Local Extreme Values

Let $f(x,y)$ be defined on a region R containing the point (a,b). Then,

(i): $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

(ii) $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

1st Derivative Test for Local Extreme Values

If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first partial derivatives exist there, then,

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0$$

*) Critical Point:

An interior point of the domain of a function $z = f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y don't exist is a critical point of f.

*) Saddle Point:

A differentiable function $f(x,y)$ has a saddle point at critical point at (a,b) if in every open disk centered at (a,b), there are domain points ~~corresponding~~ (x,y) where, $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$.

The corresponding point $(a,b,f(a,b))$ on the surface ~~(a,b)~~ $z = f(x,y)$ is called saddle point of the surface.

2nd Derivatives Test for Local Extreme Values

Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and $f_x(a,b) = f_y(a,b) = 0$. Then,

(i) f has a local maximum at (a,b) if
$f_{xx} < 0$
and
$f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b)

(ii) f has a local minimum at (a,b) if
$f_{xx} > 0$
and
$f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b)

(iii) f has a saddle point at (a,b) if
$f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b)

(iv) the test is inconclusive at (a,b) if
$f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b).

We use any other way to determine the behaviour of f at (a,b).

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called discriminant or Hessian of f.

i.e, $f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

<Q>: Find the local maxima, local minima and saddle point (if exist) of the following functions.

(i): $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Soln:

Given,
$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

We know,
The above function is differentiable for all x and y. Hence, the function has extreme points where f_x and f_y are simultaneously zero.

So,

$f_x = y - 2x - 2 = 0$ —(i) and $f_y = x - 2y - 2 = 0$. —(ii)

Solving (i) & (ii),

$(x, y) = (-2, 2)$

Local extrema point only exists on $\cancel{(x,y)}$ $(-2, 2)$

So,

$f_{xx} = -2$ $\quad f_{yy} = -2$

$(f_x)_y = 1$

Now,

$f_{xx} f_{yy} - f_{xy}^2 = (-2 \times -2) - 1$
$= 3$

Here,

$f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

So, f has local maximum at $(-2, 2)$

i.e. $f_{max} = 8$ and saddle point doesn't exist.

(ii): $f(x,y) = x^3 - y^3 - 2xy + 6$

Soln.

Given,

$f(x,y) = x^3 - y^3 - 2xy + 6$

We know, the above function is differentiable on all x and y. Hence, the function has local maximum and local minima where f_x and f_y are simultaneously zero.

So,

$f_x = 3x^2 - 2y = 0$ —(i) $\quad f_y = -3y^2 - 2x = 0$

Solving (i) and (ii), or $3y^2 + 2x = 0$

~~$(3x^2 - 2y) = (-3y^2 - 2x)$~~

or ~~$3x^2 + 3y^2 + 2x - 2y = 0$~~

from eqn (i), $y = \frac{3x^2}{2}$

~~So~~

~~$3x^2 + 3\left(\frac{3x^2}{2}\right)^2 + 2x - 2\left(\frac{3x^2}{}\right.$~~

So, putting in eqn (ii),

$3\left(\frac{3x^2}{2}\right)^2 + 2\left(\frac{3x}{2}\right) = 0$

$3\left(\frac{3x^2}{2}\right)^2 + 2x = 0$

or $\frac{27x^4}{4} + 2x = 0$

or, $27x^4 + 8x = 0$

or, $x(27x^3 + 8) = 0$

either,

$x = 0$ $\quad 27x^3 + 8 = 0$ $\quad \therefore x = -\frac{2}{3}$

When $x = 0, y = 0$

when $x = -2/3, y = 2/3$.

Hence, the critical points are $(0, 0)$ and $(-2/3, 2/3)$

Now,

$f_{xx} = 6x$ $\quad f_{xy} = -2$

$f_{yy} = -6y$

At $(0,0)$,

$f_{xx} = 0$
$f_{yy} = 0$
$f_{xy} = -2$

Now,

$f_{xx}f_{yy} - f_{xy}^2 = 0 \times 0 - (-2)^2$
$= -4$

Since, $f_{xx}f_{yy} - f_{xy}^2 < 0$, $(0,0)$ is the saddle point.

At $(-2/3, 2/3)$,

$f_{xx} = -4$ $\quad f_{xy} = -2$
$f_{yy} = -4$

Now, $f_{xx}f_{yy} - f_{xy}^2 = (-4)\times(-4) - (-2)^2$
$= 16 - 4 = 12$

Here, $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$. So, f has local maximum at $(-2/3, 2/3)$.

Now,

$f_{max} = \left(\frac{-2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 - 2 \times \frac{-2}{3} \times \frac{2}{3} + 6$

$= \frac{170}{27}$

(iii) $f(x,y) = e^{2x}\cos y$.

Sol^n:

Given,

$f(x,y) = e^{2x}\cos y$

Now,

$f_x = \frac{\partial e^{2x}}{\partial 2x} \times \frac{\partial 2x}{\partial x} \times \cos y = 2e^{2x}\cos y = 0$ —(i)

$f_y = -e^{2x}\sin y = 0$ —(ii)

~~Solving (i) and (ii)~~

$\cancel{2e^{2x}\cos y = -e^{2x}\sin y}$

or $\cancel{2e^{2x}\cos y + e^{2x}\sin y = 0}$

or $\cancel{e^{2x}(2\cos y + \sin y) = 0}$

~~Now,~~

$f_{xx} = 4e^{2x}\cos y$
$f_{yy} = -e^{2x}\cos y$

Now,

$f_x = 0$

or, $2e^{2x}\cos y = 0$

and

$f_y = 0$

or $e^{2x}\sin y = 0$

Here, we can't find critical points thus we cannot calculate local maxima and minima