General Physics I (PHYS 101)

Lecture 08

Rotational Dynamics

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Outline

1. Perpendicular and Parallel axes theorem of Moment of Inertia
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5. Moment of inertia of a solid sphere

Perpendicular and Parallel axes theorem of Moment of Inertia

Perpendicular axes theorem (applicable only for plane lamina)

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia of the lamina about any two mutually perpendicular axes, passing through its own plane, intersecting each other at the point through which the perpendicular axis passes.

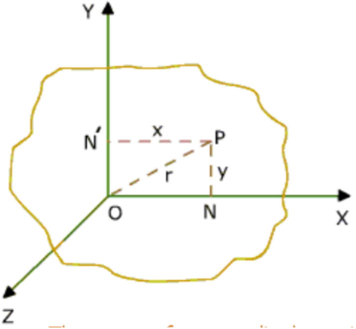

Figure 1: Theorem of perpendicular axis

Proof:-

Let us consider a plane lamina lying in the XOY plane. The lamina is made up of a large number of particles. Consider a small mass element dm at P. From P, PN and PN$'$ are drawn perpendicular to X-axis and Y-axis, respectively. Now PN$' = x$, PN $= y$

Moment of inertia of the whole of lamina about X- axis is

$$I_x = \int y^2 dm \tag{1}$$

Moment of inertia of the whole of lamina about Y- axis is

$$I_y = \int x^2 dm \tag{2}$$

Moment of inertia of the whole of lamina about Z-axis is

$$I_z = \int r^2 dm \tag{3}$$

But $r^2 = x^2 + y^2$, so that

$$I_z = \int \left(x^2 + y^2\right) dm = \int x^2 dm + \int y^2 dm = I_y + I_x$$

hence

$$I_z = I_x + I_y \tag{4}$$

Perpendicular and Parallel axes theorem of Moment of Inertia

Parallel axes theorem

The moment of inertia of a body about an axis is equal to the moment of inertia of the body about the axis through center of mass and parallel to the given axis plus the product of total mass of the body and square of the perpendicular distance between the axes.

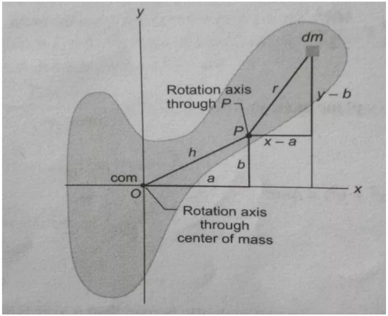

Figure 2: Theorem of parallel axis

If I be the moment of inertia of a body about an axis, I_{com} is the moment of inertia of the body about the axis passes through center of mass and parallel to the given axis, M be the total mass, and h be the perpendicular distance between the axes, then

$$I = I_{com} + Mh^2 \tag{5}$$

Proof:-

Let O be the center of mass of any arbitrarily shaped rigid body of total mass M. We place the origin at this point O. Now, we consider an axis passing through perpendicular to the plane of the body and another axis passing through point P parallel to the first axis. Let the coordinates at P be (a, b).

Let dm be an element of mass with general coordinates (x, y). Let r represents the perpendicular distance of elemental mass dm from the point P. Also, the line OP in xy-plane is perpendicular distance between two parallel axes, which is equal to h.

The moment of inertia of the rigid body about the axis passing through point P is given by

$$I = \int r^2 dm = \int \left\{ (x-a)^2 + (y-b)^2 \right\} dm$$
$$= \int (x^2 - 2xa + a^2 + y^2 - 2yb + b^2) dm$$

Rearranging, we have

$$I = \int (x^2 + y^2) dm + \int (a^2 + b^2) dm - 2a \int x dm - 2b \int y dm \qquad (6)$$

However, the coordinates of the center of mass by definition are given as $x_{cm} = \dfrac{1}{M}\int xdm$ and $y_{cm} = \dfrac{1}{M}\int ydm$.
But, x- and y-coordinates of center of mass are zero as it lies on z-axis. It means that $x_{cm} = \dfrac{1}{M}\int xdm = 0 \implies \int xdm = 0$, and similarly $\int ydm = 0$
Thus, the equation (6) for the moment of inertia of the rigid body about the axis parallel to an axis passing through center of mass 0 is

$$I = \int (x^2 + y^2)dm + \int (a^2 + b^2)dm \tag{7}$$

From the figure 2, $x^2 + y^2 = R^2$ where R is the distance from 0 to dm and $a^2 + b^2 = h^2$.

Substituting in the equation (7) , we have

$$I = \int R^2 dm + \int h^2 dm \tag{8}$$

We, however, note that R is variable, but h is constant. Taking the constant out of the integral sign

$$I = \int R^2 dm + h^2 \int dm = \int R^2 dm + Mh^2 \tag{9}$$

The integral on right hand side is the expression of moment of inertia of the rigid body about the axis passing through center of mass.

Hence,

$$I = I_{com} + Mh^2 \tag{10}$$

Where the term I_{com} is the moment of inertia/rotational inertia of the rigid body about an axis through its center of mass.

Consider a slender rod of length L and mass M. AB is the axis through center of mass (mid point) and perpendicular to the rod, and A'B' is the axis through one end of the rod and perpendicular to the rod. To find the moment of inertia about AB, the rod can be divided into a number of pieces with elemental length dx.

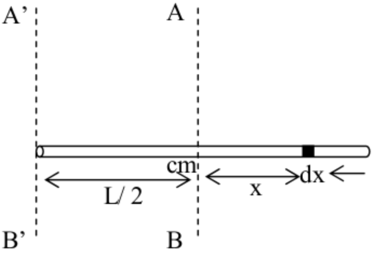

Figure 3: A slender rod

Let's take elemental length dx at a distance x from center of mass.

The mass of the elemental length dx is $dm = \frac{M}{L} dx$

Moment of inertia of a slender rod (contd.)

The moment of inertia of the slender rod about AB is

$$I_{cm} = \int_{-L/2}^{+L/2} x^2 dm = \int_{-L/2}^{+L/2} x^2 \frac{M}{L} dx = \frac{M}{L} \int_{-L/2}^{+L/2} x^2 dx = \frac{M}{L} \left[\frac{x^3}{3} \right]_{-L/2}^{L/2}$$

$$= \frac{M}{3L} \left(\frac{L^2}{8} + \frac{L^3}{8} \right) = \frac{1}{12} ML^2$$

Using parallel axes theorem, the moment of inertia about the axis A'B' is

$$I_{\text{A'B'}} = I_{cm} + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2$$

Therefore, the radius of gyration about AB is $K_{AB} = \sqrt{\frac{I_{cm}}{M}} = \frac{L}{2\sqrt{3}}$ and about A'B' is $K_{A'B'} = \frac{L}{\sqrt{3}}$

Moment of inertia of circular ring

Consider a circular ring of radius R and mass M on xy-plane with center at origin O. Here z-axis is perpendicular to the plane of the ring and passing through center of mass of the ring. To calculate the moment of inertia of the ring about the z-axis, let's take a elemental mass dm on the ring. The mass dm lies at a distance R from z-axis.

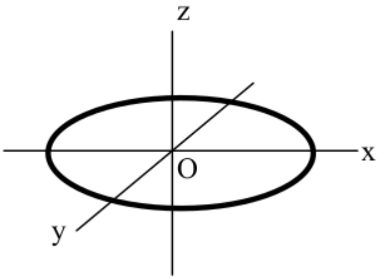

Figure 4: Moment of inertia of ring

Moment of inertia of circular ring (contd.)

The moment of inertia of the ring about z-axis is

$$I_z = \int R^2 dm$$

Since the distance R is the same for all elemental mass dm, it can be taken out side from integral sign. So that

$$I_z = R^2 \int dm = R^2 M = MR^2 \tag{11}$$

The ring is symmetrical about both x- and y- axes. So the moment of inertia about x- and y-axes are the same i.e. $I_x = I_y$. Using perpendicular axes theorem of moment of inertia

$$I_x + I_y = I_z \implies I_x + I_x = MR^2 \implies 2I_x = MR^2 \implies I_x = \frac{1}{2}MR^2$$

Moment of inertia of circular ring (contd.)

Hence the moment of inertia of the ring about x- or y-axis i.e. diameter is $\frac{1}{2}MR^2$

Using the parallel axes theorem the moment of inertia about a tangent is

$$I_T = I_x + MR^2 = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$$

Moment of inertia of homogenous circular disc

Consider a circular disc of radius R and mass M on xy-plane with center at origin O. As result z-axis is perpendicular to the disc and passing through the center of mass O of the disc. To calculate the moment of inertia about the z-axis, the disc can be divided into a number of concentric circular ring each of elemental thickness dr.

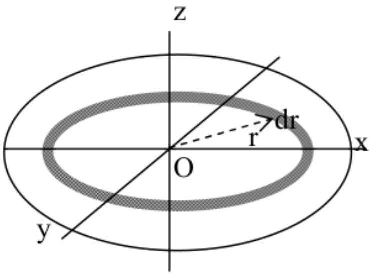

Figure 5: Moment of inertia of disc

Let's take a ring of elemental thickness dr and radius r. The elemental

Moment of inertia of homogenous circular disc (contd.)

area of the ring is $da = 2\pi r dr$ and the elemental mass

$$dm = \frac{M}{\pi R^2} da = \frac{M}{\pi R^2} 2\pi r dr = \frac{2M}{R^2} r dr$$

The moment of inertia of the ring about the z-axis is

$$dI_z = dm r^2 = \left(\frac{2M}{R^2} r dr \right) r^2 = \frac{2M}{R^2} r^3 dr$$

The moment of inertia of the whole disc about the z-axis is

$$I_z = \frac{2M}{R^2} \int_0^R r^3 dr = \frac{2M}{R^2} \frac{R^4}{4} = \frac{1}{2} M R^2$$

Moment of inertia of homogenous circular disc (contd.)

The disc is symmetrical about both x- and y- axes. So the moment of inertia about x- and y-axes are the same i.e. $I_x = I_y$ Using perpendicular axes theorem of moment of inertia

$$I_x + I_y = I_z \implies I_x + I_x = \frac{1}{2}MR^2 \implies 2I_x = \frac{1}{2}MR^2 \implies I_x = \frac{1}{4}MR^2$$

Hence the moment of inertia of the ring about x- or y-axis i.e. diameter is $\frac{1}{4}MR^2$

Using the parallel axes theorem the moment of inertia about a tangent is

$$I_T = I_x + MR^2 = \frac{1}{4}MR^2 + MR^2 = \frac{5}{4}MR^2$$

Moment of inertia of a solid sphere

Consider a solid sphere of radius R and mass M with center at origin O as in figure 6. The sphere is symmetric about all axes. So that, the moment of inertia about all axes is the same. To calculate the moment of inertia about z-axis as in figure 6, the sphere can be divided into the coaxial circular discs of various radius each of with elemental thickness dz.

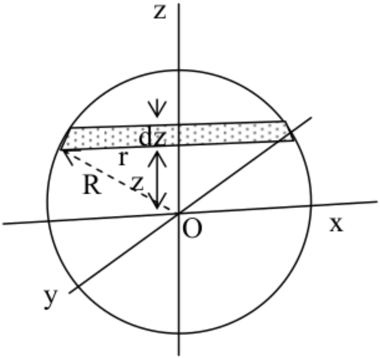

Figure 6: Moment of inertia of solid sphere

let's take a circular disc of radius r at a distance z from O. The disc is

Moment of inertia of a solid sphere (contd.)

parallel to xy-plane. The elemental volume of the disc is $dv = \pi r^2 dz$ and the elemental mass of the disc is

$$dm = \frac{M}{\frac{4}{3}\pi R^3} dv = \frac{3M}{4\pi R^3} \pi r^2 dz = \frac{3M}{4R^3} r^2 dz$$

Here z-axis is perpendicular to the disc and through its center, so the moment of inertia of the disc about z-axis is

$$dI_z = \frac{1}{2}(dm) r^2 = \frac{1}{2}\left(\frac{3M}{4R^3} r^2 dz\right) r^2 = \frac{3M}{8R^3} r^4 dz$$

But from figure 6, $R^2 = r^2 + z^2 \implies r^2 = R^2 - z^2$, therefore

$$dI_z = \frac{3M}{8R^3}\left(r^2\right)^2 dz = \frac{3M}{8R^3}\left(R^2 - z^2\right)^2 dz = \frac{3M}{8R^3}\left(R^4 - 2R^2 z^2 + z^4\right) dz$$

Moment of inertia of a solid sphere (contd.)

Now the moment of inertia of the whole solid sphere about z-axis i.e. diameter is

$$I_z = \frac{3M}{8R^3}\int_{-R}^{R}\left(R^4 - 2R^2z^2 + z^4\right)dz$$

$$= \frac{3M}{8R^3}\left[R^4 z - 2R^2\frac{z^3}{3} + \frac{z^5}{5}\right]_{-R}^{R}$$

$$= \frac{3M}{8R^3}\left(R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5 + R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right)$$

$$= \frac{3M}{8R^3}R^5\left(2 - \frac{4}{3} + \frac{2}{5}\right) = \frac{3}{8}MR^2\left(\frac{30-20+6}{15}\right) = \frac{3}{8}MR^2\frac{16}{15}$$

$$\therefore I_z = \frac{2}{5}MR^2$$

Moment of inertia of a solid sphere (contd.)

Using parallel axis theorem, the moment of inertia about a tangent is

$$I_T = I_z + MR^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$