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Basis Vectors:

Let S be the subspace of $\mathbb{R}^n$.
The set of vectors $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}\}$ in S is called basis of S if.

i) the vectors $\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n}$ are linearly independent.
ii) S should be spanned by $\vec{u_1}, \vec{u_2}, \ldots \vec{u_n}$.

Let $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$ be the column vectors of identity matrix.

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The set $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}\}$ is standard basis for $\mathbb{R}^n$.

(X) Notes:

i) If the row rank is equal to the number of vectors, the vectors are linearly independent.

ii) if the row rank is less than the number of vectors, the vectors are linearly dependent.

iii) if $|V| \neq 0$, the vectors are linearly independent.

Basis can be understood as the minimum set of vectors that span the subspace.

<Q>: Show that the vectors $\vec{u_1} = (1,0,1)$, $\vec{u_2} = (0,1,1)$, $\vec{u_3} = (1,1,0)$ are basis for $\mathbb{R}^3$.

Soln:

Given,

$\vec{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\vec{u_3} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

So, the matrix represented by $\vec{u_1}, \vec{u_2}$ & $\vec{u_3}$,

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Here, row rank of the U is equal to the number of vectors. Thus, the vectors are linearly independent.

Now,

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_1$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Applying $R_3 \to -\frac{1}{3} R_3$.

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Applying.

$R_2 \to R_2 - R_3$ and $R_1 \to R_1 - R_3$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since all the pivot elements exist, in RREF, $\mathbb{R}^3$ is a span of $\vec{u_1}, \vec{u_2}$ & $\vec{u_3}$

Since the vectors are linearly independent and span $\mathbb{R}^3$,

$\vec{u_1}, \vec{u_2}, \vec{u_3}$ are basis for $\mathbb{R}^3$.

<Q>: show that $\vec{u_1} = (3,0,-6)$, $\vec{u_2} = (-4,1,7)$ and $\vec{u_3} = (-2,1,5)$ are basis for $\mathbb{R}^3$.

Soln,

Given,

$\vec{u_1} = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $\vec{u_3} = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$

So, the matrix represented by $\vec{u_1}, \vec{u_2}$ & $\vec{u_3}$,

$$U = \begin{pmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{pmatrix}$$

Here, row rank of v is equal to the numbers of vectors. Thus, the vectors are linearly independent.

Now,

$$v = \begin{pmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{pmatrix}$$

~~Applying~~ and,

$$|v| = \begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} + 4 \begin{vmatrix} 0 & 1 \\ -6 & 5 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ -6 & 7 \end{vmatrix}$$

$$= 3(5-7) + 4(0+6) - 2(0+6)$$

$$= -6 + 24 - 12 = 6 \neq 0.$$

Since $|v| \neq 0$,

$\vec{u_1}, \vec{u_2}, \vec{u_3}$ spans $\mathbb{R}^3$.

Hence, $\vec{u_1}, \vec{u_2}$ & $\vec{u_3}$ are basis for $\mathbb{R}^3$.

<Q>: Find basis for given subspace of $\mathbb{R}^3$ that the plane with $3x - 2y + 5z = 0$.

Soln.

Given,

$$3x - 2y + 5z = 0$$

$$\text{or, } x = \frac{2y - 5z}{3}$$

Here, y and z are free variables.

Putting $y = s$ and $z = t$,

$$x = \frac{2}{3}s - \frac{5}{3}t.$$

Writing in vector form.

Let $\vec{u} = (x, y, z)$

So,

$$\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}$$

or, $\vec{u} = s\vec{u_1} + t\vec{u_2}$

where,

$\vec{u_1} = (2/3, 1, 0)$ and $\vec{u_2} = (-5/3, 0, 1)$

Thus $\{\vec{u_1}, \vec{u_2}\}$ spans $\mathbb{R}^3$.

and

Putting $s = t = 0$, we get.

$\vec{u} = 0$, $\vec{u_1} = 0$, $\vec{u_2} = 0$

So, $\vec{u_1}$ and $\vec{u_2}$ are linearly independent.

Thus ~~$\{\vec{u_1}, \vec{u_2}\}$~~ $\vec{u_1}$ & $\vec{u_2}$ are basis for $\mathbb{R}^3$.

<Q>: Find the basis for column space and null space of matrix. $A = \begin{bmatrix} -2 & -2 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

Soln:

Given

$$A = \begin{bmatrix} -2 & -2 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_4$

$$\sim \begin{bmatrix} -2 & -2 & 8 & 0 & -17 \\ 0 & -4 & 8 & -4 & 8 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Applying $R_3 - 3R_4$.

$$\sim \begin{bmatrix} -2 & -2 & 8 & 0 & -17 \\ 0 & -4 & 8 & -4 & 8 \\ 0 & -10 & 20 & -8 & 10 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Applying, $R_2 \to 2R_2 + R_1$
$R_3 \to 2R_3 + 3R_1$
$R_4 \to 2R_4 + R_1$

$$\sim \begin{bmatrix} -2 & -2 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 7 & -14 & 14 & -49 \\ 0 & 9 & -18 & 10 & -23 \end{bmatrix}$$

Applying $R_3 \to R_3 - 7R_2$
$R_4 \to R_4 - 9R_2$

$$\sim \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 40 \end{bmatrix}$$

Here, pivot positions are in x_1, x_2 and x_4.
Hence, x_3 and x_5 are free variables.

So, $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right.$ are basis of col(A).

Now, $N(A) = N(\text{rref}(A))$.

Interchanging R_3 and R_4,

$$\sim \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 8 & -40 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Applying $R_1 \to -1/2 \times R_1$ and $R_3 \to -1/8 \times R_3$.

$$\sim \begin{bmatrix} 1 & 5/2 & 4 & 0 & 17/2 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 5/2 & 4 & 0 & 17/2 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

Applying $R_1 \to R_1 + 2R_2$

$$\sim \begin{bmatrix} 1 & 9/2 & 0 & 0 & 29/2 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

$$\text{rref}(A) = \begin{pmatrix} 1 & 9/2 & 0 & 0 & 29/2 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 9/2 & 0 & 0 & 29/5 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From matrix multiplication,
We get,

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$x_4 - 5x_5 = 0$
$x_2 - x_3 + 3x_5 = 0$
$x_1 - 9/2\, x_2 + 29/5\, x_5 = 0$

Here, x_3 and x_5 are free variables. Let $x_3 = s$, $x_5 = t$.

So, $x_4 = 5x_5$
$\therefore x_4 = 5t$
$x_2 = x_3 - 3x_5$
$\therefore x_2 = s - 3t$

$x_1 = \frac{9}{2} x_2 - \frac{29}{2} x_5$

$\therefore x_1 = -9s - t$

Writing in vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$N(A) = N(\text{rref}(A)) = x_3 \vec{a} + x_5 \vec{b}$

where,
$\vec{a} = (-9, 1, 1, 0, 0)$ and $\vec{b} = (-1, -3, 0, 5, 1)$

$= \text{span}\{\vec{a}, \vec{b}\} = \text{span}\left\{ \begin{bmatrix} -9 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

~~Here, $\vec{a}$ and $\vec{b}$~~

~~Hence, $\vec{u}$~~ Hence, $\vec{u_1}$ and $\vec{u_2}$ are basis for Null (A).