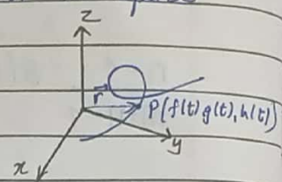
Unit: 5

VECTOR FUNCTIONS AND THEIR DERIVATIVES

o) Vector Valued Functions:

A vector valued function on a domain set D is a rule that assigns a vector in space to each element in D

eg $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$



o) Limits:

Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D and $\vec{L}$ be a vector. We say that r has limit $\vec{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\vec{r}(t) - \vec{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta$$

If $\vec{L} = L_1\vec{i} + L_2\vec{j} + L_3\vec{k}$ then, it can be shown that $\lim_{t \to t_0} \vec{r}(t) = \vec{L}$ precisely when

$$\lim_{t \to t_0} f(t) = L_1, \quad \lim_{t \to t_0} g(t) = L_2, \quad \lim_{t \to t_0} h(t) = L_3$$

<Q>: Find $\lim_{t \to \pi/2}$ for $\vec{r}(t) = t\,\vec{i} + \sin t\,\vec{j} + \cos t\,\vec{k}$.

Soln.

$\vec{r}(t) = t\,\vec{i} + \sin t\,\vec{j} + \cos t\,\hat{k}$

Now,

$$\lim_{t \to \pi/2} \vec{r}(t) = \lim_{t \to \pi/2} t\,\vec{i} + \lim_{t \to \pi/2} \sin t\,\vec{j} + \lim_{t \to \pi/2} \cos t\,\hat{k}$$

$$\therefore \lim_{t \to \pi/2} \vec{r}(t) = \frac{\pi}{2}\vec{i} + \vec{j}$$

<Q>: Find $\lim_{t \to \pi/2}$ for $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t\,\hat{k}$

Soln.

$\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t\,\hat{k}$

Now,

$$\lim_{t \to \pi/2} \vec{r}(t) = \lim_{t \to \pi/2} 2\cos t\,\vec{i} + \lim_{t \to \pi/2} 2\sin t\,\vec{j} + \lim_{t \to \pi/2} t\,\vec{k}$$

$$\therefore \lim_{t \to \pi/2} \vec{r}(t) = 2\vec{j} + \frac{\pi}{2}\vec{k}$$

(*) Note:

(i): Helix: $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + t\,\vec{k}$

(ii) Circle: $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

(iii) Ellipse: $\vec{r}(t) = a\cos t\,\vec{i} + b\sin t\,\vec{j}$

(iv) Hyperbola $\vec{r}(t) = a\cos t\,\vec{i} - b\sin t\,\vec{j}$

Date. No.

(*) Continuity:

A vector function $\vec{r}(t)$ is continuous at a point $t = t_0$ in its domain. If

$$\lim_{t \to t_0} r(t) = r(t_0)$$

The function is continuous if it is continuous at every point in its domain.

Eg: function $\vec{g}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + \lfloor t \rfloor \vec{k}$ is discontinuous at every integer, where the greatest integer function ~~is~~ $\lfloor t \rfloor$ is discontinuous.

(Q): Is function $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t\,\vec{k}$ continuous at $t = \pi/2$?

Soln:

Here,

$\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t\,\vec{k}$ $\quad t_0 = \pi/2$.

Now,

$$r(t_0) = r(\pi/2) = 2\cos\frac{\pi}{2}\,\vec{i} + 2\sin\frac{\pi}{2}\,\vec{j} + \frac{\pi}{2}\,\vec{k}$$

$$\therefore r(t_0) = 2\vec{j} + \frac{\pi}{2}\vec{k}.$$

Here,

$f(t) = 2\cos t$ $\quad \therefore \lim_{t \to \pi/2} f(t) = 0$

$g(t) = 2\sin t$ $\quad \therefore \lim_{t \to \pi/2} g(t) = 2$

$h(t) = t$ $\quad \therefore \lim_{t \to \pi/2} h(t) = \pi/2$.

Date. No.

So,

$$\lim_{t \to \pi/2} \vec{r}(t) = \lim_{t \to \pi/2} f(t)\vec{i} + \lim_{t \to \pi/2} g(t)\vec{j} + \lim_{t \to \pi/2} h(t)\vec{j}$$

$$\therefore \lim_{t \to \pi/2} \vec{r}(t) = 2\vec{j} + \frac{\pi}{2}\vec{k}.$$

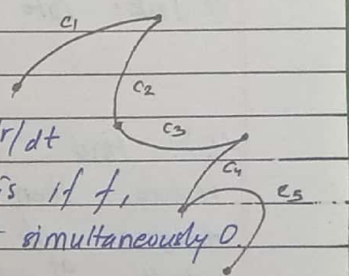
Thus, $\vec{r}(t)$ is continuous at $t = \pi/2$.

(*) Derivative:

The vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ has derivative at t if f, g, h have derivatives at t. The derivative is the vector function.

$$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k}$$

(*) Smooth curve:



A vector function r is differentiable if it is differentiable at every point of its domain. The curve traced by r is smooth if dr/dt is continuous and never 0, that is if f, g, h have continuous 1st derivatives not simultaneously 0.

(*) Piecewise Smooth Curve:

A curve made up of a finite number of smooth curves pieced together in continuous fashion is called piecewise smooth curve.

Date. No.

(*) Terminologies:

$\vec{r}$ = position vector of a particle moving along a smooth curve in space

(i) Velocity: ~~$(\vec{v}(t))$~~ $\vec{v}(t) = \frac{d\vec{r}}{dt}$

(ii): speed $= |\vec{v}(t)|$

(iii) Direction of Motion $= \frac{\vec{v}}{|\vec{v}|}$

Since

velocity = speed x direction.

$= |\vec{v}| \times \frac{\vec{v}}{|\vec{v}|}$

(iv) Acceleration $(\vec{a}(t)) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$

v) Jerk: rate of change of acceleration.

<Q>: Find the particle's velocity and acceleration vectors. Then, find the particle's speed and direction of motion at t ~~=~~. Write the particle's velocity at that time as a product of its speed and direction.

Date. No.

(i): $\vec{r}(t) = (2\cos t)\vec{i} + (3\sin t)\vec{j} + (4t)\vec{k}$, $t = \pi/2$.

Soln,

Given,

$\vec{r}(t) = (2\cos t)\vec{i} + (3\sin t)\vec{j} + (4t)\vec{k}$

$\therefore \frac{d\vec{r}(t)}{dt} = \vec{v}(t) = -2\sin t\,\vec{i} + 3\cos t\,\vec{j} + 4\vec{k}$

$\therefore \vec{a}(t) = \frac{d\vec{v}(t)}{dt} = -2\cos t\,\vec{i} - 3\sin t\,\vec{j}$

$\vec{v}(\pi/2) = -2\vec{i} + 4\vec{k}$

$|\vec{v}(\pi/2)| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}$ = speed.

Direction of motion $= \frac{\vec{v}(\pi/2)}{|\vec{v}(\pi/2)|} = \frac{-2}{\sqrt{20}}\vec{i} + \frac{4}{\sqrt{20}}\vec{k}$

Velocity $(\vec{v}) = \sqrt{20} \times \left(\frac{-2\vec{i}}{\sqrt{20}} + \frac{4\vec{k}}{\sqrt{20}}\right)$

$\therefore \vec{v} = -2\vec{i} + 4\vec{k}$ at $t = \pi/2$.

Date. No.

(ii): $\vec{r}(t) = (2\ln(t+1))\vec{i} + t^2\vec{j} + \frac{t^2}{2}\vec{k}$

Soln:

Given,

$\vec{r}(t) = (2\ln(t+1))\vec{i} + t^2\vec{j} + \frac{t^2}{2}\vec{k}$

$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\,2\ln(t+1)}{d(t+1)} \times \frac{d(t+1)}{dt}\vec{i} + \frac{dt^2}{dt}\vec{j} + \frac{d\,t^2/2}{dt}\vec{k}$

$= 2 \cdot \frac{t}{(t+1)}\vec{i} + 2t\vec{j} + t\vec{k}$

$\therefore \vec{v}(t) = \frac{2t}{(t+1)}\vec{i} + 2t\vec{j} + t\vec{k}$

$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{1}{(1+t)^2} + 2\vec{j} + \vec{k}$

When $t=1$,

$\vec{v}(1) = \vec{i} + 2\vec{j} + \vec{k}$

Speed $= |\vec{v}(1)| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$

direction of motion. $= \frac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}$

Velocity = Speed × Motion's direction

$= \sqrt{6} \times \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}$

$\vec{v}(1) = \vec{i} + 2\vec{j} + \vec{k}$

Date. No.

(*) Rules of Differentiation:

If u and v are differentiable functions of t, C is a constant vector, c is any scalar, f any differentiable scalar functions.

(i): Constant function rule: $\frac{d}{dt}C = 0$

(ii) Scalar multiple Rules: $\frac{d}{dt}[cu(t)] = cu'(t)$

$\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$

(iii) Sum rule $= \frac{d}{dt}[u(t) \pm v(t)] = u'(t) \pm v'(t)$

(iv) Dot product rule: $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t)v'(t)$

(v): Chain rule. $\frac{d}{dt}[u(f(t))] = f'(t) \cdot u'(f(t))$

(vi) Cross product rule: $\frac{d}{dt}[u(t) \times v(t)]$

$= u'(t) \times v(t) + u(t) \times v'(t)$.

Date. No.

(*) Vector Functions at Constant length:

If r is a differentiable vector function of t & constant length, then

$$r \cdot \frac{dr}{dt} = 0.$$

By direction calculation:

$r(t) \cdot r(t) = c^2$ [$\because |r(t)| = c$]

or $\frac{d}{dt}[r(t) \cdot r(t)] = 0$ [$\because$ Differentiating both sides w.r.t t]

or $r'(t) \cdot r(t) + r(t) \cdot r'(t) = 0$

or $2\, r'(t) \cdot r(t) = 0$

Hence, $r'(t) \perp r(t)$ because their dot product is 0.

Date. No.

Integrals / Antiderivatives

The indefinite integral of r with respect to t is the set of all antiderivatives of r, denoted by $\int r(t)\, dt$.

If R is any antiderivative of r, then

$$\int r(t) \cdot dt = R(t) + C$$

If the components of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a,b]$ then so is r, and the definite integral of r from a and b is

$$\int_a^b r(t)\, dt = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)\, dt\right)\vec{j} + \left(\int_a^b h(t)\, dt\right)\vec{k}$$

<Q7: If the velocity of a particle moving in space is $\frac{d\vec{r}}{dt} = (\cos t)\vec{i} - (\sin t)\vec{j} + \vec{k}$. Find the particle's position as a function of t if $\vec{r} = 2\vec{i} + \vec{k}$ when $t = 0$.

Soln:

Given,

$$\frac{d\vec{r}}{dt} = (\cos t)\vec{i} - (\sin t)\vec{j} + \vec{k}$$

On integration w.r.t t,

Date. No.

$$\vec{r}(t) = \int(\cos t\, dt)\vec{i} - \int(\sin t\, dt)\vec{j} + \int(dt)\vec{k}$$

$$= \sin t\,\vec{i} + \cos t\,\vec{j} + t\,\vec{k} + C$$

At $t=0$, $\vec{r}(t) = 2\vec{i} + \vec{k}$

$$2\vec{i} + \vec{k} = \sin 0\,\vec{i} + \cos 0\,\vec{j} + 0\vec{k} + \vec{c}$$

$$\therefore \vec{c} = 2\vec{i} - \vec{j} + \vec{k}$$

$$\therefore \vec{r}(t) = (\sin t + 2)\,\vec{i} + (\cos t - 1)\,\vec{j} + (t+1)\,\vec{k}$$

(Q) Solve IVP:

(i) $\frac{d\vec{r}}{dt} = -t\vec{i} - t\vec{j} - t\vec{k}$ $\quad \vec{r}(0) = \vec{i} + 2\vec{j} + 3\vec{k}$

Soln:

Given,

$$\frac{d\vec{r}}{dt} = -t\vec{i} - t\vec{j} - t\vec{k}$$

On integrating w.r.t t,

$$\vec{r}(t) = -\int(t\,dt)\vec{i} - \int(t\cdot dt)\vec{j} - \int(t\cdot dt)\vec{k}$$

$$= -\frac{t^2}{2}\vec{i} - \frac{t^2}{2}\vec{j} - \frac{t^2}{2}\vec{k} + \vec{c}$$

Now,

$$\vec{r}(0) = \vec{i} + 2\vec{j} + 3\vec{k}$$

Date. No.

Now,

$$\vec{i} + 2\vec{j} + 3\vec{k} = 0 + 0 + 0 + \vec{c}$$

$$\therefore \vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\therefore \vec{r}(t) = \left(\frac{-t^2}{2} + 1\right)\vec{i} + \left(\frac{-t^2}{2} + 2\right)\vec{j} + \left(\frac{-t^2}{2} + 3\right)\vec{k}$$

(ii): $\frac{d^2\vec{r}}{dt^2} = -32\vec{k}$, $\quad \vec{r}(0) = 100\vec{k}$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\vec{i} + 8\vec{j}$$

Soln

Given,

$$\frac{d^2\vec{r}}{dt^2} = -32\vec{k}$$

On integrating w.r.t t,

$$\frac{d\vec{r}}{dt} = -32t\,\vec{k} + \vec{c}$$

At $t=0$,

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = 8\vec{i} + 8\vec{j}$$

or $0 + \vec{c} = 8\vec{i} + 8\vec{j}$ $\quad \therefore \vec{c} = 8\vec{i} + 8\vec{j}$

So,

$$\frac{d\vec{r}}{dt} = 8\vec{i} + 8\vec{j} - 32\vec{k}\,t$$

On integrating w.r.t t,

$$\vec{r}(t) = 8t\,\vec{i} + 8t\,\vec{j} - 16t^2\,\vec{k} + \vec{c}$$

We know,

$$\vec{r}(0) = 100\,\vec{k}$$

So,

$$0+0+0+\vec{c} = 100\,\vec{k} \qquad \therefore \vec{c} = 100\,\vec{k}$$

$$\therefore \vec{r}(t) = 8t\,\vec{i} + 8t\,\vec{j} + (-16t^2+100)\,\vec{k}$$

(*) Arc length: $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, $t_0 \le t \le t_1$

$$\text{Arc length } (L) = \int_{t_0}^{t_1} \sqrt{\left(\frac{df(t)}{dt}\right)^2 + \left(\frac{dg(t)}{dt}\right)^2 + \left(\frac{dh(t)}{dt}\right)^2}\, dt$$

$$L = \int_{t_0}^{t_1} |v|\, dt$$

(*) Arc length parameter:

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$$

(Q): Find the length of the indicated portion of the curve

$$\vec{r}(t) = (4\cos t)\,\vec{i} + (4\sin t)\,\vec{j} + (3t)\,\vec{k}, \quad 0 \le t \le \pi/2.$$

Soln.:

Given,

$$\vec{r}(t) = (4\cos t)\,\vec{i} + (4\sin t)\,\vec{j} + (3t)\,\vec{k}, \quad 0 \le t \le \pi/2.$$

$$\frac{df}{dt} = \frac{d(4\cos t)}{dt} = -4\sin t$$

$$\frac{dg}{dt} = \frac{d(4\sin t)}{dt} = 4\cos t$$

$$\frac{dh}{dt} = \frac{d(3t)}{dt} = 3$$

Now,

$$L = \int_0^{\pi/2} \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{16\sin^2 t + 16\cos^2 t + 9}\, dt$$

$$= \int_0^{\pi/2} 5\, dt = \frac{5\pi}{2} \text{ units.}$$

<Q>: Find the length of the curve $\vec{r}(t) = (\sqrt{2}t)\vec{i} + (\sqrt{2}t)\vec{j} + (1-t^2)\vec{k}$ from $(0,0,1)$ to $(\sqrt{2},\sqrt{2},0)$.

Soln:

Given,

$\vec{r}(t) = (\sqrt{2}t)\vec{i} + (\sqrt{2}t)\vec{j} + (1-t^2)\vec{k}$

from $(0,0,1)$ to $(\sqrt{2},\sqrt{2},0)$

When $t=0$, $\vec{r}(0) = 0\vec{i} + 0\vec{i} + 1\vec{k} = (0,0,1)$

when $t=1$, $\vec{r}(1) = \sqrt{2}\vec{i} + \sqrt{2}\vec{j} + 0\cdot\vec{k} = (\sqrt{2},\sqrt{2},0)$

Now, $\vec{r}(t) = \sqrt{2}t\vec{i} + \sqrt{2}t\vec{j} + (1-t^2)\vec{k}$, $0 \le t \le 1$.

So,

$$\frac{df}{dt} = \frac{d\sqrt{2}t}{dt} = \sqrt{2}$$

$$\frac{dg}{dt} = \frac{d\sqrt{2}t}{dt} = \sqrt{2}$$

$$\frac{dh}{dt} = \frac{d(1-t^2)}{dt} = -2t$$

Now,

$$L = \int_0^1 \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$$

$[\because \int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)) + C]$

$$= \int_0^1 \sqrt{2+2+4t^2}\, dt$$

$$= \int_0^1 2\sqrt{1+t^2}\, dt$$

Let $t = \tan\theta$. when $t=0$, $\theta = 0°$

So, $dt = \sec^2\theta\, d\theta$ when $t=1$, $\theta = \pi/4$.

$$= 2\int_0^{\pi/4} \sqrt{1+\tan^2\theta}\, \sec^2\theta\, d\theta$$

$$= 2\int_0^{\pi/4} \sec^3\theta\, d\theta$$

$$= 2\int_0^{\pi/4} \sec\theta \cdot \sec^2\theta\, d\theta$$

$$= 2\left[\sec\theta \int \sec^2\theta\, d\theta - \int\left(\left(\frac{d\sec\theta}{d\theta}\right)\left(\int \sec^2\theta\, d\theta\right)\right)d\theta\right]_0^{\pi/4}$$

$$= 2\left[\sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta\right]_0^{\pi/4}$$

$$= 2\left[\sec\theta\tan\theta - \left(\int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta\right)\right]_0^{\pi/4}$$

$$= 2\left[\sec\theta\tan\theta + \int \sec\theta\, d\theta - \int \sec^3\theta\, d\theta\right]_0^{\pi/4}$$

$$= \cancel{2} \times \frac{1}{\cancel{2}}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\pi/4}$$

$\therefore L = \sqrt{2} + \ln|\sqrt{2}+1|$ units

<Q>: Find the arc length parameter along the curve

$$\vec{r}(t) = (4\cos t)\vec{i} + (4\sin t)\vec{j} + (3t)\vec{k} \cdot \quad 0 \le t \le \pi/2$$

from point $t=0$.

Sol^n:

Given,

$$\vec{r}(t) = (4\cos t)\vec{i} + (4\sin t)\vec{j} + (3t)\vec{k}, \quad 0 \le t \le \pi/2$$

Now,

$$\frac{df}{dt} = \frac{d\,4\cos t}{dt} = -4\sin t$$

$$\frac{dg}{dt} = \frac{d(4\sin t)}{dt} = 4\cos t$$

$$\frac{dh}{dt} = \frac{d\,3t}{dt} = 3$$

So,

Arc length parameter $L = \int_0^t \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$

$$= \int_0^t \sqrt{(-4\sin t)^2 + (4\cos t)^2 + (3)^2}\, dt$$

$$= \int_0^t \sqrt{16\sin^2 t + 16\cos^2 t + 9}\, dt$$

$$= \int_0^t 5\, dt = 5t \text{ units.}$$