Unit: 2

DERIVATES

Derivatives:

The rate of change of a function is called derivative.

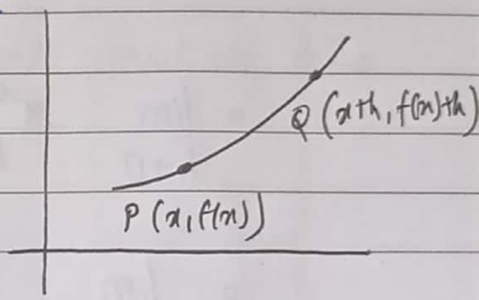
OR,

The slope of the tangent line at a particular point on a function is called derivative.

From first principle,

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or, $$\lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x}$$



The derivatives of a function at point $x = x_0$ is defined as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

provided that limit exists.

Eg: $f(x) = \frac{x}{(x-1)}$, ~~check~~ if find $f'(x)$.

Soln:

Given,

$$f(x) = \frac{x}{(x-1)}$$

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Now,
Let 'h' be the small change in value of h.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(x-1) - x(x+h-1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{x} + \cancel{xh} - h - \cancel{x^2} - \cancel{xh} + \cancel{x}}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(x+h-1)(x-1)}$$

$$= \frac{-1}{(x-1)^2}$$

$$\therefore f'(x) = -(x-1)^{-2}$$

One-Sided Derivative

A function $y = f(x)$ is differentiable on $[a,b]$ if it is differentiable on (a,b) and exists at end point.

Right hand derivative of a function $f(x)$ at $x=a$ is

$$R\,f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Left hand derivative of a function $f(x)$ at $x=a$ is

$$L\,f'(a) = \lim_{h \to 0^-} \frac{f(a-h) - f(a)}{-h}$$

A function is said to differentiable at $x=a$ if

~~Theorem:~~ $Rf'(a) = Lf'(a)$

If $Rf'(a) \neq Lf'(a)$, then derivative doesn't exist at $x=a$.

Theorem: Every differentiable function are continuous and

If f has a derivative at $x=c$, then f is continuous at $x=c$. But converse may not always be true.

Proof:

Since $f'(x)$ exists at $x=c$ and taking $h>0$, we have

$$f(c+h) - f(c) = \{f(c+h) - f(c)\} \times \frac{h}{h}$$

Taking $\lim\limits_{h \to 0}$ ~~on both sides,~~ on both sides,

$$\lim_{h \to 0} \cancel{h \to 0}\; f(c+h) - f(c)$$

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$$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \times \lim_{h \to 0} h$$

$= Rf'(c) \times 0$

$= \{\text{finite value}\} \times 0 \quad = 0$

$\therefore \lim_{h \to 0} f(c+h) - f(c) = 0$ so, $\lim_{h \to 0} f(c+h) = f(c)$

Similarly, if we take.

$$f(c-h) - f(c) = \{f(c-h) - f(c)\} \times \frac{h}{h}$$

Taking $\lim_{h \to 0}$ on both sides, we get.

$$\lim_{h \to 0} f(c-h) - f(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{h} \times \lim_{h \to 0} h$$

$= Lf'(c) \times 0$

$= \{\text{finite value}\} \times 0$

$= 0$

$\therefore \lim_{h \to 0} f(c-h) - f(c) = 0$ so, $\lim_{h \to 0} f(c-h) = f(c)$.

Hence,

$$\lim_{h \to 0} f(c+h) = \lim_{h \to 0} f(c-h) = f(c)$$

ie, f is continuous at $x=c$

For converse part;

Let us consider $f(x) = |x|$ at $x = 0$

For continuity at $x = 0$,

$$LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$$

$$RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

$\therefore f(0) = 0.$

f is continuous at $x = 0$.

At $x = 0$,

$$LHD = \lim_{h \to 0} \frac{f(0-h) - f(0)}{h}$$

$$= \lim \frac{-h}{h} = -1$$

$$RHD = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \frac{h}{h} = 1$$

Here, $LHD \neq RHD$.

So, $f'(0)$ doesn't exist.

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<Q>: $f(x) = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 \text{ for } & x = 0 \end{cases}$

Find $f'(x)$ exists.

Soln:

$$LHD = \lim_{h \to 0} \frac{(0-h)^2 \sin\left(\frac{1}{0-h}\right)}{-h}$$

$$= \lim_{h \to 0} (-h)^2 \times \frac{\sin\left(\frac{1}{0-h}\right)}{-h} = 0$$

Also,

$$RHD = \lim_{h \to 0} \frac{(0+h)^2 \sin\left(\frac{1}{0+h}\right)}{h}$$

$$= 0$$

Since, $Rf'(0) = Lf'(0)$ so, $f'(x)$ exists.

<Q>: $f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \leq x \leq 2 \\ x - x^2/2 & \text{for } x > 2 \end{cases}$

Does $f'(x)$ exist at $x = 1$ and $x = 2$?

Soln:

At $x = 1$, $f(x) = 2 - x$.

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$$\cancel{LHD = \lim_{h\to 0} \frac{f(1-h) - f(1)}{h}}$$

$$\cancel{= 2 - (1-h)}$$

$$LHD = \lim_{h\to 0} \cancel{2-(1-h)} \; (1-h)$$

$$= \cancel{2-1=1} \; 1-0=1$$

$$RHD = \lim_{h\to 0} 2-(1+h)$$

$$= 2-1=1$$

Here, LHD = RHD
so, $f'(x)$ exists at $x=1$.

At $x=2$, $f(x) = 2-x$.

$$LHD = \lim_{h\to 0} 2-(2-h)$$

$$= 0$$

$$RHD = \lim_{h\to 0} (2+h) - \frac{(2+h)^2}{2}$$

$$= 2 - \frac{4}{2} = 0$$

~~At~~ Here, LHD = RHD
So, $f'(x)$ exists at $x=2$.

Derivative formulae:

(i): $\frac{de^{ax}}{dx} = ae^{ax}$

(ii) $\frac{de^{x}}{dx} = e^{x}$

(iii): $\frac{d}{dx} a^{x} = a^{x} \cdot \ln a$

(iv) $\frac{d}{dx} \ln(x) = \frac{1}{x}$

(v): $\frac{d}{dx} \log_a x = \frac{1}{x \ln(a)}$

(vi) $\frac{d}{dx}(\sinh x) = \cosh x$

(vii) $\frac{d}{dx}(\cosh x) = \sinh x$

(viii) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(ix) $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

(x) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

(xi) $\frac{d}{dx}(\operatorname{cosech}) = -\operatorname{cosech} x \cdot \coth x$

(xii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(xiii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(xiv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xvi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

(xvii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

(xviii) $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{x^2+1}}$

(xix) $\frac{d}{dx}(\cosh^{-1}x) = \frac{-1}{\sqrt{x^2-1}}$ $\{x > 1\}$

(xx) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$ $\{|x| < 1\}$

(xxi) $\frac{d}{dx}(\coth^{-1}x) = \frac{1}{1-x^2}$ $\{|x| > 1\}$

(xxii) $\frac{d}{dx}(\operatorname{cosech}^{-1}x) = \frac{-1}{|x|\sqrt{x^2+1}}$ $\{x \neq 0\}$

(xxiii) $\frac{d}{dx}(\operatorname{sech}^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}$ $\{0 < x < 1\}$

⟨Q⟩: Find dy/dx.

(a): $y = \ln x + \sqrt{1-x^2} \cdot \sinh^{-1}x$.

Soln:

Let $u = \ln x$ and $v = \sqrt{1-x^2} \cdot \sinh^{-1}x$

So $y = u + v \Rightarrow dy/dx = du/dx + dv/dx$ —(i)

$\frac{du}{dx} = \ln x$

$\therefore \frac{du}{dx} = \frac{1}{x}$ —(ii)

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$$= \frac{dv}{dx}\left(\sqrt{1-x^2}\right) \times \sinh^{-1}x\,)$$

$$= \sqrt{1-x^2} \cdot \frac{d\sinh^{-1}x}{dx} + \sinh^{-1}x \cdot \frac{d\sqrt{1-x^2}}{d(1-x^2)} \times \left(\frac{d\cdot 1}{dx} - \frac{dx^2}{2}\right)$$

$$= \sqrt{1-x^2} \times \frac{1}{\sqrt{1+x^2}} + \sinh^{-1}x \cdot \frac{1}{\cancel{2}\sqrt{1-x^2}} \times -\cancel{2}x$$

$$= \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}} - \frac{x\sinh^{-1}x}{\sqrt{1-x^2}} \quad -(iii)$$

So, eqn (i) becomes,

$$\frac{dy}{dx} = \frac{1}{x} + \sqrt{\frac{1-x^2}{1+x^2}} - \frac{x\sinh^{-1}x}{\sqrt{1-x^2}}$$

(b): $y = \sinh^{-1}(x^2)$

Soln.

Diffrentiating both sides w.r.t x,

$$\frac{dy}{dx} = \frac{d\sinh^{-1}(x^2)}{dx^2} \times \frac{dx^2}{dx}$$

$$= \frac{1}{\sqrt{x^4+1}} \times 2x$$

$$\therefore \frac{dy}{dx} = \frac{2x}{\sqrt{x^4+1}}$$

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(c): $y = 2\sqrt{t} \cdot \tanh\sqrt{t}$

Sol^n:

Differentiating both sides wrt t

$$\frac{dy}{dt} = \frac{d}{dt}\left(2\sqrt{t} \cdot \tanh\sqrt{t}\right)$$

$$= 2\sqrt{t} \cdot \frac{d}{dt}(\tanh\sqrt{t}) + \tanh\sqrt{t} \times \frac{d\,2\sqrt{t}}{dt}$$

$$= 2\sqrt{t} \cdot \frac{d\tanh\sqrt{t}}{d\sqrt{t}} \times \frac{d\sqrt{t}}{dt} + 2\tanh\sqrt{t} \times \frac{dt^{\frac{1}{2}}}{dt}$$

$$= \cancel{2\sqrt{t}} \times \sec^2h^2\sqrt{t} \times \frac{1}{\cancel{2\sqrt{t}}} + \cancel{2}\tanh\sqrt{t} \times \frac{1}{\cancel{2}\sqrt{t}}$$

$$\therefore \frac{dy}{dx} = \operatorname{sech}^2\sqrt{t} + \tanh\sqrt{t}$$

(d): $y = \log\left(\cos\left(e^{\sqrt{\sin h}}\right)\right)$

Sol^n:

Differentiating both sides w.r.t h,

$$\frac{dy}{dh} = \frac{d\log(\cos(e^{\sqrt{\sin h}}))}{d(\cos(e^{\sqrt{\sin h}}))} \times \frac{d\cos(e^{\sqrt{\sin h}})}{de^{\sqrt{\sin h}}} \times \frac{de^{\sqrt{\sin h}}}{d\sqrt{\sin h}} \times \frac{d\sqrt{\sin h}}{d\sin h} \times \frac{d\sin h}{dh}$$

$$= \frac{1}{\cos(e^{\sqrt{\sin h}})} \times -\sin(e^{\sqrt{\sin h}}) \times e^{\sqrt{\sin h}} \times \frac{1}{2\sqrt{\sin h}} \times \cos h$$

$$\therefore \frac{dy}{dx} = \frac{-\cos h \times e^{\sqrt{\sin h}} \times \sin(e^{\sqrt{\sin h}})}{2\sqrt{\sin h} \times \cos(e^{\sqrt{\sin h}})}$$

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(e): $y = (\ln x)^{\ln x}$

Sol^n:

Taking log on both sides,

$\log y = \ln x \log(\ln x)$

Differentiating both sides w.r.t x,

$$\frac{d\log y}{dy} \times \frac{dy}{dx} = \ln x \times \frac{d\log(\ln x)}{d\ln x} \times \frac{d\ln x}{dx} + \log(\ln x) \times \frac{d\ln x}{dx}$$

or $$\frac{1}{y} \times \frac{dy}{dx} = \cancel{\ln x} \times \frac{1}{\cancel{\ln x}} \times \frac{1}{x} + \log.(\ln x) \times \frac{1}{x}$$

or, $$\frac{1}{y} \times \frac{dy}{dx} = \left(\frac{1 + \log(\ln x)}{x}\right)$$

$$\therefore \frac{dy}{dx} = (\ln x)^{\ln x} \left\{\frac{1 + \log(\ln x)}{x}\right\}$$

Tangent and Normal lines

Eq^n for a tangent at point (x_1, y_1)

$$y - y_1 = m(x - x_1)$$

~~Normal~~ Eq^n for normal at point (x_1, y_1)

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$m = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x}$$

⟨Q⟩: Find the tangent line of $y = 3/x$ at $[3, 1]$

Soln:

Here, $x_0 = 3.$

$f(x) = y = \frac{3}{x}$

We know,

$$m = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x}$$

$$= \lim_{x \to 3} \frac{\frac{3}{3} - 3/x}{3 - x}$$

$$= \lim_{x \to 3} \frac{1 - 3/x}{3 - x}$$

$$= \lim_{x \to 3} \frac{-\cancel{(x-3)}}{x\cancel{(x-3)}}$$

$$\therefore m = -\frac{1}{3}$$

Now

The eqn of tangent is, $y - y_1 = m(x - x_1)$

$$y - 1 = -\frac{1}{3}(x - 3)$$

or $3y - 3 = -x + 3$

or $x + 3y - 6 = 0$

which is the reqd eqn of tangent.

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Angle between Two Curves

Let 'θ' be the angle between two curves having slope 'm_1' and 'm_2'.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\therefore \theta = \tan^{-1}\left(\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right)$$

Related Rates

<Q>: A spherical ballon is inflated with helium at a rate of $100\pi\ ft^3/min$. How fast is the balloon Radius increasing at the instant radius is 5 ft? How fast is the surface area increasing?

Soln:

Given,

$$\frac{dv}{dt} = 100\pi\ ft^3/min$$

radius at instance $(r) = 5\ ft$.

We know,

$$V = \frac{4}{3}\pi r^3$$

or, $$\frac{dv}{dt} = d\left(\frac{4}{3}\pi r^3\right)/dt$$

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or, $100\pi = \cancel{8 \times \frac{4}{3} \pi r^2 \frac{dr}{dt}}\ d\left(\frac{4}{3}\pi r^3\right)/dr \times \frac{dr}{dt}$

or, $\overset{25}{\cancel{100}}\cancel{\pi} = 4\cancel{\pi} r^2 \times \frac{dr}{dt}$

or $\frac{dr}{dt} = \frac{25}{25}$ $\therefore \frac{dr}{dt} = 1$ ft/sec

We also now,

$SA = 4\pi r^2$

$\frac{dSA}{dt} = \frac{d(4\pi r^2)}{dr} \times \frac{dr}{dt}$

or $\frac{dSA}{dt} = 8\pi r \times 1$

or $\frac{dSA}{dt} = 40\pi$ ft^2/sec.

<Q7: When a circular plate in metal heated in an oven, its radius increases at a rate of 0.01 cm/min. At what rate is the plate's area increasing when radius ~~of~~ is 50 cm?

Soln,

~~We know~~ Given,

$\frac{dr}{dt} = 0.01$ cm/min.

radius at instance $(r) = 50$ cm.

P.T.O.

We know,

$$SA = \cancel{4}\pi r^2$$

or, $$\frac{dSA}{dt} = \frac{d}{dr}(\cancel{4}\pi r^2) \times \frac{dr}{dt}$$

or, $$\frac{dSA}{dt} = 2\pi r \times \frac{dr}{dt}$$

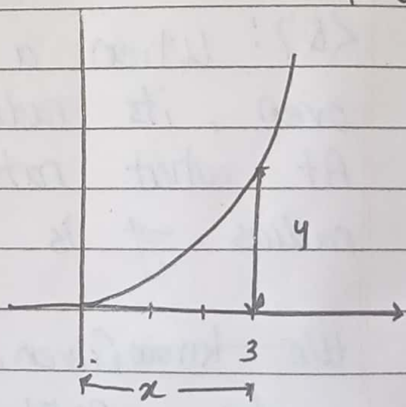
or $$\frac{dSA}{dt} = 2 \times \pi \times 50 \times 0.01$$

$$\therefore \frac{dSA}{dt} = \pi \ cm^2/min.$$

<Q>: A particle moves ~~mass~~ along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate increases at steady 10 m/sec. How fast is the angle of inclination joining the particle to the origin changing when $x = 3$ m.

Sol^n.

1st Q.



Given,

$$\frac{dx}{dt} = 10 \ m/sec$$

x at instance $= 3$ m

We know,

$$y = x^2$$

or $$\frac{dy}{dt} = \cancel{dx}\ \frac{d(x^2)}{dx} \times \frac{dx}{dt}$$

or $$\frac{dy}{dt} = 2x \times \frac{dx}{dt}$$

$$\therefore \frac{dy}{dt} = 2 \times 3 \times 10$$

$$\therefore \frac{dy}{dt} = 60 \ m/s.$$

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~~Also,~~

$$\cancel{\frac{dy}{dx} = \tan\theta}$$

$$\cancel{\text{or } \frac{(dy/dt)}{(dx/dt)} = \tan\theta}$$

If $x = 3$, $y = 9$. $[\because y = x^2]$

From figure,

$\tan\theta = \frac{y}{x}$ So, $\tan\theta = 9/3$ $\therefore \theta = \tan^{-1}(3)$
$= 71.56°$

or $\tan\theta = \frac{x^2}{x}$ or $\tan\theta = x$

Diffrentiating both sides w.r.t ~~x~~ t,

$$\frac{d\tan\theta}{d\theta} \times \frac{d\theta}{dt} = \frac{dx}{dt}$$

or, $\sec^2\theta \times \frac{d\theta}{dt} = 10$

or $\frac{d\theta}{dt} = \frac{10}{\sec^2(71.56)}$

$\therefore \frac{d\theta}{dt} = 1$ rad/sec

Linearization & Diffrentiation

If 'f' is differentiable at $x = a$ then the approxiamating function is
$L(x) = f(a) + f'(a)(x-a)$ is a linearization of f at a.

<Q>: Find linearization of f(x) at $f(x) = \sqrt{1+x}$ at $x=0$.

Sol^n:

Given,
$f(x) = \sqrt{1+x}$
$f(0) = 1$

$f'(x) = \frac{d(\sqrt{1+x})}{dx}$

$\therefore f'(x) = \frac{1}{2\sqrt{1+x}}$ $\qquad \therefore f'(0) = \frac{1}{2}$

Thus, the linearization of f(x) at $x=0$ is
$L(x) = f(a) + f'(a)(x-a)$
$= 1 + \frac{1}{2} \times (x-0)$

$\therefore L(x) = 1 + \frac{x}{2}$

<Q7: Find linearization of $f(x)$ at ~~x = 0~~ $f(x) = \cos x$ at $x = \pi/2$.

Soln:

Given,

$f(x) = \cos x$

So,

$f(\pi/2) = \cos \pi/2 = 0$

$f(x) = \cos x$

$f'(x) = -\sin x$ $\therefore f'(\pi/2) = -1$

Thus, the linearization of $f(x)$ at $x = \pi/2$ is

$L(x) = f(a) + f'(a)(x-a)$

$= 0 + -1(x - \pi/2)$

$\therefore L(x) = \frac{\pi}{2} - x.$