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UNIT: 4

BETA & GAMMA FUNCTIONS

(x) Beta Functions

The Beta Function or first Eulerian Integral denoted by $B(m,n)$.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \left(\begin{matrix} m>0, \\ n>0 \end{matrix}\right)$$

(x) Gamma Functions

The Gamma Function or second Eulerian integral denoted by $\Gamma(p)$.

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1}\,dx \qquad (p>0)$$

⟺ Properties

(i) $B(m,n) = B(n,m)$

Soln:

We know,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $1-x=t$. then, $dx = -dt$. So,

$$B(m,n) = \int_0^1 (1-t)^{m-1}\, t^{n-1}\,dx \qquad = B(n,m).$$

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(ii) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Soln:

We know,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $x = \sin^2\theta$

So, $dx = 2\sin\theta\cos\theta\, d\theta$

When $x=0,\ \theta=0$

When $x=1,\ \theta=\pi/2$

$$B(m,n) = \int_0^{\pi/2} \sin^{2(m-1)}\theta\,(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \sin\theta \cdot \cos^{2n-2}\theta \cdot \cos\theta\, d\theta$$

$$\therefore B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

(iii): $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Soln:

We know,

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\,dx$$

Let $x = \frac{y}{1-y}$. So, $1+x = 1+\frac{y}{1-y} = \frac{1}{1-y}$.

Therefore,

$$dx = \frac{1}{(1-y)^2}\,dy$$

When $x=0,\ y=0$.

when $x=\infty,\ y=1$.

Therefore,

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{y^{m-1}}{(1-y)^{m-1}} \cdot \frac{(1-y)^{m+n}}{(1-y)}\,dy$$

$$= \int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

$$\therefore B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx.$$

(iv): $\Gamma(n+1) = n\Gamma(n)$.

Soln:

We know,

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1}\,dx$$

$$= \int_0^\infty e^{-x} x^n\,dx \quad = \lim_{b\to\infty}\left[-x^n e^{-x}\right]_0^b + n\int_0^\infty e^{-x} x^{n-1}\,dx$$

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~~But $\lim_{b\to\infty} \frac{b^n}{e^b} = 0$. {since $n>0$}~~

~~Therefore, $\Gamma(n+1) = n\Gamma(n)$.~~

i.e.,

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx$$

$$= \lim_{a\to\infty} \left[x^n \int_0^a e^{-x} dx - \int_0^a \left(\frac{dx^n}{dx} \int_0^a e^{-x} dx \right) dx \right]$$

$$= \lim_{a\to\infty} \left[x^n \int_0^a e^{-x} dx - \int_0^a nx^{n-1} \cdot \frac{e^{-x}}{-1} dx \right]$$

$$= x^n \cdot \frac{e^{-x}}{-1} \Big|_0^\infty + n \int_0^\infty x^{n-1} e^{-x} dx$$

$$\therefore \Gamma(n+1) = n\Gamma(n).$$

(v): $\Gamma(n+1) = n!$ (If $n \in Z^{+ve}$)

Soln.

We know,

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n\Gamma[(n-1)+1]$$

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$$\text{or, } \Gamma(n+1) = n(n-1)\Gamma(n-1)$$
$$= n(n-1)\Gamma[(n-2)+1]$$
$$= n(n-1)(n-2)\ldots\ldots \Gamma(n-2)$$
$$= n(n-1)(n-2)\ldots\ldots\Gamma(1)$$
$$= n!$$

$$\therefore \Gamma(n+1) = n!$$

(*) Prove that $\Gamma(1) = 1$

Soln:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

So,

$$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx$$

$$= \int_0^\infty e^{-x} dx = \lim_{a\to\infty} \left(\frac{e^{-x}}{-1} \right)_0^a$$

$$= \frac{e^0}{1} - \frac{e^{-a}}{1}$$

$$\therefore \Gamma(1) = 1$$

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(vi): $\Gamma(m)\,\Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$.

~~Sol~~^{n}

When $m = 1/2$,

$$\Gamma(1/2)\,\Gamma(1-1/2) = \frac{\pi}{\sin 1/2\,\pi}$$

$$\text{or, } \Gamma(1/2)\,\Gamma(1/2) = \frac{\pi}{\sin \pi/2}$$

$$\text{or, } \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$$

$$\therefore \Gamma(1/2) = \sqrt{\pi}$$

Now,

(a): $\Gamma\left(\dfrac{3}{2}\right) = \Gamma\left(\dfrac{1}{2}+1\right) = \dfrac{1}{2}\,\Gamma\left(\dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2}$

(b) $\Gamma\left(\dfrac{7}{2}\right) = \Gamma\left(\dfrac{5}{2}+1\right) = \dfrac{5}{2}\,\Gamma\left(\dfrac{5}{2}\right)$

$= \dfrac{5}{2}\,\Gamma\left(\dfrac{3}{2}+1\right) = \dfrac{5}{2}\times\dfrac{3}{2}\,\Gamma\left(\dfrac{3}{2}\right)$

$= \dfrac{5}{2}\times\dfrac{3}{2}\times\dfrac{\sqrt{\pi}}{2} = \dfrac{5\cancel{15}\sqrt{\pi}}{\cancel{8}\,2} = \dfrac{5\sqrt{\pi}}{2}$

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(c): $\Gamma\left(\dfrac{1}{3}\right)\Gamma\left(\dfrac{2}{3}\right)$

Sol^n.

$= \Gamma\left(\dfrac{1}{3}\right)\Gamma\left(1-\dfrac{1}{3}\right)$

We know,

$$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

So, $m = 1/3$.

$$\therefore \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{\pi}{\sin 1/3\,\pi}$$

$$\text{or } \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$$

(d): $\Gamma\left(\dfrac{1}{4}\right)\Gamma\left(\dfrac{3}{4}\right)$

Sol^n:

$= \Gamma\left(\dfrac{1}{4}\right)\Gamma\left(1-\dfrac{1}{4}\right)$

We know,

$$\Gamma(m)\,\Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

So,

$$\Gamma(1/4)\,\Gamma(1-1/4) = \frac{\pi}{\sin \pi/4}$$

$$\text{or } \Gamma(1/4)\,\Gamma(3/4) = \frac{\pi}{1/\sqrt{2}}$$

$$\therefore \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$$

(Q) Evaluate:

(a) $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\ldots\ldots\Gamma\left(\frac{8}{9}\right)$

Soln.

$$= \Gamma\left(1-\frac{8}{9}\right)\Gamma\left(1-\frac{7}{9}\right)\Gamma\left(1-\frac{6}{9}\right)\Gamma\left(1-\frac{5}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(\frac{6}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(\frac{8}{9}\right)$$

$$= \Gamma\left(\frac{8}{9}\right)\Gamma\left(1-\frac{8}{9}\right)\Gamma\left(\frac{7}{9}\right)\Gamma\left(1-\frac{7}{9}\right)\Gamma\left(\frac{36}{69}\right)\Gamma\left(1-\frac{6}{9}\right)\Gamma\left(\frac{5}{9}\right)\Gamma\left(1-\frac{5}{9}\right)$$

$$= \frac{\pi}{\sin 8\pi/9} \times \frac{\pi}{\sin 7\pi/9} \times \frac{\pi}{\sin 6\pi/9} \times \frac{\pi}{\sin 5\pi/9}$$

$$= \frac{\pi^4}{\sin 160 \times \sin 140 \times \sin 120 \times \sin 100} = \frac{16\pi^4}{3}$$

(*) Important Properties.

(i): $\int_0^{\pi/2} \sin^p\theta \, d\theta = \int_0^{\pi/2} \cos^p\theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right)$

(ii) $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

(iii) $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

<Q7: $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma(p+1/2)\,\Gamma(q+1/2)}{2\Gamma(p+q+2/2)}$

Soln.

We know,

$$= \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

Let $p = 2m-1$ $\quad q = 2n-1$

$\therefore m = (p+1)/2$ $\quad \therefore n = (q+1)/2$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

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<Q>: $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Soln.

Sol: Let $x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2\sqrt{t}}$

When $x = 0$, $t = 0$.

When $x = \infty$, $t = \infty$.

$= \int_0^\infty \frac{e^{-t}}{t^{1/2}} dt$

~~Let p~~ ~~then~~, Let $p-1 = -1/2$ $\therefore p = 1/2$

$\int_0^\infty e^{-t} t^{p-1} dt$

$= \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

<Q>: Evaluate:

(i): $\int_0^{\pi/2} \sin^3\theta \cos^4\theta \, d\theta$

Soln:

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$= \int_0^{\pi/2} \sin^3\theta \cos^4\theta \, d\theta$

$= \int_0^{\pi/2} \sin^{4-1}\theta \cos^{5-1}\theta \, d\theta$ $= \int_0^{\pi/2} \sin^{2\times2-1}\theta \cdot \cos^{2\times 5/2-1}\theta \, d\theta$

Here,

$2M-1 = 3$ $\therefore M = 2$

$2N-1 = 4$ $\therefore N = 5/2$

So,

$\left[\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m+n) = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$

$= \frac{1}{2} \frac{\Gamma(M)\Gamma(N)}{\Gamma(M+N)}$

$= \frac{1}{2} \frac{\Gamma(2)\Gamma(5/2)}{\Gamma(2+5/2)}$

$= \frac{1}{2} \frac{\Gamma(1+1)\Gamma(5/2)}{\Gamma(9/2)} = \frac{1}{2} \frac{\Gamma(5/2)}{\Gamma(9/2)}$

$= \frac{1}{2} \frac{\Gamma(3/2+1)}{\Gamma(7/2+1)} = \frac{1}{2} \times \frac{3}{2} \times \frac{7}{2} \times \frac{\Gamma(3/2)}{\Gamma(7/2)}$

$= \frac{1}{2} \times \frac{3}{2} \times \frac{7}{2} \times \frac{\sqrt{\pi}}{2} \times \frac{2}{5\sqrt{\pi}} = \frac{1}{2} \times \frac{1 \times 3/2 \times 1/2 \times 1}{7/2 \times 5/2 \times 3/2 \times 1/2 \times 1}$

$= \frac{2}{35}$

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<Q7: Prove that: $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Soln.

We know,

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt \quad (m>0) \quad -(i)$$

Let $t = zx$ then,

$dt = z\,dx$

when $t=0$, $x=0$.

when $t=\infty$, $x=\infty$

From (i),

$$\Gamma(m) = \int_0^\infty e^{-zx} (zx)^{m-1} z\,dx$$

$$= \int_0^\infty z^m e^{-zx} x^{m-1} dx$$

or, $$\Gamma(m)\left[\int_0^\infty e^{-z} z^{n-1} dz\right] = \int_0^\infty z^m e^{-zx} x^{m-1} dx \left(\int_0^\infty e^{-z} z^{n-1} dz\right)$$

or, $$\Gamma(m)\Gamma(n) = \int_0^\infty\int_0^\infty z^{m+n-1} e^{-zx-z} x^{m-1} dx\,dz$$

or, $$\Gamma(m)\Gamma(n) = \int_0^\infty\int_0^\infty z^{m+n-1} e^{-z(1+x)} x^{m-1} dx\,dz.$$

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Integrating z from 0 to ∞, we have.

or, $$\Gamma(m)\Gamma(n) = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz\right] x^{m-1} dx$$

or, $$\Gamma(m)\Gamma(n) = \int_0^\infty \frac{\Gamma(m,n) \cdot x^{m-1}}{(1+x)^{m+n}} dx.$$

We know,

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

So,

$$\Gamma(m)\Gamma(n) = \Gamma(m,n)\, B(m,n)$$

$$\therefore B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m,n)}$$

<Q>: Prove that:

(i) $B(m,n)\, B(m+n,l) = B(n,l)\, B(n+l, m)$

Soln.

$LHS = B(m,n)\, B(m+n,l)$

$$= \frac{\Gamma(m)\Gamma(n)}{\cancel{\Gamma(m+n)}} \times \frac{\cancel{\Gamma(m+n)}\Gamma(l)}{\Gamma(m+n+l)}$$

$$= \frac{\Gamma(n)\Gamma(l)}{\Gamma(n+l)} \times \frac{\Gamma(n+l)\Gamma(m)}{\Gamma(m+n+l)}$$

$= B(n,l)\, B(n+l, m)$ Hence, proved

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(ii): $\frac{B(m,n+1)}{n} = \frac{B(m+1,n)}{m} = \frac{B(m,n)}{m+n}$

Soln:

We have

$$\frac{B(m,n+1)}{n} = \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)} \times \frac{1}{n}$$

$$= \frac{\Gamma(m)\,\cancel{n}\,\Gamma(n)}{(m+n)\,\Gamma(m+n)} \times \frac{1}{\cancel{n}}$$

$$= \frac{B(m,n)}{(m+n)}$$

$$\therefore \frac{B(m,n+1)}{n} = \frac{B(m,n)}{(m+n)}$$

Also,

$$\frac{B(m+1,n)}{m} = \frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)} \times \frac{1}{m}$$

$$= \frac{\Gamma(m) \times \cancel{m} \times \Gamma(n)}{(m+n)\,\Gamma(m+n)} \times \frac{1}{\cancel{m}}$$

$$\therefore \frac{B(m+1,n)}{m} = \frac{B(m,n)}{(m+n)}.$$

Hence, proved.

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(Q): Prove that:

$B(m,n) = \frac{n-1}{m+n-1} B(m,n-1) = \frac{m-1}{m+n-1} B(m-1,n)$

Soln:

$$\frac{n-1}{m+n-1} B(m,n-1) = \frac{(n-1)}{(m+n-1)} \times \frac{\Gamma(m)\,\Gamma(n-1)}{\Gamma(m+n-1)}$$

$$= \frac{\Gamma(m)\cdot(n-1)\,\Gamma(n-1)}{(m+n-1)\,\Gamma(m+n-1)}$$

$$= \frac{\Gamma(m)\,\Gamma(n-1+1)}{\Gamma(m+n-1+1)} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \frac{(n-1)}{(m+n-1)} B(m,n-1) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = B(m,n).$$

and.

$$\frac{m-1}{m+n-1} B(m-1,n) = \frac{(m-1)}{(m+n-1)} \frac{\Gamma(m-1)\,\Gamma(n)}{\Gamma(m+n-1)}$$

$$= \frac{(m-1)}{(m+n-1)} \cdot \frac{\Gamma(m-1)}{\Gamma(m+n-1)} \Gamma(n)$$

$$= \frac{\Gamma(m-1+1)\,\Gamma(n)}{\Gamma(m+n-1+1)}$$

$$\therefore \frac{m-1}{m+n-1} B(m-1,n) = B(m,n)$$

Hence, proved.