General Physics I (PHYS 101)

Lecture 01

Dynamics of system of particles

Keshav Raj Sigdel

Assistant Professor

Department of Physics

Kathmandu University

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Outline

Potential Energy

The energy stored in a body or system by virtue its position or configuration.

For eg: a stretched catapult, water collected in dam

Depending on the nature of force operating in the system the potential energy of the system can be different types:

(a) Gravitational potential energy

(b) Elastic potential energy

(c) Electrostatic potential energy

The potential energy is a function of position whose negative derivative gives the force

$$F(x) = -\frac{dU(x)}{dx}$$

Potential Energy (contd.)

When work is done in a system by a conservative force the configuration of its parts change and so the potential energy from initial value U_i to its final value U_f. The change in potential energy due to change in configuration $\Delta U = U_f - U_i$
In an isolated system in which conservative force acts

$$K + U = \text{constant} \implies \Delta K + \Delta U = 0 \implies \Delta U = -\Delta K \tag{1}$$

According work-energy theorem

$$\Delta K = W \tag{2}$$

From equation(3) and equation(4)

$$\boxed{\Delta U = -W}$$

Expression for Potential Energy at a point:

We have $\Delta K = -\Delta U$

Work done by resultant force

$W = \int f(x)dx = -\Delta U \implies \Delta U = -\int f(x)dx$

So gain in potential energy is equal to work done against the conservative force.

While stretching the spring, work done against the elastic force is equal to change in potential energy.

i.e. $\Delta U = U(b) - U(a) = -\int_a^b f(x)dx$

$\implies U(b) = -\int_a^b f(x)dx + U(a)$

We assume the potential energy at the initial position $x = a$, $U(a) = 0$.

Then $U(b) = -\int_a^b f(x)dx$

Expression for Potential Energy at a point: (contd.)

So potential energy of a body (system) at a point is the amount of work done against the conservative force in taking the body (system) fromposition where potential energy is taken to be zero $(x = a, U(a) = 0)$ to the present position of the point at $x = b$.

Conservative force as negative gradient of potential energy:

We have under the action of conservative force mechanical energy remains constant.

$K + U = \text{constant} \implies \Delta K + \Delta U = 0$

$\implies \Delta U = -\Delta K$ = work done against conservative force

$= -\int_{x_0}^{x} F_x dx - \int_{y_0}^{y} F_y dy - \int_{z_0}^{z} F_z dz$

In vector form

$-\int_{r_0}^{r} dU = \int_{r_0}^{r} \vec{F}.d\vec{r}$

Where r_0 refers to the position of zero potential energy.

$\implies -dU = \vec{F}.d\vec{r}$

$\implies -dU = (\hat{i}F_x + \hat{j}F_y + \hat{k}F_z).(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$\implies -dU = F_x dx + F_y dy + F_z dz$

Conservative force as negative gradient of potential energy: (contd.)

If we keep y and z constant and let only x changes then using the partial differential. We get

$F_x = -\dfrac{\partial U}{\partial x}$

Similarly, $F_y = -\dfrac{\partial U}{\partial y}$ (keeping z and x constant)

and $F_z = -\dfrac{\partial U}{\partial z}$ (keeping x and y constant)

Hence the force $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z = -\left(\hat{i}\dfrac{\partial U}{\partial x} + \hat{j}\dfrac{\partial U}{\partial y} + \hat{k}\dfrac{\partial U}{\partial z}\right)$

The quantity $\left(\hat{i}\dfrac{\partial U}{\partial x} + \hat{j}\dfrac{\partial U}{\partial y} + \hat{k}\dfrac{\partial U}{\partial z}\right)$ is called the gradient of U or grad$U$ or ∇U or (del$U$). The symbol ∇ (del) stands

Conservative force as negative gradient of potential energy: (contd.)

$\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)$ is a vector operator that converts a scalar function into a vector function.

Then $\vec{F} = -\text{grad}U \implies \vec{F} = -\nabla U$.

i.e. Conservative force $\vec{F}$ is equal to negative gradient of potential energy.

Conservation of energy

Conservation of mechanical energy Consider a particle is acted by both conservative and non-conservative forces. Let, W_c be the work done by the conservative forces and W_n is that of by non-conservative forces. Hence the total work done by both the forces is

$$W = W_c + W_n \tag{3}$$

But from the work-kinetic energy theorem,

$$W = \Delta K \tag{4}$$

Conservation of energy (contd.)

And from the definition of potential energy

$$W_c = -\Delta U \tag{5}$$

Using Equations (3), (4) and (5), we get

$$\begin{aligned} \Delta K &= -\Delta U + W_n \\ \implies W_n &= \Delta K + \Delta U \\ \implies W_n &= \Delta E_0 \end{aligned} \tag{6}$$

where, $E_0 = K + U$ is the mechanical energy and ΔE_0 is the change in mechanical energy. Hence, **work done**

by non-conservative forces is equal to the change in mechanical energy.

If non-conservative forces do not act or contribute to the work done, i.e. W_n =0, then

$$\Delta E_0 = 0 \implies E_0 = \text{constant}$$

That means, if a body is not acted by non-conservative forces or non-conservative forces do not contribute to the work done, then the total mechanical energy of the body remains conserved.

Conservation of total energy Work done against non-conservative forces results to change the other form of energy except mechanical energy, i.e.

$$-W_n = \Delta Q \tag{7}$$

where, Q is the other form of energy, e.g. heat, sound, light, etc., except mechanical energy.

From Equation (6) and (7)

$$-\Delta Q = \Delta E_0 \implies \Delta E_0 + \Delta Q = 0$$

$$\implies \Delta K + \Delta U + \Delta Q = 0 \implies \Delta K + \Delta U = -\Delta Q$$

Conservation of energy (contd.)

$$\text{Change in mechanical energy} = -\Delta Q$$

$$\implies K + U + Q = \text{constant}$$

Thus under the action of non-conservative force mechanical energy changes into other forms of energy such as heat or some other form of energy. But the total energy of the isolated system remains constant.

Energy can not be created nor be destroyed but can be changed from one form to other. So total energy of the isolated system remains constant. This is the principle of conservation of energy.

Center of mass

When a body moves every particle covers the same displacement in the same interval of time by the external force then the motion is called translational motion.

When a body rotates or vibrates as it moves there is one point on the body called the center of mass that move in the same way that a single particle subjected to the same external force would move.

Thus, the center of mass of a body is a point at which whole mass of the body is supposed to be concentrated and when the line of action of a force passes through the point then the body accelerates without rotation. The center of mass of a body represents purely the translation motion of the body even though the body rotates during motion.

Consider a system of n-point masses m_1, m_2, ..., m_n are located on x-axis with positions x_1, x_2, ..., x_n, respecitively, as shown in figure 1.

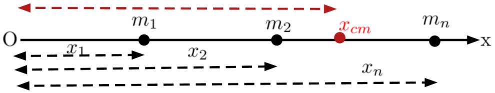

Figure 1: Linear distribution of point masses

The center of mass of the system is defined as

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \tag{8}$$

or

$$x_{cm} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i \tag{9}$$

where, $M = \sum_{i=1}^{n} m_i$ is the total mass of the system.

Center of mass for point mass distribution (contd.)

If the point masses are distributed in xy-plane with coordinates (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), respectively, then position of center of mass of the system (x_{cm}, y_{cm}) is defined as

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

$$y_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i$$

Center of mass for point mass distribution (contd.)

Similarly, if the point masses are distributed in space, the position of center of mass is given by

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \quad y_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \quad z_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

In vector notation, each particle can be described by position vector as $\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ and the position vector of center of mass $\vec{r}_{cm}$ is

Center of mass

Center of mass for point mass distribution (contd.)

given by

$$\begin{aligned}\vec{r}_{cm} &= x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k} \\ &= \left(\frac{1}{M}\sum_{i=1}^{n} m_i x_i\right)\hat{i} + \left(\frac{1}{M}\sum_{i=1}^{n} m_i y_i\right)\hat{j} + \left(\frac{1}{M}\sum_{i=1}^{n} m_i z_i\right)\hat{k} \\ &= \frac{1}{M}\sum_{i=1}^{n} m_i\left(x_i\hat{i} + y_i\hat{j} + z_i\hat{k}\right) = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i\end{aligned}$$

Therefore, in term of position vector the center of mass is given by

$$\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i \tag{10}$$

Center of mass

Center of mass for point mass distribution (contd.)

For a continuous body, the center of mass can be obtained by subdividing the body into n-numbers of small elements by mass Δm_i located approximately at point (x_i, y_i, z_i).

The co-ordinate of center of mass are given approximately by

$$x_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i x_i}{\sum_{i=1}^{n} \Delta m_i}, \quad y_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i y_i}{\sum_{i=1}^{n} \Delta m_i}, \quad z_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i z_i}{\sum_{i=1}^{n} \Delta m_i}$$

If the mass element Δm_i tends to zero and the number of elements n tends to infinity, then the co-ordinate of centre of mass are now given more precisely as

$$x_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum\limits_{i=1}^{n} \Delta m_i x_i}{\sum\limits_{i=1}^{n} \Delta m_i}, \quad y_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum\limits_{i=1}^{n} \Delta m_i y_i}{\sum\limits_{i=1}^{n} \Delta m_i}, \quad z_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum\limits_{i=1}^{n} \Delta m_i z_i}{\sum\limits_{i=1}^{n} \Delta m_i}$$

Thus

$$x_{cm} = \frac{\int x dm}{\int dm} = \frac{1}{M} \int x dm, \quad y_{cm} = \frac{1}{M} \int y dm, \quad z_{cm} = \frac{1}{M} \int z dm$$

Where dm is the differential element of mass and $\int dm$ total mass of the object.

In vector form

$$\begin{aligned}\vec{r}_{cm} &= x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k} \\ &= \frac{1}{M}\int x dm\hat{i} + \frac{1}{M}\int y dm\hat{j} + \frac{1}{M}\int z dm\hat{k} \\ &= \frac{1}{M}\int \left(x\hat{i} + y\hat{j} + z\hat{k}\right) dm = \frac{1}{M}\int \vec{r} dm\end{aligned}$$

Consider the motion of group of particles whose masses are $m_1, m_2, ...m_n$ and whose total mass is M.

Now, equation of center of mass can be written as

$$M\vec{r_{cm}} = m_1\vec{r_1} + m_2\vec{r_2} + ... + m_n\vec{r_n} \tag{11}$$

Where $\vec{r_{cm}}$ is the position vector of center of mass.

Differentiating equation (11) with respect to time t we get

$$M\frac{d\vec{r_{cm}}}{dt} = m_1\frac{d\vec{r_1}}{dt} + m_2\frac{d\vec{r_2}}{dt} + ... + m_n\frac{d\vec{r_n}}{dt}$$

Center of mass

Motion of center of mass (contd.)

$$\implies M\vec{v_{cm}} = m_1\vec{v_1} + m_2\vec{v_2} + ... + m_n\vec{v_n} \tag{12}$$

Where $\vec{v_{cm}}$ is the velocity of center of mass and $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ are the velocities of individual particles.

Differentiating (12) with respect to time t we get

$$M\frac{d\vec{v_{cm}}}{dt} = m_1\frac{d\vec{v_1}}{dt} + m_2\frac{d\vec{v_2}}{dt} + ... + m_n\frac{d\vec{v_n}}{dt}$$

$$\implies M\vec{a_{cm}} = m_1\vec{a_1} + m_2\vec{a_2} + ... + m_n\vec{a_n} \tag{13}$$

Where $\vec{a_{cm}}$ is the acceleration of center of mass and $\vec{a}_1, \vec{a}_2, \ldots.., \vec{a}_n$ are the acceleration of individual particles.

From Newton's second law $\vec{F} = m\vec{a}$ equation (13) can be written as

$$\implies M\vec{a_{cm}} = \vec{F}_1 + \vec{F}_2 + \ldots\ldots + \vec{F}_n \tag{14}$$

$\vec{F}_1, \vec{F}_2, \ldots.., \vec{F}_n$ are the forces acting on the individual particles.

i.e. the total mass of group of particle times the acceleration of center of mass is equal to the vector sum of all the forces acting on the group of velocity.

Equation (14) can be written as $M\vec{a_{cm}} = \vec{F_{ext}}$

This states that the center of mass of a system of particles moves if all the mass of the system were concentrated at the center of mass and all the external forces were applied at that point.

If $\vec{F_{ext}} = 0$, then $M\vec{a_{cm}} = 0 \implies \vec{a_{cm}} = 0 \implies \vec{v_{cm}} = \text{constant}$

Under such condition the center of mass either remains at rest or moves with constant speed.