Lecture 08

Electrostatic Field in Matter (Contd.)

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Outline

Dielectrics

- A dielectric is insulator (nonconducting material) such as air, oil, rubber, glass, plastic,or waxed paper etc.
- In dielectrics, all charges are tightly bound to specific atoms or molecules, and all they can move a bit within the atom of molecules. So there no free electrons to carry currents on the dielectric.
- An applied field causes a displacement of charges but there is no flow of charges in dielectrics.

Dielectrics (contd.)

The molecules of a dielectric may be classified as **polar** or **nonpolar**.

- In polar molecules, the "centers of gravity" of the positive and negative charge distributions do not coincide at a point.
- A polar molecule has a permanent dipole moment, even in the absence of a polarizing filed.
  Examples:
  H_2O, HCl, NH_3 etc.

- In nonpolar molecules, the "centers of gravity" of the positive and negative charge distributions coincide at a point.
- A nonpolar molecule does not have a permanent dipole moment.
  Examples:
  H_2, N_2, O_2, CO_2, He, Ne, Ar etc.

Induced Dipole Moment and Atomic Polarizability(α)

In non-polar molecules and in every isolated atom, the centers of the positive and negative charges coincide (Figure 1a) and thus no dipole moment is set up. However, if we place an atom or a non-polar molecule in an external electric field, the field distorts the electron orbits and separates the centers of positive and negative charge (Figure 1b). Because the electrons are negatively charged, they tend to be shifted in a direction opposite the field. This shift sets up a dipole moment $\vec{p}$ that points in the direction of the field. This dipole moment is said to be induced by the field, and the atom or molecule is then said to be *polarized* by the field.

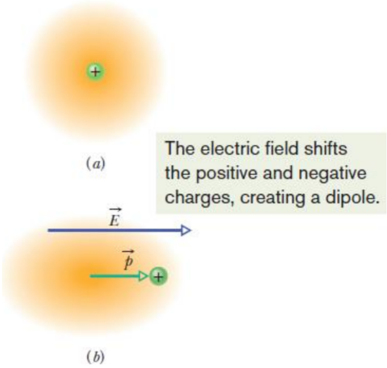

Figure 1

Induced Dipole Moment and Atomic Polarizability(α) (contd.)

If the electric field is so large the electrons are seperated from the atom and the atom become ionized. But for small electric field, the atom just becomes a dipole, this induced dipole moment is approximately proportional to the applied electric field:

$$\vec{p} = \alpha \vec{E} \tag{1}$$

The constant of proportionality α is called atomic polarizability. Its value depends on the detailed structure of the atom. Atomic Polarizability is defined as the electric dipole moment induced in the atom by an electric field of unit strength.

$$\boxed{\alpha = \frac{p}{E}} \tag{2}$$

The unit of α is

$\frac{\mathrm{C \cdot m}}{\mathrm{V \cdot m^{-1}}} = \mathrm{C \cdot m^2 \cdot V^{-1}} = \left(\mathrm{C \cdot V^{-1}}\right) \cdot \mathrm{m^2} = \mathrm{F \cdot m^2}$ (Farad meter squared).

Induced Dipole Moment and Atomic Polarizability(α) (contd.)

Example:- **A primitive model for an atom consists of a point nucleus $(+q)$ surrounded by a uniformly charged spherical cloud $(-q)$ of radius a [Figure 2. Calculate the atomic polarizability of such an atom.**

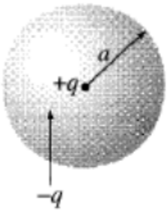

Figure 2: Neutral atom

Induced Dipole Moment and Atomic Polarizability(α) (contd.)

Solution:- When an atom is placed in an external electric field $\vec{E}$ directed from left to right, the nucleus will be shifted slightly to the right and the electron cloud to the left, as shown in Figure 3.

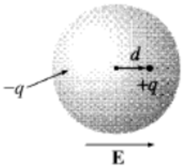

Figure 3: Induced dipole

If E_e be the field at a distance 'd' from the center of electron cloud (sphere) when equilibrium occurs.

Induced Dipole Moment and Atomic Polarizability(α) (contd.)

So, at equilibrium,

$$E = E_e = \frac{\rho d}{3\varepsilon_0} = \frac{q}{\frac{4}{3}\pi a^3}\frac{d}{3\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3} = \frac{1}{4\pi\varepsilon_0}\frac{p}{a^3}$$

$$\therefore \boxed{p = \left(4\pi\varepsilon_0 a^3\right) E} \tag{3}$$

Therefore, the atomic polarizability is

$$\alpha = 4\pi\varepsilon_0 a^3 = 3\varepsilon_0\left(\frac{4}{3}\pi a^3\right) = 3\varepsilon_0 v \qquad [\because p = \alpha E]$$

$$\therefore \boxed{\alpha = 3\varepsilon_0 v} \tag{4}$$

where v is the volume of the atom.

Polarization

When a piece of dielectric material is placed in an electric field, a lot of little dipoles pointing along the direction of the field and the material is said to be *polarized*. The dipole moment per unit volume of the polarized material is called polarization and dented by $\vec{P}$. For an elemental volume $d\tau$ of the material with net dipole moment $d\vec{p}$, the polarization is

$$\vec{P} = \frac{d\vec{p}}{d\tau} \tag{5}$$

or

$$d\vec{p} = \vec{P} d\tau \tag{6}$$

The SI unit of polarization is coulomb per meter squared C·m^{-2}.

Polarization:- Bound charges

Suppose we have a long string of dipoles, as shown in Figure 4. Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends, there are two charges left over: plus at the right end and minus at the left. The net charge at the ends is called **bound charge** to remind ourselves that it cannot be removed; in a dielectric, every electron is attached to a specific atom or molecule.

Figure 4

Polarization:-Physical Interpretation of Bound Charges

For uniform polarization

To calculate the bound charge, consider a "tube" of dielectric parallel to uniform polarization $\vec{P}$ (Figure 5). The dipole moment of tiny chunk shown in Figure 5 is

$$p = P(Ad) \tag{7}$$

where A is the cross-sectional area of the tube and d is the length of the chunk. In terms of the charge (q) at the end, this same dipole moment can be written as

$$p = qd \tag{8}$$

From equations (7) and (8), we get

$$q = PA \tag{9}$$

Therefore, the bound charge that piles up at the right end of the tube is $q = PA$.

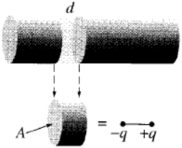

Figure 5

Polarization:-Physical Interpretation of Bound Charges

For uniform polarization (contd.)

For the ends sliced off perpendicularly, the surface charge density is

$$\sigma_b = \frac{q}{A} = P \tag{10}$$

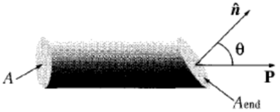

Figure 6: An oblique cut

i.e. **surface charge density of bound charge = uniform polarization**

For an oblique cut (Figure 6), the charge is still the same, but $A = A_{end}\cos\theta$.

So, the surface charge density is $\sigma_b = \dfrac{q}{A_{end}} = \dfrac{q}{A/\cos\theta} = P\cos\theta = \vec{P}\cdot\hat{n}$

The effect of the polarization, then, is to paint a bound charge $\sigma_b = \vec{P}\cdot\hat{n}$ over the surface of the material. Thus uniform polarization produces the surface charge density at the end. There is no volume bound charge for uniform polarization.

Polarization:-Physical Interpretation of Bound Charges

For nonuniform Polarization

If the polarization is nonuniform, we get accumulations of bound charge within the material as well as on the surface.

Figure 7 suggests that a diverging $\vec{P}$ results in a pile up of negative charge. Indeed, the net bound charge $\int_V \rho_b d\tau$ in a given volume is equal and opposite to the amount that has been pushed out through the surface $\oint_S \sigma_b da$.

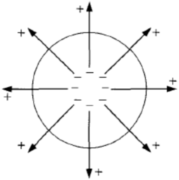

Figure 7

i.e. $$\int_V \rho_b d\tau = -\oint_S \sigma_b da = -\oint_S \left(\vec{P}\cdot\hat{n}\right) da = -\int_V \left(\nabla\cdot\vec{P}\right) d\tau$$

Since this is true for any volume, we have

$$\boxed{\rho_b = -\nabla\cdot\vec{P}} \qquad (11)$$

Gauss's Law in the Presence of Dielectrics

We know that, the effect of polarization is to produce accumulations of bound charge, $\rho_b = -\nabla \cdot \vec{P}$ within the dielectric and $\sigma_b = \vec{P} \cdot \hat{n}$ on the surface. Within the dielectric, the total volume charge density can be written as

$$\rho = \rho_b + \rho_f$$

where ρ_b is the bound volume charge density and ρ_f is the free volume charge density.

We know the differential form of Gauss's law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \nabla \cdot \vec{E} = \frac{\rho_b + \rho_f}{\varepsilon_0} \qquad \text{where } \vec{E} \text{ is the total field.}$$

Gauss's Law in the Presence of Dielectrics (contd.)

$$\Rightarrow \; \varepsilon_0 \nabla \cdot \vec{E} = \rho_b + \rho_f$$

$$\Rightarrow \; \nabla \cdot \varepsilon_0 \vec{E} = -\nabla \cdot \vec{P} + \rho_f$$

$$\Rightarrow \; \nabla \cdot \left(\varepsilon_0 \vec{E} + \vec{P} \right) = \rho_f \quad \therefore \; \boxed{\nabla \cdot \vec{D} = \rho_f}$$

Where $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ is the **electric displacement**.

In integral form,

$$\int \left(\nabla \cdot \vec{D} \right) d\tau = \int \rho_f d\tau$$

$$\oint \vec{D} \cdot d\vec{a} = \mathrm{Q}_{f_{enc}}$$

where $\mathrm{Q}_{f_{enc}}$ denotes the total free charge enclosed in the volume. This is a particularly useful way to express Gauss's law, in the context of dielectrics, because it refers only to free charges.

Example 1:- **A long straight wire, carrying line charge λ, surrounded by rubber insulation out to a radius a (Figure 8). Find the electric displacement.**

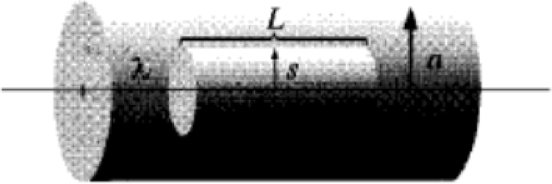

Figure 8

Solution: We have, Gauss's law in the presence of dielectric in integral form:

$$\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}} \tag{12}$$

where $Q_{f_{enc}}$ denotes the total free charge enclosed in the volume.

Drawing a cylindrical Gaussian surface, of radius s of length L, and applying Eq. (12), we find:

$$D \oint da = Q_{f_{enc}}$$

$$D\,(2\pi s L) = \lambda L \implies \boxed{\vec{D} = \frac{\lambda}{2\pi s}\hat{s}}$$

Gauss's Law in the Presence of Dielectrics (contd.)

Where λL = charge enclosed by closed Gaussian surface.

We know that $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

Here $\vec{P} = 0$ ($\because$ there is no dielectric material)

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$$

Susceptibility, Permittivity, Dielectric Constant

When the dielectric material is placed in the electric field, the dielectric material get polarized. The polarization results from the lining up the atomic or molecular dipole along the direction of electric field. In many dielectric substance, the polarization is proportional to the the electric field, provided $\vec{E}$ is not too strong: i.e. $\vec{P} \propto \vec{E}$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \tag{13}$$

The constant of proportionality, χ_e is called the electric susceptibility of the medium (a factor of ε_0 has been extracted to make χ_e dimensionless). The materials that obey Eq. (13) are called linear dielectrics.

Susceptibility, Permittivity, Dielectric Constant (contd.)

In linear media we have

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 \left(1 + \chi_e\right) \vec{E} \tag{14}$$

So, $\vec{D}$ is also proportional to $\vec{E}$:

$$\vec{D} = \varepsilon \vec{E} \tag{15}$$

$$\text{where} \quad \varepsilon = \varepsilon_0 \left(1 + \chi_e\right) \tag{16}$$

This new constant ε is called the permittivity of the material. In vacuum, where there is no matter to polarize, the susceptibility is zero, and the permittivity is ε_0. That's why ε_0 is called the permittivity of free space. The dimensionless quantity $\varepsilon_r \equiv \left(1 + \chi_e\right) = \dfrac{\varepsilon}{\varepsilon_0}$ is called the relative permittivity, or dielectric constant (K), of the material.

Susceptibility, Permittivity, Dielectric Constant (contd.)

Also,

$$\rho_b = -\nabla \cdot \vec{P} = -\nabla \cdot \left(\varepsilon_0 \chi_e \vec{E}\right) = -\nabla \cdot \left[\varepsilon_0 \chi_e \frac{\vec{D}}{\varepsilon_0 (1+\chi_e)}\right] = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f$$

$$\therefore \boxed{\rho_b = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f}$$

Hence, in homogeneous linear dielectric, the bound volume charge density (ρ_b) is proportional to free volume charge density (ρ_f). If there is not the free charge inside the material then volume bound charge would be zero. If there are positive free charges then the bound charge accumulated inside the region would be negative and vice versa.

Example 2:- **A metal sphere of radius a, carries a charge Q [Figure 9]. It is surrounded, out to radius b, by linear dielectric material of permittivity ε. Find the potential at the center (relative to infinity).**

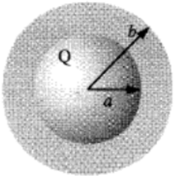

Figure 9

Solution:

Drawing a spherical Gaussian surface of radius $r(r > a)$ and applying Eq.: $\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$, we get

$$D\left(4\pi r^2\right) = Q$$

$$\therefore \vec{D} = \frac{Q}{4\pi r^2}\hat{r} \qquad \text{for all points } r > a$$

Since, $\vec{D} = \varepsilon\vec{E} \implies \vec{E} = \dfrac{\vec{D}}{\varepsilon}$

$$\vec{E} = \begin{cases} \frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2}\hat{r}, & \text{for } r > b \\ \frac{1}{\varepsilon}\frac{Q}{4\pi r^2}\hat{r}, & \text{for } a < r < b \\ 0, & \text{for } r < a \end{cases} \tag{17}$$

Susceptibility, Permittivity, Dielectric Constant (contd.)

The potential at the centre relative to infinity is

$$V = -\int_{\infty}^{0} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{b} Edr - \int_{b}^{a} Edr - \int_{a}^{0} Edr$$

$$= -\int_{\infty}^{b} \left(\frac{1}{\varepsilon_0} \frac{Q}{4\pi r^2} \right) dr - \int_{b}^{a} \left(\frac{1}{\varepsilon} \frac{Q}{4\pi r^2} \right) dr - \int_{a}^{0} (0) dr$$

$$\therefore \boxed{V = \frac{Q}{4\pi} \left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b} \right)}$$

End of Lecture 08

Thank you