Unit: 3

MULTIPLE INTEGRALS

Double Integrals:

$R: a \le x \le b \quad , \quad c \le y \le d$

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be integrable and the limit is called 'double integral' of f over R written as.

$$\iint_R f(x,y)\, dA \quad \text{or} \quad \iint_R f(x,y)\, dx\, dy.$$

R = region of integration.

⟨Q⟩: Find the values of the following integrals.

⟨a⟩: $\displaystyle\int_{-1}^{0}\int_{-1}^{1} (x+y+1)\, dx\, dy$

Solution.

$$= \int_{-1}^{0} \left[\int_{-1}^{1} (x+y+1)\, dx\right] dy$$

$$= \int_{-1}^{0} \left[\frac{x^2}{2} + xy + x\right]_{-1}^{1} dy$$

$$\int_{-1}^{0} \left[\left(\frac{1^2}{2} + 1xy + 1\right) - \left(\frac{(-1)^2}{2} + (-1)xy + (-1)\right)\right] dy$$

$$= \int_{-1}^{0} \left(\cancel{\frac{1}{2}} + y + 1 - \cancel{\frac{1}{2}} + 1 + y\right) dy$$

$$= \int_{-1}^{0} (2y+2)\, dy$$

$$= \left[\frac{\cancel{2} \times y^2}{\cancel{2}} + 2y\right]_{-1}^{0}$$

$$= \left(0^2 + 2\times 0\right) - \left((-1)^2 + 2\times(-1)\right)$$

$$= 0 + 0 - 1 \cancel{-1} + 2$$

$$= 1$$

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(b): $\int_{-1}^{1}\int_{-1}^{0} (x+y+1)\, dy\, dx$

Sol^n:

$$= \int_{-1}^{1}\left[\int_{-1}^{0}(x+y+1)\,dy\right] dx$$

$$= \int_{-1}^{1}\left[xy + \frac{y^2}{2} + y \right]_{-1}^{0} dx$$

$$= \int_{-1}^{1}\left[\left(x\times(0) + \frac{0^2}{2} + 0\right) - \left(x\times(-1) + \frac{(-1)^2}{2} + (-1)\right)\right] dx$$

$$= \int_{-1}^{1}\left(0+0+0 + x - \frac{1}{2} + 1\right) dx$$

$$= \int_{-1}^{1}\left(x + \frac{1}{2}\right) dx$$

$$= \left(\frac{x^2}{2} + \frac{x}{2}\right)_{-1}^{1}$$

$$= \left(\frac{1^2}{2} + \frac{1}{2}\right) - \left(\frac{(-1)^2}{2} + \frac{(-1)}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1 //$$

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*) Properties of Double Integrals

If $f(x,y)$ and $g(x,y)$ are continuous on the bounded region R, then the following properties hold.

1: Constant multiple: $\iint_R c f(x,y)\,dA = c\iint_R f(x,y)\,dA$

2: Sum and diffuence:

$$\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

3: Domination:

(a): $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

(b) $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ over R.

(4): Additivity: If $R = R_1 \cup R_2$.

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

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(*) Fubini's Theorem (1st form) {For rectangular domain}

If $f(x,y)$ is continuous throughout the rectangular region.

$R: a \le x \le b, \; c \le y \le d$ then,

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

<Q>: Verify Fubini's theorem of first form:

a) $\iint_R (x+y-1)\, dA \qquad R: 0 \le x \le 1, \; 1 \le y \le 2$

Soln.

1st case,

$$= \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

$$= \int_1^2 \int_0^1 \left[(x+y-1)\, dx\right] dy$$

$$= \int_1^2 \left[\frac{x^2}{2} + xy - x\right]_0^1 dy$$

$$= \int_1^2 \left[\left(\frac{1^2}{2} + 1\times y - 1\right) - \left(\frac{0^2}{2} + 0\times y - 0\right)\right] dy$$

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$$= \int_1^2 \left(y - \frac{1}{2}\right) dy$$

$$= \left(\frac{y^2}{2} - \frac{y}{2}\right)_1^2 = \left(\frac{4}{2} - \frac{2}{2}\right) - \left(\frac{1^2}{2} - \frac{1}{2}\right)$$

$= 1$ //

2nd case: $\int_a^b \int_c^d f(x,y)\, dy\, dx$

$$= \int_0^1 \int_1^2 (x+y-1)\, dy\, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} - y\right]_1^2 dx = \int_0^1 \left[\left(2x + \frac{4}{2} - 2\right) - \left(x + \frac{1}{2} - 1\right)\right] dx$$

$$= \int_0^1 \left[x + \frac{1}{2}\right] dx$$

$$= \left[\frac{x^2}{2} + \frac{x}{2}\right]_0^1 = \left[\left(\frac{1}{2} + \frac{1}{2}\right) - \left(\frac{0}{2} + \frac{0}{2}\right)\right]$$

$= 1$

$\therefore \iint_R f(x,y)\, dx\, dy = \iint_R f(x,y)\, dy\, dx$

Hence, proved.

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(b): $\int_1^2 \int_0^x (x^2+y^2)\,dy\,dx$

Soln:

$$= \int_1^2 \int_0^x (x^2+y^2)\,dy\,dx$$

$$= \int_1^2 \left[x^2 y + \frac{y^3}{3} \right]_0^x dx$$

$$= \int_1^2 \left[x^3 + \frac{x^3}{3} \right] dx$$

$$= \left[\frac{x^4}{4} + \frac{x^4}{12} \right]_1^2$$

$$= \left(\frac{2^4}{4} + \frac{2^4}{12} \right) - \left(\frac{1^4}{4} + \frac{1^4}{12} \right)$$

$$= \frac{16}{4} + \frac{16}{12} - \frac{1}{4} - \frac{1}{12}$$

$$= \frac{15}{4} + \frac{15}{12} = \frac{60}{12} = 5$$

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(*): Fubini's theorem (Stronger form): {Non-rectangular domain}

Let $f(x,y)$ be continuous on a region R.

(i): If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with $g_1(x)$ and $g_2(x)$ continuous on $[a,b]$ then.

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(ii): If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with h_1 and h_2 continuous on $[c,d]$

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

(Q): Integrate $f(x,y) = x/y$ over the region in the first quadrant

$y = x$, $y = 2x$, $x = 1$, $x = 2$.

Soln:

from Fubini's theorem in stronger form,

$$\iint_R f(x,y)\,dA = \int_1^2 \int_x^{2x} \frac{x}{y}\,dy\,dx$$

$$= \int_1^2 \left[x \ln y \right]_x^{2x} dx = \int_1^2 x \ln 2\,dx$$

$$= \ln 2 \left[\frac{x^2}{2} \right]_1^2 = \left[\frac{2^2}{2} - \frac{1^2}{2} \right] \ln 2 = \frac{3}{2} \ln 2$$

(*) Note:

<Q>: $$\int_0^1\int_1^2 xy\,dx\,dy = \left[\int_0^1 y\,dy\right]\left[\int_1^2 x\,dx\right]$$

$$= \left[\frac{y^2}{2}\right]_0^1 \left[\frac{x^2}{2}\right]_1^2$$

$$= \left(\frac{1}{2}-\frac{0}{2}\right)\left(\frac{4}{2}-\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$$

(*) Finding limits of integration:

<a>: Vertical cross-section:

While evaluating $\iint_R f(x,y)\,dA$, integrating first w.r.t y and then w.r.t x.

(i): Sketch the region of integration.

(ii) Find y-limits of integration:

→ Imagine a vertical line L cutting through R in direction of increasing y.

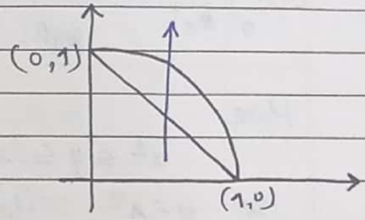
→ Marking where L enters and leaves. They are usually functions of x.

iii) Finding x-limits of integration:

→ x-limits that include all the vertical lines through R.

Eg: For region: $x^2+y^2=1$ and $x+y=1$ in 1^{st} q.

→ Using vertical cross-section



i) Enters through $y = 1-x$

ii) Exists through $y = \sqrt{1-x^2}$

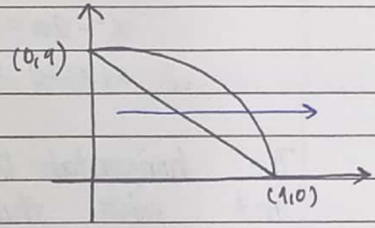
iii) y continuous $[0,1] \Rightarrow x$ interval.

Minimum $x = 0$ So,

Maximum $x = 1$.

$$\int_0^1\int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

→ Using horizontal cross-section:



i) Enters through $x = 1-y$

ii) Exists through $x = \sqrt{1-y^2}$

iii) x is continuous on $[0,1] \Rightarrow y$-interval.

Minimum $y = 0$, Maximum $y = 1$.

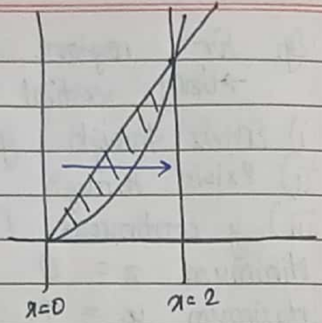
So

$$\int_0^1\int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

<Q>: Sketch the region of integration and write an equivalent integral with the order of integration reversed.

(a): $\int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx$

Soln:



Here,

$x^2 \le y \le 2x$.

So, $y = x^2$ —(i)

$y = 2x$ —(ii)

Solving (i) and (ii), we get.

when $x=0, y=0$

when $x=2, y=4$

$x^2 - 2x = 0$

or $x(x-2) = 0$ $\therefore x = 0,\ x = 2$

The horizontal line enters through $x = y/2$ and exists through $x = \sqrt{y}$.

So equivalent integral when the order of integration is reversed.

$$\int_0^2\int_{y/2}^{\sqrt{y}}(4x+2)\,dx\,dy$$

(b): $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\,dx\,dy$.

Soln.

Here,

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

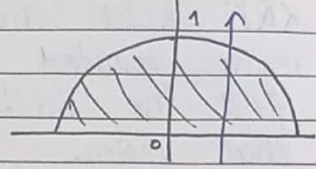
So, $x = -\sqrt{1-y^2}$ —(i)

$x = \sqrt{1-y^2}$ —(ii)

Solving (i) and (ii), we get.

$2\sqrt{1-y^2} = 0$

$y = \pm 1$.



when

Arranging eq^n (i),

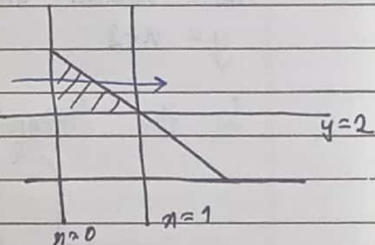
$x^2 = 1 - y^2$ or $x^2 + y^2 = 1$

The vertical line enters through $y=0$ and exists through $y = \sqrt{1-x^2}$.

So equivalent integral when the order of integration is reversed.

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} 3y\,dy\,dx$$

(c): $\int_0^1\int_2^{4-2x} dy\,dx$



Soln:

R: $2 \le y \le 4-x$

So, $y = 2$ —(i)

$y = 4-2x$ —(ii)

Solving (i) and (ii),

$4 - 2x = 2$

$\therefore x = 1$

when $x=0, y=4$

when $x=1, y=2$.

The horizontal line enters through $x = 0$, and exists through $x = (4-y)/2$

So equivalent integral.

$$\int^{4}\int^{(4-y)/2} dx\,dy.$$

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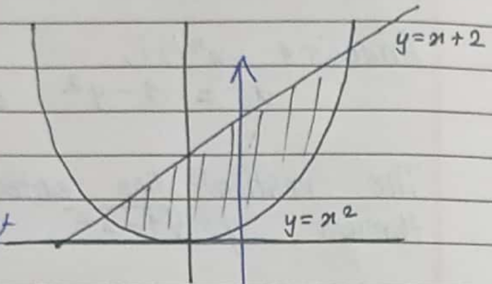
<Q>: Sketch the region of integration for the region bounded by $y = x^2$ and $y = x+2$. Also, find the limits of integration $f(x,y)$ over that region.

Soln.



Given,

$y = x^2$ — (i)

$y = x+2$ — (ii)

Solving (i) and (ii), we get

$x^2 = x+2$

or ~~$x^2 - x = 2$~~ or, $x^2 - x - 2 = 0$

~~or $x(x-1) = 2$~~

$\therefore$ ~~$x = 2$, $x = 3$~~ $\therefore x = -1, 2$

when $x = -1$, $y = 1$.

when $x = 2$, $y = 4$

The vertical line enters through $y = x^2$ and exists through $y = x+2$.

So, the integral is,

$$\int_{-1}^{2}\int_{x^2}^{x+2} f(x,y)\,dy\,dx$$

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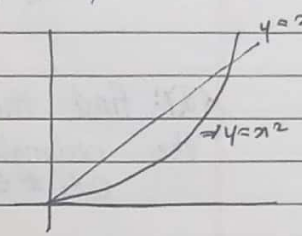
Area of Plane Region

The area of closed and bounded plane region R is

$$A = \iint_R dA$$

<Q>: Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Soln.



Given,

$y = x$ — (i)

$y = x^2$ — (ii)

Solving (i) and (ii), we get

$x^2 = x$ or $x(x-1) = 0$

$\therefore x = 0, x = 1$

Now,

$$\text{Area} = \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units.}$$

Average Value:

Average value of $f(x,y)$ over R

$$= \frac{1}{\text{area of } R} \iint_R f \, dA$$

$$= \frac{\iint_R f(x,y)\, dA}{\iint_R dA}$$

Q: Find the average value of $f(x,y) = \sin(x+y)$ over the rectangle
$0 \le x \le \pi$, $0 \le y \le \pi/2$.

Sol^n:

Now,

(i): $\iint_R dA = \int_0^{\pi}\int_0^{\pi/2} dy\, dx$

$$= \int_0^{\pi} [y]_0^{\pi/2} dx$$

$$= \int_0^{\pi} \frac{\pi}{2} dx$$

$$= \frac{\pi}{2}[x]_0^{\pi} = \frac{\pi^2}{2}$$

(ii): $\iint_R f(x,y)\, dA = \int_0^{\pi}\int_0^{\pi/2} \sin(x+y)\, dy\, dx$

$$= \int_0^{\pi} \left[-\cos(x+y)\right]_0^{\pi/2} dx$$

$$= \int_0^{\pi} \cos(x+0) - \cos(x+\pi/2) \; dx$$

$$= \int_0^{\pi} (\cos x + \sin x)\, dx$$

$$= \cancel{\sin x} \; \sin x - \cos x \Big|_0^{\pi}$$

$$= (\sin\pi - \cos\pi) - (\sin 0 - \cos 0)$$

$$= 0 - (-1) - (0 - 1) = 2$$

$\therefore$ Average value $= \dfrac{\iint_R f(x,y)\, dA}{\iint_R dA} = \dfrac{4}{\pi^2}$

Q: Find the average value of $f(x,y) = x\cos xy$ over the reactangle.
$R: 0 \le x \le \pi$, $0 \le y \le 1$.

Sol^n:

(i): $\iint_R \cancel{x\cos xy}\, dA = \int_0^{\pi}\int_0^{1} dy\, dx$

$$= \int_0^{\pi} [y]_0^1 dx \qquad [x]_0^{\pi} = \pi.$$

(ii): $\iint_R f(x,y)\,dA = \int_0^{\pi}\int_0^{1} x\cos xy \; dy\,dx$

$= \int_0^{\pi} \left(x \sin xy\right)_0^1 dx$

$= \int_0^{\pi} x \sin x \; \cancel{-x\sin}\; dx$

$= \frac{x}{x}\int_0^{\pi} \sin x\, dx - \int\left(\frac{dx}{dx}\int_0^{\pi} \sin x\, dx\right) dx$

$= \frac{x}{x}\left[-\cos x\right]_0^{\pi} - \int 1\cdot\left[-\cos x\right]_0^{\pi} dx$

$= \frac{x}{x}\left[-\cos \pi + \cos 0\right] - \int\left[-\cos\pi + \cos 0\right] dx$

$= \frac{2x}{x}$

$\therefore$ Average value $= \dfrac{\iint_R f(x,y)\,dA}{\iint_R dA} = \dfrac{2}{\pi}$ ~~sq. units~~

Double Integral in Polar Form:

We know,

$\Delta A_k = r_k \,\Delta r\, \Delta\theta_k.$

ie, $dA = r\,dr\,d\theta$

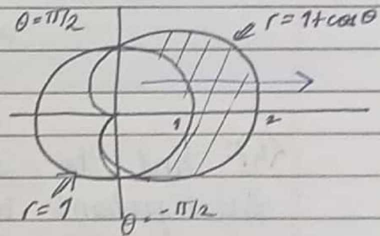
Q7: Find the limits of integration for integrating $f(r,\theta)$ over the region R lies inside the cardiod $r = 1+\cos\theta$ and outside the circle $r = 1$.

Soln:

~~r = 1~~ Given,

$r = 1$ —— (i)

$r = 1+\cos\theta$ —— (ii)



Solving (i) and (ii), we get

$1+\cos\theta = 1$

or, $\cos\theta = 0$

$\therefore \theta = \pm\pi/2$

The horizontal line enters at $r = 1$ and ~~leav~~ exits at $r = 1+\cos\theta$

The integral is:

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$

$\frac{1}{2}$

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(x) Area in Polar Coordinate:

The area of closed and bounded plane region R in polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta$$

(x) Average value in Polar Coordinates.

The area of closed and bounded plane region R is polar coordinate plane is.

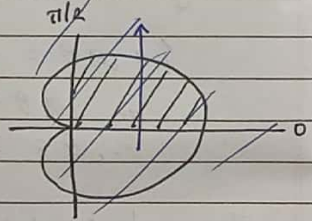
$$A = \frac{1}{\text{area of } R} \iint_R f(r,\theta)\, r\,dr\,d\theta$$

<Q>: Find the area of the region cut from the first quadrant by the cardiod $r = 1+\sin\theta$.

Soln:

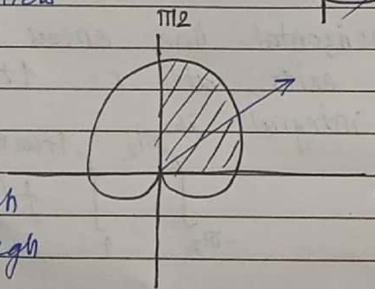
Given,

$r = 1+\sin\theta$



~~The vertical line enters~~

~~Here,~~ $\int_0^{\pi/2}$



The line enters through $r=0$ and exists through $r = 1+\sin\theta$.

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Now,

$$\text{Area} = \int_0^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{1+\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (1+\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1+2\sin\theta+\sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1\,d\theta + \int_0^{\pi/2} 2\sin\theta\,d\theta + \int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \left(\frac{\pi}{2} + 2\left[-\cos\theta\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\,d\theta\right)\times\frac{1}{2}$$

$$= \left(\frac{\pi}{2} + 2\left[\cos 0 - \cos\pi/2\right] + \int_0^{\pi/2}\frac{1}{2}\,d\theta - \int_0^{\pi/2}\cos 2\theta\,d\theta\right)\frac{1}{2}$$

$$= \left(\frac{\pi}{2} + 2 + \frac{\pi}{4} - \left[\sin 2\theta\right]_0^{\pi/2}\right)\frac{1}{2}$$

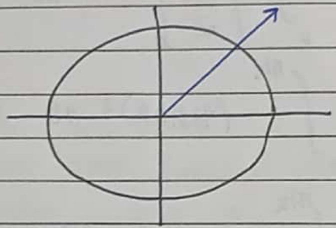
$$= \left(\frac{3\pi}{4} + 2 - \left[\sin\frac{2\times\pi}{2} - \sin\frac{2\times 0}{2}\right]\right)\frac{1}{2}$$

$$= \left(\frac{3\pi}{4}+2\right)\frac{1}{2}\ \cancel{\text{sq units}} = \frac{3\pi}{8} + 1 \text{ sq units.}$$

<Q>: Find the average ~~value~~ distance from a point $P(x,y)$ in the disk $x^2+y^2 \le a^2$ to the origin.

Soln.

We know,



Area of circle $= \pi a^2$

Average $= \dfrac{1}{\pi a^2} \displaystyle\int_0^{2\pi}\int_0^{a} r^2\,dr\,d\theta$

$= \dfrac{1}{\pi a^2} \displaystyle\int_0^{2\pi} \left[\dfrac{r^3}{3}\right]_0^a d\theta$

$= \dfrac{1}{\pi a^2} \displaystyle\int_0^{2\pi} \dfrac{a^3}{3}\,d\theta$

$= \dfrac{1}{\pi a^2} \times \dfrac{a^3}{3} \times 2\pi = \dfrac{2a}{3}.$

<Q>: Change into cartesian form:

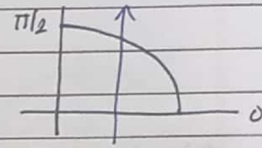
(i): $\displaystyle\int_0^{\pi/2}\int_0^{1} r^3 \cos\theta \sin\theta\, dr\, d\theta$

Soln:

$= \displaystyle\int_0^{\pi/2}\int_0^{1} (r\cos\theta)(r\sin\theta)\, r\,dr\,d\theta$

$= \displaystyle\int_0^{\pi/2}\int_0^{1} xy\, dA$

Here, region of integration: $R: 0 \le r \le 1,\ 0 \le \theta \le \pi/2$



The vertical line enters through $y=0$ and exists through $y=\sqrt{1-x^2}$

Here, maximum $x = 1$

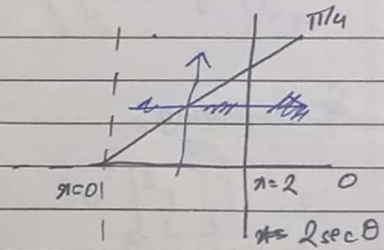
minimum $x = 0$.

So, $\displaystyle\int_0^{\pi/2}\int_0^{1} r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{1}\int_0^{\sqrt{1-x^2}} xy\,dy\,dx.$

(ii) $\displaystyle\int_0^{\pi/4}\int_0^{2\sec\theta} r^5 \sin^2\theta\, dr\, d\theta$

Soln.

Here, the region: $0 \le \theta \le \pi/4$ and $0 \le r \le 2\sec\theta$



Here, $\tan\theta = \dfrac{y}{x}$

Since, $\theta = \pi/4$. So, $y = x$.

The vertical line enters through $y=0$ and exits through $y=x$.

Maximum $x = 2$ So,

Minimum $x = 0$

Note: $a\sec\theta$ → Vertical line
$a\csc\theta$ → horizontal straight l
a = distance from axis.

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$$\int_0^{\pi/4}\int_0^{2\sec\theta} r^5\sin^2\theta\, dr\, d\theta = \int_0^2\int_0^x (r^2(r\sin\theta)^2(r\,dr\,d\theta))$$

$$= \int_0^2\int_0^x (x^2+y^2)(y)^2\, dy\, dx.$$

(Q): Change into equivalent polar integral.

(i): $\displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dy\, dx$

Sol^n:

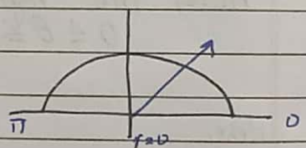
Given,

R: $-1 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2}$

From here,

the region is bounded by x-axis and $y = \sqrt{1-x^2}$ or, $x^2+y^2 = 1$ or, $r = 1$.

$$\therefore \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dy\, dx = \int_0^{\pi}\int_0^1 r\, dr\, d\theta$$



Eq. Evaluating,

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta$$

$$= \frac{1}{2}\int_0^{\pi} d\theta = \frac{\pi}{2}$$

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(ii): $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$

Sol^n.

Given,

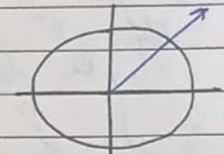
Region: R: $-a \le x \le a$ and $-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

Here,

~~the line enters through r=0 and~~

the region is bounded within.

$y = -\sqrt{a^2-x^2}$ on $x^2+y^2 = a^2$ in all quadrants.



$$\text{So, } \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx = \int_0^{2\pi}\int_0^a r\, dr\, d\theta$$

Evaluating,

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^a d\theta = \int_0^{2\pi}\frac{a^2}{2}\, d\theta$$

$$= \pi a^2.$$

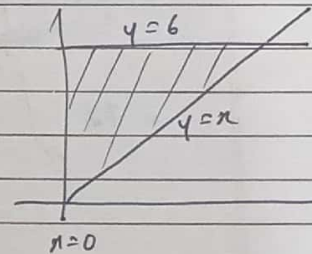
(iii): $\displaystyle\int_0^6\int_0^y x\, dx\, dy$

Sol^n:

The region is bounded within 1st quadrant

$\therefore$ R: $0 \le y \le 6$

$0 \le x \le y$



Now, $y = x$ $\therefore \theta = \pi/4$.

(iii): $\int_0^6\int_0^y x\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r\cos\theta\, r\,dr\,d\theta$

Evaluating

$$= \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\,dr\cos\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{6\csc\theta}\cos\theta\,d\theta$$

$$= \frac{216}{3}\int_{\pi/4}^{\pi/2}\csc^3\theta\cos\theta\,d\theta$$

$$= 72\int_{\pi/4}^{\pi/2}\csc^2\theta\cot\theta\,d\theta$$

Let $u=\cot\theta$.

$\therefore \frac{du}{d\theta} = -\csc^2\theta$ or, $du = -\csc^2\theta\,d\theta$.

when $\theta=\pi/4$, $u=1$

when $\theta=\pi/2$, $u=0$

So,

$$= 72\int_1^0 -u\,du$$

$$= 72\int_0^1 u\,du = 72\left[\frac{u^2}{2}\right]_0^1 = 36$$

(iv): $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dy\,dx.$

Here,

$\ln(x^2+y^2+1)\,dy\,dx = \ln(r^2+1)\,r\,dr\,d\theta.$

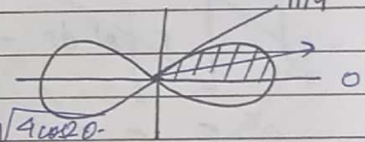
Here, the region is bounded within, $x=-\sqrt{1-y^2}$

$x^2+y^2=1.$

$$\therefore \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dy\,dx = \int_0^{2\pi}\int_0^1 \ln(r^2+1)\,r\,dr\,d\theta.$$

Q): find the area enclosed by $r^2=4\cos2\theta$.

Soln:



Here,

the line enters through $r=0$ and exits through $r=\sqrt{4\cos2\theta}$

Now,

$$\text{Area} = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4}\frac{4\cos2\theta}{2}\,d\theta = 8\int_0^{\pi/4}\cos2\theta\,d\theta$$

$$= \frac{8}{2}\sin2\theta\Big|_0^{\pi/4}$$

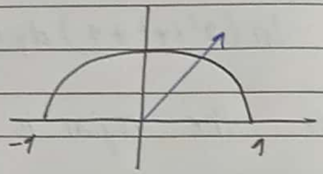
$$= \frac{8}{2}\times1 = 4$$

(Q): Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$, where R is the semi-circular region bounded by x-axis and the curve $y=\sqrt{1-x^2}$.

Sol^n.

Here,

$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$$



Enter through $y=0$
and exits through $y=\sqrt{1-x^2}$

Converting to polar form,

$$= \int_0^{\pi}\int_0^1 e^{r^2}\, r\,dr\,d\theta$$

$$\cancel{= \int_0^{\pi}\left[r\int_0^1 e^r\,dr - \int_0^1\left[\frac{dr}{dr}\int_0^1 e^r\,dr\right]d\theta r\right]d\theta}$$

$$\cancel{= \int_0^{\pi}\left[r\,[e-1] - \int_0^1 1\cdot[e-1]\,dr\right]d\theta}$$

$$\cancel{= \int_0^{\pi}\left[re - r - \left[\int_0^1 e\,dr - \int_0^1 1\,dr\right]\right]d\theta}$$

$$\cancel{= \int_0^{\pi}(re - r - e + 1)\,d\theta}$$

Let $u=r^2$

$\therefore \frac{du}{2} = r\,dr.$

So,

$$\int_0^{\pi}\int_0^1 e^u\,\frac{du}{2}\,d\theta$$

$$= \int_0^{\pi}\left[\frac{e^u}{2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\left[\frac{e}{2} - \frac{1}{2}\right]d\theta$$

$$= \frac{e}{2}\int_0^{\pi} d\theta - \frac{1}{2}\int_0^{\pi} d\theta$$

$$= \frac{e}{2}\pi - \frac{\pi}{2}$$

$$= \frac{\pi}{2}(e-1).$$