General Physics I (PHYS 101)

Lecture 18

Diffraction (Contd.)

Keshav Raj Sigdel

Assistant Professor

Department of Physics

Kathmandu University

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Outline

1. Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating):

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating):

An arrangement consisting of a large number of parallel slits of equal width and separated from one another by equal opaque spaces is called a diffraction grating.
Let a plane wave-front of monochromatic light be incident normally on N parallel slits each of width $'a'$ and separated by opaque distance $'b'$. The light diffracted through N slits is focused by lens L on the screen XY placed on the focal plane of lens L.

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

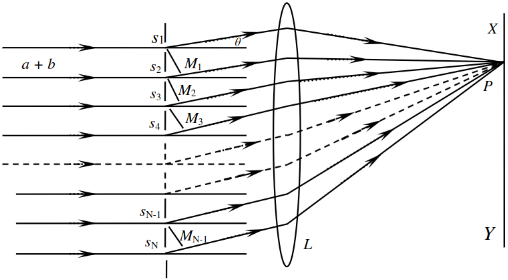

Figure 1

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

When the wave-front reaches the plane of the slits, each point on the slits sends out secondary wavelets in all directions. From the theory of Fraunhofer diffraction at a single slit, the wavelets proceeding from all points in a slit in direction θ are equivalent to a single wave of amplitude $R = \dfrac{A\sin\alpha}{\alpha}$ starting from the center of the slits where $\alpha = \dfrac{\pi a \sin\theta}{\lambda}$. Thus, the waves diffracted from all the slits in direction θ are equivalent to N parallel waves, each wave starting from the middle points $S_1, S_2, S_3, \ldots, S_{N-1}, S_N$ of the slits.

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

Let $S_1M_1, S_2M_2, ..., S_{N-1}M_{N-1}$ be the perpendiculars drawn from $S_1, S_2, ..., S_{N-1}$ on $S_2M_1, S_3M_2, ..., S_NM_{N-1}$ respectively. The path difference between the waves from S_1 and S_2 is

$$S_2M_1 = (a+b)\sin\theta$$

The path difference between the waves from S_2 and S_3 is

$$S_3M_2 = (a+b)\sin\theta$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

The path difference between the waves from S_{N-1} and S_N is

$$S_N M_{N-1} = (a+b)\sin\theta$$

Thus, as we pass from one vibration to another, the path goes on increasing by same amount

$$(a+b)\sin\theta$$

The corresponding phase difference is

$$= \frac{2\pi}{\lambda}(a+b)\sin\theta$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

Thus, as we pass from one vibration to another, the phase goes on increasing by same amount

$$\frac{2\pi}{\lambda}(a+b)\sin\theta$$

Thus in order to find the amplitude in a direction θ, we have to find the resultant amplitude of N waves, each having amplitude R and common phase difference

$$\frac{2\pi}{\lambda}(a+b)\sin\theta = 2\beta \text{ (say)}$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

Then the resultant amplitude in a direction θ is given by

$$R^{'} = R\frac{\sin N\beta}{\sin\beta} = \frac{A\sin\alpha}{\alpha}\frac{\sin N\beta}{\sin\beta}$$

The resultant intensity at P is then

$$I = R^{'2}$$

$$\therefore I = \frac{A^2\sin^2\alpha}{\alpha^2}\frac{\sin^2 N\beta}{\sin^2\beta} \qquad (1)$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

Here the factor $\frac{A^2 \sin^2\alpha}{\alpha^2}$ gives the intensity distribution in the diffraction pattern due to single slit. And it has minima at points $\alpha = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ and maxima at the points $\alpha = 0, \pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$ where $\alpha = 0$ corresponds to the principal maximum and $\alpha = \pm(2m+1)\frac{\pi}{2}$ corresponds to the secondary maxima.

The second factor $\frac{\sin^2 N\beta}{\sin^2\beta}$ gives the distribution of intensity in the pattern due to interference between the waves from all N slits. Both effects combine together give the pattern of the light differected by plane transmission grating.

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

Principal Maxima: When $\sin\beta = 0$ i.e. $\beta = \pm n\pi\,;\, n = 0, 1, 2, \ldots$ we have $\sin N\beta = 0$ so $\sin N\beta / \sin\beta$ becomes indeterminate. Now we evaluate the value of $\sin N\beta / \sin\beta$ at $\beta \to \pm n\pi$ as

$$\lim_{\beta\to\pm n\pi} \frac{\sin N\beta}{\sin\beta} = \lim_{\beta\to\pm n\pi} \frac{\frac{d}{d\beta}(\sin N\beta)}{\frac{d}{d\beta}(\sin\beta)} = \lim_{\beta\to\pm n\pi} \frac{N\cos N\beta}{\cos\beta} = N$$

Substituting this value in equation (1), we get

$$I = \frac{A^2\sin^2\alpha}{\alpha^2} N^2 \tag{2}$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

And this has maximum value. Then the resultant intensity of maxima is $\dfrac{A^2 \sin^2 \alpha}{\alpha^2} N^2$. These maxima are called the principal maxima. Thus, in order to find the resultant intensity of any of the principal maxima in the diffraction pattern, we have to multiply N^2 by the factor $\dfrac{A^2 \sin^2 \alpha}{\alpha^2}$. Hence if we increase the number of slits, the intensity of principal maxima increases. Thus, the directions of principal maxima are given by

$$\sin \beta = 0$$

$$\text{or, } \beta = \pm n\pi$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

$$\text{or, } \frac{\pi(a+b)\sin\theta}{\lambda} = \pm n\pi$$

$$\therefore\ (a+b)\sin\theta = \pm n\lambda \tag{3}$$

If we put $n = 0$, we get $\theta = 0^\circ$. And this is the direction for central maximum. The direction of central maximum for diffraction is also $\theta = 0^\circ$, so they coincide with each other and is called zero order principal maxima. If we put $n = 1, 2, 3, \ldots$ we obtain first, second, third,... order principal maxima respectively.

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

Minima:

From eq (1), for minima, we must have

$$\sin N\beta = 0 \text{ and } \sin\beta \neq 0$$

$$\text{or, } N\beta = \pm m\pi \quad \Rightarrow N\frac{\pi(a+b)\sin\theta}{\lambda} = \pm m\pi$$

$$\therefore N(a+b)\sin\theta = \pm m\lambda \tag{4}$$

Here m can have all integral values except 0, N, $2N$, ... for these values give $\sin\beta = 0$ which gives the positions of principal maxima.

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

From eq (4), it is clear that $m = 0$ gives principal maximum of zero order and $m = 1, 2, 3, ..., N-1$ give minima and then $m = N$ gives principal maxima of first order. So there must be $N-1$ minima between zero order and first order principal maxima.

Secondary maxima: As there are $N-1$ minima between two consecutive principal maxima, there must be $N-2$ secondary maxima between them. Now for secondary maxima,

$$\frac{dI}{d\beta} = 0$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

$$\text{or, } \frac{d}{d\beta}\left[\frac{A^2\sin^2\alpha}{\alpha^2}\frac{\sin^2 N\beta}{\sin^2\beta}\right] = 0$$

$$\text{or, } \frac{A^2\sin^2\alpha}{\alpha^2}\frac{2\sin N\beta}{\sin\beta}\left[\frac{N\sin\beta\ \cos N\beta - \sin N\beta\ \cos\beta}{\sin^2\beta}\right] = 0$$

$$\text{or, } N\cos N\beta\ \sin\beta - \sin N\beta\ \cos\beta = 0$$

$$\text{or, } \frac{\sin N\beta\ \cos\beta}{\cos N\beta\ \cos\beta} = \frac{N\cos N\beta\ \sin\beta}{\cos N\beta\ \cos\beta}$$

$$\therefore\ \tan N\beta = N\tan\beta$$

Then

$$\sin N\beta = \frac{\tan N\beta}{\sec N\beta} = \frac{\tan N\beta}{\sqrt{1+\tan^2 N\beta}}$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

$$= \frac{N \tan\beta}{\sqrt{1 + N^2 \tan^2\beta}}$$

$$\therefore \frac{\sin^2 N\beta}{\sin^2\beta} = \frac{N^2 \tan^2\beta}{\sin^2\beta\,(1 + N^2 \tan^2\beta)} = \frac{N^2}{\cos^2\beta\,(1 + N^2 \tan^2\beta)}$$

$$= \frac{N^2}{\cos^2\beta + N^2 \sin^2\beta} = \frac{N^2}{1 + (N^2 - 1)\sin^2\beta}$$

$$\therefore \frac{\sin^2 N\beta}{\sin^2\beta} = \frac{N^2}{1 + (N^2 - 1)\sin^2\beta}$$

Thus, intensity of secondary maxima is then

$$I' = \frac{A^2 \sin^2\alpha}{\alpha^2} \frac{N^2}{1 + (N^2 - 1)\sin^2\beta}$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

Now the ratio of intensities of secondary maxima to principal maxima is

$$\frac{I'}{I} = \frac{\text{intensity of secondary maxima}}{\text{intensity of principal maxima}} = \frac{\dfrac{A^2\sin^2\alpha}{\alpha^2}\dfrac{N^2}{1+(N^2-1)\sin^2\beta}}{\dfrac{A^2\sin^2\alpha}{\alpha^2}\cdot N^2}$$

$$\therefore \frac{I'}{I} = \frac{1}{1+(N^2-1)\sin^2\beta}$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

Hence as N increases, the intensity of secondary maxima decreases. When N is very large; as in the case of diffraction grating, the secondary maxima are not visible in the spectrum. In such cases, there is uniform darkness between any two consecutive principal maxima. The intensity distribution is as shown.

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

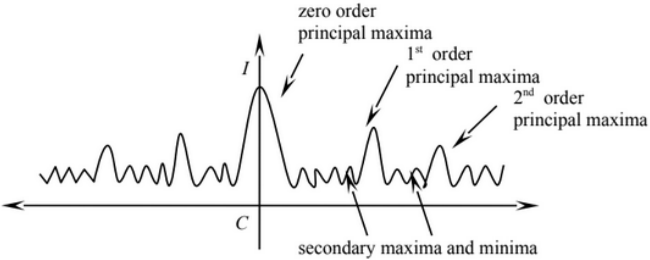

Figure 2

Maximum number of orders with a diffraction grating

For a diffraction grating,

$$(a+b)\sin\theta = n\lambda$$

Where a is width of slit and b is that of opaque portion.

$$\therefore\ n = \frac{(a+b)\sin\theta}{\lambda}$$

Fraunhofer diffraction due to N parallel slits (Theory of diffraction grating): (contd.)

For normal incidence, maximum possible values of angle of diffraction is $\theta = 90^{\circ}$. So, the maximum number of possible orders is

$$n_{\max} = \frac{(a+b)\sin 90^{\circ}}{\lambda} \qquad \therefore\ n_{\max} = \frac{a+b}{\lambda}$$

This relation gives the number of orders with a diffraction grating.

For $a+b < 3\lambda$, only up to 2^{nd} order are observed.