Lecture 15

Electromagnetic Induction (Contd.)

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Keshav Raj Sigdel

Assistant Professor

Department of Physics

School of Science

Kathmandu University

Outline

Mutual induction

We consider two coils $C_1$and C_2 placed very close to each other as shown in figure 1. When switch S is closed, current starts rising in coil C_1 and hence the flux linked with it increases. This increase in flux induces back *emf* in coil C_1 due to self induction and also causes the change in flux linked with coil C_2. The change in flux in coil C_2 induces an *emf* in it.

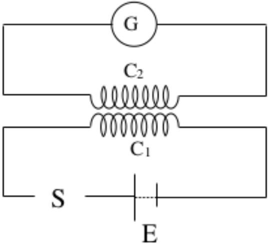

Figure 1

Mutual induction (contd.)

Suppose I_1 be the current flowing through C_1; then it is found that the flux linked with coil C_2 (i.e. Φ_2) is proportional to the current I_1

$$\text{i.e. } \Phi_2 \propto I_1 \implies \Phi_2 = M_{21} I_1$$

Here M_{21} is a constant and is called the coefficient of mutual induction of coil C_2 with respect to coil C_1.

Now form Faraday's law, the induced *emf* is

$$\mathscr{E} = -\frac{d\Phi_2}{dt} = -\frac{d(M_{21} I_1)}{dt}$$
$$\therefore\ \mathscr{E} = -M_{21} \frac{dI_1}{dt}.$$

This gives the induced *emf* in coil C_2.

Neumann formula and Reciprocity theorem

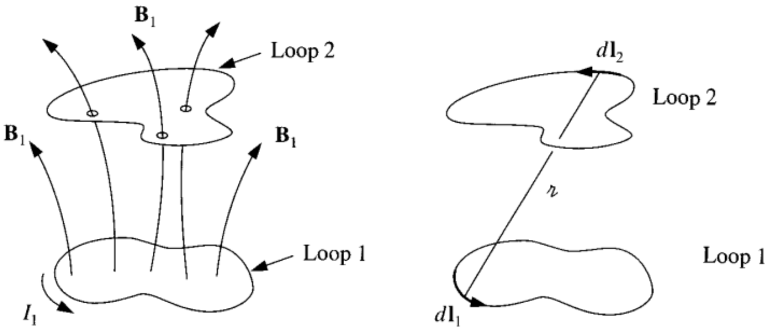

Figure 2

Consider two loops labeled 1 and 2 as shown in in figure 2. The elemental lengths $d\vec{l}_1$ on loop 1 and $d\vec{l}_2$ on loop 2 is separated by the separation vector $\vec{\mathscr{r}}$. If a steady current I_1 is allowed to flow on loop 1, then it produces the magnetic field $\vec{B}_1$ around it. Some of the field lines pass through loop 2 and resulting the flux Φ_2 through loop 2.

Using Biot-Savart law, the magnetic field due to loop 1 is

$$\vec{B}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{\text{Loop 1}} \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

and the corresponding vector potential is

$$\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint_{\text{Loop 1}} \frac{d\vec{l}_1}{\mathscr{r}}$$

Neumann formula and Reciprocity theorem (contd.)

The magnetic flux through Loop 2 due to $\vec{B}_1$ is

$$\Phi_2 = \int_{S_2} \vec{B}_1 \cdot d\vec{a}_2 = \int_{S_2} (\nabla \times \vec{A}_1) \cdot d\vec{a}_2 = \oint_{\text{Loop 2}} \vec{A}_1 \cdot d\vec{l}_2$$

Substituting the expression of $\vec{A}_1$, we have

$$\Phi_2 = \oint_{\text{Loop 2}} \left(\frac{\mu_0 I_1}{4\pi} \oint_{\text{Loop 1}} \frac{d\vec{l}_1}{\mathscr{r}} \right) \cdot d\vec{l}_2 = \left(\frac{\mu_0}{4\pi} \oint_{\text{Loop 2}} \oint_{\text{Loop 1}} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}} \right) I_1 = M_{21} I_1$$

with

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{\text{Loop 2}} \oint_{\text{Loop 1}} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}} \tag{1}$$

This is known as the Neumann formula for mutual induction and it shows that coefficient of mutual induction depends up on geometrical quantities (shape, size, position) of the loops not on the current flowing through it. Similarly, we can show

$$\Phi_1 = \left(\frac{\mu_0}{4\pi} \oint_{\text{Loop 1}} \oint_{\text{Loop 2}} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{\mathscr{r}} \right) I_1 = M_{12} I_1$$

with

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{\text{Loop 1}} \oint_{\text{Loop 2}} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{\mathscr{r}}$$

Since $d\vec{l}_1 \cdot d\vec{l}_2 = d\vec{l}_2 \cdot d\vec{l}_1$, we get $M_{12} = M_{21}$. This relation is called the reciprocity theorem. According to this theorem:

Neumann formula and Reciprocity theorem (contd.)

Whatever the shapes and positions of the loops, the flux through loop 2 when a current I flows around loop 1 is exactly the same as the flux through loop 1 when the same current flows around loop 2.

Energy stored in a magnetic field of coil

We suppose a coil is connected in series with a battery as shown in figure 3. As soon as the switch S is closed, the current starts rising in the circuit. The back emf produced in the coil opposes this rise in current.

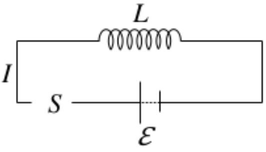

Figure 3

So, work must be done against the back emf to get the current going. At steady current, no further work is done against back emf. Suppose current increases from zero to I and L is self inductance of coil, then emf produced is

$$\mathscr{E} = -L\frac{dI}{dt}$$

Energy stored in a magnetic field of coil (contd.)

and the back emf is $-\mathscr{E} = L\frac{dI}{dt}$ So, the rate of doing work against the emf i.e. power delivered by the battery is

$$P = -\mathscr{E}I = LI\frac{dI}{dt}$$

Thus the total work done is

$$W = \int_0^I P dt = \int_0^I LI\frac{dI}{dt}dt = L\int_0^I I\,dI$$

$$\therefore W = \frac{1}{2}LI^2 \tag{2}$$

Energy stored in a magnetic field of coil (contd.)

Also, the magnetic flux linked with coil is $\Phi = LI$ and we know

$$\Phi = \int_s \vec{B} \cdot d\vec{a} = \int_s (\nabla \times \vec{A}) \cdot da = \oint_c \vec{A} \cdot d\vec{l}$$

Where C is path of integration for loop and S is area bounded by it.

$$\therefore \; LI = \oint_c \vec{A} \cdot d\vec{l} \tag{3}$$

From equations (2) and (3),

$$W = \frac{1}{2} I \oint_c \vec{A} \cdot d\vec{l}$$

$$\therefore \; W = \frac{1}{2} \int_{\text{volume}} \vec{A} \cdot \vec{J} \, d\tau \tag{4}$$

Energy stored in a magnetic field of coil (contd.)

From Ampere's law,

$$\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow \vec{J} = \frac{1}{\mu_0}(\nabla \times \vec{B})$$

so, equation (4) becomes

$$W = \frac{1}{2\mu_0} \int_{\text{volume}} \vec{A} \cdot (\nabla \times \vec{B})\, d\tau \tag{5}$$

For vectors $\vec{A}$ and $\vec{B}$ we know that

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\Longrightarrow \vec{A} \cdot (\nabla \times \vec{B}) = B \cdot (\nabla \times \vec{A}) - \nabla \cdot (\vec{A} \times \vec{B})$$

$$\Longrightarrow \vec{A} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot \vec{B} - \nabla \cdot (\vec{A} \times \vec{B}) = B^2 - \nabla \cdot (\vec{A} \times \vec{B})$$

Energy stored in a magnetic field of coil (contd.)

So, equation (5) reduces to

$$W = \frac{1}{2\mu_0}\left[\int_{\text{volume}} B^2\,d\tau - \int_{\text{volume}} \nabla\cdot(\vec{A}\times\vec{B})d\tau\right]$$

Applying Divergence theorem in the second term of right hand side

$$W = \frac{1}{2\mu_0}\left[\int_{\text{volume}} B^2\,d\tau - \oint_{\text{surface}} (\vec{A}\times\vec{B})\cdot d\vec{a}\right] \tag{6}$$

At far from current loop both $\vec{A}$ & $\vec{B}$ decreases and vanishes at infinite distance. So the surface integral over surface of infinitely large radius vanishes. The total energy over all space is only due to the volume integral part. So for integration over all space we get,

$$W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 \, d\tau \tag{7}$$

This gives magnetic energy stored in the given volume.

Thus, energy stored per unit volume (i.e. energy density) is equal to

$$u_m = \frac{B^2}{2\mu_0} \, . \tag{8}$$

1. **A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length) as shown in Figure 4. Current I flows in the short solenoid. What is the flux through the long solenoid? What is the mutual induction?**

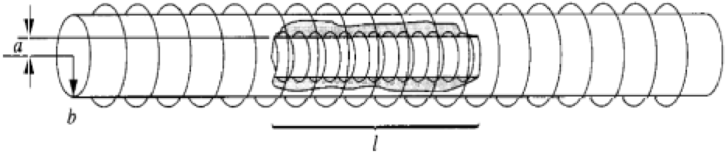

Figure 4

Hint:

Even though the detail information is given to the short solenoid, it is very difficult to find the magnetic field and hence the flux linked to it. But using reciprocity theorem we can resolve the problem by sending the same current through long solenoid.

Then we can have $\Phi_{\text{short}} = \Phi_{\text{long}}$.

Now, the magnetic field in long solenoid is $B = \mu_0 n_2 I$

The flux through the short solenoid due to the field of long solenoid is

$$\Phi_{\text{short}} = (\mu_0 n_2 I)(n_1 l)(\pi a^2) = \left(\mu_0 \pi a^2 n_1 n_2 l\right) I$$

Problems (contd.)

Hence, the flux through long solenoid when the same current flows through the short solenoid is

$$\Phi_{\text{long}} = \Phi_{\text{short}} = (\mu_0 n_2 I)(n_1 l)(\pi a^2) = \left(\mu_0 \pi a^2 n_1 n_2 l\right) I$$

Therefore, Mutual inductance $\quad M = \mu_0 \pi a^2 n_1 n_2 l$

2. **A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a , and back along the outer cylinder, radius b) as shown in Figure 5. Find the magnetic energy stored in a section of length l. Also find coefficient of self-induction.**

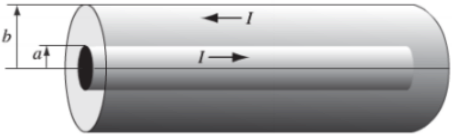

Figure 5

Hint;

First of all we have to calculate the magnetic field due to the system of current. For this purpose, let's take a coaxial Amperian loop (blue circle) of radius r passing through point P as shown in figure 6

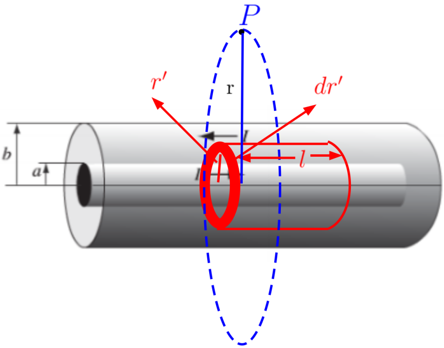

Figure 6

Ampere's law reads,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Problems (contd.)

$$\implies B(2\pi r) = \mu_0 I_{\text{enc}}$$
$$\implies B = \frac{\mu_0 I_{\text{enc}}}{2\pi r} \tag{9}$$

But

$$I_{\text{enc}} = \begin{cases} I - I = 0, & \text{for } P \text{ lies outside of the outer cylinder i.e. } r > b \\ I, & \text{for } P \text{ lies in between two cylinders i.e. } a < r < b \\ 0, & \text{for } P \text{ lies inside of the inner cylinder i.e } r < a \end{cases}$$

Therefore, the magnetic field is

$$B = \begin{cases} 0, & \text{for } P \text{ lies outside of the outer cylinder i.e. } r > b \\ \dfrac{\mu_0 I}{2\pi r} & \text{for } P \text{ lies in between the two cylinders i.e. } a < r < b \\ 0, & \text{for } P \text{ lies inside of the inner cylinder i.e } r < a \end{cases} \tag{10}$$

To calculate the magnetostatic energy stored in the system, we are going to use

$$W_m = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau$$

For this purpose, let's take an annular cylindrical portion of radius r', thickness dr' and length l as shown in figure 6.

Problems (contd.)

The elemental volume $d\tau = 2\pi r' dr' l$

Therefore, the energy stored is

$$
\begin{aligned}
W_m &= \frac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau \\
&= \frac{1}{2\mu_0} \int_0^\infty B^2 (2\pi r' dr' l) \\
&= \frac{\pi l}{\mu_0} \left[\underbrace{\cancelto{0}{\int_0^a B^2 r' dr'}}_{B=0 \text{ for } r'<a} + \int_a^b B^2 r' dr' + \underbrace{\cancelto{0}{\int_b^\infty B^2 r' dr'}}_{B=0 \text{ for } r'>b} \right] \\
&= \frac{\pi l}{\mu_0} \int_a^b \left(\frac{\mu_0 I}{2\pi r'} \right)^2 r' dr' \\
&= \frac{\mu_0 l I^2}{4\pi} \int_a^b \left(\frac{1}{r'} \right) dr'
\end{aligned}
$$

Problems (contd.)

$$\therefore W_m = \frac{\mu_0 l I^2}{4\pi} \ln\left(\frac{b}{a}\right)$$

Comparing this with $W = \dfrac{1}{2}LI^2$, the self-inductance is given by

$$L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$$

3. **Find the energy stored in a section of length l of a long solenoid (radius R, current I , and n turns per unit length) and hence obtain the expression for energy density.**

   Hint:

   The magnetic field inside the solenoid is $B = \mu_0 n I$ and constant everywhere but outside the solenoid it is zero.

   The volume of the solenoid $V_{\text{sol}} = \pi R^2 l$

   The energy stored

$$W_m = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 d\tau$$

$$= \frac{1}{2\mu_0} \int_{\text{solenoid}} B^2 d\tau$$

Problems (contd.)

$$= \frac{1}{2\mu_0} B^2 \int_{\text{solenoid}} d\tau$$

$$= \frac{1}{2\mu_0} B^2 V_{\text{sol}}$$

$$= \frac{1}{2\mu_0} (\mu_0 n I)^2 \pi R^2 l$$

$$W_m = \frac{1}{2} \pi \mu_0 n^2 R^2 l I^2$$

Comparing this with $W = \frac{1}{2} L I^2$, the self inductance is

$$L = \pi \mu_0 n^2 R^2 l$$

End of Lecture 15

Thank you