Lecture 05

Electrostatic Field (Contd.)

Outline

1. Electric Field Lines
2. Electric flux (Φ)
3. GAUSS'S LAW
   - Applications of Gauss's Law

Electric Field Lines

- Electric field lines describe an electric field in any region of space.
- The electric field vector $\vec{E}$ is tangent to the electric field line at each point.
- The number of lines per unit area through a surface perpendicular to the lines is proportional to the magnitude of $\vec{E}$ in that region.

Electric Field Lines (contd.)

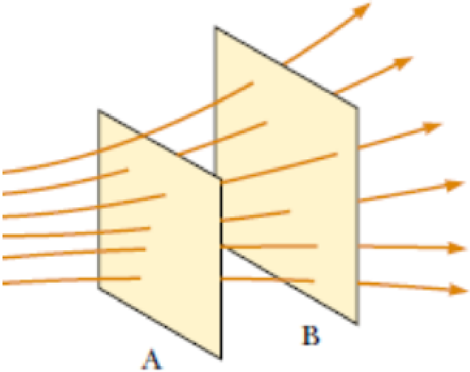

Figure 1: Electric field lines penetrating two surfaces. The magnitude of the field is greater on surface A than on surface B.

- Representative electric field lines for the field due to a single positive point charge are shown in Figure 2 :

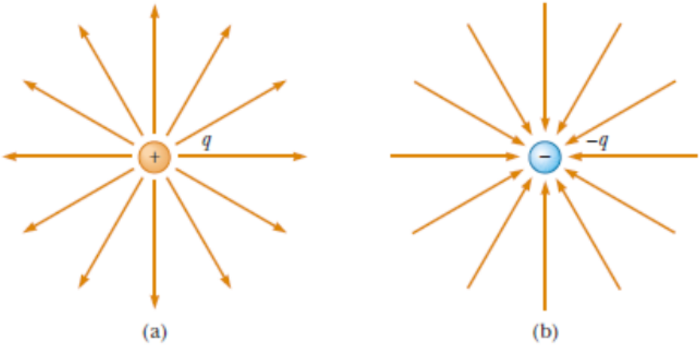

Figure 2

Electric Field Lines (contd.)

- The rules for drawing electric field lines are as follows:
  - The lines must begin on a positive charge and terminate on a negative charge.
  - The number of lines drawn leaving a positive charge or approaching a negative charge is proportional to the magnitude of the charge.
  - No two field lines can cross.
- The electric field lines for two point charges of equal magnitude but opposite signs (an electric dipole) are shown in Figure 3.

Electric Field Lines (contd.)

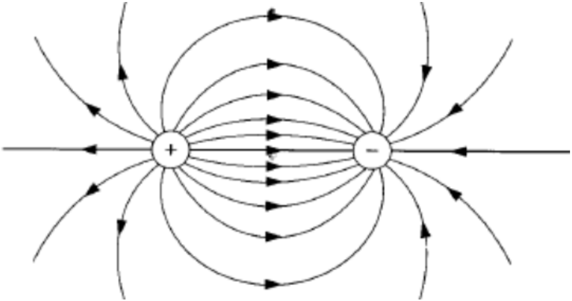

Figure 3: Equal but opposite charges

- The electric field lines for two equal positive point charges are shown in Figure 4.

Electric Field Lines (contd.)

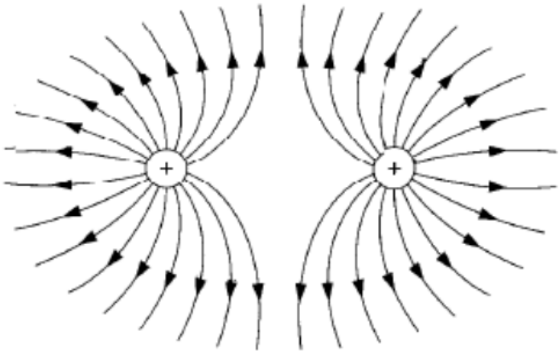

Figure 4: Equal charges

Electric flux (Φ)

- Electric flux is proportional to the number of electric field lines that penetrate a surface.
- The electric flux through a surface S is

$$\Phi_E \equiv \int_S \vec{E} \cdot d\vec{a}$$

- The SI unit of electric flux (Φ_E) is $\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{C}^{-1}$.
- The electric flux through any closed surface is a measure of the total charged inside.

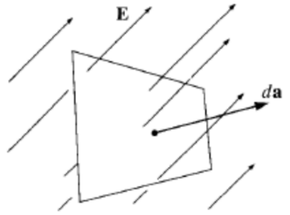

Figure 5: Field lines representing an electric field penetrating an area that is at an angle θ to the field

[Formulated by Carl Friedrich Gauss (1777–1855), Greatest mathematicians]

- The total electric flux through any closed surface is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface:

$$\boxed{\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \mathrm{Q}_{enc}}$$

- Consider an arbitrary shaped closed surface S, which encloses a point charge q at the origin O [Figure 6].

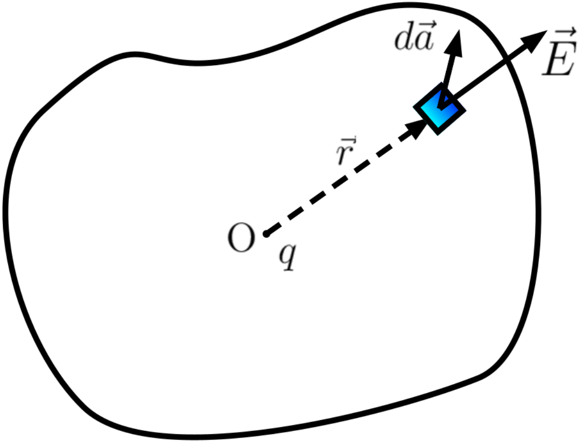

Figure 6

GAUSS'S LAW (contd.)

The electric field at the position vector $\vec{r}$ due to the point charge q located at the origin is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

The total electric flux through a closed surface $S$is

$$\begin{aligned}
\oint_S \vec{E}\cdot d\vec{a} &= \oint_S \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\right)\cdot\left(da_r\hat{r} + da_\theta\hat{\theta} + da_\phi\hat{\phi}\right) \\
&= \oint_S \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right) da_r \\
&= \int \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)\left(r^2 \sin\theta d\theta d\phi\right)
\end{aligned}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\left\{\int_0^{\pi}\sin\theta d\theta\right\}\left\{\int_0^{2\pi}d\phi\right\}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[(2)(2\pi)\right]$$

$$= \frac{q}{\varepsilon_0}$$

$$\boxed{\therefore \oint_S \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon_0}} \tag{1}$$

GAUSS'S LAW (contd.)

- If the surface S encloses several point charges $q_1, q_2, \ldots, q_n$, the net electric field is

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i$$

with $\vec{E}_i = \dfrac{q_i}{4\pi\varepsilon_0 r_i^2}\hat{r}_i$. Now the total flux through the closed surface S is

$$\oint_S \vec{E}\cdot d\vec{a} = \oint_S \left(\sum_{i=1}^{n} \vec{E}_i\right)\cdot d\vec{a} = \sum_{i=1}^{n}\left(\oint_S \vec{E}_i \cdot d\vec{a}\right) = \sum_{i=1}^{n}\left(\frac{q_i}{\varepsilon_0}\right)$$

$$\Longrightarrow \oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \qquad (2)$$

with

$$Q_{\text{enc}} = \sum_{i=1}^{n} q_i \tag{3}$$

as the total point charges enclosed by the surface. For the the continuous charge distribution

$$Q_{\text{enc}} = \int dq = \underbrace{\int_L \lambda dl'}_{\text{Line charge}} = \underbrace{\int_S \sigma da'}_{\text{Surface charge}} = \underbrace{\int_V \rho d\tau'}_{\text{Volume charge}}$$

GAUSS'S LAW (contd.)

- If there is a continuous distribution of charge with a charge density ρ, then the total charge enclosed by the surface S is $Q_{enc} = \int\limits_V \rho \; d\tau$. Gauss's Law reads

$$\boxed{\oint\limits_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \int\limits_V \rho \; d\tau} \tag{4}$$

- By applying the divergence theorem in Eq. (4), we get

$$\int\limits_V \left(\nabla \cdot \vec{E} \right) \, d\tau = \int\limits_V \left(\frac{\rho}{\varepsilon_0} \right) \, d\tau$$

GAUSS'S LAW (contd.)

Since this holds for any volume, the integrands must be equal:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$$

This is the **differential form of Gauss's Law.**

GAUSS'S LAW:-Applications of Gauss's Law

Gauss's law is usually applied for calculating the electric field for the given system of charge distribution. Even though the Gauss's law is true for all type of charge distribution, its integral form is applicable only for some symmetrical type of charge distribution in order to calculate the electric field. To find the electric at a give point, a Gaussian surface through the point is drawn such that:

(i) the electric field is normal to the surface at every point, i.e.

$$\oint_S \vec{E} \cdot d\vec{a} = \oint_S E da$$

(ii) the electric field is same at every point on the surface, i.e.

$$\oint_S E da = E \oint_S da = E \times \text{total area of Gaussian surface}$$

Therefore, for the appropriately drawn Gaussian surface, we can have

$$\oint_S \vec{E} \cdot d\vec{a} = E \times \text{total area of Gaussian surface} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

1. **Use Gauss's Law to find the electric field outside, on and inside a spherical shell of radius R, which carries a uniform surface charge density σ.**

Solution:

Consider a spherical shell of radius R and center O, which carries a uniform surface charge density σ. Let P be a point at a distance r from O. To find the electric field at P, let's construct a concentric Gaussian sphere of radius r passing through P Figure 7(a).

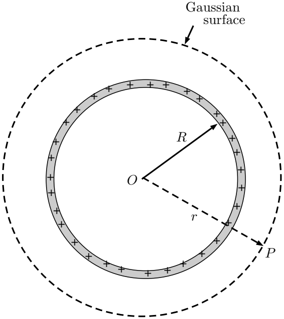

Figure 7: (a)

Then Gauss's law reads

$$\oint_S \vec{E}\cdot d\vec{a} = E(4\pi r^2) = \frac{Q_{\rm enc}}{\varepsilon_0}$$

$$\Longrightarrow E = \frac{Q_{\rm enc}}{4\pi\varepsilon_0 r^2} \tag{5}$$

(i) Electric Field Outside the Sphere

If the point P outside the sphere, i.e. $r > R$, then the Gaussian surface encloses all the charges. Therefore,

$$Q_{\rm enc} = \sigma(4\pi R^2) = q \qquad \text{[the total charge]}$$

The electric field from equation (5) is

$$E_{\text{out}} = \frac{\sigma(4\pi R^2)}{4\pi\varepsilon_0 r^2} = \frac{\sigma R^2}{\varepsilon_0 r^2}$$

In term of total charge

$$E_{\text{out}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

Therefore, the field at a point outside the shell is equivalent to a point charge q located at the center.

Electric field on the surface of a spherical shell $(r = R)$

$$E_{\text{on}} = \frac{\sigma}{\varepsilon_0} = \frac{\left(\frac{q}{4\pi R^2}\right)}{\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$

(ii) Electric field inside a spherical shell

If the point P lies inside the charged sphere i.e. $r < R$, the Gaussian surface lies inside the sphere as in Figure 8(b).The total charge enclosed by the Gaussian surface S is

$$Q_{\text{enc}} = 0$$

and hence

$$E_{\text{in}} = 0$$

Therefore the electric field inside a spherical shell is zero.

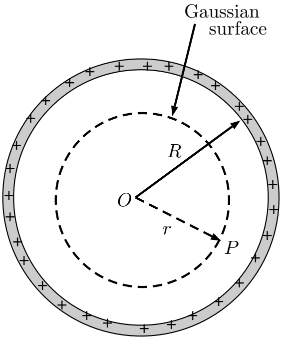

Figure 8: (b)

Hence

$$E = \begin{cases} \dfrac{\sigma R^2}{\varepsilon_0 r^2} = \dfrac{q}{4\pi\varepsilon_0 r^2}, & \text{for } r > R \\ \dfrac{\sigma}{\varepsilon_0} = \dfrac{q}{4\pi\varepsilon_0 R^2}, & \text{for } r = R \\ 0, & \text{for } r < R \end{cases}$$

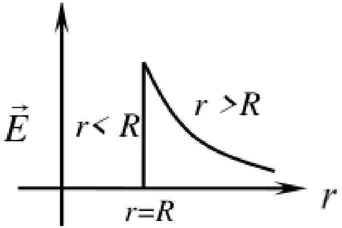

Figure 9: A plot of E versus r for a spherical shell. The electric field inside the spherical shell $(r < R)$ is zero. The electric field outside the sphere$(r > R)$ is the same as that of a point charge q located at $r = 0$

2. **Use Gauss's Law to find the electric field outside, inside and on the surface of a uniformly charged solid sphere of radius R, which carries a uniform volume charge density ρ.**

Solution:

Consider a solid sphere of radius R and center O, which carries a uniform volume charge density ρ. Let P be a point at a distance r from O. To find the electric field at P, let's construct a concentric Gaussian sphere of radius r passing through P Figure 10(a).

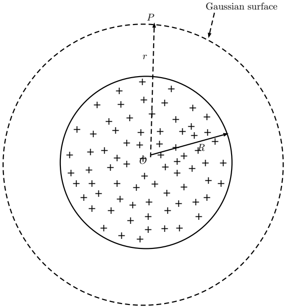

Figure 10: (a)

Then Gauss's law reads

$$\oint_S \vec{E} \cdot d\vec{a} = E(4\pi r^2) = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

$$\implies E = \frac{Q_{\text{enc}}}{4\pi\varepsilon_0 r^2} \tag{6}$$

(i) Electric Field Outside the Sphere

If the point P outside the sphere, i.e. $r > R$, then the Gaussian surface encloses all the charges. Therefore,

$$Q_{\text{enc}} = \rho\left(\frac{4}{3}\pi R^3\right) = q \qquad \text{[the total charge]}$$

The electric field from equation (6) is

$$E_{\text{out}} = \frac{\rho\left(\frac{4}{3}\pi R^3\right)}{4\pi\varepsilon_0 r^2} = \frac{\rho R^3}{3\varepsilon_0 r^2}$$

In term of total charge

$$E_{\text{out}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

Therefore, the field at a point outside the solid sphere is equivalent to a point charge q located at the center.

Electric field on the surface of a solid $(r = R)$

$$E_{\text{on}} = \frac{\rho R}{3\varepsilon_0} = \frac{\left(\frac{q}{\frac{4}{3}\pi R^3}\right)R}{3\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$$

(ii) Electric field inside the sphere

If the point P lies inside the charged sphere i.e. $r < R$, the Gaussian surface lies inside the sphere as in Figure 11(b).The total charge enclosed by the Gaussian surface S is

$$Q_{\text{enc}} = \rho\left(\frac{4}{3}\pi r^3\right)$$

and hence

$$E_{\text{in}} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{4\pi\varepsilon_0 r^2} = \frac{\rho r}{3\varepsilon_0}$$

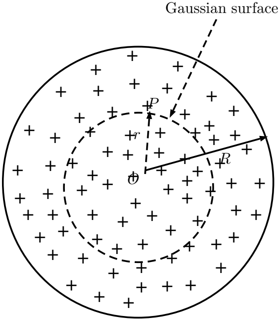

Figure 11: (b)

$$E = \begin{cases} \dfrac{\rho R^3}{3\varepsilon_0 r^2} = \dfrac{q}{4\pi\varepsilon_0 r^2}, & \text{for } r > R \\ \dfrac{\rho R}{3\varepsilon_0} = \dfrac{q}{4\pi\varepsilon_0 R^2}, & \text{for } r = R \\ \dfrac{\rho r}{3\varepsilon_0}, & \text{for } r < R \end{cases}$$

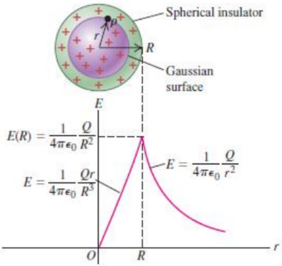

Figure 12: A plot of E versus r for a uniformly charged sphere. The electric field inside the sphere $(r < R)$ varies linearly with r. The field outside the sphere $(r > R)$ is the same as that of a point charge Q located at $r = 0$.

3. **Find the electric field outside and inside a sphere, which carries a charge density proportional to the distance from the origin, $\rho = kr$, for some constant k.**

Solution:

Consider a solid sphere of radius R and center O, which carries a volume charge density $\rho = kr'$ with k is a constant and r' is the distance from center and inside the sphere. Let P be a point at a distance r from O. To find the electric field at P, let's construct a concentric Gaussian sphere of radius r passing through P Figure 13(a).

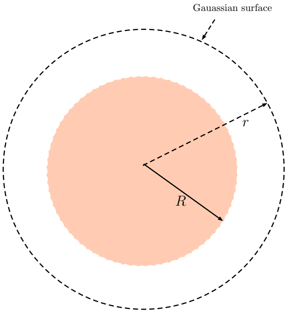

Figure 13: (a)

Then Gauss's law reads

$$\oint_S \vec{E} \cdot d\vec{a} = E(4\pi r^2) = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

$$\implies E = \frac{Q_{\text{enc}}}{4\pi\varepsilon_0 r^2} \tag{7}$$

(i) Electric Field Outside the Sphere

If the point P outside the sphere, i.e. $r > R$, then the Gaussian surface encloses all the charges. Therefore,

$$Q_{\text{enc}} = \int_V \rho d\tau' = \int_V (kr')(r'^2 dr' \sin\theta d\theta d\phi)$$
$$= k \int_0^R r'^3 dr' \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = k\pi R^4$$

The electric field from equation (7) is

$$E_{\text{out}} = \frac{k\pi R^4}{4\pi\varepsilon_0 r^2} = \frac{kR^4}{4\varepsilon_0 r^2}$$

(ii) Electric field inside the sphere

If the point P lies inside the charged sphere i.e. $r < R$, the Gaussian surface lies inside the sphere as in Figure 14(b).The total charge enclosed by the Gaussian surface S is

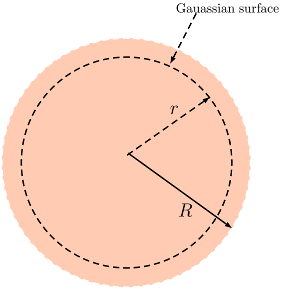

Figure 14: (b)

$$Q_{\text{enc}} = \int_V \rho d\tau' = \int_V (kr')(r'^2 dr' \sin\theta d\theta d\phi)$$

$$= k\int_0^r r'^3 dr' \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = k\pi r^4$$

The electric field from equation (7) is

$$E_{\text{out}} = \frac{k\pi r^4}{4\pi\varepsilon_0 r^2} = \frac{kr^2}{4\varepsilon_0}$$

$$E = \begin{cases} \dfrac{kR^4}{4\varepsilon_0 r^2} = \dfrac{q}{4\pi\varepsilon_0 r^2}, & \text{for } r > R \\ \dfrac{kR^2}{4\varepsilon_0} = \dfrac{q}{4\pi\varepsilon_0 R^2}, & \text{for } r = R \\ \dfrac{kr^2}{4\varepsilon_0}, & \text{for } r < R \end{cases}$$

End of Lecture 05

Thank you