Lecture 06

Electrostatic Field (Contd.)

Outline

The electric field is the negative gradient of a scalar potential

The potential difference between two points a and b :

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} \tag{1}$$

The fundamental theorem for gradients states that:

$$V(b) - V(a) = \int_a^b (\nabla V) \cdot d\vec{l} \tag{2}$$

So

$$-\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b (\nabla V) \cdot d\vec{l}$$

Since this is true for any points a and b, the integrands must be equal:

$$\boxed{\vec{E} = -\nabla V}$$

Poisson's Equation and Laplace's Equation

Gauss's law in differential form:

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho$$

The electric field can be written as the gradient of a scalar potential i.e. $\vec{E} = -\nabla V$.

$$\therefore\ \nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho$$

$$\Rightarrow \nabla \cdot (-\nabla V) = \frac{1}{\varepsilon_0}\rho$$

$$\therefore \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$

This is known as Poisson's equation.

Poisson's Equation and Laplace's Equation (contd.)

In regions where there is no charge, so that $\rho = 0$, Poisson's equation reduces to **Laplace's equation**, $\boxed{\nabla^2 V = 0}$

Example 1

The expression for electric field in a region where potential $V = -kxy$:

Hint:

$$\begin{aligned}\vec{E} = -\nabla V &= -\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right] \\ &= -\left[\hat{i}\frac{\partial\left(-kxy\right)}{\partial x} + \hat{j}\frac{\partial\left(-kxy\right)}{\partial y} + \hat{k}\frac{\partial\left(-kxy\right)}{\partial z}\right] \\ &= -\left(-ky\hat{i} - kx\hat{j}\right) = ky\,\hat{i} + kx\,\hat{j}\end{aligned}$$

Expression of electric potential

Consider a point charge q at origin O. The electric field at a point P with position vector $\vec{r}$ is

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

Using $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$, we can have

$$\vec{E} = -\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r}\right)$$

Comparing this result with $\vec{E} = -\nabla V$, we have

$$\boxed{V = \frac{q}{4\pi\varepsilon_0 r}}$$

Expression of electric potential (contd.)

This is the required expression for electric potential at any point due to a point charge at origin. For the a system of point charges, the potential reads

$$\boxed{V(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}}$$

For the the continuous charge distribution

$$\boxed{V(P) = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\mathscr{r}}}$$

Here, $\mathscr{r}$ is the separation distance from the elemental source charge to the field point P and $dq = \lambda dl'$ for line charge; $dq = \sigma da'$ for surface charge; $dq = \rho d\tau'$

Electric potential due to spherical charge

Problem:-

Find the potential of a uniformly charged spherical shell of radius R carrying a surface charge density σ.

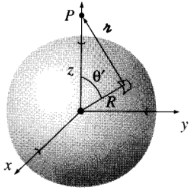


Figure 1

Solution:-

Consider an elemental area $da' = R^2 \sin\theta' d\theta' d\phi$; on the surface of the spherical shell. The elemental charge on the area da' is now

$dq = \sigma da' = \sigma R^2 \sin\theta' d\theta' d\phi'$

Let's take a point P at a distance z from the center of the spherical shell. $\mathscr{r}$ is the separation distance of P from da' as shown in Figure 1.

Electric potential due to spherical charge (contd.)

From figure and using cosine law, we can have

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos'\theta'$$

$$\implies 2\mathscr{r}d\mathscr{r} = 2Rz\sin\theta' d\theta'$$

$$\implies \sin\theta' d\theta' = \frac{\mathscr{r}d\mathscr{r}}{Rz}$$

The potential at P due to the elemental charge dq is

$$\begin{aligned} dV &= \frac{dq}{4\pi\varepsilon_0\mathscr{r}} \\ &= \frac{\sigma R^2\sin\theta' d\theta' d\phi'}{4\pi\varepsilon_0\mathscr{r}} = \frac{\sigma R^2}{4\pi\varepsilon_0\mathscr{r}}\frac{\mathscr{r}d\mathscr{r}}{Rz}d\phi' \\ dV &= \frac{\sigma R}{4\pi\varepsilon_0 z}d\mathscr{r}d\phi' \qquad (3) \end{aligned}$$

Electric potential due to spherical charge (contd.)

Case (I): Potential outside the spherical shell

If P lies outside of the spherical shell, the potential at P can be obtained by integration equation (3) from the limit $z-R$ to $z+R$ for $\mathscr{r}$ and 0 to 2π for ϕ', i.e.

$$\begin{aligned} V_{\text{out}} &= \frac{\sigma R}{4\pi\varepsilon_0 z}\int_{z-R}^{z+R} d\mathscr{r}\int_0^{2\pi} d\phi' \\ &= \frac{\sigma R}{4\pi\varepsilon_0 z}[(z+R)-(z-R)](2\pi) = \frac{\sigma R}{4\pi\varepsilon_0 z}(2R)(2\pi) \end{aligned}$$

Therefore, the potential due to the spherical charge out side of it is

$$\boxed{V_{\text{out}} = \frac{\sigma R^2}{\varepsilon_0 z}}$$

In term of total charge, $q = \sigma(4\pi R^2)$

$$\boxed{V_{\text{out}} = \frac{\sigma(4\pi R^2)}{4\pi\varepsilon_0 z} = \frac{q}{4\pi\varepsilon_0 z}}$$

Case (II): Potential inside the spherical shell

If P lies inside of the spherical shell, the potential at P can be obtained by integration equation (3) from the limit $R - z$ to $R + z$ for $\mathscr{r}$ and 0 to 2π for ϕ', i.e.

$$\begin{aligned} V_{\text{in}} &= \frac{\sigma R}{4\pi\varepsilon_0 z} \int_{R-z}^{R+z} d\mathscr{r} \int_0^{2\pi} d\phi' \\ &= \frac{\sigma R}{4\pi\varepsilon_0 z} [(R+z) - (R-z)](2\pi) \end{aligned}$$

Electric potential due to spherical charge (contd.)

$$= \frac{\sigma R}{4\pi\varepsilon_0 z}(2z)(2\pi)$$

Therefore, the potential due to the spherical charge out side of it is

$$\boxed{V_{\text{in}} = \frac{\sigma R}{\varepsilon_0}}$$

In term of total charge, $q = \sigma(4\pi R^2)$

$$\boxed{V_{\text{in}} = \frac{\sigma(4\pi R^2)}{4\pi\varepsilon_0 R} = \frac{q}{4\pi\varepsilon_0 R}}$$

The Work Done to Move a Charge

Suppose we have a stationary configuration of source charges, and we want to move a test charge Q from a point a to point b [Figure 2].

At any point along the path, the electric force on Q is $\vec{F} = Q\vec{E}$. The force we exert, in opposition to this electrical force is $-Q\vec{E}$.

The work done to move a test charge Q from a point a to point b is

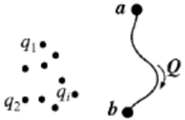


Figure 2

$$W = \int_a^b \vec{F} \cdot d\vec{l} = \int_a^b \left(-Q\vec{E}\right) \cdot d\vec{l} = Q\left[-\int_a^b \vec{E} \cdot d\vec{l}\right] = Q\left[V(b) - V(a)\right]$$

The Work Done to Move a Charge (contd.)

$$\therefore \boxed{V(b) - V(a) = V(\vec{r}_b) - V(\vec{r}_a) = \frac{W}{Q}}$$

The potential difference between points a and b is equal to the work per unit charge required to carry a charged particle from a and b. The work done to bring the charge Q from infinity to the point $\vec{r}$ is

$$W = Q\left[V(\vec{r}) - V(\infty)\right]$$

$$\boxed{\therefore W = QV(\vec{r})}$$

The potential energy per unit charge at a point in an electric field is called the Electric potential at that point.

Work and Energy in Electrostatics

The Energy of a Point charge Distribution

Consider a number of point charges q_1, q_2, ..., q_n are to be assembled by bringing them, one by one, from infinity and placing at position vectors $\vec{r}_1$, r_2, ..., $\vec{r}_n$, respectively.

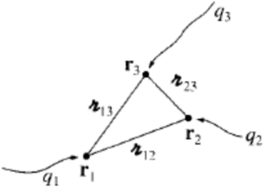


Figure 3

1. The work done to bring the charge q_1 from infinity to its location $\vec{r}_1$ is $W_1 = 0$. It is because there is no field to fight against.

The Energy of a Point charge Distribution (contd.)

2. The potential at position vector $\vec{r}_2$ due to the charge q_1 is $V_1(\vec{r}_2) = \dfrac{q_1}{4\pi\varepsilon_0 \, \mathscr{r}_{12}}$

   The work done to bring the second charge q_2 from infinity to its location $\vec{r}_2$ is

$$W_2 = q_2 \left[V_1(\vec{r}_2) \right]$$
$$= q_2 \left[\frac{1}{4\pi\varepsilon_0} \frac{q_1}{\mathscr{r}_{12}} \right]$$
$$= \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{\mathscr{r}_{12}}$$

3. The potential at the position vector $\vec{r}_3$ due to the charge q_1 and q_2 is

$$V_{1,2}(\vec{r}_3) = \frac{q_1}{4\pi\varepsilon_0 \, \mathscr{r}_{13}} + \frac{q_2}{4\pi\varepsilon_0 \, \mathscr{r}_{23}}$$

The work done to bring the third charge q_3 from infinity to its location $\vec{r}_3$ is

$$\begin{aligned} W_3 &= q_3 \left[V_{1,2}(\vec{r}_3)\right] \\ &= q_3 \left[\frac{1}{4\pi\varepsilon_0} \frac{q_1}{\mathscr{r}_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{\mathscr{r}_{23}}\right] \\ &= \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{\mathscr{r}_{23}} \end{aligned}$$

.

4. Similarly, the extra work to bring fourth charge q_4 will be

$$W_4 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{1}{4\pi\varepsilon_0}\frac{q_3 q_4}{\mathscr{r}_{34}}$$

The total work necessary to assemble the first four charges is

$$\begin{aligned} W &= W_1 + W_2 + W_3 + W_4 \\ &= 0 + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{1}{4\pi\varepsilon_0}\frac{q_3 q_4}{\mathscr{r}_{34}} \\ &= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}\right) \end{aligned}$$

$$\therefore W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{4} \sum_{\substack{j=1 \\ j>i}}^{4} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

For a system of n-point charges, we have

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{\mathscr{r}_{\mathrm{ij}}} \tag{4}$$

The stipulation $j > i$ suggests to count each pair of charges only once during calculation. If we intentionlly count the pair twice, the work done calculated become double. So, in order to manipuplate the result, it should be multiplied by $\frac{1}{2}$.

$$W = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{\mathscr{r}_{\mathrm{ij}}} = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_j}{\mathscr{r}_{\mathrm{ij}}}\right)$$

$$\therefore \boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)}$$

where $V(\vec{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_j}{\mathscr{r}_{\mathrm{ij}}}$ is the potential at point $\vec{r}_i$ (the postion of q_i) due to all other charges.

The total work necessary to assemble the n-point charges is given by

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \, V(\vec{r}_i) \tag{5}$$

where $V(\vec{r}_i)$ is the potential at point $\vec{r}_i$ (the postion of q_i) due to all other charges For a volume charge density ρ, Eq. (5) becomes

$$\begin{aligned} W &= \frac{1}{2}\int_V \rho \; V d\tau \\ &= \frac{1}{2}\int_V \left(\varepsilon_0 \nabla\cdot\vec{E}\right) \; V d\tau \qquad \left[\text{Using Gauss,s law:} \nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}\right] \\ &= \frac{\varepsilon_0}{2}\int_V V(\nabla\cdot\vec{E}) \; d\tau \qquad \left[\because \nabla\cdot(V\vec{E}) = V(\nabla\cdot\vec{E}) + (\nabla V)\cdot\vec{E}\right] \end{aligned}$$

The Energy of Continuous Charge Distribution (contd.)

$$= \frac{\varepsilon_0}{2}\left[-\int_V (\nabla V)\cdot\vec{E}\,d\tau + \int_V \nabla\cdot(V\vec{E})\,d\tau\right]$$

$\left[\text{Using } \vec{E} = -\nabla V \text{ and Gauss's Divergece theorem}\right]$

$$W = \frac{\varepsilon_0}{2}\left[\int_V \vec{E}\cdot\vec{E}\,d\tau + \oint_S (V\vec{E})\cdot d\vec{a}\right]$$

$$= \frac{\varepsilon_0}{2}\left[\int_V E^2 d\tau + \oint_S (V\vec{E})\cdot d\vec{a}\right]$$

For a point charge q at origin, the electric field and potential at a distance r are $E = \dfrac{q}{4\pi\varepsilon_0 r^2}$, and $V = \dfrac{q}{4\pi\varepsilon_0 r}$, respectively. Since $da = r^2 \sin\theta d\theta d\phi$, the surface integral

$$\int_S V\vec{E}\cdot d\vec{a} = \int_S VEda = \int_0^{\pi}\int_0^{2\pi}\left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r^3} r^2 \sin\theta d\theta d\phi = \left(\frac{q}{4\pi\varepsilon_0}\right)^2 \frac{1}{r} 4\pi \to 0$$

for $r \to \infty$. But the volume integral part is

$$\int_V E^2 d\tau = \left(\frac{q}{4\pi\varepsilon_0}\right)^2 \int_0^{\infty} \frac{1}{r^2} dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \neq 0$$

This also holds for the system of charges. Therefore, when the integration is taken over all space, the surface integral goes to zero and only the volume integral contributes to the total work done.

$$\boxed{W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 d\tau = \int_{\text{all space}} u_e d\tau}$$

where $\boxed{u_e = \frac{\varepsilon_0}{2} E^2}$ is the energy density

1.
   (a) **Three charges are situated at the corners of a square (side a), as shown in Figure 4. How much work does it take to bring in another charge, $+q$, from far away and place it in the fourth corner?**

   (b) **How much work does it take to assemble the whole configuration of four charges?**

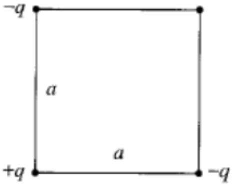


Figure 4

Hint (a):

$$
\begin{aligned}
W_4 &= qV \\
&= (+q)\left[\frac{1}{4\pi\varepsilon_0}\left\{\frac{-q}{a} + \frac{q}{a\sqrt{2}} + \frac{-q}{a}\right\}\right] \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\left[-2 + \frac{1}{\sqrt{2}}\right]
\end{aligned}
$$

Hint (b)

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q^2}{a} + \frac{q^2}{a\sqrt{2}} + \frac{-q^2}{a} + \frac{-q^2}{a} + \frac{q^2}{a\sqrt{2}} + \frac{-q^2}{a}\right] \tag{6}$$

$$= 2\frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\left[-2 + \frac{1}{\sqrt{2}}\right] \tag{7}$$

$$\tag{8}$$

$$\because \; V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right)$$

$$W = U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_1q_4}{r_{14}} + \frac{q_2q_3}{r_{23}} + \frac{q_2q_4}{r_{24}} + \frac{q_3q_4}{r_{34}}\right]$$

2. **Find the energy of a uniformly charged spherical shell of total charge q and radius R.**

   Solution:

   For a uniformly charged spherical shell, the electric field inside is $E = 0$ and Outside $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$ Therefore,

$$W_{tot} = \frac{\varepsilon_0}{2}\int\limits_{all\ \text{space}} E^2 d\tau = \frac{\varepsilon_0}{2}\int\limits_{outside}\left[\frac{1}{4\pi\varepsilon_0}\frac{\text{q}}{r^2}\right]^2 \left(r^2 \sin\theta dr d\theta d\phi\right)$$

$$= \frac{\varepsilon_0}{2}\frac{1}{(4\pi\varepsilon_0)^2}\text{q}^2\left[\left\{\int_R^\infty \frac{1}{r^2}dr\right\}\left\{\int_0^\pi \sin\theta d\theta\right\}\left\{\int_0^{2\pi} d\phi\right\}\right]$$

Solved Problems (contd.)

$$= \frac{\varepsilon_0}{2} \frac{1}{(4\pi\varepsilon_0)^2} \mathrm{q}^2 (2)(2\pi) \left[\int_R^\infty \frac{1}{r^2} dr \right]$$

$$\therefore \boxed{W_{tot} = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{2R}}$$

3. **Find the energy stored in a uniformly charged solid sphere of radius R and charge q.**

Solution:

For a uniformly charged Solid sphere of radius R, the electric field inside is $E_{\text{in}} = \frac{\rho r}{3\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}$ and outside is outside $E_{\text{out}} = \frac{1}{4\pi\varepsilon_0}\frac{\text{q}}{r^2}$ Therefore,

$$W_{tot} = \frac{\varepsilon_0}{2}\int\limits_{\text{all space}} E^2 d\tau = \frac{\varepsilon_0}{2}\int\limits_{\text{all space}} E^2\left(r^2\sin\theta dr d\theta d\phi\right)$$

$$= \frac{\varepsilon_0}{2}\left[\left\{\int_0^\infty E^2 r^2 dr\right\}\left\{\int_0^\pi \sin\theta d\theta\right\}\left\{\int_0^{2\pi} d\phi\right\}\right]$$

Solved Problems (contd.)

$$= \frac{\varepsilon_0}{2}(4\pi)\left[\int_0^R (E_{\text{in}})^2 r^2 dr + \int_R^\infty (E_{out})^2 r^2 dr\right]$$

$$= 2\pi\varepsilon_0 \left[\int_0^R \left(\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\right)^2 r^2 dr + \int_R^\infty \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)^2 r^2 dr\right]$$

$$= 2\pi\varepsilon_0 \left(\frac{1}{4\pi\varepsilon_0} q\right)^2 \left[\frac{1}{R^6}\int_0^R r^4 dr + \int_R^\infty \frac{1}{r^2} dr\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left[\frac{1}{R^6}\frac{R^5}{5} + \frac{1}{R}\right]$$

$$\therefore \boxed{W_{tot} = \frac{6}{5}\left[\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2R}\right]}$$

CONDUCTORS AND INSULATORE

Conductors are substances, like the metals, which contain large numbers of essentially free charge carriers. These charge carriers (electrons in most cases) are free to wander throughout the conducting material; they respond to almost infinitesimal electric fields, and they continue to move as long as they experience a field. These free carriers carry the electric current when a steady electric field is maintained in the conductor by external source of energy.

Insulators (Dielectrics) are substances in which all charged particles are bound rather strongly to constituent molecules. The charged particles may shift their positions slightly in response to an electric field, but they do not leave the vicinity of their molecules.

A Perfect conductor is a material containing an unlimited supply of completely free charges.

In real life there are no perfect conductors, but many substances come amazingly close.

1. $E = 0$ **inside a conductor.**
When a conductor is placed into an external electric field $\vec{E}_0$, this electric field will drive free positive charges to the right, and negative charges to the left. When they come to the edge of the material, the charges pile up: plus on the right side, minus on the left. Now, these induced charges produce a field of their own, $\vec{E}_1$, which is in the opposite direction to $\vec{E}_0$ [Figure 5]. The field of the induced charges tends to cancel off the original field. Charges will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely zero.

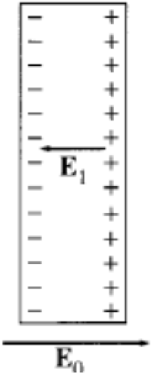


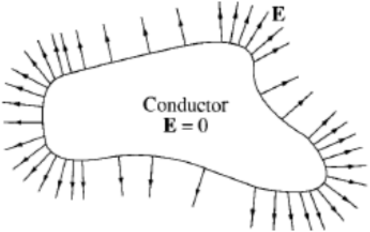
Figure 5

2. **$\rho = 0$ inside a conductor.**

   From Gauss's law: $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$. Hence, $\vec{E} = 0$ inside a conductor $\Rightarrow \rho = 0$ inside a conductor.

3. **Any net charge resides on the surface.**

4. **$\vec{E}$ is perpendicular to the surface, just outside a conductor.**



5. **A conductor is an equipotential.**

   For any two points a and $b$within (or at the surface of) a given conductor,

$$V(a) - V(b) = -\int_a^b \vec{E} \cdot d\vec{l} = 0$$

$$\Rightarrow V(a) = V(b)$$

End of Lecture 06

Thank you