General Physics I (PHYS 101)

Lecture 04

Dynamics of system of particles

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Outline

Problem: Find the centre of mass of semi-circular plate of radius R.

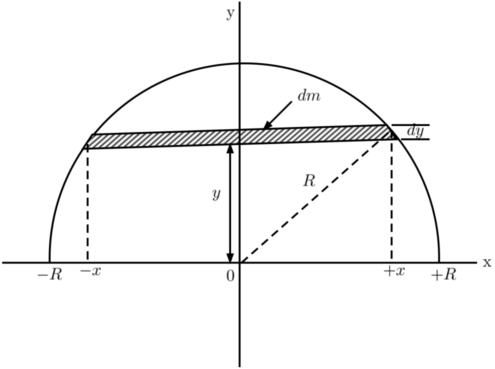

Figure 1: Semi-circular plate

Let σ be the mass per unit area of the plate. i.e.$\sigma = \frac{M}{\frac{\pi R^2}{2}}$

The homogeneous semicircular plate has rotational symmetry about the y-axis so that the center of mass must lie on the y-axis.

Consider a thin strip of mass dm of this homogeneous semicircular plate.

Area of the thin shaded strip, $da = 2\,x\,dy$

Mass of the thin strip,

$$dm = \sigma 2\,x\,dy = \frac{M}{\pi R^2/2} 2\,x\,dy = \frac{4M}{\pi R^2} x\,dy$$

Problem Center of mass (contd.)

Since the center of mass of all shaded strips lies some where y-axis i.e. $X_{cm} = 0$.

$$y_{cm} = \frac{\int ydm}{\int dm} = \frac{1}{M}\int \mathrm{y}\; dm$$

$$= \frac{1}{M}\int \mathrm{y}\left[\frac{4M}{\pi R^2}x\; dy\right] = \frac{4}{\pi R^2}\int \mathrm{y}\; x\; dy$$

$$= \frac{4}{\pi R^2}\int_0^R \mathrm{y}\sqrt{R^2 - y^2}\; dy$$

Put, $R^2 - y^2 = t^2$

$\Rightarrow -2ydy = 2tdt$

$\therefore$ when y= 0, then t = R

when y= R, then t = 0

$$\therefore\ y_{cm} = \frac{4}{\pi R^2} \int_0^R \sqrt{R^2 - y^2}\, \mathrm{y}dy$$

$$= \frac{4}{\pi R^2} \int_R^0 \mathrm{t}\,(-tdt) = \frac{4}{\pi R^2} \int_0^R t^2 dt = \frac{4}{\pi R^2} \frac{R^3}{3} = \frac{4R}{3\pi}$$

Problem Center of mass (contd.)

Thus, the center of mass of the homogeneous semicircular plate lies on the y-axis at a distance of $\frac{4R}{3\pi}$ from origin. i.e. the co-ordinate of center of mass

$$(X_{cm}, Y_{cm}) = \left(0, \frac{4R}{3\pi}\right)$$

Linear moment of system of particles

The linear momentum $\vec{P}$ of a single particle is defined as the product of its mass m and its velocity $\vec{v}$.   i.e. $\vec{P} = m\vec{v}$

From Newton's second law which states that the rate of change of momentum of a body is proportional to the resultant force acting on the body and is in the direction of force.

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}m\vec{v} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Now instead of single particle, consider system of n-particles with masses $m_1, m_2, ..., m_n$. Suppose the total mass $M = \sum m_i$ of the system remains constant with time. Where $i = 1, 2, ..., n$. The particles interact each other and suppose the external forces also act on them. The total momentum $\vec{P}$ of the system is given by

$$\begin{aligned} \vec{P} &= \vec{P}_1 + \vec{P}_2 + ... + \vec{P}_n \\ \vec{P} &= m_1\vec{v_1} + m_2\vec{v_2} + ... + m_n\vec{v_n} \\ \vec{P} &= M\vec{v_{cm}} \quad [\because M\vec{v_{cm}} = m_1\vec{v_1} + m_2\vec{v_2} + ... + m_n\vec{v_n}] \end{aligned} \tag{1}$$

i.e. total momentum of a system of particles is equal to the product of total mass of the system and the velocity of its centre of mass.

Linear moment of system of particles (contd.)

For a system of particles, Newton's second law can be written as

$$\vec{F_{ext}} = M\vec{a_{cm}} \tag{2}$$

Where $\vec{F_{ext}}$ is the vector sum of all the external forces acting on the system. The internal forces acting between particles cancel in pairs due to Newton's third law.

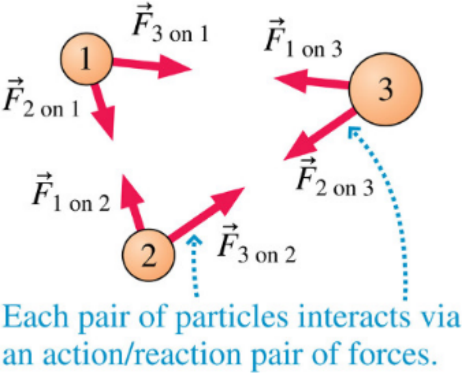

Figure 2: System of three particles

In this system of three masses the internal forces are cancel each other only the external force provides the acceleration of center of mass. Differentiating equation (1) with respect to time t. We get

$$\frac{d\vec{P}}{dt} = M\frac{dv_{cm}}{dt} = M\vec{a_{cm}} \tag{3}$$

Hence from equations (2) and (3), Newton's second law for the system of particle can be written as

$$\vec{F_{ext}} = \frac{d\vec{P}}{dt} \tag{4}$$

Conservation of linear momentum

Linear moment of system of particles (contd.)

Suppose that the sum of the external forces acting on a system is zero. Then (4) becomes

$$\frac{d\vec{P}}{dt} = 0 \implies \vec{P} = \text{Constant}$$

i.e. when the resultant external force is zero, the total momentum of the system remains constant. This is called the conservation of linear momentum.

System of variable mass

We consider a system in which mass enters or leaves the system while we are observing it, $\dfrac{dM}{dt}$ being positive in the former case and negative in the later case.

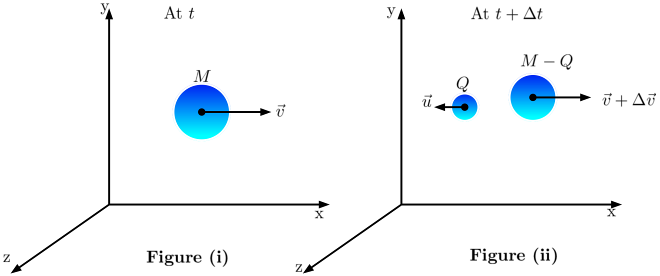

Figure 3: System of variable mass

System of variable mass (contd.)

Consider a system of mass M whose center of mass is moving with a velocity $\vec{v}$ at time t. An external force $\vec{F_{ext}}$ acts on the system. At a time Δt later a mass ΔM has been ejected from the system, its center of mass moving with a velocity $\vec{u}$. The system is reduced to $M - \Delta M$ and the velocity $\vec{v}$ of the center of mass of the system is changed to $\vec{v} + \Delta\vec{v}$.

We have, $\vec{F_{ext}} = \frac{d\vec{P}}{dt}$

For finite time interval Δt

$$\vec{F_{ext}} \approx \frac{\Delta P}{\Delta t} = \frac{\vec{P}_f - \vec{P}_i}{\Delta t}$$

System of variable mass (contd.)

Where $\vec{P}_f$ is the final system momentum in figure (ii) and $\vec{P}_i$ is the initial system momentum in figure (i).

Since, $\vec{P}_f = (M - \Delta M)(\vec{v} + \Delta\vec{v}) + \Delta M\vec{u}$ and $\vec{P}_i = M\vec{v}$

The above equation becomes

$$\begin{aligned} \vec{F_{ext}} &= \frac{[(M - \Delta M)(\vec{v} + \Delta\vec{v}) + \Delta M\vec{u}] - [M\vec{v}]}{\Delta t} \\ &= M\frac{\Delta\vec{v}}{\Delta t} + [\vec{u} - (\vec{v} + \Delta\vec{v})]\frac{\Delta M}{\Delta t} \end{aligned} \tag{5}$$

If Δt approaches to zero, $\frac{\Delta\vec{v}}{\Delta t}$ approaches $\frac{d\vec{v}}{dt}$ the acceleration of the body. The quantity ΔM is the mass ejected in Δt; this leads to the decrease in mass M of the original body. Since the change in mass of the body with time is negative, $\frac{\Delta M}{\Delta t}$ is replaced by $-\frac{dM}{dt}$ as Δt approaches zero.

Finally $\Delta\vec{v}$ goes to zero as Δt approaches zero. Hence the above equation becomes

$$\vec{F_{ext}} = M\frac{d\vec{v}}{dt} + \vec{v}\frac{dM}{dt} - \vec{u}\frac{dM}{dt} \tag{6}$$

$$\implies \vec{F_{ext}} = \frac{d}{dt}(M\vec{v}) - \vec{u}\frac{dM}{dt}$$

This is the Newton's second law for a system of variable mass.

System of variable mass (contd.)

For a constant mass $\dfrac{dM}{dt} = 0$ and hence equation (6) reduces to

$$\vec{F_{ext}} = M\frac{d\vec{v}}{dt} = M\vec{a} \tag{7}$$

Now rearrange equation (6)

$$M\frac{d\vec{v}}{dt} = \vec{F_{ext}} + (\vec{u} - \vec{v})\,\frac{dM}{dt} = \vec{F_{ext}} + \vec{v_{rel}}\frac{dM}{dt} \tag{8}$$

Where $\vec{v_{rel}}$ is the velocity of the ejected mass relative to the main body.

The rocket is the most interesting example of a system of variable mass.

Rocket: A rocket carries both the fuel and the oxidizer which born in a combustion chamber with in the rocket. When the rocket is fired that exhaust gases rush downward at a high speed and push the rocket upward. Thus the thrust on the rocket is supplied by the reaction forces of the high speed gases exhausted at the rear.

The last term in equation (8) $\vec{v_{rel}}\dfrac{dM}{dt}$ is the rate at which momentum is being transferred into (or out of) the system by the mass that system has ejected (or collected). It is reaction force exerted on the system by mass that leave it (or join it).

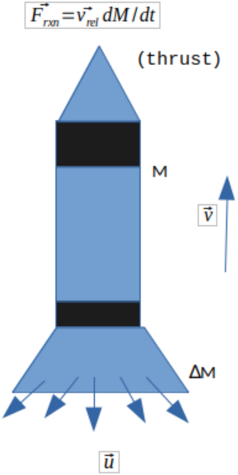

Figure 4: Rocket

For a rocket this term is called thrust. The above equation (8) can be written as

$$M\frac{d\vec{v}}{dt} = \vec{F_{ext}} + \vec{F_{rxn}}$$

If all the external forces including gravity and air resistance acting on the rocket is neglected. i.e. $\vec{F_{ext}} = 0$, then

$$M\frac{d\vec{v}}{dt} = \vec{F_{rxn}} = \vec{v_{rel}}\frac{dM}{dt}$$

$$\implies d\vec{v} = \vec{v_{rel}}\frac{dM}{M}$$

Integrating this equation from the instant, the velocity is $\vec{v}_0$ and the mass of the rocket is M_0 to the instant when the velocity is $\vec{v}$ and the mass of the rocket is M. We get

$$\int_{\vec{v}_0}^{\vec{v}} d\vec{v} = \vec{v}_{rel} \int_{M_0}^{M} \frac{dM}{M}$$

$$\text{or,} \quad \vec{v} - \vec{v}_0 = \vec{v}_{rel} \log_e \left(\frac{M}{M_0} \right)$$

$$\text{or,} \quad \vec{v} = \vec{v}_0 + \vec{v}_{rel} \log_e \left(\frac{M}{M_0} \right)$$

$$\text{or,} \quad \vec{v} = \vec{v}_0 - \vec{v}_{rel} \log_e \left(\frac{M_0}{M} \right)$$

For a rocket moving upward

$$\text{or,} \quad \vec{v} = \vec{v}_0 + \vec{v}_{rel} \log_e \left(\frac{M_0}{M} \right)$$

Hence the change in speed of the rocket in any interval of time depends only on exhaust velocity (being opposite in direction from it) and on the fraction of mass exhausted during that time interval.

If the rocket starts from rest $(\vec{v}_0 = 0)$ with an initial mass M_0 and reaches a final velocity $\vec{v}_f$ at burn out when its mass is M_f, then the above rocket equation becomes

$$
\begin{aligned}
\mathrm{v_f} =&\ \mathrm{v_{rel}} \log_e \left(\frac{M_0}{M_\mathrm{f}}\right) \\
\text{or}, \frac{\mathrm{v_f}}{\mathrm{v_{rel}}} =& -\log_e \left(\frac{M_\mathrm{f}}{M_0}\right) \\
\text{or}, -\frac{\mathrm{v_f}}{\mathrm{v_{rel}}} =&\ \log_e \left(\frac{M_\mathrm{f}}{M_0}\right) \\
\therefore M_f =&\ M_0 e^{\left(-\frac{v_f}{v_{rel}}\right)}
\end{aligned}
$$

In case the weight of a rocket taken into account the above equation for a rocket moving vertically upwards becomes

$$\text{or,} \quad \vec{v} = \vec{v}_0 - \vec{v}_{rel} \log_e \left(\frac{M_0}{M} \right) - gt$$

Problem:

A rocket moving in space, far from all other objects, has a speed of 3.0×10^3 m s^{-1} relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m s^{-1} relative to the rocket.

(i) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to half its mass before ignition?

(ii) What is the thrust on the rocket if it burns fuel at the rate50 kg s^{-1}?

Hints:

(i) $v_f = v_i + v_{rel} \log_e \left(\frac{M_i}{M_f}\right)$

$$= 3.0 \times 10^3 + \left(5.0 \times 10^3\right) \ln \left(\frac{M_i}{0.5M_i}\right) = 6.5 \times 10^3 \text{ m s}^{-1}$$

(ii) Thrust$= v_{rel} \frac{dM}{dt} = \left(5.0 \times 10^3\right)(50) = 2.5 \times 10^5 N$