General Physics I (PHYS 101)

Lecture 06 Dynamics of system of particles

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Febrary 27, 2023

Outline

1. Two dimensional elastic collision

2. Central force, Two-body problems and reduced mass
   - Central force
   - Two-body problems and reduced mass

Two dimensional elastic collision

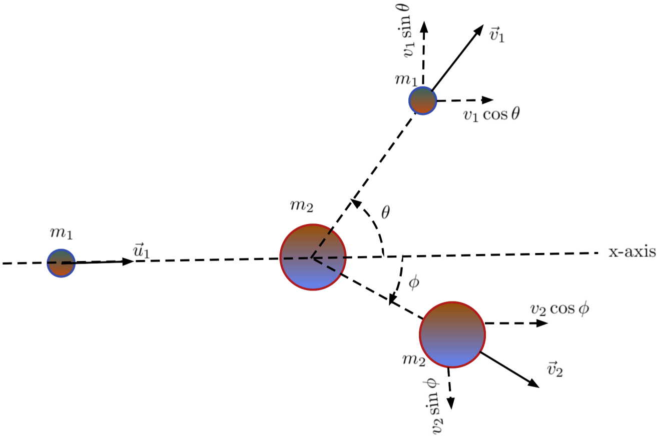

Figure 1: Two dimensional collision

Two dimensional elastic collision

(contd.)

Consider a body of mass m_1 moving with velocity $\vec{u}_1$ along x-axis elastically collides with another body of mass m_2 remains rest at origin. After collision, the first body scatters with the velocity $\vec{v}_1$ making an angle θ with its initial direction i.e. x-axis and the second body recoils with velocity $\vec{v}_2$ making an angle ϕ on the opposite side of x-axis as shown in figure 1. From the conservation linear momentum

$$m_1\vec{u}_1 = m_1\vec{v}_1 + m_2\vec{v}_2 \tag{1}$$

Two dimensional elastic collision

(contd.)

Taking x-component of equation (1)

$$m_1u_1 = m_1v_1\cos\theta + m_2v_2\cos\phi$$
$$\implies m_2v_2\cos\phi = m_1u_1 - m_1v_1\cos\theta \qquad (2)$$

Taking y-component of equation (1)

$$0 = m_1v_1\sin\theta - m_2v_2\sin\phi$$
$$\implies m_2v_2\sin\phi = m_1v_1\sin\theta \qquad (3)$$

Two dimensional elastic collision

(contd.)

Squaring and adding equations (2) and (3), we get

$$m_2^2 v_2^2 = m_1^2 u_1^2 + m_1^2 v_1^2 - 2m_1^2 u_1 v_1 \cos\phi \tag{4}$$

Since the collision is elastic, the total kinetic energy remains conserved before and after collision, i.e.

$$\frac{1}{2}m_1 u_1^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$\implies m_1 u_1^2 = m_1 v_1^2 + m_2 v_2^2 \tag{5}$$

Two dimensional elastic collision

(contd.)

Multiplying equation (5) by m_2 and using equation (4), we get

$$m_1 m_2 u_1^2 = m_1 m_2 v_1^2 + m_1^2 u_1^2 + m_1^2 v_1^2 - 2m_1^2 u_1 v_1 \cos\theta$$

$$(m_1 + m_2)\, v_1^2 - 2m_1 u_1 v_1 \cos\theta + (m_1 - m_2)\, u_1^2 = 0 \qquad (6)$$

In general the scattering angle θ can be measured and solving equation (6), we can calculate v_1 in term of initial information. After substituting the value of v_1 in equation (5), the value of v_2 can be calculated and using either equation (2) or (3) we can determine the recoil angle ϕ.

Two dimensional elastic collision

(contd.)

For equal masses i.e. $m_1 = m_2$, equation (6) reduces to

$$v_1 = u_1 \cos\theta \tag{7}$$

Using equation (5)

$$v_2 = u_1 \sin\theta \tag{8}$$

and using equation (3)

$$\sin\phi = \cos\theta \implies \phi = \sin^{-1}\cos\theta = 90^\circ - \theta$$

i.e. $\phi + \theta = 90^\circ$

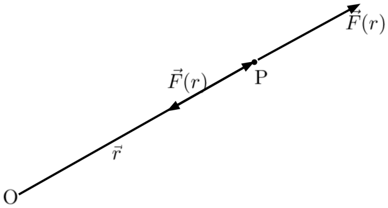

Figure 2: A central force

A force at a point is said to be a central force whose magnitude is the function of distance from a fixed and lies on the line joining the point and the fixed point. Let, O be a fixed point and P be a point with position vector $\vec{r}$ as shown in figure 2. The force $\vec{F}(r)$ is said to be the central force if it is the function of r and whose direction is either toward or away from the fixed point O, i.e.

$$\vec{F}(r) = \pm F(r)\hat{r} \tag{9}$$

$F(r)$ is the magnitude of the central force. $+$ve sign indicates that the central force is directed away from the fixed point and $-ve$ sign shows that the force is directed towards the fixed point.

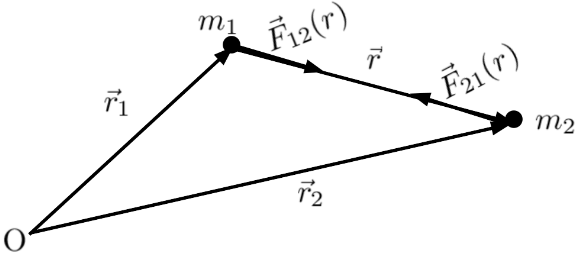

Figure 3: Two-body problem

Consider two bodies of masses m_1 and m_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$ with respect to origin O as shown in figure 3. $\vec{r}$ is the separation vector between the directed from first body to the second. These two bodies are acted by the force due to their mutual interaction. $\vec{F}_{12}(r)$ is the force acted on first body due to the second and $\vec{F}_{21}(r)$ is force on the second body due to the first. These forces are the function of separation distance r and lie on the line joining the two bodies. So that these are central force with respect to the fixed as the position of either body i.e. $\vec{F}_{12}(r)$ is the central force for the position of m_2 as the fixed point and $\vec{F}_{21}(r)$ is that of for the position of m_1 as the fixed point. From Newton's third law,

$$\vec{F}_{12}(r) = -\vec{F}_{21}(r) = F(r)\hat{r} \tag{10}$$

Two-body problems and reduced mass (contd.)

Here, $F(r)$ is the magnitude of the force of mutual interaction.

Applying Newton's second law first body

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \vec{F}_{12}(r) \implies \frac{d^2\vec{r}_1}{dt^2} = \frac{1}{m_1} F(r)\hat{r} \tag{11}$$

Applying Newton's second law for the second body

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = \vec{F}_{21}(r) \implies \frac{d^2\vec{r}_2}{dt^2} = -\frac{1}{m_2} F(r)\hat{r} \tag{12}$$

Subtracting equation (11) from equation (13), we get

$$\frac{d^2\left(\vec{r}_2 - \vec{r}_1\right)}{dt^2} = -\left(\frac{1}{m_2} + \frac{1}{m_1}\right) F(r)\hat{r} \tag{13}$$

From triangle law of vector addition, $\vec{r}_1 + \vec{r} = \vec{r}_2 \implies \vec{r}_2 - \vec{r}_1 = \vec{r}$. So

$$\frac{d^2\vec{r}}{dt^2} = -\frac{1}{\mu}F(r)\hat{r} \tag{14}$$

$$\implies \mu\frac{d^2\vec{r}}{dt^2} = -F(r)\hat{r} \tag{15}$$

where, $\dfrac{1}{m_1} + \dfrac{1}{m_2} = \dfrac{1}{\mu}$ or

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{16}$$

is called the reduced mass of the two-body system.

According to equation (15), the two-body problem can be reduced to a single body problem under the central force of mutual interaction with the reduced mass μ placed at the position of m_2 setting the position of m_1 as the fixed point or vice-versa.