General Physics I (PHYS 101)

Lecture 13

Viscosity

Keshav Raj Sigdel

Assistant Professor

Department of Physics

Kathmandu University

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Outline

Fluids

Fluids play a vital role in many aspects of everyday life. We drink them, breathe them, swim in them. They circulate through our bodies and control our weather. Airplanes fly through them; ships float in them.

A fluid is any substance that can flow; we use the term for both liquids and gases.

1. **Fluid flow can be steady or nonsteady.**

   Steady Flow: Flow speed - low

   The velocity of the moving fluid at any fixed point remains constant in time.

   Each particle of the fluid follows a smooth path, such that the paths of different particles never cross each other.

   The gentle flow of water near the center of a quiet stream is steady.

Some General Characteristics of Fluid Flow (contd.)

Nonsteady Flow: Flow speed – high

The velocities of the moving fluid vary erratically from point to point as well as from time to time (The velocities are functions of time).

Unsteady flow – a waterfall description

2. **Fluid flow can be compressible or incompressible.**

   If the density ρ of a fluid is a constant, independent of $x, y, z,$ and t, its flow is called incompressible flow.

   Liquids can usually be considered as flowing incompressibly.

   However, even for a highly compressible gas, the variation in density may be insignificant, and for practical purpose, we can consider its flow to be incompressible.

   For example, in flight at speeds much lower than the speed of sound in air, the flow of the air over the wings is nearly incompressible.

3. **Fluid flow can be viscous or nonviscous.**

   Viscosity in fluid flow is similar to friction in the motion of solid bodies.

   The greater the viscosity, the greater the external force or pressure that must be applied to maintain the flow; under similar conditions, honey and motor oil are more viscous than water and air.

   Although viscosity is present in all fluid flow, in some cases its effects may be negligible, in which case we can regard the flow as being nonviscous.

4. **Fluid flow can be rotational or irrotational.**

   Imagine a tiny bit of matter, such as a small insect, that is carried along by a flowing stream. If the particle, as it moves with the stream, does not rotate about an axis through its centre of mass, the flow is irrotational; otherwise, it is rotational.

Stream line (laminar) and Turbulent Flow

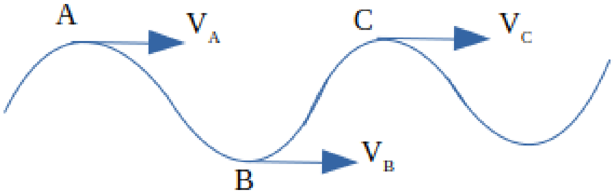


Figure 1

Let us consider a fluid flowing along the path ABC. The velocities of a fluid particles at A, B, and C is V_A, V_B and V_C respectively.

We now assume that the velocity of all the fluid particle reaching point A is V_A in magnitude and direction. Likewise velocity of all the particles at B is V_B and at C is V_C.

In such a case the motion of a fluid is said to be streamline.

i.e in stream line flow speed is sufficiently low.

The velocity of the moving fluid at any fixed point remains constant in time. Each particle of the fluid follows a smooth path, such that the paths of the different particles never cross each other.

Stream line (laminar) and Turbulent Flow (contd.)

The path taken by a fluid particle under steady flow is called a streamline. The velocity of the particle is tangent to the streamline. A set of streamlines is called a tube of flow.

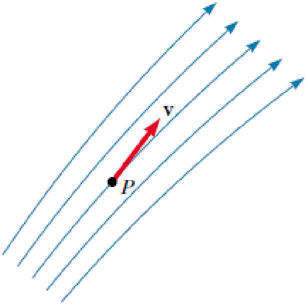


Figure 2: A streamline flow

Thus the stream line flow is the ordered motion of the fluid particles. Fluid has stream line flow if its velocity is less than the certain critical value v_c called as critical velocity.

Example: The gentle flow of water near the center of a quiet stream

Stream line (laminar) and Turbulent Flow (contd.)

Above the critical velocity (v_c) fluid losses its orderliness and the motion is said to be turbulent.

i.e in turbulent flow speed is sufficiently large.

There is great disorder and a constantly changing flow pattern.

The velocities vary erratically from point to point as well as from time to time.

Example: a waterfall- Figure 3 shows the contrast between laminar and turbulent flow for the flow of water from a faucet.

Stream line (laminar) and Turbulent Flow (contd.)

Figure 3: The flow of water from a faucet is (a) laminar at low speeds but (b) turbulent at sufficiently high speeds.

Stream line (laminar) and Turbulent Flow (contd.)

Figure 4 shows the contrast between laminar and turbulent flow for smoke rising in air.

Figure 4: Hot gases from a cigarette made visible by smoke particles. The flow of smoke rising from a cigarette is laminar up to a certain point, and then becomes turbulent.

Turbulence: When the speed of a flowing fluid exceeds a certain critical value, the flow is no longer laminar. Instead, the flow pattern becomes extremely irregular and complex, and it changes continuously with time; there is no steady-state pattern. This irregular, chaotic flow is called turbulence.

Bernoulli's equation is not applicable to regions where there is turbulence because the flow is not steady.

Application:

Listening for Turbulent Flow

Normal blood flow in the human aorta (The aorta is the main artery that carries blood away from your heart to the rest of your body.) is

Stream line (laminar) and Turbulent Flow (contd.)

laminar, but a small disturbance such as a heart pathology can cause the flow to become turbulent. Turbulence makes noise, which is why listening to blood flow with a stethoscope is a useful diagnostic technique.

Viscosity

Viscosity is the property of a liquid by virtue of which it opposes the relative motion between the different layers of liquid.
All fluids offer resistance to any force tending to cause one layer to move another. Viscosity is internal friction in a fluid.

The greater the viscosity, the greater the external force or pressure that must be applied to maintain the flow; under similar conditions, honey and motor oil are more viscous than water and air.
Viscous effects are important in the flow of fluids in pipes, the flow of blood, the lubrication of engine parts, and many other situations.

Viscosity (contd.)

Viscosities of all fluids are strongly temperature dependent, increasing for gases and decreasing for liquids as the temperature increases (Figure 5).

Figure 5: Lava is an example of a viscous fluid. The viscosity decreases with increasing temperature: The hotter the lava, the more easily it can flow.

Newton's Law of Viscosity / Coefficient of Viscosity

Newton (1642-1727) postulated that, for the straight and parallel motion of a given fluid, the tangential stress between two adjoining layers is proportional to the velocity gradient in a direction perpendicular to the layers.

We now consider a flow of liquid over a flat plate PQ as shown in figure. Obviously different layers at different distances from PQ have different velocities. The layer which is in contact with PQ has zero velocity (rest) and velocity goes on increasing upwards.

Newton's Law of Viscosity / Coefficient of Viscosity (contd.)

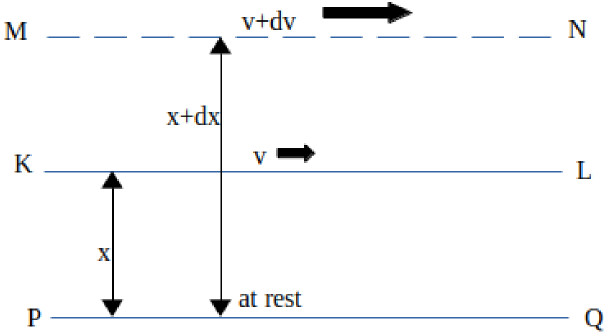


Figure 6

Newton's Law of Viscosity / Coefficient of Viscosity (contd.)

Now according to the Newton's law of viscus flow for stream line motion, the tangential viscous drag F acting between two layers of area A at a distance dx apart moving with relative velocity dv is directly propotional to

(i) area A of the layers

(ii) the rate of change of velocity with distance i.e. velocity gradient $\frac{dv}{dx}$. Such that $\quad F \propto A \quad$ and $\quad F \propto \frac{dv}{dx}$

Combining these two we get, $F \propto A\frac{dv}{dx} \implies F = -\eta A\frac{dv}{dx}$

Newton's Law of Viscosity / Coefficient of Viscosity (contd.)

where η is a constant for a particular fluid at a particular temperature. This constant of proportionality η is called the coefficient of viscosity (or absolute viscosity) of the fluid. This equation is called the Newton's law of viscosity for one dimensional flow.

If $A = 1$ and $\frac{dv}{dx} = 1$ then $F = -\eta$ i.e $\eta = -F$

Thus the **coefficient of viscosity** of fluid is defined as the viscus force per unit area of contact between two layers having a unit velocity gradient between them.

The SI unit of viscosity is $\mathrm{N \cdot s/m^2}$ or Pa s or Decapoise

The equivalent cgs unit is $\mathrm{dyne \cdot s\ /cm^2}$, which is called the poise.

Newton's Law of Viscosity / Coefficient of Viscosity (contd.)

The unit is named for the French Physician Jean-Louis Poiseuille (1799-1869), who investigated the flow of viscous fluids through tube.

$$1 \text{ poise} = 0.1 \text{ N} \cdot \text{s/m}^2$$

The viscosity is large for fluids that offer a large resistance to flow and small for fluids that flow easily. The viscosity of water at $\left(20^0 C\right)$ is 1.0×10^{-3} N $\cdot$ s/m^2 or 0.01 *poise*.

Newton's Law of Viscosity / Coefficient of Viscosity (contd.)

Table 1: Viscosities of Selected Fluids

Fluid	$\eta \ (\mathrm{N \cdot s/m^2})$
Glycerin (20^0C)	1.5
Motor oil (0^0C)	0.11
Motor oil (20^0C)	0.03
Blood (37^0C)	4.0×10^{-3}
Water (20^0C)	1.0×10^{-3}
Water (90^0C)	0.32×10^{-3}

Reynolds Number

In 1883, Osborne Reynolds proved experimentally that in case of viscous liquid, the critical speed at which the flow becomes turbulent is given by

$$v_c = R\frac{\eta}{\rho D} \tag{1}$$

where ρ is the density of the fluid, D diameter of the pipe and η viscosity of the fluid

The Reynolds number R (dimensionless constant) for any flow speed

$$R = \frac{\rho D v}{\eta}$$

Reynolds Number (contd.)

The Reynolds number can be used to characterize any flow, and we can determine by experiment the value of the Reynolds number at which the flow becomes turbulent.

For cylindrical pipes, the Reynolds number corresponding to the critical speed is about 2000.

Thus for water flowing through a pipe of diameter 2 cm, the critical speed is

$$v_c = R\frac{\eta}{\rho D} = 2000\frac{1 \times 10^{-3}\ \mathrm{N \cdot s\ /m^2}}{\left(10^3\ \mathrm{kg\ /m^3}\right)(0.02\ \mathrm{m})} = 0.1\ \mathrm{m\ /s} = 10\ \mathrm{cm\ /s}$$

This is quite a low speed, which suggests that the flow of water is turbulent in ordinary household plumbing. (The flow speed from a

Reynolds Number (contd.)

typical household tap is about1 m /s.)

Note:

Reynolds number R (0 - 2000) = laminar

Reynolds number R (2000 - 3000) = unstable

Reynolds number R (3000 - and above) = turbulent

Flow of Ideal Fluids

An ideal fluid is incompressible and non-viscous, and its flow is steady and irrotational.

Volume flow rate (or volume flux) of the fluid (R_v):

$$R_{\mathrm{V}} = Av = \text{constant} \qquad \text{[Equation of Continuity]}$$

As the liquid is incompressible the volume of fluid that enters one end of a tube is equal to the volume leaving the other end of the tube in the same interval of time.

Mass flow rate (mass flux) of the fluid (R_m): $R_m = \rho Av = \text{constant}$.

This results expresses the law of conservation of mass in fluid dynamics.

Consider an ideal fluid flowing with steady flow through a pipe of varying cross-sectional area, as illustrated in Figure 7.

Let v_1 and v_2 be the velocities of the fluid at cross-section A_1 and A_2 of the tube respectively.

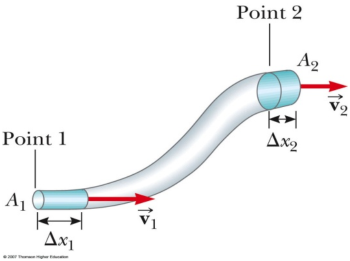


Figure 7

Equation of Continuity (contd.)

In a time Δt, the fluid at the bottom end of the pipe moves a distance $\Delta x_1 = v_1 \Delta t$. Then the mass of fluid contained in the left shaded region in Figure 7 is

$$m_1 = \rho A_1 \Delta x_1 = \rho A_1 v_1 \Delta t$$

where ρ is the (non changing) density of the ideal fluid.

Similarly, the fluid that moves through the upper end of the pipe in the time Δt has a mass

$$m_2 = \rho A_2 \Delta x_2 = \rho A_2 v_2 \Delta t$$

Since mass is conserved and the flow is steady, the mass that crosses A_1 in a time Δt must equal the mass that crosses A_2 in the time Δt.

$$
\begin{aligned}
\text{i.e.,} \quad m_1 = m_2 \implies & \rho A_1 v_1 \Delta t = \rho A_2 v_2 \Delta t \\
\implies \quad & A_1 v_1 = A_2 v_2 \\
\therefore \quad & Av = \text{constant} \qquad (2)
\end{aligned}
$$

This equation (2) is called the equation of continuity.

It states that, "The product of the area and the fluid speed at all points along the pipe is a constant for an incompressible fluid".

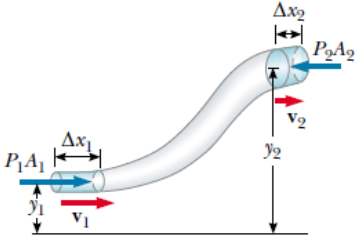
This equation tells us that the speed is high where the tube is constricted (small A) and low where the tube is wide (large A)

Bernoulli's equation

The Swiss physicist Daniel Bernoulli first derived the relationship between fluid speed, pressure, and elevation in 1738.

For steady flow the total energy of a fluid remains constant this is called Bernoullis theorem. Total energy means the sum of pressure energy, potential energy and kinetic energy.

Consider the flow of an ideal fluid through a nonuniform pipe in a time Δt, as illustrated in Figure 8.



Bernoulli's equation (contd.)

In Figure 8, we consider the element of fluid that at some initial time lies between the two cross sections a and c. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval Δt, the fluid that is initially at a moves to b, a distance $\Delta x_1 = v_1 \Delta t$, and the fluid that is initially at c moves to d, a distance $\Delta x_2 = v_2 \Delta t$. The cross-sectional areas at the two ends are A_1 and A_2 as shown in Figure 8.

From the continuity equation,

$$A_1 v_1 = A_2 v_2$$

$$\implies A_1 v_1 \Delta t = A_2 v_2 \Delta t$$

$$\implies A_1 \Delta x_1 = A_2 \Delta x_2 = \Delta V$$

Bernoulli's equation (contd.)

The volume of the fluid ΔV passing any cross section during time Δt is the same.

The net work done on a fluid element by the pressure of the surrounding fluid is

$$
\begin{aligned}
W &= P_1 A_1 \Delta x_1 - P_2 A_2 \Delta x_2 \\
\therefore W &= (P_1 - P_2)\,\Delta V
\end{aligned} \tag{3}
$$

The work W is due to forces other than the conservative force of gravity, so it equals the change in the total mechanical energy (kinetic energy plus gravitational potential energy) associated with the fluid element.

The net change in kinetic energy ΔK during time Δt is

$$\Delta K = \frac{1}{2}(\rho \Delta V)\, {v_2}^2 - \frac{1}{2}(\rho \Delta V)\, {v_1}^2$$
$$\therefore \Delta K = \frac{1}{2}\rho \Delta V \left({v_2}^2 - {v_1}^2\right) \tag{4}$$

The net change in gravitational potential energy ΔU during time Δt is

$$\Delta U = (\rho \Delta V)\, g y_2 - (\rho \Delta V)\, g y_1$$
$$\therefore \Delta U = \rho \Delta V g \left(y_2 - y_1\right) \tag{5}$$

Bernoulli's equation (contd.)

Combining Eqs. (3), (4), and (5) in the energy equation

$W = \Delta K + \Delta U$, we obtain

$$
\begin{aligned}
(P_1 - P_2)\,\Delta V &= \frac{1}{2}\rho \Delta V\left({v_2}^2 - {v_1}^2\right) + \rho \Delta V g\,(y_2 - y_1) \\
or, (P_1 - P_2) &= \frac{1}{2}\rho\left({v_2}^2 - {v_1}^2\right) + \rho g\,(y_2 - y_1) \\
\therefore P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g y_1 &= P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g y_2
\end{aligned}
$$

This is Bernoulli's equation as applied to an ideal fluid.

It is often expressed as

$$P + \frac{1}{2}\rho v^2 + \rho g y = constant$$

Bernoulli's equation (contd.)

This expression specifies that, in laminar flow, the sum of the pressure (P), kinetic energy per unit volume $\left(\frac{1}{2}\rho v^2\right)$ and gravitational potential energy per unit volume $(\rho g y)$ has the same value at all points along a streamline.

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula]

Liquid flow through narrow tube:

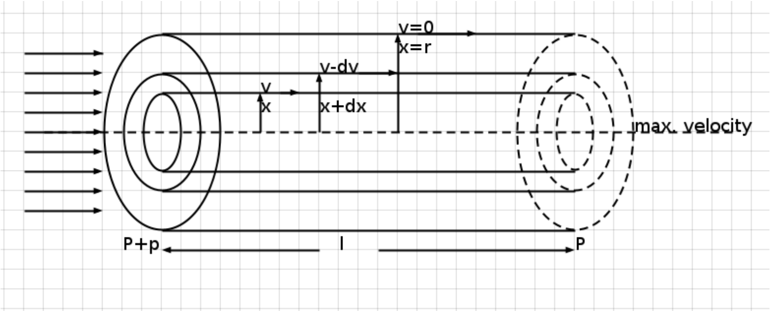


Figure 9: Fluid flows through a cylindrical pipe

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

Let us consider a steady laminar flow of fluid of viscosity η through a horizontal capillary tube of radius r and length l, under a constant pressure difference p as shown in Figure 9.The pressure $P+p$ at the left end is maintained greater than the pressure P at the right end so that the fluid flows from left to right. The flow is axis-symmetric.

When the flow is fully developed the velocity profile is constant along the pipe axis. In laminar flow, the paths of individual particles of fluid do not cross, and so the pattern of flow may be imagined as a number of thin concentric cylinders of varying radii, which slide over one

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

another. The flow velocity varies with the radius; its maximum value occurs on the axis. The velocity of the liquid goes on decreasing from the center to the surface of the tube. Its minimum value, which we assume to be zero, at the walls.

Consider an arbitrary co-axial cylinder of fluid of radius x and thickness dx.

The surface of the layer of this cylinder is $2\pi xl$.

Let v and $v - dv$ are the velocities of two layers one of radius x and other of radius $x + dx$.

$$\therefore \quad \text{Velocity gradient} = -\frac{dv}{dx}$$

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

The viscus drag experienced by the layer of radius x is

$$-\eta A\frac{dv}{dx} = -\eta 2\pi x l\frac{dv}{dx}$$

The force experienced by the layer of the radius x due to pressure difference p is = $p\times$ cross section area = $p \times \pi x^2$.

For steady laminar flow of fluid,

$$F_{vis}{=}F_{ext}$$

$$\implies -\eta\left(2\pi x l\right)\frac{dv}{dx}{=}p\pi x^2$$

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

$$\implies \frac{dv}{dx} = -\frac{px}{2\eta l}$$

$$\therefore \quad dv = -\frac{p}{2\eta l} x dx \tag{6}$$

Integrating equation (6), we get

$$\int dv = -\frac{p}{2\eta l} \int x dx$$

$$\implies v = -\frac{p}{2\eta l}\frac{x^2}{2} + C$$

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

where C is the constant of integration

$$\therefore v = -\frac{p}{4\eta l}x^2 + C \tag{7}$$

The constant of integration, C, is evaluated by using the available boundary condition at the pipe wall: i.e. at $x = r, v = 0$. Consequently,

$$0 = -\frac{p}{4\eta l}r^2 + C$$

This gives

$$C = \frac{p}{4\eta l}r^2$$

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

Substituting the value of C in equation (7), we get

$$v = -\frac{p}{4\eta l}x^2 + \frac{p}{4\eta l}r^2$$
$$\therefore v = \frac{p}{4\eta l}\left(r^2 - x^2\right) \tag{8}$$

Equation (8) shows that the velocity decreases from maximum value $v_0 = \dfrac{p}{4\eta l}r^2$ at the centre to zero at the wall of the pipe.
This shows the velocity distribution curve for laminar flow of a viscous fluid in a long cylindrical pipe is parabolic as shown.

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

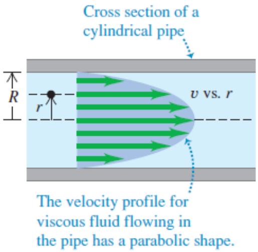


Figure 10

The cross section area of this elemental cylindrical layer of liquid flowing through the layer of radius x and $x+dx$ is $dA = 2\pi x dx$. The volume of fluid flowing through this elemental cylindrical tube of fluid per unit time is

$$
\begin{aligned}
&= dA.v \\
&= 2\pi x dx.\frac{p}{4\eta l}\left(r^2 - x^2\right) \\
\therefore dV &= \frac{\pi p}{2\eta l}\left(r^2 - x^2\right) x dx \qquad (9)
\end{aligned}
$$

The volume of fluid flowing through the pipe per unit time is therefore

$$V = \frac{\pi p}{2\eta l}\int_0^r \left(r^2 - x^2\right) x dx$$

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

$$= \frac{\pi p}{2\eta l}\left[r^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^r$$

$$= \frac{\pi p}{2\eta l}\left[\frac{r^4}{2} - \frac{r^4}{4}\right]$$

$$\therefore \boxed{V = \frac{\pi p r^4}{8\eta l}} \tag{10}$$

This equation is known as Poiseuille's formula. Equation (10) is applicable only to the fully developed laminar flow of constant-density fluids.

Steady Laminar Flow of Fluid in Pipe [Poiseuille's Formula] (contd.)

Therefore coefficient of viscosity is $\boxed{\eta = \dfrac{\pi p r^4}{8Vl}}$.

Limitation or Assumptions:

1. The flow of liquid should be stream lined.

2. The velocity of layer in contact with the walls of the tube should be zero.

3. The acceleration of liquid should be zero.

4. The formula is not applicable for gases.