Electric Potential

Potential Energy per unit charge at a point in an electric field is called electric potential (V) of that point.

Mathematically,

$$V = \frac{U}{q}$$

→ It is a scalar quantity.

→ SI unit: 1 volt = 1 J/C

The electric potential at an arbitrary point P in an electric field equals the work required per unit charge to bring a positive test charge from infinity to that point.

Mathematically,

$$V(\vec{r}) = V_P = W_{(unit)\,\infty \to P} = -\int_{\infty}^{P} \vec{E} \cdot \vec{dl}$$

Electric potential obeys the superposition principle.

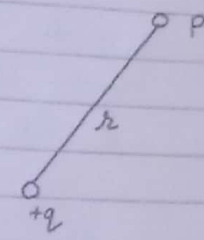
Mathematically,

$$V = V_1 + V_2 + \ldots$$

The potential at any given point is the sum of the potentials due to all source charges separately.

(x) Expression for Electric Potential

The electric potential of a point charge at a point P,



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}$$

The potential of collection of charges.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i}$$

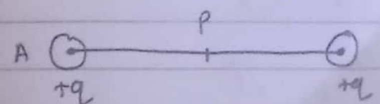
The potential for a continuous distribution of charges

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r}$$

If 'σ' is the surface charge density,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma\, da'}{r}$$

(Q):



$\Rightarrow \varepsilon = 0, V \neq 0$

(X) Potential Difference:

The potential difference ~~from~~ between two points a and b is equal to the workdone per unit charge required to carry a charged particle from a and b.

$$V(b) - V(a) = W(unit)_{a \to b} = -\int_a^b \vec{E} \cdot d\vec{l}$$

⟺ Electric field is the Negative Gradient of Scalar Potential

The potential difference between two points a and b

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} \quad \text{——(i).}$$

The fundamental theorem for gradients states that,

$$V(b) - V(a) = \int_a^b (\nabla V) \cdot d\vec{l} \quad \text{——(ii)}$$

From eq^n (i) and (ii), we get

$$\text{or} \quad -\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b (\nabla V) \cdot d\vec{l}$$

Since this is true for any points a and b, the integrands must be equal.

$$\therefore \vec{E} = -\nabla V. \quad \text{——(iii)}$$

{3 marks}

Thus, we can say, electric field is the negative gradient of scalar potential.

Now,
we have, gauss's law in differential form,

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\text{or,} \quad \nabla(-\nabla \cdot V) = \frac{\rho}{\varepsilon_0}$$

$$\therefore \nabla^2 V = \frac{\rho}{\varepsilon_0} \quad \text{——(iv)}$$

Eq^n (iv) is known as Poission's equation.

In the regions where there is no charge, $\rho = 0$. Then, the Poission's equation reduces to Laplace's equation.

$$\therefore \nabla^2 V = 0 \quad \text{——(v)}$$

Eq^n (v) is known as Laplace's equation.

<Q>: Find the expression for electric fields in a region where potential : $V = -kxy$.

Sol^n

We know,

$$V = -kxy$$

So,

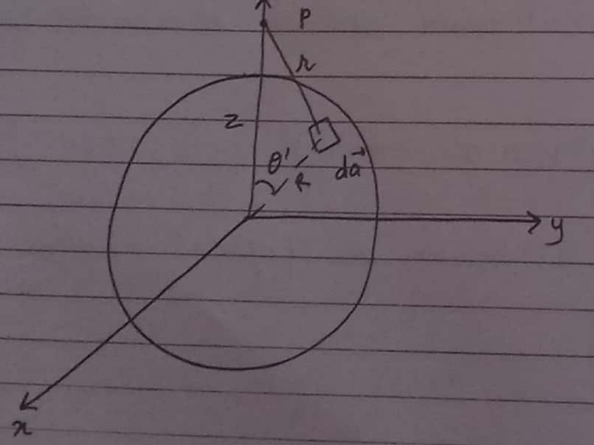
$$\vec{E} = -\nabla V = -\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right]$$

$$= -\left[\hat{i}\frac{\partial(-kxy)}{\partial x} + \hat{j}\frac{\partial(-kxy)}{\partial y} + \hat{k}\frac{\partial(-kxy)}{\partial z}\right]$$

$$= ky\,\hat{i} + kx\,\hat{j}$$

<Q>: Find the potential of a uniformly charged spherical shell of radius R.

Sol^n:



Let us consider a uniformly charged solid sphere having surface charge density (σ), radius (R).

Let us consider an elemental area $d\vec{a}$ on the surface that produces electric potential at P z distance from the center.

Now, we know,

The electric potential for a surface charge is

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, da'}{r} \quad \text{—(i)}$$

From the law of cosines,

$$r^2 = R^2 + z^2 - 2Rz\cos\theta'$$

We know, $d\vec{a} = R^2 \sin\theta'\, d\theta'\, d\phi'$

Now eq^n (i) can be written as,

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma R^2 \sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= \frac{\sigma R^2}{4\pi\varepsilon_0}\int \frac{\sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}$$

$$= \frac{\sigma R^2}{4\pi\varepsilon_0}\left[\left\{\int_0^{\pi} \frac{\sin\theta'\, d\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}}\right\}\left\{\int_0^{2\pi} d\phi'\right\}\right]$$

$$= \frac{\sigma R^2}{2\varepsilon_0}\int_0^{\pi} \frac{\sin\theta'\, d\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}} \quad \text{—(ii)}$$

Put $R^2+z^2-2Rz\cos\theta' = t^2$

Then,

$\cancel{2R} + \cancel{2t}\ 2Rz \sin\theta' d\theta' = 2t\,dt$

or, $\sin\theta' d\theta = \dfrac{t \cdot dt}{Rz}$

When $\theta'=0$, $t^2 = (R-z)^2$

$\therefore t = \sqrt{(R-z)^2}$

when $\theta' = \pi$, $t^2 = (R+z)^2$

$\therefore t = \sqrt{(R+z)^2}$

So, eqⁿ (ii) can be written as,

$$V = \frac{\sigma R^2}{2\varepsilon_0} \int_{\sqrt{(R-z)^2}}^{\sqrt{(R+z)^2}} \frac{\cancel{t} \cdot dt}{Rz \cdot \cancel{t}}$$

$$= \frac{\sigma R}{2\varepsilon_0 z} \int_{\sqrt{(R-z)^2}}^{\sqrt{(R+z)^2}} 1 \cdot dt$$

$$\therefore V = \frac{\sigma R}{2\varepsilon_0 z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

$$= \frac{\sigma R}{2\varepsilon_0 z}\left[(R+z) - (R-z)\right] \quad \text{—— (iii)}$$

For points outside the spherical shell, $z > R$.

$$V_{outside} = \frac{\sigma R}{2\varepsilon_0 z}\left[(R+z) - (z-R)\right]$$

$$= \frac{\sigma R^2}{\varepsilon_0 z} = \frac{\frac{q}{4\pi \cancel{R^2}} \times \cancel{R^2}}{\varepsilon_0 \cdot z}$$

$$\therefore V_{outside} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z}$$

For points on the spherical surface, $z = R$.

$$V_{surface} = \frac{\sigma R}{2\varepsilon_0 z}\left[(R+R) - (z-z)\right]$$

$$= \frac{\cancel{2}\sigma R^{\cancel{2}}}{\cancel{2}\varepsilon_0 R} \qquad [\because z = R]$$

$$\therefore V_{surface} = \frac{\sigma R}{\varepsilon_0} = \frac{\frac{q}{4\pi R^2} \times R}{\varepsilon_0}$$

$$\therefore V_{surface} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}.$$

For points inside the spherical surface $z < R$.

So,

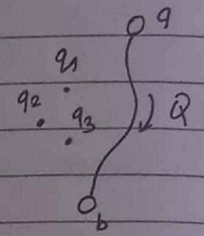
$$V_{inside} = \frac{\sigma R}{2\varepsilon_0 z}\left[(R+z) - (R-z)\right]$$

$$= \frac{\sigma R \cdot 2\cancel{z}}{\cancel{2}\varepsilon_0 \cancel{z}} = \frac{\sigma R}{\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

Here, the potential on the surface and inside of the surface are same.

Workdone to Move a Charge

Let us consider a stationary configuration of source charges and move the test charge from a to b.



At any point along path, the electric force on Q is $\vec{F} = Q\vec{E}$.

Hence, the force we exert on the charge is opposite to electric force. $= -Q\vec{E}$

Hence, the workdone to move test charge from point a to point b is.

$$W = \int_a^b \vec{F} \cdot d\vec{l} = \int_a^b (-Q\vec{E}) \cdot d\vec{l}$$

$$= Q\left[-\int_a^b \vec{E} \cdot d\vec{l}\right]$$

$$= Q\left[V(b) - V(a)\right]$$

$$\therefore V(b) - V(a) = V(\vec{r_b}) - V(\vec{r_a}) = \frac{W}{Q}$$

The potential difference between points a and b is equal to the work per unit charge required to carry a charged particle from a and b.

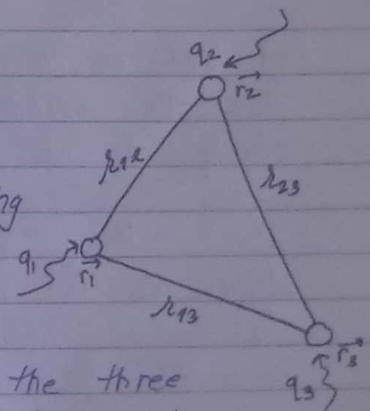
The workdone to bring the charge Q from infinity to point $\vec{r}$ is,

$$W = Q\left[V(\vec{r}) - V(\infty)\right]$$

$$\therefore W = Q\,V(\vec{r})$$

The potential energy per unit charge at a point in an electric field is called the Electric Potential at that point.

(X) Electric Potential Energy



Consider that three point charges q_1, q_2 and q_3 are lying at locations $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$ respectively.

First of all, let us remove all the three charges to infinite distance from each other.

(i): Let us move the charge q_1 from infinity to its location $\vec{r_1}$.
Here, the workdone in moving the charge from infinity to $\vec{r_1}$ ($W_1 = 0$)

(ii) Let us move the charge q_2 from infinity to its location $\vec{r_2}$.
Here, the workdone to move the charge q_2 from infinity to its location $\vec{r_2}$ is

$$W_2 = q_2 [V_1(\vec{r_2})]$$

Here $V_1(\vec{r_2})$ is the potential at $\vec{r_2}$ due to q_1.

$$\therefore W_2 = q_2 \left[\frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{12}}\right]$$

(iii) Let us move the charge q_3 to its location $\vec{r_3}$.
Here, the workdone to move the charge q_3 from infinity to its location $\vec{r_3}$ is.

$$W_3 = q_3 [V_{1,2}(\vec{r_3})]$$

where, $V_{1,2}(\vec{r_3})$ is the potential at $\vec{r_3}$ due to q_1 and q_2.

$$\therefore W_3 = q_3 \left[\frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2}{r_{23}}\right]$$

$\therefore$ The total work necessary to assemble the first three charges,
$W = W_1 + W_2 + W_3$ and is equal to potential energy U.

$$\therefore U = W = W_1 + W_2 + W_3$$

$$= 0 + \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{r_{23}}$$

$$\therefore U = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

$$= \frac{1}{2} \times \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{3} \sum_{\substack{j=1 \\ i\neq j}}^{3} \frac{q_i q_j}{r_{ij}}$$

For a system of n-point charges, we have

$$U = \frac{1}{2} \times \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j}}^{n} \frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i \left(\frac{1}{4\pi\varepsilon_0} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_j}{r_{ij}} \right)$$

$$\therefore U = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r_i}) \quad \text{--- (i)}$$

where,
$V(\vec{r_i}) = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{j=1 \\ i\neq j}}^{n} \frac{q_j}{r_{ij}}$ is the potential at point $\vec{r_i}$ due to all other charges.

For a volume charge density (ρ), eq^n (i) becomes.

$$W = \frac{1}{2}\int \rho\, V \cdot d\tau$$

$$= \frac{1}{2}\int (\varepsilon_0 \nabla \cdot \vec{E}) \cdot V \cdot d\tau$$

$$= \frac{\varepsilon_0}{2}\int V\,(\nabla \cdot \vec{E})\, d\tau$$

$$= \frac{\varepsilon_0}{2}\left[-\int (\nabla V) \cdot \vec{E} \cdot d\tau + \int \nabla \cdot (V\vec{E}) \cdot d\tau\right] \quad [\because \vec{E} = -\nabla V]$$

$$= \frac{\varepsilon_0}{2}\left[\int \vec{E} \cdot \vec{E} \cdot d\tau + \oint_S (V\vec{E}) \cdot d\vec{a}\right]$$

When the integration is taken over all space, the surface integral goes to zero.

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\, d\tau = \int_{\text{all space}} u_E\, d\tau$$

where,

$$u_E \text{ (Energy density)} = \frac{\varepsilon_0}{2} E^2$$

(x) Note:

(i) Workdone to move a charge Q from point a and point b : $W = Q\,[V(b) - V(a)]$

(ii) Workdone to move a charge Q from ∞ to point a : $W = Q\,[V(a)]$

(iii) Energy of a continuous charge distribution.

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\, d\tau = \int_{\text{all space}} u_E\, d\tau.$$

Energy density $(u_E) = \frac{\varepsilon_0}{2} E^2$ {Unit: $J m^{-3}$}.

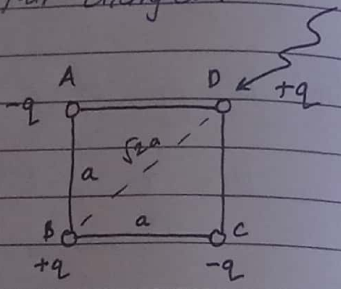
(iv) The electrostatic potential energy of configuration of three charges q_1, q_2, q_3 at locations $\vec{r}_1, \vec{r}_2$ and $\vec{r}_3$ respectively.

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

<Q7>:(a) Three charges are situated at corners of a square. How much work does it take to bring in another charge (+q) from far away to fourth corner?

(b) How much work does it take to assemble the whole configuration of four charges?

Soln:



(a): Now, at point D,

$$W_D = (+q)V$$

$$= (+q)\left[\frac{1}{4\pi\varepsilon_0}\left\{\frac{-q}{a} + \frac{q}{\sqrt{2}a} - \frac{q}{a}\right\}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\left[-2 + \frac{1}{\sqrt{2}}\right]$$

(b): The workdone to assemble the whole configuration of four charges $(W) = U$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_1q_4}{r_{14}} + \frac{q_2q_3}{r_{23}} + \frac{q_2q_4}{r_{24}} + \frac{q_3q_4}{r_{34}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a} - \frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} - \frac{q^2}{a}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-4q^2}{a} + \frac{2q^2}{\sqrt{2}a}\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0 a}\left[-4 + \frac{2}{\sqrt{2}}\right]$$

$$\therefore U = \cancel{\frac{2q^2}{4\pi\varepsilon_0}}\; 2 \times \frac{1}{4\pi\varepsilon_0} \times \frac{q^2}{a}\left[-2 + \frac{1}{\sqrt{2}}\right]$$