Advanced Calculus
Multiple Integrals

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Lecture 13

Triple Integrals

October 31, 2023

Triple Integral in Rectangular Coordinates

DEFINITION

$$\iiint\limits_D F(x, y, z)\, dV \qquad \text{or} \qquad \iiint\limits_D F(x, y, z)\, dx\, dy\, dz.$$

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint\limits_D dV.$$

Examples

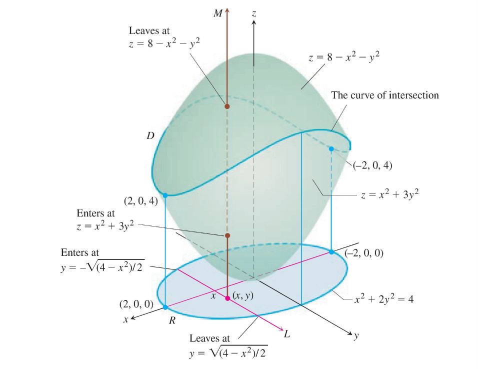
1. Evaluate $\int_0^1 \int_0^2 \int_0^3 (x+y+z)\, dx\, dy\, dz$.
2. Evaluate $\int_0^1 \int_0^2 \int_0^3 x\, y\, z\, dx\, dy\, dz$.
3. Evaluate $\int_1^e \int_1^e \int_1^e \frac{1}{xyz}\, dx\, dy\, dz$.
4. Evaluate $\int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$ Ans: $8\pi\sqrt{2}$

Triple Integrals

1. Integrate $g(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$ and $z = 2$.

2. Find the volume of cube in (1).

3. Find the volume of the region D enclosed by the surface $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



$$\text{Average Value of } F \text{ over } D = \frac{1}{\text{Volume of } D}\iiint_D F dv$$
$$= \frac{1}{\iiint_D dv}\iiint_D F dv.$$

- Example:

  Find the average value of $F(x, y, z) = xyz$ over the cube bounded by the coordinate planes and the planes $x = 2$, $y = 2$ and $z = 2$ in the first octant. (1)

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals.

Jacobian Determinant

Jacobian of the transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Problems

Find the Jacobian determinants of the following transformations

1. $x = r\cos\theta$, $y = r\sin\theta$.
2. $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
3. $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

Triple Integral in Cylindrical and Spherical Coordinates

In Cylindrical Coordinate System

$$dv = dz\, r\, dr\, d\theta$$

$$\iiint_D F(x, y, z) dv = \iiint_G F(r, \theta, z) dz\, r\, dr\, d\theta$$

Evaluate: $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^{r^2} z\, dz\, r\, dr\, d\theta.$ Ans: $\dfrac{32\pi}{3}$.

In Spherical Coordinate System

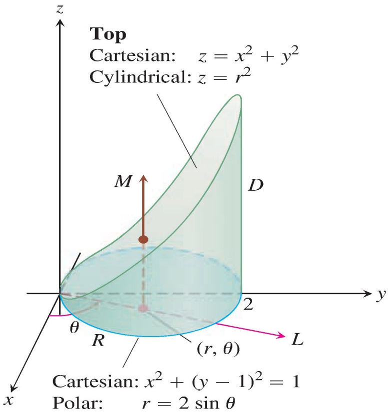
$$dv = \rho^2 \sin\phi d\rho d\phi d\theta$$

$$\iiint_D F(x, y, z) dv = \iiint_G F(\rho, \phi, \theta) \rho^2 \sin\phi d\rho d\phi d\theta$$

Evaluate: $\displaystyle\int_0^{2\pi} \int_0^{\pi} \int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$ Ans: $\dfrac{\pi}{3}$.

Finding Limits of Integration

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.



$$\boxed{\iiint_D f(r,\theta,z)\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\,dz\,r\,dr\,d\theta.}$$

End of Unit 3

Substitutions in Multiple Integrals

Evaluate the following integrals

1. Evaluate
$$\int_0^3 \int_0^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dxdydz$$
by applying the transformation
$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}, \quad w = \frac{w}{3}$$
and integrating over an appropriate region in uvw - space. Ans: 12.
2. Use the transformation $u = x + 2y$, $v = x - y$ to evaluate the integral
$$\int_0^{2/3} \int_y^{2-2y} (x+2y)e^{y-x} dxdy$$
by writing it as an integral over the region G in the uv - plane.
Ans: $\frac{1}{3}\left(1 + \frac{3}{e^2}\right) \approx 0.4687.$

Evaluate the following integrals

3. Evaluate

$$\int_0^1 \int_0^{1-x} (y-2x)^2 \sqrt{x+y}\, dy dx$$

(Hints: Integrand suggests the transformation $u = x + y,\ v = y - 2x$)

Ans: $\frac{2}{9}$.