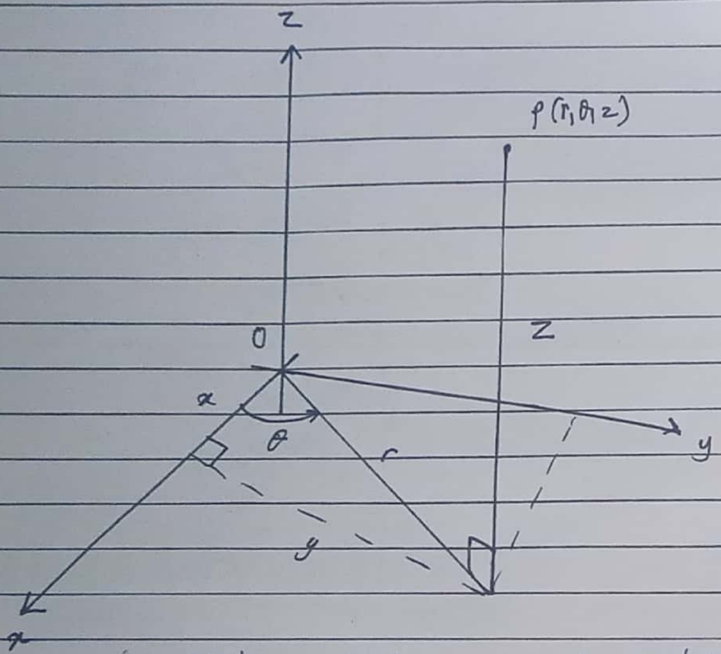
Cylindrical Coordinates



Cylindrical coordinates represents a point P in space by ordered triplets (r, θ, z) in which

i) r and θ are polar coordinates for the vertical projection of P on xy-plane.

(ii) z is the rectangular vertical coordinate.

<Q>: What do the following represent?

(i): $r=0$ $\Rightarrow$ It represents the z-axis

(ii): $r=a$ $\Rightarrow$ cylinder about z-axis

(iii) $\theta=\theta_0$ $\Rightarrow$ plane lamina containing z-axis, making angle θ_0 at x-axis.

(iv): $z=z_0$ $\Rightarrow$ plane $\perp^r$ to z-axis

* Equations relating cylindrical coordinates with Cartesian coordinates

$$x = r\cos\theta \qquad r^2 = x^2+y^2 \quad \text{or,} \quad r=\sqrt{x^2+y^2}$$
$$y = r\sin\theta$$
$$z = z \qquad \tan\theta = y/x$$

<Q>: Find the cartesian form of $z=r^2$.

Soln:

Given,

$$z = x^2+y^2$$

<Q> Find the circular cylinder in cylindrical coordinates $4x^2+4y^2=9$

Soln:

Given,

$$4x^2+4y^2=9$$

or $$x^2+y^2 = \frac{9}{4}$$

ie, $$r^2 = \frac{9}{4} \qquad \therefore r = \frac{3}{2}$$

<Q> Find the corresponding cylindrical point for Cartesian coordinate point $(3,-3,-7)$

Soln:

Given,

$$(x,y,z) = (3,-3,-7)$$

We know,

$z = -7.$

Also,

$$\theta = \tan^{-1}\left(\frac{-3}{3}\right)$$

or, $\theta = \tan^{-1}(-1)$ $\therefore \theta = -\pi/4.$

and,

$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{3^2+3^2} = \sqrt{18}$$

$\therefore r = 3\sqrt{2}$

Hence,

$(3, -3, -7) \approx (3\sqrt{2}, -\pi/4, -7)$

<Q>: Find the rectangular coordinate point for the cylindrical coordinate point. $(2, 2\pi/3, 1)$

Soln:

Given,

$(2, 2\pi/3, 1)$

$r = 2$

$\theta = 2\pi/3$

$z = 1.$

So,

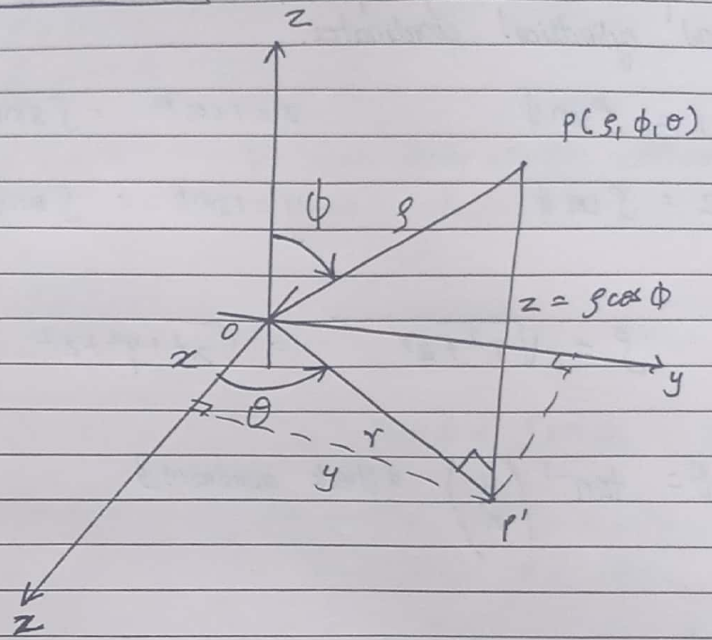
$x = r\cos\theta = 2\times\cos(2\pi/3) = -1$

$y = r\sin\theta = 2\times\sin(2\pi/3) = \sqrt{3}$

Hence,

$(2, 2\pi/3, 1) \approx (-1, \sqrt{3}, 1)$

Spherical Coordinates



Spherical coordinates represent a point P in space is by ordered triplets (ρ, ϕ, θ) in which

i) ρ is the distance from P to the origin.

ii) ϕ = angle $\vec{OP}$ makes with +ve z-axis $(0 \le \phi \le z)$

iii) θ = angle from cylinder coordinates $(\theta \le \theta \le 2\pi)$

<Q>: What do the following represents?

i) $\rho = 0$ ⇒ It gives a point.

ii) $\rho = a$ ⇒ Sphere of radius a at origin.

iii) $\phi = \phi_0$ ⇒ Cone at z-axis with vertex at origin.

iv) $\phi > \pi/2$ ⇒ Cone facing downwards.

v) $\theta = \theta_0$ ⇒ half plane containing z-axis making θ_0 angle with positive x-axis.

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(*) Equations relating spherical coordinates to Cartesian and cylindrical coordinates.

$r = \rho \sin\phi$ $\qquad x = r\cos\theta = \rho\sin\phi\cos\theta$

$z = \rho\cos\phi$ $\qquad y = r\sin\theta = \rho\sin\phi\sin\theta$

$\rho = \sqrt{r^2+z^2} = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ {check quadrant}

$\phi = \cancel{\tan}\ \cos^{-1}\left(\frac{z}{\rho}\right)$

If $\cancel{\phi}$ $\cancel{\phi > \pi/2}$ $\pi/2 < \phi < \pi$, it is obtuse.

<Q> Find the spherical coordinate equations for the equations:

i) $x^2+y^2+(z-1)^2 = 1$.

Soln:

$x^2+y^2+z^2-2z+1 = 1$

or, $\rho^2 = 2z$

or, $\rho^2 - 2\rho\cos\phi = 0$

or, $\rho(\rho - 2\cos\phi) = 0$

since $\rho = 0$,

$\therefore \rho - 2\cos\phi = 0$

(ii) $z = \sqrt{x^2+y^2}$. $\Rightarrow$ Cone facing upwards with vertex at pole.

Soln,

Given,

$z = \sqrt{x^2+y^2}$

or, $z = \sqrt{x^2+y^2}$

or, $\rho\cos\phi = r$ $\qquad$ or $\rho\cos\phi = \rho\sin\phi$

or, $\tan\phi = 1$ $\qquad \therefore \phi = \pi/4$.

ie, ϕ = constant gives cone.

(iii): $z = x^2+y^2$ $\Rightarrow$ Gives paraboloid.

<Q> Find the spherical coordinate for the Cartesian coordinate point. $(0, 2\sqrt{3}, -2)$

Soln:

Given,

$x = 0$ $\qquad y = 2\sqrt{3}$ $\qquad z = -2$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0^2+(2\sqrt{3})^2+(-2)^2} = 4$

$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{-2}{4}\right) = 2\pi/3$

$\theta = \tan^{-1}\left(\cancel{\frac{2}{2\sqrt{3}}}\ \frac{y}{x}\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{0}\right) = \pi/2$

So,

$$(0, 2\sqrt{3}, -2) \approx (4, 2\pi/3, \pi/2)$$

<Q>: Find the rectangular coordinate point for the spherical coordinate point $(2, \pi/4, \pi/3)$.

Sol^n:

Given,

$\rho = 2 \qquad \phi = \pi/4 \qquad \theta = \pi/3$

We know,

$$x = \rho \sin\phi \cos\theta = 2 \times \sin\frac{\pi}{4} \times \cos\frac{\pi}{3} = 1/\sqrt{2}$$

$$y = \rho \sin\phi \sin\theta = 2 \times \sin\frac{\pi}{4} \times \sin\frac{\pi}{3} = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\phi = 2 \times \cos \pi/4 = \sqrt{2}$$

$\therefore \cancel{(\rho, \phi, \theta)\ (1/\sqrt{2}, \sqrt{3}}$

$$\therefore (2, \pi/4, \pi/3) \approx (1/\sqrt{\cancel{3}2}, \sqrt{3/2}, \sqrt{2})$$