Integral Test:

Let $\{a_n\}$ be the sequence of positive terms.

Suppose $a_n = f(x)$ where f is continuous positive decreasing function of x for all $x > N$.

Thus, if integral $\int_n^{\infty} f(x)\,dx$ converges then $\sum_{n=N}^{\infty} a_n$ also converges.

Questions:

(i): $\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$.

Soln.

The corresponding function is,

$$f(x) = \frac{1}{(x+2)^2}$$

For interval $[1, \infty)$, $f(x)$ is always positive and continuous.

Here, and

$$f'(x) = \frac{-2}{(x+2)^3} < 0$$, it is always decreasing.

We can use integral test.

$$\int_1^{\infty} f(x)\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{(x+2)^2}\,dx$$

$$= \lim_{b\to\infty} \left.\frac{-1}{(x+2)}\right|_1^b = \lim_{b\to\infty}\left[\frac{-1}{(b+2)} + \frac{1}{3}\right]$$

$$= 0 + \frac{1}{3} = \frac{1}{3}$$ ie, finite value.

Since the integral converges, the series also converges.

(ii): $\sum_{n=1}^{\infty} \frac{2n}{3n^2+4}$

Soln:

The corresponding function is,

$$f(x) = \frac{2x}{3x^2+4}$$

For interval $[1, \infty)$, $f(x)$ is always positive and continuous and,

$$f'(x) = \frac{-6x^2+8}{(3x^2+4)} < 0$$, it is always decreasing.

We can use integral test.

$$= \int_1^{\infty} f(x)\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{2x}{3x^2+4}\,dx = \frac{1}{3}\lim_{b\to\infty} \ln|3x^2+4|\Big|_1^b$$

$$= \frac{1}{3}\left[\lim_{b\to\infty} \ln|3b^2+4| - \ln|3+4|\right]$$

$$= \infty$$

Since, the integral diverges the series also diverges.

(iii) $\sum_{n=1}^{\infty} \frac{1}{n}$

Solⁿ:

The corresponding function is,

$f(x) = \frac{1}{x}$

For interval $[1, \infty)$, $f(x)$ is positive and continuous.

and,

$f'(x) = \frac{-1}{x^2} < 0$, it is always decreasing.

We can use integral test:

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} \ln b - \ln 1$$

$= \infty$

Since the integral diverges, the series diverges.

(iv): $\sum_{n=1}^{\infty} \frac{2}{n^2+6n+10}$

Solⁿ:

The corresponding function is

$$f(x) = \frac{2}{n^2+6n+10} = \frac{2}{(n+3)^2+1^2}$$

For interval $[1, \infty)$, $f(x)$ is positive and continuous

and

$f'(x) = \frac{-4}{(x+3)^3} < 0$, it is always decreasing.

We can use integral test.

$$\int_1^{\infty} \frac{2}{(x+3)^2+1^2}$$

$$= 2 \lim_{b\to\infty} \int_1^b \frac{1}{(x+3)^2+1^2} = 2 \lim_{b\to\infty} \tan^{-1}(x+3)\Big|_1^b$$

$= 2\left[\tan^{-1}(b+3) - \tan^{-1}(1+3)\right]$

$= 28.07$ i.e, finite value.

Since the integral converges, the series also converges.

(v): $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Solⁿ:

The corresponding function is,

$f(x) = \frac{\ln(x)}{x}$

For interval $[1, \infty)$, it is always positive and continuous.

and

$f'(x) = \cancel{x^2 - \ln}\ \frac{x^2 - \ln x}{x^4} < 0$, it is always decreasing.

We can use integral test.

$$\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^{\infty} \frac{\ln x\,dx}{x} = \lim_{b\to\infty} \int_1^{\infty} \ln x\, d\ln x$$

$$= \frac{1}{2} \lim_{b\to\infty} \frac{(\ln x)^2}{1}\Big|_1^b$$

$$= \frac{1}{2}\left[\lim_{b\to\infty} (\ln b)^2 - (\ln 1)^2\right]$$

$$= \infty$$

Since the integral is divergent, the series also diverges.

p-series test:

The infinite series in the form $\sum_{n=1}^{p} \frac{1}{n^p}$ is called p-series.

~~If n ≥~~ If $p \leq 1$, series diverges
If $p > 1$, series converges.

Proof:

Here,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

The corresponding function is $f(x) = \frac{1}{x^p}$

Here, for interval $[1, \infty)$, we can see that the function is positive, continuous and decreasing.

We can use integral test.

$$\int_1^{\infty} \frac{1}{x^p} = \lim_{b\to\infty} \int_1^b x^{-p}\, dx$$

$$= \lim_{b\to\infty} \frac{x^{1-p}}{\cancel{p-1}\ 1-p}\Big|_1^b = \frac{1}{(1-p)} \lim_{b\to\infty} b^{1-p}\Big|_1^b$$

$$= \frac{1}{(1-p)} \times \infty^{1-p}$$

When $p = 1$,

$$= \left(\frac{1}{1-1}\right) \times (\infty^{1-1}) = \text{doesn't exist.}$$

Since integral diverges, the series also diverges.

Let $1-p = a$ such that $p-1 = a$. for ~~a <~~ $p<1$

When $p < 1$,

$$= \frac{1}{a} \times \infty^{a} = \infty$$

Since the integral is divergent, the series also diverges.

When $p > 1$

$$= \frac{1}{-a} \times \infty^{-a} = \frac{1}{-a} \times \frac{1}{\infty^a} = 0$$

Since the integral converges, the series also converges.