Nabla.

~~Gradient~~: (∇)

~~Gradient is denoted by vector differential~~ operator del / nabla is defined in Cartesian coordinates as.

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

~~A gradient~~ ∇ doesn't have any specific meaning until we provide a function for it to act upon.

*) Acting of operator ∇

i) Gradient : ∇f

ii) Divergence : $\nabla \cdot \vec{V}$

iii) Curl : $\nabla \times \vec{V}$

Gradient:

Suppose we have a function of three variables, the temperature in a room $T(x, y, z)$

We know, small change in temperature (dT)

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz \quad -(i)$$

$$= \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$= (\nabla T) \cdot dl$

where

$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$ is gradient of T.

Geometrical intepretation:

$dT = (\nabla T) \cdot (dl)$

$= |\nabla T| \, |dl| \cos\theta$

Here,

θ = angle between ∇T and dl

When $\theta = 0$, dT = maximum $= |\nabla T| \cdot |d\vec{l}|$

Note: (i): Gradient shows in what direction change occurs from particular point

(ii) Gradient points in the direction of maximum increase of function.

(iii) $|\nabla T|$ is rate of maximum increase.

(iv): Gradient turns scalar field into vector field.

Example: Suppose that the temperature T at point (x, y, z) is given by equation $T = x^2 - y^2 + xyz + 273$. In which direction is the temperature increase most rapidly at $(-1, 2, 3)$ and at what rate?

Soln:

Given,

$T = x^2 - y^2 + xyz + 273$

Now,

$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$

$= \frac{\partial (x^2 - y^2 + xyz + 273)}{\partial x}\hat{i} + \frac{\partial (x^2 - y^2 + xyz + 273)}{\partial y}\hat{j} + \frac{\partial (x^2 - y^2 + xyz + 273)}{\partial z}\hat{k}$

$= (2x + yz)\hat{i} + (-2y + xz)\hat{j} + (xy)\hat{k}$

At $(-1, 2, 3)$

$\nabla T = 4\hat{i} - 7\hat{j} - 2\hat{k}$

Rate $= |\nabla T| = \sqrt{4^2 + (-7)^2 + (-2)^2}$

$= \sqrt{69}$

Example: The gradient of the function $t = x^2 y + e^z$ at point $(1, 5, -2)$ is:

Soln:

Given,

$t = x^2 y + e^z \quad \text{---(i)}$

Now,

$\nabla t = \frac{\partial t}{\partial x}\hat{i} + \frac{\partial t}{\partial y}\hat{j} + \frac{\partial t}{\partial z}\hat{k}$

$$= \frac{\partial(x^2y+e^z)}{\partial x}\hat{i} + \frac{\partial(x^2y+e^z)}{\partial y}\hat{j} + \frac{\partial(x^2y+e^z)}{\partial z}$$

$$= 2xy\,\hat{i} + x^2\hat{j} + e^z\hat{k}$$

At $(1, 5, -2)$

$$\nabla t = 10\hat{i} + \hat{j} + \frac{1}{e^2}\hat{k}$$

Example: Find gradient of function $f(x,y,z) = x^2+y^2+z^2$.

Soln:

Given

$$f = x^2+y^2+z^2$$

Now,

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{(x^2+y^2+z^2)}{\partial x}\hat{i} + \frac{(x^2+y^2+z^2)}{\partial y}\hat{j} + \frac{(x^2+y^2+z^2)}{\partial z}\hat{k}$$

$$\therefore \nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Example: Find gradient of function $f(x,y,z) = e^x \sin y \ln z$

Soln.

Given,

$$f = e^x \sin y \ln z$$

Now,

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial(e^x \sin y \ln z)}{\partial x}\hat{i} + \frac{\partial(e^x \sin y \ln z)}{\partial y}\hat{j} + \frac{\partial(e^x \sin y \ln z)}{\partial z}\hat{k}$$

$$= (e^x \sin y \ln z)\hat{i} + (e^x \cos y \ln z)\hat{j} + \frac{e^x \sin y}{z}\hat{k}$$

Example: The magnitude of position vector $r = \sqrt{x^2+y^2+z^2}$. find ∇r, $\nabla(1/r)$, ∇r^n.

Soln.

We know,

$$\nabla r^n = n r^{n-2}\vec{r} = n r^{n-1}\hat{r}$$

Given, $r = \sqrt{x^2+y^2+z^2}$

or, $r^2 = x^2+y^2+z^2$

$$\frac{\partial r^2}{\partial x} = \frac{\partial(x^2+y^2+z^2)}{\partial x}$$

$$\text{or } 2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r^2}{\partial y} = \frac{\partial(x^2+y^2+z^2)}{\partial y}$$

or $$2r\frac{\partial r}{\partial y} = 2y.$$

$$\therefore \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r^2}{\partial z} = \frac{\partial(x^2+y^2+z^2)}{\partial z}$$

or $$2r\frac{\partial r}{\partial z} = 2z \qquad \therefore \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now,

(i) $$\nabla r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r} = \hat{r}$$

$$\therefore \nabla r = \hat{r}$$

(ii) $$\nabla\left(\frac{1}{r}\right) = \hat{i}\frac{\partial(r^{-1})}{\partial x} + \hat{j}\frac{\partial(r^{-1})}{\partial y} + \hat{k}\frac{\partial(r^{-1})}{\partial z}$$

$$= \cancel{\hat{i}\left[-1\cdot r^{-2}\right]} + \cancel{\hat{j}\left[-1\cdot r^{-2}\right]} + \cancel{\hat{k}\left[-1\cdot r^{-2}\right]}$$

$$= \hat{i}\left[-1\cdot r^{-2}\cdot\frac{\partial r}{\partial x}\right] + \hat{j}\left[-1\cdot r^{-2}\cdot\frac{\partial r}{\partial y}\right] + \hat{k}\left[-1\cdot r^{-2}\cdot\frac{\partial r}{\partial z}\right]$$

$$= \frac{-1}{r^2}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$$

$$= \frac{-1}{r^2}\left[\frac{\vec{r}}{r}\right] = \frac{-\vec{r}}{r^3} = \frac{-\hat{r}}{r^2}$$

Divergence:

The divergence of a vector is the limit of its surface integral per unit volume as the volume enclosed by the surface goes to zero.

Mathematically,

$$\text{div}\,\vec{F} = \lim_{V\to 0}\frac{1}{V}\oint_S \vec{F}\cdot\vec{da}$$

Divergence of vector function $\vec{v}$

$$\text{div}\cdot\vec{v} = \nabla\cdot\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Divergence of a vector function is a scalar.

If $\nabla\cdot\vec{v} > 0$, acts as source

If $\nabla\cdot\vec{v} < 0$, acts as sink

If $\nabla\cdot\vec{v} = 0$, solenoidal {source or sink}.

Example: Calculate the divergence of a vector function $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$.

Soln:

Given,

$$\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$$

Now,

$$\nabla\cdot\vec{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$\therefore \nabla\cdot\vec{v} = 3$$

Example: If $\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla\cdot\vec{A}$ at $(1,-1,1)$.

Soln:

Given,

$$\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$$

Now,

$$\nabla\cdot\vec{A} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial(x^2z)}{\partial x} + \frac{\partial(-2y^3z^2)}{\partial y} + \frac{\partial(xy^2z)}{\partial z}\hat{k}$$

$$= 2xz - 6y^2z^2 + xy^2$$

At $(1,-1,1)$,

$$= 2 - 6 + 1$$

$$= -3$$

Example: If vector r is a position vector, show that $\nabla\cdot\hat{r} = \frac{2}{r}$

Soln.

Given,

$$= \nabla\cdot\hat{r}$$

$$= \nabla\cdot\frac{\vec{r}}{r}$$

$$= \frac{1}{r}(\nabla\cdot\vec{r}) + \nabla\left(\frac{1}{r}\right)\vec{r}$$

$$= \frac{1}{r}\times 3 + \left(\frac{-\vec{r}}{r^3}\right)\cdot\vec{r} \qquad [\because \nabla\cdot\vec{r} = 3,\ \text{position vector}] \quad [\because \nabla r^n = n r^{n-2}\vec{r}\,]$$

$$= \frac{3}{r} - \frac{1}{r^3}(\vec{r}\cdot\vec{r})$$

$$\therefore \nabla\cdot\hat{r} = \frac{2}{r}.$$

Hence, proved

Curl

The component of curl $\vec{F}$ in the direction of the unit vector $\hat{n}$ is the limit of a line integral per unit area as the enclosed area goes to zero, this area being perpendicular to $\hat{n}$.

$$\hat{n} \cdot \text{curl } \vec{F} = \lim_{s \to 0} \frac{1}{s} \oint_c F \cdot \vec{dl}$$

The curl of a vector $\vec{v}$ gives a vector.

$$\text{curl } \vec{v} = \nabla \times \vec{v}$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) - \hat{j} \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z} \right) + \hat{k} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

Geometrical Interpretation:

The curl of vector function $\vec{v} = \nabla \times \vec{v}$ gives measure of how much vector $\vec{v}$ "curls around" the particular point.

If $\nabla \times \vec{v} = 0$, irrotational.

Example: Calculate the curl of vector $\vec{v} = x\hat{j} - y\hat{i}$

Soln.

Given,

$$\vec{v} = -y\hat{i} + x\hat{j} + 0 \cdot \hat{k}$$

Now,

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial \cdot 0}{\partial y} - \frac{\partial x}{\partial z} \right) - \hat{j} \left(\frac{\partial \cdot 0}{\partial x} + \frac{\partial \cdot y}{\partial z} \right) + \hat{k} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right)$$

$$= 2\hat{k}$$

Example: If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find $\nabla \times \vec{A}$ at $(1, -1, 1)$.

Soln:

Given,

$$\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$$

We know,

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial 2yz^4}{\partial y} + \frac{\partial (2x^2yz)}{\partial z}\right) - \hat{j}\left(\frac{\partial 2yz^4}{\partial x} - \frac{\partial xz^3}{\partial z}\right)$$
$$+ \hat{k}\left(\frac{\partial (2x^2yz)}{\partial x} + \frac{\partial xz^3}{\partial y}\right)$$

$$= \hat{i}(2z^4 + 2x^2y) - \hat{j}(0 - 3xz^2) + \hat{k}(4xyz)$$

At $(1,-1,1)$,

$$\therefore \nabla \times \vec{A} = 3\hat{j} + 4\hat{k}$$

***) Note:** i) $\nabla(fg) = f(\nabla g) + (\nabla f)g$

ii) $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + (\nabla f)\cdot\vec{A}$

iii) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\cdot\vec{A}) - \vec{A}(\nabla\times\vec{B})$

Example: Calculate divergence and curl of the vector function.

$$\vec{V} = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$$

Soln:

Given,

$$\vec{V} = y^2\hat{i} + (2xy+z^2)\hat{j} + 2yz\hat{k}$$

Now, Divergence $= \nabla\cdot\vec{V}$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial y^2}{\partial x} + \frac{\partial (2xy+z^2)}{\partial y} + \frac{\partial\, 2yz}{\partial z}$$

$$= 0 + 2x + 2y$$

$$= 2(x+y)$$

$$\therefore \nabla\cdot\vec{V} = 2(x+y).$$

Now,

curl $= \nabla\times\vec{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & (2xy+z^2) & 2yz \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial (2yz)}{\partial y} - \frac{\partial (2xy+z^2)}{\partial z}\right) - \hat{j}\left(\frac{\partial (2yz)}{\partial x} - \frac{\partial y^2}{\partial z}\right)$$
$$+ \hat{k}\left(\frac{\partial (2xy+z^2)}{\partial x} - \frac{\partial y^2}{\partial y}\right)$$

$$= \hat{i}(2z - 2z) - \hat{j}(0 - 0) + \hat{k}(\cancel{2x - y + 2z}\ 2y - 2y)$$

$$= 0 \quad \text{ie, irrotational.}$$

Five Species of Second Derivative

i) The gradient ∇T is a vector.
Divergence of gradient $\nabla \cdot (\nabla T) \Rightarrow$ ~~vector~~ scalar
Curl of gradient $\nabla \times (\nabla T) \Rightarrow$ vector

ii) The divergence $\nabla \cdot \vec{v}$ is a scalar.
Gradient of Divergence $\nabla(\nabla \cdot \vec{v}) \Rightarrow$ vector

iii) The curl $\nabla \times \vec{v}$ is vector
Divergence of curl $\nabla \cdot (\nabla \times \vec{v}) \Rightarrow$ scalar
Curl of curl $\nabla \times (\nabla \times v) \rightarrow$ vector.

*) Note:
i) Curl of gradient is always zero. $\nabla \times \nabla T = 0$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial T/\partial x & \partial T/\partial y & \partial T/\partial z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 T}{\partial y \partial z} - \frac{\partial^2 T}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 T}{\partial x \partial z} - \frac{\partial^2 T}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 T}{\partial x \partial y} - \frac{\partial^2 T}{\partial y \partial x}\right)$$

$= 0.$

(ii): Divergence of a curl is always zero. $\nabla(\nabla \times \vec{V}) = 0.$

Laplacian operator:

Laplacian operator: $\nabla \cdot \nabla = \nabla^2$ or Δ

The laplacian of scalar T is a scalar.

Laplacian of T: $\nabla \cdot (\nabla T) = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

The laplacian of ~~sca~~ vector $\vec{V}$ is a vector

Laplacian of $\vec{V}$: $\nabla^2 \vec{V} = (\nabla^2 V_x)\hat{i} + (\nabla^2 V_y)\hat{j} + (\nabla^2 V_z)\hat{k}$

Example: Calculate the laplacian of a function
$T_a = \sin x \sin y \sin z$

Soln.

Given,
$T_a = \sin x \sin y \sin z$

We know,

$$\nabla^2 T_a = \frac{\partial^2 T_a}{\partial x^2} + \frac{\partial^2 T_a}{\partial y^2} + \frac{\partial^2 T_a}{\partial z^2}$$

$$= \frac{\partial^2 (\sin x \sin y \sin z)}{\partial x^2} + \frac{\partial^2 (\sin x \sin y \sin z)}{\partial y^2} + \frac{\partial (\sin x \sin y \sin z)}{\partial z^2}$$

$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$

$= -3 \sin x \sin y \sin z$

Example: Calculate the Laplacian of the function $T_b = e^{-5x}\sin 4y \cos 3z$.

Sol^n.

Given,

$$T_b = e^{-5x}\sin 4y \cos 3z$$

We know,

$$\nabla^2 T_b = \frac{\partial^2 T_b}{\partial x^2} + \frac{\partial^2 T_b}{\partial y^2} + \frac{\partial^2 T_b}{\partial z^2}$$

$$= \frac{\partial^2(e^{-5x}\sin 4y\cos 3z)}{\partial x^2} + \frac{\partial^2(e^{-5x}\sin 4y\cos 3z)}{\partial y^2} + \frac{\partial^2(e^{-5x}\sin 4y\cos 3z)}{\partial z^2}$$

$$= 25e^{-5x}\sin 4y\cos 3z - 16e^{-5x}\sin 4y\cos 3z - 9e^{-5x}\sin 4y\cos 3z$$

$$\therefore \nabla^2 T_b = 25e^{-5x}\sin 4y\cos 3z - 16e^{-5x}\sin 4y\cos 3z - 9e^{-5x}\sin 4y\cos 3z$$

$$= 0.$$

Example: Calculate the Laplacian of the function $\vec{V} = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$

Sol^n.

Given,

$$\vec{V} = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$$

We know,

$$\nabla^2\vec{V} = (\nabla^2 V_x)\hat{i} + (\nabla^2 V_y)\hat{j} + (\nabla^2 V_z)\hat{k}$$

$$= (\nabla^2 x^2)\hat{i} + (\nabla^2 \cdot 3xz^2)\hat{j} + (\nabla^2 \cdot (-2xz))\hat{k}$$

$$= \hat{i}\left(\frac{\partial^2 x^2}{\partial x^2} + \frac{\partial^2 x^2}{\partial y^2} + \frac{\partial^2 x^2}{\partial z^2}\right) + \hat{j}\left(\frac{\partial^2 3xz^2}{\partial x^2} + \frac{\partial^2 3xz^2}{\partial y^2} + \frac{\partial^2 3xz^2}{\partial z^2}\right)$$
$$+ \left(\frac{\partial^2(-2xz)}{\partial x^2} + \frac{\partial^2(-2xz)}{\partial^2 y} + \frac{\partial^2(-2xz)}{\partial^2 z}\right)\hat{k}$$

$$= (2+0+0)\hat{i} + (0+0+6x)\hat{j} + (0+0+0)\hat{k}$$
$$= 2\hat{i} + 6x\hat{j}$$

Example: Calculate the Laplacian of function $\vec{V} = x^2y\,\hat{i} + (x^2-y)\hat{k}$

Sol^n.

Given,

$$\vec{V} = x^2y\,\hat{i} + (x^2-y)\hat{k}$$

We know,

$$\nabla^2\vec{V} = (\nabla^2 V_x)\hat{i} + (\nabla^2 V_y)\hat{j} + (\nabla^2 V_z)\hat{k}$$

$$= (\nabla^2(x^2y))\hat{i} + (\nabla^2(x^2-y))\hat{k}$$

$$= \left(\frac{\partial^2 x^2y}{\partial x^2} + \frac{\partial^2 x^2y}{\partial y^2} + \frac{\partial^2 x^2y}{\partial z^2}\right)\hat{i} + \left(\frac{\partial^2(x^2-y)}{\partial x^2} + \frac{\partial^2(x^2-y)}{\partial y^2} + \frac{\partial^2(x^2-y)}{\partial z^2}\right)\hat{k}$$

$$= (2y+0+0)\hat{i} + (2-0+0-0+0-0)\hat{k}$$

$$= 2y\hat{i} + 2\hat{k}$$

The curl of a gradient is always zero.

$$\nabla \times (\nabla T) = 0$$

Sol^n:

We know,

$$\nabla \times (\nabla T) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 T}{\partial y \partial z} - \frac{\partial^2 T}{\partial y \partial z}\right)\hat{i} - \left(\frac{\partial^2 T}{\partial x \partial z} - \frac{\partial^2 T}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 T}{\partial x \partial y} - \frac{\partial^2 T}{\partial x \partial y}\right)\hat{k}$$

$$= 0.$$

The divergence of a curl is always zero.

$$\nabla \cdot (\nabla \times \vec{v}) = 0.$$

Sol^n:

We know,

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Now,

$$\nabla \cdot (\nabla \times \vec{v}) = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left[\hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\therefore \nabla \cdot (\nabla \times \vec{v}) = 0.$$

Integral Calculus

(i): line integral:

If $\vec{F}$ is vector, a line integral of $\vec{F}$ is written

$$\int_{a\,C}^{b} \vec{F} \cdot \vec{dl}$$

Line integral over a closed curve: $\oint_C \vec{F} \cdot \vec{dl}$

(ii): Surface integral:

If $\vec{F}$ is vector, a surface integral of $\vec{F}$ is

$$\int_S \vec{F} \cdot d\vec{a}.$$

Surface integral over a closed surface

$$= \oint_S \vec{F} \cdot d\vec{a}$$

(iii): Volume integral

$$\int_V T \cdot d\tau$$

~~Fundamental~~ Fundamental Theorem of ~~Calculus~~ Gradient

Suppose, we have a scalar function of three ~~functions~~ variables $f(x,y,z)$.

The total change in f in going from a to b is

$$\int_a^b (\nabla f) \cdot d\vec{l} = f(b) - f(a).$$

Fundamental Theorem of Divergence (Gauss Theorem)

The fundamental theorem of divergences states that

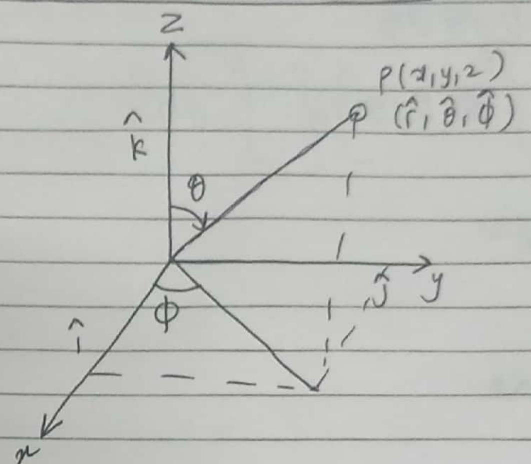
$$\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

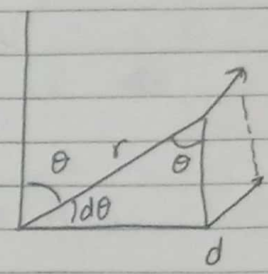
Fundamental Theorem of Curl (~~Gauss theorem~~ Stoke's Theorem)

The fundamental theorem of curls states that

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

Spherical Polar Coordinates





Now,

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

or,

$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$$

$$\vec{dl} = dl_r\, \hat{r} + dl_\theta\, \hat{\theta} + dl_\phi\, \hat{\phi}$$

$$= dr\, \hat{r} + r.\, d\theta\, \hat{\theta} + r\sin\theta\, d\phi\, \hat{\phi}$$

$$\vec{da} = dl_\theta \times dl_\phi$$

$$= (r\,d\theta).(r\sin\theta)\,d\phi = r^2\sin\theta\, d\theta\, d\phi$$

$\cancel{d\tau}$ $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

When $r \Rightarrow 0 \to \infty$, $\theta \Rightarrow 0 - \pi$, $\phi = 0 - 2\pi$

(*): Surface area of sphere

$$A = \int da$$

$$= \int r^2 \sin\theta \, d\theta \, d\phi$$

~~wh~~ $$= R^2 \left\{ \int_0^{\pi} \sin\theta \, d\theta \right\} \left\{ \int_0^{2\pi} d\phi \right\}$$

$$\therefore A = 4\pi R^2$$

(*): Volume of sphere

$$V = \int d\tau$$

$$= \int r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \left(\int_0^{R} r^2 dr \right) \times \left(\int_0^{\pi} \sin\theta \, d\theta \right) \times \left(\int_0^{2\pi} d\phi \right)$$

$$\therefore V = \frac{4}{3} \pi R^3$$