General Physics II

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Course Outline

Differential Calculus

- Product Rules
- Second Derivatives

Integral Calculus

- Line, Surface, and Volume Integrals
- The Fundamental Theorem of Calculus
- The Fundamental Theorem for Gradients
- The Fundamental Theorem for Divergences
- The Fundamental Theorem for Curls

Spherical Polar Coordinates

Product Rules

- **The General Rules**

The Product Rule:	$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$
The Quotient Rule:	$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$

- **Four Product Rules**

1. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
2. $\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + (\nabla f) \cdot \vec{A}$
3. $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
4. $\nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) + (\nabla f) \times \vec{A} = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f)$

Second Derivatives

Five Species of Second Derivatives

- **The gradient ∇T is a vector .**

1. Divergence of gradient: $\nabla \cdot (\nabla T)$ $\rightarrow$ a scalar
2. Curl of Gradient: $\nabla \times (\nabla T)$ $\rightarrow$ a vector

- **The divergence $\nabla \cdot \vec{v}$ is a scalar .**

3. Gradient of Divergence: $\nabla (\nabla \cdot \vec{v})$ $\rightarrow$ a vector

- **The curl $\nabla \times \vec{v}$ is a vector .**

4. Divergence of curl: $\nabla \cdot (\nabla \times \vec{v})$ $\rightarrow$ a scalar
5. Curl of curl: $\nabla \times (\nabla \times \vec{v})$ $\rightarrow$ a vector

Laplacian

- Laplacian Operator: $\nabla \cdot \nabla = \nabla^2 = \Delta$

- The Laplacian of a Scalar T is a scalar.

  The Laplacian of $T: \nabla \cdot (\nabla T) = \nabla^2 T = \Delta T = \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}$

- The **Laplacian** of a vector $\vec{\mathrm{v}}$ is a vector.

  The **Laplacian** of $\vec{\mathrm{v}}: \nabla^2 \vec{\mathrm{v}} \equiv \left(\nabla^2 \mathrm{v}_x\right)\hat{i} + \left(\nabla^2 \mathrm{v}_y\right)\hat{j} + \left(\nabla^2 \mathrm{v}_z\right)\hat{k}$

$$\nabla^2 \vec{\mathrm{v}} = \left(\nabla \cdot \nabla\right)\vec{\mathrm{v}} \neq \nabla\left(\nabla \cdot \vec{\mathrm{v}}\right)$$

Example - I

- Calculate the Laplacian of the function $T_a = \sin x \sin y \sin z$

Solution:

$$\nabla^2 T_a = \frac{\partial^2 T_a}{\partial x^2} + \frac{\partial^2 T_a}{\partial y^2} + \frac{\partial^2 T_a}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}(\sin x \sin y \sin z) + \frac{\partial^2}{\partial y^2}(\sin x \sin y \sin z) + \frac{\partial^2}{\partial z^2}(\sin x \sin y \sin z)$$

$$= -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z$$

$$= -3 \sin x \sin y \sin z$$

Calculate the Laplacian of the function $T_b = e^{-5x} \sin 4y \cos 3z$.

Example -2

- **Calculate the Laplacian of the function** $\vec{\mathrm{v}} = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$

$$\nabla^2\vec{\mathrm{v}} = \left(\nabla^2 \mathrm{v}_x\right)\hat{i} + \left(\nabla^2 \mathrm{v}_y\right)\hat{j} + \left(\nabla^2 \mathrm{v}_z\right)\hat{k}$$

$$= \left[\nabla^2\left(x^2\right)\right]\hat{i} + \left[\nabla^2\left(3xz^2\right)\right]\hat{j} + \left[\nabla^2\left(-2xz\right)\right]\hat{k}$$

$$= \left[\frac{\partial^2}{\partial x^2}\left(x^2\right) + \frac{\partial^2}{\partial y^2}\left(x^2\right) + \frac{\partial^2}{\partial z^2}\left(x^2\right)\right]\hat{i} + \left[\frac{\partial^2}{\partial x^2}\left(3xz^2\right) + \frac{\partial^2}{\partial y^2}\left(3xz^2\right) + \frac{\partial^2}{\partial z^2}\left(3xz^2\right)\right]\hat{j}$$

$$+\left[\frac{\partial^2}{\partial x^2}\left(-2xz\right) + \frac{\partial^2}{\partial y^2}\left(-2xz\right) + \frac{\partial^2}{\partial z^2}\left(-2xz\right)\right]\hat{k}$$

$$= \left(2+0+0\right)\hat{\mathrm{i}} + \left(0+0+6\mathrm{x}\right)\hat{j} + \left(0+0+0\right)\hat{k}$$

$$= 2\hat{\mathrm{i}} + 6\mathrm{x}\hat{\mathrm{j}}$$

Calculate the Laplacian of the function $\vec{v} = x^2 y\hat{i} + (x^2 - y)\hat{k}$.

Example -3

- **The curl of a gradient is always zero:** $\nabla\times(\nabla T)=0$

Solution:

$$\nabla\times(\nabla T)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}\frac{\partial T}{\partial x}+\hat{j}\frac{\partial T}{\partial y}+\hat{k}\frac{\partial T}{\partial z}\right)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z}\end{vmatrix}$$

$$=\hat{i}\left(\frac{\partial^2 T}{\partial y\partial z}-\frac{\partial^2 T}{\partial z\partial y}\right)-\hat{j}\left(\frac{\partial^2 T}{\partial x\partial z}-\frac{\partial^2 T}{\partial z\partial x}\right)+\hat{k}\left(\frac{\partial^2 T}{\partial x\partial y}-\frac{\partial^2 T}{\partial y\partial x}\right)$$

$$=0 \qquad \left[\because \frac{\partial^2 T}{\partial x\partial y}=\frac{\partial^2 T}{\partial y\partial x}\right]$$

Example -4

- **The divergence of a curl is always zero:** $\nabla\cdot\left(\nabla\times\vec{v}\right)=0$ **.**

  Hint:

- $$\nabla\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}=\hat{i}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-\hat{j}\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)+\hat{k}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

- $$\nabla\cdot\left(\nabla\times\vec{v}\right)=\left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]\cdot\left[\hat{i}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-\hat{j}\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)+\hat{k}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

$$=0 \qquad \left[\because \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x}\right)\right]$$

Notes:

1. If the curl of a vector field $\left(\vec{F}\right)$ vanishes (everywhere), $\vec{F}$ then can be written as the gradient of a ***scalar function*** (*V*):

$$\nabla \times \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla V$$

Curl-less (or "irrotational") fields:

- $\nabla \times \vec{F} = 0$ everywhere.
- $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path, for any given end points.
- $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop.
- $\vec{F}$ is the gradient of some scalar $\vec{F} = -\nabla V$.

Notes:

2. If the divergence of a vector field $\left(\vec{F}\right)$ vanishes (everywhere), then $\vec{F}$ can be written as the curl of a vector ***function*** $\vec{A}$:

$$\nabla \cdot \vec{F} = 0 \Leftrightarrow \vec{F} = \nabla \times \vec{A}$$

Divergence-less (or "solenoidal") fields:

- $\nabla \cdot \vec{F} = 0$ everywhere.
- $\int \vec{F} \cdot d\vec{a}$ is independent of surface, for any given boundary line.
- $\oint \vec{F} \cdot d\vec{a} = 0$ for any closed surface.
- $\vec{F}$ is the curl of some vector, $\vec{F} = \nabla \times \vec{A}$

For any Vector Field $\vec{F}$:

$$\boxed{\vec{F} = -\nabla V + \nabla \times \vec{A} \qquad \text{(always)}}$$

Integral Calculus

Line Integral:

If $\vec{\mathrm{F}}$ is a vector, a line integral of $\vec{\mathrm{F}}$ is written

$$\int_{a\ C}^{b} \vec{\mathrm{F}}\cdot d\vec{\mathrm{l}},$$

where C is the curve along which the integration is performed,

a and b the initial and final points on the curve,

and $d\vec{\mathrm{l}}$ is the infinitesimal displacement vector along the curve C.

- The line integral is a scalar.
- Line integral over a closed curve: $\oint_C \vec{\mathrm{F}}\cdot d\vec{\mathrm{l}}$
- Example of a line integral:

  The work done by a force $\vec{F}$: $\boxed{W = \int \vec{F}\cdot d\vec{l}}$
- For *conservative force*:

$$\oint \vec{F}\cdot d\vec{l} = 0$$

Integral Calculus

Surface Integral:

If $\vec{F}$ is a vector, a surface integral of $\vec{F}$ is written

$$\int_S \vec{F} \cdot d\vec{a},$$

where S is the surface over which the integration is to be performed, and $d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface.

- Surface integral over a closed surface $\oint_S \vec{F} \cdot d\vec{a}$.
- The Flux of $\vec{E}$ through a surface S : $\Phi_E = \int_S \vec{E} \cdot d\vec{a}$
- If $\vec{v}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \vec{v} \cdot d\vec{a}$ represents the total mass per unit time passing through the surface [or flux].

Integral Calculus

Volume Integral:

A volume integral is an expression of the form

$$\int_V T \, d\tau,$$

where T is a scalar function and is $d\tau$ an infinitesimal volume element.

Total charge q : $\boxed{q = \int_V \rho \, d\tau}$

where ρ is the volume charge density.

Fundamental Theorem of Calculus

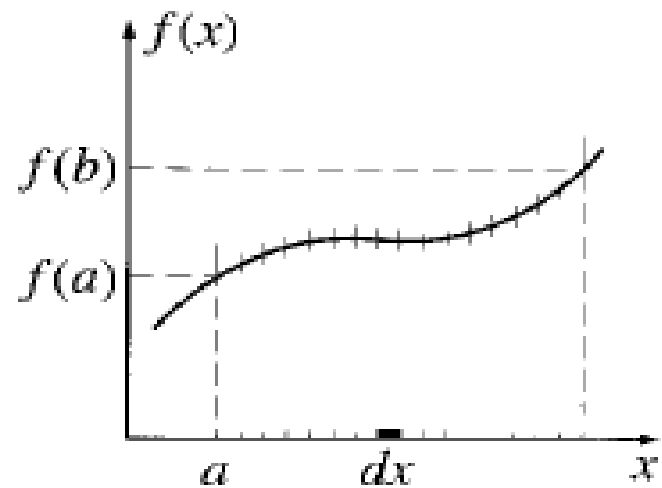
- Suppose $f(x)$ is a function of one variable.
  The fundamental theorem of calculus states:

$$\int_a^b \frac{df}{dx}dx = f(b) - f(a) \qquad \dots\dots\dots \quad (F-1)$$

Here, $\left(\frac{df}{dx}\right)dx$ is the infinitesimal change in f when you go form

(x) to $(x+dx)$.

The fundamental theorem (F-1) says that there are two ways to determine the total change in the function: *either* subtract the values at the ends *or* go step-by-step, adding up all the tiny increments as you go.
You'll get the same answer either way.



The Fundamental Theorem for Gradients

- Suppose we have a scalar function of three variables $f(x, y, z)$.

The total change in f in going from a to b is

$$\boxed{\int_a^b (\nabla f)\cdot d\vec{l} = f(\mathrm{b}) - f(\mathrm{a})}$$

- ***Geometrical Interpretation:***

Suppose you want to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up, or you could place altimeters at the top and the bottom, and subtract the two readings; you should get the same answer either way.

The Fundamental Theorem for Divergences [Gauss's Theorem]

- The fundamental theorem for divergences states that:

$$\boxed{\int_V \left(\nabla \cdot \vec{\mathrm{v}}\right) d\tau = \oint_S \vec{\mathrm{v}} \cdot d\vec{a}} \qquad \ldots\ldots\ldots\ldots\ldots(\mathrm{F-2})$$

Geometrical Interpretation:

If $\vec{\mathrm{V}}$ represents the flow of an incompressible fluid, then the flux of $\vec{\mathrm{V}}$ (the right side of Eq.(F-2)) is the total amount of fluid passing out through the surface, per unit time. Now, the divergence measures the "spreading out" of the vectors from a point – a place of high divergence is like a "faucet," pouring out liquid.

If we have lots of faucets in a region filled with incompressible fluid, an equal amount of liquid will be forced out through the boundaries of the region.

$$\boxed{\int \left(\text{faucets within the volume}\right) = \oint \left(\text{flow out through the surface}\right)}$$

The Fundamental Theorem for Curls [Stoke's Theorem]

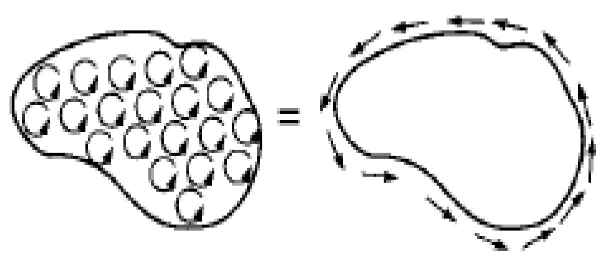
- The fundamental theorem for curls states that:

$$\boxed{\int_S (\nabla \times \vec{\mathrm{v}}) \cdot d\vec{a} = \oint_P \vec{\mathrm{v}} \cdot d\vec{l}} \qquad ....(\mathrm{F}-3)$$

Geometrical Interpretation:

The curl measures the "twist" of the vectors ; a region of high curl is a whirlpool if you put a tiny paddle wheel there, it will rotate.

Now, the integral of the curl over some surface represents the "total amount of swirl," and we can determine that swirl just as well by going around the edge and finding how much the flow is following the boundary (Figure F-B).



Spherical Polar Coordinates

- The spherical polar coordinates of a point (r,θ,ϕ) are defined in Figure S-A.

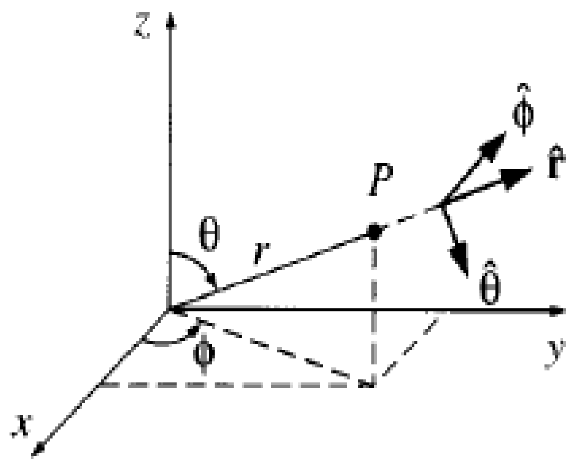


Figure S-A

$r \rightarrow$ the distance from the origin

(the magnitude of the position vector)

$\theta \rightarrow$ the polar angle

(the angle down from the axis)

$\phi \rightarrow$ the azimuthal angle

(the angle around from the axis)

$(\hat{r},\hat{\theta},\hat{\phi})$ constitute an orthogonal basis set just like $(\hat{i},\hat{j},\hat{k})$

For a vector $\vec{A}$, $\boxed{\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}}$

where $A_r, A_\theta,$ *and* A_ϕ are the radial, polar, and azimuthal components of $\vec{A}$.

Spherical Polar Coordinates

- **Infinitesimal Displacement**

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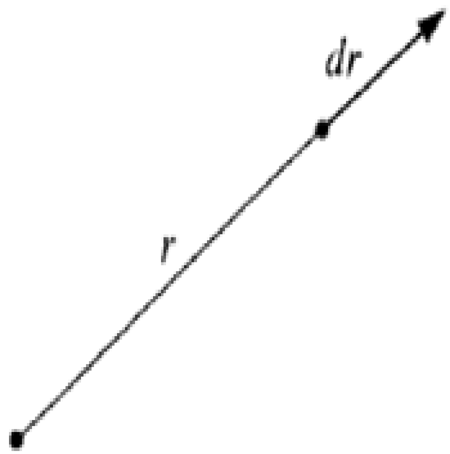
<td>

Figure S-1

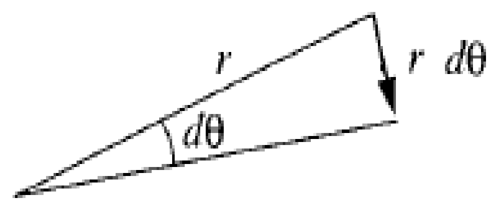
An infinitesimal displacement in the $\hat{r}$ direction: $\boxed{dl_r = dr}$</td>
<td>

Figure S-2

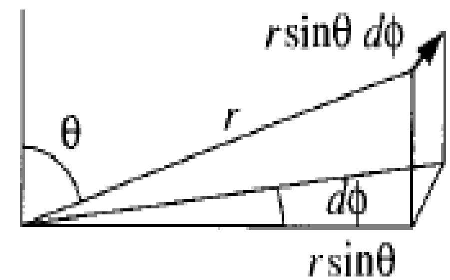
An infinitesimal displacement in the $\hat{\theta}$ direction: $\boxed{dl_\theta = r d\theta}$</td>
<td>

Figure S-3

An infinitesimal displacement in the $\hat{\theta}$ direction: $\boxed{dl_\phi = r \sin\theta d\phi}$</td>
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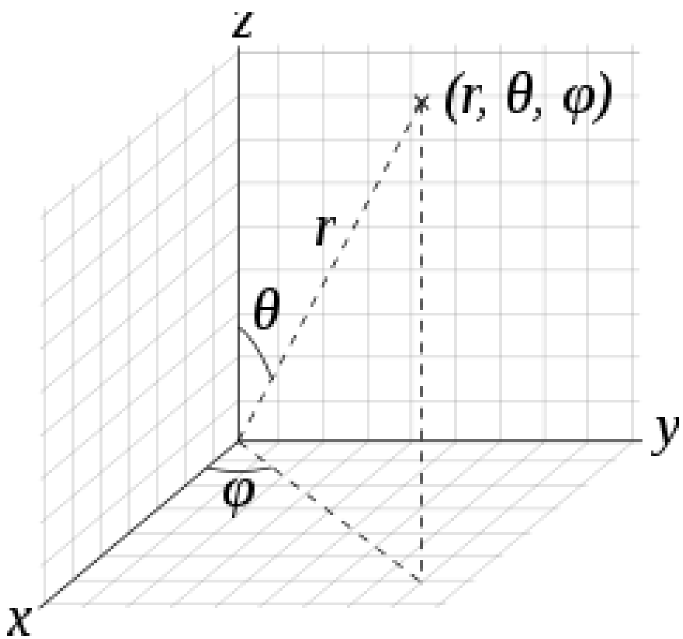
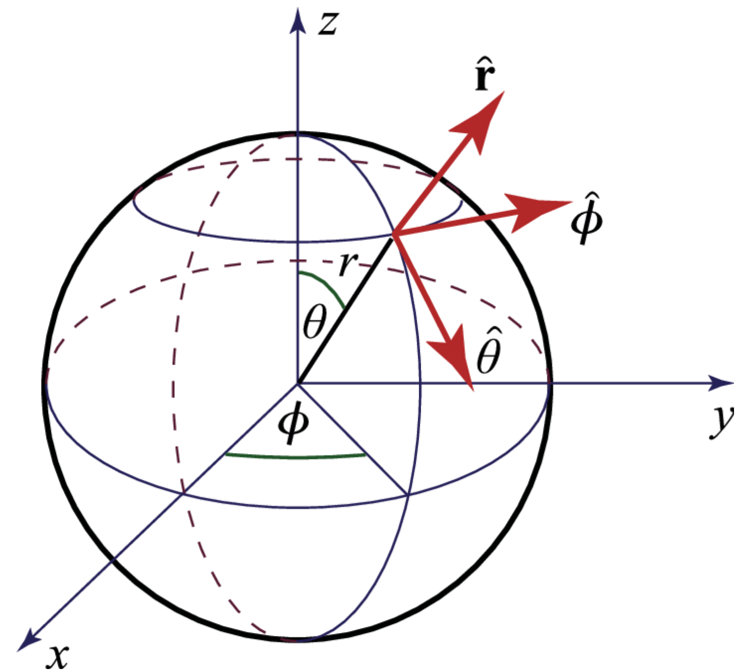
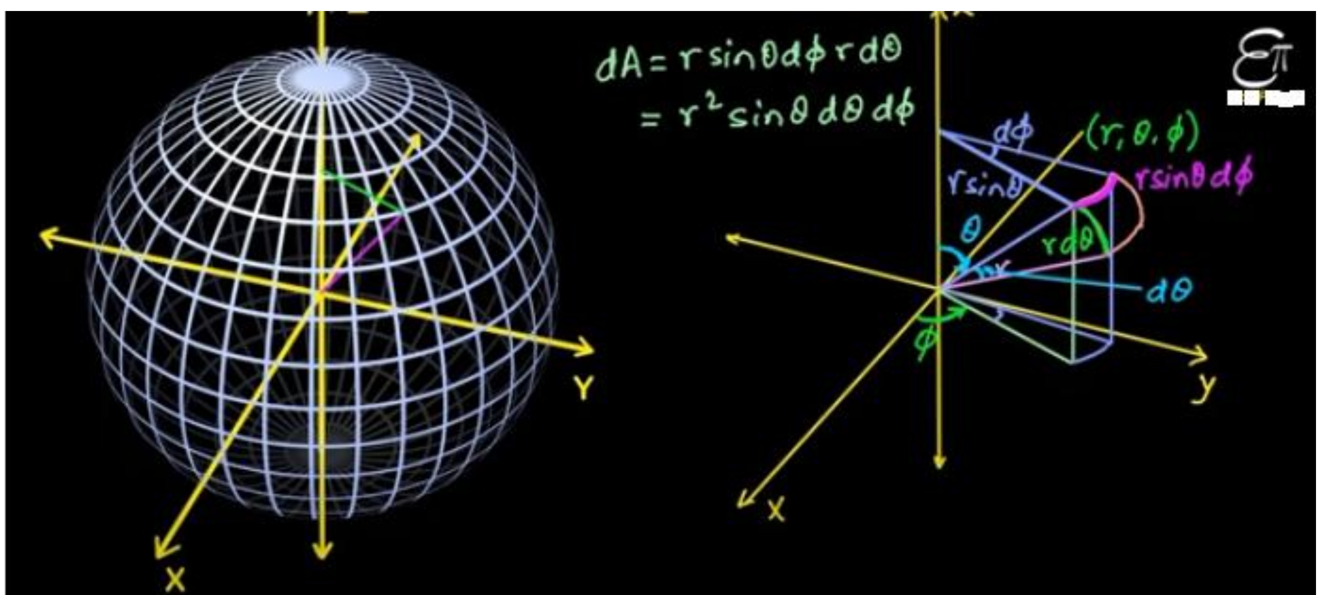
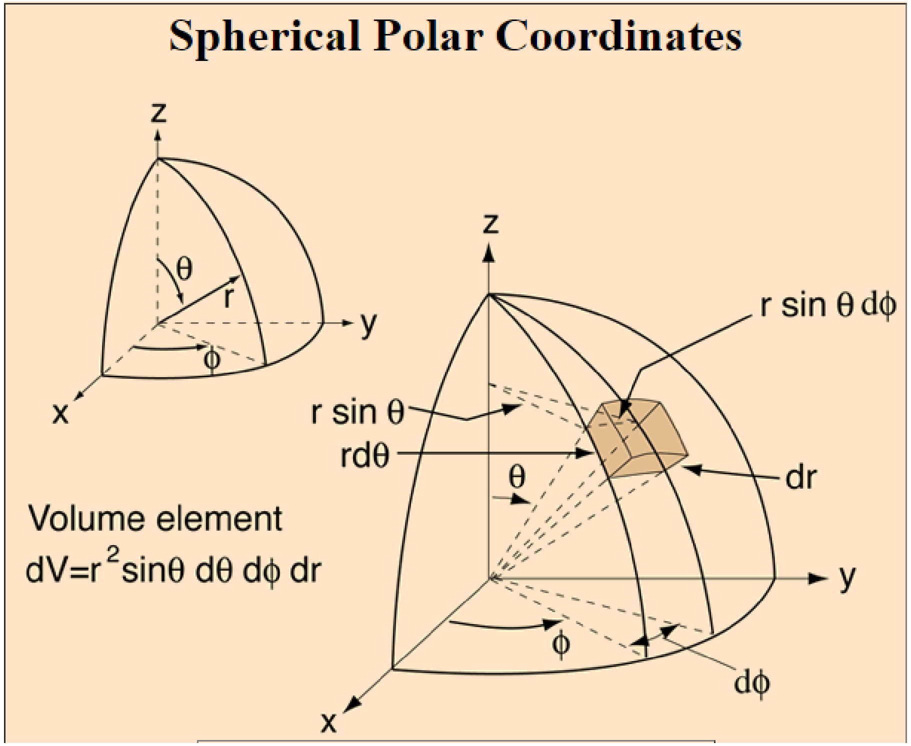
Spherical Polar Coordinates







Spherical Polar Coordinates



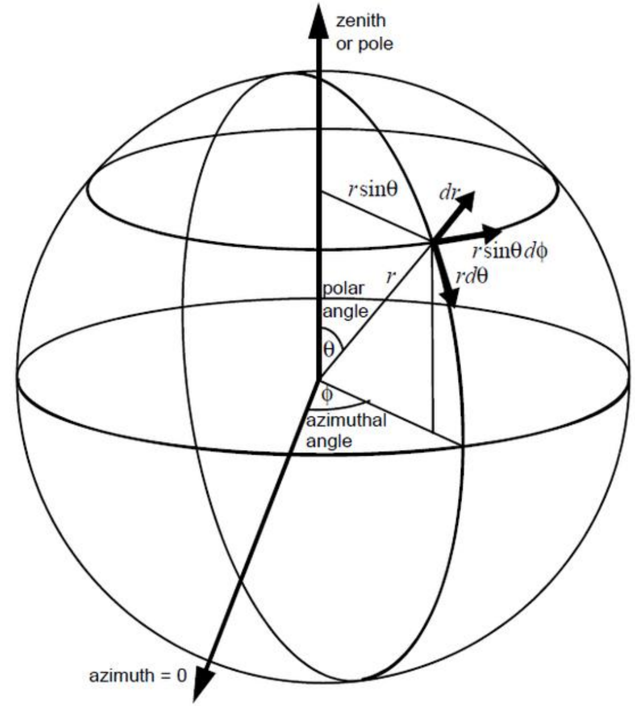


Figure 3.4: Spherical coordinates, in the physics convention.

$$da = r^2 \sin\theta \; d\theta \; d\phi$$

$$d\tau = r^2 \sin\theta \; dr \; d\theta \; d\phi$$

Spherical Polar Coordinates

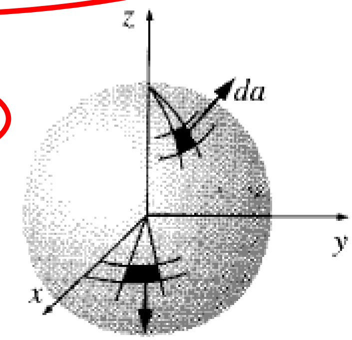
Notes:

- **The general infinitesimal displacement vector:**

$$d\vec{l} = dl_r \hat{r} + dl_\theta \hat{\theta} + dl_\phi \hat{\phi}$$
$$= dr\, \hat{r} + r\, d\theta\ \hat{\theta} + r \sin\theta\, d\phi\ \hat{\phi}$$

- **The surface element:**

$$da = r^2 \sin\theta\ d\theta\ d\phi$$



- **The infinitesimal volume element:**

$$d\tau = r^2 \sin\theta\ dr\ d\theta\ d\phi$$

$r \to 0$ to ∞ $\qquad$ $\theta \to 0$ to π $\qquad$ $\phi \to 0$ to 2π

Example:

- **Find the volume of a sphere of radius of R.**

Solution:

$$V = \int d\tau$$

$$= \int r^2 \sin\theta \; dr \; d\theta \; d\phi$$

$$= \left(\int_{r=0}^{R} r^2 dr\right)\left(\int_{\theta=0}^{\pi} \sin\theta \; d\theta\right)\left(\int_{\phi=0}^{2\pi} d\phi\right)$$

$$= \left(\frac{R^3}{3}\right)(2)(2\pi)$$

$$= \frac{4}{3}\pi R^3$$

Find the surface area of a sphere of radius of R.

$$\mathrm{A} = \int da$$

$$= \int R^2 \sin\theta \; d\theta \; d\phi$$

$$= \;\; ....$$

Text Books & References

1. **David J. Griffith**, Introduction to Electrodynamics
2. **R. A. Serway and J.W. Jewett**, Physics for Scientist and Engineers with Modern Physics
3. **Halliday and Resnick**, Fundamental of Physics
4. **www.fibreoptics4sale.com**
5. **http://hyperphysics.phy-astr.gsu.edu**


Thank you