Advanced Calculus

Functions of Several Variables

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Function of several Variables

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

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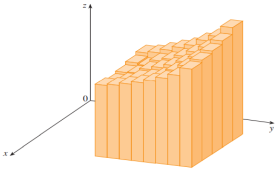

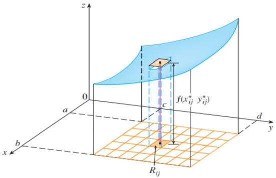

Finding Domains and Ranges of Functions

Function	Domain	Range
$z = \sqrt{y - x^2}$		
$z = \dfrac{1}{xy}$		
$z = \sin xy$		
$w = \sqrt{x^2 + y^2 + z^2}$		
$w = \dfrac{1}{x^2 + y^2 + z^2}$		
$w = xy \ln z$		

Finding Domains and Ranges of Functions

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	
$z = \dfrac{1}{xy}$	$xy \neq 0$	
$z = \sin xy$	Entire plane	
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	
$w = xy \ln z$	Half-space $z > 0$	

Finding Domains and Ranges of Functions

Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
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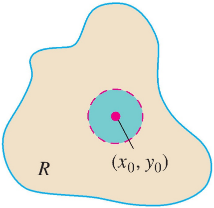
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

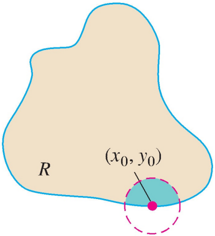
Interior and Boundary Points

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R.

A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)



(a) Interior point

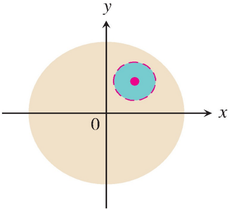


(b) Boundary point

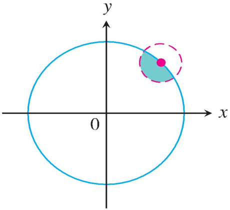
Open and Closed Sets

DEFINITIONS

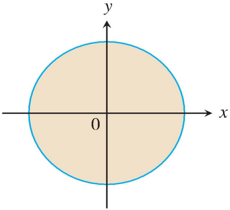
The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk. Every point an interior point.



$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)



$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk. Contains all boundary points.

Bounded and Unbounded Regions in Plane

A region in plane is bounded if it lies inside a disk of fixed radius.

Examples (Bounded Sets in Plane): Line Segments, Triangles, Rectangles, Disks etc.

A region is unbounded if it is not bounded.

Examples (Unbounded Sets in Plane): Lines, Coordinate Axes, Half Planes, Planes etc.

Define the terms for 3D ???

Open Sets in Space: Open Balls, Open Half Space ($z > 0$), Space itself etc.

Closed Sets in Space: Lines, Planes, Closed Half Space ($z \geq 0$) etc.

Neither Open nor Closed: Cube with missing face.

> **DEFINITIONS** The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

EXAMPLE 3 Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Solution The domain of f is the entire xy-plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, the positive portion of which is shown in Figure .

The level curve $f(x, y) = 0$ is the set of points in the xy-plane at which

$$f(x, y) = 100 - x^2 - y^2 = 0, \qquad \text{or} \qquad x^2 + y^2 = 100,$$

which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ are the circles

$$f(x, y) = 100 - x^2 - y^2 = 51, \qquad \text{or} \qquad x^2 + y^2 = 49$$

$$f(x, y) = 100 - x^2 - y^2 = 75, \qquad \text{or} \qquad x^2 + y^2 = 25.$$

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.)

If $x^2 + y^2 > 100$, then the values of $f(x, y)$ are negative. For example, the circle $x^2 + y^2 = 144$, which is the circle centered at the origin with radius 12, gives the constant value $f(x, y) = -44$ and is a level curve of f. ■

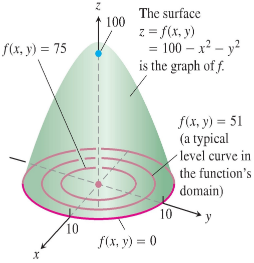


FIGURE The graph and selected level curves of the function $f(x, y)$ in Example 3.

The curve in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represents a function $f(x, y) = c$, called the contour line.

Level Surface of function of three variables

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called level surface of f.

Describe the level surface of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

Problems

Find **a.** domain **b.** range **c.** describe function's level curves **d.** find the boundary of the function's domain **e.** determine if the domain is open, closed or neither **f.** decide if the domain is bounded or unbounded.

1. $f(x, y) = \sqrt{y - x}$
2. $f(x, y) = \ln(x^2 + y^2)$
3. $f(x, y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$

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Solutions:

1. $y-x \geq 0$; $z \geq 0$ i.e., $[0,\infty)$; $y-x=c, c \geq 0$; $y=x$; closed; unbounded

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2. $(x, y) \neq (0, 0)$; all real no.s; $x^2 + y^2 = e^c$, $r > 0$; single point $(0, 0)$; open; unbounded

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2. $(x, y) \neq (0, 0)$; all real no.s; $x^2 + y^2 = e^c$, $r > 0$; single point $(0, 0)$; open; unbounded
3. $x^2 + y^2 < 16$; $z \geq 1/4$; $x^2 + y^2 = 16 - c^2$ i.e., circles centered at origin with radius < 4; $x^2 + y^2 = 4$; open; bounded

Problems

Find an equation for the level curve/surface of the given function which passes through the given point

1. $f(x, y) = 16 - x^2 - y^2$ at $(2\sqrt{2}, \sqrt{2})$

   Ans: $x^2 + y^2 = 10.$

2. $f(x, y) = \sqrt{x - y} - \ln z$ at $(3, -1, 1)$

   Ans: $f(x, y) = \sqrt{x - y} - \ln z = 2.$

2. $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ at $(-1, 2, 1)$

   Ans: $x^2 + y^2 + z^2 = 6.$

Limits in Higher Dimensions

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

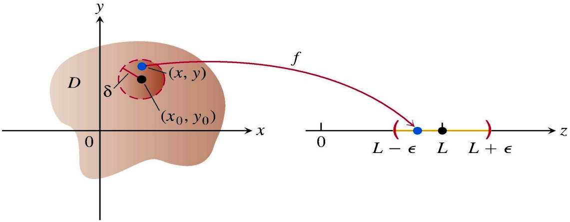

FIGURE In the limit definition, δ is the radius of a disk centered at (x_0, y_0). For all points (x, y) within this disk, the function values $f(x, y)$ lie inside the corresponding interval $(L - \epsilon, L + \epsilon)$.

Limits in Higher Dimensions

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \qquad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,

   n a positive integer, and if n is even, we assume that $L > 0$.

Problems

Evaluate the following limits:

1. $$\lim_{(x,y)\to(0,1)} \frac{x+y}{2xy+x+y}$$
2. $$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$$
3. $$\lim_{\substack{(x,y)\to(2,2)\\ x-y\neq 0}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$
4. $$\lim_{(x,y)\to(\pi/2,0)} \frac{\cos y+1}{y-\sin x}$$
5. $$\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2-xy+4x^2-4x}$$
6. $$\lim_{P\to(\pi/2,\pi/2,0)} \sin^2 x+\sin y\cos z$$

Continuity

Definition

A function $f(x, y)$ is continuous at the point (x_0, y_0) if

i. f is defined at (x_0, y_0)

ii. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists.

iii. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

A function is continuous if it is continuous at every point of the domain.

Two - Path Test for Non-existence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0) then.

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

does not exist.

Examples

Show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous at origin. (Limit of the function does not exist at origin)

Along the path, $y = mx$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x.mx}{x^2+(mx)^2} = \frac{2m}{1+m^2}$$

For $k = 1$ i.e., along the path $y = x$, the limit is 1 and for $k = -1$ i.e., along the path $y = -x$, the limit is -1.

By Two Path Test, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist at origin.

Hence, the function is not continuous at origin though $f(0,0) = 0$.

Problems

A. Find the limits of the given functions at origin if they exist.

1. $f(x, y) = \dfrac{x^4 - y^2}{x^4 + y^2}$
2. $f(x, y) = \dfrac{x^2 - y^6}{x^2 + y^6}$
3. $f(x, y) = \dfrac{x}{x^2 + y^2}$ (Change to polar form)

B. At what points, given functions are continuous?

1. $f(x, y) = \sin(x + y)$

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1. $f(x, y) = \sin(x + y)$ $\qquad \rightarrow$ all (x, y)
2. $f(x, y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$

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3. $f(x,y,z) = \dfrac{1}{|y| + |z|}$

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3. $f(x,y,z) = \dfrac{1}{|y| + |z|}$ $\rightarrow$ all (x,y,z) except $(x,0,0)$