Lecture 03

Vector Analysis (Contd.)

August 13, 2020

Outline

Second derivatives

By applying ∇ twice we can construct five species of second derivatives.

The gradient ∇T is a vector.

1. Divergence of gradient: $\nabla \cdot (\nabla T)$ $\rightarrow$ a scalar
2. Curl of gradient: $\nabla \times (\nabla T)$ $\rightarrow$ a vector

The divergence $\nabla \cdot \vec{v}$ is a scalar.

3. Gradient of divergence: $\nabla (\nabla \cdot \vec{v})$ $\rightarrow$ a vector

The curl $\nabla \times \vec{v}$ is a vector.

4. Divergence of curl: $\nabla \cdot (\nabla \times \vec{v})$ $\rightarrow$ a scalar
5. Curl of curl: $\nabla \times (\nabla \times \vec{v})$ $\rightarrow$ a vector

Second derivatives:- Laplacian

- Laplacian Operator: $\nabla \cdot \nabla = \nabla^2$
- The Laplacian of a scalar T is a scalar.

  The Laplacian of T: $\nabla \cdot (\nabla T) = \nabla^2 T = \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}$
- The Laplacian of a vector $\vec{v}$ is a vector.

  The Laplacian of $\vec{v}$: $\nabla^2 \vec{v} = (\nabla^2 v_x)\hat{i} + (\nabla^2 v_y)\hat{j} + (\nabla^2 v_z)\hat{k}$

  The Laplacian of $\vec{v}$: $\nabla^2 \vec{v} = (\nabla \cdot \nabla)\vec{v} \neq \nabla(\nabla \cdot \vec{v})$

Integral Calculus:- Line Integrals

- If $\vec{F}$ is a vector, a line integral of $\vec{F}$ is written as

$$\int_{a\,L}^{b} \vec{F} \cdot d\vec{l}$$

  where L is the curve along which the integration is performed, a and b the initial and final points on the curve, and $d\vec{l}$ is the infinitesimal displacement vector along the curve L.

- The line integral is a scalar.
- Line integral over a closed curve is given by $\oint \vec{F} \cdot d\vec{l}$
- Example of a line integral: The work done by a force $\vec{F}$:
  $W = \int \vec{F} \cdot d\vec{l}$
- For *conservative force*: $\oint \vec{F} \cdot d\vec{l} = 0$

- If $\vec{F}$ is a vector, a surface integral of $\vec{F}$ is written as

$$\int_S \vec{F} \cdot d\vec{a}$$

where S is the surface over which the integration is to be performed, and $d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface.

- Example: The flux of $\vec{E}$ through a surface S: $\Phi_E = \int_S \vec{E} \cdot d\vec{a}$
- Surface integral over a closed surface: $\oint_S \vec{F} \cdot d\vec{a}$
- If $\vec{v}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \vec{v} \cdot d\vec{a}$ represents the total mass per unit time passing through the surface [or flux].

- A volume integral is an expression of the form

$$\int_V T d\tau$$

where T is a scalar function and $d\tau$ is an infinitesimal volume element.

- Total charge q: $\boxed{q = \int_V \rho d\tau}$ where ρ is the volume charge density.

The Fundamental Theorem of Calculus

Suppose $f(x)$ is a function of one variable.

The fundamental theorem of calculus states:

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a) \qquad (1)$$

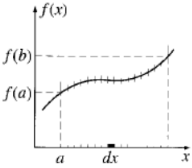


Figure 1

Here, $\left(\frac{df}{dx}\right) dx$ is the infinitesimal change in f when you go form (x) to $(x + dx)$.

The fundamental theorem (1) says that there are two ways to determine the total change in the function: *either* subtract the values at the ends *or* go step-by-step, adding up all the tiny increments as you go. You'll get the same answer either way.

Suppose, we have a scalar function of three variables $f(x,y,z)$.

The total change in f in going a from b is

$$\int_a^b (\nabla f) \cdot d\vec{l} = f(b) - f(a)$$

Geometrical Interpretation:

Suppose you want to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up, or you could place altimeters at the top and the bottom, and subtract the two readings; you should get the same answer either way (that's the fundamental theorem)

Corollary 1: $\int_a^b (\nabla f) \cdot d\vec{l}$ is independent of path taken from a to b.

Corollary 2: $\oint (\nabla f) \cdot d\vec{l} = 0$

The fundamental theorem for divergences states that: "The volume integral of divergence of a vector is equal to the surface integral of the vector over the closed surface which bound the volume"

$$\int_V (\nabla \cdot \vec{v}) d\tau = \oint_S \vec{v} \cdot d\vec{a} \tag{2}$$

Geometrical Interpretation:

If $\vec{v}$ represents the flow of an incompressible fluid, then the flux of $\vec{v}$ (the right side of Eq. (2)) is the total amount of fluid passing out through the surface, per unit time. Now, the divergence measures the "spreading out" of the vectors from a point – a place of high

The Fundamental Theorem for Divergences [Gauss's Theorem] (contd.)

divergence is like a "faucet," pouring out liquid. If we have lots of faucets in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region.

The Fundamental Theorem for Curls [Stoke's Theorem]

The fundamental theorem for curls states that: "The surface integral of curl of a vector over a surface is equal to the line integral of the vector along the closed loop which bound the surface."

$$\int_s (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_C \vec{v} \cdot d\vec{l} \tag{3}$$

Geometrical Interpretation:

The curl measures the "twist" of the vectors $\vec{v}$; a region of high curl is a whirlpool if you put a tiny paddle wheel there, it will rotate. Now, the integral of the curl over some surface represents the "total amount of swirl," and we can determine that swirl just as well by going around the edge and finding how much the flow is following the boundary (Figure F-B).

The Fundamental Theorem for Curls [Stoke's Theorem] (contd.)

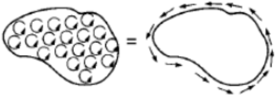


Figure F-B

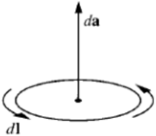


Figure F-C

You may find this a rather forced interpretation of Stokes' theorem, but it's a helpful mnemonic, if nothing else.

For a *closed* surface, $d\vec{a}$ points in the direction of the outward normal; but for an **open** surface, the direction of $d\vec{a}$ is given by right-hand rule: If your fingers point in the direction of the line integral, then your thumb fixes the direction of $d\vec{a}$ (Figure-F-C)

The Fundamental Theorem for Curls [Stoke's Theorem] (contd.)

Corollary 1: $\int_a^b (\nabla \times \vec{v}) \cdot d\vec{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint (\nabla \times \vec{v}) \cdot d\vec{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of Eq. ((3)) vanishes.

Spherical Polar Coordinates

The spherical polar coordinates (r, θ, ϕ) of a point are defined in Figure S-A.

$r \rightarrow$ the distance from the origin (the magnitude of the position vector)

$\theta \rightarrow$ the polar angle (the angle down from the axis)

$\phi \rightarrow$ the azimuthal angle (the angle around from the axis)

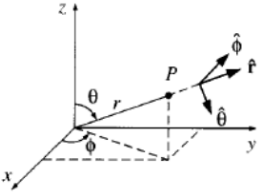


Figure S-A

Figure S-A also shows unit vectors, $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal basis set just like $(\hat{\imath}, \hat{\jmath}, \hat{k})$.

For a vector $\vec{A}$:

$$\boxed{\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}}$$

where A_r, A_θ, and A_ϕ the radial, polar and azimuthal components of $\vec{A}$

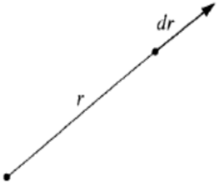


Figure S-1

An infinitesimal displacement in the $\hat{r}$ direction: $\boxed{dl_r = dr}$

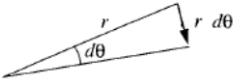


Figure S-2

An infinitesimal displacement in the $\hat{\theta}$ direction: $\boxed{dl_\theta = r d\theta}$

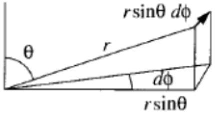
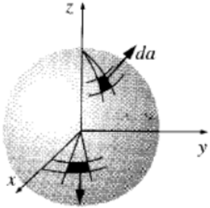


Figure S-3

An infinitesimal displacement in the $\hat{\theta}$ direction: $\boxed{dl_\phi = r\sin\theta d\phi}$

Spherical Polar Coordinates (contd.)

The general infinitesimal displacement vector:	The infinitesimal volume element:	The surface element:
$d\vec{l} = dl_r \hat{r} + dl_\theta \hat{\theta} + dl_\phi \hat{\phi}$ $= dr\, \hat{r} + r\, d\theta\, \hat{\theta} + r \sin\theta\, d\phi\, \hat{\phi}$	$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$	$da = r^2 \sin\theta\, d\theta\, d\phi$ 
$r \rightarrow 0$ to ∞	$\theta \rightarrow 0$ to π	$\phi \rightarrow 0$ to 2π

Example:

Find the volume of a sphere of radius of R.

Solution:

$$
\begin{aligned}
V &= \int d\tau \\
&= \int r^2 \sin\theta \; dr \; d\theta \; d\phi \\
&= \left(\int_{r=0}^{R} r^2 dr \right) \left(\int_{\theta=0}^{\pi} \sin\theta \; d\theta \right) \left(\int_{\phi=0}^{2\pi} d\phi \right) \\
&= \left(\frac{R^3}{3} \right) (2) \, (2\pi) = \frac{4}{3} \pi R^3
\end{aligned}
$$

1. The curl of gradient of a scalar is always zero i.e. $\nabla \times (\nabla T) = 0$

Proof:

Let T is a scalar function, then

$$\nabla \times (\nabla T) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial T}{\partial x} & \dfrac{\partial T}{\partial y} & \dfrac{\partial T}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 T}{\partial y \partial z} - \frac{\partial^2 T}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 T}{\partial x \partial z} - \frac{\partial^2 T}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 T}{\partial x \partial y} - \frac{\partial^2 T}{\partial y \partial x}\right)$$

$$= 0 \qquad \because \boxed{\frac{\partial^2 T}{\partial x \partial y} = \frac{\partial^2 T}{\partial y \partial x}}$$

Miscellaneous (contd.)

1. If the curl of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be written as the gradient of a *scalar function* i.e.

$$\nabla \times \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla V$$

2. For the curl-less (or "irrotational fields"), the following conditions are equivalent:
   - $\nabla \times \vec{F} = 0$ everywhere.
   - $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path, for any given end points.
   - $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop.
   - $\vec{F}$ is the gradient of some scalar function, $\vec{F} = \nabla V$

2. The divergence of curl of a vector is always zero, i.e.

   $\nabla \cdot (\nabla \times \vec{v}) = 0$

   Proof:

   Let $\vec{v}$ is a vector, then

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\therefore \nabla \cdot \nabla \times \vec{v} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \left[\hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]$$

Miscellaneous (contd.)

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\
&= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_z}{\partial y \partial x} + \frac{\partial^2 v_x}{\partial y \partial z} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_x}{\partial z \partial y} \\
&= 0, \qquad \because \boxed{\frac{\partial^2 v_z}{\partial x \partial y} = \frac{\partial^2 v_z}{\partial y \partial x}}
\end{aligned}
$$

1. If the divergence of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be expressed as the curl of another **vector field**:

$$\nabla \cdot \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla \times \vec{A}$$

2. For the divergence-less (or “solenoidal”) fileds, the following conditions are equivalent:

- $\nabla \cdot \vec{F} = 0$ everywhere
- $\int \vec{F} \cdot d\vec{a}$ is independent of surface, for any given boundary line
- $\oint \vec{F} \cdot d\vec{a} = 0$ for any closed surface
- $\vec{F}$ is the curl of some vector, $\vec{F} = \nabla \times \vec{A}$

In all cases (*whatever* its curl and divergence may be) a vector field $\vec{F}$ can be written as the gradient of a scalar plus the curl of a vector.

$$\vec{F} = \nabla V + \nabla \times \vec{A} \qquad \text{(always)}$$

Problems

1. Find the gradients of the following functions:
   1. $f(x,y,z) = x^2 + y^2 + z^2$
   2. $f(x,y,z) = e^x \ln(y)\sin(z)$
2. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at $(1,-2,-1)$.
3. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector and let r be its length.

   Show that
   1. $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3}$

      Hint:

$$\nabla\left(\frac{1}{r}\right) = \hat{i}\frac{\partial r^{-1}}{\partial x} + \hat{j}\frac{\partial r^{-1}}{\partial y} + \hat{k}\frac{\partial r^{-1}}{\partial z}$$

But,

$$\frac{\partial r^{-1}}{\partial x} = \frac{\partial r^{-1}}{\partial r}\frac{\partial r}{\partial x} = -r^{-2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}$$
$$= -\frac{1}{r^2}\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}(2x) = -\frac{x}{r^3}$$

Similarly, $\dfrac{\partial r^{-1}}{\partial y} = -\dfrac{y}{r^3}$ and $\dfrac{\partial r^{-1}}{\partial z} = -\dfrac{z}{r^3}$, therefore

$$\nabla\left(\frac{1}{r}\right) = -\hat{i}\frac{x}{r^3} - \hat{j}\frac{y}{r^3} - \hat{k}\frac{z}{r^3}$$
$$= -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^3} = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$$

2. $\nabla r^n = nr^{n-2}\vec{r} = nr^{n-1}\hat{r}$

   Hint:

$$\nabla (r^n) = \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

   But,

$$\frac{\partial r^n}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r} = nr^{n-2}x$$

   Similarly, $\frac{\partial r^n}{\partial y} = nr^{n-2}y$ and $\frac{\partial r^n}{\partial z} = nr^{n-2}z$. Hence

$$\nabla (r^n) = nr^{n-2}(\hat{i}x + \hat{j}y + \hat{k}) = nr^{n-2}\vec{r} = nr^{n-1}\hat{r}$$

4. Calculate the divergence of the following vector functions:

1. $\vec{v}_a = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$
2. $\vec{v}_c = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$

5. If $\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla \cdot \vec{A}$ at the point $(1, -2, 1)$.
6. Show that $\vec{A} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.
7. Calculate the curl of following vector function
   1. $\vec{v}_a = -y\hat{i} + x\hat{j}$
   2. $\vec{v}_b = x^2y\hat{i} + (x - y)\hat{k}$
8. If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, then find $\nabla \times \vec{A}$ at the point $(1, -1, 1)$.
9. Calculate the Laplacian of the following functions:
   1. $T = \sin x \sin y \sin z$

Problems (contd.)

2. $T = e^{-5x} \sin 4y \cos 3z$
3. $\vec{v} = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$

10. Prove that the divergence of a curl is always zero. Check it for the function $\vec{v} = x^2\hat{i} + 2xz^2\hat{j} - 2xz\hat{k}$.
11. Prove that the curl of a gradient is always zero. Check it for the function $f(x,y,z) = x^2y^3z^4$.
12. Let $\vec{F}_1 = x^2\hat{k}$ and $\vec{F}_2 = x\hat{i} + y\hat{j} + z\hat{k}$. Calculate the divergence and curl of $\vec{F}_1$ and $\vec{F}_2$. Which one can be written as gradient of scalar? Which one can be written as curl of a vector?
13. Show that $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ can be written both as the gradient of a scalar and as the curl of a vector.

End of Lecture 03

Thank you