CHAPTER-1: VECTOR ANALYSIS

Scalars

Scalars have magnitude only.

They are specified by a number with a unit ($10^0 C$) and obey the rules of arithmetic and ordinary algebra.

Examples: mass, temperature, charge, electric potential, work, energy etc.

Vectors

Vectors have both magnitude and direction (5m, north) and obey the rules of vector algebra.

Examples: displacement, velocity, force, momentum, torque, electric field, magnetic field etc

In diagrams, vector is denoted by arrow: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction.

In texts, we shall denote a vector by putting an arrow over the letter ($\vec{A}$, $\vec{B}$, and so on).

The magnitude of a vector $\vec{A}$ is written $|\vec{A}|$ or more simply A.

Negative of a vector:

Minus $\vec{A}$ ($-\vec{A}$) is a vector with the same magnitude as $\vec{A}$ but of opposite direction [Figure 1.1].

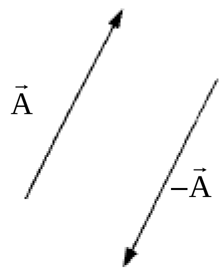
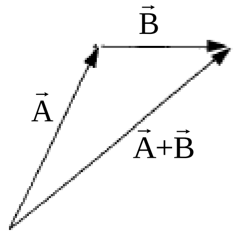


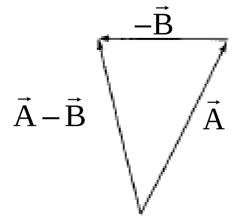
Figure 1.1

FOUR VECTOR OPERATIONS

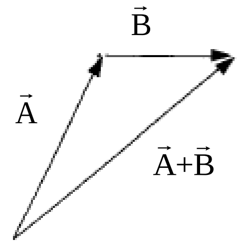
1) **Addition of Two Vectors**

- Place the tail of $\vec{B}$ at the head of $\vec{A}$; the sum, $\vec{A}+\vec{B}$, is the vector from the tail of $\vec{A}$ to the head of $\vec{B}$.

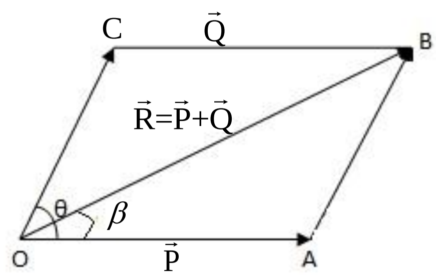




- **Triangle Law of Vector Addition**
  If two sides of a triangle taken in the same order represent the two vectors in magnitude and direction, then the third side in the opposite order represents the resultant of two vectors.



- **Parallelogram Law of Vector Addition**
  If two vectors are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, then their resultant is given in magnitude and direction by the diagonal of the parallelogram passing through that point.

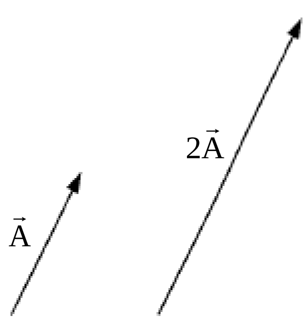


- $R = \left|\vec{P}+\vec{Q}\right| = \sqrt{P^2+Q^2+2PQ\cos\theta}$
- $\tan\beta = \dfrac{Q\sin\theta}{P+Q\cos\theta}$

- Addition is ***commutative***: $\vec{A}+\vec{B} = \vec{B}+\vec{A}$
- Addition is ***associative***: $\left(\vec{A}+\vec{B}\right)+\vec{C} = \vec{A}+\left(\vec{B}+\vec{C}\right)$

2) Multiplication by a Scalar

- Multiplication of a vector by a positive scalar a multiples the magnitude but leaves the direction unchanged.(If a is negative, the direction is reversed)



- Scalar multiplication is ***distributive***: $a\left(\vec{A}+\vec{B}\right) = a\vec{A}+a\vec{B}$

3) Dot Product (Scalar Product) of Two Vectors

- The dot product of two vectors is defined by

$$\vec{A}\bullet\vec{B} \equiv AB\ \cos\theta \qquad (\text{a scalar}) \qquad \left[W = \vec{F}\bullet\vec{S}\right]$$

where θ is the angle they form when placed tail-to-tail (Figure D-1).

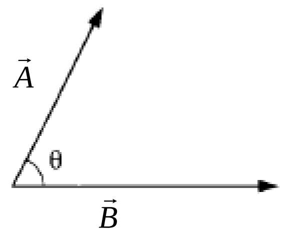


Figure D-1

- The dot product is ***commutative***: $\vec{A}\bullet\vec{B} = \vec{B}\bullet\vec{A}$
- The dot product is ***distributive***: $\vec{A}\bullet\left(\vec{B}+\vec{C}\right) = \vec{A}\bullet\vec{B}+\vec{A}\bullet\vec{C}$
- Geometrically, $\vec{A}\bullet\vec{B}$ is the product of B times the projection of $\vec{A}$ along $\vec{B}$.
  $\left[\vec{A}\bullet\vec{B} = B(A\cos\theta)\right]$
- If the two vectors are parallel, then $\vec{A}\bullet\vec{B} = AB$.
  If two vectors are perpendicular, then $\vec{A}\bullet\vec{B} = 0$.
- For any vector $\vec{E}$,

$$\vec{E}\bullet\vec{E} = E^2$$
$$\Rightarrow E = \sqrt{\vec{E}\cdot\vec{E}}$$

Example 1:

Let $\vec{C} = \vec{A}-\vec{B}$ (Figure D-2), and calculate $\vec{C}\bullet\vec{C}$

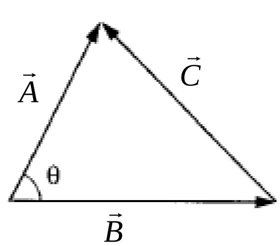


Figure D-2

Solution:

$$\vec{C}\bullet\vec{C} = \left(\vec{A}-\vec{B}\right)\bullet\left(\vec{A}-\vec{B}\right)$$
$$= \vec{A}\bullet\vec{A}-\vec{A}\bullet\vec{B}-\vec{B}\bullet\vec{A}+\vec{B}\bullet\vec{B}$$
$$\therefore \quad \boxed{C^2 = A^2+B^2-2AB\cos\theta}$$

This is the ***law of cosines***.

4) Cross Product (Vector Product) of Two Vectors

- The cross product of two vectors is defined by

$$\vec{A}\times\vec{B} = AB\,\sin\theta\ \hat{n} \qquad \text{(a vector)} \qquad \left[\tau = \vec{r}\times\vec{F}\right]$$

where $\hat{n}$ is a unit vector pointing perpendicular to the plane of $\vec{A}$ and $\vec{B}$.

The direction of $\hat{n}$ is determined by using **right-hand rule**: let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{n}$.

In Figure V-1, $\vec{A}\times\vec{B}$ points into the page; $\vec{B}\times\vec{A}$ points out of the page.

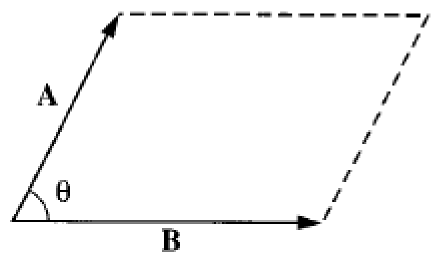


Figure V-1

- The cross product is **not *commutative***: $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$
- The cross product is ***distributive***: $\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$
- Geometrically,
  $\vec{A}\times\vec{B}$ gives the area of the parallelogram generated by $\vec{A}$ and $\vec{B}$ (Figure V-1).
- If the two vectors are parallel, then $\vec{A}\times\vec{B} = 0$.
  If two vectors are perpendicular, then $\left|\vec{A}\times\vec{B}\right| = AB$.

Vector Algebra: Component Form

Let $\hat{i}, \hat{j},$ and $\hat{k}$ be unit vectors parallel to $x, y,$ and z axes respectively (Figure $\mathrm{V_A} - 1$).

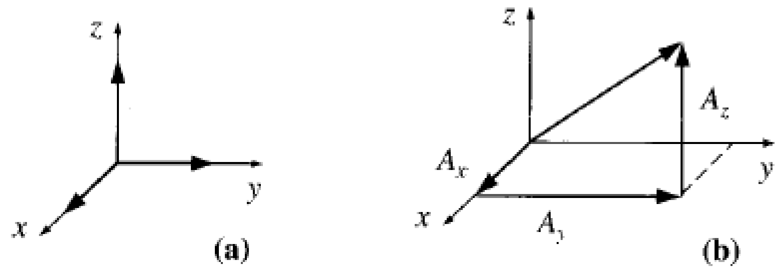


Figure $\mathrm{V_A} - 1$

Vectors $\vec{A}$ and $\vec{B}$ can be expressed in terms of **basis vectors** $\hat{i}, \hat{j},$ and $\hat{k}$ **:**

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \quad \text{and} \quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

1) **Addition of Two Vectors**

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{\mathrm{k}}$$

2) **Multiplication by a Scalar**

$$a\vec{A} = (aA_x)\hat{i} + (aA_y)\hat{j} + (aA_z)\hat{k}$$

3) **Dot Product of Two Vectors**

$$\vec{A}\bullet\vec{B} = A_xB_x + A_yB_y + A_zB_z \qquad \left[\ \hat{i}\bullet\hat{i} = \hat{j}\bullet\hat{j} = \hat{k}\bullet\hat{k} = 1\,; \quad \hat{i}\bullet\hat{j} = \hat{j}\bullet\hat{k} = \hat{k}\bullet\hat{i} = 0\ \right]$$

For any vector $\vec{A}$: $A = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$

4) **Cross Product of Two Vectors**

$$\mathbf{A}\times\mathbf{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

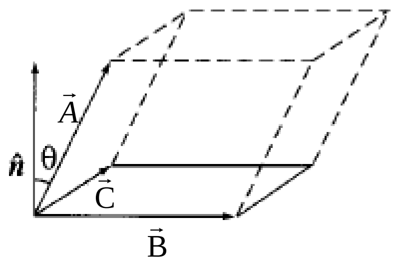
$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

$$\begin{bmatrix} \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0 & & \\ \hat{i}\times\hat{j} = \hat{k}, & \hat{j}\times\hat{k} = \hat{i}, & \hat{k}\times\hat{i} = \hat{j} \\ \hat{j}\times\hat{i} = -\hat{k}, & \hat{k}\times\hat{j} = -\hat{i}, & \hat{i}\times\hat{k} = -\hat{j} \end{bmatrix}$$

TRIPLE PRODUCTS

Scalar triple product: $\vec{A}\bullet(\vec{B}\times\vec{C})$

- For a parallelepiped generated by $\vec{A},\vec{B}$ and $\vec{\mathrm{C}}$.



$$\vec{A}\bullet(\vec{B}\times\vec{C}) = \left|\vec{B}\times\vec{C}\right|\left(A\cos\theta\right)$$

= Area of the base of parallelepiped $\times$ Altitude of the parallelepiped

=Volume of the parallelepiped generated by $\vec{A},\vec{B}$ and $\vec{C}$

$\therefore$ **Geometrically,** $\vec{A}\bullet(\vec{B}\times\vec{C})$ is the volume of the parallelepiped generated by $\vec{A},\vec{B}$ and $\vec{\mathrm{C}}$.

- $\vec{A}\bullet(\vec{B}\times\vec{C}) = \vec{B}\bullet(\vec{C}\times\vec{A}) = \vec{C}\bullet(\vec{A}\times\vec{B})$
- <u>In component form,</u>

$$\vec{A}\bullet(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & \mathrm{A}_y & \mathrm{A}_z \\ B_x & \mathrm{B}_y & \mathrm{B}_z \\ C_x & \mathrm{C}_y & \mathrm{C}_z \end{vmatrix}$$

- The dot and cross can be interchanged:

$$\vec{A}\bullet(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\bullet\vec{C}$$

Vector triple product: $\vec{A}\times(\vec{B}\times\vec{C})$

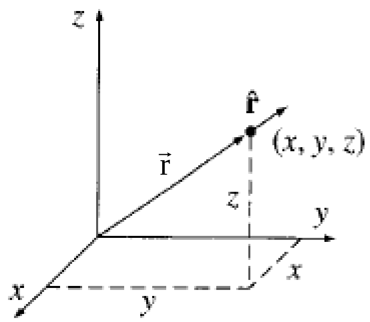
- The vector triple product can be simplified by the **BAC-CAB** rule:

$$\boxed{\vec{A}\bullet(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\bullet\vec{C}) - \vec{C}(\vec{A}\bullet\vec{B})}$$

POSITION, DISPLACEMENT, AND SEPARATION VECTORS

Position Vector:

- The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z).



Position Vector: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

- ➢ $r = \sqrt{x^2 + y^2 + z^2} = \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}$
- ➢ $\hat{r} = \dfrac{\vec{r}}{r} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

Infinitesimal Displacement Vector

- The **infinitesimal displacement vector**, from (x, y, z) to $((x+dx, y+dy, z+dz)$, is

$$\boxed{d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}}$$

Separation Vector

- In electrodynamics we frequently encounter problems involving *two points* – typically, a **source point**, $\vec{r}'$, where an electric charge is located, and a **field point**, $\vec{r}$, at which we are calculating the electric or magnetic field (Figure S-1).

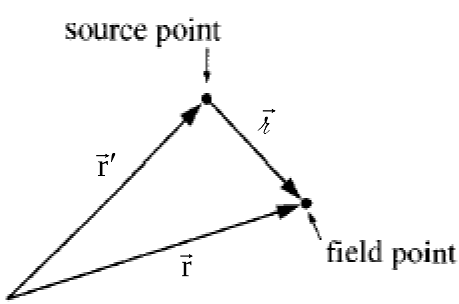


Figure S-1

The **separation vector** from the source point to the field point is

$$\vec{\mathscr{r}} = (\vec{r} - \vec{r}')$$
$$= (x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}$$

THE OPERATOR ∇

- The vector differential operator ***del (nabla)***, defined in Cartesian coordinates as

$$\boxed{\nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}}.$$

  Of course, ***del*** is not a vector, in the usual sense. Indeed, it is without specific meaning until we provide it with a function to act upon.

- There are three ways the operator ∇ can act:

  1. On a scalar function T : ∇T (the gradient);
  2. On a vector function $\vec{\mathrm{v}}$, via the dot product: $\nabla \cdot \vec{\mathrm{v}}$ (the divergence);
  3. On a vector function $\vec{\mathrm{v}}$, via the cross product: $\nabla \times \vec{\mathrm{v}}$ (the curl).

GRADIENT

Suppose that we have a function of three variables $-$ say, the temperature $T(x, y, z)$ in a room.
A theorem on partial derivatives states that

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz \qquad \ldots\ldots\ldots\ldots\ldots.. \quad \text{(G-1)}$$

This tells us how T changes when we alter all three variables by the infinitesimal amount dx, dy, dz.

Equation G-1 can be written as

$$dT = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)\cdot\left(dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k}\right)$$

$$= (\nabla T)\cdot\left(d\vec{l}\right)$$

where $\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$ is the gradient of T.

Geometrical Interpretation of the Gradient*:*

$$dT = (\nabla T)\cdot\left(d\vec{l}\right) = |\nabla T|\left|d\vec{l}\right|\cos\theta,$$

where θ is the angle between ∇T and $d\vec{l}$.

Now, if we fix the magnitude $\left|d\vec{l}\right|$ and search around in various directions, the maximum change in T evidently occurs when $\theta = 0$ (for then $\cos\theta = 1$). That is, for a fixed distance $\left|d\vec{l}\right|$, dT is greatest when we move in the same direction as ∇T. Thus:

The gradient ∇T points in the direction of maximum increase of the function T.

Moreover:

The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

Example 1

Suppose that the temperature T at the point (x, y, z) is given by the equation $T = x^2 - y^2 + xyz + 273$. In which direction is the temperature increasing most rapidly at $(-1, 2, 3)$ and at what rate?

Solution:

Here, $T = x^2 - y^2 + xyz + 273$

$$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}\left(x^2 - y^2 + xyz + 273\right)\hat{i} + \frac{\partial}{\partial y}\left(x^2 - y^2 + xyz + 273\right)\hat{j} + \frac{\partial}{\partial z}\left(x^2 - y^2 + xyz + 273\right)\hat{k}$$

$$= (2x + yz)\hat{i} + (-2y + xz)\hat{j} + (xy)\hat{k}$$

$$= 4\hat{i} + -7\hat{j} - 2\hat{k} \qquad \text{at } (-1, 2, 3)$$

The increase in temperature is fastest in the direction of this vector.

The rate of increase is $|\nabla T| = \sqrt{(4)^2 + (-7)^2 + (-2)^2} = \sqrt{69}$

Note:

Gravitational Potential Energy near the Earth

$$U = mgz$$

where z is the height from some arbitrary reference level.

- $$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

  $$= \frac{\partial}{\partial x}(mgz)\hat{i} + \frac{\partial}{\partial y}(mgz)\hat{j} + \frac{\partial}{\partial z}(mgz)\hat{k}$$

  $$= mg\hat{k}$$

- Gravitational force, $\vec{F} = mg\hat{k} = -(-mg)\hat{k} = -\nabla U$

So, the maximum change in gravitational potential energy is vertically upwards the centre of Earth.

Gradient of a scalar field T

$$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

- Grad $= \nabla$, turns a scalar field into a vector field.
- ∇T points in the direction of maximum increase of T.
- $|\nabla T|$ is the rate of maximum increase.

THE DIVERGENCE

- The divergence of a vector function $\vec{v}$:

$$\text{div}\,\vec{v} = \nabla \cdot \vec{v} \qquad \rightarrow \text{a scalar}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

- **Geometrical Interpretation**:
  The **divergence** of a vector function $\vec{v}$, $\nabla \cdot \vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point in question.

  For example,
  The vector function in Figure D_1- a has a large positive divergence.

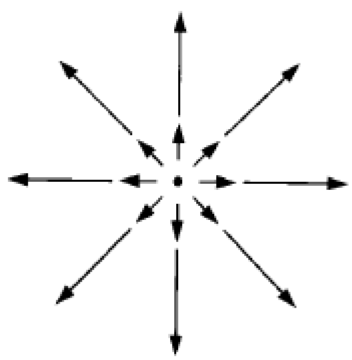


Figure D_1- a

The vector function in Figure D_1- b has zero divergence.

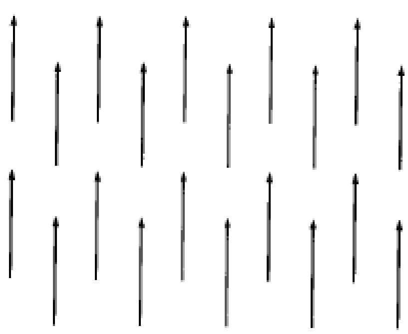


Figure D_1- b

- Imagine standing at the edge of a pond. Sprinkle some sawdust or pine needles on the surface. If the material spreads out, then you dropped it at a point of positive divergence; if it collects together, you dropped it at a point of negative divergence. (The vector function $\vec{v}$ in this model is the velocity of water.)

- A point of positive divergence is a source, or "faucet"; a point of negative divergence is a sink, or "drain".

- If at some point P,

  $\nabla \cdot \vec{v} > 0$, then $\vec{v}$ has a source at P.

  $\nabla \cdot \vec{v} < 0$, then $\vec{v}$ has a sink at P.

  $\nabla \cdot \vec{v} = 0$, then $\vec{v}$ is said to be solenoidal.

<u>Example</u>:

(1) Calculate the divergence of vector function $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution:

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1+1+1$$

$$= 3$$

(2) If $\vec{A} = x^2 z\hat{i} - 2y^3 z^2 \hat{j} + xy^2 z\hat{k}$, **find** $\nabla \cdot \vec{A}$ **at point** $(1,-1,1)$.

Solution:

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(-2y^3 z^2) + \frac{\partial}{\partial z}(xy^2 z)$$

$$= 2xz - 6y^2 z^2 + xy^2$$

$$= 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 \qquad \text{at } (1,-1,1)$$

$$= 2 - 6 + 1$$

$$= -3$$

THE CURL

- The curl of a vector function $\vec{v}$:

$$\text{curl } \vec{v} = \nabla \times \vec{v} \qquad \rightarrow \text{a vector}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

- **Geometrical Interpretation**:
  The **curl** of a vector function $\vec{v}$, $\nabla \times \vec{v}$ is a measure of how much the vector $\vec{v}$ "curls around" the point in question.

  For example,
  The vector function in Figure $C-1$ has a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest.

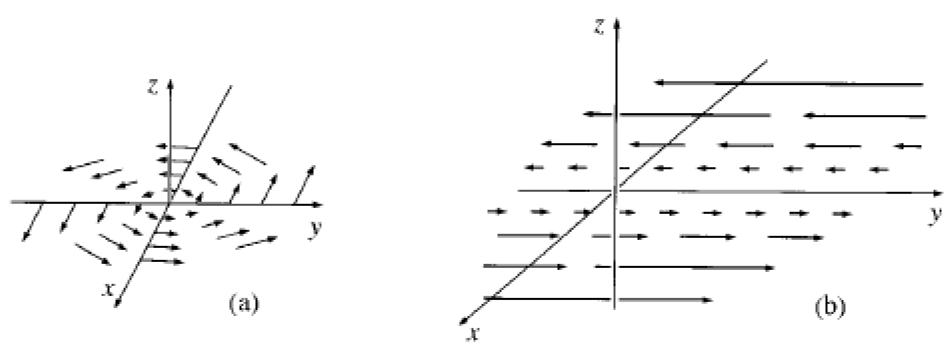


Figure $C-1$

- Imagine you are standing at the edge of a pond. Float a small paddlewheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of nonzero curl. (The vector function $\vec{v}$ in this model is the velocity of water.)

  A whirlpool would be a region of large curl.

- If $\nabla \times \vec{v} = 0$, then $\vec{v}$ is **irrotational**.

Example:

1. **Calculate curl of the vector function** $\vec{v} = x\hat{j} - y\hat{i}$.

   Solution:

$$\nabla \times \vec{v} = \nabla \times \left[x\hat{j} - y\hat{i} \right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x) \right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(-y) \right] + \hat{k}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right]$$

$$= \hat{i}[0] - \hat{j}[0-0] + \hat{k}\left[1-(-1)\right]$$

$$= 2\hat{k}$$

2. **If** $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, **find** $\nabla \times \vec{A}$ **at point** $(1,-1,1)$.

   Solution:

$$\nabla \times \vec{v} = \nabla \times \left[xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k} \right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(2yz^4\right) - \frac{\partial}{\partial z}\left(-2x^2yz\right) \right] - \hat{j}\left[\frac{\partial}{\partial x}\left(2yz^4\right) - \frac{\partial}{\partial z}\left(xz^3\right) \right] + \hat{k}\left[\frac{\partial}{\partial x}\left(-2x^2yz\right) - \frac{\partial}{\partial y}\left(xz^3\right) \right]$$

$$= \hat{i}\left[2z^4 + 2x^2y \right] - \hat{j}\left[0 - 3xz^2 \right] + \hat{k}\left[-4xyz - 0\right]$$

$$= \left[2z^4 + 2x^2y \right]\hat{i} + 3xz^2\,\hat{j} - 4xyz\hat{k}$$

$$= \left[2(1)^4 + 2(1)^2(-1) \right]\hat{i} + 3(1)(1)^2\,\hat{j} - 4(1)(-1)(1)\hat{k} \qquad \text{at } (1,-1,1)$$

$$= 3\hat{j} + 4\hat{k}$$

SECOND DERIVATIVES

By applying ∇ *twice* we can construct five species of ***second derivatives***.

The gradient ∇T ***is a vector***.

1. Divergence of gradient: $\nabla \cdot (\nabla T)$ $\rightarrow$ a scalar
2. Curl of gradient: $\nabla \times (\nabla T)$ $\rightarrow$ a vector

The divergence $\nabla \cdot \vec{v}$ ***is a scalar.***

3. Gradient of divergence: $\nabla (\nabla \cdot \vec{v})$ $\rightarrow$ a vector

The curl $\nabla \times \vec{v}$ ***is a vector.***

4. Divergence of curl: $\nabla \cdot (\nabla \times \vec{v})$ $\rightarrow$ a scalar
5. Curl of curl: $\nabla \times (\nabla \times \vec{v})$ $\rightarrow$ a vector

LAPLACIAN

- Laplacian Operator: $\nabla \cdot \nabla = \nabla^2$
- The **Laplacian** of a scalar T is a scalar.

  The **Laplacian** of $T : \nabla \cdot (\nabla T) = \nabla^2 T = \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}$
- The **Laplacian** of a vector $\vec{v}$ is a vector.

  The **Laplacian** of $\vec{v}$ $: \nabla^2 \vec{v} \equiv (\nabla^2 v_x)\hat{i} + (\nabla^2 v_y)\hat{j} + (\nabla^2 v_z)\hat{k}$

  The **Laplacian** of $\vec{v} : \nabla^2 \vec{v} = (\nabla \cdot \nabla)\vec{v} \neq \nabla (\nabla \cdot \vec{v})$

SUMMARY

- **Gradient**
  - **Grad =** ∇ **,**

    $\nabla T \equiv \hat{i}\,\frac{\partial T}{\partial x} + \hat{j}\,\frac{\partial T}{\partial y} + \hat{k}\,\frac{\partial T}{\partial z}$ [T = a scalar function]
  - ∇T is a vector quantity.
  - The gradient ∇T points in the direction of maximum increase of the function T .

    - $\nabla r = \nabla\left(\sqrt{x^2+y^2+z^2}\right) = \hat{r}$
    - $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3}$
    - $\nabla(r^n) = nr^{n-1}\hat{r} = nr^{n-2}\vec{r}$

- **The Divergence:**
  - **div** = $\nabla\bullet$ **,**

    $\nabla\cdot\vec{\mathrm{v}} \equiv \frac{\partial \mathrm{v}_x}{\partial x} + \frac{\partial \mathrm{v}_y}{\partial y} + \frac{\partial \mathrm{v}_z}{\partial z}$
  - $\nabla\cdot\vec{\mathrm{v}}$ is a scalar.
  - $\nabla\cdot\vec{\mathrm{v}}$ is a measure of how much the vector $\vec{\mathrm{v}}$ spreads out from the point in question.
    - $\nabla\cdot\hat{r} = \frac{2}{r}$.

- **The Curl:**
  - Curl = $\nabla\times$ **,**

    $$\nabla\times\vec{\mathrm{v}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{v}_x & \mathrm{v}_y & \mathrm{v}_z \end{vmatrix} = \hat{i}\left(\frac{\partial \mathrm{v}_z}{\partial y} - \frac{\partial \mathrm{v}_y}{\partial z}\right) - \hat{j}\left(\frac{\partial \mathrm{v}_z}{\partial x} - \frac{\partial \mathrm{v}_x}{\partial z}\right) + \hat{k}\left(\frac{\partial \mathrm{v}_y}{\partial x} - \frac{\partial \mathrm{v}_x}{\partial y}\right)$$
  - $\nabla\times\vec{\mathrm{v}}$ is a measure of how much the vector $\vec{\mathrm{v}}$ “curls around” the point in question.

    - $\nabla\times\frac{\hat{r}}{r^2} = \nabla\times\frac{\vec{r}}{r^3} = 0$.

LAPLACIAN

- The **Laplacian** of T : $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$
- The **Laplacian** of $\vec{\mathrm{v}}$: $\nabla^2\vec{\mathrm{v}} \equiv \left(\nabla^2 \mathrm{v}_x\right)\hat{i} + \left(\nabla^2 \mathrm{v}_y\right)\hat{j} + \left(\nabla^2 \mathrm{v}_z\right)\hat{k}$

PROBLEMS

(1) Calculate the gradient of the function $f(x,y,z) = e^x \sin y \ \ln z$ **.**

Solution:

$$\nabla f = \hat{i}\,\frac{\partial f}{\partial x} + \hat{j}\,\frac{\partial f}{\partial y} + \hat{k}\,\frac{\partial f}{\partial z}$$

$$= \hat{i}\,\frac{\partial}{\partial x}\left(e^x \sin y \ \ln z\right) + \hat{j}\,\frac{\partial}{\partial y}\left(e^x \sin y \ \ln z\right) + \hat{k}\,\frac{\partial}{\partial z}\left(e^x \sin y \ \ln z\right)$$

$$= \left(e^x \sin y \ \ln z\right)\hat{i} + \left(e^x \cos y \ \ln z\right)\hat{j} + \left(e^x \sin y \ \frac{1}{z}\right)\hat{k}$$

(2) Let $\vec{r}$ **be the position vector and** r **be its length. Show that**

(a) $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3}$.

(b) $\nabla r^n = n r^{n-2}\vec{r} = n r^{n-1}\hat{r}$.

Solution:

(a)
$$\nabla\left(\frac{1}{r}\right) == \hat{i}\,\frac{\partial}{\partial x}\left(r^{-1}\right) + \hat{j}\,\frac{\partial}{\partial y}\left(r^{-1}\right) + \hat{k}\,\frac{\partial}{\partial z}\left(r^{-1}\right)$$

$$= \hat{i}\left[(-1)r^{-2}\,\frac{\partial r}{\partial x}\right] + \hat{j}\left[(-1)r^{-2}\,\frac{\partial r}{\partial y}\right] + \hat{k}\left[(-1)r^{-2}\,\frac{\partial r}{\partial z}\right]$$

$$= \hat{i}\left[(-1)r^{-2}\left(\frac{x}{r}\right)\right] + \hat{j}\left[(-1)r^{-2}\left(\frac{y}{r}\right)\right] + \hat{k}\left[(-1)r^{-2}\left(\frac{z}{r}\right)\right] \qquad \left[\begin{array}{l} \because r^2 = x^2 + y^2 + z^2 \\ \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r} \end{array}\right]$$

$$= -\frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{r^3}$$

$$\therefore \boxed{\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}}$$

(b)
$$\nabla r^n = \hat{i}\,\frac{\partial}{\partial x}\left(r^n\right) + \hat{j}\,\frac{\partial}{\partial y}\left(r^n\right) + \hat{k}\,\frac{\partial}{\partial z}\left(r^n\right) = \hat{i}\left[n r^{n-1}\,\frac{\partial r}{\partial x}\right] + \hat{j}\left[n r^{n-1}\,\frac{\partial r}{\partial y}\right] + \hat{k}\left[n r^{n-1}\,\frac{\partial r}{\partial z}\right]$$

$$= \hat{i}\left[n r^{n-1}\left(\frac{x}{r}\right)\right] + \hat{j}\left[n r^{n-1}\left(\frac{y}{r}\right)\right] + \hat{k}\left[n r^{n-1}\left(\frac{z}{r}\right)\right] \qquad \left[\begin{array}{l} \because r^2 = x^2 + y^2 + z^2 \\ \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r} \end{array}\right]$$

$$= n r^{n-1}\,\frac{\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{r}$$

$$\therefore \boxed{\nabla r^n = n r^{n-2}\vec{r} = n r^{n-1}\hat{r}}$$

(3) Calculate the divergence of vector function $\vec{v} = xyz(x\hat{i} + y\hat{j} + z\hat{k})$.

Solution:

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(x^2yz\right) + \frac{\partial}{\partial y}\left(xy^2z\right) + \frac{\partial}{\partial z}\left(xyz^2\right)$$

$$= 2xyz+2xyz+2xyz$$

$$= 6xyz$$

(4) If $\vec{r}$ **is the position vector, then show that** $\nabla \cdot \hat{r} = \frac{2}{r}$.

Solution:

$$\nabla \cdot \hat{r} = \nabla \cdot \frac{\vec{r}}{r} = \frac{1}{r}\left(\nabla \cdot \vec{r}\right) + \nabla\left(\frac{1}{r}\right) \cdot \vec{r}$$

$$= \frac{1}{r}(3) + \left[(-1)r^{-1-2}\vec{r}\right] \cdot \vec{r} \qquad \left[\because \nabla \cdot \vec{r} = 3; \quad \nabla r^n = nr^{n-2}\vec{r}\right]$$

$$= \frac{3}{r} - \frac{1}{r} \qquad \left[\because \vec{r} \cdot \vec{r} = r^2\right]$$

$$= \frac{2}{r}$$

(5) Calculate the divergence and curl of the vector function $\vec{v} = y^2\hat{i} + \left(2xy + z^2\right)\hat{j} + 2yz\hat{k}$.

Solution:

(a) $$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(y^2\right) + \frac{\partial}{\partial y}\left(2xy + z^2\right) + \frac{\partial}{\partial z}\left(2yz\right)$$

$$= 0+2x+2y$$

$$= 2(x+y)$$

(b) $$\nabla \times \vec{v} = \nabla \times \left[y^2\hat{i} + \left(2xy + z^2\right)\hat{j} + 2yz\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}\left(2xy + z^2\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}\left(y^2\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(2xy + z^2\right) - \frac{\partial}{\partial y}\left(y^2\right)\right]$$

$$= \hat{i}\left[2z - 2z\right] - \hat{j}\left[0 - 0\right] + \hat{k}\left[2y - 2y\right]$$

$$= 0$$

(6) Calculate the Laplacian of the following functions:

(a) $T_a = \sin x \, \sin y \, \sin z$ (b) $T_b = e^{-5x} \sin 4y \, \cos 3z$ (c) $\vec{\text{v}} = x^2 \hat{i} + 3xz^2 \hat{j} - 2xz\hat{k}$

Solution:

$$(a)\nabla^2 T_a = \frac{\partial^2 T_a}{\partial x^2} + \frac{\partial^2 T_a}{\partial y^2} + \frac{\partial^2 T_a}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}(\sin x \, \sin y \, \sin z) + \frac{\partial^2}{\partial y^2}(\sin x \, \sin y \, \sin z) + \frac{\partial^2}{\partial z^2}(\sin x \, \sin y \, \sin z)$$

$$= -\sin x \, \sin y \, \sin z - \sin x \, \sin y \, \sin z - \sin x \, \sin y \, \sin z$$

$$= -3\sin x \, \sin y \, \sin z$$

$$(b)\nabla^2 T_b = \frac{\partial^2 T_b}{\partial x^2} + \frac{\partial^2 T_b}{\partial y^2} + \frac{\partial^2 T_b}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}\left(e^{-5x} \sin 4y \, \cos 3z\right) + \frac{\partial^2}{\partial y^2}\left(e^{-5x} \sin 4y \, \cos 3z\right) + \frac{\partial^2}{\partial z^2}\left(e^{-5x} \sin 4y \, \cos 3z\right)$$

$$= 25e^{-5x} \sin 4y \, \cos 3z - 16e^{-5x} \sin 4y \, \cos 3z - 9e^{-5x} \sin 4y \, \cos 3z$$

$$= 0$$

$$(c)\nabla^2 \vec{\text{v}} = \left(\nabla^2 \text{v}_x\right)\hat{i} + \left(\nabla^2 \text{v}_y\right)\hat{j} + \left(\nabla^2 \text{v}_z\right)\hat{k}$$

$$= \left[\nabla^2\left(x^2\right)\right]\hat{i} + \left[\nabla^2\left(3xz^2\right)\right]\hat{j} + \left[\nabla^2\left(-2xz\right)\right]\hat{k}$$

$$= \left[\frac{\partial^2}{\partial x^2}\left(x^2\right) + \frac{\partial^2}{\partial y^2}\left(x^2\right) + \frac{\partial^2}{\partial z^2}\left(x^2\right)\right]\hat{i} + \left[\frac{\partial^2}{\partial x^2}\left(3xz^2\right) + \frac{\partial^2}{\partial y^2}\left(3xz^2\right) + \frac{\partial^2}{\partial z^2}\left(3xz^2\right)\right]\hat{j}$$

$$+ \left[\frac{\partial^2}{\partial x^2}(-2xz) + \frac{\partial^2}{\partial y^2}(-2xz) + \frac{\partial^2}{\partial z^2}(-2xz)\right]\hat{k}$$

$$= (2+0+0)\hat{\text{i}} + (0+0+6\text{x})\hat{j} + (0+0+0)\hat{k}$$

$$= 2\hat{\text{i}} + 6\text{x}\hat{\text{j}}$$

❖ **The curl of a gradient is always zero**: $\nabla\times(\nabla T)=0$

Hint:

$$\nabla\times(\nabla T)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z}\end{vmatrix}=\hat{i}\left(\frac{\partial^2 T}{\partial y\partial z}-\frac{\partial^2 T}{\partial z\partial y}\right)-\hat{j}\left(\frac{\partial^2 T}{\partial x\partial z}-\frac{\partial^2 T}{\partial z\partial x}\right)+\hat{k}\left(\frac{\partial^2 T}{\partial x\partial y}-\frac{\partial^2 T}{\partial y\partial x}\right)=0$$

$$\left[\because \frac{\partial^2 T}{\partial x\partial y}=\frac{\partial^2 T}{\partial y\partial x}\right]$$

❖ **The divergence of a curl is always zero**: $\nabla\cdot(\nabla\times\vec{\mathrm{v}})=0$

Hint:

- $$\nabla\times\vec{\mathrm{v}}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \mathrm{v}_x & \mathrm{v}_y & \mathrm{v}_z\end{vmatrix}=\hat{i}\left(\frac{\partial \mathrm{v}_z}{\partial y}-\frac{\partial \mathrm{v}_y}{\partial z}\right)-\hat{j}\left(\frac{\partial \mathrm{v}_z}{\partial x}-\frac{\partial \mathrm{v}_x}{\partial z}\right)+\hat{k}\left(\frac{\partial \mathrm{v}_y}{\partial x}-\frac{\partial \mathrm{v}_x}{\partial y}\right)$$
- $$\nabla\cdot\nabla\times\vec{\mathrm{v}}=\left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\hat{i}\left(\frac{\partial \mathrm{v}_z}{\partial y}-\frac{\partial \mathrm{v}_y}{\partial z}\right)-\hat{j}\left(\frac{\partial \mathrm{v}_z}{\partial x}-\frac{\partial \mathrm{v}_x}{\partial z}\right)+\hat{k}\left(\frac{\partial \mathrm{v}_y}{\partial x}-\frac{\partial \mathrm{v}_x}{\partial y}\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial \mathrm{v}_z}{\partial y}-\frac{\partial \mathrm{v}_y}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \mathrm{v}_z}{\partial x}-\frac{\partial \mathrm{v}_x}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \mathrm{v}_y}{\partial x}-\frac{\partial \mathrm{v}_x}{\partial y}\right)$$

$$=0 \qquad \left[\because \frac{\partial}{\partial x}\left(\frac{\partial \mathrm{v}_z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial \mathrm{v}_z}{\partial x}\right)\right]$$

INTEGRAL CALCULUS

a) Line Integrals:

- If $\vec{F}$ is a vector, a line integral of $\vec{F}$ is written

$$\int_{a\ C}^{b} \vec{F}\cdot d\vec{l}\ ,$$

  where C is the curve along which the integration is performed, a and b the initial and final points on the curve, and $d\vec{l}$ is the infinitesimal displacement vector along the curve C.

- The line integral is a scalar.
- Line integral over a closed curve: $\oint_C \vec{F}\cdot d\vec{l}$
- Example of a line integral: The work done by a force $\vec{F}$: $W = \int \vec{F}\cdot d\vec{l}$
- For ***conservative force***: $\oint \vec{F}\cdot d\vec{l} = 0$

b) Surface Integrals:

- If $\vec{F}$ is a vector, a surface integral of $\vec{F}$ is written

$$\int_S \vec{F}\cdot d\vec{a}\ ,$$

  where S is the surface over which the integration is to be performed, and $d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface.

- Example: The flux of $\vec{E}$ through a surface S : $\Phi_E = \int_S \vec{E}\cdot d\vec{a}$
- Surface integral over a closed surface: $\oint_S \vec{F}\cdot d\vec{a}$
- If $\vec{v}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \vec{v}\cdot d\vec{a}$ represents the total mass per unit time passing through the surface [or flux].

c) Volume Integrals:

- A volume integral is an expression of the form

$$\int_V T\ d\tau$$

  where T is a scalar function and $d\tau$ is an infinitesimal volume element.

- Total charge q : $\boxed{q = \int_V \rho\ d\tau}$ where ρ is the volume charge density.

THE FUNDAMENTAL THEOREM OF CALCULUS

Suppose $f(x)$ is a function of one variable.

The fundamental theorem of calculus states:

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a) \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(F-1)}$$

Here, $\left(\frac{df}{dx}\right)dx$ is the infinitesimal change in f when you go form (x) to $(x+dx)$.

The fundamental theorem (F-1) says that there are two ways to determine the total change in the function: *either* subtract the values at the ends *o*r go step-by-step, adding up all the tiny increments as you go. You'll get the same answer either way.

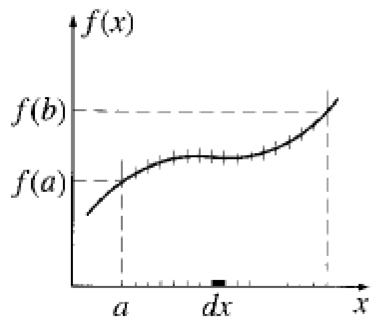


Figure F-A

The Fundamental Theorem for Gradients

Suppose we have a scalar function of three variables $f(x, y, z)$.

The total change in f in going from a to b is

$$\boxed{\int_a^b (\nabla f)\cdot d\vec{l} = f(\text{b}) - f(\text{a})}$$

Geometrical Interpretation:

Suppose you want to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up, or you could place altimeters at the top and the bottom, and subtract the two readings; you should get the same answer either way (that's the fundamental theorem)

Corollary 1: $\int_a^b (\nabla f)\cdot d\vec{l}$ is independent of path taken from a to b.

Corollary 2: $\oint (\nabla f)\cdot d\vec{l} = 0$

The Fundamental Theorem for Divergences [Gauss's Theorem]

The fundamental theorem for divergences states that:

$$\boxed{\int_V \left(\nabla\cdot\vec{v}\right)\,d\tau = \oint_S \vec{v}\cdot d\vec{a}} \qquad \text{................(F-2)}$$

Geometrical Interpretation:

If $\vec{v}$ represents the flow of an incompressible fluid, then the flux of $\vec{v}$ (the right side of Eq.(F-2)) is the total amount of fluid passing out through the surface, per unit time. Now, the divergence measures the "spreading out" of the vectors from a point – a place of high divergence is like a "faucet," pouring out liquid. If we have lots of faucets in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region.

$$\boxed{\int\left(\text{faucets within the volume}\right) = \oint\left(\text{flow out through the surface}\right)}$$

The Fundamental Theorem for Curls [Stoke's Theorem]

The fundamental theorem for curls states that:

$$\boxed{\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}} \qquad \text{................(F-3)}$$

Geometrical Interpretation:

The curl measures the "twist" of the vectors $\vec{v}$; a region of high curl is a whirlpool – if you put a tiny paddle wheel there, it will rotate. Now, the integral of the curl over some surface represents the "total amount of swirl," and we can determine that swirl just as well by going around the edge and finding how much the flow is following the boundary (Figure F-B).

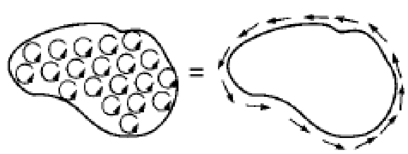


Figure F-B

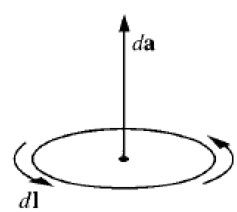


Figure F-C

You may find this a rather forced interpretation of Stokes' theorem, but it's a helpful mnemonic, if nothing else.

For a *closed* surface $d\vec{a}$ points in the direction of the outward normal; but for an *open* surface, the direction of $d\vec{a}$ is given by right-hand rule: If your fingers point in the direction of the line integral, then your thumb fixes the direction of $d\vec{a}$ (Figure-F-C)

Corollary 1: $\int_a^b \left(\nabla\times\vec{v}\right)\cdot d\vec{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint\left(\nabla\times\vec{v}\right)\cdot d\vec{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of Eq. (F-3) vanishes.

SPHERICAL POLAR COORDINATES

The spherical polar coordinates (r, θ, ϕ) of a point are defined in Figure S-A.

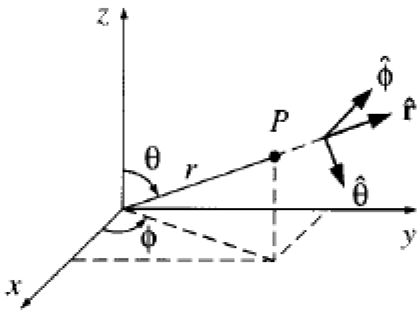


Figure S-A

$r \rightarrow$ the distance from the origin (the magnitude of the position vector)
$\theta \rightarrow$ the polar angle (the angle down from the z axis)
$\phi \rightarrow$ the azimuthal angle (the angle around from the x axis)

Figure S-A also shows unit vectors, $\hat{r}, \hat{\theta}, \hat{\phi}$, pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal basis set just like $(\hat{i}, \hat{j}, \hat{k})$.

For a vector $\vec{A}$:

$$\boxed{\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}}$$

where $A_r, A_\theta,$ and A_ϕ are the radial, polar, and azimuthal components of $\vec{A}$.

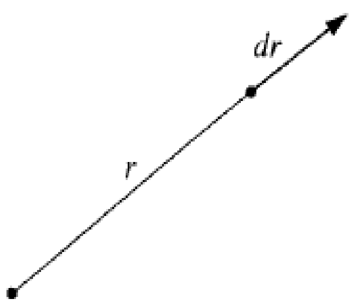


Figure S-1

An infinitesimal displacement in the $\hat{r}$ direction: $\boxed{dl_r = dr}$

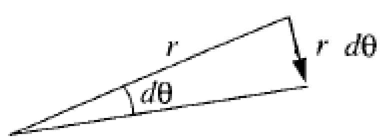


Figure S-2

An infinitesimal displacement in the $\hat{\theta}$ direction: $\boxed{dl_\theta = r d\theta}$

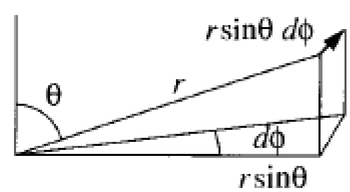
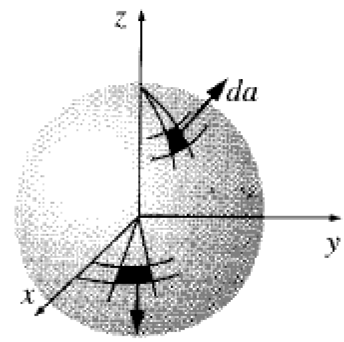


Figure S-3

An infinitesimal displacement in the $\hat{\theta}$ direction: $\boxed{dl_\phi = r \sin\theta d\phi}$

The general infinitesimal displacement vector:	**The infinitesimal volume element**:	**The surface element**:
$d\vec{l} = dl_r\hat{\text{r}} + dl_\theta\hat{\theta} + dl_\phi\hat{\phi}$ $= dr\,\hat{\text{r}} + \text{r d}\theta\ \hat{\theta} + \text{ r sin}\theta\text{ d}\phi$	$d\tau = r^2\sin\theta\ dr\ d\theta\ d\phi$	$da = r^2\sin\theta\ d\theta\ d\phi$ 
$r \to 0$ to ∞	$\theta \to 0$ to π	$\phi \to 0$ to 2π

Example S-A:

Find the volume of a sphere of radius of R.

Solution:

$$
\begin{aligned}
V &= \int d\tau \\
&= \int r^2 \sin\theta\ dr\ d\theta\ d\phi \\
&= \left(\int_{r=0}^{R} r^2 dr\right)\left(\int_{\theta=0}^{\pi} \sin\theta\ d\theta\right)\left(\int_{\phi=0}^{2\pi} d\phi\right) \\
&= \left(\frac{R^3}{3}\right)(2)(2\pi) \\
&= \frac{4}{3}\pi R^3
\end{aligned}
$$

POTENTIALS:

[1] If the curl of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be written as the gradient of a ***scalar potential*** (V):

$$\nabla \times \vec{F} = 0 \Leftrightarrow \vec{F} = -\nabla V$$

The minus sign is purely conventional.

Curl-less (or "irrotational") fields:

The following conditions are equivalent:

- $\nabla \times \vec{F} = 0$ everywhere.
- $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path, for any given end points.
- $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop.
- $\vec{F}$ is the gradient of some scalar, $\vec{F} = -\nabla V$

[2] If the divergence of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be expressed as the curl of a **vector potentials** ($\vec{A}$):

$$\nabla \cdot \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla \times \vec{A}$$

Divergence-less (or "solenoidal") fields:

The following conditions are equivalent:

- $\nabla \cdot \vec{F} = 0$ everywhere.
- $\int \vec{F} \cdot d\vec{a}$ is independent of surface, for any given boundary line.
- $\oint \vec{F} \cdot d\vec{a} = 0$ for any closed loop.
- $\vec{F}$ is the curl of some vector, $\vec{F} = \nabla \times \vec{A}$.

In all cases (*whatever* its curl and divergence may be) a vector field $\vec{F}$ can be written as the gradient of a scalar plus the curl of a vector:

$$\vec{F} = -\nabla V + \nabla \times \vec{A} \qquad \text{(always).}$$

SUMMARY

PRODUCT OF VECTORS

Scalar Product: $\vec{A}\bullet\vec{B} \equiv \text{AB}\cos\theta$

Vector Product: $\vec{A}\times\vec{B} \equiv \text{AB}\sin\theta\,\hat{n}$

Scalar Triple Product: $\vec{A}\bullet(\vec{B}\times\vec{C}) = \vec{B}\bullet(\vec{C}\times\vec{A}) = \vec{C}\bullet(\vec{A}\times\vec{B}) = \begin{vmatrix} A_x & \text{A}_y & \text{A}_z \\ B_x & \text{B}_y & \text{B}_z \\ C_x & \text{C}_y & \text{C}_z \end{vmatrix}$

Vector Triple Product: $\vec{A}\bullet(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\bullet\vec{C}) - \vec{C}(\vec{A}\bullet\vec{B})$

VECTOR DERIVATIVES

Cartesian

The Infinitesimal displacement vector: $d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Gradient: $\nabla T \equiv \hat{i}\,\frac{\partial T}{\partial x} + \hat{j}\,\frac{\partial T}{\partial y} + \hat{k}\,\frac{\partial T}{\partial z}$

Divergence: $\nabla\cdot\vec{\text{v}} \equiv \frac{\partial \text{v}_x}{\partial x} + \frac{\partial \text{v}_y}{\partial y} + \frac{\partial \text{v}_z}{\partial z}$

Curl: $\nabla\times\vec{\text{v}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{v}_x & \text{v}_y & \text{v}_z \end{vmatrix} = \hat{i}\left(\frac{\partial \text{v}_z}{\partial y} - \frac{\partial \text{v}_y}{\partial z}\right) - \hat{j}\left(\frac{\partial \text{v}_z}{\partial x} - \frac{\partial \text{v}_x}{\partial z}\right) + \hat{k}\left(\frac{\partial \text{v}_y}{\partial x} - \frac{\partial \text{v}_x}{\partial y}\right)$

Laplacian: $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

$$\nabla^2\vec{\text{v}} \equiv \left(\nabla^2\text{v}_x\right)\hat{i} + \left(\nabla^2\text{v}_y\right)\hat{j} + \left(\nabla^2\text{v}_z\right)\hat{k}$$

Spherical

The Infinitesimal displacement vector: $d\vec{l} = d\text{r}\,\hat{\text{r}} + \text{r}\,\text{d}\theta\,\hat{\theta} + \text{r}\sin\theta\,\text{d}\phi\,\hat{\phi}$

An element of surface area on the sphere of radius R**:** $da = R^2\sin\theta\, d\theta\, d\phi$

Gradient: $\nabla f \equiv \frac{\partial f}{\partial \text{r}}\hat{\text{r}} + \frac{1}{\text{r}}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{\text{r}\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$

Vector Identities

Product Rules

1) $\nabla(fg) = f(\nabla g) + g(\nabla f)$
2) $\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + (\nabla f) \cdot \vec{A}$
3) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
4) $\nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) + (\nabla f) \times \vec{A} = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f)$

Second Derivatives

5) $\nabla \cdot (\nabla \times \vec{A}) = 0$
6) $\nabla \times (\nabla f) = 0$
7) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Fundamental Theorems

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\vec{l} = f(b) - f(a)$

Divergence Theorem: $\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$ **[Gauss's Theorem]**

Curl Theorem: $\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$ **[Stokes' Theorem**

NOTES:

- Divergence of gradient: $\nabla \cdot \left(\nabla T\right) \equiv \nabla^2 T \to$ a scalar
- Curl of gradient: $\nabla \times \left(\nabla T\right) \to$ a vector
- Gradient of divergence: $\nabla\left(\nabla \cdot \vec{v}\right) \to$ a vector
- Divergence of curl: $\nabla \cdot \left(\nabla \times \vec{v}\right) \to$ a scalar
- Curl of curl: $\nabla \times \left(\nabla \times \vec{v}\right) \to$ a vector
- $\nabla \cdot \vec{F} = 0 \quad \Rightarrow \vec{F}$ is solenoidal

  $\nabla \cdot \vec{F} = 0 \;\; \& \; \nabla \cdot \nabla \times \vec{G} \quad \Rightarrow \vec{F}$ can be written as curl of a vector $\left[\vec{F} = \nabla \times \vec{G}\right]$
- $\nabla \times \vec{F} = 0 \quad \Rightarrow \vec{F}$ is irrotational

  $\nabla \times \vec{F} = 0 \;\; \& \; \nabla \times \left(\nabla T\right) = 0 \quad \Rightarrow \vec{F}$ can be written as gradient of a scalar function $\left[\vec{F} = \nabla T\right]$

NOTES:

- Which one of the following properties of a vector $\vec{F}$ does not allow us to get the identity, $\nabla\times\left(\nabla\times\vec{F}\right)=-\nabla^2\vec{F}$?

  [a] $\vec{F}$ is solenoidal .
  [b] $\nabla\cdot\vec{F}$ is a non-zero constant.
  [c] $\vec{F}$ is expressible to the curl of another vector.
  [d] $\vec{F}$ is expressible to the gradient of some scalar function.

  Hints: [d] $\left[\because \nabla\times\left(\nabla\times\vec{F}\right)=\nabla\left(\nabla\cdot\vec{F}\right)-\nabla^2\vec{F}\right]$

- The vector function $\vec{\mathrm{v}}=yz\hat{i}+zx\hat{j}+xy\hat{k}$ is ***solenoidal as well as irrotational***.

  Hints: $\nabla\cdot\vec{\mathrm{v}}=0$ and $\nabla\times\vec{\mathrm{v}}=0$.

- In two vectors $\vec{V}_1=\nabla\times\vec{F}$ and $\vec{V}_2=\nabla\phi$, $\vec{V}_1$ ***is solenoidal and*** $\vec{V}_2$ ***is irrotational.***

- $\nabla\times(\frac{\hat{r}}{r^2})=\nabla\times(\frac{\vec{r}}{r^3})=0$

- A vector field $\vec{A}$ is conservative if $\vec{A}=\nabla\phi$

- $\nabla\cdot\hat{\mathrm{r}}=\dfrac{2}{\mathrm{r}}$ $\left[\hat{\mathrm{r}}=\dfrac{\vec{\mathrm{r}}}{\mathrm{r}}=\dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right]$

-
$$r^2=x^2+y^2+z^2$$
$$\Rightarrow \frac{\partial}{\partial x}\left(r^2\right)=\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$
$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\right)\frac{\partial r}{\partial x}=2x$$
$$\Rightarrow 2r\frac{\partial r}{\partial x}=2x$$
$$\Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$$

DO YOURSELF
CHAPTER 1: VECTOR ANALYSIS

1. If the curl of a vector function is zero, then the vector function can be expressed as
   [a] the gradient of a scalar function.
   [b] the curl of some other vector function.
   [c]the divergence of some other vector field.
   [d] the gradient of another vector function.
2. The vector function $\vec{v} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is
   [a] solenoidal but not irrotational. [b] irrotational but not solenoidal.
   [c] solenoidal as well as irrotational. [d] neither solenoidal nor irrotational.
3. The Laplacian of the vector function $\vec{v} = x^2 y\hat{i} + (x^2 - y)\hat{k}$ is
   [a] zero [b] $2y\hat{i} + 2\hat{k}$ [c] $2(y+1)$ [d] $2y\hat{i}$
4. The Laplacian of the function $\vec{v} = xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k}$ is
   [a] 6 [b] $2\hat{i} + 2\hat{j} + 2\hat{k}$ [c] $2x + 2y + 2z$ [d] $2x\hat{i} + 2y\hat{j} + 2z\hat{k}$
5. Which of the following statements is NOT CORRECT?
   [a]The gradient ∇T points in the direction of maximum increase of the function T.
   [b]The curl of a gradient is always zero.
   [c]The divergence of a curl is always zero.
   [d]The curl of a curl is always zero.
6. Which one of the following statements is NOT CORRECT?
   [a] the divergence of gradient is a scalar. [b] the gradient of divergence is a vector.
   [c] the Laplacian of a vector is a scalar. [d] the curl of curl is a vector.
7. The Laplacian of the function $T = x(x+2y) + 3z + 4$ is
   [a] $2\hat{i}$ [b] 2 [c] $2(2x+y)+3$ [d] $2(x+y)\hat{i} + 2x\hat{j} + 3\hat{k}$
8. If $\vec{F}_1 = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{F}_2 = yz\hat{i} + zx\hat{j} + xy\hat{k}$, then
   [a] $\nabla \cdot \vec{F}_1 = 0$ and $\nabla \cdot \vec{F}_2 = 0$ [b] $\nabla \cdot \vec{F}_1 = 0$ and $\nabla \times \vec{F}_2 = 0$
   [c] $\nabla \times \vec{F}_1 = 0$ and $\nabla \cdot \vec{F}_2 = 0$ [d] $\nabla \times \vec{F}_1 = 0$ and $\nabla \times \vec{F}_2 = 0$
9. If $\vec{A}$ and $\vec{B}$ are two vectors, then which of the following expression does not have any physical meaning?
   [a] $\nabla\left(\vec{A}\cdot\vec{B}\right)$ [b] $\nabla\cdot\left(\vec{A}\times\vec{B}\right)$ [c] $\nabla\times\left(\vec{A}\times\vec{B}\right)$ [d] $\nabla\times\left(\vec{A}\cdot\vec{B}\right)$
10. A vector field $\vec{A}$ is conservative if
    [a] $\nabla\times\vec{A} \neq 0$ [b] $\nabla\bullet\vec{A} = 0$ [c] $\vec{A} = \nabla\phi$ [d] $\nabla\bullet\vec{A} = 1$
11. Which one of the following properties of a vector $\vec{F}$ does not allow us to get the identity,
    $$\nabla\times\left(\nabla\times\vec{F}\right) = -\nabla^2\vec{F}\ ?$$
    [a] $\vec{F}$ is solenoidal .
    [b] $\nabla\cdot\vec{F}$ is a non-zero constant.
    [c] $\vec{F}$ is expressible to the curl of another vector.
    [d] $\vec{F}$ is expressible to the gradient of some scalar function.

12. The divergence of the vector function $\vec{F} = xe^{-x}\hat{i} + y\hat{j} - xz\hat{k}$ is

[a] $(x-1)(1+e^{-x})$ [b] $(1-x)(1+e^{-x})$ [c] $(1-x)(1-e^{-x})$ [d] $(1-x)(e^{-x}-1)$

13. Given that $\vec{F}$ and $\vec{G}$ are vector fields and f is a scalar field, which of the following statement is NOT TRUE?

[a] $\nabla\cdot(\nabla\times\vec{F}) = 0$ [b] $\nabla\times(\nabla\times\vec{G}) = \nabla(\nabla\cdot\vec{G}) - \nabla^2\vec{G}$

[c] $\nabla\times(\nabla f) = 0$ [d] $\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\nabla\times\vec{F}) + \vec{F}\cdot(\nabla\times\vec{G})$

14. If the curl of a vector field ($\vec{F}$) vanishes (everywhere), then which one of the following conditions is NOT CORRECT?

[a] $\oint \vec{F}\cdot d\vec{l} = 0$ for any closed loop

[b] $\int_a^b \vec{F}\cdot d\vec{l}$ is independent of path, for any given end points

[c] $\vec{F}$ is the gradient of some scalar, $\vec{F} = -\nabla V$

[d] $\nabla\cdot\vec{F} = 0$, everywhere

Fill in the blanks.

1. The gradient of the function $f = \sin x + \cos x$ ……
2. The gradient of the function $t = x^2 y + e^z$ at the point $p(1,5,-2)$ is ……….
3. The divergence of the vector function $\vec{D} = e^{-x}\sin y\hat{i} - e^{-x}\cos y\hat{j}$ is …….
4. The curl of the vector function $\vec{A} = (y\cos ax)\hat{i} + (y+e^x)\hat{k}$ is ……
5. The Laplacian of the function $T = \sin x \ \sin y \ \sin z$ is ……..
6. The Laplacian of the function $\vec{v} = xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k}$ is ……..
7. For a function $\mathrm{T} = e^{-5x}\sin 4y \ \cos 3z$, $\nabla\cdot\nabla\mathrm{T}$ is……..
8. For a position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\nabla\times\left[\frac{\hat{k}}{r}\right]$ is ……..
9. The curl of curl of a vector function $\vec{v} = -x^2\hat{k}$ is ……….
10. If $\vec{r}$ is the position vector, then $\nabla r =$ …, $\nabla\left(\frac{1}{r}\right) =$ …, $\nabla\times\vec{r} =$ …., $\nabla\cdot\hat{r} =$ ….. ,

    $\nabla(r^n) =$ …, $\nabla(\ln r) =$………., and $\nabla\times\frac{\hat{r}}{r^2} =$……
11. If $\vec{A} = \frac{1}{2}\mu_0 nI\left(-z\hat{j} + y\hat{k}\right)$, then $\nabla\times\vec{A} =$ ……….
12. The divergence of a vector $\vec{v} = xyz(x\hat{i} + y\hat{j} + z\hat{k})$ is ………..
13. If $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$, then $\vec{F}$ can be written as ………
14. If the divergence of a vector function is zero, then the vector function can be expressed as………..
15. In two vectors $\vec{V}_1 = \nabla\times\vec{F}$ and $\vec{V}_2 = \nabla\phi$, $\vec{V}_1$ is ………………… and $\vec{V}_2$ is ………………
16. The fundamental theorem for divergences states that: …………..

    The fundamental theorem for curls [Stokes' theorem] states that……..