Lecture 01

Vector Analysis

Outline

1. Scalars
2. Vectors
   - Negative of a vector
3. Four Vector Operations
   - Addition of Two Vectors
   - Multiplication by a Scalar
   - Dot (or Scalar) Product of Two Vectors
   - Cross (or Vector) Product of Two Vectors
4. Vector Algebra: Component Form
   - Addition of Two Vectors
   - Multiplication by a Scalar

Outline (contd.)

- Dot Product of Two Vectors
- Cross Product of Two Vectors

5. Triple products
   - Scalar triple product
   - Vector triple product

Scalars

Scalars have magnitude only. They are specified by a number with a unit (e.g. $10^0 C$) and obey the rules of arithmetic and ordinary algebra. Examples: mass, temperature, charge, electric potential, work, energy etc.

Vectors

Vectors have both magnitude and direction (5m, north) and obey the rules of vector algebra. Examples: displacement, velocity, force, momentum, torque, electric field, magnetic field etc In diagrams, vector is denoted by arrow: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction. In texts, we shall denote a vector by putting an arrow over the letter ($\vec{A}, \vec{B}$, and so on). The magnitude of a vector $\vec{A}$ is written $\left|\vec{A}\right|$ or more simply A.

Minus $\vec{A}$ ($-\vec{A}$) is a vector with the same magnitude as $\vec{A}$ but of opposite direction [Figure 1].

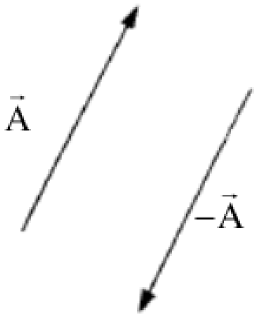


Figure 1

- Place the tail of $\vec{B}$ at the head of $\vec{A}$; the sum, $\vec{A}+\vec{B}$, is the vector from the tail of $\vec{A}$ to the head of $\vec{B}$

Figure 2

- **Triangle Law of Vector Addition**

  If two sides of a triangle taken in the same order represent the two vectors in magnitude and direction, then the third side in the opposite order represents the resultant of two vectors.

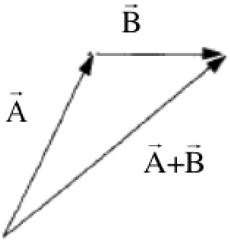
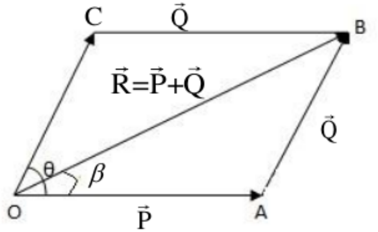


Figure 3

- **Parallelogram Law of Vector Addition** If two vectors are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, then their resultant is given in magnitude and direction by the diagonal of the parallelogram passing through that point.



- $R = \left|\vec{P}+\vec{Q}\right| = \sqrt{P^2+Q^2+2PQ\cos\theta}$
- $\tan\beta = \dfrac{Q\sin\theta}{P+Q\cos\theta}$

- Addition is commutative: $\vec{A}+\vec{B} = \vec{B}+\vec{A}$
- Addition is associative: $\left(\vec{A}+\vec{B}\right)+\vec{C} = \vec{A}+\left(\vec{B}+\vec{C}\right)$

Four Vector Operations:- Multiplication by a Scalar

- Multiplication of a vector by a positive scalar a multiplies the magnitude but leaves the direction unchanged.(If a is negative, the direction is reversed)

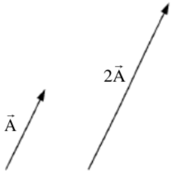


Figure 4

- Scalar multiplication is distributive: $a\left(\vec{A}+\vec{B}\right)=a\vec{A}+a\vec{B}$

Dot (or Scalar) Product of Two Vectors

- The dot product of two vectors is defined by

$$\vec{A} \cdot \vec{B} \equiv AB\cos\theta \tag{1}$$

and is a scalar. Here θ is the angle they form when placed tail-to-tail as shown in Figure 5. For example, work done by a force i.e. $W = \vec{F} \cdot \vec{S}$

Four Vector Operations:-

Dot (or Scalar) Product of Two Vectors (contd.)

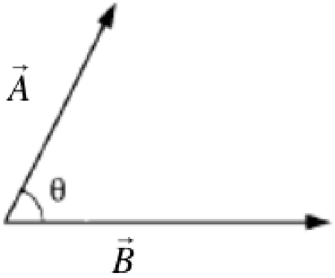


Figure 5

- The dot product is commutative: $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$
- The dot product is distributive: $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

- Geometrically, $\vec{A}\cdot\vec{B}$ is the product of B times the projection of $\vec{A}$ along $\vec{B}$. $\left[\vec{A}\cdot\vec{B} = B(A\cos\theta)\right]$
- If the two vectors are parallel, then $\vec{A}\cdot\vec{B} = AB$. If two vectors are perpendicular, then $\vec{A}\cdot\vec{B} = 0$.
- For any vector $\vec{E}$,

$$\vec{E}\cdot\vec{E} = E^2$$

$$\Rightarrow E = \sqrt{\vec{E}\cdot\vec{E}}$$

Example 1:

Let $\vec{C} = \vec{A} - \vec{B}$ [Figure 6], and calculate $\vec{C} \cdot \vec{C}$

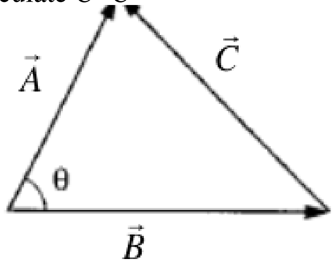


Figure 6

Solution:

$$\vec{C} \cdot \vec{C} = \left(\vec{A} - \vec{B}\right) \cdot \left(\vec{A} - \vec{B}\right)$$

$$= \vec{A} \cdot \vec{A} - \vec{A} \cdot \vec{B} - \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}$$

$$\therefore \quad \boxed{C^2 = A^2 + B^2 - 2AB\cos\theta}$$

This is the **law of cosines**.

Four Vector Operations:-

Cross (or Vector) Product of Two Vectors

- The cross product of two vectors is defined by

$$\vec{A} \times \vec{B} = AB \sin\theta \hat{n} \tag{2}$$

is a vector as an example of torque $[\vec{\tau} = \vec{r} \times \vec{F}]$. Here $\hat{n}$ is a unit vector pointing perpendicular to the plane of $\vec{A} and \vec{B}$ The direction of $\hat{n}$ is determined by using right-hand rule: let your fingers point in the direction of the first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of$\hat{n}$. In Figure 7, $\vec{A} \times \vec{B}$ points into the page; $\vec{B} \times \vec{A}$ points out of the page.

Four Vector Operations:-

Cross (or Vector) Product of Two Vectors (contd.)

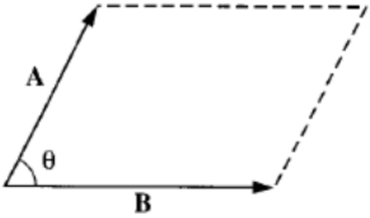


Figure 7

- The cross product is not commutative: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$
- The cross product is distributive: $\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

Four Vector Operations:-

Cross (or Vector) Product of Two Vectors (contd.)

- Geometrically, $\vec{A} \times \vec{B}$ gives the area of the parallelogram generated by $\vec{A}$ (or **A**) and $\vec{B}$ (or **B**) (Figure 7).
- If the two vectors are parallel, then $\vec{A} \times \vec{B} = 0$.
- If two vectors are perpendicular, then $\left|\vec{A} \times \vec{B}\right| = AB$.

Let $\hat{i}, \hat{j}$, and $\hat{k}$ be unit vectors parallel to $x, y,$ and z axes respectively (Figure 8).

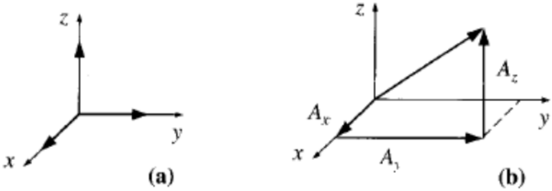


Figure 8

Vectors $\vec{A}$ and $\vec{B}$ can be expressed in terms of basis vectors $\hat{i}, \hat{j}$, and $\hat{k}$ as

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Vector Algebra: Component Form:-

Addition of Two Vectors

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

Multiplication by a Scalar

$$a\vec{A} = (aA_x)\hat{i} + (aA_y)\hat{j} + (aA_z)\hat{k}$$

Dot Product of Two Vectors

$$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_xB_x + A_yB_y + A_zB_z$$

Since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1;$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

For any vector $\vec{A}$: $A = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$

Cross Product of Two Vectors

$$
\begin{aligned}
\vec{A}\times\vec{B} &= (A_x\hat{i}+A_y\hat{j}+A_z\hat{k})\times(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}) \\
&= (A_yB_z-A_zB_y)\,\hat{i}+(A_zB_x-A_xB_z)\,\hat{j}+(A_xB_y-A_yB_x)\,\hat{k} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
\end{aligned}
$$

Cross Product of Two Vectors (contd.)

Since,

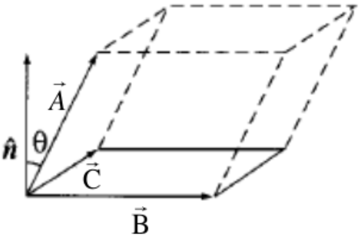
$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \ \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \ \hat{i} \times \hat{k} = -\hat{j}$$

The scalar triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is defined as $\vec{A} \cdot (\vec{B} \times \vec{C})$

- For a parallelepiped generated by $\vec{A}, \vec{B}$ and $\vec{C}$



$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \left|\vec{B}\times\vec{C}\right|(A\cos\theta)$$

$= \text{Area of the base of parallelepiped } \times \text{Altitude of the parallelepiped}$

$= \text{Volume of the parallelepiped generated by } \vec{A},\ \vec{B} \text{ and } \vec{C}$

$\therefore$ Geometrically, $\vec{A}\cdot(\vec{B}\times\vec{C})$ is the volume of the parallelepiped generated by $\vec{A},\vec{B}$ and $\vec{C}$

- $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

- In component form,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

- The dot and cross can be interchanged: $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$

The vector triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is defined as $\vec{A} \times (\vec{B} \times \vec{C})$

- The vector triple product can be simplified by the BAC-CAB rule:

$$\boxed{\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})}$$

End of Lecture 01

Thank you