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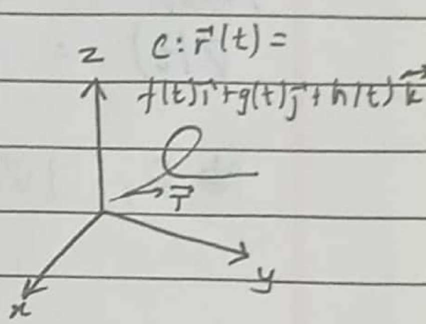
(*) Speed on a Smooth Curve

$$\frac{ds}{dt} = |v(t)|$$

Since $ds/dt > 0$, so, $|v(t)|$ is never zero for a smooth curve.

(*) Unit Tangent Vector:



It is the unit vector ~~the~~ tangent to the smooth curve called the unit tangent vector.

$$\vec{T} = \frac{d\vec{r}}{ds}.$$

Working formula: $\vec{T} = \dfrac{\vec{v}}{|\vec{v}|}$

Q7: Find the unit tangent vector of the curve.

$$\vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k}$$

Soln:

Given,

$$\vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k}$$

Now,

$$\vec{v}(t) = -3\sin t\,\vec{i} + 3\cos t\,\vec{j} + 2t\,\vec{k}$$

$$|\vec{v}(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4t^2} = \sqrt{9+4t^2}$$

Thus,

$$T = \frac{\vec{v}}{|\vec{v}|} = \frac{-3\sin t}{\sqrt{9+4t^2}}\vec{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\vec{j} + \frac{2t}{\sqrt{9+4t^2}}\vec{k}$$

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Q7: Find the unit tangent vector of the curve.
$\vec{r}(t) = (2\cos t)\vec{i} + (2\sin t)\vec{j} + (\sqrt{5}t)\vec{k}$

Soln:

Given,
$\vec{r}(t) = (2\cos t)\vec{i} + (2\sin t)\vec{j} + (\sqrt{5}t)\vec{k}$

Now,
$\vec{v}(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j} + \sqrt{5}\,\vec{k}$

So, $|\vec{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2}$
$= \sqrt{4+5} = 3$

$\therefore \vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = -\dfrac{2}{3}\sin t\,\vec{i} + \dfrac{2}{3}\cos t\,\vec{j} + \dfrac{\sqrt{5}}{3}\vec{k}$

Curvature

If T is the unit vector of a smooth curve, the curvature function of the curve is

$$K = \left|\frac{dT}{ds}\right|$$

If $\left|\frac{dT}{ds}\right|$ large, T turns sharply as particle passes through P and the curvature at P is large.

(close to 0)
If $\left|\frac{dT}{ds}\right|$ small, T turns more slowly and the curvature of P is smaller.

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Working formula: $K = \dfrac{1}{|v|}\left|\dfrac{dT}{dt}\right|$

Here,
T = unit tangent vector.

Q7: Prove that curvature of a straight line is zero and the curve of a circle is constant.

Soln.

We know that,
equation of a st. line in vector form is given by
$\vec{r}(t) = \vec{a} + \vec{b}t$

Now,
$\vec{v} = \dfrac{d\vec{r}}{dt} = \vec{b}$. $\vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{b}}{|\vec{b}|}$

$|\vec{v}| = |\vec{b}|$

$\therefore \dfrac{d\vec{T}}{dt} = 0.$

Now,
$K = \dfrac{1}{|v|}\left|\dfrac{dT}{dt}\right| = 0.$

$\therefore K = 0.$

And, eqn of circle. $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$\vec{v} = \dfrac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$

$|\vec{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a$

$\vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

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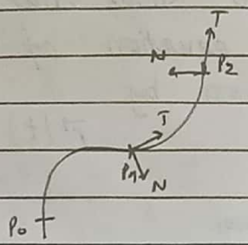
$$\frac{d\vec{T}}{dt} = -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = \frac{1}{a} = \frac{1}{\text{radius}}$$

is constant.

Hence, proved.

(*) Principal Normal



At a point where, $K \neq 0$, the principal unit normal ~~unit~~ ~~volume~~ for a smooth curve in the plane is

$$N = \frac{1}{K}\frac{dT}{ds}$$

$$N = \frac{dT/dt}{|dT/dt|}$$

(*) Circle of Curvature / Osculating Circle of a curvature

Circle of curvature or osculating circle at a point P on a plane curve where $K \neq 0$ is the circle in the plane of the curve that

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(i) is tangent to the curve at P & has same tangent line the curve has }

(ii) has the same curvature that the curve has at P.

(iii) must lie towards concave / inner side of the curve.

Radius of curvature of the curve at P is the radius of the circle of curvature.

Radius of curvature $(\rho) = \frac{1}{K}$. $(K \neq 0)$

(Q): Find the $\vec{T}, \vec{N}, K, \rho$ of the space curve.

$$\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j} + 3\vec{k}$$

Sol^n:

Given,

$$\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j} + 3\vec{k}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \cancel{-\sin t\,\vec{i}} + t\cos t\,\vec{i} + \cancel{\sin t\,\vec{i}} + \cancel{\cos t\,\vec{j}} + t\sin t\,\vec{j} \cancel{- \cos t\,\vec{j}}$$

$$\therefore \vec{v}(t) = t\cos t\,\vec{i} + t\sin t\,\vec{j}$$

$$|\vec{v}(t)| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = t$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{t}(t\cos t\,\vec{i} + t\sin t\,\vec{j}) = \cos t\,\vec{i} + \sin t\,\vec{j}$$

$$\frac{d\vec{T}}{dt} = \frac{d}{dt}(\cos t\,\vec{i} + \sin t\,\vec{j})$$

$$\therefore \frac{d\vec{T}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

Now,

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$K = \frac{1}{|v|}\cdot\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{t}\times 1 = \frac{1}{t}$$

$$\rho = \frac{1}{K} = t.$$

LQ4: Find osculating circle or find curvature K and radius of curvature of parabola $y = x^2$ at origin.

Sol^n:

Given,

$y = x^2$

Let $x = t$. Then, $y = t^2$.

Now, $\vec{r}(t) = x\vec{i} + y\vec{j} = t\,\vec{i} + t^2\,\vec{j}$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \vec{i} + 2t\,\vec{j}$$

$$|\vec{v}(t)| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

At origin, $t = 0$. So,

$|\vec{v}(t)| = 1$

$$\vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{i} + 2t\,\vec{j}}{1}$$

$$\therefore \vec{T} = \vec{i} + 2t\,\vec{j}$$

$$\frac{d\vec{T}}{dt} = 2\vec{j} \qquad \left|\frac{d\vec{T}}{dt}\right| = 2$$

Now,

$$K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{1}\times 2 = 2$$

So,

radius of curvature of parabola $= \frac{1}{K} = \frac{1}{2}$.

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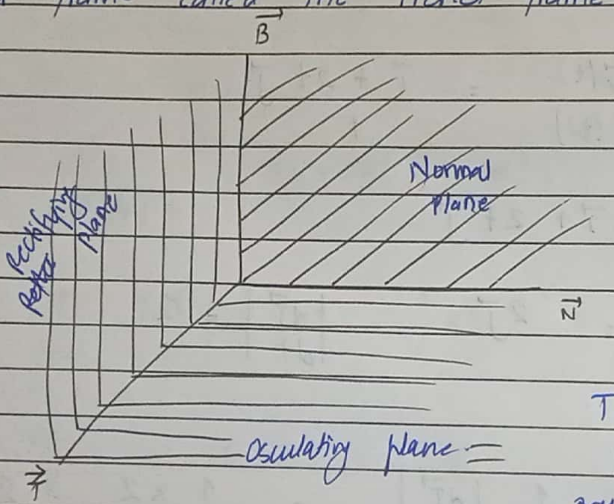
(*) Binomial Vector:

Binomial vector of a curve in space

$\vec{B} = \vec{T} \times \vec{N}$, is a unit vector orthogonal to both $\vec{T}$ and $\vec{B}$

Together $\vec{T}, \vec{N}, \vec{B}$ define a moving right-handed vector frame called the Frenet frame / TNB frame.



= TNB frame / Frenet frame / orthogonal Triad.

(*) Torsion:

Let $\vec{B} = \vec{T} \times \vec{N}$. The torsion function of a smooth curve is.

$$\vec{\tau} = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$

Torsion is the rate of change of binomial vector.

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We know,

$$\vec{B} = \vec{T} \times \vec{N}$$

or, $$\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds} + \frac{d\vec{T}}{ds} \times \vec{N}$$

$$\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds}$$

We see that, dB/ds is orthogonal to T.

Also, dB/ds is orthogonal to B., it means dB/ds is parallel to N. (dB/ds ⊥ to plane of $\vec{T} \times \vec{B}$)

$$\frac{dB}{ds} = -\tau N. \quad \text{or} \quad \tau = -\frac{d\vec{B}}{ds} \vec{N}$$

Here, τ = torsion of the curve.

(*) Tangential and Normal Components of Acceleration.

We know,

$$\vec{v} = \frac{dr}{dt} = \frac{dr}{ds} \cdot \frac{ds}{dt} = \vec{T} \cdot \frac{ds}{dt}$$

Also,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{ds}\left(T \cdot \frac{ds}{dt}\right) = \frac{d^2s}{dt^2} T + \frac{ds}{dt} \cdot \frac{dT}{dt}$$

$$= \frac{d^2s}{dt^2} T + \frac{ds}{dt}\left(\frac{dT}{ds} \cdot \frac{ds}{dt}\right) = \frac{d^2s}{dt^2} T + \frac{ds}{dt}\left(K N \frac{ds}{dt}\right)$$

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$$a = \frac{d^2s}{dt^2}\vec{T} + K\left(\frac{ds}{dt}\right)^2\vec{N}$$

If acceleration vector is written as,

$$a = a_T\vec{T} + a_N\vec{N}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|v| \quad \{\text{tangential component of acceleration}\}$$

$$a_N = K\left(\frac{ds}{dt}\right)^2 = K(|v|)^2 \quad \{\text{normal component of acceleration}\}$$

Formula: $a_N = \sqrt{(|\vec{a}|)^2 - a_T^2}$

(*) Another formula for Curvature

$$\vec{v}\times\vec{a} = \frac{ds}{dt}\vec{T} \times \left(\frac{d^2s}{dt^2}\vec{T} + K\left(\frac{ds}{dt}\right)^2\vec{N}\right)$$

$$= \frac{ds}{dt}\times\frac{d^2s}{dt^2}(\vec{T}\times\vec{T}) + K\left(\frac{ds}{dt}\right)^3(\vec{T}\times\vec{N})$$

$$= K\left(\frac{ds}{dt}\right)^3$$

So,

$$|\vec{v}\times\vec{a}| = K\left|\frac{ds}{dt}\right|^3|\vec{B}| = K|\vec{v}|^3$$

$$\therefore K = |\vec{v}\times\vec{a}| / |\vec{v}|^3$$

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<Q>: Without finding $\vec{T}$ & $\vec{N}$, find the acceleration of the motion.

$$\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}, \quad t>0.$$

in the form of

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

Sol^n.

We know,

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\vec{v}| \text{ and } a_N = K(|v|)^2$$

Given,

$$\vec{r}(t) = (\cos t + t\sin t)\vec{i} + (\sin t - t\cos t)\vec{j}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \cancel{-\sin t\,\vec{i}} + \cancel{\sin t\,\vec{i}} + t\cos t\,\vec{i} + \cancel{\cos t}\,\vec{j} - \cancel{\cos t}\,\vec{j} - t\sin t\,\vec{j}$$

$$\therefore \vec{v}(t) = t\cos t\,\vec{i} - t\sin t\,\vec{j}$$

$$|\vec{v}(t)| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = t$$

$$\vec{a}(t) = \cos t\,\vec{i} - t\sin t\,\vec{i} - \sin t\,\vec{j} - t\cos t\,\vec{j}$$
$$= (\cos t - t\sin t)\vec{i} - (\sin t + t\cos t)\vec{j}$$

$$a_T = \frac{d}{dt}|\vec{v}| = \frac{dt}{dt} = 1.$$

$$a_N = \sqrt{|\vec{a}|^2 - a_T^2} = t$$

$$|\vec{a}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{t^2+1}$$

$\therefore$ Acceleration

$$\vec{a} = \vec{T} + t\vec{N}$$