Linear Transformation:

Let T be the transformation from $\mathbb{R}^n \to \mathbb{R}^m$. So, T is linear transformation if the following properties hold true.

If $\vec{a}, \vec{b} \in \mathbb{R}^n$,

(i): $T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$

(ii) $T(c\vec{a}) = c\,T(\vec{a})$.

<Q>: If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x_1, x_2) = (x_1 + x_2, 3x_1)$. Show that T is linear transformation.

Soln:

Let $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$.

Now

$$\vec{a}+\vec{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$T(\vec{a}+\vec{b}) = T\left(\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix}\right) = \begin{bmatrix} a_1+a_2+b_1+b_2 \\ 3a_1+3b_1 \end{bmatrix}$$

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$$T(\vec{a}) = T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 \\ 3a_1 \end{bmatrix}$$

$$T(\vec{b}) = T\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} b_1 + b_2 \\ 3b_1 \end{bmatrix}$$

$$T(\vec{a}) + T(\vec{b}) = \begin{bmatrix} a_1 + a_2 \\ 3a_1 \end{bmatrix} + \begin{bmatrix} b_1 + b_2 \\ 3b_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 3a_1 + 3b_1 \end{bmatrix}$$

So,

$$T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b}) \quad \text{——— (i).}$$

Let c be any scalar.

$$c\vec{a} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}$$

$$T(c\vec{a}) = T\left[\begin{pmatrix} ca_1 \\ ca_2 \end{pmatrix}\right] = \begin{bmatrix} ca_1 + ca_2 \\ 3ca_1 \end{bmatrix} = c\begin{bmatrix} a_1 + a_2 \\ 3a_1 \end{bmatrix}$$

$$= cT(\vec{a})$$

So,

$$T(c\vec{a}) = cT(\vec{a}) \quad \text{——— (ii)}$$

From (i) and (ii), we conclude that $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 + x_2, 3x_1)$ is linear combination.

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<Q>: If $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x, y, z) = (x + y + z, 0)$ then, Show that T is linear transformation.

Soln:

Let $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$

$$\vec{a} + \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$$

$$T(\vec{a} + \vec{b}) = T\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 + a_3 + b_1 + b_2 + b_3 \\ 0 \end{bmatrix}$$

$$T(\vec{a}) = T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 + a_3 \\ 0 \end{bmatrix}$$

$$T(\vec{b}) = T\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right) = \begin{bmatrix} b_1 + b_2 + b_3 \\ 0 \end{bmatrix}$$

$$T(\vec{a}) + T(\vec{b}) = \begin{bmatrix} a_1 + a_2 + a_3 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 + b_2 + b_3 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 + b_1 + b_2 + b_3 \\ 0 \end{bmatrix}$$

So, $T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$ ——— (i)

Let $\vec{c}$ be any scalar

$$\vec{c}\,\vec{a} = c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}$$

$$T(c\vec{a}) = T\left(\begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}\right) = \begin{bmatrix} ca_1 + ca_2 + ca_3 \\ 0 \end{bmatrix}$$

$$= c\begin{bmatrix} a_1 + a_2 + a_3 \\ 0 \end{bmatrix} = c\,T(\vec{a})$$

So, $T(c\vec{a}) = c\,T(\vec{a})$ —— (ii)

From (i) and (ii), we can conclude that.

$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T(x, y, z) = (x+y+z, 0)$ is a linear transformation.