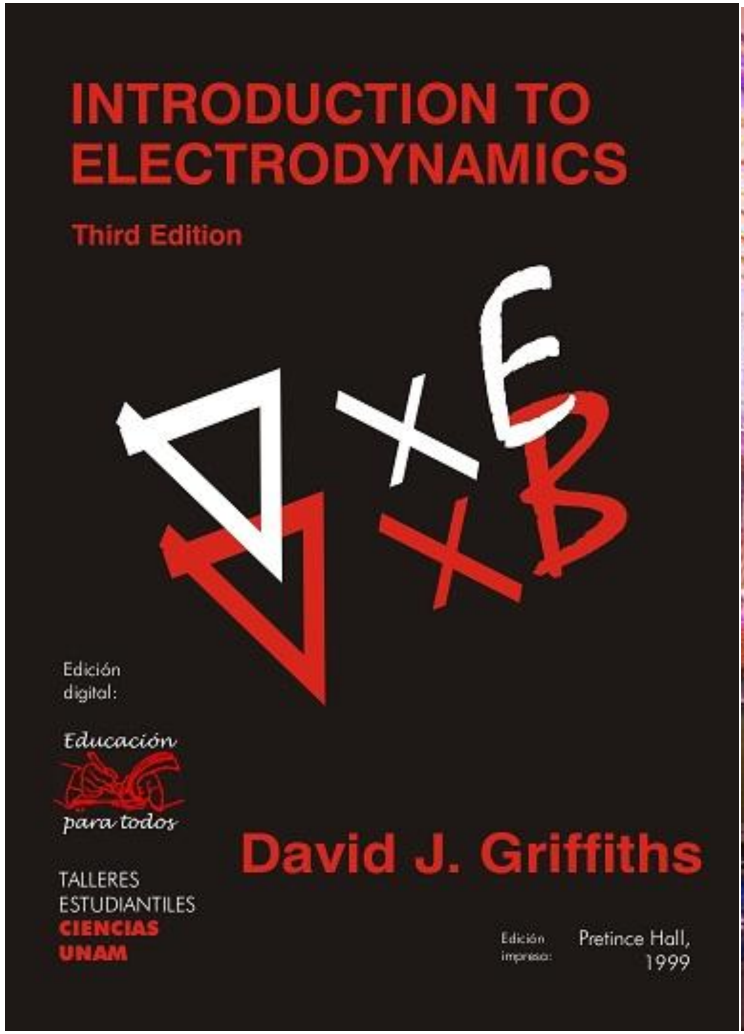
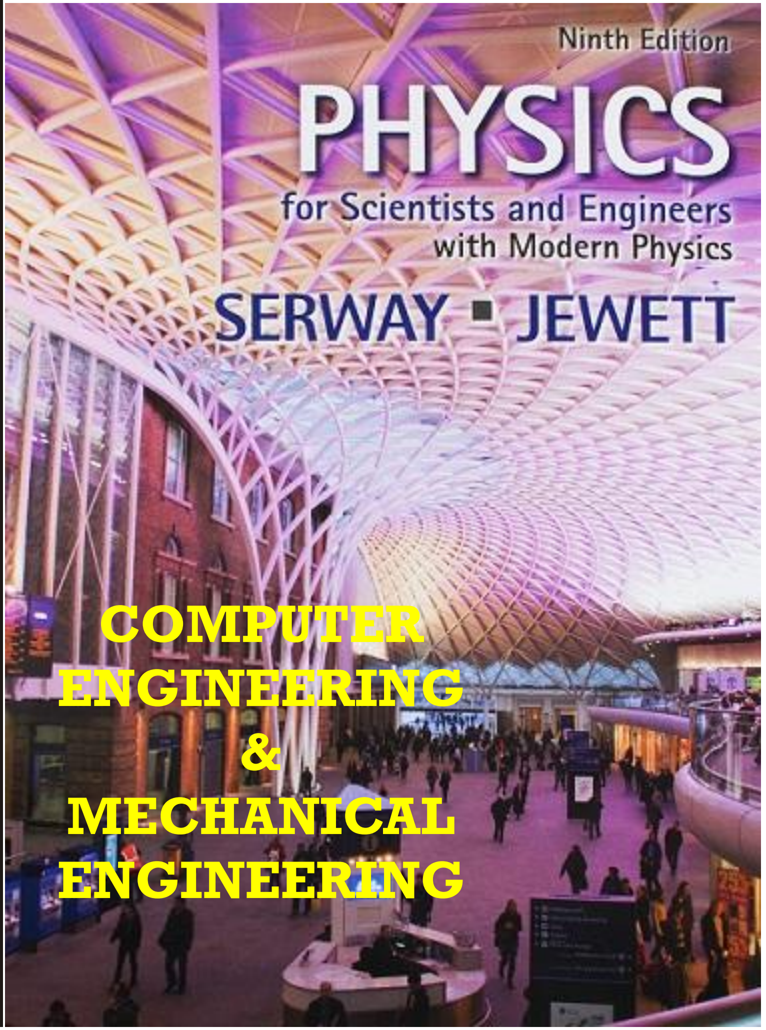
PHYSICS

General Physics II (PHYS 102)





COMPUTER ENGINEERING & MECHANICAL ENGINEERING

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Course Outline

- Polarization
- Bound Charges
- Gauss's Law in the Presence of Dielectrics
- Linear Dielectrics
- Clausius-Mossotti Relation

Polarization

Polarization:

What happens to a piece of dielectric material when it is placed in an electric field ?

If the substance consists of neutral atoms (or nonpolar molecules), the field will induce in each a tiny dipole moment, pointing in the same direction as the field.

If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction.

These two mechanisms produce the same basic result:

a lot of little dipoles pointing along the direction of the field

- the material becomes **polarized**.

A convenient measure of this effect is

$\vec{P} \equiv$ dipole moment per unit volume,

which is called the **polarization**.

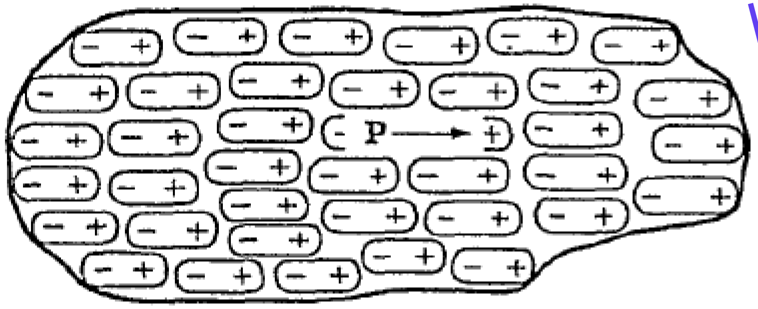


Figure Dp-1 A piece of polarized dielectric material

Polarization

- **The dipole moment per unit volume of the polarized material**

net electric dipole moment

Polarization $\Rightarrow$ $$\vec{P} = \frac{d\vec{p}}{d\tau} = \frac{1}{d\tau}\left(\sum_m \vec{p}_m\right)$$

an elemental volume of the material

SI unit of polarization	$\Rightarrow$ C m^{-2}

Bound Charges

Bound Charge

- The electric field of a polarized object (polarization $\vec{P}$ ≡ electric dipole moment per unit volume) is equivalent to the field produced by surface and volume "bound" charges

$$\sigma_b = \vec{P}\cdot\hat{n}, \quad \rho_b = -\nabla\cdot\vec{P}$$

where $\hat{n}$ is a unit vector perpendicular to the surface (pointing outward).

This is easy to understand: polarization results in perfectly genuine accumulations of charge, differing from "free" charge only in the sense that each electron is attached to a particular atom.

So, **Bound charge** is a useful construct for calculating the electrostatic field of polarized material, and it represents a perfectly genuine accumulation of charge.

Physical Interpretation of Bound Charge

- Suppose we have a long string of dipoles, as shown in Figure B_C-1.

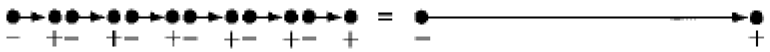


Figure B_C-1:

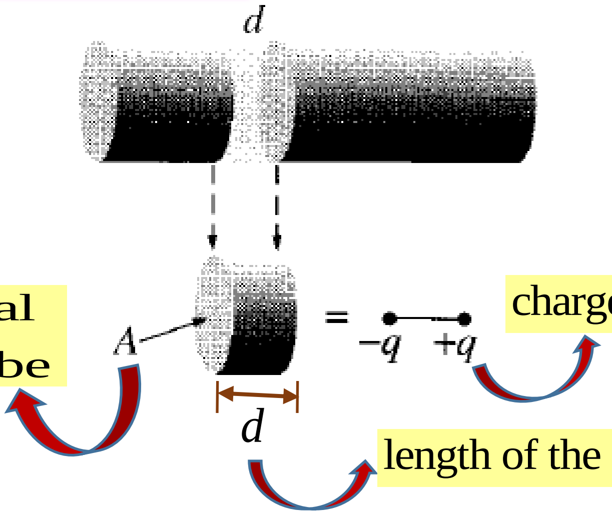
- Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends, there are two charges left over: plus at the right end and minus at the left.
- The net charge at the ends is called ***bound charge*** to remind ourselves that it cannot be removed; in a dielectric, every electron is attached to a specific atom or molecule.

Physical Interpretation of Bound Charges

Calculation of the actual amount of bound charge resulting from a polarization

For Uniform Polarization

- Consider a "tube" of dielectric parallel to uniform polarization $\vec{P}$ (Figure Bc-2)



cross-sectional area of the tube

charge at the end

length of the chunk

Figure Bc-1

- The dipole moment of tiny chunk shown in Figure Bc-2 is

- $p = P(Ad)$
- $p = qd$

$$q = PA$$

Therefore, the bound charge that piles up at the right end of the tube is $q = PA$.

For the ends sliced off perpendicularly,

Surface charge density:

$$\sigma_b = \frac{q}{A} = P$$

For an oblique cut (Figure Bc-2).

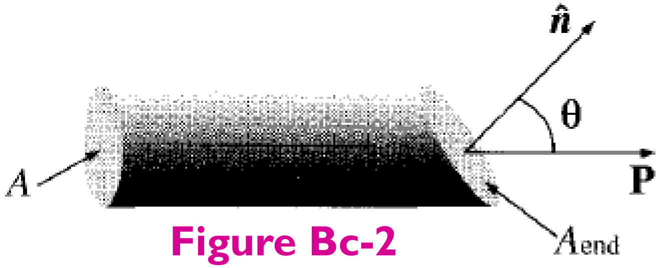
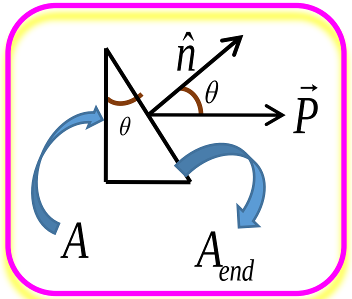


Figure Bc-2



Surface charge density:

$$\sigma_b = \frac{q}{A_{end}} = \frac{q}{A/\cos\theta} = P\cos\theta = \vec{P}\cdot\hat{n}$$

The effect of the polarization, then, is to paint a bound charge $\sigma_b = \vec{P}\cdot\hat{n}$ over the surface of the material.

Physical Interpretation of Bound Charges

Calculation of the actual amount of bound charge resulting from a polarization

ELECTRIC FIELDS IN MATTER

For Nonuniform Polarization

- If the polarization is nonuniform, we get accumulations of bound charge within the material as well as on the surface.

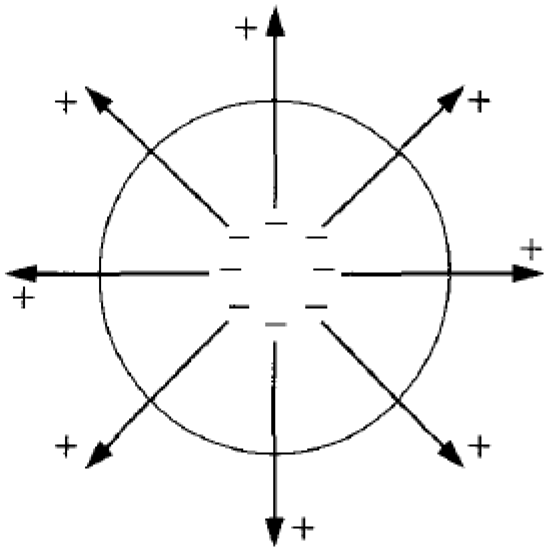


Figure Bc-1

- Figure B_C-4 suggests that a diverging $\vec{P}$ results in a pile up of negative charge.

The net bound charge in a given volume is equal and opposite to the amount that has been pushed out through the surface

i.e. $$\int_V \rho_b d\tau = -\oint_S \sigma_b da$$

$$= -\oint_S \sigma_b da$$

$$= -\oint_S \left(\vec{P}\cdot\hat{n}\right) da$$

$$\therefore \boxed{\int_V \rho_b d\tau = -\int_V \left(\nabla\cdot\vec{P}\right) d\tau}$$

Since this is true for any volume, we have

$$\boxed{\rho_b = -\nabla\cdot\vec{P}}$$

Gauss's Law in the Presence of Dielectrics

Gauss's Law in the Presence of Dielectrics

- We know that, the effect of polarization is to produce accumulations of bound charge, $\rho_b = -\nabla \cdot \vec{P}$ within the dielectric and $\sigma_b = \vec{P} \cdot \hat{n}$ on the surface.

  The field due to polarization of the medium is just the filed of this bound charge.

- Gauss's Law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$\left[\vec{E} \Rightarrow \text{total field}\right]$

$$\Rightarrow \quad \nabla \cdot \vec{E} = \frac{\rho_b + \rho_f}{\varepsilon_0}$$

[Within the dielectric,
total charge density, $\rho = \rho_b + \rho_f$]

bound volume charge density

free volume charge density

$$\Rightarrow \quad \varepsilon_0 \nabla \cdot \vec{E} = \rho_b + \rho_f$$

$$\Rightarrow \quad \nabla \cdot \varepsilon_0 \vec{E} = -\nabla \cdot \vec{P} + \rho_f$$

$$\Rightarrow \quad \nabla \cdot \left(\varepsilon_0 \vec{E} + \vec{P}\right) = \rho_f$$

$$\therefore \quad \boxed{\nabla \cdot \vec{D} = \rho_f}$$

where $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ is the electric displacement.

In integral form,

$$\oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$$

total free charge enclosed in the volume

$$\int_V \left(\nabla \cdot \vec{D}\right) d\tau = \int_V \rho_f \, d\tau$$

Gauss's divergence theorem

$$\Rightarrow \oint \vec{D} \cdot d\vec{a} = Q_{f_{enc}}$$

This is a particularly useful way to express Gauss's law, in the context of dielectrics, because it refers only to free charges.

Linear Dielectrics

Susceptibility, Permittivity, Dielectric Constant

- The polarization of a dielectric ordinarily results from an electric field, which lines up the atomic or molecular dipoles.

- For many substances, the polarization is *proportional* to the field, provided E is not too strong:

$$\boxed{\vec{P} = \varepsilon_0 \chi_e \vec{E}} \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathrm{L_D}-1)$$

The constant of proportionality, χ_e is called the **electric susceptibility** of the medium .

The materials that obey Eq. L_D-I are called **linear dielectrics.**

- In Linear media

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$$
$$= \varepsilon_0 (1+\chi_e) \vec{E} \quad \ldots\ldots\ldots\ldots\ldots \quad (\mathrm{L_D}-2)$$
$$= \varepsilon \vec{E}$$

permittivity of free space

$$\text{where } \varepsilon = \varepsilon_0 (1+\chi_e)$$

permittivity of the material

- **Dielectric constant or relative permittivity of the material:**

$$k \equiv (1+\chi_e) = \frac{\varepsilon}{\varepsilon_0} \equiv \varepsilon_r$$

- **In homogeneous linear dielectric ,**

the bound volume charge density is proportional to free volume charge density :

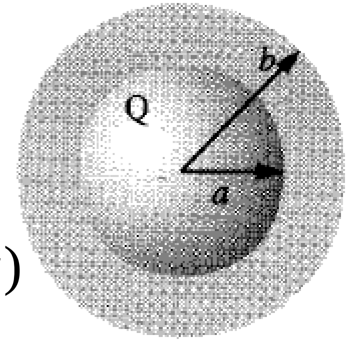
$$\rho_b = -\nabla \cdot \vec{P}$$
$$= -\nabla \cdot \varepsilon_0 \chi_e \vec{E}$$
$$= -\nabla \cdot \varepsilon_0 \chi_e \left[\frac{\vec{D}}{\varepsilon_0 (1+\chi_e)} \right]$$
$$= -\left(\frac{\chi_e}{1+\chi_e} \right) \rho_f$$
$$\therefore \boxed{\rho_b = -\left(\frac{\chi_e}{1+\chi_e} \right) \rho_f}$$

a measure of how easily a dielectric polarizes in response to an electric field

Problem

A metal sphere of radius *a*, carries a charge Q [Figure D-2]. It is surrounded, out to radius *b*, by linear dielectric material of permittivity ε_0 . Find the potential at the center (relative to infinity).

Solution:



- Gauss's law in the presence of dielectric:

$$\oint \vec{D}\cdot d\vec{a} = Q_{f_{enc}} \quad .... \quad (1)$$

- Drawing **a spherical Gaussian surface** of radius $r(r>a)$ and applying Eq. (1), we get

$$D\left(4\pi r^2\right) = Q$$

$$\therefore \boxed{\vec{D} = \frac{Q}{4\pi r^2}\hat{r}} \qquad \text{for all points } r>a$$

- **So** ,

$$\vec{E} = \frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2}\hat{r}, \qquad \text{for } r>b$$

$$= \frac{1}{\varepsilon}\frac{Q}{4\pi r^2}\hat{r}, \qquad \text{for } a<r<b \qquad \left[\begin{array}{l} \because \vec{D} = \varepsilon_0\left(1+\chi_e\right)\vec{E} = \varepsilon\vec{E} \\ \Rightarrow \vec{E} = \dfrac{1}{\varepsilon_0\left(1+\chi_e\right)}\vec{D} = \dfrac{1}{\varepsilon}\vec{D} \end{array}\right]$$

$$= 0, \qquad \text{for } r<a$$

$$\int \vec{E}\cdot d\vec{l} = \int \left(E\,\hat{r}\right)\cdot\left(\mathrm{dr}\,\hat{\mathrm{r}} + \mathrm{r}\,\mathrm{d}\theta\,\hat{\theta} + \mathrm{r}\sin\theta\,\mathrm{d}\phi\,\hat{\phi}\right)$$
$$= \int E\,dr$$

- The potential at the centre relative to the infinity is

$$V = -\int_{\infty}^{0}\vec{E}\cdot d\vec{l}$$

$$= -\int_{\infty}^{b}Edr - \int_{b}^{a}Edr - \int_{a}^{0}Edr$$

$$= -\int_{\infty}^{b}\left(\frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2}\right)dr - \int_{b}^{a}\left(\frac{1}{\varepsilon}\frac{Q}{4\pi r^2}\right)dr - \int_{a}^{0}(0)dr$$

$$\therefore \boxed{V = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right)}$$

Problem

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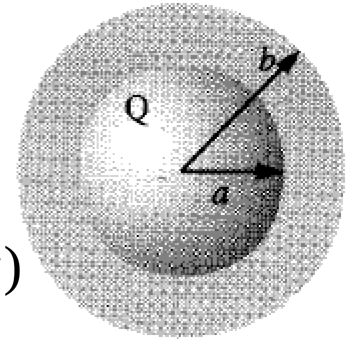
Solution:



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- **So** ,

$$\vec{E} = \frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2}\hat{r}, \qquad \text{for } r > b$$

$$= \frac{1}{\varepsilon}\frac{Q}{4\pi r^2}\hat{r}, \qquad \text{for } a < r < b$$

$$\left[\begin{array}{l} \because \vec{D} = \varepsilon_0\left(1+\chi_e\right)\vec{E} = \varepsilon\vec{E} \\ \Rightarrow \vec{E} = \dfrac{1}{\varepsilon_0\left(1+\chi_e\right)}\vec{D} = \dfrac{1}{\varepsilon}\vec{D} \end{array}\right]$$

$$= 0, \qquad \text{for } r < a$$

$$\int \vec{E}\cdot d\vec{l} = \int\left(E\,\hat{r}\right)\cdot\left(\mathrm{dr}\,\hat{r} + \mathrm{r}\,d\theta\,\hat{\theta} + \mathrm{r}\sin\theta\,d\phi\,\hat{\phi}\right)$$
$$= \int E\,dr$$

- The potential at the centre relative to the infinity is

$$V = -\int_{\infty}^{0}\vec{E}\cdot d\vec{l}$$

$$= -\int_{\infty}^{b}Edr - \int_{b}^{a}Edr - \int_{a}^{0}Edr$$

$$= -\int_{\infty}^{b}\left(\frac{1}{\varepsilon_0}\frac{Q}{4\pi r^2}\right)dr - \int_{b}^{a}\left(\frac{1}{\varepsilon}\frac{Q}{4\pi r^2}\right)dr - \int_{a}^{0}(0)dr$$

$$\therefore \boxed{V = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right)}$$

Clausius-Mossotti Equation

An expression for the electric field at the centre of a spherical cavity inside a polarized dielectric due to the charges on the wall of the cavity

- When a spherical cavity of radius *r* is made inside a uniformly polarized dielectric medium with polarization P directed from left to right as shown in **Figure C-1(a)**, then negative bound charges induce on the right hemisphere and positive bound charges on the left hemisphere.

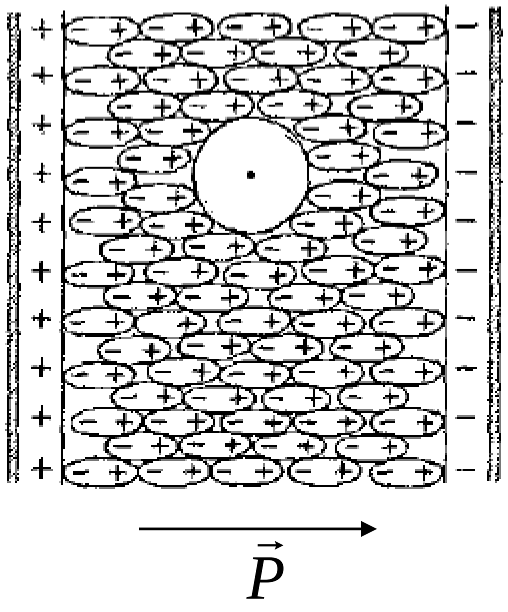


Figure C-1 (a)

Figure C-1(b) is the magnified view of the cavity sphere with induced bound charges.

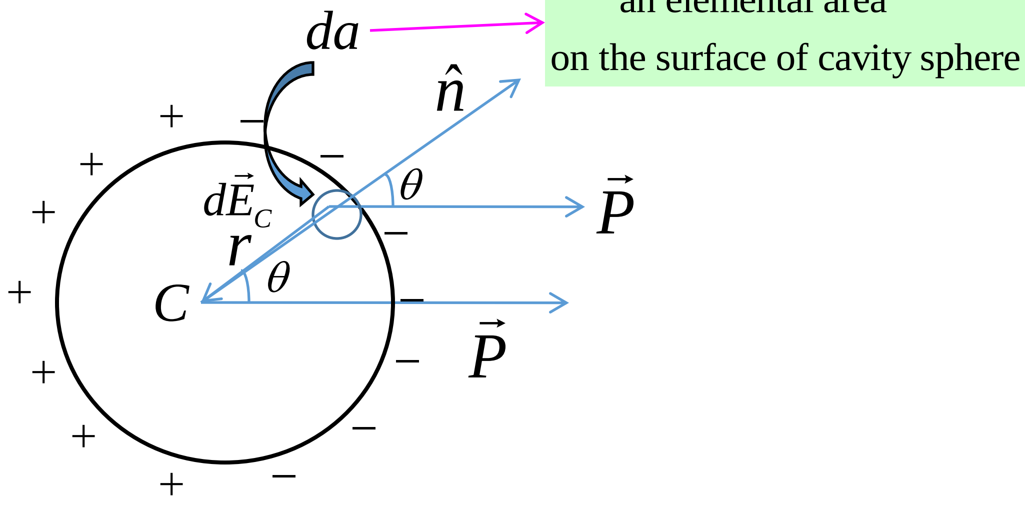


Figure C-1 (b)

The charge on an elemental area da is

$$dq = -\sigma_b da = -\left(\vec{P}\cdot\hat{n}\right)da$$

$$= -P\cos\theta\left(r^2 \sin\theta \, d\theta \, d\phi\right)$$

Clausius-Mossotti Equation

ELECTRIC FIELDS IN MATTER

An expression for the electric field at the centre of a spherical cavity inside a polarized dielectric due to the charges on the wall of the cavity

- The electric field at the centre of the cavity due to the charge dq

$$d\vec{E}_C = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^3}\vec{r} = \frac{1}{4\pi\varepsilon_0}\frac{\left(-P\cos\theta\, r^2\sin\theta\, d\theta\, d\phi\right)}{r^3}\vec{r}$$

$\vec{r}$: the vector from the surface to the centre of the sphere

$$\therefore \boxed{d\vec{E}_C = \frac{P}{4\pi\varepsilon_0}\left(\cos\theta\sin\theta\, d\theta\, d\phi\right)\hat{n}}$$

The component of $d\vec{E}_C$ along the direction of $\vec{P}$ is

$$dE_C\cos\theta = \frac{P}{4\pi\varepsilon_0}\cos^2\theta\sin\theta\, d\theta\, d\phi$$

- **Due to symmetry of the cavity, the components of $d\vec{E}_C$ along the direction perpendicular to $\vec{P}$ is zero.**

Therefore, the electric field at the centre C of the spherical cavity due to the entire surface charge on the cavity surface is

$$E_C = \int \frac{1}{4\pi\varepsilon_0}P\cos^2\theta\sin\theta\, d\theta\, d\phi$$

$$= \frac{P}{4\pi\varepsilon_0}\left\{\int_0^{\pi}\cos^2\theta\sin\theta\right\}\left\{\int_0^{2\pi}d\phi\right\}$$

$$= \frac{P}{4\pi\varepsilon_0}\left\{\int_0^{\pi}\cos^2\theta\sin\theta\right\}\{2\pi\} = \frac{P}{2\varepsilon_0}\left(\frac{2}{3}\right)$$

$$\therefore \boxed{\vec{E}_C = \frac{\vec{P}}{3\varepsilon_0}}$$

$$\left\{\begin{array}{l} \text{put } \cos\theta = x \\ -\sin\theta d\theta = dx \\ \text{when } \theta=0, \text{ then } x=1 \\ \text{when } \theta=\pi, \text{ then } x=-1 \end{array}\right\} \Rightarrow \int_0^{\pi}\cos^2\theta\sin\theta = \int_1^{-1}-x^2dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$$

Clausius-Mossotti Equation

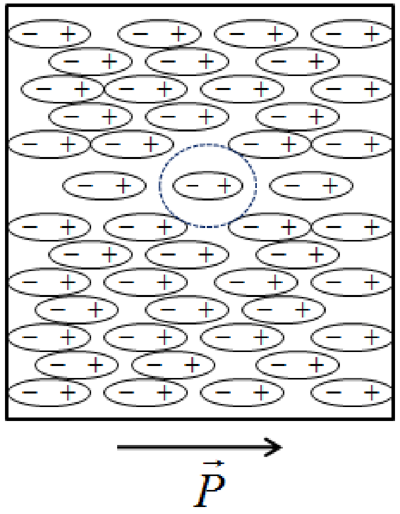
Clausius – Mossotti Equation

- Clausius and Mossotti established a relation between the dielectric constant and the molecular polarizability of a dielectric. This relation is known as **Clausius-Mossotti Equation.**

- **Clausius and Mossotti assumed that each molecule of a uniformly polarized dielectric medium lies at the centre of the cavity sphere.**

  Therefore, **the net electric field experienced by the molecule** (also called molecular field) is the sum of electric field due to the bound charge on the cavity surface and resultant of all other fields except due to the bound charges on the cavity surface.

  $$i.e.\ \vec{E}_m = \vec{E}_C + \vec{E} \quad .... \quad (1)$$



- The dipole moment of a molecule per unit molecular field is called its polarizability, α_m . In other words,

  $$\vec{P}_m = \alpha_m \vec{E}_m \quad .... \quad (2)$$

- If there are N molecules per unit volume, then the **polarization**

$$\vec{P} = Np_m = N\alpha_m \vec{E}_m = N\alpha_m\left[\vec{E}_C + \vec{E}\right]$$

$$\text{or, } \vec{P} = N\alpha_m\left[\frac{\vec{P}}{3\varepsilon_0} + \frac{\vec{P}}{\chi_e\varepsilon_0}\right] \qquad \left[\because \vec{P} = \varepsilon_0\chi_e\vec{E}\right]$$

$$\text{or, } 1 = N\alpha_m\left[\frac{1}{3\varepsilon_0} + \frac{1}{\chi_e\varepsilon_0}\right]$$

$$\text{or, } 1 = N\alpha_m\left[\frac{\chi_e + 3}{3\varepsilon_0\chi_e}\right]$$

$$\text{or, } \alpha_m = \frac{3\varepsilon_0}{N}\left[\frac{\chi_e}{\chi_e + 3}\right]$$

$$\text{or, } \alpha_m = \frac{3\varepsilon_0}{N}\left[\frac{K-1}{(K-1)+3}\right] \qquad \left[\because 1 + \chi_e = K\right]$$

$$\therefore \boxed{\alpha_m = \frac{3\varepsilon_0}{N}\left[\frac{K-1}{K+2}\right]}$$

$$\therefore \boxed{\alpha_m = \frac{3\varepsilon_0}{N}\left[\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right]}$$

Text Books & References

1. **David J. Griffith**, Introduction to Electrodynamics
2. **R.A. Serway and J.W. Jewett**, Physics for Scientist and Engineers with Modern Physics
3. **Halliday and Resnick**, Fundamental of Physics
4. **John R. Reitz, Frederick J. Milford, Robert W. Christy,** Foundations of Electromagnetic Theory


Thank you