KATHMANDU UNIVERSITY

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Subject: MATHS 101

Assignment No: 2

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<Q.1> Find differential coefficients w.r.t x.

(a): $y = \sec^3(\tan\sqrt{x^3})$

Sol^n:

Given,

$$y = \sec^3(\tan\sqrt{x^3})$$

Differentiating both sides w.r.t x,

$$\frac{dy}{dx} = \frac{d(\sec^3(\tan\sqrt{x^3})}{dx}$$

$$= \frac{d(\sec^3(\tan\sqrt{x^3})}{d(\sec(\tan\sqrt{x^3})} \times \frac{d(\sec(\tan\sqrt{x^3})}{d(\tan\sqrt{x^3})} \times \frac{d(\tan\sqrt{x^3})}{d\sqrt{x^3}} \times \frac{d\sqrt{x^3}}{dx^3} \times \frac{dx^3}{dx}$$

$$= 3\sec^2(\tan\sqrt{x^3}) \times \sec(\tan\sqrt{x^3})\tan(\tan\sqrt{x^3}) \times \sec^2\sqrt{x^3} \times \frac{1}{2\sqrt{x^3}} \times 3x^2$$

$$= \frac{9}{2}\left\{\frac{\sec^2(\tan\sqrt{x^3})\sec(\tan\sqrt{x^3})\cdot\tan(\tan\sqrt{x^3})\cdot\sec^2(\sqrt{x^3})x^2}{\sqrt{x^3}}\right\}$$

(b) $y = e^{\sin(\sqrt{\ln x})}$

Sol^n:

Given,

$$y = e^{\sin(\sqrt{\ln x})}$$

Differentiating both sides w.r.t x,

$$\frac{dy}{dx} = \frac{de^{\sin(\sqrt{\ln x})}}{dx}$$

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or, $\frac{dy}{dx} = \frac{d e^{\sin(\sqrt{\ln x})}}{d(\sin\sqrt{\ln x})} \times \frac{d\sin(\sqrt{\ln x})}{d\sqrt{\ln x}} \times \frac{d\sqrt{\ln x}}{d\ln x} \times \frac{d\ln x}{dx}$

$= e^{\sin(\sqrt{\ln x})} \times \cos(\sqrt{\ln x}) \times \frac{1}{2\sqrt{\ln x}} \times \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{e^{\sin(\sqrt{\ln x})}\cos(\sqrt{\ln x})}{2x\sqrt{\ln x}}$

(c): $y = x^{\sqrt{x}}$

soln:

Given,

$y = x^{\sqrt{x}}$

Taking ~~log~~ ln on both sides.

$\ln y = \sqrt{x} \cdot \ln x$

Differentiating both sides w.r.t x,

$\frac{d\ln y}{dx} = \frac{d}{dx}\left(\sqrt{x} \times \ln x\right)$

or $\frac{d\ln y}{dy} \times \frac{dy}{dx} = \sqrt{x} \cdot \frac{d\ln x}{dx} + \ln x \times \frac{d\sqrt{x}}{dx}$

or $\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = x^{\sqrt{x}}\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)$

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(d): $y = \frac{\ln x}{1 + \ln x}.$

Sol^n:

Given,

$$y = \frac{\ln x}{1 + \ln x}$$

Differentiating both sides w.r.t x,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{1 + \ln x}\right)$$

$$= \frac{(1 + \ln x) \cdot \frac{d \ln x}{dx} - \ln x \cdot \frac{d(1 + \ln x)}{dx}}{(1 + \ln x)^2}$$

$$= \frac{\frac{(1 + \ln x)}{x} - \ln x \cdot \frac{1}{x}}{(1 + \ln x)^2}$$

$$= \frac{1 + \cancel{\ln x} - \cancel{\ln x}}{x(1 + \ln x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x(1 + \ln x)^2}$$

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(c): $ax^2 + 2hxy + by^2 = 1$

Soln:

Given,

$ax^2 + 2hxy + by^2 = 1$

Differentiating both sides w.r.t x,

$$\frac{d}{dx}\left(ax^2 + 2hxy + by^2\right) = \frac{d.1}{dx}$$

or $$a \cdot \frac{dx^2}{dx} + 2h \cdot \frac{dxy}{dx} + b \cdot \frac{dy^2}{dx} = 0$$

or, $$2ax + 2h \cdot \left(x \cdot \frac{dy}{dx} + y \cdot \frac{dx}{dx}\right) + b \cdot \left(\frac{dy^2}{dy} \times \frac{dy}{dx}\right) = 0$$

or $$2ax + 2hx\frac{dy}{dx} + 2hy + b\left(2y \times \frac{dy}{dx}\right) = 0$$

or, $$2ax + 2hy + 2hx\frac{dy}{dx} + 2yb \times \frac{dy}{dx} = 0$$

or $$\frac{dy}{dx}\left(2hx + 2yb\right) = -\left(2ax + 2hy\right)$$

or $$\frac{dy}{dx} = -\frac{\cancel{2}(ax+hy)}{\cancel{2}(hx+yb)}$$

$$\therefore \frac{dy}{dx} = -\frac{(ax+hy)}{(hx+yb)}$$

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(f): $y^2 - x^2 - \sin xy = 0$

Soln:

Given,

$$y^2 - x^2 - \sin xy = 0$$

or, $y^2 - \sin xy = x^2$

Differentiating both sides w.r.t x,

$$\frac{dy^2}{dx} - \frac{d\sin(xy)}{dx} = \frac{dx^2}{dx}$$

or $\frac{dy^2}{dy} \times \frac{dy}{dx} - \frac{d\sin(xy)}{dxy} \times \frac{dxy}{dx} = 2x$

or $2y \cdot \frac{dy}{dx} - \cos xy \times \left(x \cdot \frac{dy}{dx} + y \cdot \frac{dx}{dx} \right) = 2x$

or, $2y \cdot \frac{dy}{dx} - x\cos xy \cdot \frac{dy}{dx} - y\cos xy = 2x$

or $2y \cdot \frac{dy}{dx} - x\cos xy \cdot \frac{dy}{dx} = 2x + y\cos xy$

or $\frac{dy}{dx}(2y - x\cos xy) = 2x + y\cos xy$

$\therefore \frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$.

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<Q.2>: Find dy/dx.

(a): $x = \cos(\ln t)$, $y = \ln(\cos t)$.

Sol^{n}:

We know,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{— (i).}$$

Now,

$x = \cos(\ln t)$

Differentiating both sides w.r.t. t,

$$\frac{dx}{dt} = \frac{d\cos(\ln t)}{d(\ln t)} \times \frac{d(\ln t)}{dt}$$

$$= -\sin(\ln t) \times \frac{1}{t}$$

$$\therefore \frac{dx}{dt} = \frac{-\sin(\ln t)}{t}$$

Also, $y = \ln(\cos t)$

Differentiating both sides w.r.t t,

$$\frac{dy}{dt} = \frac{d\ln(\cos t)}{d(\cos t)} \times \frac{d\cos t}{dt}$$

$$= \frac{1}{\cos t} \times -\sin t$$

$$\therefore \frac{dy}{dt} = \frac{-\sin t}{\cos t}$$

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Putting in eq^n (i);

$$\frac{dy}{dx} = \frac{\cancel{-\sin(\ln t)}/\cancel{t}}{\cancel{-\sin t}/\cancel{\cos t}} \qquad \frac{-\sin t/\cos t}{-\sin(\ln t)/t} \qquad \frac{\sin t}{\cos t} \times \frac{t}{\sin(\ln t)}$$

$$= \cancel{\frac{\sin(\ln t)}{t}} \times \cancel{\frac{\cos t}{\sin t}} \qquad \therefore \frac{dy}{dx} = \frac{\cancel{\sin(\ln t)\cos t}}{\cancel{t \sin t}} \; \frac{t\tan t}{\sin(\ln t)}$$

(b): $x = 2a\tan\theta$, $y = a\sec^2\theta$

Sol^n:

We know,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Now,

$$x = 2a\tan\theta$$

Differentiating both sides w.r.t θ,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(2a\tan\theta)$$

$$= 2a\frac{d\tan\theta}{d\theta} \qquad \therefore \frac{dx}{d\theta} = 2a\sec^2\theta$$

Also,

$$y = a\sec^2\theta$$

Differentiating both sides w.r.t θ,

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sec^2\theta)$$

$$= a\frac{d\sec^2\theta}{d\sec\theta} \times \frac{d\sec\theta}{d\theta} \qquad \therefore \frac{dy}{d\theta} = 2a\sec\theta\sec\theta\tan\theta$$

$$= 2a\sec^2\theta\tan\theta$$

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Putting in eq^n(i), we get.

$$\frac{dy}{dx} = \frac{\cancel{2}\cancel{a}\sec^2\theta \tan\theta}{\cancel{2}\cancel{a}\sec^2\theta \cancel{\tan\theta}} \qquad \therefore \frac{dy}{dx} = \cancel{\cot\theta}\ \tan\theta$$

(c): $x = e^{\ln\cos 4\theta}$, $y = e^{\ln\sin 4\theta}$

Sol^n:

We know,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Now,

$$x = e^{\ln\cos 4\theta}$$

Differentiating both sides w.r.t θ,

$$\frac{dx}{d\theta} = \frac{de^{\ln(\cos 4\theta)}}{d\theta}$$

$$= \frac{de^{(\ln(\cos 4\theta))}}{d(\ln(\cos 4\theta))} \times \frac{d\ln(\cos 4\theta)}{d\cos 4\theta} \times \frac{d\cos 4\theta}{d\,4\theta} \times \frac{d\cdot 4\theta}{d\theta}$$

$$= e^{\ln(\cos 4\theta)} \times \frac{1}{\cos 4\theta} \times -\sin 4\theta \cdot 4$$

$$\therefore dx/d\theta = -4e^{\ln(\cos 4\theta)} \cdot \tan 4\theta$$

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Again,

$$y = e^{\ln(\sin 4\theta)}$$

Differentiating both sides w.r.t θ,

$$\frac{dy}{d\theta} = \frac{de^{\ln(\sin 4\theta)}}{d\theta}$$

$$= \frac{de^{\ln(\sin 4\theta)}}{d(\ln(\sin 4\theta))} \times \frac{d\ln(\sin 4\theta)}{d\sin 4\theta} \times \frac{d\sin 4\theta}{d4\theta} \times \frac{d4\theta}{d\theta}$$

$$= e^{\ln(\sin 4\theta)} \times \frac{1}{\sin 4\theta} \times \cos 4\theta \times 4$$

$$\therefore \frac{dy}{d\theta} = 4e^{\ln(\sin 4\theta)} \times \cot 4\theta$$

Putting in eq^n(i),

$$\frac{dy}{dx} = \frac{\cancel{4}e^{\ln(\sin 4\theta)} \times \cot 4\theta}{-\cancel{4}e^{\ln(\cos 4\theta)} \times \tan 4\theta}$$

$$= \frac{-\sin 4\theta}{\cos 4\theta} \times \frac{\cot 4\theta}{\tan 4\theta} = -\cancel{\tan 4\theta} \times \frac{\cot 4\theta}{\cancel{\tan 4\theta}}$$

[$\because e^{\ln(\text{any quantity})}$ = that quantity]

$$\therefore \frac{dy}{dx} = \cot 4\theta$$

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3: If $f(x) = \begin{cases} x & \text{for } 0<x<1 \\ 2-x & \text{for } 1\le x\le 2 \\ x-\frac{x^2}{2} & \text{for } x>2 \end{cases}$. Does $f'(x)$ exists at $x=1$ and $x=2$.

Sol^n:

Checking for $f'(x)$ existence at $x=1$.

$$LHD = \lim_{h\to 0^-} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0^-} \frac{\cancel{1}-h-\cancel{1}}{-h}$$

$LHD = 1$

$$RHD = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0^+} \frac{2-(x+h)-(2-x)}{h} = \lim_{h\to 0^+} \frac{\cancel{2}-\cancel{x}-h-\cancel{2}+\cancel{x}}{h}$$

$= -1$

Since, $LHD \neq RHD$
so, $f'(x)$ doesn't exist at $x=1$.

Checking for existence at $x=2$.

So $a=2$.

$$LHD = \lim_{h \to 0^-} \frac{f(a-h)-f(a)}{-h}$$

$$= \lim_{h \to 0^-} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h \to 0^-} \frac{2-(2-h)-2+2}{-h}$$

$$= \lim_{h \to 0^-} \frac{\cancel{2}-\cancel{2}+h-\cancel{2}+\cancel{2}}{-h} = -1$$

$\therefore LHD = -1$

And.

$$RHD = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h}$$

$$\cancel{= \lim_{h \to 0^+} \frac{(2+h) - \frac{(2+h)^2}{2} - \left[(2+h) - \frac{(2+h)^2}{2}\right]}{h}}$$

$$\cancel{= \lim_{h \to 0^+} \frac{4+2h-4-4h-h^2-\left[4+2h-4-4h-h^2\right]}{2h}}$$

$$\lim_{h\to 0^+} \frac{(2+h) - \frac{(2+h)^2}{2} - \left(2 - \frac{2^2}{2}\right)}{h}$$

$$= \lim_{h\to 0^+} \frac{\cancel{4} + \cancel{4h} - \cancel{4} - 4h - h^2}{2h}$$

$$= \lim_{h\to 0^+} \frac{-2h - h^2}{2h} = \lim_{h\to 0^+} \frac{-\cancel{h}}{\cancel{h}} \frac{(2-h)}{2}$$

$\therefore$ RHD $= -1$

Since, LHD $=$ RHD $= -1$.

So, $f'(x)$ exists at $x = 2$.

<Q.4>: For function $y = f(x)$ is defined as.

$$f(x) = \begin{cases} 2x+1 & \text{for } x < 1 \\ 3 & \text{for } x = 1 \\ x^2+2 & \text{for } x > 1 \end{cases}$$ Find $f'(1)$.

Soln:

Check for existence of $f'(x)$ at $x = 1$.

$a = 1$

$$\text{LHD} = \lim_{h\to 0^-} \frac{f(a-h) - f(a)}{-h}$$

$$= \lim_{h\to 0^-} \frac{2(1-h)+1 - 2\times 1+1}{-h}$$

$$= \lim_{h\to 0^-} \frac{2 - 2h + 1 - 3}{-h} = \lim_{h\to 0^-} \frac{+2\cancel{h}}{+\cancel{h}}$$

$\therefore$ LHD $= 2$.

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$$RHD = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{(1+h)^2 + 2 - (1^2 + 2)}{h}$$

$$= \lim_{h \to 0^+} \frac{1 + 2h + h^2 + 2 - 3}{h} = \lim_{h \to 0^+} \frac{\cancel{h}(2+h)}{\cancel{h}}$$

$RHD = 2$

Here, LHD $\{Lf(1)\}$ = RHD $\{R'f(1)\} = 2$. so,
$f'(1) = 2$.

<Q. 57: For what ~~th~~ values of a, m and b does the function

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \\ -x^2 + 3x + a & \text{if } 0 \leq x < 1 \\ mx + b & \text{if } 1 \leq x \leq 2 \end{cases}$$

satisfy MVT on the interval $[0, 2]$.

Sol^n:

For $f(x)$ to satisfy MVT, $f(x)$ must be continuous and differentiable in $[0, 2]$.

We know,

$f(x)$ satisfies MVT.

which is $f(x)$ is continuous on $[0,2]$ and differentiable on $(0,2)$.

~~let $c \in (a,b)$ such that~~

At $x = 0$.

$f(0) = 3$

$$RHL = \lim_{x \to 0^-} -x^2 + 3x + a$$

$$= \lim_{h \to 0^-} -(0-h)^2 + 3(0-h) + a$$

$$= a$$

Here,

$f(0) = RHL$ at $f(0)$ [$\because f(x)$ is continuous]

$\therefore a = 3$.

So, $f(x) = -x^2 + 3x + 3$ $[0,1)$.

At ~~x~~ $x = 1$,

$$LHL = \lim_{x \to 1^-} -x^2 + 3x + 3$$

$$= \lim_{h \to 0^-} -(1-h)^2 + 3(1-h) + 3$$

$= -1 + 3 + 3$

$LHL = 5$

$$RHL = \lim_{x \to 1^+} mx + b$$

$$= \lim_{h \to 0} m(1+h) + b$$

$$= m + b$$

Since, $f(x)$ is continuous,

$5 = m + b$ — (i).

Again,

At $x = 1$, $f(x)$ is also differentiable.

$\therefore a = 1$.

$$LHD = \lim_{h \to 0^-} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0^-} \frac{[-(1-h)^2 + 3(1-h) + 3] - [-1 + 3 + 3]}{-h}$$

$$= \lim_{h \to 0^-} \frac{3 + 3 - 3h - 1 + 2h - h^2 - 5}{-h}$$

$$= \lim_{h \to 0^-} \frac{-h - h^2}{h} = \lim_{h \to 0} \frac{-\cancel{h}(1-h)}{-\cancel{h}}$$

$\therefore LHD = 1$

$$RHD = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{m(1+h) + b - [m+b]}{h}$$

$$= \lim_{h \to 0^+} \frac{\cancel{m} + mh + \cancel{b} - \cancel{m} - \cancel{b}}{h}$$

$$= \lim_{h \to 0^+} \frac{m\cancel{h}}{\cancel{h}}$$

$$= m.$$

Since $f(x)$ is differentiable at $x = 1$.

$LHD = RHD = f'(x)$

$\cancel{1 = m = 1}$

$$f'(x) = \frac{df(x)}{dx} = \frac{d(mx+b)}{dx} = m.$$

$\therefore m = 1.$

We know,

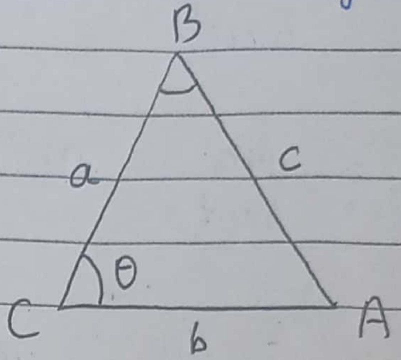
$m + b = 5$

$\therefore b = 4$

The values of m, a, and b is $1, 3, 4$.

<Q.6>: Two sides of a triangle having lengths 'a' and 'b' and the ~~gate~~ angle between them is θ. What value of θ will maximizes the triangle area?

Soln:



Given,
length of triangle sides 'a' and 'b'.

θ = angle between a & b.

We know,

$$\text{Area } (A) = \frac{1}{2} ab \sin\theta$$

Or, $A = \frac{1}{2} ab \sin\theta$ — (i)

Differentiating (i) w.r.t θ,

$$\frac{dA}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2} ab \sin\theta\right) = \frac{1}{2} ab \cdot \frac{d \sin\theta}{d\theta}$$

or, $$\frac{dA}{d\theta} = \frac{1}{2} ab \cos\theta$$

Putting $dA/d\theta = 0$

or $0 = \cos\theta$

$$\therefore \theta = \frac{\pi}{2}$$

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Now,

$$\frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left(\frac{1}{2}ab\cos\theta\right)$$

$$= -\frac{1}{2}ab\sin\theta$$

Putting $\theta = \pi/2$,

$$= -\frac{1}{2}ab \times 1 = -\frac{1}{2}ab < 0 \quad \text{(maximum)}.$$

Area is maximum when $\theta = \frac{\pi}{2}$.

$\therefore$ Maximum area $= \frac{1}{2}ab\sin\frac{\pi}{2} = \frac{1}{2}ab.$

<Q.7>: Find the local and global extreme values from the following.

(a): $f(x) = 4-x^2$ at $-3 \leq x \leq 1$

Soln:

Given,

$$f(x) = 4-x^2$$

$$\therefore f'(x) = -2x$$

Putting $f'(x) = 0$.

$$\therefore x = 0.$$

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$f(-3) = 4-(-3)^2$
$\therefore f(-3) = -5$

$f(0) = 4-(0)^2$
$\therefore f(0) = 4$

$f(1) = 4-(1)^2$
$\therefore f(1) = 3$

So, global maximum $= 4$ at $x=0$
global minimum $= -5$ at $x=-3$.

Now,
$$f''(x) = \frac{df'(x)}{dx} = \frac{d(\overset{-2x}{\cancel{4-x^2}})}{dx} = -2 < 0 \text{ (maximum)}$$
$\therefore f(x)$ has only local maximum value
Local maxima $= 4$ at $x=0$.

(b): $f(x) = -\sqrt{5-x^2}$ at $-\sqrt{5} \le x \le 0$.

Soln:

Given,
$f(x) = -\sqrt{5-x^2}$

$$\therefore f'(x) = \frac{df(x)}{dx} = -\frac{d(5-x^2)^{1/2}}{d(5-x^2)} \times \frac{d(5-x^2)}{dx}$$

$$\therefore f'(x) = \frac{x}{\sqrt{5-x^2}}$$

Putting $f'(x)=0$, $x=0$.

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So,

$$f(-\sqrt{5}) = -\sqrt{5-(-\sqrt{5})^2}$$

$$\therefore f(-\sqrt{5}) = 0$$

So, global maximum = 0 at $x = -\sqrt{5}$
global minimum = $-\sqrt{5}$ at $x = 0$

$$f(0) = -\sqrt{5-(0)^2}$$

$$\therefore f(0) = -\sqrt{5}$$

Now,

$$f''(x) = \frac{df'(x)}{dx}$$

$$= \frac{\sqrt{5-x^2}\cdot\frac{dx}{dx} - x\cdot\frac{d\sqrt{5-x^2}}{dx}}{5-x^2}$$

$$= \frac{\sqrt{5-x^2} - \frac{x^2}{\sqrt{5-x^2}}}{5-x^2}$$

$$\therefore f''(x) = \frac{5-\cancel{x^2}+\cancel{x^2}}{(\sqrt{5-x^2})(5-x^2)}$$

$$\therefore f''(0) = \frac{\cancel{5}}{\sqrt{5}\times\cancel{5}} = \frac{1}{\sqrt{5}} > 0 \text{ (minimum)}$$

f(x) has local minimum at $x = 0$.

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<c>: $f(x) = \sec x$ at $-\pi/3 \le x \le \pi/6$

Soln:

Given, $f(x) = \sec x$

$\therefore f'(x) = \cancel{\tan^2 x}. \sec x \tan x.$

Putting $f'(x) = 0$. $\sec x = 0 \Rightarrow$ undefined.

$\therefore x = 0$

$\cancel{\therefore f''(x) = \frac{d \tan^2 x}{d \tan x}} \times \frac{d \tan x}{dx}$

$\therefore f''(x) = \frac{d(\sec x \tan x)}{dx}$

$= \sec x \cdot \frac{d \tan x}{dx} + \tan x \cdot \frac{d \sec x}{dx}$

$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$

$\therefore f''(x) = \sec^3 x + \tan^2 x \sec x$

The point of inflection is not defined at this case.

$f''(0) = \sec^3 0 + \tan^2 0 \cdot \sec 0$

$= 1 > 0$ (minimum).

$f(x)$ has local minimum at $x = 0$ ie $= 1$.

Putting $x = -\pi/3$,

$f(-\pi/3) = \sec(-\pi/3)$ $\therefore f(-\pi/3) = 2$

$f(\pi/6) = \sec(\pi/6)$ $\therefore f(\pi/6) = 1.15$

$f(0) = \sec(0)$ $\therefore f(0) = 1$

So, global maximum $= 2$ at $-\pi/3$ and global minimum $= 1$ at $x = 0$.

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<Q.8>: Verify Rolle's Theorem for the following:

(a): $f(x) = x^2 - 6x + 8$ in $[2,4]$.

Soln:

Given,

$$f(x) = x^2 - 6x + 8$$

(i): $f(x)$ is continuous in $[2,4]$ as it is a polynomial function.

(ii): $f'(x) = 2x - 6$

Hence, $f(x)$ is differentiable for $x \in (2,4)$.

(iii): $f(2) = 2^2 - 6\times2 + 8 = 0$

$f(4) = 4^2 - 6\times4 + 8 = 0$

$\therefore f(2) = f(4)$.

Here, all the conditions for Rolle's theorem is satisfied.

Let 'c' be the point in (a,b) such that

$$f'(c) = 0$$

$$2c - 6 = 0$$

$$\therefore c = 3.$$

Hence, the Rolle's theorem is verified.

(b): $f(x) = x^3 - 4x$ in $[-2,2]$

Given,

$f(x) = x^3 - 4x$

(i): $f(x)$ is continuous in $[-2,2]$ as it is a polynomial function.

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(ii): $f'(x) = 3x^2$

Here, $f'(x)$ is differentiable in $(2,4)$.

(iii): $f(-2) = (-2)^3 - 4\times(-2) = 0$

$f(2) = (2)^3 - 4\times(2) = 0$

$\therefore f(-2) = f(2)$.

All the conditions for Rolle's theorem is satisfied.

Let 'c' be the point in (a,b) such that

$f'(c) = 0$

or $3c^2 = 0$

$\therefore c = 0$.

Hence, Rolle's theorem is verified.

(c): $f(x) = \sin x$ in $[-\pi, \pi]$.

Soln:

Given,

$f(x) = \sin x$

(i): ~~It~~ $f(x)$ exists for all $x \in [-\pi, \pi]$.

So, $f(x)$ is continuous ~~for all~~ in $[-\pi, \pi]$.

(ii) $f'(x) = \cos x$.

$f(x)$ exists for all $x \in [-\pi, \pi]$

So, $f(x)$ is diffrentiable ~~for~~ in $(-\pi, \pi)$

(iii): $f(-\pi) = 0$

$f(\pi) = 0$

$\therefore f(-\pi) = f(\pi)$ All conditions for Rolle's theorem is satisfied.

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Let 'c' be point in (a,b). ~~So~~, such that $f'(c) = 0$

$\cos c = 0$

$\therefore c = \frac{\pi}{2}$

Hence, Rolle's theorem is verified.

<d>: $f(x) = x(x+3)e^{-x/2}$ in $[-3,0]$

Soln

Given,

$$f(x) = \frac{x^2+3x}{e^{x/2}} \quad \text{~~or~~} = (x^2+3x)e^{-x/2}$$

(i): $f(x)$ exists for all $x \in [-3,0]$

Thus, $f(x)$ is continuous in $[-3,0]$

(ii): $$f'(x) = (x^2+3x)\cdot\frac{de^{-x/2}}{dx} + e^{-x/2}\cdot\frac{d(x^2+3x)}{dx}$$

$$= (x^2+3x)\cdot\frac{de^{-x/2}}{d(-x/2)} \times \frac{d\,-x/2}{dx} + e^{-x/2}\left(\frac{dx^2}{dx} + \frac{d3x}{dx}\right)$$

$$\therefore f'(x) = -\frac{1}{2}e^{-x/2}\cdot(x^2+3x) + (2x+3)e^{-x/2}.$$

$$\therefore f'(x) = \frac{(2x+3)}{e^{x/2}} - \frac{1(x^2+3x)}{2e^{x/2}}$$

$$\therefore f'(x) = \frac{2x+6-x^2-3x}{2e^{x/2}} = \frac{x+6-x^2}{2e^{x/2}}$$

exists for all $x \in (-3,0)$

Thus, $f'(x)$ is differentiable in $(-3,0)$

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(iii): $f(-3) = (-3)(-3+3)e^{-3/2} = 0$

$f(0) = 0 \times (0+3)e^{-0/2} = 0$

$\therefore f(-3) = f(0)$

Hence, all conditions for Rolle's theorem is satisfied.

Let 'c' be the point in $(-3, 0)$ such that

$f'(c) = 0$

or, $\cancel{\frac{c+3+c^2}{2e^{c/2}}} = 0$

or $\cancel{c^2+c+3} = 0$

$\cancel{(2c+3)}\ c+6-c^2 = 0$

or $c^2 - c - 6 = 0$

or $c^2 - 3c + 2c - 6 = 0$

or $c(c-3) + 2(c-3) = 0$

or $(c-3)(c+2) = 0$

$\therefore c = 3 \notin (-3, 0)$ and $c = 2 \in (-3, 0)$

Thus, the Rolle's theorem is verified.

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<Q-9>: verify MVT for the following:

(a): $f(x) = x^2$ in $[0,2]$.

Soln:

Given,

$f(x) = x^2$

(i): $f(x)$ is defined for all $x \in [0,2]$

$\therefore f(x)$ is continuous in $[0,2]$

(ii): $f'(x) = 2x$

$\therefore f'(x)$ is exists for all $(0,2)$.

$\therefore f'(x)$ is ~~discon~~ differentiable in $(0,2)$.

All the conditions for MVT are satisfied.

Let 'c' be the point in $(0,2)$ such that.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

or, $$2c = \frac{2^2 - 0^2}{2-0}$$

or $c = 1 \in (0,2)$

Thus, MVT is verified.

(b): $f(x) = x^2$ in $[1,2]$

Given,

$f(x) = x^2$

(i): $f(x)$ is defined for all $x \in [1,2]$

$\therefore f(x)$ is continuous in $[1,2]$.

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(ii): $f'(x) = 2x$

$f'(x)$ exists for all $(1, 2)$

So $f(x)$ is differentiable for all $(1,2)$

All the conditions for MVT is satisfied.

Let $c \in (\cancel{a,b})^{1,2}$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$\Rightarrow 2c = \frac{2^2-1^2}{2-1}$ $\therefore c = \frac{3}{2} \in (1, 2)$

Thus, MVT is verified.

~~(iii)~~: (c): $f(x) = \sqrt{x}$ in $[4, 9]$

Soln:

Given,

$f(x) = \sqrt{x}$

(i): $f(x)$ exists for all $x \in [4, 9]$

Thus $f(x)$ is continuous ~~for all~~ in $[4, 9]$

(ii): $f'(x) = \frac{1}{2\sqrt{x}}$

$f'(x)$ is differentiable for all $x \in (4, 9)$

Thus, all the conditions for MVT are satisfied.

~~Let~~ ~~Here,~~ Let $c \in (4, 9)$ ~~doesn't exist~~, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, $\frac{1}{2\sqrt{c}} = \frac{\sqrt{9}-\sqrt{4}}{9-4}$

or, $\frac{1}{\sqrt{c}} = \frac{2}{5}$ $\therefore c = \frac{25}{4} \in (4, 9)$

Hence, MVT is satisfied.

(d): $f(x) = (x-1)(x-\overset{2}{\cancel{3}})(x-3)$ in $[0,4]$

Soln:

Given,

$f(x) = (x-1)(x-2)(x-3)$

$= (x^2-2x-x+2)(x-3) = (x^2-3x+2)(x-3)$

$= x^3 - 3x^2 + 2x - 3x^2 + 6x - 6$

$\therefore f(x) = x^3 - 6x^2 + \overset{11}{\cancel{8}}x - 6$

(i): $f(x)$ is a polynomial function.
So, $f(x)$ is continuous ~~at~~ in $[0,4]$

(ii): $f'(x) = 3x^2 - 12x + \cancel{8}11$ exists for $(0,4)$
Thus, $f(x)$ is differentiable for all $(0,4)$.
All the conditions for MVT is satisfied.

Let $c \in (0,4)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

$3c^2 - 12c + 8 = \cancel{\dfrac{8-8}{4-0}}$

or $3c^2 - 12c + \overset{11}{\cancel{8}} = \dfrac{\{(4-1)(4-2)(4-3)\} - \{(0-1)(0-2)(0-3)\}}{4-0}$

$3c^2 - 12c + 11 = 3$

or $3c^2 - 12c + 8 = 0$

Solving, we get.

$c = \dfrac{6 \pm 2\sqrt{3}}{2} \in (0,4)$

Hence, MVT is verified.

<Q7: Define different types of asymptotes with examples.

Soln:

A asymptote is a straight line that continually approaches a given curve but doesn't meet it at any finite distance.

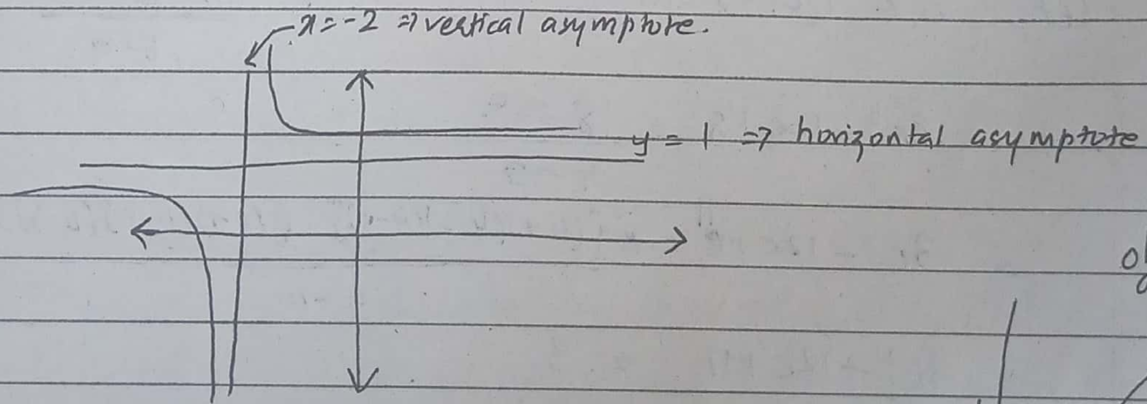
The types of asymptotes are as follows:

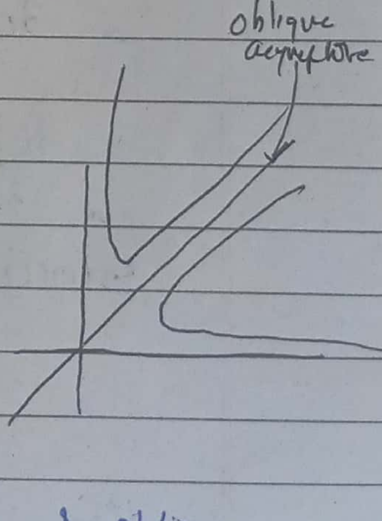
(i): Horizontal asymptote:

A line $y = b$ is a horizontal asymptote of $f(x)$ if $\lim_{x \to \pm\infty} f(x) = b$.

(ii) Vertical asymptote.

A line $x = a$ is vertical asymptote of $f(x)$ if $\lim_{x \to \pm a} f(x) = \pm\infty$.





(iii) Oblique asymptotes:

It is a slanted asymptote.

if $\lim_{x \to \infty} f(x) = \infty$ (when in $f(x)$, $n^{\circ} > d^{\circ}$ by 1)

↳ shows possibility of oblique asymptote.

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<Q·11>: Sketch the graph of the following:

(a): $f(x) = x^4 - 4x^3 + 10$

Soln.

Given,

$f(x) = x^4 - 4x^3 + 10$

(i): For symmetry,

Putting $x = -x$ in $f(x)$,

$f(-x) = (-x)^4 - 4(-x)^3 + 10$

$\therefore f(-x) = x^4 + 4x^3 + 10 \neq -f(x)$

$\neq f(x)$

This function has no symmetry.

(ii): For critical points:

$f(x) = x^4 - 4x^3 + 10$

$\therefore f'(x) = 4x^3 - 12x^2 \cancel{+10}$

Putting $f'(x) = 0$,

$0 = 4x^2(x-3)$

$\therefore x = 0 \qquad \therefore x = 3.$

(iii) For increasing or decreasing.

int:	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$f'(x)$	$^-$ve	$^-$ve	$^+$ve
	decreasing	decreasing	increasing
	$f' < 0$	$f' < 0$	$f' > 0$.

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The first derivative test for increasing and decreasing functions shows f' ~~st~~ changes from $-ve$ to $+ve$ at $x=3$,

ie, f has local minimum at $x=3$.

$\therefore f_{min} = -17$.

Here, there is no local maximum.

(iv): For point of inflection,

$$f''(x) = 12x^2 - 24x$$

Putting $f''(x)=0$,

$$12x(x-2)=0$$

$$\therefore x=0, \quad x=2.$$

(v): For concavity: ~~of~~

Interval.	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
$f''(x)$	$+ve$	$-ve$	$+ve$
	increasing	decreasing	increasing
	$f''>0$	$f''<0$	$f''>0$.

The second derivative test for concavity shows f is concave upwards for $(-\infty,0)$ and $(2,\infty)$ and concave downward for $(0,2)$.

We have,
three points,

$(0,10)$, $(2,-6)$, $(3,-17)$

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(vi) Asymptotes:

Putting, $\lim_{x \to \pm\infty} x^4 - 4x^3 + 10 = \infty$

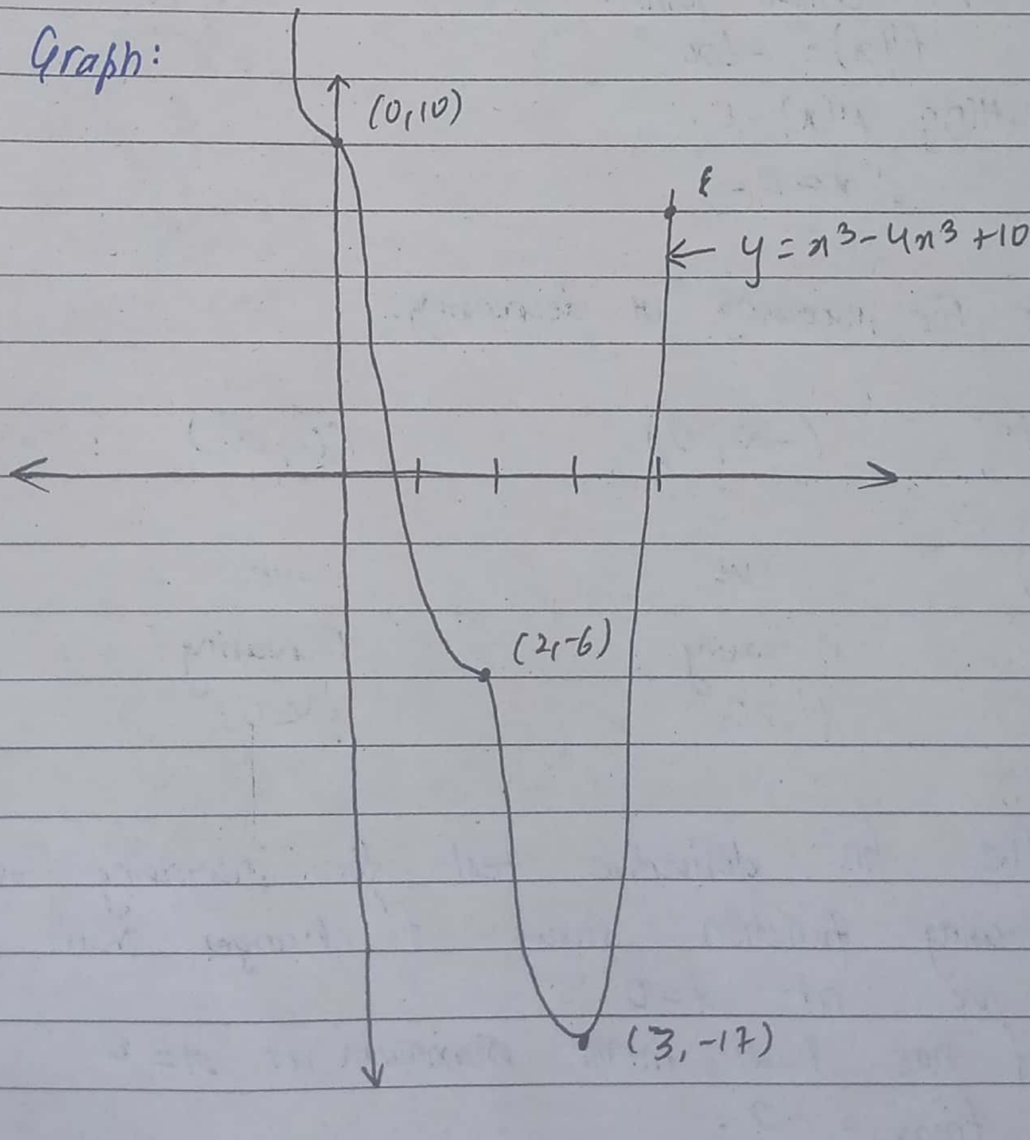
(No ~~ver~~ horizontal asymptote)

Putting $\lim_{x \to 0} x^4 - 4x^3 + 10 = 10$

(No vertical asymptote).

Since $n^\circ = 4$ and $d^\circ = 0$ i.e. $n^\circ - d^\circ \neq 1$ so.
oblique asymptote doesn't exist.

(vii) : Graph:



<b>: $f(x) = x^2 - 4x^2 - 2$

Soln:

Given, $f(x) = -3x^2 - 2$.

(i): For symmetry.

~~f(x) =~~ Putting $x = -x$.

$f(x) = -3(-x)^2 - 2$

$= -3x^2 - 2 \quad = f(x)$

The function is symmetrical about y-axis.

(ii): For critical point:

$f'(x) = -6x$

putting $f'(x) = 0$.

$\therefore x = 0$.

(iii): For increasing or decreasing.

Int:	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	+ve	-ve
	increasing $f' > 0$	decreasing. $f' < 0$

The first derivative test for increasing or decreasing function shows f' changes from +ve to -ve at $x = 0$.

$\therefore f$ has local ~~minim~~ maximum at $x = 0$

$f_{max} = -2$.

Here, there's no local minimum.

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(iv): For point of inflection,

$f''(x) = -6$.

Here, there's no point of inflection. ~~ie, asymptote exist~~.

(v): Concavity test:

~~At $x=0$, $f''(0) = -6 < 0$ (maximum).~~

Here, point of inflection doesn't exist.

So, there's no change in concavity.

(vi): Asymptotes.

$\lim_{x \to \pm\infty} f(x) = x^2 - 4x^2 - 2 = \infty$

H.A. doesn't exist.

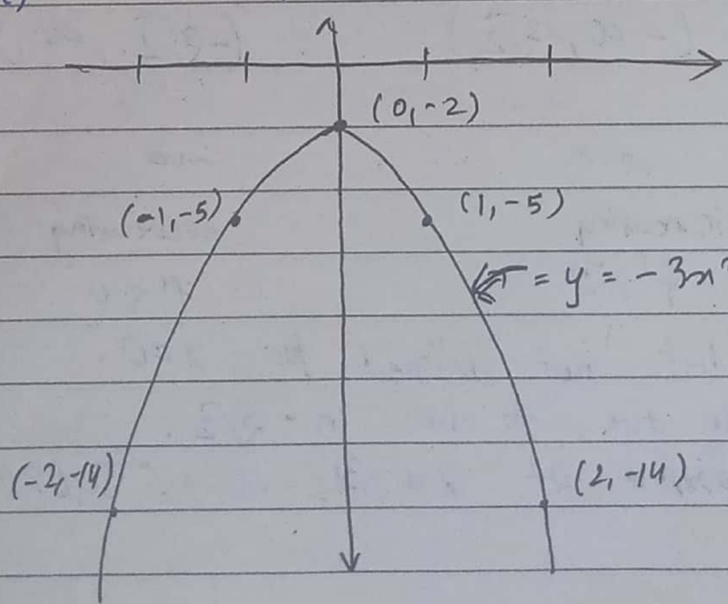
$\lim_{x \to 0} f(x) = 0^2 - 4\times0^2 - 2 = -2$

V.A. doesn't exist.

Here, $n^\circ = 2$, $d^\circ = 0$ $\therefore n^\circ - d^\circ \neq 1$. O.A doesn't exist.

(vii) Graph.

We have, $(0,-2)$, $(1,-5)$, $(-1,-5)$, $(2,-14)$, $(-2,-14)$



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(c): $f(x) = \frac{-x^3+1}{x^2}$

Sol^n:

Given, $f(x) = \frac{-x^3+1}{x^2}$

(i): For symmetry;

Putting $x = -x$.

$f(-x) = \frac{-(-x)^3+1}{(-x)^2} = \frac{x^3+1}{x^2} \neq -f(x)$
$\neq f(x)$.

This function has no symmetry.

(ii): For critical points,

$f'(x) = \frac{-2-x^3}{x^3} = \frac{-2}{x^3} - 1$

Putting $f'(x) = 0$.

$0 = 2 - x^3 \quad \therefore x = -\sqrt[3]{2}$

(iii): For increasing or decreasing function

Int.	$(-\infty, -\sqrt[3]{2})$	$(-\sqrt[3]{2}, \infty)$
$f'(x)$	+ve	-ve
	increasing $f' > 0$	decreasing $f' < 0$

but not defined for $x = 0$.

f' changes from +ve to -ve in $-\sqrt[3]{2}$.

So, local maxima at $x = -\sqrt[3]{2}$ is -0.629

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(iv): For point of inflection:

$$f''(x) = \frac{-6}{x^4}$$

Point of inflection doesn't exist.

(v): Concavity:

Here, point of inflection doesn't exist.
Hence, concavity doesn't change.

(vi) Asymptote:

$$\lim_{x \to \infty} f(x) = -x + \frac{1}{x^2} = \infty \quad \text{(HA doesn't exist)}.$$

$$\lim_{x \to 0} f(x) = -x + \frac{1}{x^2} = \infty \quad \text{(V.A. exist) at } x = 0.$$

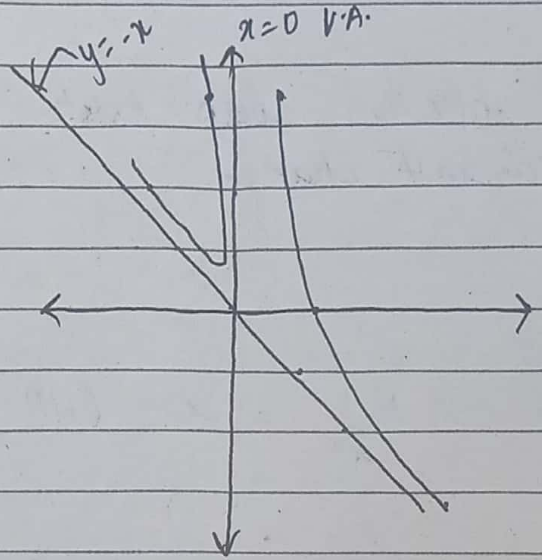
Here, $n^\circ = 3$, $d^\circ = 2$ ie, $n^\circ - d^\circ = 1$.
O.A exist.

$$x^2 \,\big)\, -x^3 + 1 \,\big(\, -x + 1/x^2$$
$$\cancel{-x^3}$$
$$1$$
$$\cancel{-1}$$
$$0$$

Here,
$y = -x$ is oblique asymptote.

(vii): Graph:

x	1	-1	2	0.5	-0.5
y	0	-2	-3.5	3.5	3.5



<d>: $f(x) = \dfrac{2x^2 - 3}{7x + 4}.$

Soln.

Given,

$$f(x) = \frac{2x^2 - 3}{7x+4}$$

(i): For symmetry.

Putting $x = -x$.

$$f(-x) = \frac{2(-x)^2 - 3}{7(-x)+4} = \frac{2x^2 - 3}{-7x+4} \neq -f(x)$$
$$\neq f(x)$$

This function has no symmetry.

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(ii): For critical point:

$$f'(x) = \frac{14x^2+16x+21}{(7x+4)^2}$$

Putting $f'(x) = 0$.

$14x^2+16x+21 = 0$

This has no real solution.

<Q.12>: Sketch the general shape of graph of f if.

(a): $y' = 2+x-x^2$.

Soln:

(i): For critical point:

Putting $y' = 0$.

$0 = 2+x-x^2$

or $x^2 - x - 2 = 0$

or $x^2 - (2-1)x - 2 = 0$

or $x^2 - 2x + x - 2 = 0$

or $x(x-2) + 1(x-2) = 0$ or, $(x-2)(x+1) = 0$

either, or,

$x = 2$ $x = -1$.

(ii): For increasing or decreasing.

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Int:	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
y'	-ve	+ve	-ve
	decreasing	increasing	decreasing
	$f' < 0$	$f' > 0$	$f' < 0$

From first derivative test for local extreme value at $x = -1$ and $x = 2$.
when f' changes from -ve to +ve at $x = -1$:, it has local ~~maximum~~ minimum value.
~~At $x=-1$~~, when f' changes from +ve to -ve at $x = 2$, it has local maximum value.

(iii) For point of inflection.
$$f''(x) = 1 - 2x$$
Putting $f''(x) = 0$.
$$0 = 1 - 2x \qquad \therefore x = 1/2.$$

(iv): For concavity:

Int	$(-\infty, 1/2)$	$(1/2, \infty)$
$f''(x)$	+ve	-ve

From second derivative test for concavity, it shows that:
f is concave upwards for $(-\infty, 1/2)$
f is concave downwards for $(1/2, \infty)$

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$(2, \infty)$

-ve
decreasing
$f' < 0$

value at

$x = -1$:,

to -ve at

ity,

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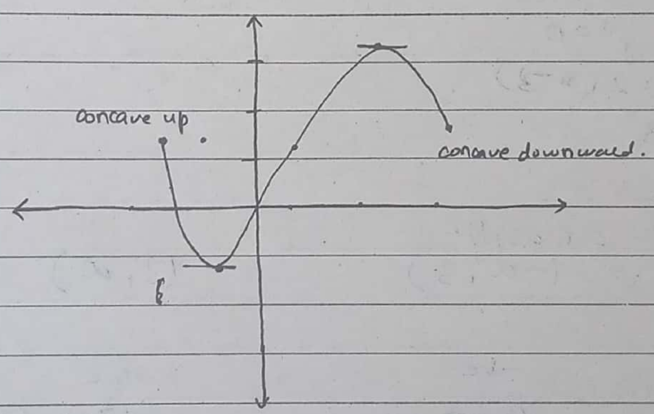
~~(v): Asymptote:~~ (vi): The function;

$$y = \int (2 + x - x^2)\, dx$$

$$y = 2x + \frac{x^2}{2} - \frac{x^3}{3} + c$$

Now,

x	2	-1	1/2	3	-2
y	3.33	-1.16	1.083	1.5	0.67



(b): ~~Sol~~ $y' = (x-3)^2$

Soln:

(i): For critical point,

Put, $y' = 0$

$0 = x - 3$ $\therefore x = 3$.

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(ii) For increasing or decreasing.

Int.	$(-\infty, 3)$	$(3, \infty)$
y'	+ve	+ve
	$f' > 0$	$f' > 0$
	increasing	increasing.

So, $f(x)$ is increasing in $(-\infty, 3) \cup (3, \infty)$

(iii): for point of inflection:

$y'' = 2(x-3)$.

putting $y'' = 0$.

$0 = 2(x-3)$

$\therefore x = 3$

(iv): For concavity;

Int	$(-\infty, 3)$	$(3, \infty)$
y''	−ve	+ve
	$f'' < 0$	$f'' > 0$

For second derivative test for concavity,
f is concave downwards at $(-\infty, 3)$
f is concave upwards at $(3, \infty)$

(v): The function,

$y = \int y' \, dx = \int (x-3)^2 \, dx$

$\therefore y = \frac{(x-3)^3}{3} + c$

(vi): ~~We have,~~ For symmetry;

x	3	0	6	2
y				

$$\cancel{y = \frac{(x-3)^3}{3} + c}$$

~~Putting~~ $\cancel{x = -x}$

$$\cancel{y = \frac{(-x-3)^3}{3} + c}$$

(vi): We have;

x	3	0	6	2	1
y	0	-9	9	-0.33	-2.66