Dimension of a sub-space:

The numbes of vectors in a basis for sub-space 'S' is called dimension of S.

Dimension of column space:

The numbes of vectors in any basis for the column space. This is called the rank of the column space.

or

The numbes of linearly independent column vectors that can span your column space.

<Q>: Find the rank of $A = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & -2 & 5 & 1 & -5 \\ 1 & -1 & -3 & -2 & 9 \end{bmatrix}$

Soln.

Given,

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 2 & 1 & 0 & 0 & 9 \\ -1 & -2 & 5 & 1 & -5 \\ 1 & 1 & -3 & -2 & 9 \end{bmatrix}$$

Here,

$$col(A) = c(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ -5 \\ 9 \end{bmatrix} \right\}$$

Now, to find basis for col(A).

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & -1 & -2 & -2 & 5 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$ and $R_4 \to R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -2 & 6 \end{bmatrix}$$

Applying $R_4 \to R_4 + 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $rref(A) = \begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Here, x_1, x_2 & x_4 are pivot elements.

So,

So,

$$\begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$ are the basis of col (A).

Hence

Dimension of col(A) = 3 = rank of col space of A.

Dimension of Null space:

The number of vectors in any basis for the null space is called dimension of null space.

It is also known as nullity.

Nullity of a matrix A = no. of non-pivot columns in rref (A)

= no. of free variables in rref(A)

Theorem: Row rank + nullity = no. of columns.

It can also be defined as the dimension of null space of matrix.

Q7: Find the nullity of $B = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 3 & 1 & 4 \end{bmatrix}$

Soln:

Given,

$$B = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 3 & 1 & 4 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & -2 & 2 \end{bmatrix}$$

Applying $R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 7 & -2 \\ 0 & 0 & 1 & -2 & 2 \end{bmatrix}$$

So,

$$\text{rref}(B) = \begin{bmatrix} 1 & 1 & 0 & 7 & -2 \\ 0 & 0 & 1 & -2 & 7 \end{bmatrix}$$

Here, x_1 and x_3 are pivot elements.

Let $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$ such that

$$\begin{bmatrix} 1 & 1 & 0 & 7 & -2 \\ 0 & 0 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By matrix - multiplication, we get,

$$x_1 + x_2 + 7x_4 - 2x_5 = 0.$$
$$x_3 - 2x_4 + 2x_5 = 0$$

or $x_1 = -x_2 - 7x_4 + 2x_5$

or $x_3 = 2x_4 - 2x_5$

Writing in vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Here, $\vec{a} = (-1, 1, 0, 0, 0)$

$\vec{b} = (-7, 0, 2, 1, 0)$

$\vec{c} = (2, 0, -2, 0, 1)$

$N(B) = \text{span}\{\vec{a}, \vec{b}, \vec{c}\}.$

Thus,

nullity $= 3.$