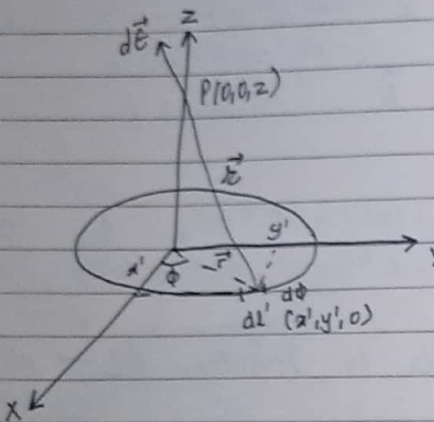
Example: Find the electric field at distance z above center of a circular loop of radius 'r' which carries a uniform line charge λ.

Sol^n:



Here, the figures illustrates the geometry and coordinates to be used.

From figure,

$$\vec{\mathscr{r}} = -x'\hat{i} - y'\hat{j} + z\hat{k}$$
$$= -r\cos\phi\,\hat{i} - r\sin\phi\,\hat{j} + z\hat{k}$$

$$\therefore \mathscr{r} = (r^2+z^2)^{1/2}$$

The charge on an elemental length dl' along a circular loop is,

$$dq = \lambda\, dl'$$
$$\therefore dq = \lambda r\, d\phi$$

The electric field at P due to the charge dq is,

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\mathscr{r}^3}\vec{\mathscr{r}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda r\, d\phi}{(r^2+z^2)^{3/2}}\left(-r\cos\phi\,\hat{i} - r\sin\phi\,\hat{j} + z\hat{k}\right)$$

$\therefore$ The net electric field at P due to the charge on whole circular loop is.

$$\vec{E} = \int d\vec{E}$$

$$= \frac{\lambda r}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}\int_0^{2\pi}\left(-r\cos\phi\,\hat{i} - r\sin\phi\,\hat{j} + z\hat{k}\right)d\phi$$

$$= \frac{\lambda r}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}\left[\left(-r\int_0^{2\pi}\cos\phi\, d\phi\right)\hat{i} + \left(-r\int_0^{2\pi}\sin\phi\, d\phi\right)\hat{j} + \left(z\int_0^{2\pi}d\phi\right)\hat{k}\right]$$

$$= \frac{\lambda r}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}} \times z \times (2\pi)\,\hat{k} \quad \left[\because \int_0^{2\pi}\cos\phi\, d\phi = 0,\ \int_0^{2\pi}\sin\phi\, d\phi = 0\right]$$

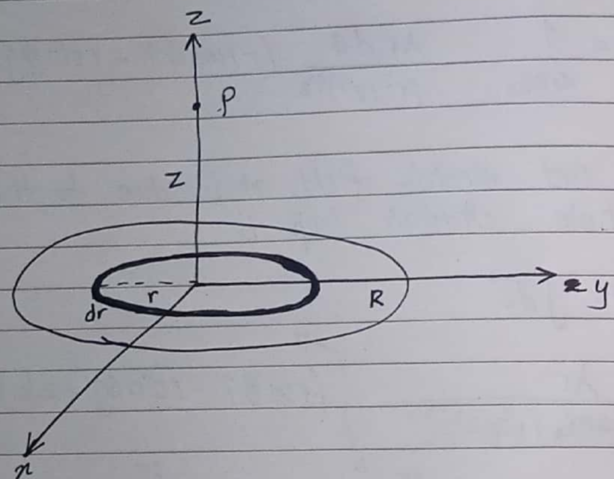
$$= \frac{1}{4\pi\varepsilon_0}\lambda(2\pi r)\cdot\frac{z}{(r^2+z^2)^{3/2}}\hat{k}$$

$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\, q\, \frac{z}{(r^2+z^2)^{3/2}}\hat{k}$$

Date No.

Example: Find the electric field at a distance z above the center of a flat circular disc of radius R which carries uniform surface charge σ.

Soln:



Here, the figure illustrates the geometry and the coordinates to be used.

The disc can be considered as combination of infinite number of infinitesimally thin rings.

Consider a ring of radius r and thickness dr of this disc.

If σ is the uniform charge density, then the charge on the ring $dq = \sigma(2\pi r\,dr)$

The electric field at P due to charge dq on the ring is,

Date No.

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\,dq\,\frac{z}{(r^2+z^2)^{3/2}}\,\hat{k}$$

$$= \frac{1}{4\pi\varepsilon_0}\,\sigma(2\pi r\,dr)\,\frac{z}{(r^2+z^2)^{3/2}}\,\hat{k}$$

$$= \frac{\sigma z}{2\varepsilon_0}\,\frac{r\,dr}{(r^2+z^2)^{3/2}}\,\hat{k} \quad \text{——(i).}$$

Hence, the total electric field due to the charge on whole flat circular disc is given by,

$$\vec{E}_{disc} = \int d\vec{E}$$

$$= \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}}\,\hat{k}$$

Put $r^2+z^2 = t^2$

Then,

$$2r\,dr = 2t\,dt$$

$$\therefore r\,dr = t\cdot dt$$

When $r=0$, $t=z$

When $r=R$, $t=\sqrt{R^2+z^2}$

So,

$$\vec{E}_{disc} = \int_z^{\sqrt{R^2+z^2}} \frac{t\cdot dt}{t^3}$$

$$= \frac{\sigma z}{2\varepsilon_0} \int_{z}^{\sqrt{R^2+z^2}} \frac{dt}{t^2} \hat{k}$$

$$= \frac{\sigma z}{2\varepsilon_0} \left[-\frac{1}{t} \right]_{z}^{\sqrt{R^2+z^2}} \hat{k}$$

$$= \frac{\sigma z}{2\varepsilon_0} \left[-\frac{1}{\sqrt{R^2+z^2}} + \frac{1}{z} \right] \hat{k}$$

$$= \frac{\sigma z}{2\varepsilon_0} \left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right] \hat{k}$$

$$\therefore \vec{E}_{disc} = \frac{\sigma}{2\varepsilon_0} \left(1 - \frac{z}{\sqrt{R^2+z^2}} \right) \hat{k}$$

As $R \to \infty$,

$$\therefore \vec{E}_{disc} = \frac{\sigma}{2\varepsilon_0} \hat{k}$$

for points far from the disc, $z \gg R$.

so,

$$\vec{E}_{disc} = \frac{\sigma}{2\varepsilon_0} \left[1 - \frac{1}{(R^2+z^2/z^2)^{1/2}} \right] \hat{k}$$

~~Using binomial expansion,~~

$$= \frac{\sigma}{2\varepsilon_0} \left[1 - \left(1 + \frac{R^2}{z^2} \right)^{-1/2} \right] \hat{k}$$

Using binomial expansion,

$$= \frac{\sigma}{2\varepsilon_0} \left[1 - \left(1 - \frac{1}{2} \frac{R^2}{z^2} + \cdots \right) \right] \hat{k}$$

so,

$$\vec{E}_{disc} = \frac{\sigma}{2\varepsilon_0} \times \left(\frac{1}{2} \frac{R^2}{z^2} \right) \hat{k} = \frac{1}{4\pi\varepsilon_0} \frac{\sigma \pi R^2}{z^2} \hat{k}$$

$$\therefore \vec{E}_{disc} = \frac{1}{4\pi\varepsilon_0} q \frac{1}{z^2} \hat{k}$$

Electric Field Lines

Electric field lines describes an electric field in any region of space.

The electric field vector $\vec{E}$ is tangent to the electric field line at each point.

The number of lines per unit area through a surface perpendicular to the lines is proportional to the magnitude of electric field in that region.

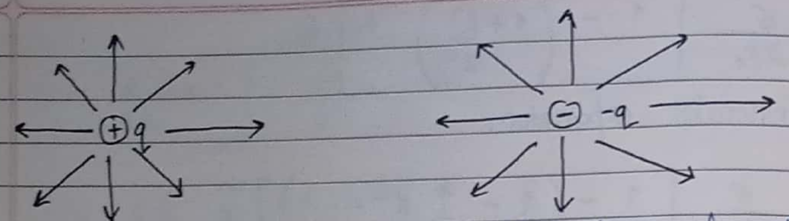
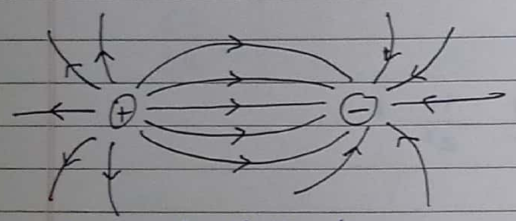
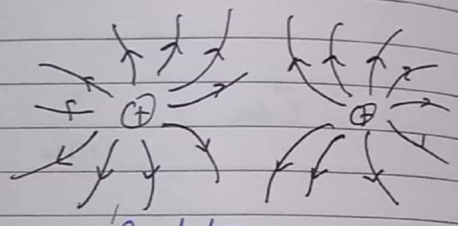


Fig: Representation Electric Field Lines for the Field due to single point charge.

For any two point charges, the electric field lines.



Equal and opposite charges.



Equal charges.

(*): Properties:

(i): Electric field lines dont intersect.

(ii): the tangent to electric field line at a point gives electric field vector $\vec{E}$ at that point

(iii) The lines begin on a positive charge and end on a negative charge.

Electric Flux

The total numbers of electric field lines passing a given area in unit time is called electric flux.

It is directly proportional to the number of electric field lines that penetrate the surface.

The electric ~~field~~ flux through a surface S is.

$$\phi_E = \int_S \vec{E} \cdot d\vec{a}$$

SI unit: $N m^2 C^{-1}$

The electric flux through any closed surface gives the measure of the total charge inside.

Gauss Law:

Gauss's law states that, "the total electric flux through any closed surface is equal to $1/\varepsilon_0$ times the total charge enclosed by the surface."

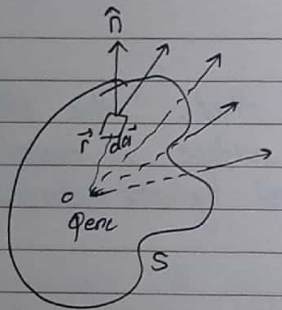
Mathematically

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

<Q>: Two concentric imaginary spherical surfaces of radii R and 2R respectively surround a positive charge q located at the centre of the surfaces. When compared to electric flux ϕ_1 through the surface of radius R, the electric flux ϕ_2 through the surface of radius 2R is.

(a): $\phi_1 = 2\phi_2$
(b) $2\phi_1 = \phi_2$
(c) $\phi_1 = 1/4\,\phi_2$
(d) $\phi_1 = \phi_2$ ⇐ Correct answer, $\phi \propto Q_{enc}$.

Let us consider an arbiteraly shaped which encloses a point charge Q_{enc}.



The electric field at vector $\vec{r}$ due to point charge located at origin is.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2}\hat{r}$$

Then,
the total electric flux passing through a closed surface S is.

$$\oint_S \vec{E}\cdot d\vec{a} = \oint_S \left(\frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2}\hat{r}\right)\cdot\left(da_r\,\hat{r} + da_\theta\,\hat{\theta} + da_\phi\,\hat{\phi}\right)$$

$$= \frac{Q_{enc}}{4\pi\varepsilon_0}\oint_S \frac{1}{r^2}\,da_r$$

$$= \frac{Q_{enc}}{4\pi\varepsilon_0}\oint_S \frac{1}{r^2}\left(r^2\sin\theta\, d\theta\, d\phi\right)$$

$$= \frac{Q_{enc}}{4\pi\varepsilon_0}\left[\int_0^{\pi}\sin\theta\, d\theta \cdot \int_0^{2\pi} d\phi\right]$$

$$= \frac{Q_{enc}}{4\pi\varepsilon_0} \times 2 \times 2\pi \qquad \therefore \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

This is integrated form of Gauss law.

For multiple charge

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} Q_i \qquad [\because \text{for discrete distribution}]$$

$$= \frac{1}{\varepsilon_0}\int \rho\, d\tau \qquad [\because \text{for continuous distribution}]$$

Applying divergence theorem,

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

This is differential form of Gauss law!