Advanced Calculus - Polar Coordinates

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Polar Coordinates

Definition

- Fix an origin O (called the pole)
- Fix an initial ray from the origin (called initial ray)
- $P(r, \theta)$ where $OP = r$, (directed distance from the point O to P.

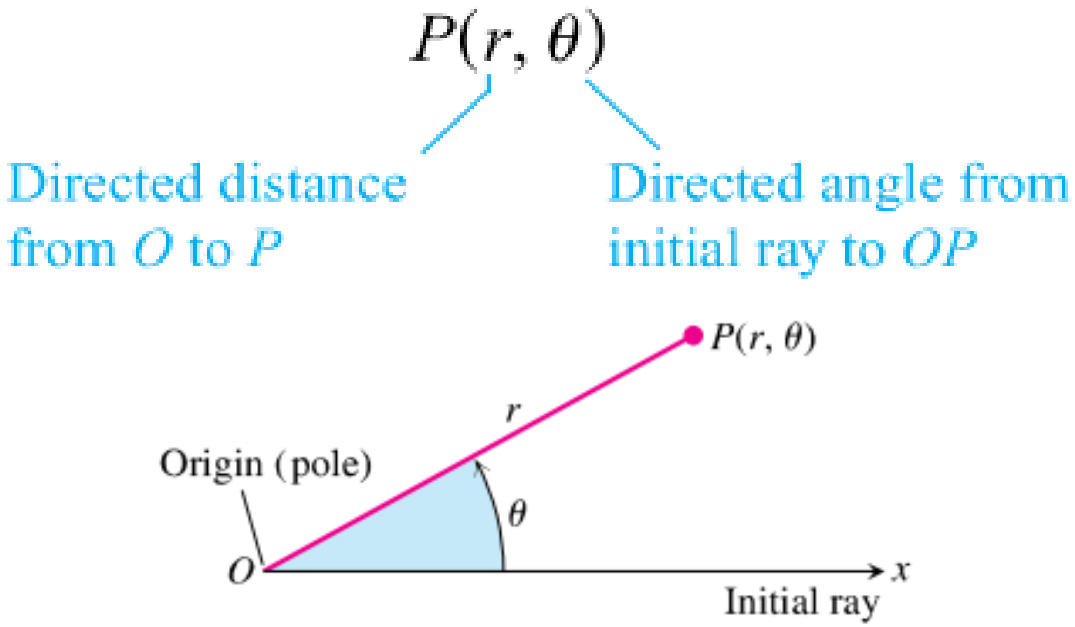
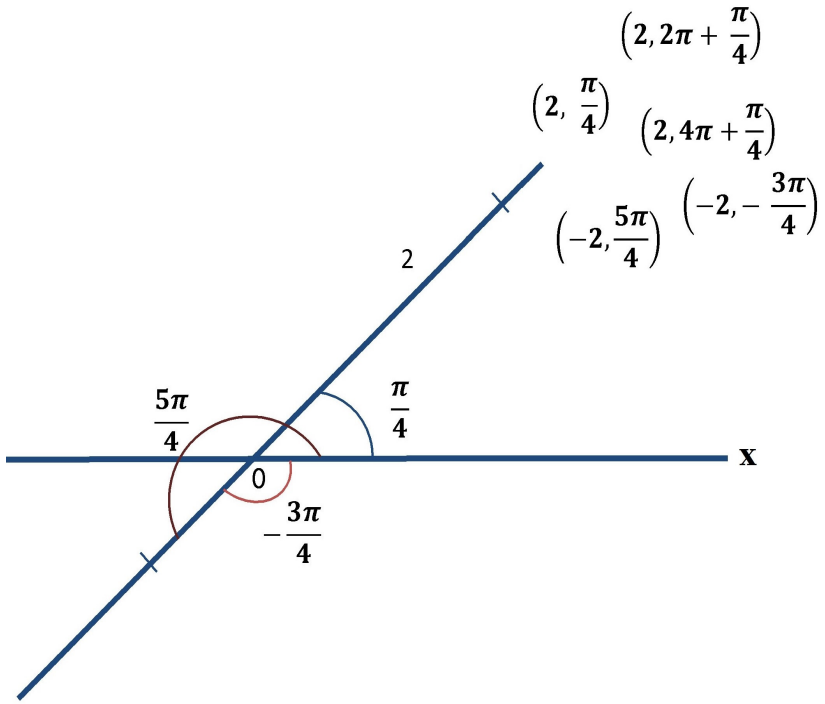


FIGURE 11.18 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

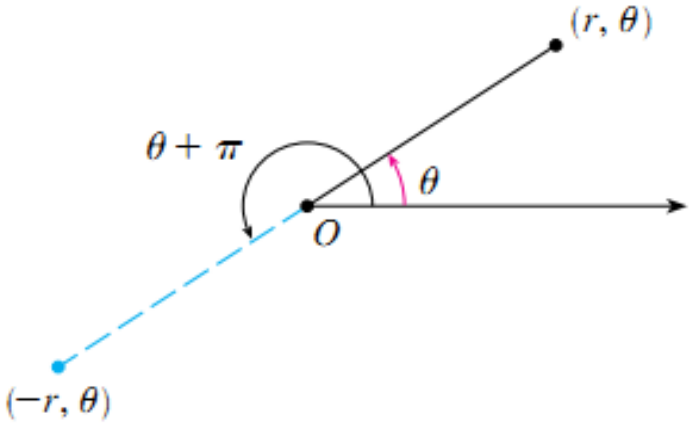
Polar coordinates are not unique...



- For $r = 2$, $\theta = \pi/4$, the complete list of angles is $\pi/4, \quad \pi/4 + 2\pi, \quad \pi/4 + 4\pi, \quad \pi/4 + 6\pi, + \cdots$
- For $r = -2$, $\theta = 5\pi/4$ the complete list of angles is $5\pi/4, \quad 5\pi/4 + 2\pi, \quad 5\pi/4 + 4\pi, \quad 5\pi/4 + 6\pi, + \cdots$

For the point $P(r, \theta)$, the equivalent polar coordinates are $P(r, \theta + 2n\pi)$, $n = 0, \pm 1, \pm 2, \pm 3, \cdots$

Polar Coordinates



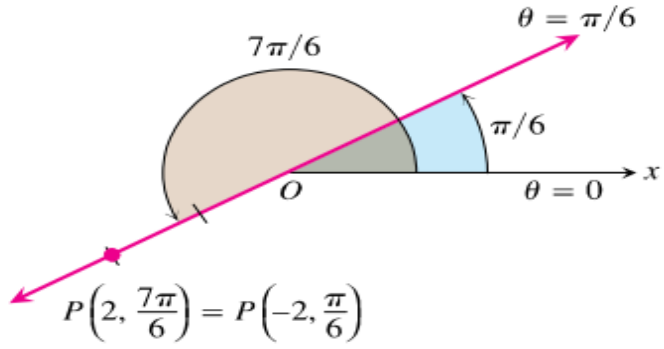


FIGURE 11.20 Polar coordinates can have negative r-values.

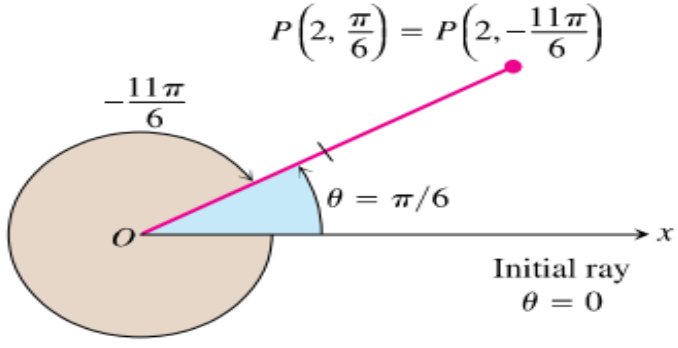
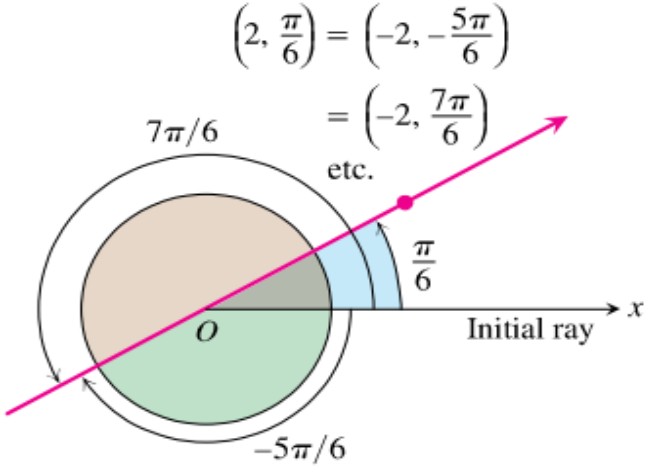


FIGURE 11.19 Polar coordinates are not unique.



Examples

1. Find all polar coordinates of the point $P(1, \pi/3)$.

2. Plot the following points.

   a. $(2, \pi/2)$ b. $(-3, \pi/4)$ c. $(-2, \pi/3)$

3. Which polar coordinate pairs label the same point?

   a. $(3, 0)$ c. $(2, 2\pi/3)$ e. $(r, \pi + \theta)$

   b. $(-3, \pi)$ d. $(-2, -\pi/3)$ f. $(-r, \theta)$

Polar Equations and Graphs

Equation	**Graph**
$r = a$	Circle of radius $\lvert a \rvert$ centered at origin.
$\theta = \theta_0$	Line through origin making an angle θ_0 with the initial ray.

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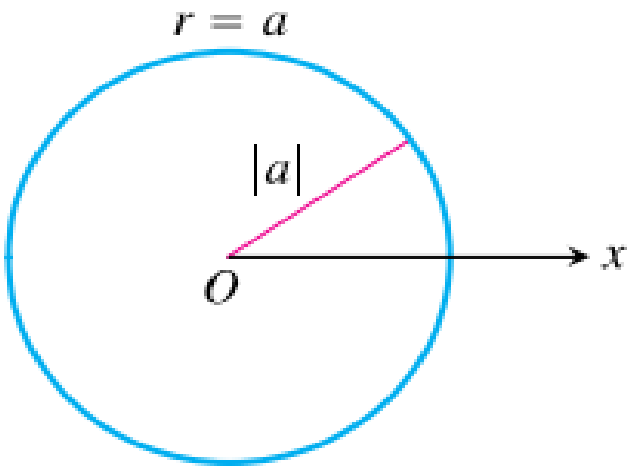

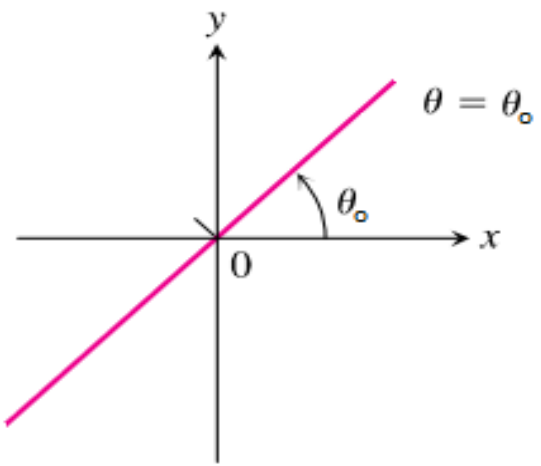

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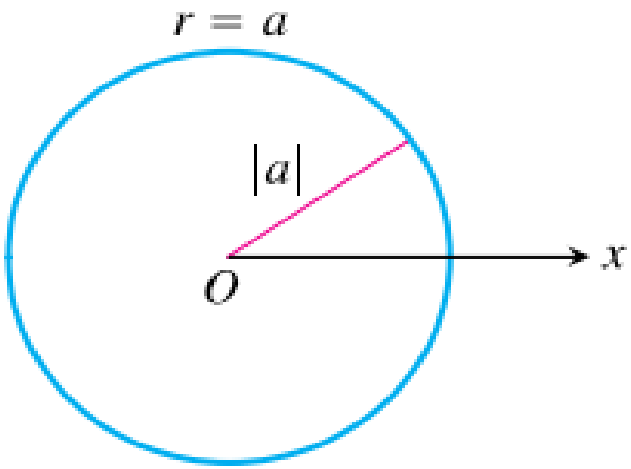

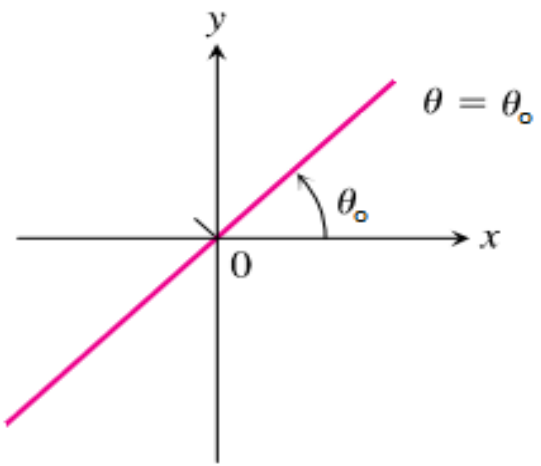

What does $r = r_0$, $\theta = \theta_0$ represent?

Equations of axes???

Graphs of Polar Equations

a. $r = 1$, $r = -1$

b. $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi/2$

c. $-3 \leq r \leq 2$ and $\theta = \pi/4$

d. $2\pi/3 \leq \theta \leq 5\pi/6$
(no restriction on r)

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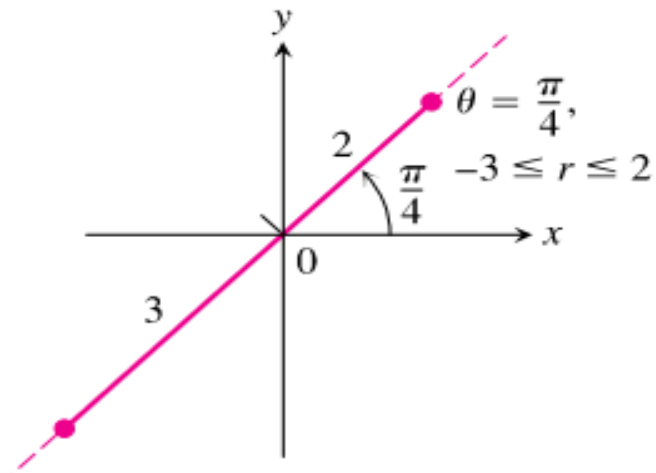

Figure: Graph of equation (c)

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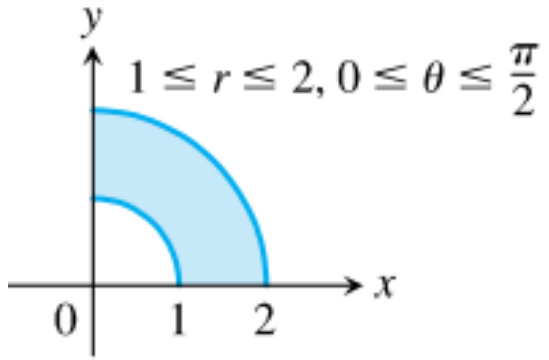
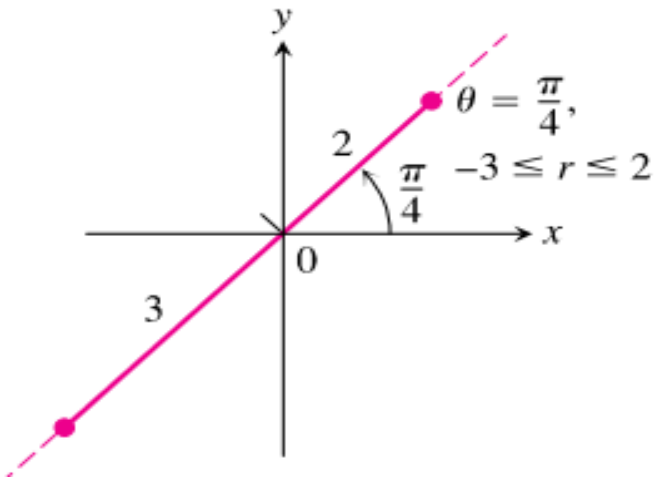


Figure: Graph of equation (b)



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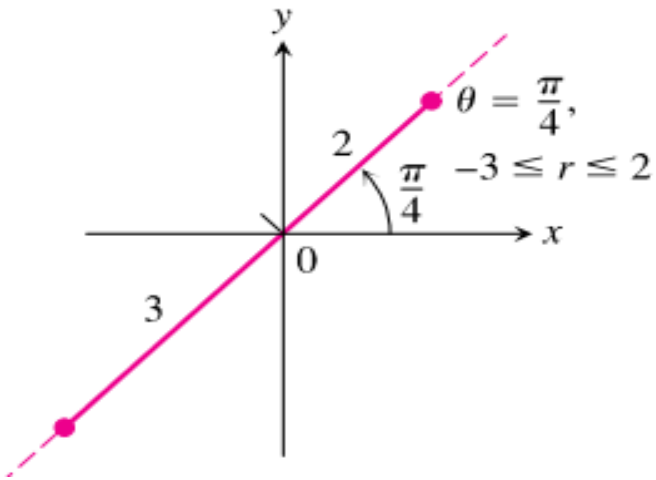


Figure: Graph of equation (c)

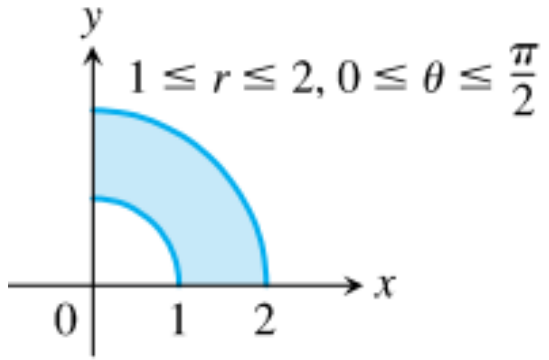


Figure: Graph of equation (b)

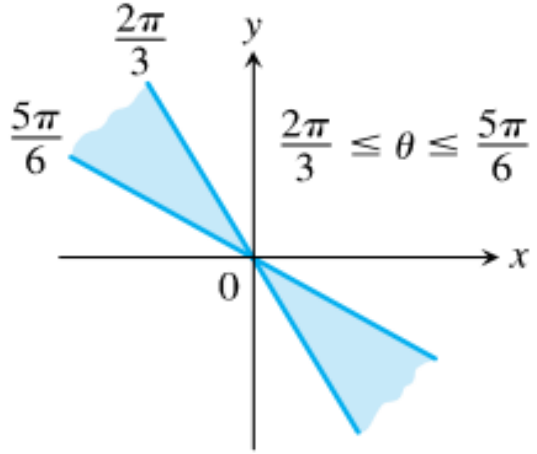


Figure: Graph of equation (d)

Examples

Graph the sets of points whose polar coordinates satisfy the following

a. $r = 2$

b. $0 \leq \theta \leq \pi/6,\ r \geq 0$

c. $-\pi/4 \leq \theta \leq \pi/4,$
$-1 \leq r \leq 1$

d. $\theta = \pi/2,\ r \geq 0$

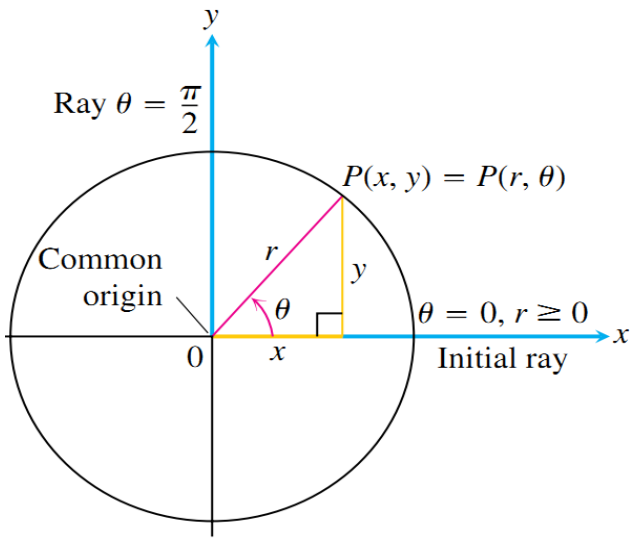
e. $0 \leq r \leq 2$

f. $\theta = \pi/2,\ r \leq 0$

g. $r \geq 1$

h. $0 \leq \theta \leq \pi,\ r = 1$

Relating Polar and Cartesian Coordinates



Relation

$x = r\cos\theta,\ y = r\sin\theta$ (Polar - Cartesian)

$r^2 = x^2 + y^2,\ \tan\theta = \dfrac{y}{x}$ (Cartesian - Polar)

Write equivalent Cartesian equations.

a. $r\sin\theta = 2$

b. $r = -3\sec\theta$

c. $r^2\sin 2\theta = 2$

d. $r = 1 - \cos\theta$

e. $r = 1 + 2r\cos\theta$

f. $r = 2\cos\theta - \sin\theta$

g. $r\sin\left(\theta + \dfrac{\pi}{6}\right) = 2$

h. $r\sin\theta = \ln r + \ln\cos\theta$

Write equivalent Polar equations.

a. $x = 1$ **b.** $x^2 + y^2 = 4$ **c.** $x^2 + xy + y^2 = 1$

THE END