Advanced Calculus - Polar Coordinates

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August 15, 2023

Graphs of polar equations

Identify the symmetries ans sketch the curves with given polar equations.

1. Cardiods: $r = a + b\cos\theta$, $r = a + b\sin\theta$ $(a = b)$
   i. $r = 1 - \cos\theta$
   ii. $r = 1 + \sin\theta$
   iii. $r = 1 - \sin\theta$

2. Limacons: $r = a + b\cos\theta$, $r = a + b\sin\theta$, $(a \neq b)$
   i. $r = 2 + 3\cos\theta$
   (Limacon with inner loop)
   ii. $r = 3 + 2\sin\theta$
   (Dimpled Limacon)
   iii. $r = 8 + 2\cos\theta$
   (Convex/Oval Limacon)

3. Rose Petal Curves: $r = a\sin n\theta$, $r = a\cos n\theta$, $(a \neq 0)$
   i. $r = 2\sin 3\theta$ (3 petals)
   ii. $r = 2\cos 4\theta$ (8 petals)

4. Lemniscates: $r^2 = a^2\sin 2\theta$, $r^2 = a^2\cos 2\theta$ $(a, n \neq 0)$
   i. $r^2 = 4\sin 2\theta$
   ii. $r^2 = 4\cos 2\theta$

Graphs of polar equations

Sketch the region bounded by inequalities;

1. $-1 \leq r \leq 2,\ -\pi/2 \leq \theta \leq \pi/2$
2. $0 \leq r \leq 2 - 2\cos\theta$

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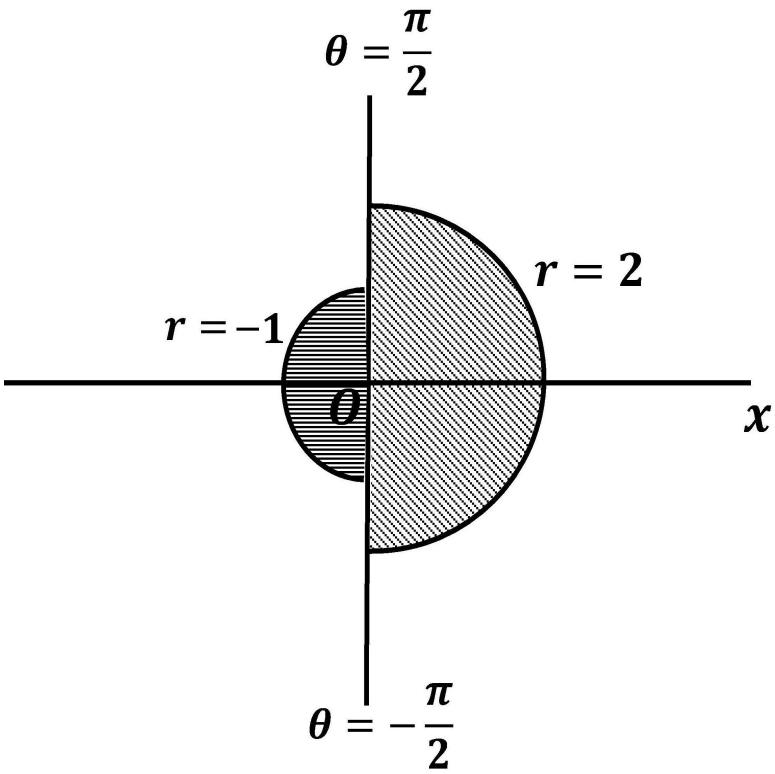
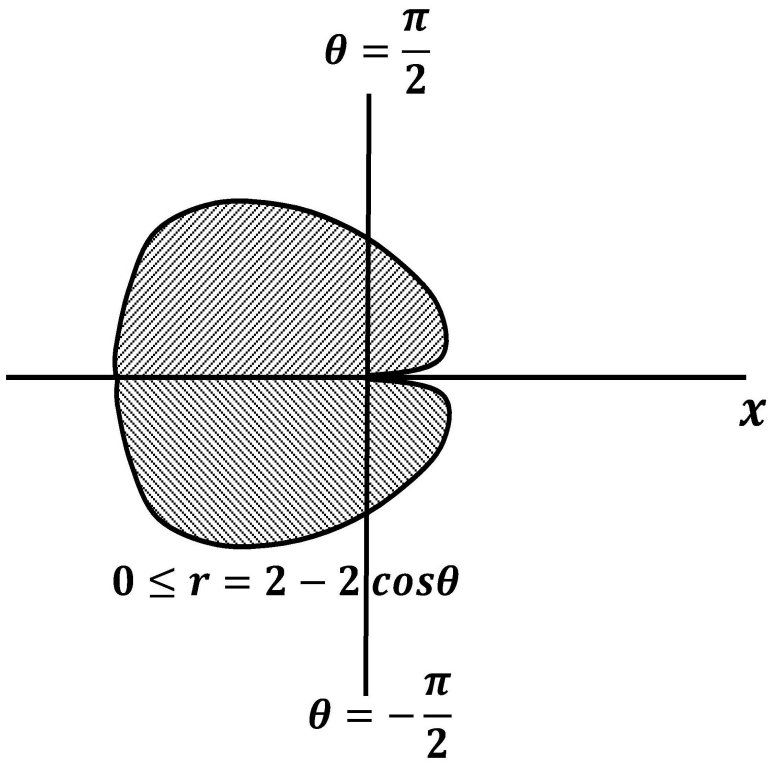




Sketch $0 \leq r \leq 2\sec\theta,\ -\pi/4 \leq \theta \leq \pi/4$.

Cylindrical Coordinates

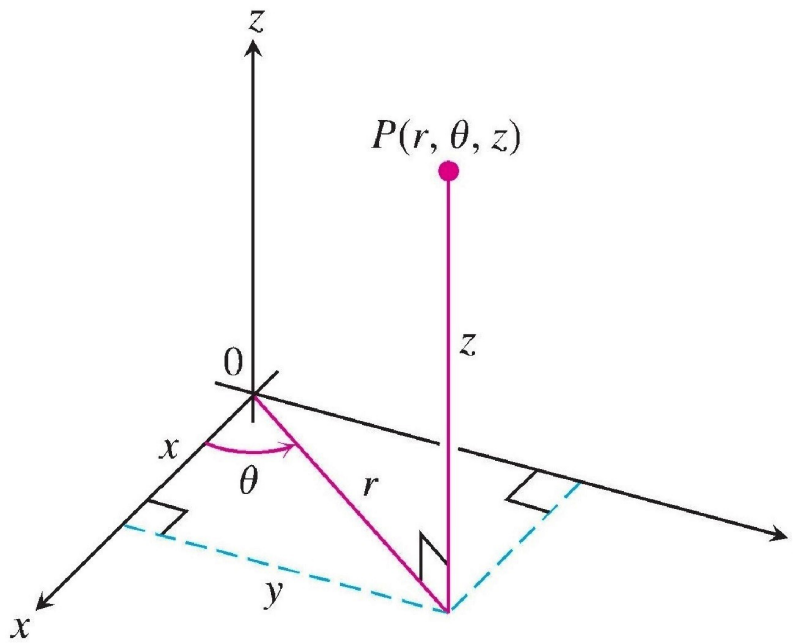


FIGURE The cylindrical coordinates of a point in space are r, θ, and z.

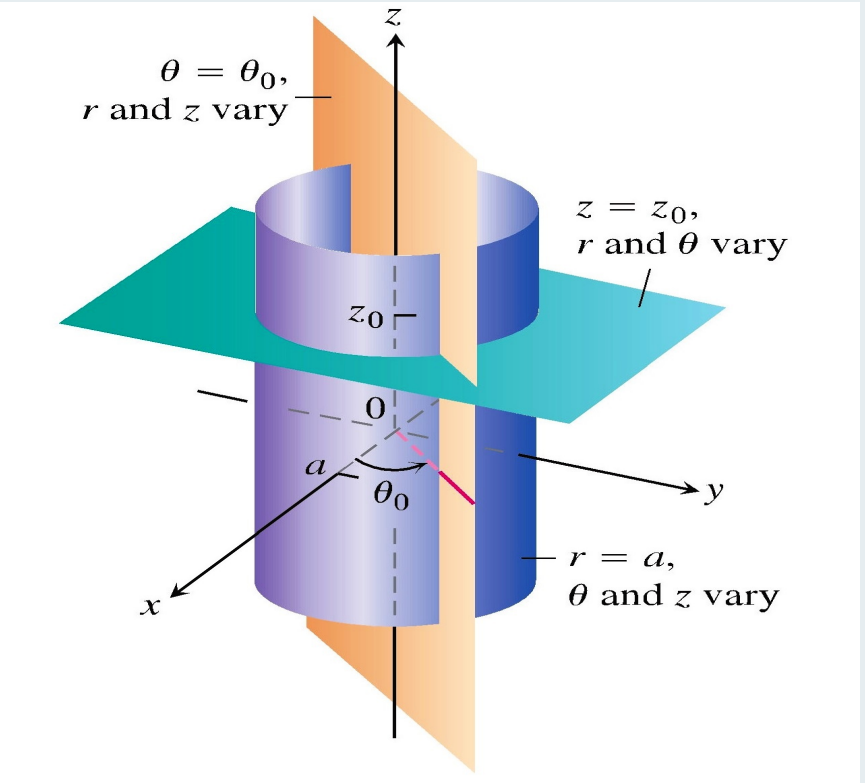
DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

Cylindrical Coordinates

What do the following equations represent?

1. $r = 0$
2. $r = a$
3. $\theta = \theta_0$
4. $z = z_0$



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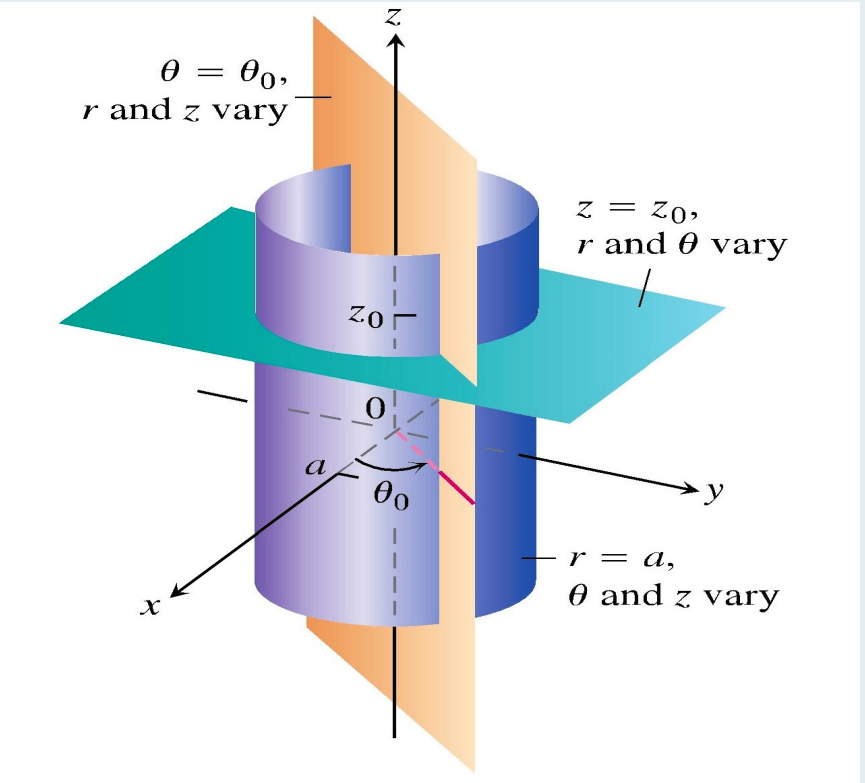
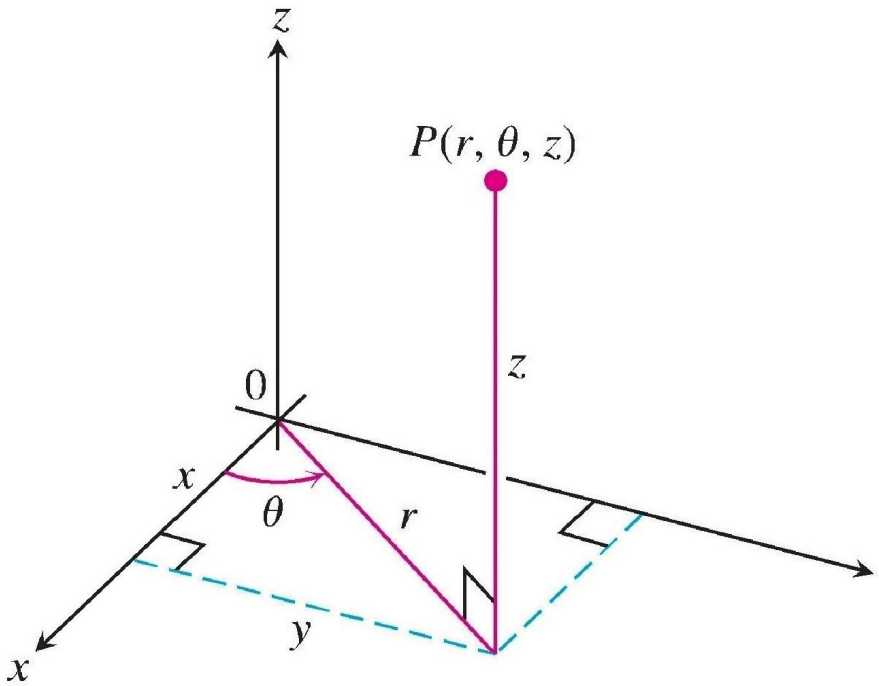


1. z-axis
2. cylinder about z-axis
3. plane containing z-axis, making angle θ_0 with x-axis
4. plane perpendicular to z-axis

Relating Cylindrical Coordinates with Cartesian Coordinates



Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

Examples

1. Find the Cartesian form of $z = r^2$

   Ans: $z = x^2 + y^2$

2. Find the circular cylinder in cylindrical coordinates $4x^2 + 4y^2 = 9$

   Ans: $r = 3/2$

3. Find the corresponding cylindrical coordinate point for the Cartesian coordinate point (3, -3, -7)

   Ans: $(3\sqrt{2}, 7\pi/4, -7)$ or, $(3\sqrt{2}, -\pi/4, -7)$

4. Find the rectangular coordinate point for the cylindrical coordinate point $(2, 2\pi/3, 1)$

   Ans: $(-1, \sqrt{3}, 1)$

Spherical Coordinates

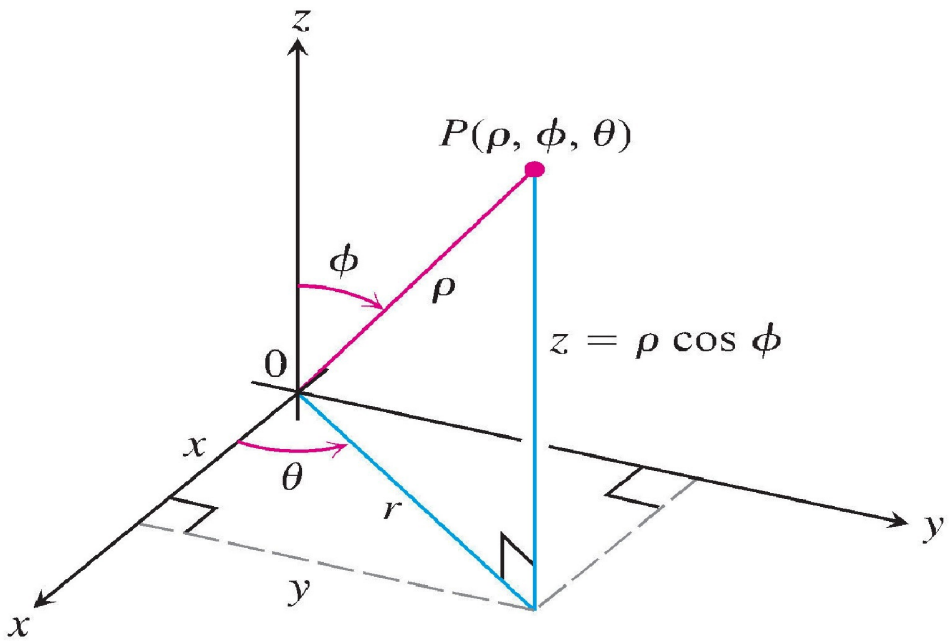


FIGURE The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

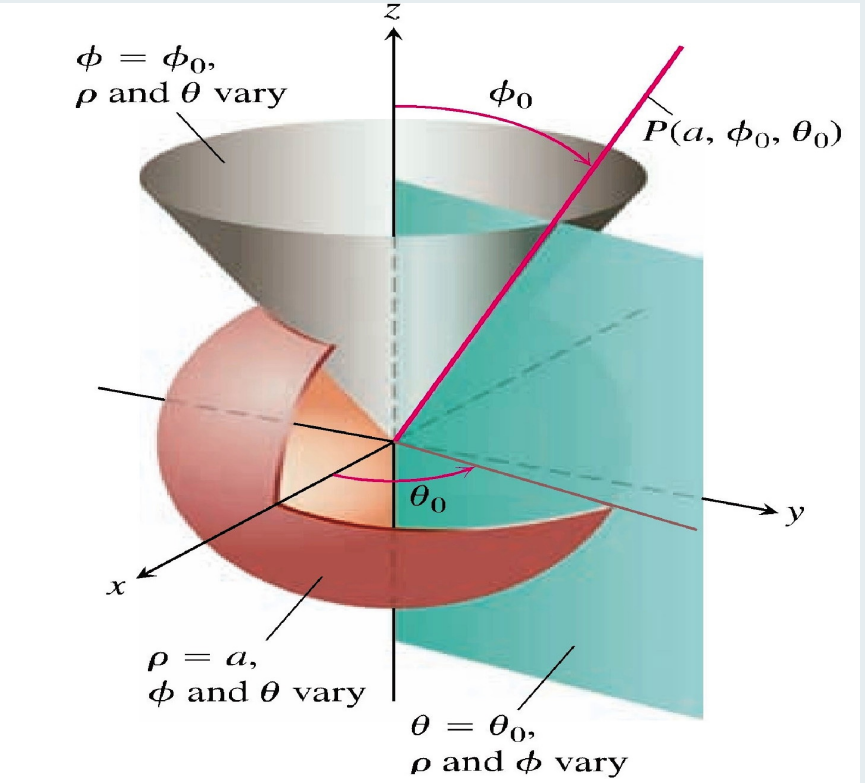
DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \le \phi \le \pi)$.
3. θ is the angle from cylindrical coordinates $(0 \le \theta \le 2\pi)$.

Spherical Coordinates

What do the following represent?

1. $\rho = 0$
2. $\rho = a$
3. $\phi = \phi_0$
4. $\phi > \pi/2$
5. $\theta = \theta_0$



1. point 2. sphere of radius a centered at origin
3. cone with vertex at origin, axis: z-axis 4. cone downwards
5. half plane containing z-axis, making angle θ_0 with +ve x-axis

Equations relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

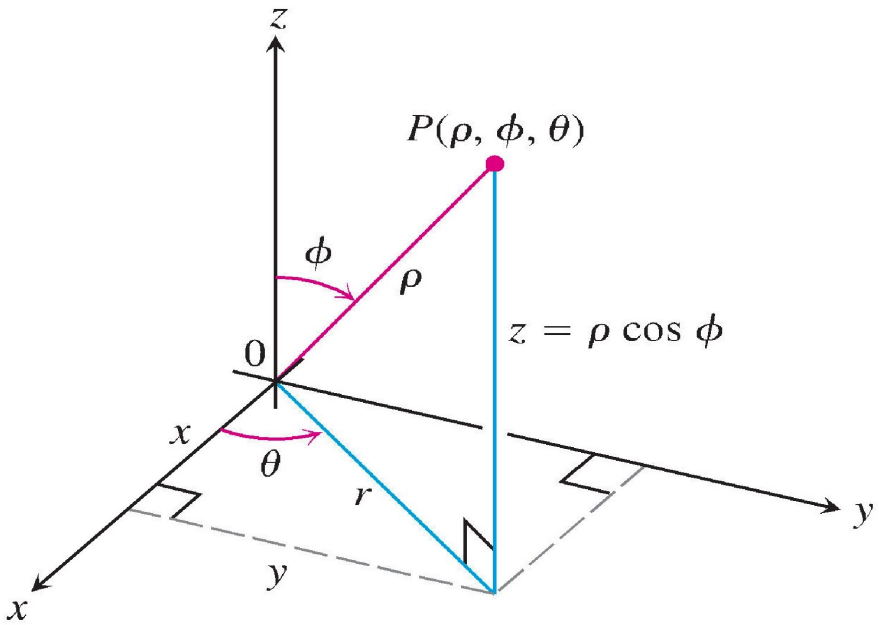


FIGURE The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin\phi, \qquad x = r\cos\theta = \rho \sin\phi \cos\theta,$$
$$z = \rho\cos\phi, \qquad y = r\sin\theta = \rho\sin\phi\sin\theta, \tag{1}$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

Examples

Solve the following problems:

1. Find the spherical coordinate equations for the equations

   i. $x^2 + y^2 + (z-1)^2 = 1$, $\rightarrow$Ans: $\rho = 2\cos\phi$

   ii. $z = \sqrt{x^2 + y^2}$ $\rightarrow$Ans: $\phi = \pi/4$

2. Find the spherical coordinate point for the Cartesian coordinate point $(0, 2\sqrt{3}, -2)$ $\rightarrow$Ans: $(4, 2\pi/3, \pi/2)$

3. Find the rectangular coordinate point for the spherical coordinate point $(2, \pi/4, \pi/3)$ $\rightarrow$Ans: $(1/\sqrt{2}, \sqrt{3/2}, \sqrt{2})$