ASSIGNMENT-IV (2023)
MATH 104

1. Define Beta Function and establish the formulae/relations:

   (i) $B(m,n) = B(n,m)$, $\quad B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

   (ii) $B(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}dx = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}dx$

   (iv) $B(m,n) = \frac{n-1}{m+n-1}B(m,n-1) = \frac{m-1}{m+n-1}B(m-1,n)$

2. Define Gamma function and prove the following

   $$\Gamma(1) = 1, \qquad \Gamma(n+1) = n\Gamma(n), \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

3. Relation between Beta and Gamma functions: Show that

   (i) $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ $\qquad (ii)$ $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

   (iii) $\int_0^{\pi/2} \sin^p\theta \sin^q\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$ iii)

4. Prove that:

   (a) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$ $\qquad (b)$ $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{3}\pi$

   (c) $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\cdots \quad \cdots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$, where the symbols have their usual meaning.

5. Evaluate the following integrals.

   (i) $\int_0^{\pi/2} \sin^6\theta \cos^4\theta d\theta$

   (ii) $\int_0^{a} x^3(a^2-x^2)^{3/2}dx$ (put $x = a\sin\theta$).

6. Define limit, Continuity and derivative of the vector function $\overrightarrow{r}(t) = f(t)\overrightarrow{i} + g(t)\overrightarrow{j} + h(t)\overrightarrow{k}$. State the component test for continuity of the vector function $\overrightarrow{r}(t) = f(t)\overrightarrow{i} + g(t)\overrightarrow{j} + h(t)\overrightarrow{k}$. Let $\overrightarrow{r}(t) = \sqrt{(1-t^2)}\overrightarrow{i} + 3t\overrightarrow{j} - 7\overrightarrow{k}$. At what value of t is the vector function $\overrightarrow{r}$ continuous? Explain reason.

7. Define the smooth curve. Is the vector function $\overrightarrow{r}(t) = (cost)\overrightarrow{i} + (sint)\overrightarrow{j} + t\overrightarrow{k}$ smooth in the interval $[-\pi, \pi]$? Explain.

8. Find $\frac{d\overrightarrow{r}}{dt}$ if

(a) $\overrightarrow{r}(t) = \ln\sqrt{(1-t)}\,\overrightarrow{i} + \sqrt{(1-t^2)}\,\overrightarrow{j}$

(b) $\overrightarrow{r}(t) = (\sin^{-1} 2t)\,\overrightarrow{i} + (\tan^{-1} 3t)\,\overrightarrow{j} + \frac{1}{t}\overrightarrow{k}$

(c) $\overrightarrow{r}(t) = (\frac{(2t-1)}{(2t+1)})\,\overrightarrow{i} + \ln(1-4t^2)\,\overrightarrow{j} + (\sec t)\,\overrightarrow{k}$.

9. Prove that: $\frac{d}{dt}(\overrightarrow{u} \times \overrightarrow{v}) = \frac{d\overrightarrow{u}}{dt} \times \overrightarrow{v} + \overrightarrow{u} \times \frac{d\overrightarrow{v}}{dt}$ for two vector functions u and v.

10. The vector $\overrightarrow{r}(t)$ defines the position of a particle moving in the plane/ space at time t. Find the particle's velocity, acceleration, speed and direction of motion of particle at time specified.

(a) $\overrightarrow{r}(t) = (t^2+1)\,\overrightarrow{i} + (2t-1)\,\overrightarrow{j}, t = \frac{1}{2}$

(b) $\overrightarrow{r}(t) = (\cos 2t)\,\overrightarrow{i} + (3\sin 2t)\,\overrightarrow{j}, t = 0$

(c) $\overrightarrow{r}(t) = (1+t)\,\overrightarrow{i} + \frac{t^2}{\sqrt{2}}\overrightarrow{j} + \frac{t^3}{3}\overrightarrow{k}, t = 1$

11. Solve the initial value problem:

$$\frac{d^2\overrightarrow{r}}{dt^2} = -32\,\overrightarrow{k}$$

with the initial conditions: $\overrightarrow{r}(0) = 100\,\overrightarrow{k}$ and $\left.\frac{d\overrightarrow{r}}{dt}\right|_{t=0} = 8\,\overrightarrow{i} + 8\,\overrightarrow{j}$.

12. Find the arc length parameter along the curve $\overrightarrow{r}(t) = (e^t\cos t)\,\overrightarrow{i} + (e^t\sin t)\,\overrightarrow{j} + (e^t)\,\overrightarrow{k}$, from the point where $t = 0$ by evaluating the integral $s = \int_0^t |v(\tau)|d\tau$ and then find the length of the curve for $-\ln 4 \leq t \leq 0$.

13. Prove the relations (a) $\kappa = \dfrac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$ (b) $\kappa = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$ where the symbols have their usual meaning.

14. Find $\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}, \kappa, \tau$ for the following space curves:

(a) $\overrightarrow{r}(t) = (e^t\cos t)\,\overrightarrow{i} + (e^t\sin t)\,\overrightarrow{j} + 2\,\overrightarrow{k}$

(b) $\overrightarrow{r}(t) = (\cos t + t\sin t)\,\overrightarrow{i} + (\sin t - t\cos t)\,\overrightarrow{j} + 3\,\overrightarrow{k}$

(c) $\overrightarrow{r}(t) = (\cos ht)\,\overrightarrow{i} + (\sin ht)\,\overrightarrow{j} + t\,\overrightarrow{k}$

*** The End ***