Advanced Calculus

Functions of Several Variables

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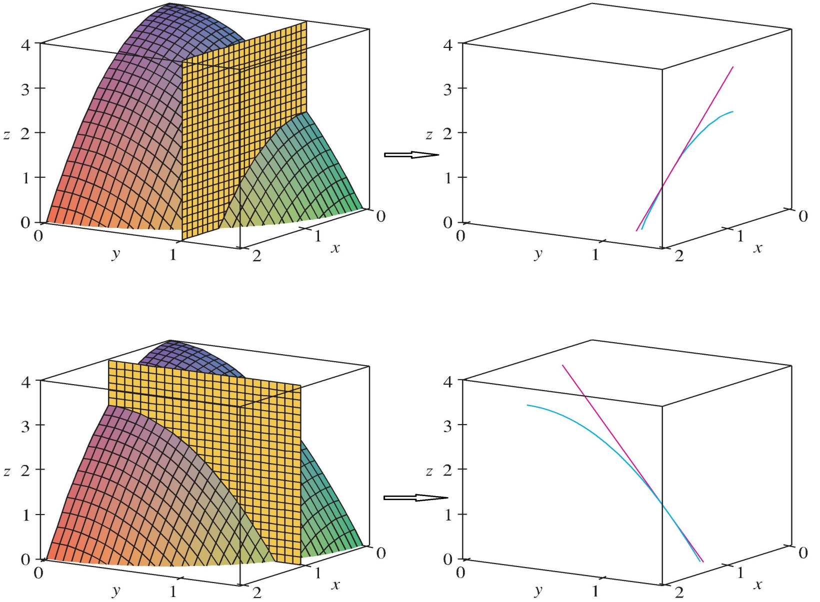
Partial Derivatives



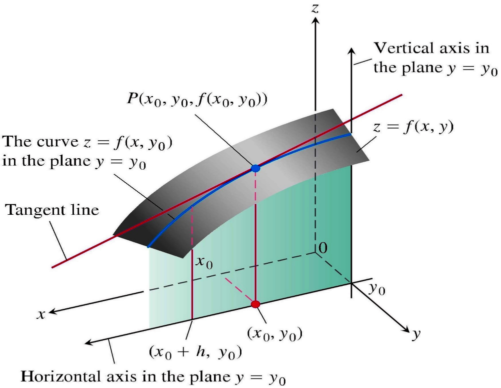


FIGURE The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$,

DEFINITION The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

An equivalent expression for the partial derivative is $\left.\dfrac{d}{dx} f(x, y_0)\right|_{x=x_0}$.

We use several notations for the partial derivative:

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0), \qquad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \qquad \text{and} \qquad f_x, \frac{\partial f}{\partial x}, z_x, \text{ or } \frac{\partial z}{\partial x}.$$

The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x. We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ with respect to y at y_0.

DEFINITION The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

Examples

A. Find the derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ using definition (from the first principle)

1. $f(x, y) = 1 - x^2 - y^2 - 2xy$ at $(1, 1)$.
2. $f(x, y, z) = x^2y^2z^2$ at $(1, 2, 3)$.

B. Find f_x, f_y if

1. $f(x, y) = xy^2$
2. $f(x, y) = e^{-x}\sin(x + y)$

C. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at (1,2) if $f(x, y) = 1 - x + y - 3x^2y$.

Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \qquad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}$$

In fact,

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y$$

- Show that $f(x, y, z) = e^{3x+4y} \cos 5z$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Mixed Derivative Theorem (Clairaut's Theorem)

Theorem:

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Example

- Verify mixed derivative theorem if
  a. $f(x, y) = y + x/y$
  b. $w = x \sin y + y \sin x + xy$

Definition

The linearization of a function $f(x, y)$ at a point (x_0, y_0) when f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation $f(x, y) \approx L(x, y)$ is the standard linear approximation of f at (x_0, y_0).

- Find the linearization $L(x, y)$ of the function

$$f(x, y) = x^2 + y^2 + 1 \text{ at } (1, 1).$$

Total Differential

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting differential in f is

$$\boxed{df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy}$$

This change in linearization of f is called the total differential of f.

Function of more thant two variables: Linearization and Differential

$$\boxed{L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0)}$$

$$\boxed{df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz}$$