Lecture 02

Vector Analysis (Contd.)

Outline

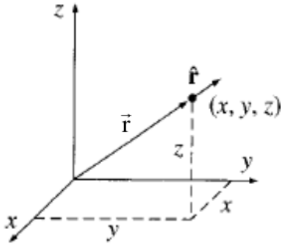
1. Position, Displacement, and Separation Vectors
   - Position Vector
   - Infinitesimal Displacement Vector
   - Separation Vector
2. The Operator ∇
3. Gradient
   - Geometrical Interpretation of the Gradient
4. The Divergence
   - Geometrical Interpretation of divergence
5. The Curl
   - Geometrical Interpretation of curl

Outline (contd.)

6. Product rules for gradient, divergence and curl

Position Vector

- The location of a point in three dimensions can be described by listing its Cartesian coordinates(x, y, z).



Position Vector: It is given by

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ with magnitude

$$r = \sqrt{x^2 + y^2 + z^2} = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}$$

The unit vector of it is

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Position, Displacement, and Separation Vectors

Infinitesimal Displacement Vector

- The infinitesimal displacement vector, from (x, y, z) to $(x + dx, y + dy, z + dz)$, is

$$\boxed{d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}}$$

Position, Displacement, and Separation Vectors

Separation Vector

- In electrodynamics we frequently encounter problems involving *two points* – typically, a source point, $\vec{r}'$, where an electric charge is located, and a field point, $\vec{r}$, at which we are calculating the electric or magnetic field (Figure 1).

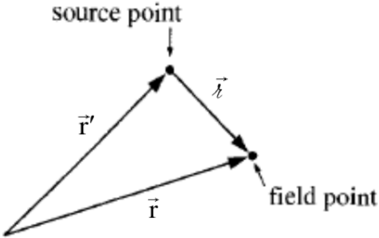


Figure 1

Position, Displacement, and Separation Vectors

Separation Vector (contd.)

- The **separation vector** from the source point to the field point is

$$\begin{aligned}\vec{\mathscr{r}} &= (\vec{r} - \vec{r}') \\ &= (x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}\end{aligned}$$

- The unit vector of separation vector is given by

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{(x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

The Operator ∇

- The vector differential operator del (nabla), defined in Cartesian coordinates as

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

  Of course, del is not a vector, in the usual sense. Indeed, it is without specific meaning until we provide it with a function to act upon.

- There are three ways the operator ∇ can act:
  1. On a scalar function T : ∇T **(the gradient)**;
  2. On a vector function $\vec{v}$, via the dot product: $\nabla \cdot \vec{v}$ **(the divergence)**;
  3. On a vector function $\vec{v}$, via the cross product: $\nabla \times \vec{v}$ **(the curl)**.

Gradient

Suppose that we have a function of three variables– say, the temperature $T(x,y,z)$ in a room. A theorem on partial derivatives states that

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz \tag{1}$$

This tells us how T changes when we alter all three variables by the infinitesimal amount dx, dy, dz.

Equation (1) can be written as

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) \\ &= (\nabla T) \cdot \left(d\vec{l}\right) \end{aligned}$$

Gradient (contd.)

where $\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$ is the gradient of T

Gradient

Geometrical Interpretation of the Gradient

$$dT = (\nabla T) \cdot \left(d\vec{l}\right) = |\nabla T| \left|d\vec{l}\right| \cos\theta$$

where θ is the angle between ∇T and $d\vec{l}$
Now, if we fix the magnitude $\left|d\vec{l}\right|$ and search around in various directions, the maximum change in T evidently occurs when $\theta = 0$ for then $\cos\theta = 1$. That is, for a fixed distance $\left|d\vec{l}\right|$, dT is greatest when we move in the same direction as ∇T. Thus:

The gradient ∇T points in the direction of maximum increase of the function T

Moreover:

The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

Example 1

Suppose that the temperature T at the point (x, y, z) is given by the equation $T = x^2 - y^2 + xyz + 273$. In which direction is the temperature increasing most rapidly at $(-1, 2, 3)$ and at what rate?

Solution:

Here, $T = x^2 - y^2 + xyz + 273$

$$\begin{aligned}
\nabla T &= \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k} \\
&= \frac{\partial}{\partial x}\left(x^2 - y^2 + xyz + 273\right)\hat{i} + \frac{\partial}{\partial y}\left(x^2 - y^2 + xyz + 273\right)\hat{j} \\
&\quad + \frac{\partial}{\partial z}\left(x^2 - y^2 + xyz + 273\right)\hat{k} \\
&= (2x + yz)\hat{i} + (-2y + xz)\hat{j} + (xy)\hat{k} \\
&= 4\hat{i} - 7\hat{j} - 2\hat{k} \quad \text{at } (-1, 2, 3)
\end{aligned}$$

Geometrical Interpretation of the Gradient (contd.)

The increase in temperature is fastest in the direction of this vector.

The rate of increase is

$$|\nabla T| = \sqrt{(4)^2 + (-7)^2 + (-2)^2} = \sqrt{69}$$

Note:

Gravitational Potential Energy near the Earth

$U = mgz$

where z is the height from some arbitrary reference level

$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$
$$= \frac{\partial}{\partial x}(mgz)\,\hat{i} + \frac{\partial}{\partial y}(mgz)\,\hat{j} + \frac{\partial}{\partial z}(mgz)\,\hat{k} = mg\hat{k}$$

Gravitational force, $\vec{F} = -mg\hat{k} = -mg\hat{k} = -\nabla U$ So, the maximum change in gravitational potential energy is vertically upwards from the centre of Earth.

Geometrical Interpretation of the Gradient (contd.)

Gradient of a scalar field T

$$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

- ∇, turns a scalar field into a vector field.
- ∇T points in the direction of maximum increase of T.
- $|\nabla T|$ is the rate of maximum increase.

The Divergence

The divergence of a vector $\vec{F}$, written div$\vec{F}$ or, $\nabla \cdot \vec{F}$ is defined as follows:

The divergence of a vector is the limit of its surface integral per unit volume as the volume enclosed by the surface goes to zero. That is,

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \lim_{V \to 0} \frac{1}{V} \oint_S \vec{F} \cdot d\vec{a}$$

The divergence is clearly a scalar point function (scalar field), and it is defined at the limit point of the surface integration.

In Cartesian coordinate it can be expressed as

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$$

The Divergence (contd.)

$$\nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \right)$$

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Geometrical Interpretation of divergence

- The divergence of a vector function $\vec{v}$, i.e. $\nabla \cdot \vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point in question.

For example,

The vector function in Figure 2 has a large positive divergence.

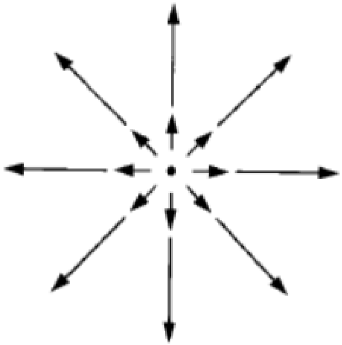


Figure 2

Geometrical Interpretation of divergence (contd.)

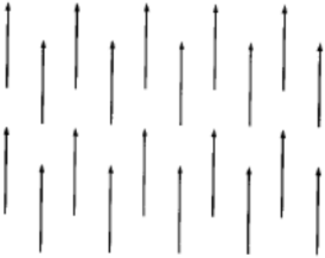


Figure 3

The vector function in Figure 3 has zero divergence.

Geometrical Interpretation of divergence (contd.)

- Imagine you are standing at the edge of a pond. Sprinkle some sawdust or pine needles on the surface. If the material spreads out, then you dropped it at a point of positive divergence; if it collects together, you dropped it at a point of negative divergence. (The vector function $\vec{v}$ in this model is the velocity of water.)
- A point of positive divergence is a source, or “faucet”; a point of negative divergence is a sink, or “drain”.

- If at some point P,

$$\nabla \cdot \vec{v} > 0, \text{ then } \vec{v} \text{ has a source at P}$$

$$\nabla \cdot \vec{v} < 0, \text{ then } \vec{v} \text{ has a sink at P.}$$

$$\nabla \cdot \vec{v} = 0, \text{ then } \vec{v} \text{ is said to be solenoidal}$$

1. Calculate the divergence of vector function $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$.

   Solution:

$$\begin{aligned}
\nabla \cdot \vec{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\
&= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

Example:

2. If $\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla \cdot \vec{A}$ at point $(1, -1, 1)$.

Solution:

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}\left(x^2 z\right) + \frac{\partial}{\partial y}\left(-2y^3z^2\right) + \frac{\partial}{\partial z}\left(xy^2 z\right)$$

$$= 2xz - 6y^2z^2 + xy^2$$

$$= 2\,(1)\,(1) - 6(-1)^2(1)^2 + (1)\,(-1)^2 \text{ at } (1, -1, 1)$$

$$= 2 - 6 + 1$$

$$= -3$$

The Curl

The curl of a vector function $\vec{F}$ is written as curl$\vec{F}$ or $\nabla \times \vec{F}$ and defined as follow.

The component of curl $\vec{F}$ in the direction of the unit vector $\hat{n}$ is the limit of a line integral per unit area , as the enclosed area goes to zero, this area being perpendicular to $\hat{n}$. That is,

$$\hat{n} \cdot \text{ curl } \vec{F} = \hat{n} \cdot (\nabla \times \vec{F}) = \lim_{S \to 0} \frac{1}{S} \oint_C \vec{F} \cdot d\vec{l}$$

where the curve C , which bounds the surface S, is in a plane normal to $\hat{n}$.

The Curl (contd.)

In Cartesian coordinate, the curl of a vector function $\vec{v}$ can be expressed as:

$$
\begin{aligned}
\text{curl } \vec{v} &= \nabla \times \vec{v} \\
&= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times \left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \right) \\
&= \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}
\end{aligned}
$$

- The curl of a vector function $\vec{v}$, $\nabla \times \vec{v}$ is a measure of how much the vector $\vec{v}$ "curls around" the point in question.
  For example,
  The vector function in Figure 4 has a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest.

Geometrical Interpretation of curl (contd.)

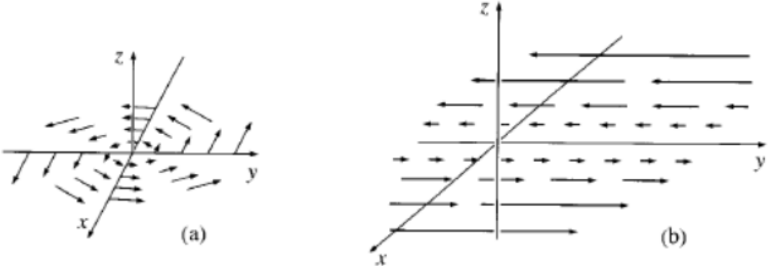


Figure 4

Geometrical Interpretation of curl (contd.)

- Imagine you are standing at the edge of a pond. Float a small paddle-wheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of nonzero curl. (The vector function $\vec{v}$ in this model is the velocity of water.)

  A whirlpool would be a region of large curl.
- If $\nabla \times \vec{v} = 0$, then $\vec{v}$ is irrotational.

The Curl

Examples:

1. Calculate curl of the vector function $\vec{v} = x\hat{j} - y\hat{i}$.

   Solution:

$$\nabla \times \vec{v} = \nabla \times \left[x\hat{j} - y\hat{i}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-y)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right]$$

$$= \hat{i}\,[0] - \hat{j}\,[0-0] + \hat{k}\,[1-(-1)] = 2\hat{k}$$

2. If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find $\nabla \times \vec{A}$ at point $(1, -1, 1)$.

   Solution:

$$\nabla \times \vec{v} = \nabla \times \left[xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(2yz^4\right) - \frac{\partial}{\partial z}\left(-2x^2yz\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(2yz^4\right) - \frac{\partial}{\partial z}\left(xz^3\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(-2x^2yz\right) - \frac{\partial}{\partial y}\left(xz^3\right)\right]$$

Examples:

$$
\begin{aligned}
&= \hat{i}\left[2z^4 + 2x^2y\right] - \hat{j}\left[0 - 3xz^2\right] + \hat{k}\left[-4xyz - 0\right] \\
&= \left[2z^4 + 2x^2y\right]\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k} \\
&= \left[2(1)^4 + 2(1)^2(-1)\right]\hat{i} + 3(1)(1)^2\hat{j} - 4(1)(-1)(1)\hat{k}, \text{ at } (1,-1,1) \\
&= 3\hat{j} + 4\hat{k}
\end{aligned}
$$

Product rules for gradient, divergence and curl

There are *six* product rules as shown in Eq. (2) to (7), two for gradients:

$$\nabla(fg) = f\nabla g + g\nabla f \tag{2}$$

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} \tag{3}$$

two for divergences:

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f) \tag{4}$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \tag{5}$$

two for curls:

$$\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f) \tag{6}$$

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) \tag{7}$$

End of Lecture 02

Thank you