Substitution in Multiple Integrals

Jacobian Determinant:

Jacobian of the transformation

$x = g(u,v)$, $\quad y = h(u,v)$ is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Converting from cartesian to polar:

$x = r\cos\theta \qquad y = r\sin\theta$.

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

So, $\quad dxdy = |J|\, dr\, d\theta$

Cartesian $\longrightarrow$ cylindrical.

Jacobian transformation of $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

Date. No.

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding from R_3,

$= r(\cos^2\theta + \sin^2\theta) = r \times 1$

$= r$

So, $dz\,dx\,dy = |J|\, r\, dz\, dr\, d\theta$

Jacobian determinant for spherical coordinate.

$x = \rho\sin\phi\cos\theta$, $y = \rho\,\cancel{\cos}\sin\phi\sin\theta$, $z = \rho\cos\phi$

$$J(\rho,\phi,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

Expanding from R_3.

$\cancel{\cos\phi\,(\rho\cos\phi\cos\theta \times \rho\sin\phi\cos\theta}$

$= \cos\phi\,(\rho\cos\phi\cos\theta \times \rho\sin\phi\cos\theta + \rho\cos\phi\sin\theta \times \rho\sin\phi\sin\theta)$

$+ \rho\sin\phi\,(\rho\sin\phi\cos\theta \times \sin\phi\cos\theta + \sin\phi\sin\theta \times \rho\sin\phi\sin\theta)$

Date. No.

$= \cos\phi\,(\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta)$

$+ \rho\sin\phi\,(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta)$

$= \{\cos\phi \times \rho^2\sin\phi\cos\phi\,(\cos^2\theta + \sin^2\theta)\} +$

$\{\rho\sin\phi \times \rho\sin^2\phi\,(\cos^2\theta + \sin^2\theta)\}$

$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi$

$= \rho^2\sin\phi$

So, $\int dz\,dy\,dx = \int |J|\, d\rho\, d\phi\, d\theta$

$= \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

Eg: $x = h(u,v,w)$, $y = g(u,v,w)$, $z = k(u,v,w)$.

$\iiint f(x,y,z)\, dx\, dy\, dz$

$= \iiint_G F(u,v,w)\, |J(u,v,w)|\, du\, dv\, dw$

$= \iiint_G F(r,\theta,z)\; r\, dr\, d\theta\, dz$

$= \iiint_G F(r,\phi,\theta)\; \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

Date. No.

Q.17: Evaluate $\int_0^3\int_0^4\int_{x=y/2}^{x=y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$ by applying the transformation $u=\frac{2x-y}{2}$, $v=\frac{y}{2}$, $w=\frac{z}{3}$ and integrating over ~~space~~ appropriate region in uvw space.

Soln:

Given,

$$\int_0^3\int_0^4\int_{x=y/2}^{x=y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

and transformation,

$$u=\frac{2x-y}{2},\quad v=\frac{y}{2},\quad w=\frac{z}{3}$$

Now,

$u=\frac{2x-y}{2}$ $\qquad$ $y=2v$

or $2u=2x-y$ $\qquad$ $z=3w$.

or $2u=2x-2v$

or $x=u+v$

Again,

$$|J|=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

Date. No.

$\frac{\partial x}{\partial u}=\frac{\partial(u+v)}{\partial u}=1$ $\qquad$ $\frac{\partial y}{\partial u}=\frac{\partial(2v)}{\partial u}=0$

$\frac{\partial z}{\partial u}=\frac{\partial(3w)}{\partial u}=0$ $\qquad$ $\frac{\partial x}{\partial v}=\frac{\partial(u+v)}{\partial v}=1$

$\frac{\partial y}{\partial v}=\frac{\partial(2v)}{\partial v}=2$ $\qquad$ $\frac{\partial z}{\partial v}=\frac{\partial(3w)}{\partial v}=0$

$\frac{\partial x}{\partial w}=\frac{\partial(u+v)}{\partial w}=0$ $\qquad$ $\frac{\partial y}{\partial w}=\frac{\partial(2v)}{\partial w}=0$ $\qquad$ $\frac{\partial z}{\partial w}=\frac{\partial(3w)}{\partial w}=3$

So,

$$|J|=\begin{vmatrix}1 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{vmatrix}=6$$

Now,

xyz eqn	uvw eqn	simplied form
$x=y/2$	$u+v=v$	$u=0$
$x=y/2+1$	$u+v=v+1$	$u=1$
$y=0$	$2v=0$	$v=0$
$y=4$	$2v=4$	$v=2$
$z=0$	$3w=0$	$w=0$
$z=3$	$3w=3$	$w=1$

So,

$$\int_0^3\int_0^4\int_{x=y/2}^{x=y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz=\int_0^1\int_0^2\int_0^1(u+w)\,6\,du\,dv\,dw$$

$$=6\int_0^1\int_0^2\int_0^1(u+w)\,du\,dv\,dw.$$

Date. No.

$$= 6\int_0^1\int_0^2 \left(\frac{u^2}{2} + uw\right)_0^1 dv\,dw$$

$$= 6\int_0^1\int_0^2 \left(\left(\frac{1}{2} + w\right) dv\right) dw$$

$$= 6\int_0^1 \left(\frac{v}{2} + vw\right)_0^2 dw$$

$$= 6\int_0^1 (1 + 2w)\, dw$$

$$= 6\int_0^1 \left(w + w^2\right)_0^1$$

$$= 6(1+1) = 12$$

(*): Steps to solve:

(i): Find x, y, z in terms of u, v, w.

(ii): Find Jacobian determinant.

(iii) xyz eqn, uvw eqn, simplied form table. Relating with integral range to find new range.

(iv): write the transformed integral

(v): solving integral.

Date. No.

<Q>: Evaluate:

(i): $$\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2} (r^2\sin^2\theta + z^2)\, dz\,dr\,d\theta$$

Soln.

$$= \int_0^{2\pi}\int_0^1 \left[r^2\sin^2\theta\, z + \frac{z^3}{3}\right]_{-1/2}^{1/2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left\{\left(r^2\sin^2\theta \times \frac{1}{2} + \frac{(1/2)^3}{3}\right) - \left(r^2\sin^2\theta \times \left(\frac{-1}{2}\right) + \frac{(-1/2)^3}{3}\right)\right\} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{r^2\sin^2\theta}{2} + \frac{1}{24} + \frac{r^2\sin^2\theta}{2} + \frac{1}{24}\right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r^2\sin^2\theta + \frac{1}{12}\right) dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^3}{3}\sin^2\theta + \frac{r}{12}\right)_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{\sin^2\theta}{3} + \frac{1}{12}\right) d\theta$$

$$= \int_0^{2\pi} \frac{\sin^2\theta}{3}\, d\theta + \int_0^{2\pi} \frac{1}{12}\, d\theta = \left[\frac{\theta}{12}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\sin^2\theta}{3}\, d\theta$$

$$= \frac{2\pi}{12} + \frac{1}{3\times 2}\int_0^{2\pi} (1 - \cos 2\theta)\, d\theta = \frac{\pi}{6} + \frac{1}{6}\left[\int_0^{2\pi} 1\cdot d\theta - \int_0^{2\pi} \cos 2\theta\, d\theta\right]$$

Date. No.

$$= \frac{\pi}{6} + \frac{1}{6} \times 2\pi - \frac{1}{6} \times \left[\cos 4\pi - \cos 0\right]$$

$$= \frac{\pi}{6} + \frac{2\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$$

(iii): $\int_0^2$