Lecture 11

Magnetostatic (Contd.)

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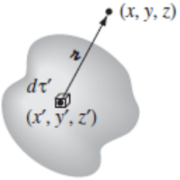
Outline

The divergence of $\vec{B}$ i.e. $(\nabla \cdot \vec{B})$

Form Biot-Savart law, the magnetic field at a point $P(x, y, z)$ in term of volume current density is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3} d\tau' \qquad (1)$$



Here, the integration is taken over the source coordinate (x', y', z'). $\vec{J}$ is purely the function of source coordinate x', y', z' and separation vector $\vec{\mathscr{r}}$ is the fuction of both (x, y, z) and (x', y', z'). After integration $\vec{B}$ comes out to be as purely the function of field coordinate (x, y, z). Now taking divergence of $\vec{B}$ with respect to (x, y, z), we get

$$\nabla \cdot \vec{B} = \nabla \cdot \left(\frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3} d\tau' \right) = \frac{\mu_0}{4\pi} \int_V \nabla \cdot \left(\frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3} \right) d\tau' \qquad (2)$$

The divergence of $\vec{B}$ i.e. $(\nabla \cdot \vec{B})$ (contd.)

Using the product rule $\nabla \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\nabla \times \vec{A}\right) - \vec{A} \cdot \left(\nabla \times \vec{B}\right)$, we can have

$$\nabla \cdot \left(\frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3}\right) = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} \cdot \left(\nabla \times \vec{J}\right) - \vec{J} \cdot \left(\nabla \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right)$$

Since $\vec{J}$ depends only on source coordinate and curl on it is taken over field coordinate, $\nabla \times \vec{J} = 0$

$$\therefore \nabla \cdot \left(\frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3}\right) = -\vec{J} \cdot \left(\nabla \times \frac{\vec{\mathscr{r}}}{\mathscr{r}^3}\right)$$

The divergence of $\vec{B}$ i.e. $(\nabla \cdot \vec{B})$ (contd.)

Using $\nabla \left(\frac{1}{\mathscr{r}} \right) = -\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$, we have

$$\nabla \cdot \left(\frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3} \right) = \vec{J} \cdot \left[\nabla \times \nabla \left(\frac{1}{\mathscr{r}} \right) \right]$$

As the curl of gradient is always zero, i.e. $\nabla \times \left[\nabla \left(\frac{1}{\mathscr{r}} \right) \right] = 0$

$$\nabla \cdot \left(\frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3} \right) = 0$$

Hence, the equation (2) reduces to

$$\nabla \cdot \vec{B} = 0 \tag{3}$$

This shows that the magnetic field is always divergence-less i.e. solenoidal. There is no point source which can produce the magnetic field i.e. magnetic monopole does not exist.

Taking the volume integral in equation (3), we get

$$\int_V \left(\nabla \cdot \vec{B}\right) d\tau = 0$$

Applying the Gauss divergence theorem, we can have

$$\oint_S \vec{B} \cdot d\vec{a} = 0 \qquad (4)$$

Therefore, the surface integral of a magnetic field over a closed surface i.e. total flux through the closed surface is always zero.

Ampere's law and curl of magnetic field

It states that "the line integral of the magnetic field $\vec{B}$ around any closed path is equal to μ_0 times the net current enclosed by the path" i.e. if I_{enc} be the current enclosed by the path and $d\vec{l}$ be an elemental length of closed path then according to this law, we can have

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \tag{5}$$

If the path of integration encloses no current i.e. if the current carrying conductor is outside of the path of integration then the line integral of the magnetic field $\vec{B}$ is zero for this integration. That means $\oint \vec{B} \cdot d\vec{l} = 0$ does not necessarily mean that $\vec{B} = 0$; but only that no current is linked by the path.

Ampere's law and curl of magnetic field (contd.)

In a region distributed current flow, the total current flowing across the surface bounded by the path is $I_{\text{enc}} = \int_S \vec{J} \cdot d\vec{a}$, where $\vec{J}$ is current density and $\hat{n}$ is unit normal to surface.

From Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \int_S \vec{J} \cdot d\vec{a}$$

Using Stoke's theorem, $\oint \vec{B} \cdot d\vec{l} = \int\limits_S (\nabla \times \vec{B}) \cdot d\vec{a}$

$$\int\limits_S (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int_S \vec{J} \cdot d\vec{a}$$

$$\therefore \boxed{\nabla \times \vec{B} = \mu_0 \vec{J}} \tag{6}$$

This is differential form of Ampere's law.

Applications of Ampere's law

In order to apply the Ampere's law (in integral form), the path of integration i.e. Amperian loop must be so chosen such that

1. The magnetic field is tangential to the path so that

$$\oint \vec{B} \cdot d\vec{l} = \oint Bdl$$

2. The magnitude of magnetic is same on all points on the path so that

$$\oint Bdl = B \oint dl$$

Therefore, for appropriately chosen Amperian loop, we can have

$$\oint \vec{B} \cdot d\vec{l} = B \oint dl = B \times \text{Total length of Amperian loop} = \mu_0 I_{\text{enc}}$$

Applications of Ampere's law (contd.)

Example-I **Find the magnetic field a distance s from a long straight wire, carrying a steady current I.**

Consider a straight wire carrying current I. To find the magnetic field at P, let's construct a circular Amperian loop of radius s passing through P with the wire as the axis as shown in the figure 1. The current enclosed by the loop $I_{\text{enc}} = I$.

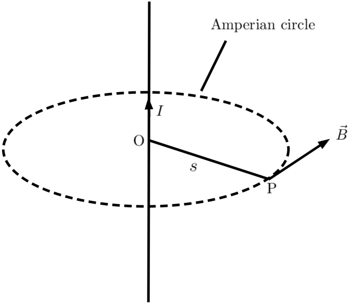


Figure 1

Applications of Ampere's law (contd.)

From Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

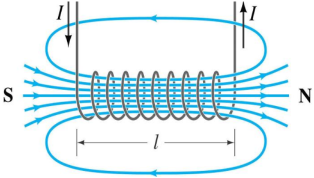
$$B \oint dl = \mu_0 I$$

$$B(2\pi s) = \mu_0 I$$

$$\therefore B = \frac{\mu_0 I}{2\pi s}$$

This is the required relation for magnetic field due to straight current carrying conductor at a distance s from it and is directed tangential along the loop.

Example-II **Find the magnetic field of a very long solenoid, consisting of closely wound turns per unit length on a cylinder of radius R and carrying a steady current I.**



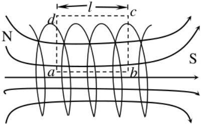


Figure 2

Applications of Ampere's law (contd.)

A solenoid is a long wire wound in a closed packed helix and carrying a current I. We assume that the solenoid is very long compared with its diameter such that the internal field near its center is almost uniform and parallel to axis and field outside it is very small. For practical solenoid, the field outside is taken as zero i.e. $B_{\text{out}} = 0$ and this is not very bad assumption. In order to calculate $\vec{B}$ inside the solenoid, let's construct rectangular Amperian loop *abcda* of length l as shown in figure 2. The number of turns enclosed by

Applications of Ampere's law (contd.)

the Amperian loop is $N = nl$ and the total current enclosed is $I_{\text{enc}} = NI = nlI$. Now from Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\implies \int_a^b \vec{B} \cdot d\vec{l} + \int_b^c \vec{B} \cdot d\vec{l} + \int_c^d \vec{B} \cdot d\vec{l} + \int_d^a \vec{B} \cdot d\vec{l} = \mu_0 Inl \quad (7)$$

where $d\vec{l}$ is element of path of integration. The section bc and da is perpendicular to the magnetic field inside the solenoid and there is no magnetic field outside. Therefore,

$$\underbrace{\int_b^c \vec{B} \cdot d\vec{l}}_{\vec{B} \perp d\vec{l}} = \underbrace{\int_c^d \vec{B} \cdot d\vec{l}}_{B_{\text{out}}=0} = \underbrace{\int_d^a \vec{B} \cdot d\vec{l}}_{\vec{B} \perp d\vec{l}} = 0$$

and

$$\underbrace{\int_a^b \vec{B} \cdot d\vec{l}}_{\vec{B}||d\vec{l} \text{ and constant inside}} = B \int_a^b dl = Bl$$

Therefore the equation (7), bocomes

$$Bl = \mu_0 Inl \implies B = \mu_0 nI$$

$$\boxed{B = \mu_0 nI} \tag{8}$$

This is the required expression for magnetic field due to solenoid.

Magnetic vector potential

We know that the divergence of magnetic field is zero i.e. $\nabla \cdot \vec{B} = 0$ and divergence of curl of a vector is always zero i.e. $\nabla \cdot (\nabla \times \vec{A}) = 0$. So, $\vec{B}$ may be represented as curl of some vector field $\vec{A}$ as

$$\vec{B} = \nabla \times \vec{A} = \text{curl}\vec{A} \tag{9}$$

The vector field $\vec{A}$ is called vector potential and defined as the vector field whose curl is equal to magnetic field $\vec{B}$. Also the curl of gradient is always zero that means vector $\vec{A}$ is not uniquely defined by equation (9). Suppose $\vec{A}'$ be a vector field given by $\vec{A}' = \vec{A} + \nabla\psi$ where ψ is scalar function. Then

$$\nabla \times \vec{A}' = \nabla \times [\vec{A} + \nabla\psi] = \nabla \times \vec{A} + \nabla \times \nabla\psi = \nabla \times \vec{A}$$

Magnetic vector potential (contd.)

Thus, for given vector function $\vec{B}$, a lot of vectors $\vec{A}'$ will satisfy equation (9). For simplicity, the vector function $\vec{A}$ is chosen such that

$$\nabla \cdot \vec{A} = 0 \tag{10}$$

Now from differential form of Ampere's law

$$\begin{aligned} \nabla \times \vec{B} &= \mu_0 \vec{J} \\ \text{or,} \nabla \times (\nabla \times \vec{A}) &= \mu_0 \vec{J} \\ \text{or,} \nabla (\cancelto{0}{\nabla \cdot \vec{A}}) - \nabla^2 \vec{A} &= \mu_0 \vec{J} \\ \implies -\nabla^2 \vec{A} &= \mu_0 \vec{J} \\ \therefore \nabla^2 \vec{A} &= -\mu_0 \vec{J} \end{aligned} \tag{11}$$

Magnetic vector potential (contd.)

In component form,

$$\nabla^2 A_x = -\mu_0 J_x \tag{12a}$$

$$\nabla^2 A_y = -\mu_0 J_y \tag{12b}$$

$$\nabla^2 A_z = -\mu_0 J_z \tag{12c}$$

Each equation of eq. (12) equivalent ot Poisson equation $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$ with solution $V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V \dfrac{\rho d\tau}{\mathscr{r}}$. Using the same token, the solutions of eq. (12) can be written as

$$A_x = \frac{\mu_0}{4\pi}\int_V \frac{J_x d\tau}{\mathscr{r}} \tag{13a}$$

$$A_y = \frac{\mu_0}{4\pi}\int_V \frac{J_y d\tau}{\mathscr{r}} \tag{13b}$$

Magnetic vector potential (contd.)

$$A_z = \frac{\mu_0}{4\pi}\int_V \frac{J_z d\tau}{\mathscr{r}} \tag{13c}$$

and

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = \frac{\mu_0}{4\pi}\int_v \frac{\left(J_x\hat{i} + J_y\hat{j} + J_z\hat{k}\right)}{\mathscr{r}} d\tau$$

$$\Longrightarrow \vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J} d\tau}{\mathscr{r}} \tag{14}$$

Equation (14) is solution of eq. (11) and this is the expression for vector potential.

Magnetic vector potential (contd.)

Alternatively

From Biot-Savart law,

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\vec{\mathscr{r}}}{\mathscr{r}^3}d\tau$$

$$= -\frac{\mu_0}{4\pi}\int_V \left[\vec{J}\times\nabla\left(\frac{1}{\mathscr{r}}\right)\right]d\tau \qquad \text{Using } \boxed{\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}}$$

Applying the vector identity

$$\nabla\times\left(\frac{\vec{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\cancelto{0}{\nabla\times\vec{J}} - \vec{J}\times\nabla\left(\frac{1}{\mathscr{r}}\right) \implies \vec{J}\times\nabla\left(\frac{1}{\mathscr{r}}\right) = -\nabla\times\left(\frac{\vec{J}}{\mathscr{r}}\right)$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \nabla\times\left(\frac{J}{\mathscr{r}}\right)d\tau$$

Magnetic vector potential (contd.)

$$\implies \vec{B} = \nabla \times \left(\frac{\mu_0}{4\pi} \int_V \frac{\vec{J}}{\mathscr{r}} d\tau \right)$$

Comparing this equation with equation with $\vec{B} = \nabla \times \vec{A}$, we get,

$$\vec{A} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} d\tau}{\mathscr{r}}$$

We have $I d\vec{l} = \vec{K} da = \vec{J} d\tau$, the expression of vector potential in term surface current density $\vec{K}$ and line current I are respectively

$$\vec{A} = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K} da}{\mathscr{r}} \tag{15}$$

and

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_L \frac{d\vec{l}}{\mathscr{r}} \tag{16}$$

Magnetic vector potential

Magnetic flux in term vector potential

The magnetic flux Φ_m passing through an area S may be given as

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a} = \int_s \left(\nabla \times \vec{A} \right) \cdot d\vec{a}$$

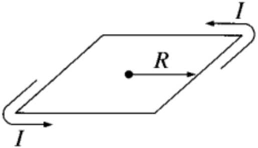
Using Stokes theorem, we have

$$\Phi_m = \int_S \vec{B} \cdot d\vec{a} = \oint_L \vec{A} \cdot d\vec{l} \tag{17}$$

That means the line integral of magnetic vector potential around a closed path which encloses the magnetic field lines is equal to the magnetic flux Φ_m.

Problems

1. a. **Find the magnetic field at the center of a square loop of side R, which carries a steady current I.**



Hint:-

Here, for a side $\theta_1 = -\dfrac{\pi}{4}$ and $\theta_2 = +\dfrac{\pi}{4}$ from figure above. Use the formula $\vec{B} = \dfrac{\mu_0 I}{4\pi r}(\sin\theta_2 - \sin\theta_1)\hat{k}$ to find the field $\vec{B}_1$ due to a side, assuming the loop in xy-plane with center at origin. Use $r = R$. Total magnetic field is $\vec{B} = 4\vec{B}_1$

- **(b)** **Find the field at the center of a regular n-sided polygon, carrying a steady current I. Again, let R be the distance from the center to any side.**

  Hint:-

  The angle subtended by a side at the center is $\frac{2\pi}{n}$ and hence, $\theta_1 = -\frac{\pi}{n}$ and $\theta_2 = +\frac{\pi}{n}$. So field due to a side is $\vec{B}_1 = \frac{\mu_0 I}{4\pi R} 2\sin\frac{\pi}{n}\hat{k}$ and for n side $\vec{B} = n\vec{B}_1 = \frac{\mu_0 I}{2R}\frac{n}{\pi}\sin\frac{\pi}{n}\hat{k}$

- **(c)** **Check that your formula reduces to the field at the center of a circular loop, in the limit $n \to \infty$**

  Hint:-

  As $n \to \infty$, $\frac{\pi}{n} \to 0$ and $\lim_{n\to\infty}\frac{n}{\pi}\sin\frac{\pi}{n} = \lim_{\frac{\pi}{n}\to 0}\frac{\sin\frac{\pi}{n}}{\pi/n} = 1$ so $\vec{B} = \frac{\mu_0 I}{2R}$ as for circular loop.

2. **Find the magnetic field on the axis of a tightly wound solenoid (helical coil) consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I [Figure 3]. Express your answer in terms of θ_1 and θ_2. Consider the turns to be essentially circular. What is the field on the axis of an infinite solenoid (infinite in both directions)?**

Hint:-

Let's take an elemental length dz on the solenoid at a distance z from the point P. The radius of the loops at this position subtends an angle θ at P..

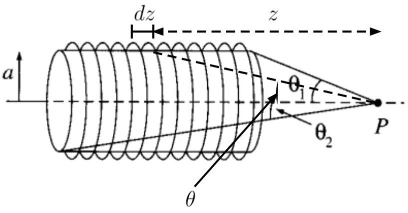


Figure 3

Problems (contd.)

The number of turns within dz is $dN = ndz$. Using the result for the magnetic field due to N number of circular loops i.e. $\vec{B} = \dfrac{\mu_0 NIR^2}{2\left(R^2+z^2\right)^{3/2}}\hat{k}$, we have the magnetic field at P due to the turns on elemental length dz as

$$d\vec{B} = \frac{\mu(dN)Ia^2}{\left(a^2+z^2\right)^{3/2}}\hat{k} = \frac{\mu_0(ndz)Ia^2}{\left(a^2+z^2\right)^{3/2}}\hat{k} = \frac{\mu_0 nIa^2}{2}\frac{dz}{\left(a^2+z^2\right)^{3/2}}\hat{k}$$

From figure, $z = a\cot\theta$ and $dz = -a\ \mathrm{cosec}^2\theta d\theta$. Now the above equation reduces to

$$d\vec{B} = \frac{\mu_0 nIa^2}{2}\frac{\left(-a\ \mathrm{cosec}^2\theta d\theta\right)}{a^3\ \mathrm{cosec}^3\theta}\hat{k} = -\frac{\mu_0 nI}{2}\sin\theta d\theta\hat{k}$$

Problems (contd.)

Therefore, the magnetic field at P due to the current on whole solenoid is

$$\vec{B} = -\frac{\mu_0 n I}{2}\int_{\theta_1}^{\theta_2} \sin\theta d\theta \hat{k} \implies \vec{B} = \frac{\mu_0 n I}{2}(\cos\theta_2 - \cos\theta_1)\hat{k}$$

For a long solenoid, $\theta_1 = \pi$ and $\theta_2 = 0$ and

$$\vec{B} = \frac{\mu_0 n I}{2}(\cos 0 - \cos\pi)\hat{k} = \mu_0 n I \hat{k}$$

Hence, the magnetic field inside the long solenoid is constant and along direction of axis of the solenoid. The magnitude is

$$B = \mu_0 n I$$

Problems (contd.)

3. A steady current I flows down a long cylindrical wire of radius a [Figure 4]. Find the magnetic field, both inside and outside the wire, if
   a. The current is uniformly distributed over the outside surface of the wire.
   b. The current is distributed in such a way that J is proportional to s, the distance from the axis.

Hint:

Consider P be a point at a distance r from the axis of the cylindrical wire. To find the magnetic field at P, let's construct the coaxial Amperian circle of radius r passing through P as shown in figure 4

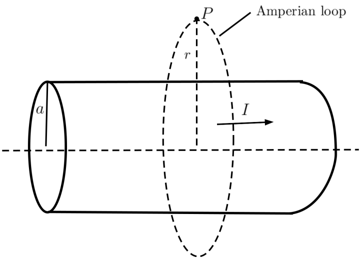


Figure 4

Now the Ampere's law reads,

$$\oint \vec{B} \cdot d\vec{l} = B(2\pi r) = \mu_0 I_{\text{enc}}$$

Problems (contd.)

$$\implies B = \frac{\mu_0 I_{\text{enc}}}{2\pi r} \tag{18}$$

a. For uniformly distributed surface current

$$I_{\text{enc}} = \begin{cases} I, & \text{if } P \text{ lies outside the cylinder} \\ 0, & \text{if } P \text{ lies inside the cylinder} \end{cases}$$

Therefore, the magnetic field outside and inside is

$$B = \begin{cases} \dfrac{\mu_0 I}{2\pi r}, & \text{if } P \text{ lies outside the cylinderi.e. } r > a \\ 0, & \text{if } P \text{ lies inside the cylinder i.e. } r < a \end{cases}$$

Problems (contd.)

- (b) For the case in which the current is distributed inside the cylinder such that J is proportional to distance from the axis i.e. $J = kr'$, with some constant k, the total current can be calculated by taking a coaxial circular ring of radius r' with elemental thickness dr'.

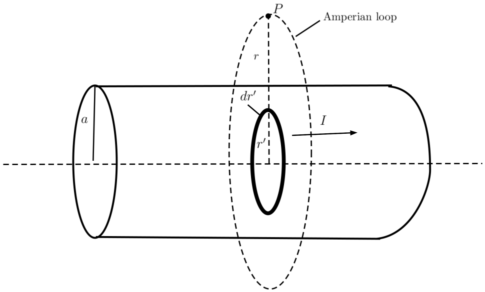


Figure 5

So that the elemental area $da = 2\pi r' dr'$ as shown in figure 5 and

$$I = \int \vec{J} \cdot d\vec{a} = \int J da = \int_0^a kr'(2\pi r' dr') = 2\pi k \int_0^a r'^2 dr' = \frac{2\pi k a^3}{3}$$

$$\implies k = \frac{3I}{2\pi a^3} \tag{19}$$

If P lies outside the cylinder, the Amperian loop enclosed total current I i.e. $I_{\text{enc}} = I$ and if P lies inside the cylinder

$$I_{\text{enc}} = \int_0^r kr' 2\pi r' dr' = 2\pi k \int_0^r r'^2 dr' = 2\pi \frac{3I}{2\pi a^3} \frac{r^3}{3} = \frac{Ir^3}{a^3}$$

So,

$$B = \begin{cases} \dfrac{\mu_0 I}{2\pi r}, & \text{if } P \text{ lies outside the cylinder i.e. } r > a \\ \dfrac{\mu_0 I r^2}{2\pi a^3}, & \text{if } P \text{ lies inside the cylinder i.e. } r < a \end{cases}$$

4. **Find the magnetic vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.**

Hint:

The magnetic vector potential $\vec{A}$ encircle the magnetic field in the same way as the magnetic field encircle the current. The magnetic field inside a long solenoid is $B = \mu_0 nI$ and parallet to the axis. The magnetic field outside is zero. In order to find the magnetic vector potential inside and outside of the solenoid, let's construct a coaxial Amperian circle of radius r passing through P, a point of our concern as shown in figure 6.

Problems (contd.)

The total flux enclosed by the Amperian loop is

$$\Phi_{\text{enc}} = \oint \vec{A} \cdot d\vec{l} = A(2\pi r) \implies A = \frac{\Phi_{\text{enc}}}{2\pi r}$$

If P is out side, the fluxed enclosed

$$\Phi_{\text{enc}} = \int_S \vec{B} \cdot d\vec{a} = (\mu_0 n I)\left(\pi R^2\right)$$

If P lies inside the solenoid

$$\Phi_{\text{enc}} = (\mu_0 n I)\left(\pi r^2\right)$$

so,

$$A = \begin{cases} \dfrac{\mu_0 n I R^2}{2r}, & \text{if } P \text{ lies outside the solenoid i.e. } r > R \\ \dfrac{\mu_0 n I r}{2}, & \text{if } P \text{ lies inside the solenoid i.e. } r < R \end{cases}$$

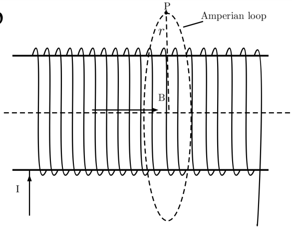


Figure 6

5. **Find the magnetic vector potential of a finite segment of straight wire, carrying a current I.**

Hint:
Consider a segment AB of a straight wire carrying current I on Y-axis with end A at position $(0, y_1, 0)$ and end B with position $(0, y_2, 0)$ as shown in figure 7. P is a point on Z-axis with coordinate $(0, 0, z)$. To find the magnetic vector potential at P, let's take an elemental length dy' on the wire with coordinate $(0, y', 0)$.

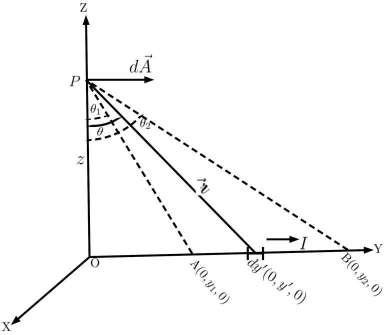


Figure 7

Problems (contd.)

So that $d\vec{l'} = dy'\hat{j}$. The separation vector of P from dy' is

$$\vec{\mathscr{r}} = -y'\hat{j} + z\hat{k}$$

and the magnitude is

$$\mathscr{r} = \sqrt{y'^2 + z^2}$$

The magnetic vector potential at P due to the current on dy' is

$$d\vec{A} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l'}}{\mathscr{r}} = \frac{\mu_0 I}{4\pi}\frac{dy'}{\sqrt{y'^2 + z^2}}\hat{j} \tag{20}$$

Problems (contd.)

From figure 7, we have, $y' = z\tan\theta$ and $dy' = z\sec^2\theta d\theta$. Now, equation (20) becomes

$$d\vec{A} = \frac{\mu_0 I}{4\pi}\frac{z\sec^2\theta d\theta}{z\sec\theta}\hat{j} = \frac{\mu_0 I}{4\pi}\sec\theta d\theta\hat{j}$$

The net vector potential due to the current through whole segment is

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\sec\theta d\theta\hat{j} = \frac{\mu_0 I}{4\pi}\ln\left(\frac{\sec\theta_2+\tan\theta_2}{\sec\theta_1+\tan\theta_1}\right)\hat{j}$$

Problems (contd.)

But $\sec\theta_1 = \dfrac{\sqrt{z^2+y_1^2}}{z}$, $\tan\theta_1 = \dfrac{y_1}{z}$, $\sec\theta_2 = \dfrac{\sqrt{z^2+y_2^2}}{z}$, $\tan\theta_2 = \dfrac{y_2}{z}$ The vector potential in term of Cartesian coordinate

$$\vec{A} = \frac{\mu_0 I}{4\pi} \ln\left(\frac{\sqrt{y_2^2+z^2}+y_2}{\sqrt{y_1^2+z^2}+y_1}\right)\hat{j} = A_y(z)\hat{j}$$

with

$$A_y(z) = \frac{\mu_0 I}{4\pi} \ln\left(\frac{\sqrt{y_2^2+z^2}+y_2}{\sqrt{y_1^2+z^2}+y_1}\right)$$

Now,

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_z}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + \frac{\partial A_y(z)}{\partial y} + 0 = 0$$

and

$$\begin{aligned}
\vec{B} &= \nabla \times \vec{A} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & A_y(z) & 0 \end{vmatrix} \\
&= -\frac{\partial A_y(z)}{\partial z}\hat{i}
\end{aligned}$$

Problems (contd.)

$$= -\frac{\partial}{\partial z}\left[\frac{\mu_0 I}{4\pi}\ln\left(\frac{\sqrt{y_2^2+z^2}+y_2}{\sqrt{y_1^2+z^2}+y_1}\right)\right]\hat{i}$$

$$= -\frac{\mu_0 I}{4\pi}\frac{\partial}{\partial z}\left[\ln\left(\sqrt{y_2^2+z^2}+y_2\right)-\ln\left(\sqrt{y_1^2+z^2}+y_1\right)\right]\hat{i}$$

$$= -\frac{\mu_0 I}{4\pi}\left[\frac{1}{\sqrt{y_2^2+z^2}+y_2}\left(\frac{1}{2}\frac{2z}{\sqrt{y_2^2+z^2}}\right)-\frac{1}{\sqrt{y_1^2+z^2}+y_1}\left(\frac{1}{2}\frac{2z}{\sqrt{y_1^2+z^2}}\right)\right]\hat{i}$$

$$= -\frac{\mu_0 I}{4\pi}\left[\frac{\cos\theta_2}{z\sec\theta_2+z\tan\theta_2}-\frac{\cos\theta_1}{z\sec\theta_1+z\tan\theta_1}\right]\hat{i}$$

$$= -\frac{\mu_0 I}{4\pi z}\left[\frac{\cos^2\theta_2}{1+\sin\theta_2}-\frac{\cos^2\theta_1}{1+\cos\theta_1}\right]\hat{i}$$

$$= -\frac{\mu_0 I}{4\pi z}[(1-\sin\theta_2)-(1-\sin\theta_1)]\hat{i}$$

$$\therefore \vec{B} = \frac{\mu_0 I}{4\pi z}(\sin\theta_2-\sin\theta_1)\hat{i}$$

6. **If $\vec{B}$ is *uniform*, show that $\vec{A}(\vec{r}) = -\frac{1}{2}\left(\vec{r}\times\vec{B}\right)$ works. That is, check that $\nabla\cdot\vec{A} = 0$ and $\nabla\times\vec{A} = \vec{B}$**

   Hint:-
   Consider a closed loop C with origin O as shown in figure 8 in a constant magnetic field $\vec{B}$. Let's take an elemental vector length $\vec{PQ} = d\vec{l}$ on the loop with position vector $\vec{OP} = \vec{r}$.

   The area of traingle OPQ is equal to
   $d\vec{a} = \frac{1}{2}(\vec{OP}\times\vec{PQ}) = \frac{1}{2}(\vec{r}\times d\vec{l})$.

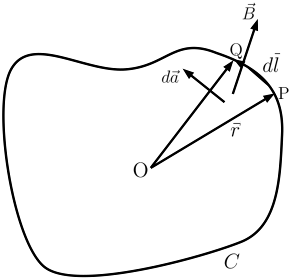


Figure 8

The magnetic flux crossing the triangle OPQ is

$$d\Phi = \vec{B}\cdot d\vec{a} = \vec{B}\cdot\frac{1}{2}\left(r\times d\vec{l}\right) = \frac{1}{2}\left(\vec{B}\times\vec{r}\right)\cdot d\vec{l}$$

The magnetic flux crossing the whole loop is

$$\Phi = \frac{1}{2}\oint_C \left(\vec{B}\times\vec{r}\right)\cdot d\vec{l}$$

In term of vector potential, the flux through the whole loop is

$$\Phi = \oint_C \vec{A}\cdot d\vec{l}$$

So that

$$\vec{A} = \frac{1}{2}\left(\vec{B}\times\vec{r}\right) = -\frac{1}{2}\left(\vec{r}\times\vec{B}\right) \tag{21}$$

Problems (contd.)

Now,

$$\nabla \cdot \vec{A} = -\frac{1}{2}\nabla \cdot \left(\vec{r} \times \vec{B}\right)$$
$$= -\frac{1}{2}\left[\vec{B} \cdot (\nabla \times \vec{r}) - \vec{r} \cdot \left(\nabla \times \vec{B}\right)\right]$$

Since $\nabla \times \vec{r} = 0$ and $\nabla \times \vec{B} = 0$ for constant $\vec{B}$

$$\therefore \nabla \cdot \vec{A} = 0$$

and

$$\vec{r} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ B_x & B_y & B_z \end{vmatrix} = (yB_z - zB_y)\hat{i} + (zB_x - xB_z)\hat{j} + (xB_y - yB_x)\hat{k}$$

Problems (contd.)

$$
\begin{aligned}
\therefore \nabla \times \vec{A} &= -\frac{1}{2} \nabla \times \left(\vec{r} \times \vec{B} \right) \\
&= -\frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yB_z - zB_y & zB_x - xB_z & xB_y - yB_x \end{vmatrix} \\
&= -\frac{1}{2} \left[(-B_x - B_x)\hat{i} + (-B_y - B_y)\hat{j} + (-B_z - B_z)\hat{k} \right] \\
&= \frac{1}{2} 2 \left(B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \right) \\
\therefore \nabla \times \vec{A} &= \vec{B}
\end{aligned}
$$

So $\vec{A} = -\frac{1}{2} \left(\vec{r} \times \vec{B} \right)$ meets all the criteria imposed during its definition.

End of Lecture 11

Thank you