \ldots \vec{u}_k$ where $c_1, c_2, c_3 \ldots c_k$ are any scalars.

eg: prove that $\mathbb{R}^3$ is the span of vectors $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$

proof:

Let $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$ be an linear combination of $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

$= c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$

$= (c_1,0,0) + (0,c_2,0) + (0,0,c_3)$

$= (c_1, c_2, c_3) \in \mathbb{R}^3$

$\therefore \mathbb{R}^3$ is a span of e_1, e_2, e_3.

note: for a ~~vector~~ matrix to span any dimension, pivot elements must be present in same number of dimension in reduced row-echelon form.

OR

Let $(a, b, c) \in \mathbb{R}^3$

$$(a, b, c) = a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$$
$$= a(1,0,0) + b(0,1,0) + c(0,0,1)$$
$$= (a, b, c)$$

Vector sub-space:

A non-empty subset 'S' of $\mathbb{R}^n$ is called a sub-space if the following properties are satisfied.

(i) Zero vector is in 'S'.
(ii) If $\vec{u}$ and $\vec{v}$ are in 'S', then $\vec{u} + \vec{v} \in S$.

(iii) If $\vec{u}$ is in 'S' and c be the any scalar then $c\vec{u} \in S$.

eg: prove that $(4t, -2t, t)$ is a sub space of $\mathbb{R}^3$.

proof:

Let $S = (4t, -2t, t)$

$P_1 : (0,0,0) = (4 \times 0, -2 \times 0, 0) \in S$

$\vec{u}$ $\vec{v}$

P_2: Let $(4t_1, -2t_1, t_1), (4t_2, -2t_2, t_2) \in S$.

then, $\vec{u} + \vec{v} = (4t_1, -2t_1, t_1) + (4t_2, -2t_2, t_2)$

$= (4(t_1+t_2), -2(t_1+t_2) + (t_1+t_2))$
$\in S$

$\therefore \vec{u} + \vec{v} \in S$

P_3: Let $\vec{u} = (4t_1, -2t_1, t_1) \in S$
$\vec{v} = (4t_2, -2t_2, t_2)$

c be any scalar, then,

$c\vec{u} = c(4t_1, -2t_1, t_1)$
$= (4(ct_1), -2(ct_1), (ct_1))$
$= \in S$

$\therefore$ S is a sub-space of $\mathbb{R}^3$.

Nullspace

The nullspace of an ~~$\vec{M} \times \vec{N}$~~ $m \times n$ matrix, A, is the set of all vectors $\vec{x}$ such that $A\vec{x} = 0$.

Eg: Show that the nullspace of the matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 3 & -7 \end{bmatrix}$$ is the span of a single vector.

Soln

Let $\vec{x} = (x_1, x_2, x_3)$

such that $A\vec{x} = 0$

The augmented matrix is;

$$\left[\begin{array}{ccc:c} 1 & -2 & 4 & 0 \\ -1 & 3 & -7 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\left[\begin{array}{ccc:c} 1 & -2 & 4 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$R_1 \to R_1 + 2R_2 \Rightarrow \left(\begin{array}{ccc:c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right)$

now,

$$x_1 - 2x_3 = 0$$
$$x_2 - 3x_3 = 0$$

Let $x_3 = t$

then, $x_1 = 2t$
$x_2 = 3t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t \\ 3t \\ t \end{pmatrix}$$

$$= t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$\therefore \vec{x} = t(2,3,1)$

Hence, $\vec{x}$ is a span of a vector $(2,3,1)$

Columnspace

The column space of a matrix A is defined to be the subspace spanned by column vectors of A. That is, if A has the column vectors $\vec{a_1}, \vec{a_2}, \vec{a_3} \ldots \vec{a_k}$ then, the columnspace of A is the set of all linear combinations.

$$c_1 \vec{a_1} + c_2 \vec{a_2} + c_3 \vec{a_3} + \ldots + c_k \vec{a_k}$$

where $c_1, c_2, c_3 \ldots c_k$ are any constants.

eg 1. find the column space of matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

The column vector of A are;

$$\vec{a_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{a_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{a_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

now, column space is,

$$\vec{V} = c_1 \vec{a_1} + c_2 \vec{a_2} + c_3 \vec{a_3}$$

$$= c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

where c_1, c_2, c_3 are arbitrary constants.

NOTE:-

The linear system $A\vec{x} = \vec{b}$ is consistent if and only if (iff) $\vec{b}$ is in the column space.

example:-

Describe the span of $\vec{a_1} = (1,0,1)$
$\vec{a_2} = (1,1,3)$
$\vec{a_3} = (2,3,7)$

or,

if $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 3 & 7 \end{bmatrix}$, find column space.

Soln

Let $\vec{b} = (x, y, z)$

such that $\vec{b} = c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3}$

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$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ The augmented matrix is;

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & x \\ 0 & 1 & 3 & y \\ 1 & 3 & 7 & z \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & x \\ 0 & 1 & 3 & y \\ 0 & 2 & 5 & z-x \end{array}\right]$$

$R_1 \to R_1 - R_2$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & x-y \\ 0 & 1 & 3 & y \\ 0 & 0 & -1 & -2y+z-x \end{array}\right]$$

$R_3 \to -R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & x-y \\ 0 & 1 & 3 & y \\ 0 & 0 & 1 & x+2y-z \end{array}\right]$$

$R_1 \to R_1 + R_3$
$R_2 \to R_2 - 3R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2x+y-z \\ 0 & 1 & 0 & -3x-5y+3z \\ 0 & 0 & 1 & x+2y-z \end{array}\right]$$

Span or column space is,

$$\vec{b} = (2x+y-z)\vec{a_1} + (-3x-5y+3z)\vec{a_2} + (x+2y-z)\vec{a_3}$$

Describe the span of vectors.

$\vec{a_1} = (1,0,1)$
$\vec{a_2} = (1,1,3)$
$\vec{a_3} = (2,3,8)$

Soln: Let $\vec{b} = (x, y, z)$

such that,
$\vec{b} = c_1\vec{a_1} + c_2\vec{a_2} + c_3\vec{a_3}$

where c_1, c_2, c_3 are arbitrary constants

now,

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 3 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The augmented matrix is;

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & x \\ 0 & 1 & 3 & y \\ 1 & 3 & 8 & z \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & x \\ 0 & 1 & 3 & y \\ 0 & 2 & 6 & z-x \end{array}\right]$$

$R_1 \to R_1 - R_2$, $R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & x-y \\ 0 & 1 & 3 & y \\ 0 & 0 & 0 & z-x-2y \end{array}\right]$$

This shows that the system is consistent only if $z - x - 2y = 0$

Let $c_3 = t$

$$c_1 - c_3 = x - y$$
$$c_1 = x - y + t$$

also,

$$c_2 + 3c_3 = y$$
$$\therefore c_2 = y - 3t$$

if $t = 0$, $c_3 = 0$

$$c_1 = x - y$$
$$c_2 = y$$

Hence, $\vec{b} = (x-y)\vec{a_1} + y\,\vec{a_2}$

$$= (x-y)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

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Linearly dependent and linearly independent

The set of K (where $K \geq 2$) vectors, $\vec{u_1}, \vec{u_2} \ldots \vec{u_K}$ in R^n is said to be linearly dependent if one or more of vectors can be written as a linear combination of other vectors. The set of vectors is called linearly independent if it is not linearly dependent.

Theorem (Criteria)

The vectors $\vec{u_1}, \vec{u_2} \ldots \vec{u_K}$ in R^n are linearly dependent if and only if there exists scalars $c_1, c_2, c_3 \ldots c_K$ with at least one scalar not zero such that

$$c_1 \vec{u_1} + c_2 \vec{u_2} + c_3 \vec{u_3} + \ldots c_K \vec{u_K} = 0 \rightarrow (1)$$

if (1) is followed if all scalars are zero, then linearly independent.

eg:- show that vectors $\vec{u_1} = (-6, 2)$ and $\vec{u_2} = (9, -3)$ are linearly dependent.

Soln $\vec{u_1} = (-6, 2) = 2(-3, 1)$

$\vec{u_2} = (9, -3) = -3(-3, 1)$

$= -3 \frac{\vec{u_1}}{2}$

Hence, $\vec{u_2} = -\frac{3}{2} \vec{u_1}$

Hence, $\vec{u_1}$ and $\vec{u_2}$ are linearly dependent.

The criteria for linearly dependent and independent

Note :- The homogeneous system of equation has non-zero solution or zero solution.

⇒ dependent/independent test for given vector $\{\vec{u_1}, \vec{u_2}, \vec{u_3} \ldots \vec{u_k}\}$

Suppose that;

$$c_1 \vec{u_1} + c_2 \vec{u_2} + \ldots + c_k \vec{u_k} = 0$$

or, $U \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_k \end{bmatrix} = 0$

now, reducing U to row-echelon form, say E.

(i) If each column of E has leading 1, then the vectors are linearly independent.

(ii) If some column of E do not have leading 1, then the vectors are linearly dependent.

eg. $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix} \Rightarrow$ dependent.

this column don't have leading 1

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$ independent

leading 1

leading 1.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ independent

eg: $\mathbb{R}^3 \Rightarrow$ 3 dimension

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Other notes

⇒ If the row rank of U is same as the number of vectors, then the vectors are linearly independent.

⇒ If the row rank of U is less than the number of vectors, then the vectors are linearly dependent.

⇒ if no. of vectors (k) > n (dimension), then the vectors are linearly dependent.

⇒ if A be the $\overset{r}{n} \times \overset{c}{n}$ matrix, then the vectors are linearly independent iff

$|U| \neq 0$. (Can calculate inverse)
(invertable)

Example 1: Determine the following vectors if they are dependent or independent.

$\vec{u_1} = (1, 2, 1)$
$\vec{u_2} = (3, 1, 2)$
$\vec{u_3} = (2, 5, 1)$

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Soln

The matrix of column vectors are;

$$U = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 1 & 2 & 1 \end{pmatrix}$$

Now,

$$|U| = 1(1-10) - 3(2-5) + 2(4-1)$$
$$= -9 + 9 + 6$$
$$= 6 > 0$$

$\therefore$ The given vectors are linearly independent.

Basis Vectors

$\boxed{\text{Def}^n}$: Let S be a subspace of $\mathbb{R}^n$. The set of vectors $\{u_1, u_2, \ldots u_k\}$ ~~of the vectors~~ in S is called basis of S if

(i) The vectors $\vec{u_1}, \vec{u_2}, \ldots \vec{u_k}$ are linearly independent.

(ii) S should be spanned by $\vec{u_1}, \vec{u_2} \ldots \vec{u_k}$.

Here, each vector in basis is called a basis vectors.

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Note: The standard basis for vector space $\mathbb{R}^n$ are

$\vec{e_1} = (1, 0, 0 \ldots 0)$

$\vec{e_2} = (0, 1, 0, 0 \ldots 0)$

$\vec{e_3} = (0, 0, 1, \ldots 0)$

$\vec{e_n} = (0, 0, 0, \ldots 1)$

Here, $e_1, e_2, \ldots e_n$ are linearly independent.

If $(x_1, x_2, x_3 \ldots x_n) \in \mathbb{R}^n$, then

$$(x_1, x_2, x_3 \ldots x_n) = x_1 e_1 + x_2 e_2 + x_3 e_3 + \ldots + x_n e_n$$

eg: The set S, of vectors $v = (x, y, z)$ in $\mathbb{R}^3$ such that $x - 2y - z = 0$, is a sub-space of $\mathbb{R}^3$. find the basis for S.

Soln

$$x - 2y - z = 0$$

$$\text{or, } \begin{pmatrix} 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

Let $z = t_1$

$y = t_2$

$x = 2t_2 + t_1$

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Let $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t_2 + t_1 \\ t_2 \\ t_1 \end{pmatrix}$

$= \begin{pmatrix} t_1 \\ 0 \\ t_1 \end{pmatrix} + \begin{pmatrix} 2t_2 \\ t_2 \\ 0 \end{pmatrix}$

$\vec{v} = t_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Here, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent and S is spanned by $(1,0,1)$ and $(2,1,0)$.

Hence $(1,0,1)$ and $(2,1,0)$ are ~~span~~ basis vectors of S.

Q. Show that the vectors $\vec{u_1} = (1,0,1)$, $\vec{u_2} = (0,1,1)$ and $u_3 = (1,1,0)$ are basis for $\mathbb{R}^3$.

Soln

first criteria

for vector $\vec{u_1}, \vec{u_2}, \vec{u_3}$ to be basis for $\mathbb{R}^3$, they should be linearly independent.

$$\vec{U} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Here, ~~roco~~ rank of U is equal to the number of vectors. So, given vectors are linearly independent.

given to RREF

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in RREF, pivot elements are present in three rows so span $\mathbb{R}^3$.

<u>Second criteria</u>

Let S be subspace of $\mathbb{R}^3$

Let c_1, c_2, c_3 be any ~~scalars~~

$$S = c_1\vec{u_1} + c_2\vec{u_2} + c_3\vec{u_3}$$

$$= c_1(1,0,1) + c_2(0,1,1) + c_3(1,1,0)$$

$$= (c_1, 0, c_1) + (0, c_2, c_2) + (c_3, c_3, 0)$$

$$= (c_1 + c_3, c_2 + c_3, c_1 + c_2)$$

$$= \in \mathbb{R}^3$$

Hence, $(1,0,1)$, $(0,1,1)$ and $(1,1,0)$ are basis for $\mathbb{R}^3$.

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Theorem:-

The column vectors of an $n \times n$ matrix form a basis for $\mathbb{R}^n$ iff the matrix is invertable i.e. $|U| \neq 0$.

Dimension of sub-space or vector-space

Let 'S' be a subspace of $\mathbb{R}^n$, then every basis for S has same number of vectors

def^n:- The number of vectors in a basis for sub-space 'S' is called dimension of S.

Column rank

The column rank of a matrix is the dimension of column space. ((The row rank and column rank of matrix are always equal.))

Basis selection

eg: Select a basis for the column space of the matrix

$$\begin{bmatrix} 0 & 2 & 4 & -2 \\ 2 & -1 & 0 & 5 \\ 1 & -2 & -3 & 4 \end{bmatrix}$$

can make 1 in any column then make the column other element zero.

uniaue format like (1,0,0), (0,1,0), (0,0,1).

can't make same format.

$R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 0 & 2 & 4 & -2 \\ 0 & 3 & 6 & -3 \\ 1 & -2 & -3 & 4 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$

$$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & -3 \\ 1 & -2 & -3 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

Here, first and second column became identical.

so, $\{(0,2,1), (2,-1,-2)\}$ is a basis for column space.

The Nullity of a matrix

The nullity of a matrix is defined to be the dimension of the nullspace of the matrix.

Theorem:

Let A be a matrix of n-columns then,
Row Rank + nullity = n.

Hence, $\boxed{\text{nullity} = n - \text{Row Rank}}$

eg 1: find the rank and nullity of the matrix.

$$\begin{bmatrix} 1 & -2 & -1 & 2 & 8 \\ 1 & -2 & 0 & 1 & 5 \\ 2 & -4 & 0 & 3 & 12 \\ -2 & 4 & 1 & -1 & -9 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 + 2R_1$

$$\begin{bmatrix} 1 & -2 & -1 & 2 & 8 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 2 & -1 & -4 \\ 0 & 0 & -1 & 3 & 7 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & -2 & -1 & 2 & 8 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\begin{bmatrix} 1 & -2 & -1 & 2 & 8 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, Row Rank $= 3$

Columns of matrix $(n) = 5$

nullity $= n -$ Row Rank

$= 5 - 3$

$= 2$ #

Eigenvalues and Eigenvectors

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A number λ is called an eigen-value of the square matrix A if there exist a non-zero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}.$$

If λ is eigenvalue of matrix A, then any vector $\vec{x} \neq 0$, that satisfy

$A\vec{x} = \lambda\vec{x}$, $\vec{x}$ is called eigen vector of A, associated with λ.

example:-

Show that $\vec{x} = (2, 3)$ is an eigenvector of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

Also find eigenvalue of $\vec{x}$.

Soln

$$\vec{x} = (2, 3)$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

now,

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 12 \end{bmatrix} = 4(2, 3)$$

Hence, $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

so, $(2,3)$ is an eigenvector of matrix $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

also, eigenvalue $\lambda = 4$.

finding eigenvalues

<u>Theorem</u>: A number λ is an eigenvalue of the square matrix A iff it is a root of the equation. $\det(A - \lambda I) = 0$.

Defn:

The polynomial equation of λ from $\det(A - \lambda I) = 0$ is called characteristic equation.

example:- Find the eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}.$$

soln

$$A - \lambda I = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{bmatrix}$$

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now,

$$|A - \lambda I| = 0$$

or, $(1-\lambda)(2-\lambda) - 12 = 0$

or, $2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$

or $\lambda^2 - 3\lambda - 10 = 0$

or $\lambda^2 - 5\lambda + 2\lambda - 10 = 0$

or $\lambda(\lambda - 5) + 2(\lambda - 5) = 0$

or $(\lambda + 2)(\lambda - 5) = 0$

Hence, $\lambda = 5$ or -2.

finding eigenvectors

If λ is an eigenvalue of a matrix A, then non-zero solution of the homogeneous system of eqn

$(A - \lambda I)\vec{x} = 0$, are eigen ~~values~~ vectors.

The solution space of the system which is same as the null space is called eigen space for the eigen value λ.

example :-

If $\lambda = 5$ is an eigen value of matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, find the associated eigen vectors.

Soln

$\lambda = 5$, $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

$\vec{x} = ?$

i.e have,

$$(A - \lambda I)\vec{x} = 0$$

$$\left\{ \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \vec{x} = 0$$

or $\begin{pmatrix} 1-5 & 4 \\ 3 & 2-5 \end{pmatrix} \vec{x} = 0$

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or, $\begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or, $\begin{bmatrix} -4x_1 + 4x_2 \\ 3x_1 - 3x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence,

$\cancel{-4x_1 + 4x_2 = 0}$

$$\left[\begin{array}{cc:c} -4 & 4 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$R_1 \to \frac{R_1}{-4}$

$$\left[\begin{array}{cc:c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$$\left[\begin{array}{cc:c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $x_2 = t$

$x_1 - x_2 = 0$

$x_1 = t$

also eigenspace

$\therefore \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

which is eigenvector of eigenvalue $\lambda = 5$ with matrix A. $(t \neq 0)$

example :-

Find the eigenvalue and the corresponding eigenspace of the matrix.

$$A = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & 2 \\ 3 & -3 & -2 \end{bmatrix}$$

Soln

$A - \lambda I$

$$= \begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & 2 \\ 3 & -3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & -2 & -2 \\ -2 & 3-\lambda & 2 \\ 3 & -3 & -2-\lambda \end{bmatrix}$$

Hence,

$$|A - \lambda I| = 0$$

or $3-\lambda \begin{vmatrix} 3-\lambda & 2 \\ -3 & -2-\lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & 2 \\ 3 & -2 \end{vmatrix}$

$$-2 \begin{vmatrix} -2 & 3-\lambda \\ 3 & -3-\lambda \end{vmatrix} = 0$$

or $3-\lambda[-6+2\lambda+6+2\lambda] + 2[-2] - 2[6+2\lambda-9+3\lambda]$
$= 0$

or $3-\lambda[4\lambda] - 4 - 2[5\lambda - 3] = 0$

or $12\lambda - 4\lambda^2 - 4 - 10\lambda + 6 = 0$

or $-4\lambda^2 + 2\lambda + 2 = 0$

or, $4\lambda^2 - 2\lambda - 2 = 0$

or, $2\lambda^2 - \lambda - 1 = 0$

or $2\lambda^2 - 2\lambda + \lambda - 1 = 0$

or $2\lambda(\lambda-1) + 1(\lambda-1) = 0$

or $(2\lambda+1)(\lambda-1) = 0$

Either $\lambda = 1$ or $-\frac{1}{2}$

solving, we get,

$\lambda = 1$ or -2

for $\lambda = 1$, the eigen space,

$$(A - \lambda I)\vec{x} = 0$$

or, $\begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & 2 \\ 3 & -3 & -3 \end{bmatrix} \vec{x} = 0$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The augmented matrix is,

$$\left[\begin{array}{ccc:c} 2 & -2 & -2 & 0 \\ -2 & 2 & 2 & 0 \\ 3 & -3 & -3 & 0 \end{array}\right]$$

$R_1 \to \frac{R_1}{2}$

$$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 0 \\ -2 & 2 & 2 & 0 \\ 3 & -3 & -3 & 0 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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Let $x_2 = t$, $x_3 = s$.

$$x_1 - x_2 - x_3 = 0$$

or $x_1 - t - s = 0$

$\therefore x_1 = t + s$

Hence,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t+s \\ t \\ s \end{bmatrix}$$

$$= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix}$$

$t \neq 0$
$s \neq 0$

$$= t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, this is a required space for $\lambda = 1$ eigen

for $\lambda = 2$ eigen value,

$$[A - \lambda I]\vec{x} = 0$$

or $$\left\{\begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & 2 \\ 3 & -3 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right\}\vec{x} = 0$$

or, $$\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & 2 \\ 3 & -3 & -4 \end{bmatrix}\vec{x} = 0$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The augmented matrix is;

$$\left[\begin{array}{ccc:c} 1 & -2 & -2 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & -3 & -4 & 0 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & -2 & -2 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc:c} 1 & -2 & -2 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let

$x_3 = t$

so,

$-3x_2 - 2x_3 = 0$

or, $3x_2 = -2x_3$

$\therefore x_2 = \frac{-2t}{3}$

$x_1 - 2x_2 - 2x_3 = 0$

or $x_1 = 2t - 2\left(-\frac{2t}{3}\right)$

$\therefore x_1 = 2t + \frac{4t}{3} = \frac{10t}{3}$

So,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{10}{3}t \\ -\frac{2}{3}t \\ t \end{bmatrix}$$

$$= t \begin{bmatrix} \frac{10}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Hence, this is required space for $\lambda = 2$.

Linear transformation or linear mapping

Let V and W be two vector spaces. A transformation or mapping T from V to W is denoted by $T: V \to W$ is called a linear transformation or linear mapping if

(i) for any $\vec{v_1}, \vec{v_2}, \in V$,

$$T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$$

and

(ii) for any scalar K and $\vec{v_1} \in V$,

$$T(K\vec{v_1}) = KT(\vec{v_1})$$

A transformation T from a vectorspace V to itself is known as a linear operator.

A linear transformation $T: V \to W$ is called a isomorphism if T is one to one onto

The vector spaces V and W are said to be isomorphic if there is an isomorphism of V onto W.

eg: Check if the following function is linear transformation or not.

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x, y) = (x+y, x-y)$

Soln

Let $\vec{v_1} = (x_1, y_1)$ and $\vec{v_2} = (x_2, y_2) \in \mathbb{R}^2$.

$\vec{v_1} + \vec{v_2} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

now,

$T(\vec{v_1} + \vec{v_2}) = T(x_1 + x_2, y_1 + y_2)$

$= ((x_1 + x_2) + (y_1 + y_2), (x_1 + x_2) - (y_1 + y_2))$

$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2)$

also,

$T(\vec{v_1}) + T(\vec{v_2})$

$= T(x_1, y_1) + T(x_2, y_2)$

$= (x_1 + y_1, x_1 - y_1) + (x_2 + y_2, x_2 - y_2)$

$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2)$

Hence, $T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$

again, let $\vec{v_1} = (x_1, y_1)$ and k be any scalar.
then,

$$T(k\vec{v_1}) = T(k(x_1, y_1))$$

$$= T(kx_1, ky_1)$$

$$= (kx_1 + ky_1, kx_1 - ky_1)$$

$$= k(x_1 + y_1, x_1 - y_1)$$

$$= k\,T(x_1, y_1)$$

Hence, $T(k\vec{v_1}) = k\,T(\vec{v_1})$

Hence, T is a linear transformation.

eg 1:

$T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$T(x, y) = (x, y, xy)$

Soln

Let $\vec{v_1} = (x_1, y_1)$ $\in \mathbb{R}^2$

and $\vec{v_2} = (x_2, y_2)$

Hence, $\vec{v_1} + v_2 = (x_1 + x_2, y_1 + y_2)$

$T(\vec{v_1} + \vec{v_2}) = T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2, y_1 + y_2, (x_1 + x_2)(y_1 + y_2))$

also,

$T(\vec{v_1}) + T(\vec{v_2})$

$= T(x_1, y_1) + T(x_2, y_2)$

$= (x_1, y_1, x_1 y_1) + (x_2, y_2, x_2 y_2)$

$= (x_1 + x_2, y_1 + y_2, x_1 y_1 + x_2 y_2)$

$\neq T(\vec{v_1} + \vec{v_2})$

Hence, T is not a linear ~~combination~~ transformation