Beta and Gamma functions

Syllabus: Beta and Gamma functions; Properties of the functions; Transformation of Gamma functions; relations between the functions.

Definition (Beta Function): The Beta function or the first Eulerian integral, denoted by B(m , n) is defined as B(m , n) = $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ (m > 0, n > 0)

Definition (Gamma Function): The Gamma function or the second Eulerian integral, denoted by Γ(p) is defined as Γ(p) = $\int_0^\infty e^{-x}x^{p-1}dx$ (p > 0).

Properties:

1. B(m, n) = B (n, m)

We have B(m, n) = $\int_0^1 x^{m-1}(1-x)^{n-1}dx$. We put 1 – x = t, then dx = - dt, so B(m, n) = $\int_0^1 (1-t)^{m-1}t^{n-1}dt$ = B (n, m)

2. (Another form of Beta function): B(m, n) = $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

For this, we put $x = \frac{y}{1-y}$, so that $1+x = 1+\frac{y}{1-y} = \frac{1}{1-y}$. Therefore, $dx = \frac{1}{(1-y)^2}dy$.

Also x = 0 implies y = 0 and x → ∞ ⇒ y =1.

Therefore $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^1 \frac{y^{m-1}}{(1-y)^{m-1}}\frac{(1-y)^{m+n}}{(1-y)}dy$ = $\int_0^1 y^{m-1}(1-y)^{n-1}dy$ = B (m, n)

3. Γ(n + 1) = nΓ(n)

We have Γ(n + 1) = $\int_0^\infty e^{-x}x^n dx$ = $\lim_{b\to\infty}[-x^n e^{-x}]_0^b + n\int_0^\infty e^{-x}x^{n-1}dx$

But $\lim_{b\to\infty}\frac{b^n}{e^b} = 0$ (since n > 0). Therefore, Γ(n + 1) = n Γ(n).

In particular, if n is a positive number, we have Γ(n + 1) = n (n - 1) (n - 2) …3. 2. 1 Γ(1) = n! [Since Γ(1) = $\int_0^\infty e^{-x}dx$ = $\lim_{b\to\infty} -e^{-x}]_0^b$ = 1].

Therefore Γ(n + 1) = n! (If n is a positive integer).

4. **Relation between Beta and Gamma Functions:** B (m, n) = $\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

We have $\dfrac{\Gamma(m)}{z^m} = \int_0^\infty e^{-zx} x^{m-1} dx$ (To get it put x = zy in Γ(m)) ⇒Γ(m) = $\int_0^\infty z^m e^{-zx} x^{m-1} dx$

$\Rightarrow \Gamma(m) e^{-z} z^{n-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1} dx$

Now integrating with respect to z from 0 to ∞, we have

$\Gamma(m)\int_0^\infty e^{-z} z^{n-1} dz = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} dz\right] x^{m-1} dx$

$\Rightarrow \Gamma(m)\Gamma(n) = \int_0^\infty \dfrac{\Gamma(m+n)}{(1+x)^{m+n}}\, x^{m-1} dx$ = Γ(m + n) B (m , n). Hence B (m , n) = $\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

\# **Γ(m) Γ(1 - m) =** $\dfrac{\pi}{\sin m\pi}$ **(0 < m < 1) (Without proof).**

If m = ½, then $\left[\Gamma\left(\dfrac{1}{2}\right)\right]^2 = \pi$. Therefore, $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(\dfrac{3}{2}\right) = ?$, $\Gamma\left(\dfrac{9}{2}\right) = ?$

5. $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{q+1}{2}\right)}{2\Gamma\left(\dfrac{p+q+2}{2}\right)}$

We put $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$. Therefore,

$\int_0^1 x^{m-1}(1-x)^{n-1} dx = 2\int^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1} d\theta$

$\Rightarrow$ B (m, n) = $2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$. But B (m, n) = $\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, so

$\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \dfrac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$. Now taking p = 2m –1, q = 2n – 1, the above result may be expressed as follows:

$\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \dfrac{\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{q+1}{2}\right)}{2\Gamma\left(\dfrac{p+q+2}{2}\right)}$.

In particular, if we take q = 0, we get

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \, d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} = \int_0^{\frac{\pi}{2}} \cos^p \theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right)$$

6. B (m, n) = $\frac{n-1}{m+n-1} B(m, n-1) = \frac{m-1}{m+n-1} B(m-1, n)$

Proof: By definition

B (m, n) = $\int_0^1 x^{m-1}(1-x)^{n-1} dx$

$$= (1-x)^{n-1} \frac{x^m}{m}]_0^1 + \int_0^1 (n-1)(1-x)^{n-2} \frac{x^m}{m} dx \text{ (using integration by parts)}$$

$$= \frac{n-1}{m} \int_0^1 \left[x^m (1-x)^{n-2} \right] dx$$

$$= \frac{n-1}{m} \int_0^1 \left[x^{m-1} - x^{m-1}(1-x) \right] (1-x)^{n-2} dx$$

$$= \frac{n-1}{m} \int_0^1 x^{m-1}(1-x)^{n-2} dx - \frac{n-1}{m} \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$= \frac{n-1}{m} \int_0^1 x^{m-1}(1-x)^{n-2} dx - \frac{n-1}{m} B(m, n)$$

$\Rightarrow$ B(m, n) + $\frac{n-1}{m} B(m,n) = \frac{n-1}{m} \int_0^1 x^{m-1}(1-x)^{n-2} dx$

$\Rightarrow \frac{m+n-1}{m} B(m,n) = \frac{n-1}{m} B(m, n-1)$

$\Rightarrow B(m,n) = \frac{n-1}{m+n-1} B(m, n-1)$

But B(m, n) = B (n, m), so

$$B(m,n) = \frac{n-1}{m+n-1} B(m, n-1) = \frac{m-1}{m+n-1} B(m-1, n)$$

In particular, is n is a positive integer, then

$$B(m,n) = \frac{n-1}{m+n-1} B(m, n-1)$$

$$= \frac{n-1}{m+n-1}\frac{n-2}{m+n-2}\cdots\frac{1}{m+1}B(m,1) \text{ (using formula repeatedly)}$$

$$= \frac{(n-1)!}{(m+n-1)(m+n-2)\cdots(m+1)}B(m,1)$$

$$= \frac{(n-1)!}{(m+n-1)(m+n-2)\cdots(m+1)}\frac{1}{m} \text{ (since B(m, 1) = 1/m)}$$

Similarly if m is a positive integer, then

$$\text{B (m, n)} = \frac{(m-1)!}{n(n+1)(n+2)\cdots(m+n-1)}$$

If both m and n are positive integers, then

$$\text{B(m, n)} = \frac{(n-1)!}{m(m+1)(m+2)\cdots(m+n-1)}$$

$$= \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Example: $B(5, 3) = \frac{(5-1)!(3-1)!}{(5+3-1)!} = \frac{4!2!}{7!}$

Some important properties

1. $\int_0^{\frac{\pi}{2}} \sin^p\theta\, d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)} = \int_0^{\frac{\pi}{2}} \cos^p\theta\, d\theta$

2. $\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

3. $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Problems on Beta and Gamma functions:

1. Evaluate: B (7, 4), Γ(8), Γ(3/2), Γ(5/2), Γ(7/2)

2. Prove that: (a) Γ(1/4) Γ(3/4) = $\sqrt{2}\pi$ (b) Γ(1/3) Γ(2/3) = $\frac{2}{\sqrt{3}}\pi$

(c) Γ(1/9) Γ(2/9) Γ(3/9)… Γ(8/9) = $\frac{3}{16}\pi^4$

3. Prove that: $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \pi$

4. Prove that:

(a) B (m, n) = B(m + 1, n) + B (m , n + 1) for m >0, n > 0

(b) $\frac{B(m,n-1)}{n} = \frac{B(m+1,n)}{m} = \frac{B(m,n)}{m+n}$

5. Evaluate:

(a) Γ(1/3) Γ(2/3)

(b) $\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^6\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^6\theta \cos^4\theta\, d\theta$ [Ans: (3/512)π]

(c) $\int_0^1 \frac{x}{\sqrt{1-x^5}}\, dx$

(d) $\int_0^1 x^{3/2}(1-x)^{3/2}\, dx$

(e) $\int_0^a x^3(a^2-x^2)^{5/2}\, dx$

6. Show that $\int_0^1 x^{m-1}(1-x^2)^{n-1}\, dx$ = (1/2) B((1/2)m, n)

7. Show that $\int_0^{\pi/2} \sin^3 x \cos^5 x dx$ = 1/24

8. Show that $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\, dx$ = B (m, n)