Chapter 1: Mechanics:

DYNAMICS OF SYSTEM OF PARTICLES

A) Concept on Vector:

→ vector consists of both magnitude and direction.
→ Defn: vector states the exact position in co-ordinate space/ plane.
ie, distance from X-, Y-, Z- axis.

Eg:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

x) Unit vector: $\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\text{vector}}{\text{its magnitude}}$.

Along x-axis, $\hat{i}$
Along y-axis, $\hat{j}$
Along z-axis, $\hat{k}$

y) Scalar product: ~~$\vec{A}\times$~~ $\vec{A}\cdot\vec{B} = A\cdot B\cdot\cos\theta$ (Projection of a vector on another)
called scalar product or dot product because product of two products is a scalar
ie, solution doesn't have a vector.

$\vec{i}\cdot\vec{j} = 0$ $\quad$ $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative) $\quad$ $\vec{i}\cdot\vec{i} = 1$
$\vec{i}\cdot\vec{k} = 0$ $\quad$ $\vec{j}\cdot\vec{j} = 1$
$\vec{j}\cdot\vec{k} = 0$ $\quad$ $\vec{k}\cdot\vec{k} = 1$

x) Vector product: $\vec{A} \times \vec{B} = A \cdot B \cdot \sin\theta\, (\hat{n})$ → Unit vector giving direction to plane containing A and B.

also called cross-product.

it indicates area swept up by the rotational motion.

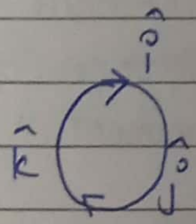
$$\vec{A} \times \vec{B} = - \vec{B} \times \vec{A} \qquad \text{ie, } \vec{B} \times \vec{A} = A \cdot B \sin\theta\, (-\hat{n}).$$

for unit vectors;

$\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

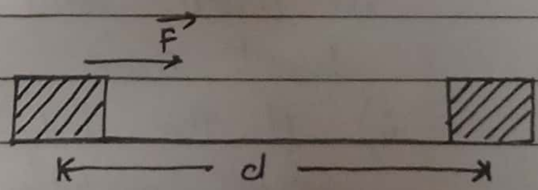


(Note: initial background vector concept needed for Ch: L)

<u>Workdone By a Constant Force:</u> SI unit: Joule
CGS unit: erg

When a body moves through a distance (or displacement $\vec{d}$) by a force $\vec{F}$, then workdone

$$W = \vec{F} \cdot \vec{d}$$



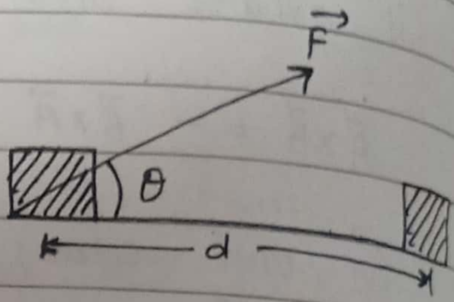
Here,

$W = Fd \cos 0°$

ie, applied force is in the same direction as the displacement

So, workdone = Force x displacement.

If the applied force is not in the same direction of the displacement, then workdone



$W = \vec{F} \cdot \vec{d}$
$= (F\cos\theta) \cdot d$
= the component of the force in the direction of displacement × the displacement.

If $\theta < \pi/2$, workdone is positive ie, the body gains its K.E.

If $\theta > \pi/2$, workdone is negative ie, the K.E. of the body decreases.

If $\theta = \pi/2$, workdone is zero $(W = 0)$.

- Examples of no workdone:
  i) When we push the wall and the wall doesn't move.
  ii) If the body is moving in a circular path, then overall workdone is zero.

Workdone by a Number of Constant Forces

Let us consider a system of 'n' number of forces ~~of~~ $F_1, F_2, \ldots, F_n$ are applied on a body and the body displaces a certain distance 'd'.

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Then, the workdone by a force F_1 is $W_1 = \vec{F_1} \cdot \vec{d}$

the workdone by a force F_2 is $W_2 = \vec{F_2} \cdot \vec{d}$

$\vdots$

the workdone by a force F_n is $W_n = \vec{F_n} \cdot \vec{d}$

Therefore, the total workdone

$$
\begin{aligned}
W &= W_1 + W_2 + W_3 + \dots + W_n \\
&= \vec{F_1} \cdot \vec{d} + \vec{F_2} \cdot \vec{d} + \dots + \vec{F_n} \cdot \vec{d} \\
&= (\vec{F_1} + \vec{F_2} + \dots + \vec{F_n}) \cdot \vec{d} \\
&= \vec{F_r} \cdot \vec{d}
\end{aligned}
$$

= resultant force x displacement

Thus, the resultant workdone is equal to the product of the sum of all forces (ie, resultant force) and the displacement of the body.

ie, resultant workdone = sum of all the individual workdone by the system of forces.

Num. No. 1 A block of mass 10 kg is to be raked from the bottom to the top of an inclined plane which is 5 m long and 3m up the ground. Assuming frictionless surface, how much work must be done by a force parallel to the incline pushing the block upto the constant speed.

Soln:

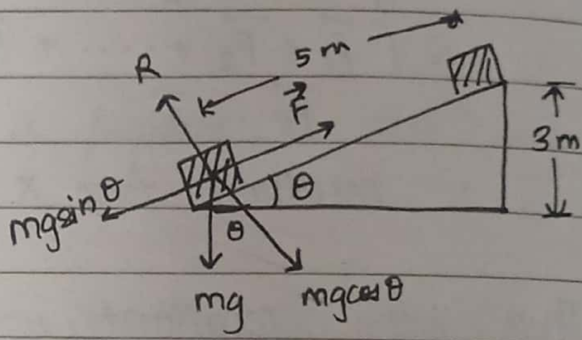
Given;

mass of block (m) = 10 kg

displacement (d) = 5 m

height (h) = 3 m

Workdone by force parallel to the surface (W) = ?



Now,

i) From the figure; $\sin\theta = \frac{3}{5}$

So,

$\cos\theta = \frac{4}{5}$ [∵ By Pythagorus theorem]

So,

ii) From newton's second law of motion;

$F_{ext} - mg\sin\theta = ma$

Since, the block moves up in constant speed, ie, $a = 0$.

So,

$F_{ext} = mg\sin\theta$

$= \overset{2}{\cancel{10}} \times 9.8 \times \frac{3}{\cancel{5}}$

$\therefore F_{ext} = 58.8$ N

Now,

Workdone $(W) = \vec{F}_{ext} \cdot \vec{d}$

$= 58.8 \times 5 = 294 \text{ J}$

For verification, we calculate the P.E. at the height of 3 meters. as at that height when the body stops,

KE = 0 and PE is maximum.

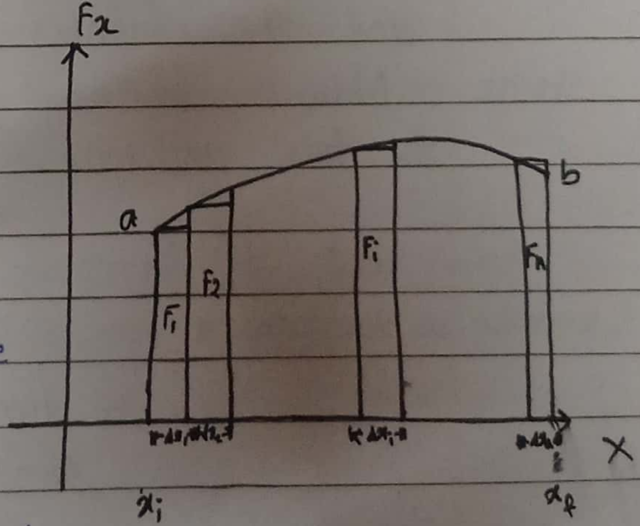
So,

$PE = mgh$

$= 10 \times 9.8 \times 3 = 294 \text{ J}.$

Thus, the workdone by the force parallel to the incline pushing the block upto the constant speed.

Workdone by Variable Force in One Dimension



Consider the case where the force varies only in magnitude.

Let a force 'F_x' on the body is applied and the body moves in the direction of the force along x-axis.

We assume the whole displacement to be divided into large number of small parts $\Delta x_1, \Delta x_2 \ldots$ $\ldots, \Delta x_i \ldots, \Delta x_n$.

In the first step, F_1 is assumed to be constant

$\therefore$ Workdone $(\Delta W_1) = F_1 \cdot \Delta x_1$

In the second step, F_2 is assumed to be constant

$\therefore$ Workdone $(\Delta W_2) = F_2 \cdot \Delta x_2$

$\vdots$

In the i^{th} step, F_i is assumed to be constant

$\therefore$ workdone $(\Delta W_i) = F_i \cdot \Delta x_i$

$\vdots$

In the n^{th} step, F_n is assumed to be constant.

$\therefore$ Workdone $(\Delta W_n) = F_n \cdot \Delta x_n$.

Now, the total workdone (W)

$$= \Delta W_1 + \Delta W_2 + \cdots + \Delta W_i + \cdots + \Delta W_n$$

$$= F_1 \cdot \Delta x_1 + F_2 \cdot \Delta x_2 + \cdots + F_i \cdot \Delta x_i + \cdots + F_n \cdot \Delta x_n$$

An exact result can be obtained if each $\Delta x \to 0$ and the number of intervals tends to infinity (∞)

Now,

the total workdone $= \lim_{\Delta x \to 0} F_1 \cdot \Delta x_1 + \lim_{\Delta x \to 0} F_2 \cdot \Delta x_2 + \cdots$

$$\cdots + \lim_{\Delta x \to 0} F_i \cdot \Delta x_i + \cdots + \lim_{\Delta x \to 0} F_n \cdot \Delta x_n$$

$$= \lim_{\Delta x \to 0} \sum_{i=1}^{n} F_i \cdot \Delta x_i$$

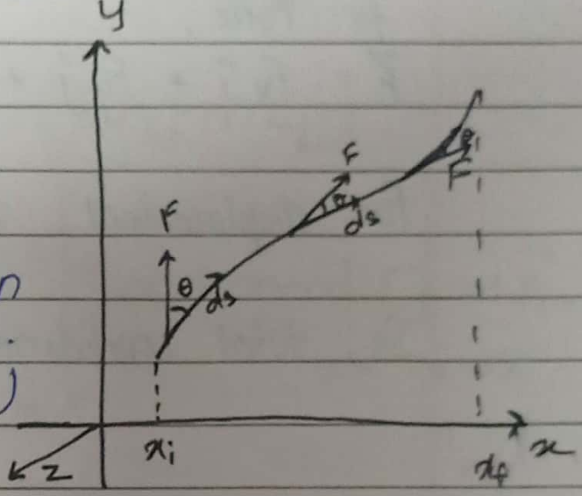
$$= \int_{x_i}^{x_f} F_x \cdot dx$$

= Area under the loop.

The total workdone in one-d by the variable force is equal to the magnitude of area ~~into~~ covered by the loop.

Workdone by Variable Force in Three-Dimension

Consider a body is moved along a curve path by a force.



When the magnitude and direction of a force vary in 3-dimension, it can be expressed as a function $\vec{F}(\vec{r})$ or as coordinate $\vec{F}(x,y,z)$.

We assume the whole displacement to be divided into large number of small parts $\vec{ds}$.

So, workdone ~~in~~ by the force in the small displacement $\vec{ds}$ is

$$\Delta W = \vec{F} \cdot \vec{ds}$$

Now, total workdone is the sum of all ΔW ie, workdone on displacing body by variable ~~so~~ force in small displacement ds

So,

total workdone ~~(W)~~ = from initial point x_i to final point x_f is. (taking only x coordinate as reference)

$$W_{x_i \to x_f} = \int_{x_i}^{x_f} \vec{F} \cdot \vec{ds}$$

$$= \int_{x_i}^{x_f} (F \cdot \cos\theta)\, ds$$

In terms of rectangular components,
for Force,

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

For displacement, $\vec{ds} = dx \hat{i} + dy \hat{j} + dz \hat{k}$

So, total workdone $(W_{x_i \to x_f}) = \int_{x_i}^{x_f} \vec{F} \cdot \vec{ds}$

$$= \int_{x_i}^{x_f} (F_x \hat{i} + F_y \hat{j} + F_z \hat{k}) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

Since, we know,

$\hat{i} \cdot \hat{i} = 1$, $\hat{j} \cdot \hat{j} = 1$, $\hat{k} \cdot \hat{k} = 1$

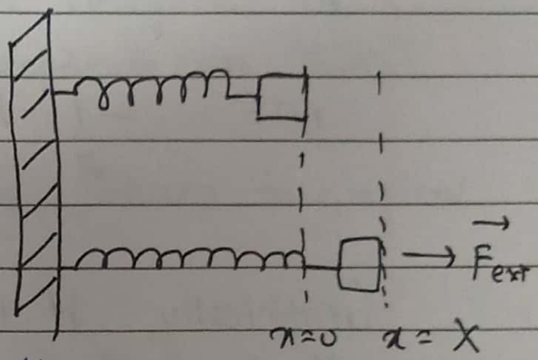
$\hat{i} \cdot \hat{j} = 0$, $\hat{i} \cdot \hat{k} = 0$, $\hat{j} \cdot \hat{k} = 0$

So,

$$W = \int_{x_i}^{x_f} F_x \, dx + \int_{y_i}^{y_f} F_y \, dy + \int_{z_i}^{z_f} F_z \cdot dz$$ [$\because$ taking all 3 cordinate axes]

Thus, the total workdone in 3-d can be calculated.

Workdone by a spring:



Initially, the spring is at $x=0$.
If certain force $\vec{F}_{ext}$ is applied on the string horizontally and it displaces by distance ~~x~~ X.

According to Hooke's law, the force exerted by the ~~string~~ spring is directly proportional to the force applied.

ie, $F_{\cancel{ext}s} \propto x$

or $F_s = -kx$. [$\because$ opposing force]

So, the force applied on the spring,

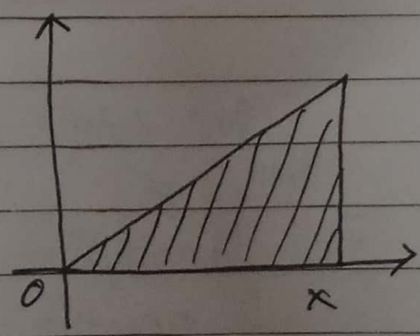
$$F_{ext} = kx.$$

Now, workdone by spring for when it is displaced to 'x' distance is,

$$W = \int_0^x F_s \, dx = -\int_0^x kx \, dx$$

$$= -k\left[\frac{x^2}{2}\right]_0^x$$

$$\therefore W = -\frac{1}{2}kx^2$$

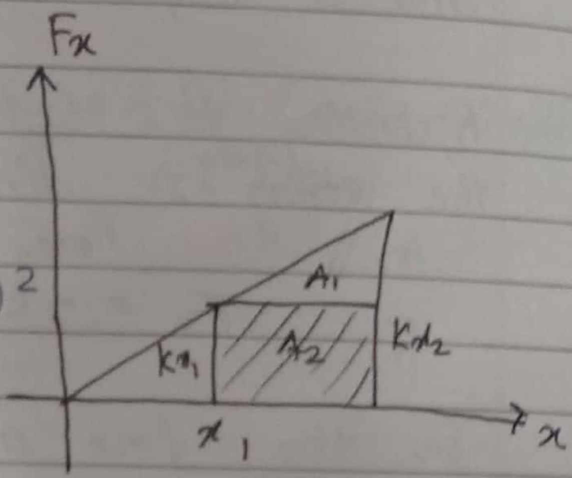


Now, if the spring has been displaced from x_i to x_f then,

$$W = -\frac{1}{2} k \left(x_f^2 - x_i^2 \right)$$

Graphically, if the spring is stretched from X_1 to X_2 then,

total workdone (w) = $A_1 + A_2$



Here,

$$A_1 = \frac{1}{2} \times (x_2 - x_1) \times (kx_2 - kx_1)^2$$

$$A_2 = kx_1 \times (x_2 - x_1)$$

So,

$$W = \frac{1}{2} (x_2 - x_1)(kx_2 - kx_1)^2 + kx_1 \times (x_2 - x_1)$$

$$= \cancel{\frac{1}{2} kx_2^2} = \frac{1}{2} k (x_2^2 - x_1^2)$$

$$= \left(\frac{1}{2} kx_2^2 - \frac{1}{2} kx_1^2 \right)$$

So, total workdone (w) = $-\left(\frac{1}{2} kx_2^2 - \frac{1}{2} kx_1^2 \right)$

because workdone by a spring force is negative.

Note:

i) workdone by a spring force is negative

ii) Workdone on spring force depends on initial and final points.

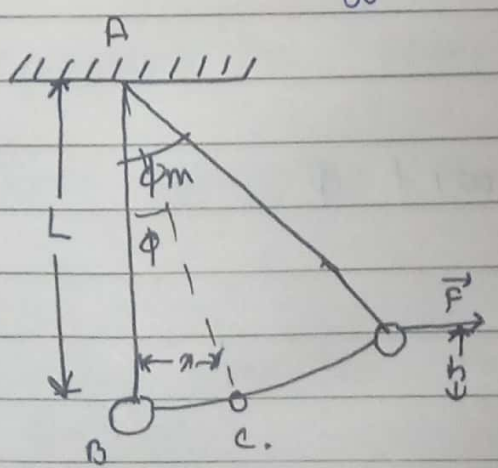
iii) Net workdone when spring returns to original position is zero.

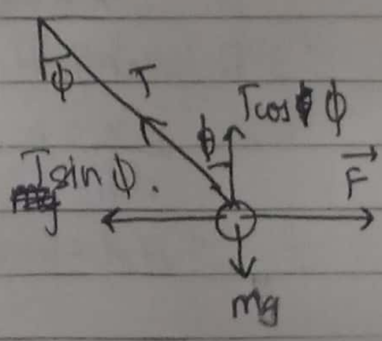
Here, the total workdone in spring by external for $\vec{F}_{ext}$ will be

$$W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2.$$

Q.2: A small object of mass 'm' is suspended from a string of length 'L'. The spring is pulled sideways by a force F that is also horizontal until the string finally makes angle of ϕm with vertical. The displacement is accomplished of a small constant speed. Find the workdone by all forces acting on that object.

Sol^n:





From free body diagram,

$$T\cos\phi = mg \quad \text{— (i)}$$
$$T\sin\phi = F \quad \text{— (ii)}$$

Dividing (i) from (ii),

$$\frac{\cancel{T}\sin\phi}{\cancel{T}\cos\phi} = \frac{F}{mg}$$

$$F = mg\tan\phi \quad \text{— (iii)}$$

We know,

Total workdone by force F is.

$$W = \int F \cdot dx$$
$$= \int mg\tan\phi \, dx. \quad \text{— (iv)}$$

In ΔABC,

$$\sin\phi = \frac{x}{L}$$

or, $x = L\sin\phi$

Differentiating both sides w.r.t ϕ,

$$\frac{dx}{d\phi} = L\cos\phi$$

$$\therefore dx = L\cos\phi \, d\phi$$

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Replacing the value of dx in (iv).

$$W = \int mg \tan\phi \, L\cos\phi \, d\phi$$

or, $$W = \int mgL \sin\phi \, d\phi$$

Here, work is done from $0°$ to ϕ_m.

so, total workdone $(W) = \int_0^{\phi_m} mgL\sin\phi \, d\phi$

$$= Lmg \left[-\cos\phi \right]_0^{\phi_m}$$

$$= Lmg \left[-\cos\phi_m + \cos 0 \right]$$

$$= Lmg \left[1 - \cos\phi_m \right] \quad \text{--- (v)}$$

From figure;

$$\cos\phi_m = \frac{L-h}{L}$$

or, $$h = L\left[1 - \cos\phi_m\right] \quad \text{--- (vi)}$$

Replacing the value of h to eq^n (v).

Total workdone by all forces on object = mgh.

$\therefore W_f = mgh.$

[Verified by the fact that at height 'h', the PE = mgh as KE = 0]

Work-Energy Theorem:

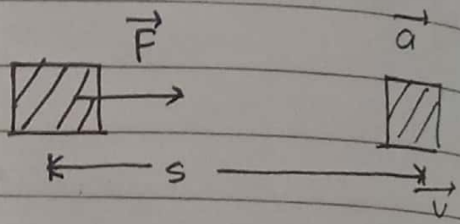
*) Statement:

Workdone by the resultant force is equal to change in KE.

ie,

$$W_{net} = \Delta KE.$$

*) For constant force:



We know,

$$W = \vec{F} \cdot \vec{s}$$

$$= m\vec{a} \cdot \vec{s}$$

$$= m\left(\frac{\vec{v}-\vec{u}}{t}\right) \cdot \left(\frac{\vec{u}+\vec{v}}{2}\right) \times t$$

$$= \frac{1}{2} m (v^2 - u^2)$$

$$= \Delta KE$$

$\therefore$ $W = \Delta KE$ [Work Energy theorem is verified]

*) For variable force:

We know,

$$W = \int F \cdot ds$$

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$$= \int m a \, ds$$

$$= \int m \cdot \frac{dv}{dt} \cdot ds$$

$$= \int m v \cdot dv$$

$$= m \left[\frac{v^2}{2} \right]_u^v$$

$$= \frac{1}{2} m (v^2 - u^2) = \Delta KE$$

$\therefore W = \Delta KE$ (work-energy theorem is verified).

*> Physical Significance:

(i): From W.E. theorem, we easily defined work and kinetic energy and derive relation using them from Newton's 2^{nd} law of motion.

(ii) It is used to calculate the workdone by resultant force and calculating speed at that distance.

* Limitation:

(i) Since, it is derived from Newton's 2^{nd} law of motion, it is only applicable on particles. Hence, we consider whole object as a single particle if all of its object particles behave like particles.

(ii) Direction of velocity cannot be determined.

Q.3: A body of mass 4.5 gm is dropped from rest at height 10.5 m above earth's surface. Neglecting air resistance, what will be its speed just before it strikes the ground.

Soln:

Given;

mass (m) = 4.5 gm = 4.5×10^{-3} kg.

height (h) = 10.5 m.

initial velocity (u) = 0 m/s.

According to work-energy theorem;

$$W_{net} = \Delta KE$$

or, $$mgh = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

or, $$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2 \qquad [\because u=0]$$

or, $$v = \sqrt{2gh}$$

or, $$v = \sqrt{2 \times 9.8 \times 10.5} = 14.34 \text{ m/s.}$$

The speed just before striking the ground is 14.34 m/s.

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Q.4: A block of mass 3.63 kg slides on a horizontal frictionless table with speed v = 1.22 m/s. It is brought to rest in compressing a ~~string~~ spring in its path. By how much is the spring compressed if ~~total~~ ~~workdone~~ spring constant is 135 N/m?

Soln:

Given,

mass (m) = 3.63 kg

initial speed (u) = 1.22 m/s

final velocity (v) = 0

spring constant (K) = 135 N/m.

According to Work-Energy theorem;

$$W_{net} = \Delta KE$$

or, $$-\frac{1}{2} k x^2 = \frac{1}{2} mv^2 - \frac{1}{2} mu^2$$

or, $$+\frac{1}{\cancel{2}} kx^2 = +\frac{1}{\cancel{2}} mu^2 \quad [\because v = 0, \text{ brought to rest}]$$

$$x = \sqrt{\frac{mu^2}{K}} = \sqrt{\frac{3.63 \times (1.22)^2}{135}}$$

$$\therefore x = 0.20 \text{ m}$$

The spring gets compressed by 0.20 m.

Conservative forces: & Non-conservative forces

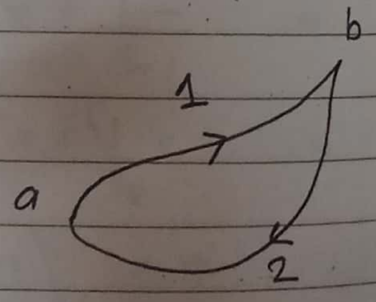
If the workdone by a force during a round trip of a system is always zero, the force is said to be conservative.

OR,

If the workdone by force in moving the body from initial to final position is independent of path betn two points, then force is said to be conservative.

If both of the conditions is not satisfied, the force is said to be non-conservative.

Now,
Here, suppose,
a particle goes from A → B through path 1 and from B → A through path 2.

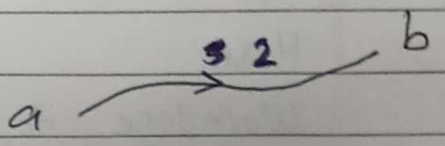


If conservative force is acting on the particle, the total workdone must be zero.

ie, $W_{AB_1} + W_{BA_2} = 0$

$W_{AB_1} = -W_{BA_2}$ — (i)

Again, if the work is done along same path.



$W_{AB_2} + W_{BA_2} = 0$

$\therefore W_{AB_2} = -W_{BA_2}$ — (ii)

From eq^n (i) and (ii);

$W_{AB_1} = W_{AB_2}$

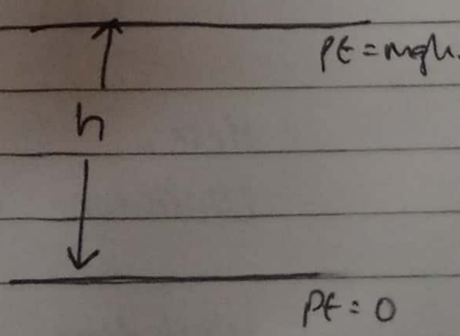
ie, the workdone by the force in moving from 'a' to 'b' from either part is same.

So, ~~conso~~ the workdone by conservative force only depends upon the ~~path~~ ~~taken~~ position of 'a' and 'b' and not the path followed by it.

Examples of Conservative Forces:

a) Gravitational Force:

Here,



G.F. acting upward $= -mgh = a$

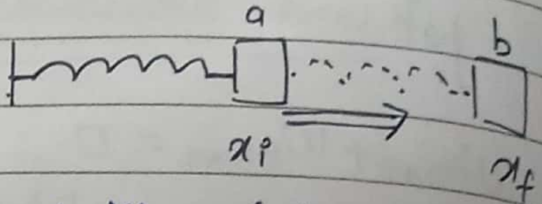
G.F. acting downward $= mgh = b$

Total workdone $= a + b$

$= -mgh + mgh$

$= 0$ ie, conservative.

(b): Elastic spring force.



Here,
Workdone to take body from initial to final position $(a) = \frac{1}{2}k(x_f^2 - x_i^2)$

Workdone to take body from final to initial position $(b) = \frac{1}{2}k(x_i^2 - x_f^2)$

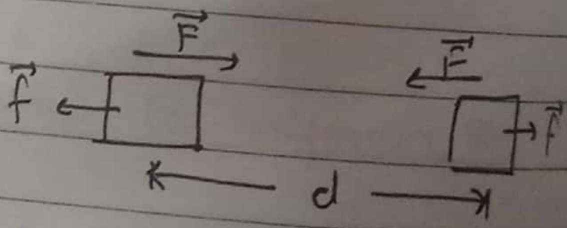
$= -\frac{1}{2}k(x_f^2 - x_i^2)$

Total workdone $(W_{net}) = a + b$

$= 0$ ie, conservative

(*): Example of Non-conservative force:

a) Frictional force:
We know;
Workdone by frictional force $(W) = -fd$.
[∵ opposing force is always negative]



Here,
[∵ $W = Fd\cos\theta$]

Workdone from A to B by frictional force $(W_{AB}) = fd\cos 180° = -fd$.

Workdone from B to A $(W_{BA}) = fd\cos 180° = -fd$.

$\therefore$ Total workdone ~~from~~ $(W_{net}) = W_{AB} + W_{BA}$

$= -2fd.\ \cancel{x},\ \neq 0$

i.e., non-conservative.

Note:

(i): Workdone by conservative force is regainable and there's no loss of energy when conservative forces do work.

(ii): Workdone by non-conservative force is not regainable and there's loss of energy when non-conservative forces are involved.

Non-conservative forces are also called dissipative force.