General Physics I (PHYS 101)

Lecture 21

Heat Transfer

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Outline

Outline (contd.)

- Derivation of Stefan’s law from Planck’s law

Thermodynamics is a branch of physics which deals with the transformation of heat into mechanical work. It involves the study of interaction of one body on another which is described in terms of quantities of heat and work. Thus thermodynamics describe the interrelationship between heat and mechanical work.

Heat Flux (Φ)

Heat flux is defined as the amount of heat transmitted per unit area per unit time form or to a surface from one place to the other by three different ways. They are conduction, convection and radiation.

It is the process in which heat is transmitted from one point to the other through the substance without the actual motion of the particles. When one end of metal bar is heated, the molecules at the hot end vibrate with higher amplitude (kinetic energy) and transmit the heat energy from one particle to the next and so on. However, the particles remain in their mean positions.

For example: Heat transfer in a metal bar.

It is the process in which heat is transmitted from one place to the other by the actual movement of the heated particles. It is prominent in the case of liquids and gases. The heated particles go farther away from the heating source while the less heated particles come to fulfill the vacant space in the process.

For example: Hot air and hot water supply on home heating system, the flow of blood in the body

It is the process in which heat is transmitted from one place to the other directly without the necessity of the intervening medium. Heat radiations can pass through vacuum. These are the part of electromagnetic radiations and have properties similar to light radiations.

For example: Heat obtain from the sun.

We suppose a cube of side x and area of cross-section A. Let θ_1 and θ_2 be the temperatures of any two opposite faces where $\theta_1 > \theta_2$.

Suppose an amount of heat Q flows through the cube in time t in the direction θ_1 to θ_2 as shown in figure 1.

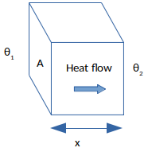

Figure 1: Thermal Conductivity

Then the rate of flow of heat across the rod is given by

$$\frac{Q}{t} \propto A\frac{\theta_1 - \theta_2}{x} \implies \frac{Q}{t} = \kappa A\frac{\theta_1 - \theta_2}{x} \tag{1}$$

Where κ is a constant called the coefficient of thermal conductivity of the material of the cube.

$$\therefore \kappa = \frac{Qx}{A\left(\theta_1 - \theta_2\right)t} \tag{2}$$

Thus, the coefficient of thermal conductivity is defined as the amount of heat flowing in one second across the opposite faces of the cube of side one unit maintained at temperature different of 1^0C.

SI unit of κ is $Wm^{-1}K^{-1}$.

Planck's radiation law

The Planck's law is obtained by using the following postulates:

1. A black body radiation chamber is filled up not only with radiation; but also with simple harmonic oscillators or resonators of the molecular dimension; which cannot have any value of energy but only energies given by
   $E = nh\nu$; $n = 0, 1, 2, 3, \ldots$where ν is frequency of the oscillator and h is Planck's constant.
2. The oscillators cannot radiate or absorb energy continuously; but an oscillator of frequency ν can only radiate or absorb energy in units or quanta of magnitude $h\nu$.

Planck's radiation law (contd.)

3. The average energy of Planck's oscillator is given by

$$\overline{E} = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{3}$$

k is Boltzmann constant

The number of resonators per unit volume in the frequency range ν and $\nu + d\nu$ is given by

$$N_\nu d\nu = \frac{8\pi\nu^2}{c^3} d\nu \tag{4}$$

Planck's radiation law (contd.)

The energy density belonging to range $d\nu$ can be obtained by multiplying the average energy of a Planck's oscillator by the number of resonators per unit volume, in the frequency range ν and $\nu + d\nu$ i.e.

$$E_\nu \, d\nu = \bar{E} N_\nu d\nu = \frac{8\pi\nu^2}{c^3} \frac{h\,\nu}{e^{h\nu/kT} - 1} d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu \quad (5)$$

This is the Planck's radiation law and $E_\nu d\nu$ is the energy density i.e. total energy per unit volume. The energy density $E_\lambda d\lambda$ belonging to the range $d\lambda$ can be obtained by using the relation $\nu = c/\lambda$ and hence $|d\nu| = \left| -\frac{c}{\lambda^2} d\lambda \right|$ i.e.

$$E_\lambda d\lambda = \frac{8\pi h}{c^3} \left(\frac{c}{\lambda}\right)^3 \frac{1}{e^{hc/\lambda kT} - 1} \left| -\frac{c}{\lambda^2} d\lambda \right| = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} d\lambda \quad (6)$$

This is another form of Planck's radiation law. According to this law, the energy density radiated from a radiation chamber at a temperature T increases with increase in wavelength, becomes maximum at a certain wavelength and then decreases with increase of wavelength as shown in figure 2.

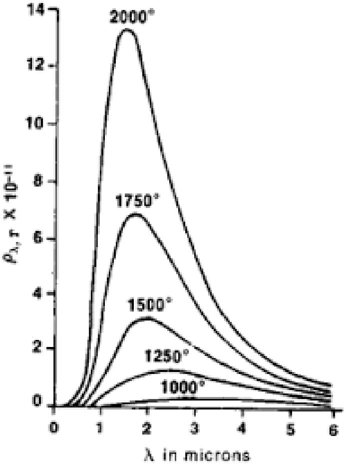

Figure 2: Planck's radiation curve

From Planck's law

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} d\lambda$$

For longer wavelengths,

$$e^{hc/\lambda kT} \approx 1 + \frac{hc}{\lambda kT}$$

and hence

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{\left(1 + \frac{hc}{\lambda kT} - 1\right)} d\lambda = \frac{8\pi hc}{\lambda^5} \frac{\lambda kT}{hc} d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda$$

This is Rayleigh-Jean's law.

Wien's displacement law

It states that "the product of the wavelength corresponding to maximum energy, $\lambda_m$and the absolute temperature T is constant" i.e.

$$\lambda_m T = \text{constant}$$

This constant is called Wien's displacement constant and has a value 0.2896 cm·K. According to this law λ_m decreases with increase in temperature.

Derivation of Wien's displacement law from Planck's law

To calculate the maximum energy radiated by the chamber, we have

$$
\begin{aligned}
& \frac{dE_\lambda}{d\lambda} = 0 \\
\implies & \frac{d}{d\lambda}\left[\frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\right] = 0 \\
\implies & \frac{d}{d\lambda}\left[\frac{1}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\right] = 0 \\
\implies & -\frac{5}{\lambda^6}\frac{1}{e^{hc/\lambda kT}-1} + \frac{1}{\lambda^5}\left(\frac{-1}{\left(e^{hc/\lambda kT}-1\right)^2}\right)\left(e^{hc/\lambda kT}\right)\left(\frac{hc}{kT}\right)\left(\frac{-1}{\lambda^2}\right) = 0 \\
\implies & 5 - \frac{hc}{\lambda kT}\frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT}-1} = 0
\end{aligned}
$$

Wien's displacement law

Derivation of Wien's displacement law from Planck's law (contd.)

Letting $\dfrac{hc}{\lambda kT} = x$, we get

$$5 - \frac{xe^x}{e^x - 1} = 0$$

$$\implies x \approx 4.98 \implies \lambda T = \frac{hc}{kx} \approx 0.002898\text{mK} = 0.2898\text{cmK}$$

Hence the product of wavelength corresponds to maximum radiation energy and the absolute temperature is constant i.e.

$$\lambda_m T = \text{constanat} = 0.2898 \text{ cmK}$$

Stefan-Boltzmann law

Stefan's law states that "the rate of emission of radiant energy by unit area of perfectly black-body is directly proportional to the fourth power of its absolute temperature" i.e.

$$I = \sigma T^4$$

Where σ is a constant and is called Stefan's constant.
A black body at absolute temperature T surrounded by another black body at absolute temperature T_0 not only losses an amount of energy σT^4 but also gains σT_0^4, thus the amount of heat lost by the former per unit time is given by

$$I = \sigma(T^4 - T_0^4)$$

The law is known as Stefan-Boltzmann law.

The total radiation energy per unit volume emitted by the black body radiation chamber over all range of frequency or wavelength can be calculated by integrating equation (5) or (6) from the limit $\nu \to 0$ to $\nu \to \infty$ or $\lambda \to 0$ to $\lambda \to \infty$, i.e.

$$\begin{aligned} E &= \int_0^\infty E_\nu d\nu \\ &= \int_0^\infty \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kt}-1} d\nu \end{aligned}$$

Derivation of Stefan's law from Planck's law (contd.)

Let, $\frac{h\nu}{kT} = x \implies \nu = \frac{kT}{h}x$, hence

$$E = \frac{8\pi h}{c^3} \int_0^\infty \left(\frac{kTx}{h}\right)^3 \frac{1}{e^x - 1} \left(\frac{kT}{h}\right) dx$$
$$= \frac{8\pi k^4 T^4}{c^3 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx$$

But the integral is the standard integral and has the value $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$,

$$\therefore \qquad E = \frac{8\pi k^4 T^4}{c^3 h^3} \frac{\pi^4}{15} = \frac{8\pi^5 k^4 T^4}{15 c^3 h^3}$$

Derivation of Stefan's law from Planck's law (contd.)

The energy per second radiated by a unit area of the black body chamber is effectively equal to $I = \frac{1}{4}Ec$. Therefore, the rate of energy radiated per unit area is

$$I = \frac{1}{4}\frac{8\pi^5 k^4 T^4}{15c^3h^3}c = \frac{2\pi^5 k^4}{15c^2h^3}T^4 = \sigma T^4$$

Where

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} = \frac{2\pi^5 \times \left(1.38 \times 10^{-23}\right)^4}{\left(3 \times 10^8\right)^2 \times \left(6.62 \times 10^{-34}\right)^3} = 5.67 \times 10^{-8} Wm^{-2}K^{-4}$$