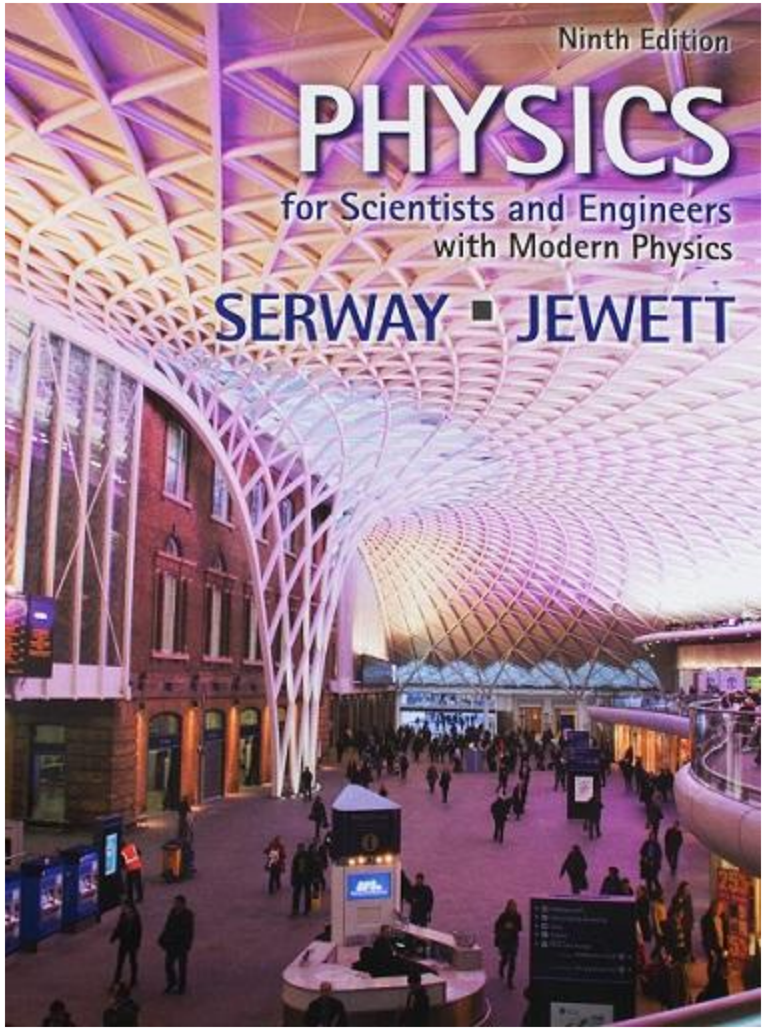
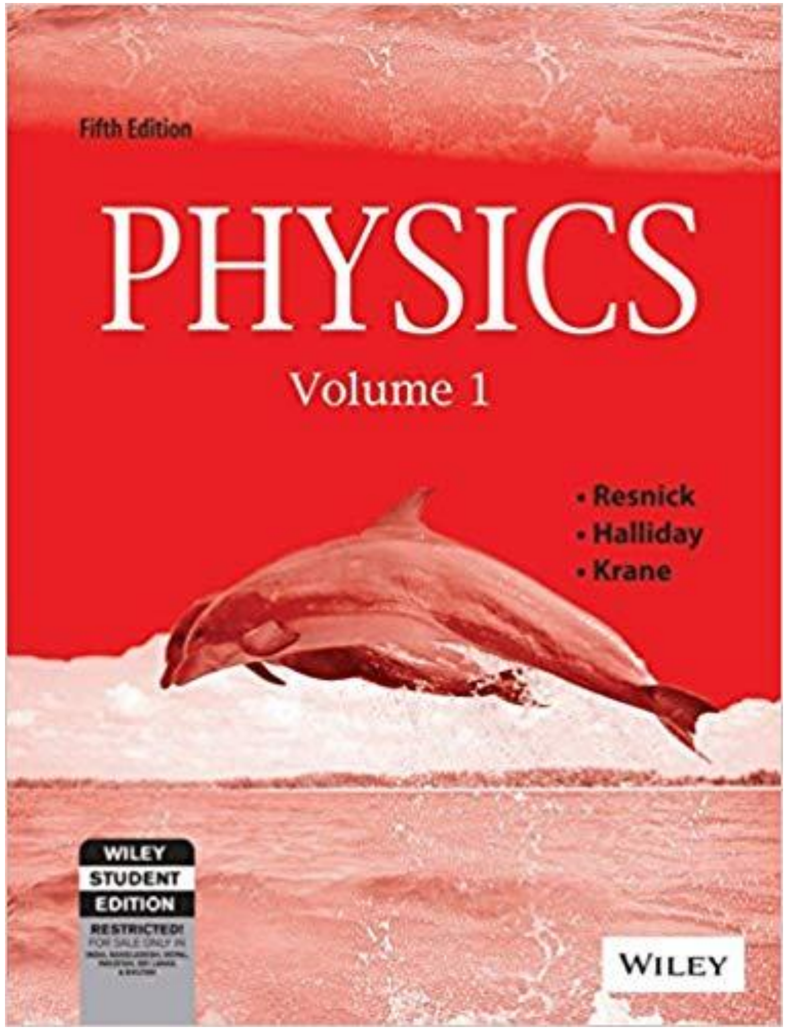
PHYSICS

General Physics I (PHYS 101)





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Course Outline

- Rigid Object
- Angular Speed
- Torque
- Angular Momentum
- Rotational Kinetic Energy
- Moment of Inertia
- Perpendicular Axes Theorem and Parallel Axes Theorem of Moment of Inertia
- Calculation of Moment of Inertia of :
    - Uniform Rigid Rod
    - Circular Ring
    - Homogeneous Circular Disc
    - Solid Sphere

Angular Speed of the Rigid Object

Angular Speed of the Rigid Object

- Figure R_D.1 illustrates a planar (flat), rigid object of arbitrary shape confined to the *xy* plane and rotating about a fixed axis through *O*.

  Let *O* be the origin of an *xy* coordinate system and axis of rotation is perpendicular to the xy-plane.

- Let us look at the motion of only one of the millions of "particles" making up this object.

  A particle at *P* is at a fixed distance *r* from the origin and rotates about it in a circle of radius *r*.

- As the particle moves along the circle from the positive *x-axis* $(\theta = 0)$ to *P*, it moves through an arc of length *s*, which is related to the angular position θ through the relationship

$$\boxed{s = r\theta}$$

$$\therefore \theta = \frac{s}{r} \qquad \text{(rad)}$$

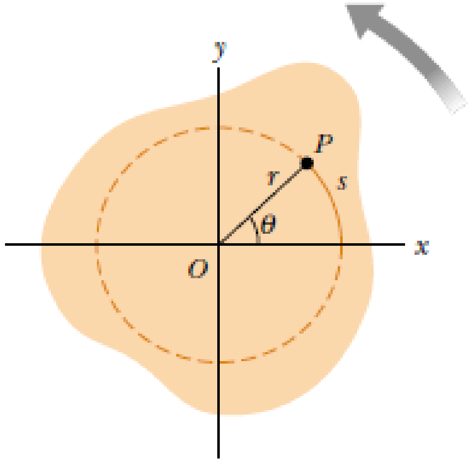


Figure R_D-1

Rigid Object:

- Rigid Object – Nondeformable
- All particles of the rigid body have the same angular velocity.

- Angular Speed: $\omega = \dfrac{d\theta}{dt}$

- Linear Speed: $v = \dfrac{ds}{dt} = r\dfrac{d\theta}{dt} \qquad \because s = r\theta$

  $\therefore \boxed{v = r\omega}$

That is, the tangential speed of a point on a rotating rigid object equals the perpendicular distance of that point from the axis of rotation multiplied by the angular speed.

Torque

Torque

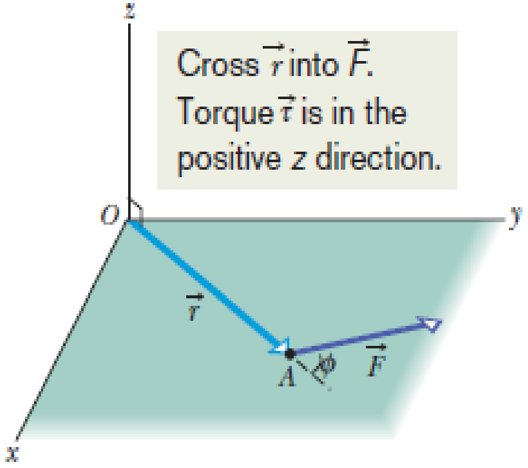
ROTATIONAL DYNAMICS

- The tendency of a force to rotate an object about some axis is measured by a vector quantity called **torque** .

  It is defined relative to a fixed point (usually an origin); it is

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

where $\vec{F}$ is a force applied to a particle and $\vec{r}$ is a position vector locating the particle relative to the fixed point.

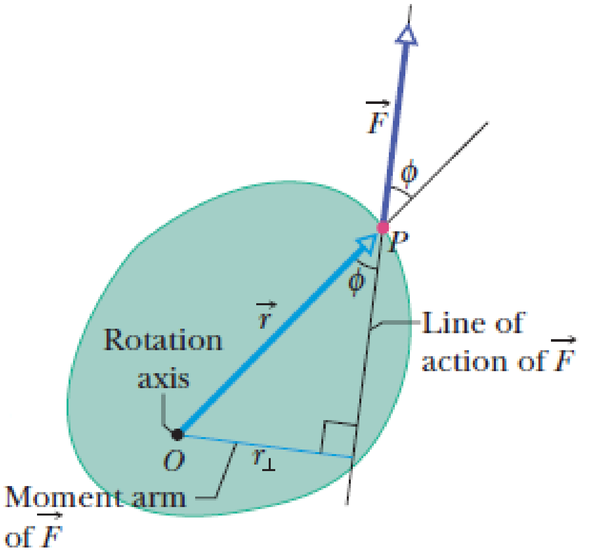


- The direction of $\vec{\tau}$ given by the right-hand rule for cross products.
- The SI unit of torque is the newton-meter (N-m).
- Magnitude of the Torque: $\boxed{\tau = rF\sin\phi = r_{\perp}F}$

where ϕ is the angle between $\vec{r}$ and $\vec{F}$.

$r_{\perp}$ is the moment of arm of $\vec{F}$

[The perpendicular distance between the rotation axis and an extended line running through the vector (line of action of $\vec{F}$).]



Angular Momentum

Angular Momentum of a Particle

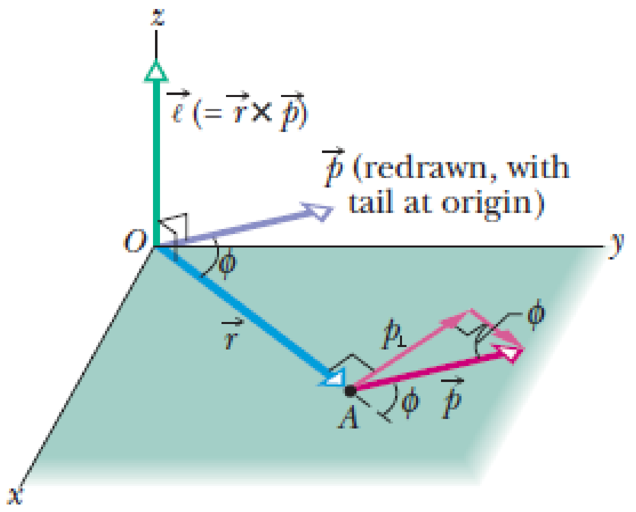
- The angular momentum of a particle $\vec{l}$ with linear momentum $\vec{p}$, mass m, and linear velocity $\vec{v}$ is a vector quantity defined relative to a fixed point (usually an origin) as

$$\boxed{\vec{l} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}}$$

- The magnitude of $\vec{l}$ is given by

$$\boxed{l = rmv\sin\phi = r_{\perp}p = rp_{\perp}}$$

where ϕ is the angle between $\vec{r}$ and $\vec{p}$, $p_{\perp}$ is the component of $\vec{p}$ perpendicular to $\vec{r}$, and $r_{\perp}$ is the perpendicular distance between the fixed point and the extension of $\vec{p}$.



(a)

- **Newton's second law** for a particle can be written in angular form as

$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt},$$

where $\vec{\tau}_{net}$ is the net torque acting on the particle and $\vec{l}$ is the angular momentum of the particle.

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of theangular momentum of that particle.

Angular Momentum

Angular Momentum of a System of Particles

- The angular momentum $\vec{L}$ of a system of particles is the vector sum of the angular momenta of the individual particles:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i$$

- The time rate of change of the angular momentum of a system of particles is equal to the net external torque on the system:

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum

- The angular momentum of a system remains constant if the net external torque acting on the system is zero:

$$\vec{L} = \text{a constant} \qquad \text{(isolated system)}$$

or,

$$\vec{L}_i = \vec{L}_f \qquad \text{(isolated system)}$$

$$\begin{pmatrix}\text{net angular momentum} \\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{net angular momentum} \\ \text{at some initial time } t_f\end{pmatrix}$$

$$\boxed{I_i\omega_i = I_f\omega_f = \text{constant}}$$

$$\boxed{L = I\omega \qquad \text{(rigid body, fixed axis)}}$$

Rotational Kinetic Energy

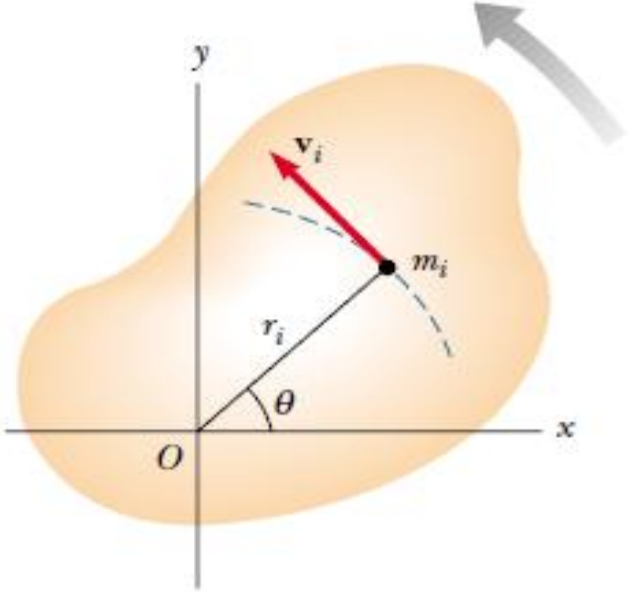
Rotational Kinetic Energy

- Let us consider a rigid object as a collection of particles and assume it rotates about a fixed z-axis with an angular speed ω.
- The kinetic energy of the *i*th particle of mass m_i and speed v_i is

$$\boxed{K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i (r_i\omega)^2 = \frac{1}{2}m_i r_i^2\omega^2}$$

where ω is the body's angular speed.

and r_i is the perpendicular distance of ith particle from the axis of rotation (z-axis).



- The rotational kinetic energy of the rigid object is the sum of the kinetic energies of the individual particles:

$$\boxed{K_R = \sum_i K_i = \sum_i \frac{1}{2}m_i r_i^2\omega^2 = \frac{1}{2}\left(\sum_i \frac{1}{2}m_i r_i^2\right)\omega^2 = \frac{1}{2}I\omega^2}$$

where $I = \sum_i \frac{1}{2}m_i r_i^2$ is the moment of inertia of the rigid object for this rotation axis.

- The SI unit of moment of inertia is the kilogram-meter² $(\mathrm{kg \cdot m^2})$.
- The greater the moment of inertia, the greater the kinetic energy of a rigid body rotating with a given angular speed.
- The greater a body's moment of inertia, the harder it is to start the body rotating if it's at rest and the harder it is to stop its rotation if it's already rotating. For this reason, *I* is also called the *rotational inertia*.

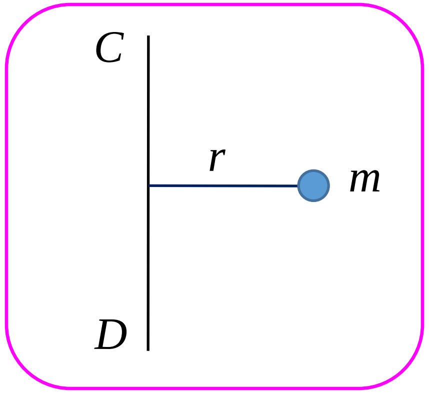
Rotational Kinetic Energy

- The **moment of inertia** is a measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion.
- **For a Particle,**

The moment of inertia of a particle of mass m about an axis of rotation (CD) is

$$\boxed{I_{CD} = mr^2}$$

where r is the perpendicular distance of the particle from the axis of rotation.



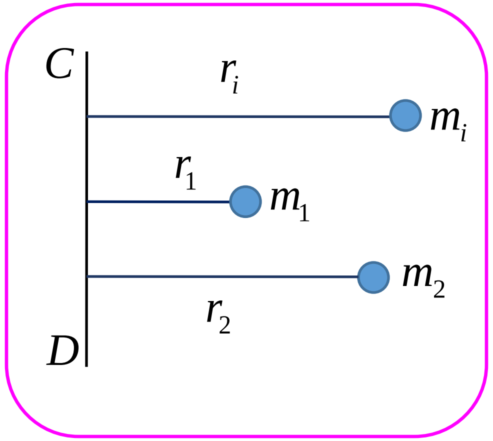
For a system of discrete particles,

The moment of inertia of a system of particles about an axis of rotation (CD) is

$$\boxed{I_{CD} = \sum_i m_i r_i^2}$$

where r_i is the mass of the ith particle and

m_i is the perpendicular distance of the ith particle from the axis of rotation



Moment of Inertia

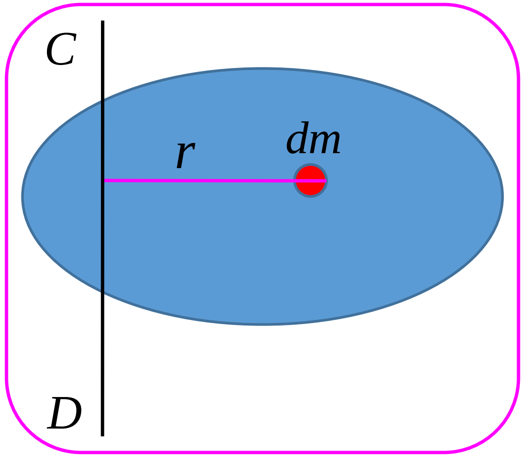
Moment of Inertia of Rigid Object

- The moment of inertia of a rigid object about an axis of rotation (CD) is

$$\boxed{I_{CD} = \int r^2 dm = \int r^2 \rho dV}$$

where dm is the mass of an infinitesimal element of the body and

r is the perpendicular distance of the dm from axis of rotation



- The moment of inertia of a body plays the same role in rotational motion as mass does in linear motion.

$$\left\{\begin{array}{l}\text{Kinetic energy associated with liner motion, } \mathrm{K}=\frac{1}{2}m\mathrm{v}^2 \\ \text{Rotational kinetic energy, } \mathrm{K}_R=\frac{1}{2}I\omega^2\end{array}\right\} \rightarrow \left\{\begin{array}{l}\text{The quantities } I \text{ and } \omega \text{ in rotational motion} \\ \quad\text{are analogous} \\ \text{to } m \text{ and v in linear motion,respectively}\end{array}\right\}$$

- Mass is an intrinsic property of an object, whereas I depends on how the body's mass is distributed in space (choice of axis of rotation).

Radius of Gyration

Radius of Gyration

- The **radius of gyration**, k, of a body about a given axis of rotation is that radial distance from the axis the square of which on being multiplied by the total mass of the body gives the moment of inertia of the body about that axis.

  Thus
  $$\boxed{I = Mk^2 = \sum mr^2}$$

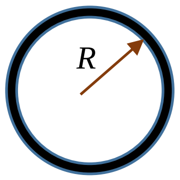
  Where M is the total mass of the body. This means that

  $$k = \sqrt{\left(\frac{I}{M}\right)}$$

- **Radius of Gyration** of a body about an axis of rotation is defined as the radial distance of a point from the axis of rotation at which, if whole mass of the body is assumed to be concentrated, its moment of inertia about the given axis would be the same as with its actual distribution of mass.

- The moment of inertia of the circular ring about an axis Oz passing through the centre of mass and perpendicular to the plane of the ring is

  $$I_z = MR^2$$

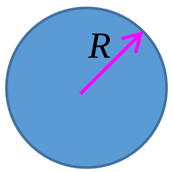


  The radius of gyration of the circular ring about an axis Oz:

  $$k = R \qquad \boxed{\because I = Mk^2 = MR^2}$$

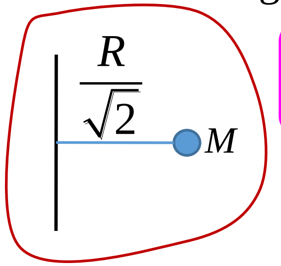
- The moment of inertia of the homogeneous circular disk about an axis Oz passing through the centre of mass and perpendicular to the plane of the disk is

  $$I_z = \frac{1}{2}MR^2$$



  The radius of gyration of the homogeneous circular disk about an axis Oz:

  $$k = \frac{R}{\sqrt{2}} \qquad \boxed{\because I = Mk^2 = \frac{1}{2}MR^2}$$



Perpendicular Axes Theorem of Moment of Inertia

Perpendicular Axes Theorem

- The moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane, and intersecting each other at the point where the perpendicular axis passes through it.
- If I_x and I_y be the moments of inertia of a plane lamina (Figure R_P.1), about the perpendicular Ox and Oy respectively, which lie in the plane of the lamina and intersect each other at O, its moment of inertia (I_z) about an axis passing through and perpendicular to its plane is given by

$$\boxed{I_z = I_x + I_y}$$

Proof:

Figure RP-1 shows a plane lamina of mass M in the xy-plane .

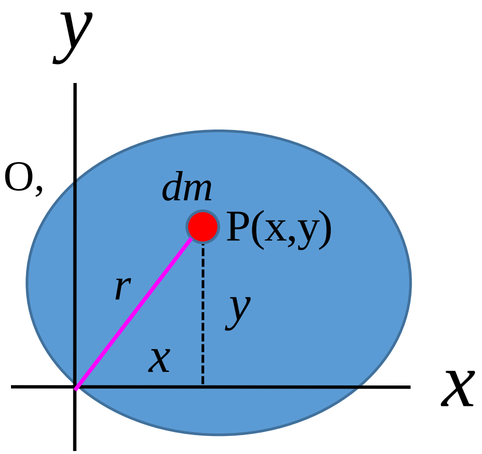


Figure RP-I

- Let Ox and Oy be the two perpendicular axes in the plane of the lamina and intersect each other at O, and Oz an axis passing through O and perpendicular to the plane of the lamina.
- Let the mass element dm have coordinates (x, y ,0).
- The moment of inertia of the plane lamina about an axis Oz is

$$I_z = \int r^2 dm = \int \left(y^2 + x^2\right)^2 dm = \int y^2 dm + \int x^2 dm$$

$$= I_x + I_y$$

where I_x is the moment of inertia of plane lamina about an axis Ox, and

I_y is the moment of inertia of plane lamina about an axis Oy

$\therefore \quad \boxed{I_z = I_x + I_y}$

$r \rightarrow$ perpendicular distance of dm to the axis Oz

Parallel Axes Theorem of Moment of Inertia

Parallel Axes Theorem

- The moment of inertia of any body about an arbitrary axis equals the moment of inertia about a parallel axis through the centre of mass plus the total mass times the squared distance between the two axes.
- Let I_{cm} be the moment of inertia of a body about some axis through its centre of mass, and I be the moment of inertia of the body about a parallel axis. Then

$$\boxed{I = I_{cm} + Md^2}$$

where M is the total mass of the body and d is the perpendicular distance between the two parallel axes.

Proof:

Figure RP-2 shows a thin slab in the xy-plane which rotates in the xy-plane about the z-axis.

$Oz \rightarrow$ an axis passing through the point O and perpendicular to the thin slab

$O'z' \rightarrow$ an axis passing through the point O' and parallel to the axis Oz

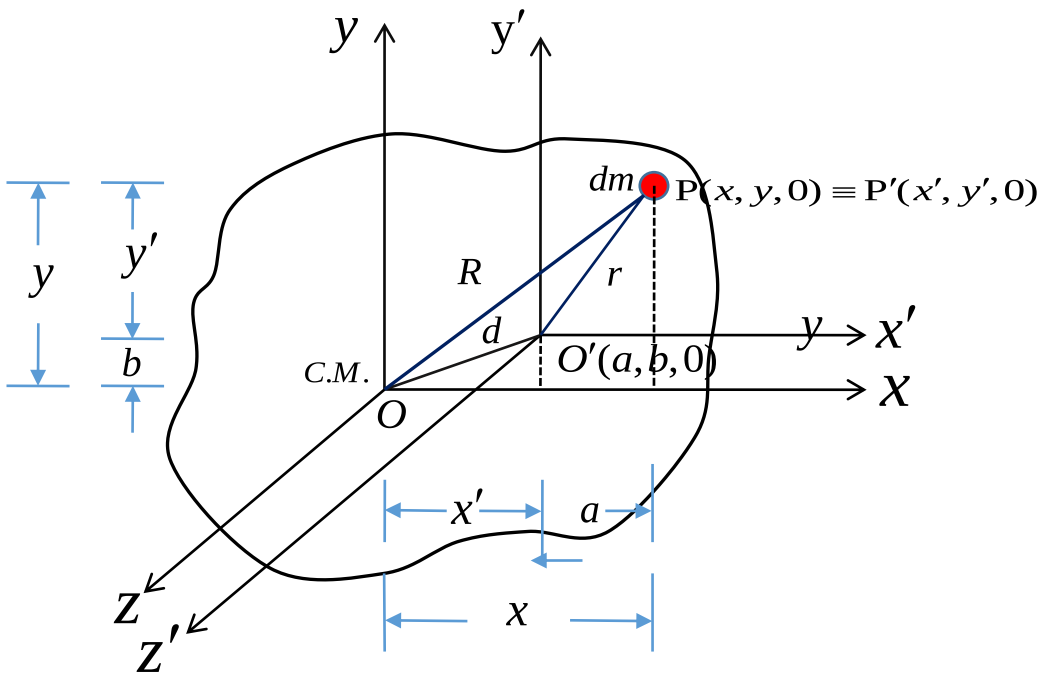


Figure RP-2

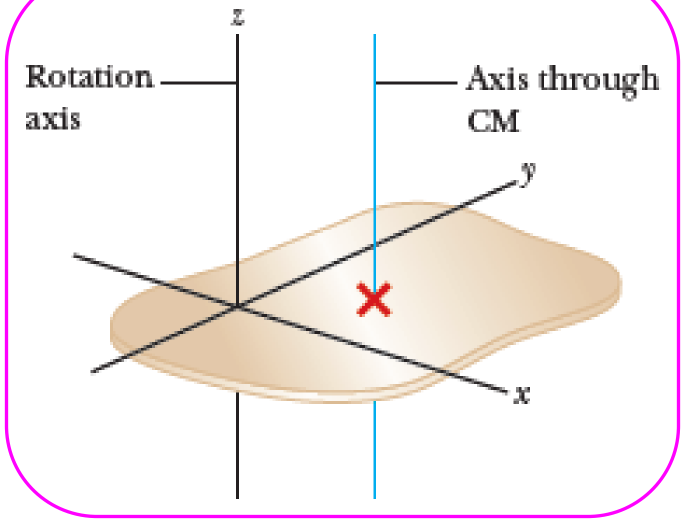
Parallel Axes Theorem of Moment of Inertia

Parallel Axes Theorem

- We take the origin of our coordinate system to be at the center of mass of the body; the coordinates of the center of mass are then $x_{cm} = y_{cm} = z_{cm} = 0$.

  The mass element dm is located at coordinates $(x, y, 0)$ relative to the origin O and at coordinates $(x', y', 0)$ relative to the origin O'. Relative to O, the point O' has the coordinate a and b.

  The moment of inertia of the body about an axis Oz is



$$
\begin{aligned}
I &= \int r^2 dm \\
&= \int \left(x'^2 + y'^2\right) dm \\
&= \int \left[\left(x-a\right)^2 + \left(y-b\right)^2\right] dm \\
&= \int \left[\left(x^2 - 2ax + a^2\right) + \left(y^2 - 2by + b^2\right)\right] dm \\
&= \int \left(x^2 + y^2\right) dm + \int \left(a^2 + b^2\right) dm - 2a\int x\, dm - 2b\int y\, dm \\
&= \int R^2 dm + d^2 \int dm \qquad \left[\because x_{cm} = \frac{1}{M}\int x\, dm \quad \& \quad y_{cm} = \frac{1}{M}\int y\, dm \right]
\end{aligned}
$$

$$\therefore \quad \boxed{I = I_{cm} + Md^2}$$

ROTATIONAL DYNAMICS

Moment of Inertia of Uniform Rigid Rod

M.I. of Uniform Rigid Rod

- Consider a uniform rigid rod of length L and mass M [Figure M-1].

$A'B' \rightarrow$ an axis passing through centre of mass and perpendicular to the rod

$A'B' \rightarrow$ an axis passing through one end of the rod and parallel to the axis

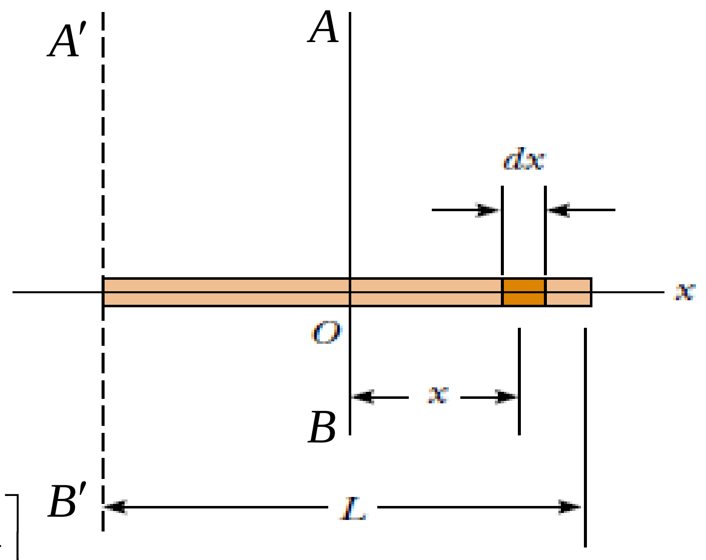


Figure M-1

- Let's take an element of the rod of length dx at a distance x from the axis AB.

  The mass of the element of the rod of length dx is

$$dm = \frac{M}{L}dx \qquad \left[\because \frac{\text{element's mass}}{\text{element's length}} = \frac{\text{rod's mass}}{\text{rod's length}} \Rightarrow \frac{dm}{dx} = \frac{M}{L}\right]$$

- The moment of inertia of the element of the rod of length dx and mass dm is

$$dI = x^2 dm$$

$$\therefore \quad \boxed{dI = x^2 \frac{M}{L} dx} \quad .... \ (1)$$

Moment of Inertia of Uniform Rigid Rod

M.I. of Uniform Rigid Rod

- The moment of inertia of the uniform rod about an axis passing through the centre of mass (AB) is

$$I_{\mathrm{AB}} = I_{cm} = \int_{L/2}^{L/2} x^2 dm = \frac{M}{L}\int_{L/2}^{L/2} x^2 dx = \frac{M}{L}\left[2\int_0^{L/2} x^2 dx\right] = 2\frac{M}{L}\left[\frac{x^3}{3}\right]_0^{L/2} = 2\frac{M}{L}\left[\frac{\left(L/2\right)^3}{3}\right]$$

$$\therefore \quad \boxed{I_{cm} = \frac{1}{12}ML^2}$$

- The moment of inertia of a uniform rigid rod about an axis $\mathrm{A'B'}$

$$I_{\mathrm{A'B'}} = I_{cm} + M\left(\frac{L}{2}\right)^2 \qquad \text{[using parallel axes theroem]}$$

$$= \frac{1}{12}ML^2 + \frac{1}{4}ML^2$$

$$\therefore \quad \boxed{I_{cm} = \frac{1}{3}ML^2}$$

Moment of Inertia of Uniform Rigid Rod

M.I. of Uniform Rigid Rod

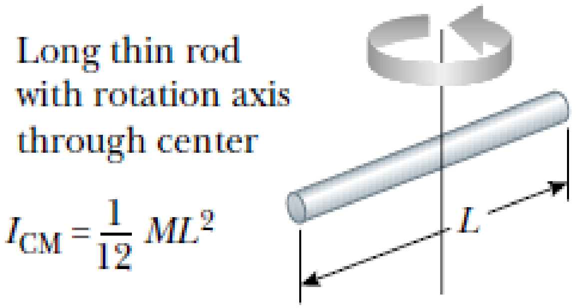
- The radius of gyration of a uniform rigid rod about an axis (AB) passing through the centre of mass and perpendicular to the rod is

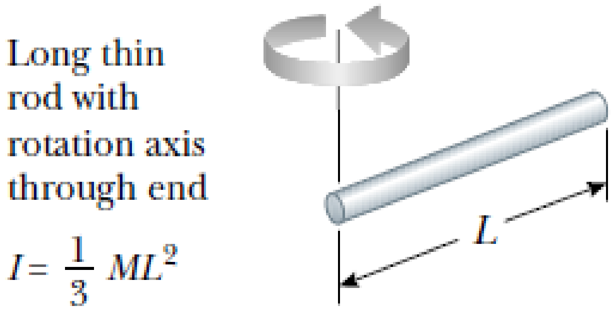
$$K_{AB} = \sqrt{\frac{I_{AB}}{M}} = \sqrt{\frac{\left(\frac{1}{12}ML^2\right)}{M}} \qquad \therefore \boxed{K_{AB} = \frac{1}{2\sqrt{3}}L}$$

- The radius of gyration of a uniform rigid rod about an axis ($A'B'$):

$$K_{A'B'} = \sqrt{\frac{I_{A'B'}}{M}} = \sqrt{\frac{\left(\frac{1}{3}ML^2\right)}{M}} \qquad \therefore \boxed{K_{A'B'} = \frac{1}{\sqrt{3}}L}$$

- Notes:





ROTATIONAL DYNAMICS

Moment of Inertia of Circular Ring

M.I. of Circular Ring

- Consider a circular ring of radius R and mass M on xy-plane with centre of mass at origin O (Figure R-1).

Oz $\rightarrow$ an axis passing through the centre of mass and perpendicular to the plane of the ring

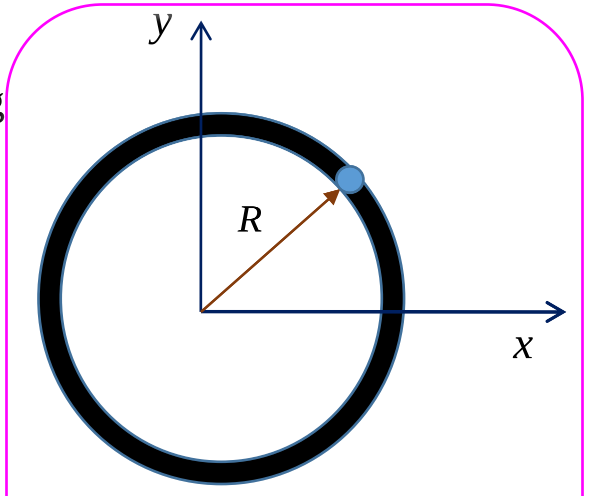


Figure R-1

- Let's take an element of the ring of mass dm . Let the mass element dm have the co-ordinates $(x, y, 0)$.

The moment of inertia of a small element of the ring about an axis Oz is

$$dI_z = R^2 dm \qquad .... \quad (1)$$

The moment of inertia of the ring about an axis (Oz) passing through the centre of mass and perpendicular to the plane of the ring is

$$I_z = \int dI_z = \int R^2 dm = R^2 \int dm$$

$$\therefore \quad \boxed{I_z = MR^2}$$

- The ring is symmetrical about both x- and y-axes. So the moment of inertia of the ring about x- and y-axes are the same.

$$i.e. \quad I_x = I_y$$

ROTATIONAL DYNAMICS

Moment of Inertia of Circular Ring

M.I. of Circular Ring

- Using perpendicular axes theorem of moment of inertia,

$$I_x + I_y = I_z$$

$$\text{or,} \quad 2I_x = I_z$$

$$\text{or,} \quad I_x = \frac{1}{2} I_z = \frac{1}{2}\left[MR^2\right]$$

$$\therefore \quad \boxed{I_x = \frac{1}{2} MR^2}$$

- Hence the moment of inertia of the ring about *x*- and *y*-axes i.e. diameter is

$$\boxed{I_d = \frac{1}{2} MR^2}$$

- The moment of inertia of the ring about a tangent and parallel to the diameter of the ring is

$$I_T = I_{cm} + M(R)^2 \qquad \text{[using parallel axes theroem]}$$

$$= I_d + MR^2 = \frac{1}{2} MR^2 + MR^2$$

$$\therefore \quad \boxed{I_T = \frac{3}{2} MR^2}$$

ROTATIONAL DYNAMICS

Moment of Inertia of Homogeneous Circular Disc

M.I. of Homogeneous Circular Disc

- Consider a homogeneous circular disc of radius and mass on xy-plane with centre of mass at origin O [Figure MD-1].

  Oz $\rightarrow$ an axis passing through the centre of mass and perpendicular to the plane of the disc

- Let us take a ring of radius r thickness dr as shown in Figure M_D-1.

  The mass of the ring,

$$dm = \frac{M}{\pi R^2}(2\pi \mathrm{r}\ \mathrm{dr}) = \frac{2M}{R^2}\mathrm{r}\ \mathrm{dr} \qquad \left[\because \frac{\text{ring's mass}}{\text{ring's area}} = \frac{\text{disc's mass}}{\text{disc's area}} \Rightarrow \frac{dm}{2\pi \mathrm{r}\ \mathrm{dr}} = \frac{M}{\pi R^2}\right]$$

- The moment of inertia of the ring about an axis Oz is

$$dI_z = \mathrm{r}^2 dm = \mathrm{r}^2\left[\frac{2M}{R^2}\mathrm{r}\ \mathrm{dr}\right]$$

$$\therefore \boxed{dI_z = \frac{2M}{R^2}\mathrm{r}^3\ \mathrm{dr}} \qquad \text{..........} \quad (1)$$

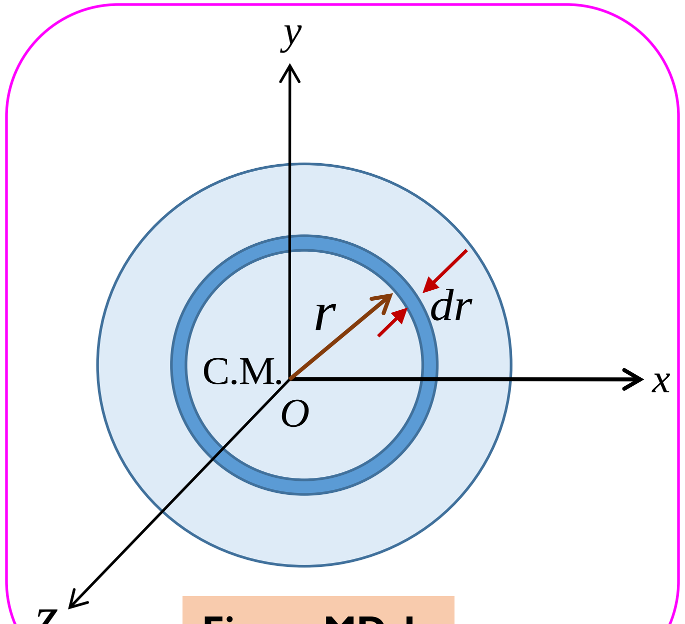


Figure MD-I

Moment of Inertia of Homogeneous Circular Disc

M.I. of Homogeneous Circular Disc

- The moment of inertia of the homogeneous circular disc about an axis (Oz) passing through the centre of mass and perpendicular to the plane of the disc is

$$I_z = \int dI_z = \int_0^R \left[\frac{2M}{R^2} r^3 \text{ dr}\right] = \frac{2M}{R^2}\int_0^R r^3 \text{ dr} = = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R = \frac{2M}{R^2}\left[\frac{R^4}{4}\right]$$

$$\therefore \boxed{I_z = \frac{1}{2}MR^2}$$

- The disc is symmetrical about both x- and y-axes .

  Therefore, the moment of inertia of the homogeneous circular disc about x- and y-axes are the same.

$$i.e.\ \ I_x = I_y$$

- Using perpendicular axes theorem of moment of inertia,

$$I_x + I_y = I_z$$

$$\text{or,}\qquad 2I_x = I_z$$

$$\text{or,}\qquad I_x = \frac{1}{2}I_z = \frac{1}{2}\left[\frac{1}{2}MR^2\right]$$

$$\therefore\qquad \boxed{I_x = \frac{1}{4}MR^2}$$

ROTATIONAL DYNAMICS

Moment of Inertia of Homogeneous Circular Disc

M.I. of Homogeneous Circular Disc

- Hence the moment of inertia of the homogeneous circular disc about x- and y-axes , i.e. diameter is

$$\boxed{I_d = \frac{1}{4}MR^2}$$

- The moment of inertia of the homogeneous circular disc about a tangent is

$$I_T = I_{cm} + M\left(R\right)^2 \qquad \text{[using parallel axes theroem]}$$

$$= I_d + MR^2$$

$$= \frac{1}{4}MR^2 + MR^2$$

$$\therefore \quad \boxed{I_T = \frac{5}{4}MR^2}$$

Hence the moment of inertia of the homogeneous circular disc about a tangent is $I_T = \frac{5}{4}MR^2$.

Moment of Inertia of a Solid Sphere

M.I. of a Solid Sphere

- Consider a solid sphere of radius and mass with centre of mass at origin O [Figure MS-1].

  The solid sphere is symmetrical about *x*-, *y*-, and *z*-axes.

  Therefore, the moment of inertia of solid sphere about all axes is the same.

- Let us take a circular disc of radius *r* thickness *dy* at a distance *y* from O

  as shown in Figure MS-1.

  The mass of the disc,

$$dm = \frac{M}{\frac{4}{3}\pi R^3}\left(\pi r^2\ dy\right) \qquad \left[\because \frac{\text{disc's mass}}{\text{disc's volume}} = \frac{\text{sphere's mass}}{\text{sphere's volume}}\right.$$

$$= \frac{3M}{4R^3} r^2\ dy \qquad \left.\Rightarrow \frac{dm}{\pi r^2\ dy} = \frac{M}{\frac{4}{3}\pi R^3}\right]$$

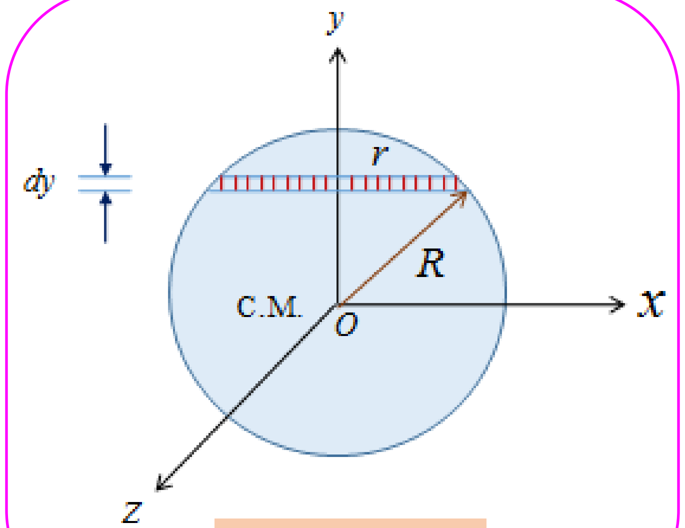


Figure MS-I

- The moment of inertia of the disc about an axis passing through the centre of mass of the disc and perpendicular to the plane of the disc (O*y*) is

$$dI_y = \frac{1}{2} dm\ r^2 = \frac{1}{2}\left[\frac{3M}{4R^3} r^2\ dy\right] r^2 = \frac{3M}{8R^3}\left(r^2\right)^2\ dy = \frac{3M}{8R^3}\left(R^2 - y^2\right)^2\ dy \qquad \left[\because r^2 = R^2 - y^2\right]$$

$$\therefore \boxed{dI_y = \frac{3M}{8R^3}\left(R^4 - 2R^2y^2 + y^4\right)dy} \quad \text{.....................} \quad (1)$$

Moment of Inertia of a Solid Sphere

M.I. of a Solid Sphere

- The moment of inertia of the solid sphere about an axis (Oy) i.e. diameter is

$$I_d = I_y = \int dI_y$$

$$= \int_{-R}^{R} \frac{3M}{8R^3}\left(R^4 - 2R^2y^2 + y^4\right) dy$$

$$= \frac{3M}{8R^3}\left[2\int_0^R \left(R^4 - 2R^2y^2 + y^4\right) dy\right]$$

$$\left[\begin{array}{l}\text{For even function } f(y): \\ \quad \int_{-\alpha}^{\alpha} f(y)\,dy = 2\int_0^{\alpha} f(y)\,dy\end{array}\right]$$

$$= \frac{3M}{4R^3}\left[R^4 y - 2R^2\frac{y^3}{3} + \frac{y^5}{5}\right]_0^R$$

$$= \frac{3M}{4R^3}\left[R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right]$$

$$= \frac{3M}{4R^3}R^5\left[1 - \frac{2}{3} + \frac{1}{5}\right]$$

$$= \frac{3}{4}MR^2\left[\frac{8}{15}\right]$$

$$\therefore \quad \boxed{I_d = \frac{2}{5}MR^2}$$

Moment of Inertia of a Solid Sphere

M.I. of a Solid Sphere

- Hence the moment of inertia of the solid sphere about diameter is $I_d = \frac{2}{5}MR^2$.

- The moment of inertia of the solid sphere about a tangent is

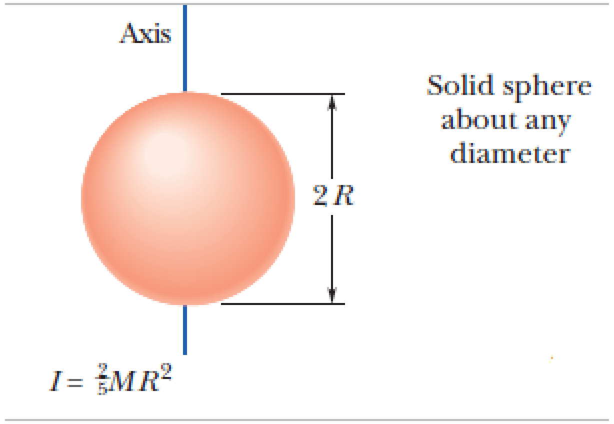
$$I_T = I_{cm} + M(R)^2 \qquad \text{[using parallel axes theroem]}$$

$$= I_d + MR^2$$

$$= \frac{2}{5}MR^2 + MR^2$$

$$\therefore \quad \boxed{I_T = \frac{7}{5}MR^2}$$

- Note:



ROTATIONAL DYNAMICS

Sample Problem

- **Two masses *M* and *m* are connected by a rigid rod of length *L* and of negligible mass, as shown in Figure MI-1. For an axis perpendicular to the rod, show that the system has the minimum moment of inertia when the axis passes through the centre of mass. Show that this moment of inertia is**

$$\boxed{I = \left(\frac{mM}{m+M}\right)L^2}$$

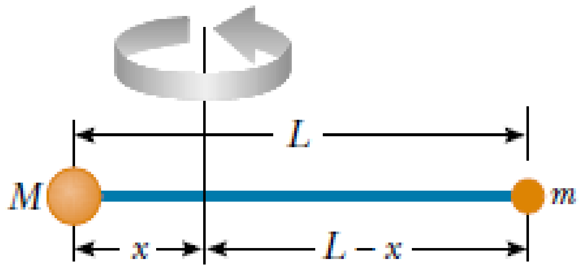


Figure MI-1

Hint:

The moment of inertia of the system of two masses M and m about an axis perpendicular to the rod and at a distance x from M is

$$I = Mx^2 + m(\mathrm{L} - \mathrm{x})^2 \quad \text{....................} \quad (1)$$

For minimum value of I,

$$\frac{dI}{dx} = 0$$

or, $2Mx + 2m(L - x)(-1) = 0$

or, $2Mx - 2mL + 2mx = 0$

or, $2x(M + m) = 2mL$

$$\therefore \boxed{x = \frac{mL}{M+m}} \quad \text{....................} \quad (2)$$

Sample Problem

- **Two masses *M* and *m* are connected by a rigid rod of length *L* and of negligible mass, as shown in Figure MI-1. For an axis perpendicular to the rod, show that the system has the minimum moment of inertia when the axis passes through the centre of mass. Show that this moment of inertia is**

$$\boxed{I = \left(\frac{mM}{m+M}\right)L^2}$$

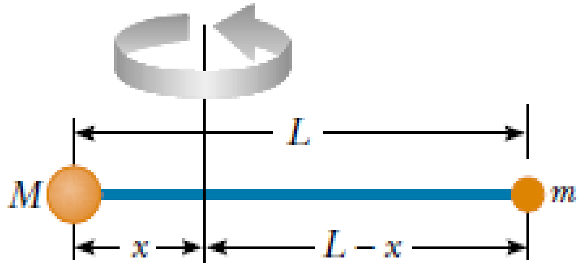
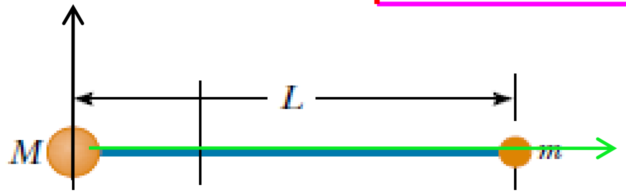


Figure MI-1

Hint:

The centre of mass of the system of two masses *M* and *m*, which are connected by a rigid rod of length *L* and of negligible mass, is

$$x_{cm} = \frac{M\times 0 + m\times L}{M+m} = \frac{mL}{M+m} \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$



$(x_1, y_1) = (0,0)$ $\quad$ $(x_2, y_2) = (L,0)$

Equations (2) and (3) shows that, for an axis perpendicular to the rod, the system has the minimum moment of inertia when the axis passes through the centre of mass.

The moment of inertia of the system of two masses *M* and *m* about an axis perpendicular to the rod and passing through the centre of mass is

$$I = Mx^2 + m(L-x)^2 = M\left(\frac{mL}{M+m}\right)^2 + m\left[L - \left(\frac{mL}{M+m}\right)\right]^2 \qquad \therefore \boxed{I = \left(\frac{mM}{m+M}\right)L^2}$$

Sample Problem

- Two identical solid spheres of mass M and radius R are joined together, and the combination is rotated about an axis tangent to one sphere and perpendicular to the line connecting them (Figure RI-2). What is the rotational inertia of the combination?

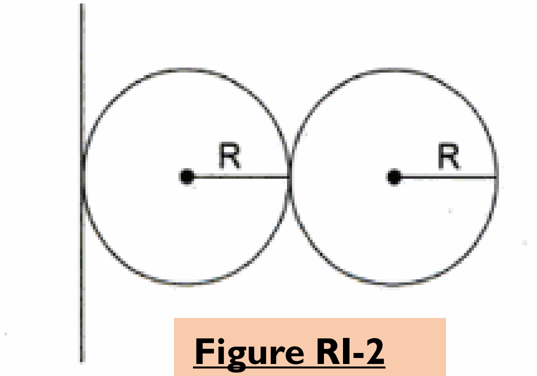
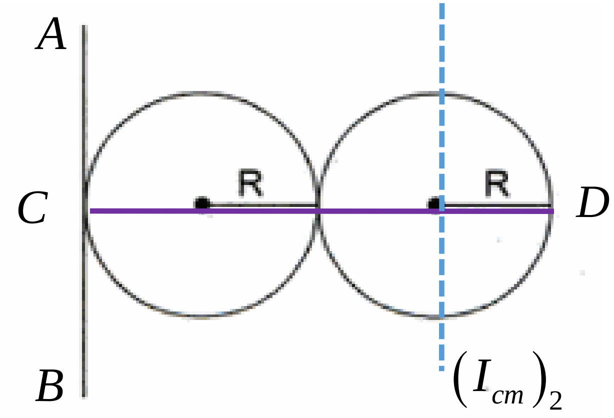


Figure RI-2

Hint:



The rotational inertia of the combination about an axis (AB) tangent to first sphere and perpendicular to the line (CD) connecting two identical spheres is

$$I = I_1 + I_2 = \frac{7}{5}MR^2 + \left[(I_{cm})_2 + M(3R)^2\right]$$

$$= \frac{7}{5}MR^2 + \left[\frac{2}{5}MR^2 + 9MR^2\right] \quad = \frac{7}{5}MR^2 + \left[\frac{47}{5}MR^2\right]$$

$$\therefore \quad \boxed{I = \frac{54}{5}MR^2}$$

Sample Problem

- **A conical pendulum consists of a bob of mass m in motion in a circular path in a horizontal plane as shown in Figure RI-3. During the motion, the supporting wire of length maintains the constant angle with the vertical. Show that the magnitude of the angular momentum of the bob about the circle's centre is**

$$L = \left(\frac{m^2 g l^3 \sin^4\theta}{\cos\theta} \right)^{\frac{1}{2}}$$

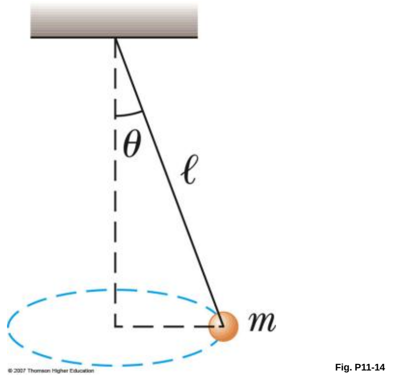
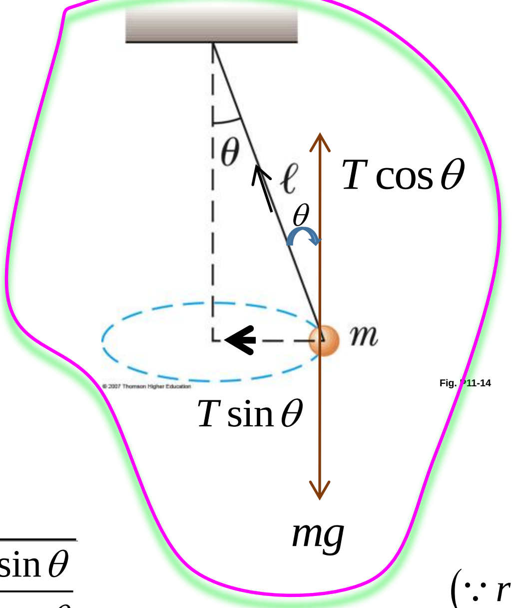


Figure RI-2

Hint:

$$\left. \begin{array}{l} T\sin\theta = \dfrac{mv^2}{r} \\ T\cos\theta = mg \end{array} \right\} \Rightarrow v = \sqrt{rg\frac{\sin\theta}{\cos\theta}}$$



The magnitude of the angular momentum of the bob about the circle's centre is

$$L = rmv\sin 90^0 \qquad \left(\because \vec{L} = \vec{r}\times\vec{v}\right)$$

$$= rm\sqrt{rg\frac{\sin\theta}{\cos\theta}} = \sqrt{m^2 r^3 g\frac{\sin\theta}{\cos\theta}} = \sqrt{m^2 (l\sin\theta)^3 g\frac{\sin\theta}{\cos\theta}} \qquad (\because r = l\sin\theta)$$

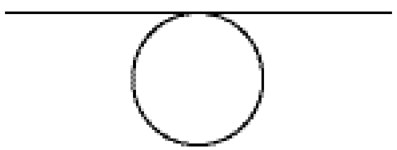
$$\therefore \boxed{L = \left(\frac{m^2 g l^3 \sin^4\theta}{\cos\theta} \right)^{\frac{1}{2}}}$$

Sample Problem

- **Two particles (m = 0.20 kg, M = 0.30 kg) are positioned at the ends of a 2.0-m long rod of negligible mass. What is the moment of inertia of this rigid body about an axis perpendicular to the rod and through the center of mass?**

[a] $0.48\ \mathrm{kg \cdot m^2}$ **[b]** $0.50\ \mathrm{kg \cdot m^2}$ **[c]** $0.80\ \mathrm{kg \cdot m^2}$ **[d]** $1.2\ \mathrm{kg \cdot m^2}$

- **A uniform sphere of radius R and mass M rotates freely about a horizontal axis that is tangent to an equatorial plane of the sphere, as shown below. The moment of inertia of the sphere about this axis is**



[a] $\frac{2}{5}MR^2$ **[b]** $\frac{5}{7}MR^2$ **[c]** $\frac{7}{5}MR^2$ **[d]** $\frac{3}{2}MR^2$

- **Stars originate as large bodies of slowly rotating gas. Because of gravity, these clumps of gas slowly decrease in size. The angular velocity of a star increases as it shrinks because of**

[a] conservation of angular momentum.

[b] conservation of linear momentum.

[c] conservation of energy.

[d] the law of universal gravitation.

ROTATIONAL DYNAMICS

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Thank you