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Unit 4:

INTEGRATION

* ~~Sim~~ Indefinite ~~Int~~ Integral:

(i): $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

ii) $\int 1 \cdot dx = x + C$

(iii) $\int \frac{1}{x} \, dx = \ln x + C$

(iv) $\int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)}$

(v) $\int \frac{1}{ax+b} \, dx = \frac{\ln |ax+b|}{a} + C$

(vi) $\int e^{ax} \, dx = \frac{e^{ax}}{a} + C$

(vii) $\int \cos x \, dx = \sin x + C$

(viii) $\int \sin x \, dx = \cos x + C$

ix) $\int \sec^2 dx = \tan x + C$

x) $\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)| + C$

xi) $\int \frac{f'(x)}{\sqrt{2 f(x)}} = 2\sqrt{f(x)} + C$

* Definite Integral:

$$\int_a^b f(x) \, dx = F(x) \Big|_a^b$$

$$= F(b) - F(a)$$

Fundamental theorem of Calculus part I (FTC 1)

If f is continuous in $[a,b]$ then ~~$F(x) = \int_a^x f(x)$~~ $F(x) = \int_a^x f(t)\cdot dt$ is also continuous on (a,b) and we get.

$$F'(x) = f(x)$$

i.e. $\frac{d}{dx}\int_a^x f(t)\cdot dt = f(x).$

<Q>: Find dy/dx if $y = \int_x^5 3t\sin t \cdot dt.$

Soln:

Given,

~~$\frac{dy}{dx} =$~~ $y = \int_x^5 3t\sin t \cdot dt$

Now,

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t \cdot dt$$

$$= -\frac{d}{dx}\int_5^x 3t\sin t \cdot dt$$

By FTC 1,

$$\therefore \frac{dy}{dx} = -3x\sin x$$

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<Q7: Evaluate ie, dy/dx if $\int_1^{x^4} \sec t \cdot dt$.

Soln.

~~Given,~~ Let us put $u = x^4$

So,

$$\frac{dy}{dx} = \frac{d}{dx}\int_1^{x^4} \sec t \cdot dt = \frac{d}{dx}\int_1^{u} \sec t \cdot dt$$

$$= \frac{d}{du}\left[\int_1^{u} \sec t \cdot dt\right] \times \frac{du}{dx}$$

By FTC 1,

$$\frac{dy}{dx} = \sec u \times \frac{du}{dx}$$

$$= \sec x^4 \times \frac{dx^4}{dx}$$

$$\therefore \frac{dy}{dx} = 4x^3 \sec x^4.$$

<Q7: Evaluate $y = \int_{1+3x^2}^{4} \frac{dt}{2+t}$.

Soln:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{1+3x^2}^{4} \frac{dt}{2+t}\right]$$

Let $1+3x^2 = u$

or, $\frac{du}{dx} = \frac{d}{dx}(1+3x^2)$ $\therefore \frac{du}{dx} = 6x$

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$\cancel{\frac{dy}{dx}} \cancel{=} \cancel{\frac{d}{du}\int}$

$$\frac{dy}{dx} = \frac{d}{du}\left[\int_u^4 \frac{dt}{2+t}\right] \times \frac{du}{dx} = \frac{-d}{du}\left(\int_4^u \frac{dt}{2+t}\right) \times \frac{du}{dx}$$

By FTC I,

$$\frac{dy}{dx} = \frac{-1}{2+u} \times \frac{du}{dx}$$

$$= \frac{-1}{2+1+3x^2} \times \frac{d(1+3x^2)}{dx}$$

$$= \frac{-1}{3(1+x^2)} \times (0+6x)$$

$$\therefore \frac{dy}{dx} = -\frac{2x}{1+x^2}$$

Fundamental Theorem of Calculus Part II (FTC II)

If f is continuous on $[a,b]$ and F is any derivative of f on $[a,b]$ then,

$$\int_a^b f(x) \cdot dx = F(x)\Big|_a^b = F(b) - F(a)$$

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<Q>: $\int_1^4 e^x \cdot dx$

Sol^n:

$= \int_1^4 e^x \, dx$

$= e^x \Big|_1^4 \quad = e^4 - e \quad = e(e^3 - 1)$

<Q>: $\int_{-\pi/4}^{0} \sec\theta \tan\theta \, d\theta = \cdots$?

Sol^n:

$= \int_{-\pi/4}^{0} \sec\theta \tan\theta \, d\theta \qquad \left[\because \frac{d\sec\theta}{d\theta} = \sec\theta \tan\theta \right]$

$= \sec\theta \Big|_{-\pi/4}^{0}$

$= \sec 0 - \sec(-\pi/4)$

$= 1 - \sqrt{2}$

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<Q>: $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2} \right) dx$

Sol^n:

$= \int_1^4 \frac{3}{2}\sqrt{x}\, dx - \int_1^4 \frac{4}{x^2}\, dx$

$= \frac{3}{2}\int_1^4 \cancel{8}\, x^{1/2}\, dx - 4\int_1^4 x^{-2}\, dx$

$= \frac{3}{2}\int_1^4 x^{1/2}\, dx - 4\int_1^4 x^{-2}\, dx$

$= \frac{\cancel{3}}{\cancel{2}}\, \frac{x^{3/2}}{} \times \frac{\cancel{2}}{\cancel{3}} \Big|_1^4 + 4\, \frac{1}{x}\Big|_1^4$

$= (4^{3/2} - 1^{3/2}) + 4\left(\frac{1}{4} - \frac{1}{1}\right)$

$= 8 - 4$

$= 4.$

<Q>: $\int_0^{\pi} (1+\cos\theta)\, d\theta$

Sol^n

$= \int_0^{\pi} (1+\cos\theta)\, d\theta$

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$$= \int_0^{\pi} 1\, d\theta + \int_0^{\pi} \cos\theta\, d\theta$$

$$= \theta \Big|_0^{\pi} + \sin\theta \Big|_0^{\pi}$$

$$= (\pi - 0) + (\sin\pi - \sin 0) \quad = \pi$$

<Q>: $\int_0^{\pi/4} \sin 2x \, dx$

Soln:

$$= \int_0^{\pi/4} \sin 2x \, dx \quad = -\frac{\cos 2x}{2}\Big|_0^{\pi/4}$$

$$= -\frac{\cos \cancel{2} \times \pi/\cancel{4}2}{2} + \frac{\cos 2\times 0}{2}$$

$$= 0 + \frac{1}{2} \quad = \frac{1}{2}$$

Eg: For $\int_{-1}^{3} \frac{1}{x^2} dx$.

Here, this cannot be solved because given $1/x^2$ is improper interval which is not continuous at $x = 0$.

Average Value of a Function

$$av(f) = \frac{1}{b-a} \int_a^b f(x) \cdot dx.$$

<Q>: find average of f if $y = \int_{-2}^{2} \sqrt{4-x^2}\, dx.$

Soln:

$$= \int_{-2}^{2} \sqrt{2^2 - x^2}\, dx$$

$$= \frac{x}{2}\sqrt{4-x^2} + \frac{x^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\Bigg|_{-2}^{2}$$

$$= \left[\frac{2}{2}\sqrt{4-2^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{2}{2}\right)\right] - \left[\frac{-2}{2}\sqrt{4-(-2)^2} + \frac{(-2)^2}{2}\sin^{-1}\left(\frac{-2}{2}\right)\right]$$

$$= 0 + 2 \times \frac{\pi}{2} + 0 + 2 \times \frac{\pi}{2}$$

$$= 2\pi$$

So,

$$\text{avg value} = \frac{1}{2+2} \times 2\pi$$

$$= \frac{\pi}{2}$$

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* Note:

(i): $$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{if } f(x) \text{ is even.}$$

(ii) $$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{if } f(x) \text{ is odd.}$$

<Q>: $\int \cos^2 x \sin x\,dx$.

Soln:

Let us put $t = \cos x$.

$\therefore \frac{dt}{dx} = -\sin x$ $\therefore dt = -\sin x\,dx$ so $-dt = \sin x \cdot dx$

So,

$$\int \cos^2 x \sin x\,dx = -\int t^2\,dt$$

$$= -\frac{t^3}{3} + C$$

$$= \frac{-\cos^3 x}{3} + C$$

<Q>: $\int \cos(7\theta+3)\,d\theta$

Let $t = 7\theta+3$

So,

$$\frac{dt}{d\theta} = \frac{d}{d\theta}(7\theta+3)$$

or, $\frac{dt}{d\theta} = 7$ $\therefore d\theta = \frac{1}{7}\,dt$

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$= \int \cos t \cdot \frac{1}{7} dt \qquad = \frac{1}{7}\int \cos t \cdot dt$

$= \frac{1}{7} \sin t + C$

$= \frac{1}{7} \sin(7\theta + 3) + C$

<Q>: $\int x\sqrt{2x+1}\, dx$

Sol^n:

Let $t = 2x+1$ or, $\frac{t-1}{2} = x$

so, $\frac{dt}{dx} = 2$

$\therefore dx = dt/2$

$= \int \left(\frac{t-1}{2}\right) \sqrt{t} \, \frac{dt}{2}$

$= \frac{1}{4} \int (t\sqrt{t} - \sqrt{t})\, dt$

$= \frac{1}{4}\int t\sqrt{t}\, dt - \frac{1}{4}\int \sqrt{t}\, dt$

$= \frac{1}{4}\int t^{3/2} dt - \frac{1}{4}\int t^{1/2} dt$

$= \frac{1}{2\cancel{4}} \times \frac{\cancel{2}}{5} t^{5/2} - \frac{1}{2\cancel{4}} \times \frac{\cancel{2}}{3} \times t^{3/2} + C$

$= \frac{1}{10} t^{5/2} - \frac{1}{6} t^{3/2} + C$

Standard Integrals

To solve for standard integrals,

i) $(a^2-x^2) \Rightarrow$ Let $x = a\sin\theta$ or $a\cos\theta$

ii) $(x^2+a^2) \Rightarrow$ Let $x = a\tan\theta$ or $a\cot\theta$

iii) $(x^2-a^2) \Rightarrow$ Let $x = a\sec\theta$ or $a\operatorname{cosec}\theta$

<Q.1>: $\int \frac{dx}{a^2+x^2}$

Sol^n:

Let us put $x = a\tan\theta$ $\therefore \theta = \tan^{-1}(x/a)$

So,

$$\frac{dx}{d\theta} = a\sec^2\theta \qquad \therefore dx = a\sec^2\theta\, d\theta$$

So, $$\int \frac{a\sec^2\theta\, d\theta}{a^2 + a^2\tan^2\theta} = \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)}$$

$$= \int \frac{1}{a}\, d\theta = \frac{1}{a}\theta + C$$

$$\therefore \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

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<Q.2>: $\int \frac{dx}{x^2 - a^2}$

Sol^n:

$$= \int \frac{1}{x^2 - a^2} dx$$

$$= \frac{1}{2a} \int \frac{(x+a) - (x-a)}{(x-a)(x+a)} dx$$

$$= \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx$$

$$= \frac{1}{2a} \left[\int \frac{1}{x-a} dx - \int \frac{1}{x+a} dx \right]$$

$$= \frac{1}{2a} \left[\ln(x-a) - \ln(x+a) \right] + C$$

$$= \frac{1}{2a} \ln\left(\frac{x-a}{x+a} \right) + C$$

<Q.3>: $\int \frac{dx}{a^2 - x^2}$

Sol^n:

$$= \int \frac{dx}{a^2 - x^2}$$

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$$= \frac{1}{2a} \int \frac{(a+x) + (a-x)}{(a+x)(a-x)} dx$$

$$= \frac{1}{2a} \left[\int \frac{1}{a-x} dx + \int \frac{1}{a+x} dx \right]$$

$$= \frac{1}{2a} \left[-\ln(a-x) + \ln(a+x) \right] + C$$

$$= \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + C \quad \therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + c$$

<Q.4>: $\int \frac{dx}{\sqrt{x^2-a^2}}$

Soln:

$$= \int \frac{dx}{\sqrt{x^2-a^2}}$$

$\therefore \theta = \sec^{-1}\left(\frac{x}{a}\right)$

Let $x = a\sec\theta$

$\frac{dx}{d\theta} = a\sec\theta \tan\theta \, d\theta$ $\quad \therefore dx = a\sec\theta \tan\theta \cdot d\theta$

So,

$$= \int \frac{a\sec\theta \tan\theta \, d\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int \frac{\cancel{a}\sec\theta \tan\theta \, d\theta}{\cancel{a}\sqrt{\sec^2\theta - 1}}$$

$$= \int \frac{\sec\theta \cancel{\tan\theta} \, d\theta}{\cancel{\tan^2\theta}} = \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left| \frac{x}{a} + \sqrt{\frac{x^2-a^2}{a}} \right| + c = \frac{\cancel{1}}{\cancel{a}} \ln|x + \sqrt{x^2-a^2}| + c$$

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$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + c$$

<Q.5>: $\int \frac{dx}{\sqrt{x^2+a^2}}$

Soln:

Let $x = a\tan\theta$ $\qquad \therefore \theta = \tan^{-1}(x/a)$

$\frac{dx}{d\theta} = a\sec^2\theta$ $\qquad \therefore dx = a\sec^2\theta\, d\theta$

Now,

$$= \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta + a^2}} = \int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\cancel{\sec\theta}}$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{x}{a} + \sqrt{1+\frac{x^2}{a^2}}\right| + c$$

$$= \ln|x+\sqrt{x^2+a^2}| - \ln|a| + c$$

$$= \ln|x+\sqrt{x^2+a^2}| + c$$

Again,

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$x = a\sinh\theta$ $\therefore \theta = \sinh^{-1}(x/a)$

or $\frac{dx}{d\theta} = a\frac{d\sinh\theta}{d\theta}$ $\therefore dx = a\sinh\theta\, d\theta$

So,

$$\int \frac{\cancel{dx}\, a\sinh\theta\, d\theta}{\sqrt{a^2\sinh^2\theta + a^2}} = \int \frac{\cancel{a}\cancel{\cosh\theta}\, d\theta}{\cancel{a}\cosh\theta} = \int 1\cdot d\theta$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + C.$$

Integration ~~of~~ by Parts

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\cdot dx\right)dx$$

It follows ILATE rule.

I = imaginary function

L = Logarithmic function

A = Arithmetic function

T = Trigonometric function

E = exponential function

<Q.67>: $\int \sqrt{x^2 - a^2}\, dx$

Soln:

$I = \int 1\cdot\sqrt{x^2-a^2}\, dx$

Integrating by parts, we get.

$$= \sqrt{x^2-a^2}\int 1\cdot dx - \int\left(\frac{d\sqrt{x^2-a^2}}{dx}\int 1\cdot dx\right)dx$$

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$$= x\sqrt{x^2-a^2} - \int \frac{2x(x^2-a^2)^{-1/2}}{2} x\, dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}} dx$$

$$= x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\, dx + \int \frac{a^2}{\sqrt{x^2-a^2}} dx$$

or $2I = x\sqrt{x^2-a^2} + a^2 \ln|x+\sqrt{x^2-a^2}| + C$

$$\therefore \int\sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$$

<Q·7>: $\int \sqrt{a^2-x^2}\, dx$

Soln.

$I = \int 1\cdot\sqrt{a^2-x^2}\, dx$

Integrating by parts,

$$= \sqrt{a^2-x^2}\int 1\cdot dx - \int\left(\frac{d\sqrt{a^2-x^2}}{dx}\int 1\cdot dx\right) dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{-2x^2}{2\sqrt{a^2-x^2}} dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}} dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{a^2}{\sqrt{a^2-x^2}} - \int\left(\sqrt{a^2-x^2}\right) dx$$

$$\therefore \int\sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

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$\langle Q.8 \rangle$: $\int \sqrt{a^2+x^2}\, dx$

Soln.

Let $I = \int \sqrt{a^2+x^2}\, dx$.

Now,

$I = \int 1 \cdot \sqrt{a^2+x^2}\, dx$

Integrating by parts,

$$I = \sqrt{x^2+a^2} \int 1 \cdot dx - \int \left(\frac{d\sqrt{x^2+a^2}}{dx} \int 1 \cdot dx \right) dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\, dx$$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\, dx + \int \frac{a^2}{\sqrt{x^2+a^2}}\, dx$$

or $2I = x\sqrt{x^2+a^2} + a^2 \ln |x + \sqrt{x^2+a^2}| + C$

$$\therefore \int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \ln |x+\sqrt{x^2+a^2}| + C$$

$\langle Q.9 \rangle$: $\int \frac{dx}{x\sqrt{x^2-a^2}}$

Soln.

Let $z^2 = x^2 - a^2$

or $\cancel{2}z \cdot \frac{dz}{dx} = \cancel{2}x$ or, $dx = \frac{z}{x}\, dz$

Now,

$$= \int \frac{\cancel{z} \cdot dz}{x \cdot x \cdot \cancel{z}} = \int \frac{dz}{x^2} = \int \frac{dz}{z^2+a^2}$$

$= \frac{1}{a} \tan^{-1}\left(\frac{3}{a}\right) + C$

$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \tan^{-1}\left(\frac{\sqrt{x^2-a^2}}{a}\right) + C$

Integration by Parts:

<Q>: $\int x \sin x \, dx$

Sol^n:

Integrating by parts,

$= x \int \sin x - \int \left[\frac{dx}{dx} \int \sin x \, dx \right] dx$

$= -x \cos x + \int \cancel{1} \cos x \, dx$

$= \cancel{si} - x \cos x + \sin x + C$

$= \sin x - x \cos x + C$

<Q>: $\int e^x \sin x \, dx.$

Sol^n

Integrating by parts;

$= \sin x \int e^x dx - \int \left(\frac{d \sin x}{dx} \int \frac{d \cdot e^x}{dx} \right) dx$

$= e^x \sin x \cancel{dx} - \int \cos x \, e^x \, dx$

Integrating by parts;

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$$= e^{x}\sin x \,\cancel{dx} - \left[\cos x \cdot \int \frac{de^{x}}{dx} - \int\left(\frac{d\cos x}{dx}\int \frac{de^{x}}{dx}\right)dx\right.$$

$$= e^{x}\sin x\,\cancel{dx} - \cos x\, e^{x} \cancel{e} - \int \sin x\, e^{x}\, dx$$

$$\therefore \int e^{x}\sin x\, dx = \frac{e^{x}}{2}\sin x - \frac{e^{x}}{2}\cos x + C$$

~~Q7: $\int \frac{1}{5}v$~~

Successive Integration

$$\int_a^b uv\,dx = [uv_1]_b^a - \int_a^b\left(\frac{du}{dx}\int_a^b v\cdot dx\right)dx.$$

$$= \left[uv_1 - u'v_2 + u''v_3 - \dots\right]_a^b$$

<Q7: $\int_0^4 x e^{-x}\, dx$

Soln:

$$= x\times(-e^{-x}) - 1\cdot e^{-x}\Big|_0^4$$

$$= \left(4\times(-e^{-4}) - 1\cdot e^{-4}\right) - \left(0\times(e^{-0}) - 1\cdot e^{-0}\right)$$

$$= -4e^{-4} - e^{-4} - 0 + 1$$

$$= 1 - 5e^{-4}$$

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How to integrate the following:

$$\int \frac{dx}{ax^2+bx+c}, \quad \int \sqrt{ax^2+bx+c}\, dx, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

i) Convert to standard form such that

$$\int \frac{b \cdot dx}{(x-a)^2+c^2}$$

We can use standard integral formula if

$$\frac{d(x-a)}{dx} = b.$$

If standard form can't be used, we use substitution.

⟨Q⟩: $\int \frac{dx}{\sqrt{x^2-2x+5}}$

Sol^n.

$$= \int \frac{dx}{\sqrt{(x)^2-2\cdot x\cdot 1+1^2-1+5}} = \int \frac{1\cdot dx}{\sqrt{(x-1)^2+2^2}}$$

Using standard integral,

$$= \ln\left|(x-1)+\sqrt{(x-1)^2+2^2}\right|+C$$

$$= \ln\left|x-1+\sqrt{x^2-2x+5}\right|+C$$

Suppose, if standard integral couldn't be used.

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$x - 1 = t$

$\therefore dx = dt.$

$$= \int \frac{dt}{\sqrt{t^2 + a^2}} = \frac{1}{a} \tan^{-1}\left(\frac{t}{a}\right) + C$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x-1}{2}\right) + C$$

How to integrate : $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$

(i): Put $px + q = 1/t$

(ii) Convert all x to t.

(iii) Use standard form.

(Q) : $\int \frac{dx}{(2x+1)\sqrt{x^2+2x+3}}$

Solⁿ:

Let $(2x+1) = \frac{1}{t}$

$2x = \frac{1}{t} - 1 \qquad \therefore x = \frac{1-t}{2t}$

$$\frac{d(2x+1)}{dx} = \frac{d(1/t)}{dt} \times \frac{dt}{dx}$$

or $2 = \ln x \times \frac{dt}{dx} \qquad$ or $\frac{2\, dx}{\ln x} = \cancel{(\ldots)}\, dx$

or $dx = \frac{\ln x}{2} \times dt$

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So,

x^2+2x+2

$= \left(\frac{1-t}{2t}\right)^2 + 2\times(1-t)+2$

$= \frac{1-2t+t^2}{4t^2} + 2-2t+2$

$= 1-2t+t^2$

$dx = \frac{\ln t}{2} \cdot dt$

So,

$= \int \frac{\ln t}{2} dt \times \frac{1}{t\sqrt{(t-1)^2}}$

$= \frac{1}{2}\int \frac{\ln t \, dt}{t\times(t-1)}$

<Q7>: $\int \frac{dx}{(2x+1)\sqrt{x^2+2x+2}}$

Soln:

Let $2x+1 = \frac{1}{t}$

or, $t = \frac{1}{2x+1}$

or, $\frac{dt}{dx} = \frac{d(2x+1)^{-1}}{d(2x+1)} \times \frac{d(2x+1)}{dx}$

$= -1(2x+1)^{-2} \times 2$

$\cancel{\frac{dt}{dx} = \frac{-2}{\sqrt{(2x+1)}}}$ $\cancel{dx = \frac{\sqrt{(2x+1)} \times dt}{-2}}$

or $\frac{dt}{dx} = \frac{-2}{(2x+1)^2}$ or, $dx = \frac{(2x+1)^2 dt}{-2}$

Now,

$$= -\frac{1}{2}\int \frac{(2x+1)^{\cancel{2}}\, dt}{\cancel{(2x+1)}(\sqrt{x^2+2x+2})}$$

$$= \frac{-1}{2}\int \frac{(2x+1)\ dt}{\sqrt{(x+1)^2+1^2}}$$

$$= \frac{-1}{2}\left[\int \frac{2x+2\ \ dt}{\sqrt{x^2+2x+2}} - \int \frac{1}{\sqrt{x^2+2x+2}}\, dt\right]$$

$$= -\frac{1}{2}\left[2(x^2+2x+2)\right] + \frac{1}{2}\int \frac{1 \cdot dx}{\sqrt{(x+1)^2+1^2}}$$

$$= -\sqrt{x^2+2x+2} + \frac{1}{2}\ln\left|x+1+\sqrt{(x+1)^2+1^2}\right| + c$$

$$= \frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x+2}\right| - \sqrt{x^2+2x+2} + c$$

How to integrate:

$$\int \frac{dx}{a+b\cos x}, \quad \int \frac{dx}{a+b\sin x}, \quad \int \frac{dx}{a+b\sin x + c\cos x}$$

We let, $t = \tan\frac{x}{2}$ So, $dt = \frac{2 \cdot dt}{1+t^2}$

So,

$\sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

<Q>: $\int \frac{dx}{4+5\sin x}$.

Sol^n:

Let $\tan\frac{x}{2} = t$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ or, $\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

or, $\frac{dt}{dx} = \frac{1}{2}(1+t^2)$ so, $dx = \frac{2\cdot dt}{(1+t^2)}$

So, $\sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

Now,

$$= \int \frac{dx}{4+5\sin x} = \int \frac{2\cdot dt}{(1+t^2)} \times \frac{1}{\left(4+\frac{10t}{1+t^2}\right)}$$

$$= \int \frac{2\cdot dt}{\cancel{(1+t^2)} \times \frac{(4+4t^2)+10t}{\cancel{(1+t^2)}}}$$

$$= \int \frac{2dt}{4+4t^2+10t} = \int \frac{2\,dt}{4t^2+10t+4}$$

$$= \int \frac{2\cdot dt}{\left(2t+\frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$$

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$$= \frac{1}{2\times\frac{3}{2}} \ln\left|\frac{2t+5/2-3/2}{2t+5/2+3/2}\right|$$

$$= \frac{1}{3} \ln\left|\frac{2t+1}{2t+4}\right| + C$$

$$= \frac{1}{3} \ln\left|\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}\right| + C$$

Integration of Rational Functions By Partial Functions

(i) For non-repeated linear factors in the denominator

$$\frac{f(x)}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

(ii) For repeated linear factors in the denominator

$$\frac{f(x)}{(x-a)(x-b)^2} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-b)^2}$$

(iii) For non-repeated and non-factorable quadratic factors in denominators.

$$\frac{f(x)}{(x-a)(ax^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{(ax^2+bx+c)}$$

(iv): For integral of $\int \frac{dx}{(x-a)^m(y-b)^n}$ where, $a<b$ and m & n are positive integers

→ Put $(x-a) = t(x-b)$

- Express x in terms of t and dx in terms of dt and integrate

- Express the result in terms of x.

<Q>: $\int \frac{(2x+1)\,dx}{(x+1)(x-2)(x-3)}$

Soln:

$$\frac{(2x+1)}{(x+1)(x-2)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)} \quad \text{— (i)}$$

or, $\frac{2x+1}{\cancel{(x+1)(x-2)(x-3)}} = \frac{A(x-2)(x-3) + B(x+1)(x-3) + C(x+1)(x-2)}{\cancel{(x+1)(x-2)(x-3)}}$

or, $2x+1 = A(x-2)(x-3) + B(x+1)(x-3) + C(x+1)(x-2)$ — (ii)

Integrating (i);

$$\int \frac{(2x+1)\,dx}{(x+1)(x-2)(x-3)} = \int \left(\frac{A}{(x+1)}\right) + \left(\frac{B}{(x-2)}\right) + \left(\frac{C}{(x-3)}\right) dx$$

$$= \int \frac{2x+1 \; dx}{(x+1)(x-2)(x-3)} = A \ln|x+1| + B \ln|x-2| + C \ln|x-3| + C \quad \text{— (iii)}$$

Putting $x=2$ in eqn (ii),

$2\times2+1 = B(2+1)(2-3)$

or, $5 = -3B$ $\therefore B = -5/3$

Putting $x=-1$ in eqn (ii)

$2\times(-1)+1 = A(-1-2)(-1-3)$

or, $-1 = 12A$ $\therefore A = \frac{-1}{12}$

Putting $x = 3$,

$2 \times 3 + 1 = C(3+1)(3-2)$

or $7 = 4C \quad \therefore C = \cancel{4} \; 7/4$

Putting values of A, B, C in eqn (iii)

$$\int \frac{(2x+1)\,dx}{(x+1)(x-2)(x-3)} = -\frac{1}{12}\ln|x+1| - \frac{5}{3}\ln|x-2| + \frac{7}{4}\ln|x-3| + C$$

$$\therefore \int \frac{(2x+1)\,dx}{(x+1)(x-2)(x-3)} = \frac{7}{4}\ln|x-3| - \frac{1}{12}\ln|x+1| - \frac{5}{3}\ln|x-2| + C$$

<Q>: $\displaystyle\int \frac{dx}{x(x^2+1)}$

Soln:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

or, $1 = A(x^2+1) + (Bx+C)x$

or, $1 = \cancel{A(x^2+1) + Bx} \; A(x^2+1) + Bx^2 + Cx \quad \text{—(i)}$

Putting $x = 0$,

or $1 = A \cancel{+B} \quad \therefore A = 1$

Putting

or, $0 \cdot x^2 + 0 \cdot x + 1 = (A+B)x^2 + Cx + A$

So, $A + B = 0. \quad A = 1 \quad \therefore B = -1$

$C = 0$

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$$\int \frac{dx}{x(x^2+1)} = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$$

$$= \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$= \ln(x) - \frac{1}{2}\ln(x^2+1) + C$$

$$\cancel{= \frac{2\ln(x) - \ln(x^2+1)}{2} + C}$$

$$\cancel{= \ln\left(\frac{x^2}{x^2+1}\right) + C}$$

$$= \ln x - \ln\left(\sqrt{x^2+1}\right) + C$$

$$= \ln\left(\frac{x}{\sqrt{x^2+1}}\right) + C$$

<Q>: $\int \frac{dx}{x(1+x)^2}$

Let $\frac{1}{x(1+x)^2} = \frac{A}{x} + \frac{B}{(1+x)} + \frac{C}{(1+x)^2}$

or, $\cancel{\frac{1}{x(1+x^2)} = \frac{A(1+x)(1+x^2) + Bx(1+x) + Cx}{x(1+x^2)}}$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{B(1+x)+C}{(1+x)^2}$$

or $\frac{1}{x(1+x^2)} = \frac{A(1+x)^2 + Bx(1+x) + Cx}{x(1+x^2)}$

or, $1 = A(1+x^2) + Bx(1+x) + Cx$

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or, $1 = A + Ax^2 + Bx + Bx^2 + Cx$

or $1 = (A+B)x^2 + (B+C)x + A$

or $0\cdot x^2 + 0\cdot x + 1 = (A+B)x^2 + (B+C)x + A$

So,

$A = 1$

$A + B = 0 \quad \therefore B = -1$

$B + C = 0 \quad \therefore C = 1$

So,

$$\frac{1}{x(1+x)^2} = \frac{1}{x} - \frac{1}{(1+x)} + \frac{1}{(1+x)^2}$$

$$\int \frac{dx}{x(1+x)^2} = \int \frac{dx}{x} - \int \frac{dx}{(1+x)} + \int \frac{dx}{(1+x)^2}$$

$$= \ln x - \ln(x+1) - \frac{1}{(1+x)} + C$$

$$= \ln\left|\frac{x}{x+1}\right| - \frac{1}{(1+x)} + C$$

How to integrate Improper Rational Functions

$\int \frac{f(x)}{g(x)}\,dx$ then, It is converted to proper form using division and then integrated.

<Q>: $\int \frac{(2x^3 - 4x^2 - x - 3)}{x^2 - 2x - 3} dx.$

Soln

$$x^2 - 2x - 3 \overline{) 2x^3 - 4x^2 - x - 3} (2x$$
$$\cancel{2x^3} - \cancel{4x^2} - 6x$$
$$(-) \quad (+) \quad (+)$$
$$\overline{\quad 5x - 3}$$

So,

$$\int \left(2x + \frac{5x - 3}{x^2 - 2x - 3} \right) dx$$

$$= \int 2x \, dx + \int \frac{5x - 3}{x^2 - 2x - 3} dx$$

For $\int 2x dx.$,

$$= 2 \times \frac{x^2}{2} = x^2$$

For $\int \frac{5x - 3 \, dx}{x^2 - 2x - 3}$

$$= \int \frac{5x - 3}{(x+1)(x-3)} dx$$

$$\frac{5x - 3}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$$

or, $5x - 3 = A(x-3) + B(x+1)$ —(i)

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Putting $x = 3$,

$12 = 4B$ $\therefore B = 3$

Putting $x = -1$

$\cancel{-5} \cdot 5\times -1 - 3 = A(-1-3)$

$\therefore A = 2.$

So,

$$\int \frac{5x-3}{(x+1)(x-3)} = \int \frac{2\,dx}{x+1} + \int \frac{3\,dx}{x \cancel{+1} - 3}$$

$$= 2\int \frac{dx}{x+1} + 3\int \frac{dx}{x-3}$$

$$= 2\ln(x+1) + 3\ln(x-3) + C$$

$\langle Q \rangle$: $\displaystyle\int_{-1}^{2} \frac{t \cdot dt}{\sqrt{2t^2+8}}$

Sol^n.

Let $x = 2t^2$

$\therefore \frac{dx}{dt} = 2.2t$ $\therefore dx = 4t\,dt$

Now,

$$= \int_{-1}^{2} \frac{t \cdot dt}{\sqrt{2t^2+8}} = \frac{1}{4}\int_{-1}^{2} \frac{dx}{\sqrt{x+8}}$$

$$= \frac{1}{4} \left| 2\sqrt{x+8} \right|_{-1}^{2} = \frac{1}{2}\sqrt{2t^2+8}$$

$= 0.4188$

$$= \cancel{\frac{1}{2}\sqrt{2+8}} - \cancel{\frac{1}{2}\sqrt{-1+8}}$$ $\cancel{= 0.2582}$

<Q>: $\displaystyle\int_0^4 \frac{dx}{x(\log x)^2}$

Sol^n.

$$= \int_0^4 \frac{\frac{1}{x}\,dx}{(\log x)^2}$$

Let $\log x = t$

on $\dfrac{1}{x} = \dfrac{dt}{dx}$ on $\dfrac{dx}{x} = dt$

$$= \int_0^4 \frac{dt}{t^2}$$

$$= \cancel{\int_0^4 - \frac{1}{t}\Big|_0^4} \quad \frac{t^{-2+1}}{-2+1}\Bigg|_0^4 = -\frac{1}{t}\Bigg|_0^4$$

$$= \cancel{cot} \quad \frac{-1}{\log x}\Bigg|_0^4$$

$$= 0 + \frac{1}{\log 4} = 0.7213$$

<Q>: $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$

Sol^n:

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$$= \int_1^2 \frac{dx}{(3-5x)^2}$$

$$= \int_1^2 \cancel{dx}\,(3-5x)^{-2}\,dx$$

$$= \left. \frac{(3-5x)^{-2+1}}{-2+1} \right|_1^2$$

$$= \left. \frac{-1}{(3-5x)} \right|_1^2$$

$$= \frac{-1}{(3-10)} + \frac{1}{(3-5)} = -\frac{5}{14}$$

Improper Integrals

An improper integral is a definite integral that has either or both limits infinite or an integrand that approaches infinity at one ~~of~~ or more points in the range of integration.

i) limit tends to ∞

ii) discontinuous in interval.

(*) Type: I

$\lim_{x \to \infty} f(x) = \infty$ ie, limit doesn't exist $\Rightarrow$ divergent

$\lim_{x \to \infty} f(x) = b$ ie, limit exists and finite $\Rightarrow$ convergent.

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(i): If f is continuous on $[a, \infty)$ then,

$$\int_a^{\infty} f(x) \cdot dx = \lim_{b \to \infty} \int_a^b f(x)\, dx$$

(ii) If f is continuous on $[-\infty, b]$ then,

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx$$

(iii) If f is continuous on $(-\infty, \infty)$ then.

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_c^{\infty} f(x)\, dx$$

$$= \lim_{a \to -\infty} \int_a^c f(x)\, dx + \lim_{b \to \infty} \int_c^b f(x)\, dx$$

<Q>: $\int_1^{\infty} \frac{1}{x}\, dx$.

Soln:

$$= \lim_{a \to \infty} \int_1^a \frac{1}{x}\, dx$$

$$= \lim_{a \to \infty} [\ln x]_1^a$$

$$= \lim_{a \to \infty} \ln a - \ln 1$$

$$= \infty \quad \text{(Doesn't exist)}$$

So, it is divergent.

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<Q7: ~~lim~~ $\int_{-\infty}^{0} xe^{x}\,dx$

Sol^n.

$$= \lim_{a\to-\infty} \int_{a}^{0} xe^{x}\,dx$$

$$= \lim_{a\to-\infty} \left[xe^{x} - e^{x} \right]_{a}^{0}$$

$$= \lim_{a\to-\infty} 0\times e^{0} - e^{0} - (ae^{a} - e^{a})$$

$$= \lim_{a\to-\infty} -1 - ae^{a} + e^{a}$$

For $\lim_{a\to-\infty} -ae^{a}$.

Using L-Hopital rule,

$\lim_{a\to-\infty} -a\times 0 = 0$

$$= \lim_{a\to-\infty} e^{a} - 1 - ae^{a}$$

$$= e^{-\infty} - 1 - 0$$

$$= \frac{1}{e^{\infty}} - 1$$

$$= -1$$

It is convergent.

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<Q7>: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Soln:

$$= \int_{-\infty}^{c} \frac{dx}{1+x^2} + \int_{c}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{a \to -\infty} \int_{a}^{c} \frac{dx}{1+x^2} + \lim_{b \to \infty} \int_{c}^{b} \frac{dx}{1+x^2}$$

$$= \lim_{a \to -\infty} \tan^{-1}(x) \Big|_{a}^{c} + \lim_{b \to \infty} \tan^{-1}(x) \Big|_{c}^{b}$$

$$= \lim_{a \to -\infty} \tan^{-1}(c) - \tan^{-1}(a) + \lim_{b \to \infty} \tan^{-1}(b) - \tan^{-1}(c)$$

$$= \cancel{\tan^{-1}(c)} - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \cancel{\tan^{-1}(c)}$$

$$= \cancel{\frac{\pi}{2}} - \pi - \cancel{\frac{\pi}{2}}$$

$= -\pi$

It is ~~diverg~~ convergent.

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(*) Type II

(i): If $f(x)$ is continuous on $(a,b]$ then,

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

(ii) If $f(x)$ is continuous on $[a,b)$ then,

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

(iii) If $f(x)$ is continuous on (a,b) then, $a<c<b$.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<Q>: $\displaystyle\int_a^5 \frac{dx}{\sqrt{x-2}}$

Soln.

$$= \lim_{c\to 2^+} \int_c^5 \frac{dx}{\sqrt{x-2}}$$

$$= \lim_{c\to 2^+} \left[\frac{(x-2)^{1/2}}{1/2}\right]_c^5 = \lim_{c\to 2^+} 2(x-2)^{1/2}\Big|_c^5$$

$$= 2(5-2)^{1/2} - \lim_{c\to 2^+} \{2(c-2)^{1/2}$$

$$= 2\times 3^{1/2} - \lim_{c} 2(2-2)^{1/2} = 2\sqrt{3} - 0 = 2\sqrt{3}$$

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<Q7: $\int_1^3 \frac{dx}{1-x}$

Sol^n:

$$= \int_0^1 \frac{dx}{1-x} + \int_1^3 \frac{dx}{1-x}$$

$$= \lim_{c\to 1^-} \int_0^c \frac{dx}{1-x} + \lim_{c\to 1^+} \int_c^3 \frac{dx}{1-x}$$

$$= \lim_{c\to 1^-} -\ln|1-x|\Big|_0^c + \lim_{c\to 1^+} -\ln|1-x|\Big|_c^3$$

$$= \lim_{c\to 1^-} -\ln|1-c| + \ln|1-0| + \lim_{c\to 1^+} +\ln|1-c| - \ln|1-3|$$

$$= \lim_{c\to 1^-} -\ln|1-C| + \lim_{c\to 1^+} \ln|1-c| + \ln 1 - \ln|-2|$$

$= \infty$ (limit doesn't exist) ie, divergent.

<Q7: For what values of p does the integral $\int_1^\infty \frac{dx}{x^p}$ is convergent.

Sol^n.

$$= \int_1^\infty \frac{dx}{x^p}$$

$$= \lim_{a\to\infty} \int_1^a \frac{dx}{x^p} = \lim_{a\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^a$$

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$$= \cancel{\lim_{a\to\infty} \frac{a^{1-p}}{}} = \lim_{a\to\infty} \frac{a^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}$$

- Here,

Case 1: $p = 1$

$$\lim_{a\to\infty} \left[\frac{a^{1-p}}{1-1} - \frac{1^{1-1}}{1-1} \right] = \infty \quad \text{(divergent)}$$

Case 2: $p < 1$.

$$\lim_{a\to\infty} \frac{a^{1-p}}{1-p} - \frac{1^{1-p}}{1-p} = -\frac{1^{1-p}}{1-p} + \lim_{a\to\infty} \frac{a^{1-p}}{1-p}$$

$$= \infty \quad \text{(divergent)}$$

Case 3: $p > 1$.

$$\cancel{\lim_{a\to\infty} \frac{a^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}} = \cancel{-\frac{1^{1-p}}{1-p}} + \cancel{\lim_{a\to\infty}} \cancel{\frac{a^{1-p}}{1-p}}$$

$$\int_1^\infty \frac{dx}{x^p} = \frac{-1}{1-p} \neq \text{finite} = \text{convergent}.$$

or,

$$\lim_{a\to\infty} \frac{a^{1-p}}{1-p} - \frac{1^{1-p}}{1-p} = -\frac{1^{1-p}}{1-p} + \lim_{a\to\infty} \frac{a^{1-p}}{1-p}$$

$$= \cancel{\text{finite}} -\frac{1^{1-p}}{1-p} + 0$$

$$= \text{finite}. \quad = \text{convergent}.$$

Integral is convergent for $p > 1$.