Moment of Inertia of Homogenous Circular Disc

Consider a circular disc of radius R and mass M on xy plane with center at origin. O.

As result z-axis is $\perp^r$ to the disc and passing through CM O of disc.

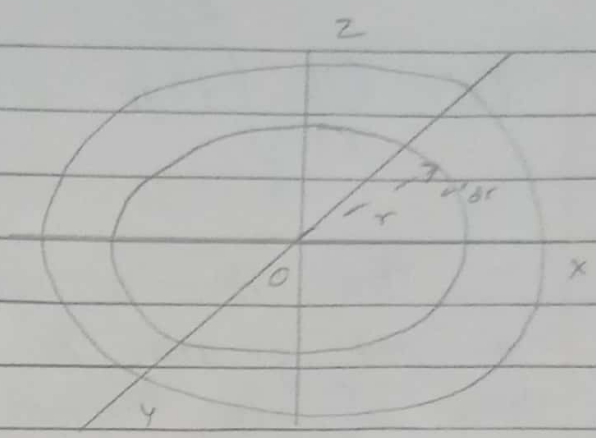

Fig. M.I of Disc

To check moment of inertia on z-axis, the disc is divided into many concentric circular ring with elemental thickness 'dr'.

Let us consider a ring of elemental thickness 'dr' and radius 'r'.

The elemental area of the ring is,

$da = 2\pi r\, dr$

and elemental mass.

$$dm = \frac{M}{\pi R^2} \times da = \frac{M}{\pi R^2} \times 2\pi r dr = \frac{2M}{R^2} r \cdot dr$$

$$\therefore dm = \frac{2M}{R^2} r dr$$

The moment of inertia of the ring about the z-axis is.

$$dI_z = dm\, r^2 = \left(\frac{2M}{R^2} r dr\right) r^2 = \left(\frac{2M}{R^2} r^3\right) dr$$

The moment of inertia of the whole disc about z-axis is,

$$I_z = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{2M}{R^2} \times \frac{R^4}{4} = \frac{1}{2} MR^2$$

The disc is symmetrical about both x- and y- axis. So, $I_x = I_y$.
Using $\perp^r$ axis theorem of M.o.I,

$$I_x + I_y = I_z$$

or, $I_x + I_x = \frac{1}{2} MR^2$

$\therefore I_x = \frac{1}{4} MR^2$

Hence, moment of inertia of the ring about x- or y-axis i.e, diameter $= \frac{1}{4} MR^2$.

Using parallel axes theorem, the moment of inertia about tangent is.

$$I_T = I_x + MR^2 = \frac{1}{4} MR^2 + MR^2$$

$\therefore I_T = \frac{5}{4} MR^2$

Moment of Inertia of a Solid Sphere

Consider a solid sphere of radius R and mass M with center at origin O. The sphere is symmetric about all axes so moment of inertia about all axes is the same.

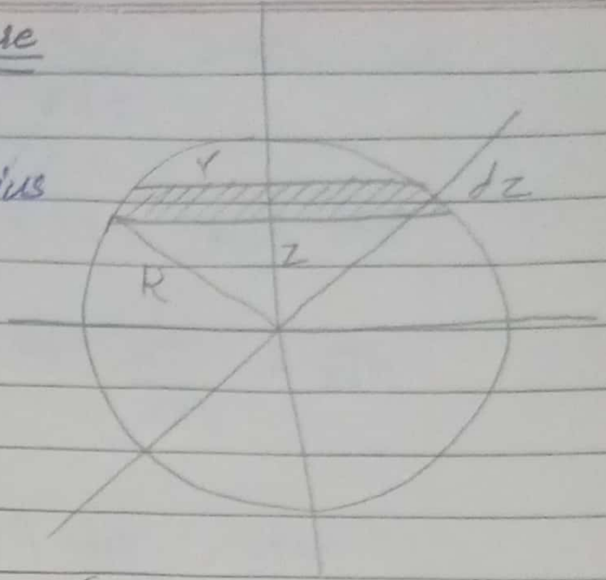

Fig: M.O.I of solid sphere

Let us divided the sphere into the coaxial circular discs of various radius with thickness dz.

Let us take a circular disc of radius 'r' at distance 'z' from O. The disc is parallel to xy-plane.

The elemental volume of disc is $dv = \pi r^2 dz$
and elemental mass of the disc

$$= \frac{M}{\frac{4}{3}\pi R^3} dv = \frac{3M}{4\pi R^3} \pi r^2 dz$$

$$\therefore dm = \frac{3M}{4R^3} r^2 dz$$

Here, z-axis is $\perp^r$ to disc and through its center. So, the moment of inertia of the disc about z-axis is:

$$dI_z = \frac{1}{2}(dm) r^2 = \frac{1}{2}\left(\frac{3M}{4R^3} r^2 dz\right) r^2 = \frac{3M}{8R^3} r^4 dz$$

$$dI_z = \frac{3M}{8R^3} r^4 dz$$

from figure, $r^2 = R^2 - z^2$ &,

$$dI_z = \frac{3M}{8R^3}(R^2-z^2)^2 dz$$

$$\therefore dI_z = \frac{3M}{8R^3}(R^4 - 2R^2z^2 + z^4)dz$$

Now, the moment of inertia of the whole solid sphere about z-axis ie, of diameter is.

$$I_z = \frac{3M}{8R^3}\int_{-R}^{+R}(R^4 - 2R^2z^2 + z^4)dz$$

$$= \frac{3M}{8R^3}\left[R^4z - 2R^2\frac{z^3}{3} + \frac{z^5}{5}\right]_{-R}^{R}$$

$$= \frac{3M}{8R^3}\left[R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5 + R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right]$$

$$= \frac{3M}{8R^3}R^5\left(2 - \frac{4}{3} + \frac{2}{5}\right)$$

$$\therefore I_z = \frac{2}{5}MR^2$$

Using parallel axes theorem, the moment of inertia about a tangent is

$$I_T = I_Z + MR^2 = \frac{2}{5}MR^2 + MR^2$$

$$\therefore I_T = \frac{7}{5}MR^2$$