Date
Page

KATHMANDU UNIVERSITY

DHULIKHEL , KAVRE

Subject: MATHS 101

Assignment No: 3

SUBMITTED BY:
Name : Ashraya Kadel
Rollno 25
Group OCE
Level UNG/I/I

SUBMITTED TO:
Hem Raj Pandey
Department of Mathematics

SUBMISSION DATE : 30 / 03 / 2023

Date
Page

<Q.1> Define indefinite integral of a function $f(x)$.
Evaluate the following:

a) ~~$\int \sin^4 x \, dx$~~

The set of all anti-derivatives of the function $f(x)$ is called indefinite integral and is denoted by $\int f(x)\,dx$.

Now,

$$\int f(x)\,dx = \cancel{F(x)}\ F(x) + C$$

where,

C = any arbitrary constant.

(a): $\int \sin^4 x \, dx$

Soln.

$$= \int \sin^2 x \sin^2 x \, dx$$

$$= \int (1-\cos^2 x)\sin^2 x \, dx$$

$$= \int \sin^2 x \, dx - \int \sin^2 x \cos^2 x \, dx$$

$$= \frac{1}{2}\int 2\sin^2 x \, dx - \frac{1}{4}\int (2\sin 2x)^2 \, dx$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx - \frac{1}{8}\int (1-\cos 4x)\,dx$$

$$= \frac{1}{2}\left[\int 1\cdot dx - \int \cos 2x \, dx\right] - \frac{1}{8}\left[\int 1\cdot dx - \int \cos 4x \, dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] - \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right]$$

Date
Page

$$= \frac{1}{2}\left[\frac{2x - \sin 2x}{2}\right] - \frac{1}{8}\left[\frac{4x - \sin 4x}{4}\right]$$

$$= \frac{(2x - \sin 2x)}{4} - \frac{(4x - \sin 4x)}{32}$$

$$= \frac{8(2x - \sin 2x) - (4x - \sin 4x)}{32} = \frac{16x - 8\sin 2x - 4x + \sin 4x + c}{32}$$

$$= \frac{12x - 8\sin 2x + \sin 4x}{32} + c$$

$$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c.$$

(b): $\int (e^{ax} + e^{-bx})\, dx$

Soln,

$$= \int e^{ax}\, dx + \int e^{-bx}\, dx$$

$$= \frac{e^{ax}}{a} + \frac{e^{-bx}}{(-b)} + c$$

$$= \frac{e^{ax}}{a} - \frac{e^{-bx}}{b} + c$$

(c): $\int \sqrt{1 - \cos x}\, dx$

Soln,

$$= \int \sqrt{2\sin^2 \frac{x}{2}}\, dx = \sqrt{2} \int \sin \frac{x}{2}\, dx$$

Date
Page

$= \sqrt{2} \int \sin \frac{x}{2} \, dx$

$= \sqrt{2} \cdot \left(- \frac{\cos \frac{x}{2}}{1/2} \right) + C$

$= \cancel{\frac{2}{\sqrt{2}} \cos \frac{x}{2} + C}$ $= -2\sqrt{2} \cos x/2 + C$

(d): $\int \sec^3 x \tan x \, dx$

Soln.

$= \int \sec^2 x \cdot \sec x \tan x \, dx$

Let $y = \sec x$

$\therefore \frac{dy}{dx} = \sec x \tan x$ or, $dy = \sec x \tan x \, dx$.

Now,

$= \int y^2 \cdot dy = \frac{y^3}{3} + C$

$= \frac{\sec^3 x}{3} + C$

(e): $\int \frac{1}{x} \sin(\ln x) \, dx$

Soln:

Let $y = \ln x$

$\therefore \frac{dy}{dx} = \frac{1}{x}$ or $dy = \frac{dx}{x}$

Now,

Date
Page

$= \int \sin y \, dy$

$= -\cos y + C$

$= -\cos(\ln x) + C$

<(f)>: $\int \frac{\sin(\tan^{-1}x)}{1+x^2} dx$

Soln.

Let $y = \tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$ or $dy = \frac{dx}{1+x^2}$

So,

$\int \sin y \, dy$

$= -\cos y + C$

$= -\cos(\tan^{-1}x) + C$

<Q.2>: Evaluate the integrals.

(a): $\int \frac{x}{(x+1)^2} dx$

Soln:

$= \int \frac{x+1-1}{(x+1)^2} dx = \int \frac{\cancel{(x+1)}}{(x+1)^{\cancel{2}}} dx - \int \frac{1 \cdot dx}{(x+1)}$

Date
Page

$$\int \frac{1}{(x+1)} dx - \int \frac{dx}{(x+1)^2}$$

$$= \ln(x+1) - \frac{(x+1)^{-2+1}}{-2+1} + C$$

$$= \ln(x+1) + \frac{1}{(x+1)} + C$$

b) $\int x^2 \ln x \, dx$

Sol^n:

$$\cancel{x^2 \int \ln x} = \ln x \int x^2 \, dx - \int \left[\frac{d \ln x}{dx} \int x^2 dx \right] dx$$

$$= \ln x \cdot \frac{x^3}{3} - \int \left[\frac{1}{\cancel{x}} \times \frac{x^{3\,2}}{3} \right] \cancel{+C} \, dx$$

$$= \frac{x^3 \cdot \ln x}{3} - \int \frac{x^2 dx}{3} \cancel{+C}$$

$$= \frac{x^3 \cdot \ln x}{3} - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3 \cdot \ln x}{3} - \frac{x^3}{9} + C.$$

(c): $\int \operatorname{cosec}^3 x \, dx$

Soln:

$I = \int \operatorname{cosec} x \cdot (1+\cot^2 x) \, dx$

$I = \int \operatorname{cosec} x + \operatorname{cosec} x \cot^2 x \, dx$

$I = \int \operatorname{cosec} x \, dx + \int \operatorname{cosec} x \cot x \cdot \cot x \, dx$

$I = \ln|\operatorname{cosec} x - \cot x| \cancel{c} + \int \operatorname{cosec} x \cdot \cot^2 x \, dx$

or, $I = \ln|\operatorname{cosec} x - \cot x| + \cot x \int \operatorname{cosec} x \cdot \cot x \, dx \cancel{- \int \frac{d\cot x}{dx}}$

$- \int \left[\frac{d\cot x}{dx} \int \operatorname{cosec} x \cot x \, dx \right] dx$

or, $I = \ln|\operatorname{cosec} x - \cot x| + \cot x \, (-\operatorname{cosec} x) - \int [\cancel{co} -\operatorname{cosec}^2 x \cdot (-\operatorname{cosec} x) \, dx$

or, $I = \ln|\operatorname{cosec} x - \cot x| - \operatorname{cosec} x \cdot \cot x - \int \operatorname{cosec}^3 x \, dx$

or, $2I = \ln|\operatorname{cosec} x - \cot x) - \cot x \operatorname{cosec} x$

$\therefore \int \operatorname{cosec}^3 x \, dx = \dfrac{\ln|\operatorname{cosec} x - \cot x) - \cot x \operatorname{cosec} x}{2}$

d) $\int e^{\sqrt{x}}\, dx$

Soln:

Let $y = \sqrt{x}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ or, $dy = \frac{dx}{2\sqrt{x}}$ on $2y\, dy = dx$

Now,

$= \int e^{y} \cdot 2y\, dy$

$= 2\int y \cdot e^{y}$

$= 2\left[y \int e^{y} dy - \int \left[\frac{dy}{dy} \int e^{y} dy \right] dy \right]$

$= 2\left[ye^{y} - \int e^{y} dy \right] \quad = 2\left[ye^{y} - e^{y} \right] + c$

$= 2ye^{y} - e^{y} + c$

$= 2\sqrt{x}\, e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$

$\cancel{e) \int \frac{1-x}{\sqrt{8+2x-x^2}}\, dx}$

~~Soln:~~

e): $\int e^{ax} \cos bx \, dx$

Sol^n:

Let $I = \int e^{ax} \cdot \cos bx \, dx$

Now,

$$\cancel{I = e^{ax} \int \cos bx \, dx - \int \left[\frac{de^{ax}}{dx} \int \cos bx \, dx \right] dx}$$

$$\cancel{= e^{ax} \cdot \frac{-\sin bx}{b} - \int \frac{e^{ax}}{a}}$$

$$I = \cos bx \int e^{ax} \, dx - \int \left[\frac{d \cos bx}{dx} \int e^{ax} \cdot dx \right] dx$$

or, $$I = \frac{\cos bx \cdot e^{ax}}{a} - \int - \sin bx \times b \times \frac{e^{ax}}{a} \, dx$$

or, $$I = \frac{\cos bx \cdot e^{ax}}{a} + \frac{b}{a} \int e^{ax} \cdot \sin bx \, dx$$

or, $$I = \frac{\cos bx \cdot e^{ax}}{a} + \frac{b}{a} \left[\sin bx \cdot \int e^{ax} dx - \int \left[\frac{d \sin bx}{dx} \int e^{ax} dx \right] dx \right.$$

or, $$I = \frac{\cos bx \cdot e^{ax}}{a} + \frac{b}{a} \left[\frac{\sin bx \cdot e^{ax}}{a} - \frac{b}{a} \int \cos bx \cdot e^{ax} \, dx \right]$$

or $$I = \frac{\cos bx \cdot e^{ax}}{a} + \frac{b \sin bx \cdot e^{ax}}{a^2} - \frac{b^2}{a^2} I$$

or, $$\left(1 + \frac{b^2}{a^2}\right) I = \frac{\cos bx \cdot e^{ax}}{a} + \frac{b \sin bx \cdot e^{ax}}{a^2}$$

$$\therefore I = \left(\frac{a^2}{a^2+b^2}\right) \left\{ \frac{\cos bx \cdot e^{ax}}{a} + \frac{b \sin bx \cdot e^{ax}}{a^2} \right\}$$

Date ______
Page ______

(f): $\int \sin^{-1}x \, dx.$

Sol^n:

$= \int \sin^{-1}x \cdot dx \qquad = \int 1\cdot \sin^{-1}x \, dx$

$= \sin^{-1}x \cdot \int 1\cdot dx - \int \left[\frac{d \sin^{-1}x}{dx} \left\{ \int 1\cdot dx \right\} \right] dx$

$= x\cdot \sin^{-1}x \cdot - \int \frac{x}{\sqrt{1-x^2}} dx$

$= x \sin^{-1}x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} dx$

$= x \sin^{-1}x + \frac{1}{\cancel{2}} \times \cancel{2}\sqrt{1-x^2} + c$

$= x \sin^{-1}x + \sqrt{1-x^2} + c.$

(3): Evaluate the following integrals.

a): $\int \frac{1}{x^2 \sqrt{36x^2+121}} dx$

Sol^n

Let $t = \frac{1}{x}$ or $x = \frac{1}{t}$

$\therefore \frac{dt}{dx} = \frac{-1}{x^2} \qquad \therefore dt = -\frac{dx}{x^2} \qquad$ or $-dt = \frac{dx}{x^2}$

Date
Page

$$= \int \frac{-dt}{\sqrt{36/t^2 + 121}}$$

$$= \int \frac{-t \cdot dt}{\sqrt{36 + 121t^2}}$$

$$= \frac{-1}{242} \int \frac{242t \cdot dt}{\sqrt{36+121t^2}}$$

$$= \frac{-1}{\cancel{242}\,121} \times 2 \times \sqrt{36+121t^2} + C$$

$$= \frac{-1}{121} \sqrt{36+121t^2} + C$$

$$= \frac{-1}{121} \sqrt{36 + \frac{121}{x^2}} + C$$

$$= \frac{-1}{121x} \sqrt{36x^2+121} + C$$

b): $\int \sqrt{25 - 9x^2}\, dx$

Sol^n:

$$= \int \sqrt{5^2 - (3x)^2}\, dx$$

$$= \frac{3x}{2} \sqrt{5^2-(3x)^2} + \frac{5^2}{2} \sin^{-1}\left(\frac{3x}{5}\right) + C$$

$$= \frac{3x}{2} \sqrt{25-9x^2} + \frac{25}{2} \sin^{-1}\left(\frac{3x}{5}\right) + C$$

$$\int f(x)\sqrt{f(x)} = \frac{2}{3} f(x)^{3/2} + C$$

Date ____
Page ____

Q7: $\int (x+2)\sqrt{x^2+10x-11}\,dx$

Soln:

Soln:

$$= \frac{1}{2}\int (2x+4)\sqrt{x^2+10x-11}\,dx$$

$$= \frac{1}{2}\int (2x+10-6)\sqrt{x^2+10x-11}\,dx$$

$$= \frac{1}{2}\int (2x+10)\sqrt{x^2+10x-11}\,dx - \frac{1\times 6}{2}\int \sqrt{x^2+10x-11}\,dx$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3} \times (x^2+10x-11)^{3/2} - 3\int \sqrt{x^2+10x+5^2-5^2-11}\,dx$$

$$= \frac{1}{3}(x^2+10x-11)^{3/2} - 3\int \sqrt{x^2+10x+25-36}\,dx$$

$$= \frac{1}{3}(x^2+10x-11)^{3/2} - 3\int \sqrt{(x-5)^2-(6)^2}\,dx$$

$$= \frac{1}{3}(x^2+10x-11)^{3/2} - 3\left[\frac{(x-5)}{2}\sqrt{(x-5)^2-6^2} + \frac{6^2}{2}\ln\left|x-5+\sqrt{(x-5)^2-6^2}\right|\right] + C$$

$$= \frac{1}{3}(x^2+10x+11)^{3/2} - \frac{3}{2}(x-5)\sqrt{x^2+10x-11} + 54\ln\left|x-5+\sqrt{x^2+10x-11}\right| + C$$

Date
Page

(e): $\int \frac{1-x}{\sqrt{8+2x-x^2}} dx$

Soln.

$= \frac{1}{2}\int \frac{\cancel{2x}\, 2-2x}{\sqrt{8+2x-x^2}} dx$

$= \frac{1}{\cancel{2}} \times \cancel{2}\sqrt{8+2x-x^2} + C$

$= \sqrt{8+2x-x^2} + C$

(f): $\int \frac{2x+3}{x^2+4x+5} dx$

$= \int \frac{2x+4-1}{x^2+4x+5} dx$

$= \int \frac{(2x+4)\,dx}{(x^2+4x+5)} - \int \frac{1\,dx}{x^2+4x+5}$

$= \ln|x^2+4x+5| - \int \frac{dx}{(x+2)^2+(1)^2}$

$= \ln|x^2+4x+5| - \frac{1}{1}\tan^{-1}\left(\frac{x+2}{1}\right) + C$

$= \ln|x^2+4x+5| - \tan^{-1}(x+2) + C$

Date
Page

<Q.4>: Evaluate the following integrals.

(a): $\int \frac{5x-13}{(x-2)(x-3)} dx$

Soln:

$$= \int \frac{5x-13}{x^2-5x+6} dx = \int \frac{5x-13}{(x-2)(x-3)} dx$$

$$\frac{A}{(x-2)} + \frac{B}{(x-3)} = \frac{5x-13}{(x-2)(x-3)}$$

or, $A(x-3)+B(x-2) = 5x-13$

Putting $x=3$,

$A(3-3) + B(3-2) = 5\times3-13$

or $\therefore B = 2$

Putting $x=2$,

$A(2-3) + B(2-2) = 5\times2-13$

or $-A = -3$ $\therefore A=3$

Then,

$$\int \frac{3}{(x-2)} dx + \int \frac{2}{(x-3)} dx$$

$$= 3\ln|x-2| + 2\ln(x-3) + C$$

Date
Page

(b): $\int \frac{x+1}{x^2(x+1)} dx$

Soln:

Using partial function,

$$\frac{x+1}{x^2(x-1)} = \frac{A}{(x-1)} + \frac{B}{x} + \frac{C}{x^2} \quad \text{--- (a)}.$$

or $x+1 = Ax^2 + Bx(x-1) + C(x-1)$ — (i)

Putting $x=0$.

$0+1 = A\times 0^2 + B\times 0\times (0-1) + C(0-1)$

$\therefore C = -1$.

Putting $x=1$,

$1+1 = A\times 1^2 + B\times 1\times (1-1) + C(1-1)$

$\therefore A = 2$

~~Putting values of A and C in eqn (i) and putting x=2.~~

$\cancel{2+1 = 2\times 2^2 + B\times 2\times(2-1) + (-1)\times 2}$

$\cancel{\text{or } 3 = 8 + 2B + 2}$

Comparing coefficients of x^2 in (1)

$A + B = 0$

$\therefore B = -2$.

Now, integrating (a).

$$\int \frac{(x+1)\,dx}{x^2(x-1)} = \int \frac{A}{(x-1)} + \int \frac{B}{x} + \int \frac{C}{x^2}$$

Date
Page

$$= A\ln|x-1| + B\ln x + \frac{C}{x} + C$$

$$= 2\ln|x-1| - 2\ln x + \frac{1}{x} + C$$

$$= 2\ln\left(\frac{x-1}{x}\right) + \frac{1}{x} + C$$

(c): $\displaystyle\int \frac{1}{1-\sin x}\,dx$

Soln:

$$= \int \frac{1}{1 - \frac{2\tan x/2}{1+\tan^2 x/2}}\,dx = \int \frac{1+\tan^2 x/2}{1-2\tan x/2 + \tan^2 x/2}\,dx$$

$$= \int \frac{\sec^2 x/2 \;\, dx}{\tan^2 x/2 - 2\tan x/2 + 1}$$

Let $y = \tan x/2$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{2}\sec^2\dfrac{x}{2}$ or $dy = \dfrac{1}{2}\sec^2\dfrac{x}{2}\cdot dx$

or $2dy = \sec^2 x/2 \cdot dx$

Now,

$$= \int \frac{2\cdot dy}{y^2-2y+1} = 2\int \frac{dy}{(y-1)^2}$$

$$= \frac{-2}{(y-1)} + C = \frac{-2}{\tan\frac{x}{2} - 1} + C$$

<d>: $\int \frac{1}{2+\cos x}\,dx$

Sol^n:

$$= \int \frac{1}{2 + \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}\,dx$$

$$= \int \frac{1+\tan^2 x/2}{2+2\tan^2 x/2 + 1 - \tan^2 x/2}\,dx = \int \frac{1+\tan^2 x/2}{3+\tan^2 x/2}\,dx$$

$$= \cancel{\int \frac{\sec^2 x/2}{3\sec^2 x/2 - 3\tan^2 x/2}} = \int \frac{\sec^2 x/2}{3+\tan^2 x/2}\,dx$$

Let $y = \tan x/2$

$\frac{dy}{dx} = \frac{1}{2}\sec^2 \frac{x}{2}$ or $2\,dy = \sec^2 \frac{x}{2} \cdot dx$

Now,

$$\int \frac{2\cdot dy}{3+y^2} = 2\int \frac{dy}{(y)^2 + (\sqrt{3})^2}$$

$$= 2\left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{y}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$$

Date
Page

(e): $\displaystyle\int \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\, dx$

Sol^n:

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{a^2\sin^2 x}{\cos^2 x} + \frac{b^2\cos^2 x}{\cos^2 x}}\, dx$$

$$= \int \frac{\sec^2 x \quad dx}{a^2\tan^2 x + b^2}$$

Let $y = \tan x$

$\therefore \frac{dy}{dx} = \sec^2 x \qquad$ or $\quad dy = \sec^2 x\, dx$

$$= \int \frac{dy}{a^2 y^2 + b^2}$$

$$= \frac{1}{b} \tan^{-1}\left(\frac{ay}{b}\right) + C$$

$$= \frac{1}{b} \tan^{-1}\left(\frac{a \tan x}{b}\right) + C$$

f): $\int \frac{dx}{4-5\sin^2 x}$

Soln:

Let $t = \tan x/2$

or $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ or $2\cdot dt = \sec^2\frac{x}{2}\cdot dx.$

Now $\frac{dt}{dx} = \frac{1}{2}(1+t^2) \Rightarrow dx = \frac{2\cdot dt}{(1+t^2)}$

$= \cancel{\int \frac{2\,dt}{4-5}}$ Now, $\sin t = \frac{2t}{1+t^2}$

$$= \int \frac{2\cdot dt}{(1+t^2)} \times \frac{1}{4+\frac{10t}{(1+t^2)}}$$

$$= \int \frac{2dt}{4+4t^2+10} = \int \frac{2\,dt}{\left(2t+\frac{5}{2}\right)^2-\left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{2\times\frac{3}{2}} \ln\left|\frac{2t+5/2-3/2}{2t+5/2+3/2}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{2t+1}{2t+4}\right| + c$$

$$= \frac{1}{3}\ln\left|\frac{2\tan x/2+1}{2\tan x/2+4}\right| + c.$$

Date ______ Page ______

<57: Evaluate the following definite integrals.

<a>: $\int_{-1}^{2} \frac{t \cdot dt}{\sqrt{2t^2+8}}$

Soln:

Let $x = 2t^2$

$\therefore \frac{dx}{dt} = 4t$ or $dx = 4t \cdot dt$

Now,

$= \int_{-1}^{2} \frac{t \cdot dt}{\sqrt{2t^2+8}} = \frac{1}{4} \int_{-1}^{2} \frac{4t \cdot dt}{\sqrt{2t^2+8}}$

$= \frac{1}{4} \int_{-1}^{2} \frac{dx}{\sqrt{x+8}}$

$= \frac{1}{4} \left. 2\sqrt{x+8} \right|_{-1}^{2}$

or $\frac{1}{2} \left. 2\sqrt{2t^2+8} \right|_{-1}^{2}$

$= \frac{1}{2} \left[\sqrt{\cancel{2t^2+} 2 \times 2^2 + 8} - \sqrt{2 \times (-1)^2 + 8} \right]$

$= 0.4188$

<b>:

$$\int_0^{\sqrt{\ln \pi}} 2x e^{x^2} \cos e^{x^2} \, dx$$

Soln.

Let $y = e^{x^2}$

$\therefore \frac{dy}{dx} = e^{x^2} \cdot 2x$ $\therefore dy = e^{x^2} \cdot 2x \cdot dx$

Then,

$$\int_0^{\sqrt{\ln \pi}} \cos y \cdot dy$$

$$= \sin y \Big|_0^{\sqrt{\ln \pi}}$$

$$= \sin e^{x^2} \Big|_0^{\sqrt{\ln \pi}}$$

$$= \sin\left(e^{(\sqrt{\ln \pi})^2}\right) - \sin\left(e^{0^2}\right)$$

$$= \sin \pi - \sin 1.$$

<c>:

$$\int_{-2\pi/3}^{\pi} \tan^3 \frac{x}{4} \sec^2 \frac{x}{4} \, dx$$

Soln:

Let $y = \sec \frac{x}{4}$ or, $dy = \frac{1}{4} \sec \frac{x}{4} \tan \frac{x}{4} \, dx.$

$\therefore \frac{dy}{dx} = \frac{1}{4} \sec \frac{x}{4} \tan \frac{x}{4}$

Now,

$$= 4\int_{-2\pi/3}^{\pi} \tan^2 \frac{x}{4} \cdot \sec\frac{x}{4} \times \frac{1}{4}\sec\frac{x}{4} \times \tan\frac{x}{4} \, dx$$

$$= 4\int_{-2\pi/3}^{\pi} \tan^2 \frac{x}{4} \times \sec\frac{x}{4} \times dy$$

$$= 4\int_{-2\pi/3}^{\pi} \left(\sec^2 \frac{x}{4} - 1\right)\sec\frac{x}{4} \, dy$$

$$= 4\int_{-2\pi/3}^{\pi} \sec^3 \frac{x}{4} - \sec\frac{x}{4} \, dy$$

$$= 4\int_{-2\pi/3}^{\pi} y^3 - y \, dy$$

$$= 4\int_{-2\pi/3}^{\pi} y^3 \, dy - 4\int_{-2\pi/3}^{\pi} y \, dy$$

$$= \frac{4}{4} y^4 \Big|_{-2\pi/3}^{\pi} - \frac{4}{2} y^2 \Big|_{-2\pi/3}^{\pi}$$

$$= \left(\sec\frac{x}{4}\right)^4 \Big|_{-2\pi/3}^{\pi} - 2\left(\sec\frac{x}{4}\right)^2 \Big|_{-2\pi/3}^{\pi}$$

$$= \left\{\left(\sec\left(\frac{\pi}{4}\right)\right)^4 - \left(\sec\frac{-2\pi/3}{4}\right)^4\right\} - 2\left[\left(\sec\frac{\pi}{4}\right)^2 - \left(\sec\frac{-2\pi/3}{4}\right)^2\right]$$

$$= \frac{20}{9} - 2\times\frac{2}{3} = \frac{20}{9} - \frac{4}{3} = \frac{20-12}{9} = \frac{8}{9}$$

(d): $\int_{-2}^{2} |x-x^2|\,dx$

Sol^n:

$$= \int_{-2}^{0} -(x-x^2)\,dx + \int_{0}^{1} (x-x^2)\,dx + \int_{1}^{2} -(x-x^2)\,dx$$

$$= \int_{-2}^{0} -x\,dx + \int_{-2}^{0} x^2\,dx + \int_{0}^{1} x\,dx - \int_{0}^{1} x^2\,dx + \int_{1}^{2} -x\,dx + \int_{1}^{2} x^2\,dx$$

$$= -\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^3}{3}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{1} - \frac{x^3}{3}\Big|_{0}^{1} - \frac{x^2}{2}\Big|_{1}^{2} + \frac{x^3}{3}\Big|_{1}^{2}$$

$$= 2 + \frac{8}{3} + \frac{1}{2} - \frac{1}{3} - \frac{3}{2} + \frac{7}{3}$$

$$= \frac{17}{3}$$

Date ______ Page ______

(Q): Evaluate : $\int_{-1}^{3} f(x)\,dx$ where $f(x) = \begin{cases} 3-x & \text{if } x \le 2 \\ x/2 & \text{if } x > 2 \end{cases}$

Soln:

$$= \int_{-1}^{3} f(x)\,dx = \int_{-1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$

$$= \int_{-1}^{2} (3-x)\,dx + \int_{2}^{3} \frac{x}{2}\,dx$$

$$= \int_{-1}^{2} 3\,dx - \int_{-1}^{2} x\,dx + \frac{1}{2}\int_{2}^{3} x\,dx$$

$$= 3\,x\Big|_{-1}^{2} - \frac{1}{2}x^2\Big|_{-1}^{2} + \frac{1}{2}\times\frac{1}{2}\,x^2\Big|_{2}^{3}$$

$$= 3\{2-(-1)\} - \frac{1}{2}\left(2^2-(-1)^2\right) + \frac{1}{4}\left(3^2-2^2\right)$$

$$= \frac{35}{4}$$

(Q.7): Find dy/dx if.

a) $y = \int_{2}^{x^2} e^{\sqrt{t}}\,dt$

Soln.

Let $u = \sqrt{t}$ $\therefore \frac{du}{dt} = \frac{1}{2\sqrt{t}}$ or, $du = \frac{dt}{2\sqrt{t}}$

or, $2u\,du = dt$

Date
Page

when $t = 2$, $u = \sqrt{2}$

when $t = x^2$, $u = x$

Then,

$$y = \int_{\sqrt{2}}^{x} e^{u}\, 2u\, du$$

$$= 2\int_{\sqrt{2}}^{x} e^{u} \cdot u\, du$$

$$= 2\left[ue^{u} - e^{u}\right]_{\sqrt{2}}^{x}$$

$$\cancel{\int e^{\sqrt{t}}\, dt} = 2\left[xe^{x} - e^{x} - \sqrt{2}e^{\sqrt{2}} + e^{\sqrt{2}}\right]$$

$$\frac{dy}{dx} = \frac{d\,2\left[xe^{x} - e^{x} - \sqrt{x}e^{\sqrt{2}} + e^{\sqrt{2}}\right]}{dx}$$

$$= 2 \cdot \frac{d(xe^{x} - e^{x})}{dx}$$

$$= 2\left[\frac{d\,xe^{x}}{dx} - \frac{d\,e^{x}}{dx}\right]$$

$$= 2\left[xe^{x} + e^{x} - e^{x}\right]$$

$$= 2xe^{x}.$$

Date
Page

(b): $y = \int_{\tan x}^{0} \frac{-\tan^{-1} t \, dt}{\sqrt{1+t^2}}$

Let $\frac{-\tan^{-1} t}{\sqrt{1+t^2}} = f(t)$

$y = \int_{\tan x}^{0} f(t) \, dt$

$\frac{d}{dx}\left[\int f(t)\, dt\right]_0^{\infty}$

$= \frac{d}{dx}\left(f(\tan x) - f(0)\right)$

$= \cancel{\frac{-\tan^{-1}\tan x}{\sqrt{1+\tan^2 x}}} = -\left(\frac{d}{dx \tan x} F(\tan x) \times \frac{d \tan x}{dx}\right)$

$= \left(\sec^2 x \cdot x - \frac{\tan^{-1}(\tan x)}{\sqrt{1+\tan^2 x}}\right)$

$= x \sec x$

Date
Page

<c>: $\int_{\sqrt{x}}^{2\sqrt{x}} \sin t^2 \, dt$

Let $f(t) = \sin t^2$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\int_{\sqrt{x}}^{2\sqrt{x}} f(t)\, dt\right]$$

$$= \frac{d}{dx}\left(F(t)\Big|_{\sqrt{x}}^{2\sqrt{x}}\right)$$

$$= \frac{d}{dx}\left\{F(2\sqrt{x}) - f(\sqrt{x})\right\}$$

$$= \frac{d\,F(2\sqrt{x})}{dx\,2\sqrt{x}} \times \frac{d(2\sqrt{x})}{dx} - \frac{d\,f(\sqrt{x})}{d\sqrt{x}} \cdot \frac{d\sqrt{x}}{dx}$$

$$= \frac{\cancel{2} \times 1}{\cancel{2}\sqrt{x}} \times \sin 4x - \frac{1}{2\sqrt{x}} \times \sin x$$

$$= \frac{\sin 4x}{\sqrt{x}} - \frac{\sin x}{2\sqrt{x}} = \frac{2\sin 4x - \sin x}{2\sqrt{x}}$$

(Q·
con
in
int
o
en
(a

(Q.8): Define impropes integrals. Investigate the convergence of the following impropes integrals.

Ans:

Impropes integrals are the limit of a definite integral as an endpoint of the interval of integration approaches either a specified real numbes or +ve or -ve infinity or in some cases, both end points approach limits.

(a): $\int_0^{\infty} x e^{-x^2} dx$.

Soln.

$= \int_0^{\infty} x e^{-x^2} dx$

Let $-x^2 = y$ $\therefore \frac{dy}{dx} = -2x$ or $dy = -2x\,dx$

$= \lim_{a\to\infty} -\frac{1}{2} \int_0^a -2x \cdot e^{-x^2} dx$

$= \lim_{a\to\infty} -\frac{1}{2} \int_0^a e^{y} \cdot dy$

$= -\frac{1}{2} \lim_{a\to\infty} e^{y} \Big|_0^a$ $= -\frac{1}{2} \lim_{a\to\infty} e^{-x^2} \Big|_0^a$

$= -\frac{1}{2} \left(\frac{1}{e^{-\infty^2}} - \frac{1}{e^{0^2}} \right) = \frac{-1}{2}$ $\Rightarrow$ convergent.

(b): $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Sol^n:

$$= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \tan^{-1}(x^2)\Big|_{-\infty}^{0} + \tan^{-1}(x)\Big|_{0}^{\infty}$$

$$= \cancel{\tan^{-1}(x^0)} - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \cancel{\tan^{-1}(0)}$$

$$- \pi - \cancel{\frac{\pi}{2}} + \frac{\pi}{2} = -\pi \Rightarrow \text{convergent.}$$

(c): $\int_{4}^{6} \frac{dx}{\sqrt{x-4}}$

$$= \int_{4}^{6} (x-4)^{1/2}\, dx$$

$$= 2\cancel{\frac{q}{2}}\,(x-4)^{1/2}\Big|_{4}^{6}$$

$$= \cancel{\frac{q}{2}}\,2 \lim_{a \to 4^+} (x-4)^{1/2}\Big|_{a}^{6}$$

$$= 2\left\{(6-4)^{1/2} - (4-4)^{1/2}\right\}$$

$$= 2\sqrt{2} \Rightarrow \text{convergent.}$$

<Q.9>: Find the average values of following functions.

(a): $f(x) = \sin x \quad [0, \pi/2]$

Soln:

We know,

$$av(f) = \frac{1}{b-a}\int_a^b \cancel{\sin x}\, f(x)\, dx$$

Now,

$$\int_0^{\pi/2} \sin x\, dx = -\cos x \Big|_0^{\pi/2}$$

$$= -\cos \pi/2 + \cos 0$$

$$= 1$$

So,

$$av(f) = \frac{1}{\pi/2 - 0} \times 1 = \frac{2}{\pi}$$

(b): $f(x) = \sqrt{2x+1} \quad , [4, 12]$

Soln:

We know,

$$\int_4^{12} \sqrt{2x+1}\, dx = \frac{(2x+1)^{3/2}}{3}\Bigg|_4^{12}$$

$$= \frac{1}{3}\left[(2\times 12+1)^{3/2} - (2\times 4+1)^{3/2}\right] = 37.94$$

Date
Page

$\therefore av(f) = \dfrac{1}{8/12 - 4} \times 37.94 = 4.08 = \dfrac{49}{12}$

(Q.10): Solve the initial value problems.

(a): $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}, \; y(0) = 0$

Sol^n:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

or, $\displaystyle\int dy = \int \frac{1}{\sqrt{1-x^2}}\, dx$

or $y = \sin^{-1}x + c$ —— (1)

We know, $y(0) = 0$,

$0 = \sin^{-1}(0) + c \quad \therefore c = 0$

$\therefore y = \sin^{-1}x$ is the required sol^n.

(b) $\frac{dy}{dx} = \frac{x}{x^2+1}, \; y(0) = 0.$

Soln.

Given,

$$\frac{dy}{dx} = \frac{x}{x^2+1}$$

or $dy = \left(\frac{x}{x^2+1}\right) dx$

$$\int dy = \frac{1}{2}\int \frac{2x}{x^2+1}\, dx$$

or $y = \frac{1}{2} \ln(x^2+1) + C$

We know,

$$y(0) = 0$$

or $\frac{1}{2} \ln|0+1| = 0 + C = 0$

$$C = 0$$

$\therefore \; y = \frac{1}{2} \ln|x^2+1|$ is the reqd soln.

Date
Page

<Q.117: Find the area bounded the two curves.

(a): $y = x^4$ and $y = 8x$

Soln:

Given,

$y = x^4$ — (i)

$y = 8x$ — (ii)

From (i) and (ii),

$x^4 - 8x = 0$

or, $x(x^3 - 8) = 0$

$\therefore x = 0, \quad x = 2.$

So, area betn curves.

$$A = \int_0^2 (8x - x^4)\, dx$$

$$= \int_0^2 \cancel{8x}\ 8x\, dx - \int_0^2 x^4\, dx$$

$$= \frac{\overset{4}{\cancel{8}}}{\cancel{2}} x^2 \Big|_0^2 - \frac{1}{5} x^5 \Big|_0^2$$

$$= 4\{2^2\} - \frac{1}{5}(2)^5$$

$$= \frac{48}{5} \text{ sq. units.}$$

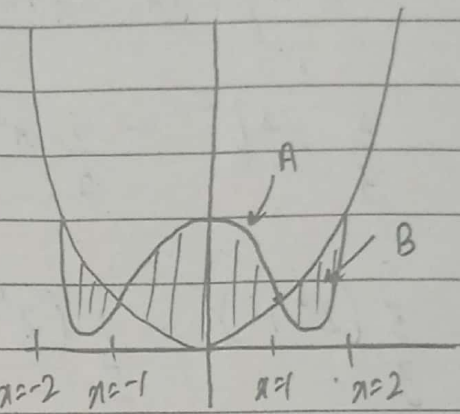
(b): ~~Soln~~ $y = x^4 - 4x^2 + 4$ and $y = x^2$.

Soln:

Given,

$y = x^4 - 4x^2 + 4$ —(i)

$y = x^2$ —(ii)



Let $t = x^2$.

$t = t^2 - 4t + 4$

or $t^2 - 5t + 4 = 0$

$\therefore t = 4, 1$

When $t = 4$, $x = \pm 2$

$t = 1$, $x = \pm 1$

Thus, the region is bounded betn (2,4) and (1,1).

So

Area betn curves = $2(A+B)$.

Now,

$$\text{Area } A = \int_0^1 \{(x^4 - 4x^2 + 4) - (x^2)\}\, dx$$

$$= \int_0^1 (x^4 - 5x^2 + 4)\, dx$$

$$= \frac{1}{5} x^5 \Big|_0^1 - \frac{5}{3} x^3 \Big|_0^1 + 4x \Big|_0^1 = \frac{1}{5} - \frac{5}{3} + 4 = \frac{38}{15}$$

sq. units.

Date
Page

$$\text{Area } B = \int_1^2 x^2 - (x^4 - 4x^2 + 4)\, dx$$

$$= \int_1^2 (5x^2 - x^4 - 4)\, dx$$

$$= \frac{5}{3} x^2 \Big|_1^2 - \frac{1}{5} x^4 \Big|_1^2 - 4x \Big|_1^2$$

$$= \frac{5}{3}(2^2 - 1^2) - \frac{1}{5}(2^4 - 1^4) - 4(2-1)$$

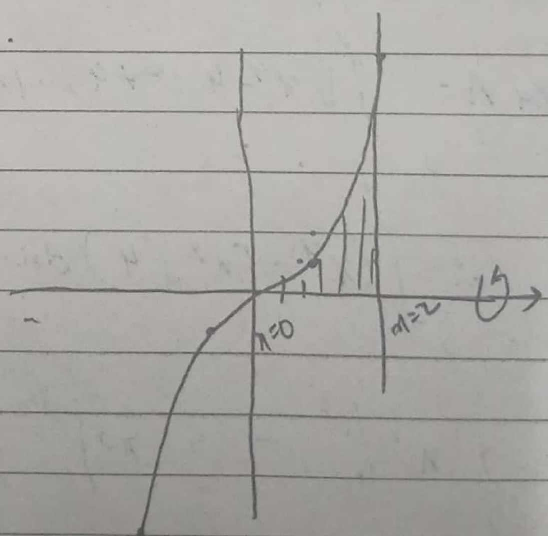
$$= \frac{22}{15}$$

$\therefore$ Total area $= 2(A+B)$
$= 8$ sq. units.

{Q.12}: Find the volume of solid of revolution about x-axis.

(a): $y = x^3$, $y = 0$, $x = 2$.
Sol^n:

Given,
$R(x) = x^3$



Using disk method,

$$V = \pi \int_0^2 R(x)^2\, dx$$

$$= \pi \int_0^2 x^6 \, dx$$

$$= \pi \frac{x^7}{7} \Big|_0^2 = \frac{128\pi}{7} \text{ cub-units.}$$

(b): $y = x - x^2$, $y = 0$.

Sol^n:

Given,

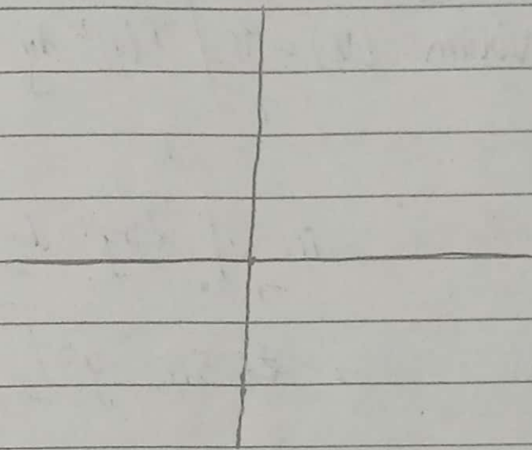
$y = x - x^2$

$(R(x)) = x - x^2$

when $y = 0$.

$x(\cancel{x}\, 1 - x) = 0$

$\therefore x = 0, 1$



Now

$$\text{Volume } (V) = \pi \int_0^1 R(x)^2 \, dx$$

$$= \pi \int_0^1 (x^2 - 2x^3 + x^4) \, dx$$

$$= \pi \left[\int_0^1 x^2 dx - \int_0^1 2x^3 dx + \int_0^1 x^4 dx \right]$$

$$= \frac{\pi}{3} x^3 \Big|_0^1 - \frac{\cancel{2}\pi}{\cancel{4}2} x^4 \Big|_0^1 + \frac{\pi}{5} x^5 \Big|_0^1 = \frac{\pi}{30} \text{ cub-units.}$$

Date
Page

<Q7: Find volume of solid of revolution about y-axis,

(a): $x = 5y^2$, $x = 0$, $y = -1$, $y = 1$

Soln:

Given,

$R(y) = 5y^2$.

~~The~~ Now,

$$\text{Volume } (V) = \pi \int_{-1}^{1} R(y)^2 \, dy$$

$$= \pi \int_{-1}^{1} 25y^4 \, dy$$

$$= \cancel{25}\, 5\pi \; y^5 \Big|_{-1}^{1}$$

$$= 5\pi \left((1)^5 - (-1)^5 \right)$$

$$= 10\pi \text{ cub. units.}$$

(b): $x = 0$, $x = 2/y$ $y \in [1, 4]$

Soln:

Given,

$R(y) = 2/y$.

Using disc. method,

$$\text{Volume } (V) = \pi \int R(y)^2 \, dy$$

$$= \pi \int^{4} \frac{4}{y^2} \, dy$$

Date
Page

$= 4\pi \left\{ -\frac{1}{y} \right|_1^4$

$= -4\pi \times \frac{1}{4} + 4\pi \times \frac{1}{1}$

$= 4\pi - \pi = 3\pi$ cub. units

<Q no 14>: Find the length of the following curves.

(a): $y = x^{3/2}$ from $x = 0$ to $x = 4$.

Soln:

$f(x) = x^{3/2}$

$\therefore f'(x) = \frac{3\sqrt{x}}{2}$

Now,

$L = \int_0^4 \sqrt{1 + \frac{9}{4}x}$

$= \frac{1}{2} \int_0^4 \sqrt{4 + 9x}$

$= \frac{1}{2} \times \left. \frac{(9x+4)^{3/2}}{9 \times \frac{3}{2}} \right|_0^4 \qquad = \frac{1}{27} \times (9x+4)^{3/2} \Big|_0^4$

$= \frac{1}{27} \times (9 \times 4 + 4)^{3/2} = 9.36$ units.

Date
Page

(b) $y = \int_0^x \sqrt{\cos 2t}\, dt$ from $x=0$ to $x=\frac{\pi}{2}$

So/n

$\frac{dy}{dx} = \frac{d}{dx}\int_0^x f(t)\cdot dt$ Let $f(t)=\sqrt{\cos 2t}$

$= \frac{d}{dx} F(t)\Big|_0^x$

$= \frac{d\left(F(x) - F(0)\right)}{dx}$

$= \sqrt{\cos 2x}$

So,

$L = \int_0^{\pi/2} \sqrt{1+f'(x)^2}\, dx$

$= \int_0^{\pi/2} \sqrt{1+\cos 2x}\, dx$

$= \sqrt{2}\int_0^{\pi/2} \sqrt{\cancel{2\cos^2 x/2}\ \cos^2 x}$

$= \sqrt{2}\ \sin x\Big|_0^{\pi/2}$

$= \sqrt{2}\sin \pi/2 - \sqrt{2}\cdot\sin 0$

$= \sqrt{2}$ unit.

Date ______
Page ______

<Q·15>: Find the area of surface generated by revolving the curves about given axes.

(a): $y = \frac{x^3}{9}$, $0 \le x \le 2$ about x-axis.

Soln:

$$f(x) = \frac{x^3}{9}$$

$$\therefore f'(x) = \frac{x^2}{3}$$

Now, surface area generated,

$$SA = \int_0^2 2\pi f(x) \sqrt{1 - (f'(x))^2}\, dx$$

$$= 2\pi \int_0^2 \frac{x^3}{9} \sqrt{1 + \frac{x^4}{9}}\, dx$$

$$= \frac{2\pi}{27} \int_0^2 x^3 \sqrt{9 + x^4}\, dx$$

$$= \frac{\pi}{54} \int_0^3 4x^3 \sqrt{9 + x^4}\, dx$$

$$= \frac{\pi}{\cancel{54}\,27} \times \frac{\cancel{2}}{3}\, (9 + x^4)^{3/2} \Big|_0^2 \cancel{+c}$$

$$= \frac{\pi}{81} \times \left[(9 + 2^4)^{3/2} - (9)^{3/2} \right] \quad \frac{98}{81}\pi \text{ sq. units.}$$

(b): $x = \frac{y^3}{3}$, $1 \le \overset{y}{\cancel{x}} \le 5$ about y-axis.

Soln:

Given,

$R(y) = \frac{y^3}{3}$

$f'(y) = y^2$

Now, the surface area generated.

$$= \int_1^5 2\pi f(y) \sqrt{1 + (f'(y))^2} \, dy$$

$$= \int_1^5 2\pi \frac{y^3}{3} \sqrt{1 + y^4} \, dy$$

$$= \frac{2\pi}{12} \int_1^5 4y^3 \sqrt{1 + y^4} \, dy$$

$$= \frac{\pi}{6} \times \frac{2}{3} (1+y^4)^{\frac{3}{2}} \Big|_1^5$$

$$= \frac{\pi}{9} \left[(1 + (5^4))^{\frac{3}{2}} - (1 + 1^4)^{\frac{3}{2}} \right]$$

$= 5466.26$ sq. units.