SYSTEM OF LINEAR EQUATIONS

A set of linear equations is called system of linear equations.

Let the 'm' system of linear equations with 'n' unknown variables is

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_2 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \text{— (i)}$$

Representing system of linear equations (i) in matrix form,

So, coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \vdots & \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

Homogeneous and Non-homogeneous System of Linear Equations.

A linear system of equations is said to be homogeneous if all the constant terms are zero.

→ Homogeneous system of equations is always consistent

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

If any of the constant terms in system of linear equations is ~~ar~~ not equal to zero, the system of linear equations is said to be non-homogeneous.

$$a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = 1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{mn}x_n = 0$$

Elementary Row operations:

The row operations in matrices used to transform a system of linear equations into new systems having the same solutions as the original one.

Elementary Row operations are.

(i): Row swap: $R_m \leftrightarrow R_n$

(ii) Scalar multiplication: $R_m \to kR_m$

(iii) Row sum/difference: $R_m \to R_m \pm R_n$

$R_m \to \cancel{k}R_m \pm kR_n$

Solutions of System of linear Equations.

System of linear equations has three types of solutions.

i) Unique solution

ii) Infinitely many solutions

iii) No solution.

If system of linear equations has solution, it is called consistent otherwise it is called inconsistent.

Let the system of equations:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

i) Writing in augmented form,

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{23} & a_{33} & : & b_3 \end{array}\right]$$

ii) Elements a_{21}, a_{31} and a_{23} are made zero using row operations.

Thus, augmented matrix obtained,

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & : & b_1 \\ 0 & c_{22} & c_{23} & : & d_1 \\ 0 & 0 & e_{33} & : & d_2 \end{array}\right]$$

Here,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$\cancel{a c_{22} + e_{23}x} \quad c_{22}y + c_{23}z = d_1$$
$$e_{33}\cancel{=d_2}\, z = d_2$$

Cases:

i) If $e_{33} \neq 0$ & $d_2 \neq 0$ or $e_{33} \neq 0$ & $d_2 = 0$, the system of linear equations has unique solution.

ii) If $e_{33} = d_2 = 0$, the system of linear equations has many solutions

iii) If $e_{33} = 0$ but $d_2 \neq 0$, the system of linear equations has no solution.

If after elementary row operations, the matrix is in the form,

$$\begin{bmatrix} 1 & a_{12} & a_{13} & b_1 \\ 0 & 1 & c_{23} & d_2 \\ 0 & 0 & 1 & d_3 \end{bmatrix}$$

ie, upper triangular matrix with all main diagonals elements unity, the matrix is called echolen form of matrix.

If row operations are further done such that the augmented matrix ~~are in the~~ is in the form.

$$\begin{bmatrix} 1 & 0 & 0 & : & e_1 \\ 0 & 1 & 0 & : & e_2 \\ 0 & 0 & 1 & : & e_3 \end{bmatrix}$$

ie, $x = e_1$, $y = e_2$, $z = e_3$,

This form is called row reduced echolen form.

Row Rank:

The number of non-zero rows in the matrix is known as its row rank.

If row rank of coefficient matrix is equal to the rank of augmented matrix, the system of linear equations is said to be consistent. {in row reduced echolen form}.

Eg: $\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & 0 & : & 1 \end{bmatrix}$

rank of coefficient matrix = 1
rank of augmented matrix = 2

$\begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 3 \\ 0 & 0 & 1 & : & 4 \end{bmatrix}$

rank of coefficient matrix = 3
rank of augmented matrix = 3

Parametrically Represented Solutions

When the augmented matrix of the system is converted to row reduced echolen form and the number of non-zero rows is less than the number of variables, parametric solution method is used.

When this happens, there are basic variables (corresponding to 1) and free variable (corresponding to 0).

~~be~~ We let the free variable be k (any value) and find the solution in terms of k.

Questions:

1:
$$\begin{aligned} x - 2y + z &= 0 \\ 2y - 8z &= 8 \\ -4x + 5y + 9z &= -9. \end{aligned}$$

Soln:

Given,
$$\begin{aligned} x - 2y + z &= 0 \\ 2y - 8z &= 8 \\ -4x + 5y + 9z &= -9 \end{aligned}$$

Since, all the constant is not equal to zero, the system of linear equations is non-homogeneous.

Writing in augmented matrix form,

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

~~[∵ R3 → R3 + 4R1]~~

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \quad [\because R_3 \to R_3 + 4R_1]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 2 & 6 \end{array}\right] \quad [\because R_3 \to 2R_3 + 3R_2]$$

~~Here~~, Applying, $R_3 \to R_3/2$ and $R_2 \to R_2/2$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

[∵ Row rank of coefficient matrix = row rank of augmented matrix = 3 i.e, consistent]

This is the echoleon form of the system of linear equations.

From R_3, $z = 3$

From R_2, $y - 4z = 4$

or $y = 4 + 4 \times 3$ $\therefore y = 16$

From R_1 $x - 2y + z = 0$

or $x = 2y - z$ $\therefore x = 29$

The required solution is $(29, 16, 3)$.

2) $x + 2y - 3z = 3$
$-2x - 5y + 4z = 5$
$-5x - 13y + 9z = 18$

Soln:

Given,

$$x + 2y - 3z = 3$$
$$-2x - 5y + 4z = 5$$
$$-5x - 13y + 9z = 18$$

Since, all the constant terms is not equal to zero, the system of linear equations is non-homogeneous.

Writing in augmented form.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ -2 & -5 & 4 & 5 \\ -5 & -13 & 9 & 18 \end{array}\right]$$

Applying $R_2 = R_2 + 2R_1$ and $R_3 \to R_3 + 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & -3 & -6 & 33 \end{array}\right]$$

Applying ~~R2 → R2 1×R2~~ $R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Applying $R_2 \to -1 \times R_2$.

$$\sim \begin{bmatrix} 1 & 2 & -3 & : & 3 \\ 0 & 1 & 2 & : & -11 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

This is the echelon form.

Here,

From R_3, $0 \times z = 0$

So, z is a free variable.

From R_2,

$y + 2z = -11$

$\therefore y = -11 - 2z$

From R_1,

$x + 2y - 3z = 3$

$\therefore x = 3 + 3z + 22 + 4z = \cancel{3} 7z + 25$

The parametric form of solution is,

Putting $z = r$.

$y = -11 - 2r$

$x = 7r + 25$

When $r = 0$, the required solⁿ is $(25, -11, 0)$

When $r = 1$, the required solⁿ is $(32, -13, 1)$

and so on.

Also, row rank of coefficient matrix $= 2$

row rank of augmented matrix $= 3$

ie, inconsistent system of linear equations.

(3): $x + 2y - z = 4$

$3x + 7y - 2z = 1$

$-2x + 3y + 3z = -1$

Solⁿ:

Given,

$x + 2y - z = 4$

$3x + 7y - 2z = 1$

$-2x + 3y + 3z = -1$

Since, all the constant terms is not equal to zero, the system of linear equations is non-homogeneous.

Writing in augmented matrix form,

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 4 \\ 3 & 7 & -2 & : & 1 \\ -2 & 3 & 3 & : & -1 \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 4 \\ 0 & 1 & 1 & : & 1 \\ 0 & 7 & 1 & : & 7 \end{bmatrix}$$

Applying $R_3 \to R_3 - 7R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & : & 4 \\ 0 & 1 & 1 & : & 1 \\ 0 & 0 & -6 & : & 0 \end{bmatrix}$$

Applying $R_3 \to -\frac{1}{6} R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This is the echolen form.

From R_3, $z = 0$

From R_2, $y + z = 1$

$\therefore y = 1$

From R_1, $x + 2y - z = 4$

$\therefore x = 2$

The reqd solution is $(2, 1, 0)$

Here, row rank of coefficient matrix $= 3$

row rank of augmented matrix $= 3$

ie, system of linear equations is consistent.

4): $2x - 2y + 4z + 6w = 8$

$-4x + 5y - 2z - 7w = -10$

$2x + y + 22z + 21w = 36$

$-3x + 5y - 42z + 11w = 10$

soln:

Given,

$2x - 2y + 4z + 6w = 8$

$-4x + 5y - 2z - 7w = -10$

$2x + y + 22z + 21w = 36$

$-3x + 5y - 42z + 11w = 10$

Here, all the constant terms are not zero, so the system is non-homogeneous.

Writing matrix in augmented form.

$$\sim \left[\begin{array}{cccc:c} 2 & -2 & 4 & 6 & 8 \\ -4 & 5 & -2 & -7 & -10 \\ 2 & 1 & +22 & 21 & 36 \\ -3 & 5 & -42 & 11 & 10 \end{array}\right]$$

Applying $R_1 \to \frac{1}{2} \times R_1$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & 3 & 4 \\ -4 & 5 & -2 & -7 & -10 \\ 2 & 1 & 22 & 21 & 36 \\ -3 & 5 & 42 & 11 & 10 \end{array}\right]$$

Applying $R_2 \to R_2 + 4R_1$, $R_3 \to R_3 - 2R_1$, $R_4 + 3R_1$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 5 & 6 \\ 0 & 3 & 18 & 15 & 28 \\ 0 & 2 & 48 & 20 & 22 \end{array}\right]$$

Applying $R_3 \to R_3 - 3R_2$ and $R_4 \to R_4 - 2R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 5 & 6 \\ 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 36 & 10 & 10 \end{array}\right]$$

Applying $R_4 \to R_4 - 6R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 5 & 6 \\ 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & -20 & -26 \end{array}\right]$$

Applying $R_4 \to -1/20 \times R_4$.

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 5 & 6 \\ 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 1 & 1.3 \end{array}\right]$$

From R_4, $w = 1.3$

From R_3, $0 \cdot z = 16$ i.e, not possible.

Thus, z has no solution.

So, the system in whole has no solution.

Hence, it is inconsistent,

(5): $-5x + 3y - 2z + 4w = 8$

$6x + 8y + 5z - 3w = 4$

$x + 2y - 7z - 5w = -1$

$-3x - 7y + 6z + 2w = 2$

Soln:

Given,

$-5x + 3y - 2z + 4w = 8$

$6x + 8y + 5z - 3w = 4$

$x + 2y - 7z - 5w = -1$

$-3x - 7y + 6z + 2w = 2$

Since, all the constant terms are not zero, the system of linear equations is non-homogeneous.

Writing into augmented matrix form,

$$\left[\begin{array}{cccc:c} -5 & 3 & -2 & 4 & 8 \\ 6 & 8 & 5 & -3 & 4 \\ 1 & 2 & -7 & -5 & -1 \\ -3 & -7 & 6 & 2 & 2 \end{array}\right]$$

Applying $R_1 \to -1/5 \times R_1$

$$\sim \left[\begin{array}{cccc:c} 1 & -3/5 & 2/5 & -4/5 & -8/5 \\ 6 & 8 & 5 & -3 & 4 \\ 1 & 2 & -7 & -5 & -1 \\ -3 & -7 & 6 & 2 & 2 \end{array}\right]$$

Applying $R_2 \to R_2 - 6R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 + \cancel{7}\,3R_1$

$$\sim \left[\begin{array}{cccc:c} 1 & -3/5 & 2/5 & -4/5 & -8/5 \\ 0 & 58/5 & 13/5 & 9/5 & 68/5 \\ 0 & 13/5 & -37/5 & -21/5 & 3/5 \\ 0 & -44/5 & 36/5 & -2/5 & -14/5 \end{array}\right]$$

Applying $R_2 \to R_2 \times 5/58$

$$\sim \left[\begin{array}{cccc:c} 1 & -3/5 & 2/5 & -4/5 & -8/5 \\ 0 & 1 & 13/58 & 9/58 & 68/58 \\ 0 & 13/5 & -37/5 & -21/5 & 3/5 \\ 0 & -44/5 & 36/5 & -2/5 & -14/5 \end{array}\right]$$

Applying $R_3 \to R_3 - 13/5 R_2$,
$R_4 \to \cancel{R_4} R_4 + 44/5 R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & -3/5 & 2/5 & -4/5 & -8/5 \\ 0 & 1 & 13/58 & 9/58 & 68/58 \\ 0 & 0 & -463/58 & -267/58 & -71/29 \\ 0 & 0 & 226/29 & 28/29 & 218/29 \end{array}\right]$$

Applying $R_3 \to -58/463 R_3$

$$\sim \left[\begin{array}{cccc:c} 1 & -3/5 & 2/5 & -4/5 & -8/5 \\ 0 & 1 & 13/58 & 9/58 & 68/58 \\ 0 & 0 & 1 & 267/463 & 142/463 \\ 0 & 0 & 226/29 & 28/29 & 218/29 \end{array}\right]$$

Applying $R_4 \to R_4 - \frac{226}{29} R_3$

$$\sim \left[\begin{array}{cccc:c} 1 & -3/5 & 2/5 & -4/5 & -8/5 \\ 0 & 1 & 13/58 & 9/58 & 68/58 \\ 0 & 0 & 1 & 267/468 & 142/463 \\ 0 & 0 & 0 & -3.52 & 5.12 \end{array}\right]$$

Applying : $R_4 \to -1/3.52 \; R_4$.

$$\sim \left[\begin{array}{cccc:c} 1 & -3/5 & 2/5 & -4/5 & -8/5 \\ 0 & 1 & 13/58 & 9/58 & 68/58 \\ 0 & 0 & 1 & 267/468 & 142/463 \\ 0 & 0 & 0 & 1 & -16/11 \end{array}\right]$$

From R_4,
$w = -16/11 = -1.454$

From R_3, $z + \frac{267}{468} w = \frac{142}{463}$

$\therefore z = 1.136$

From R_2, $y + \frac{13}{58} z + \frac{9}{58} w = \frac{68}{58}$

$\therefore y = 1.143$

From R_1, $x - \frac{3}{5} y + \frac{2}{5} z - \frac{4}{5} w = \frac{-8}{5}$

$\therefore x = -2.528$

The required solution is $(-2.528, 1.143, 1.136, -1.454)$

And
row rank of coefficient matrix = 4
row rank of augmented matrix = 4

$\therefore$ The system of linear equations is consistent.