Advanced Calculus - MATH 104

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Circles-Polar Integrals

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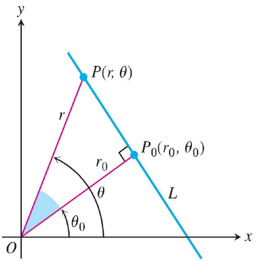
Straight Line in Polar Form



Lines

Suppose the perpendicular from the origin to line L meets L at the point $P_0(r_0, \theta_0)$, with $r_0 \geq 0$ (Figure 11.51). Then, if $P(r, \theta)$ is any other point on L, the points P, P_0, and O are the vertices of a right triangle, from which we can read the relation

$$r_0 = r\cos(\theta - \theta_0).$$

The Standard Polar Equation for Lines

If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then an equation for L is

$$r\cos(\theta - \theta_0) = r_0.$$

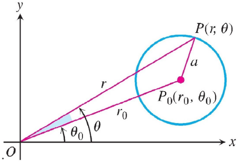
For example, if $\theta_0 = \pi/3$ and $r_0 = 2$, we find that

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

$$r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 2$$

$$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 2, \qquad \text{or} \qquad x + \sqrt{3}\,y = 4.$$

Find Cartesian form if $\theta_0 = \pi/4, r_0 = \sqrt{2}$



Circles

To find a polar equation for the circle of radius a centered at $P_0(r_0, \theta_0)$, we let $P(r, \theta)$ be a point on the circle and apply the Law of Cosines to triangle OP_0P. This gives

$$a^2 = r_0^2 + r^2 - 2r_0 r \cos(\theta - \theta_0).$$

If the circle passes through the origin, then $r_0 = a$ and this equation simplifies to

$$\begin{aligned} a^2 &= a^2 + r^2 - 2ar\cos(\theta - \theta_0) \\ r^2 &= 2ar\cos(\theta - \theta_0) \\ r &= 2a\cos(\theta - \theta_0). \end{aligned}$$

If the circle's center lies on the positive x-axis, $\theta_0 = 0$ and we get the further simplification

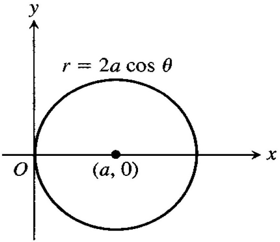
$$r = 2a\cos\theta.$$

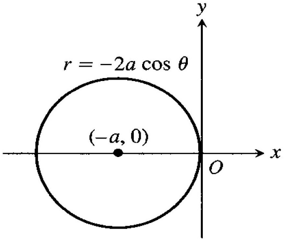
If the center lies on the positive y-axis, $\theta = \pi/2$, $\cos(\theta - \pi/2) = \sin\theta$, and the equation $r = 2a\cos(\theta - \theta_0)$ becomes

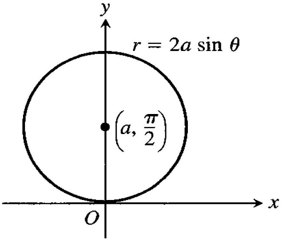
$$r = 2a\sin\theta.$$

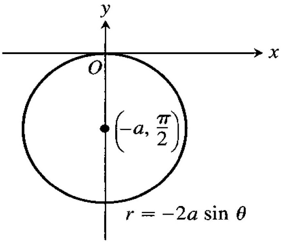
Equations for circles through the origin centered on the negative x- and y-axes can be obtained by replacing r with $-r$ in the above equations.

Circles Through the Origin Centered on the $x-$ and $y-$ axes, Radius a







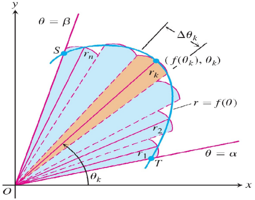


Examples

Sketch the circles, give polar coordinates for their centers and find their radii.

1. $r = 4\cos\theta$

2. $r = -2\cos\theta$



Area in the Plane

The region OTS in Figure 11.30 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2} r_k^2 \, \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k .$$

The area of region OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k .$$

If f is continuous, we expect the approximations to improve as the norm of the partition P goes to zero, where the norm of P is the largest value of $\Delta\theta_k$. We are then led to the following formula defining the region's area:

$$\begin{aligned} A &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k \\ &= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta . \end{aligned}$$

Area Bounded by Polar Curves

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta .$$

This is the integral of the **area differential** (Figure 1)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \big(f(\theta)\big)^2 \, d\theta .$$

EXAMPLE 1 Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid (Figure 11.32) and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta \\
&= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \\
&= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta \\
&= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi .
\end{aligned}$$

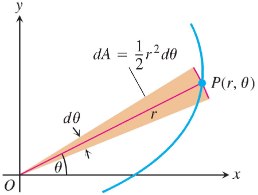


FIGURE 1 The area differential dA for the curve $r = f(\theta)$.

Find the areas of the region in the plane

1. enclosed by the cardiod $r = 2(1 + \cos\theta)$. (Ans: 6π sq. units)
2. bounded by the spiral $r = \theta$ for $0 \leq \theta \leq \pi$. (Ans: $\pi^3/6$ sq. units)
3. inside the oval limacon $r = 4 + 2cos\theta$. (Ans: 18π sq. units)
4. inside one loop of lemniscate $r^2 = 4\sin 2\theta$. (Ans: 2 sq. units)
5. inside the circle $r = a$. (Ans.: ?????????)
6. inside one petal of $r = \cos 3\theta$. (Ans: $\pi/12$ sq. units)

Examples: Areas Shared by Polar Curves

Find the areas the region in the plane

1. shared by circles $r = 2\cos\theta$ and $r = 2\sin\theta$.

   Ans: $(\pi/2 - 1)$ sq. units

2. shared by the circle $r = 2$ and cardiod $r = 2(1 - \cos\theta)$.

   Ans: $(5\pi - 8)$ sq. units

3. inside the circle $r = 4cos\theta$ and to the right of the vertical line $r = \sec\theta$.

   Ans: $(8\pi/3 + \sqrt{3})$ sq. units

4. Find the area of the region that lies inside the circle $r = 1$ outside the cardiod $r = 1 - \cos\theta$.

   Ans: $2 - \pi/4$ sq. units