Lecture 04

Electrostatic Field

Outline

1. ELECTRIC CHARGE (q)
2. COULOMB'S LAW
3. THE ELECTRIC FIELD $\left(\vec{E}\right)$
   - Electric Field of a Point Charge
   - The Line Integral of Electric Field
   - The Curl of Electric Field
   - Electric Field due to a Continuous Charge Distribution
     - Line Charge Density
     - Surface Charge Density
     - Volume Charge Density
   - Some solved problems

ELECTRIC CHARGE (q)

- Charge is a fundamental and characteristics property of the elementary particles which make up matter.
- It is a scalar quantity.
- SI unit of charge is coulomb (C).
- **Kinds of Charges:**
  1. Positive Charge
  2. Negative Charge
- **Properties of Electric charge:**
  1. Like charges repel each other and unlike charges attract each other.

ELECTRIC CHARGE (q) (contd.)

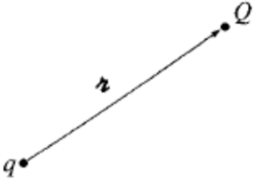
2. Electric charge is quantized. [Charge on a charged body, $\boxed{q = \pm ne}$, where n is a positive integer.]
3. Electric charge is conserved.
   [The net charge of any isolated system cannot change.]
4. The electric charge is additive in nature.
5. The charge on a body is not affected by the speed of the body.

- Elementary Charge
- The magnitude of charge on a proton or an electron is called the elementary charge and denoted by e and the value of e is 1.6×10^{-19} C.

COULOMB'S LAW

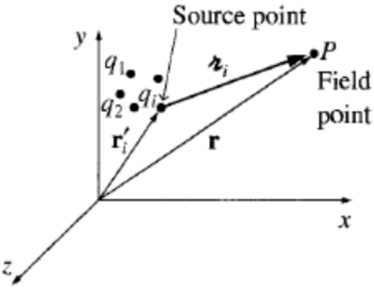
The force on a test charge Q due to a single point charge q, which is at rest a distance $\mathscr{r}$ away is given by Coulomb's law:

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}} \qquad (1)$$



The constant ε_0 is called the permittivity of free space given by $\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C}^2{\cdot}\mathrm{N}^{-1}{\cdot}\mathrm{m}^{-2}$ The magnitude of the electric force between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

- If we have several point charges $q_1, q_2, \ldots, q_n$ with position vectors $\vec{r}_1', \vec{r}_2', \ldots, \vec{r}_n'$ from origin O and with separation vectors $\vec{\mathscr{r}}_1, \vec{\mathscr{r}}_2, \ldots, \vec{\mathscr{r}}_n$ from the test charge Q, then the total force on Q due to all the source charges is given by



$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

according to Principle of Superposition:

The interaction between any two charges is completely unaffected by the presence of others

$$\implies \vec{F} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^{\,2}}\hat{\mathscr{r}}_1 + \frac{q_2 Q}{\mathscr{r}_2^{\,2}}\hat{\mathscr{r}}_2 + \ldots\right)$$

$$= Q\left(\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^{\,2}}\hat{\mathscr{r}}_i\right)$$

.

$$\therefore \boxed{\vec{F} = Q\vec{E}} \tag{2}$$

with

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^{\,2}}\hat{\mathscr{r}}_i \tag{3}$$

THE ELECTRIC FIELD $\left(\vec{E}\right)$ (contd.)

is the electric field of the source charges at a field point P

- The electric field $\vec{E}$ at a point in space is defined as the electric force $\vec{F}$ acting on a point positive test charge Q placed at that point divided by the magnitude of the test charge:

$$\boxed{\vec{E} = \frac{\vec{F}}{Q}} \tag{4}$$

- The **electric field** is a vector quantity that varies from point to point and is determined by the configuration of source charges.
- The SI unit of **electric field** is newton per coulomb ($N{\cdot}\mathrm{C}^{-1}$).

- **Principle of Superposition:**

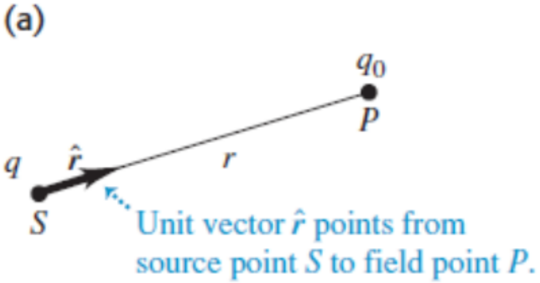
  The total field is a vector sum of their individual fields:

  $$\boxed{\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots}$$

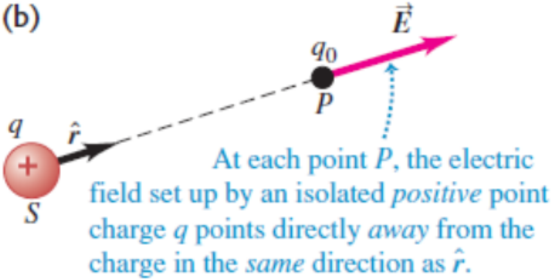
- The electric field produced at point by an isolated point charge q at the origin is given by

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}}$$

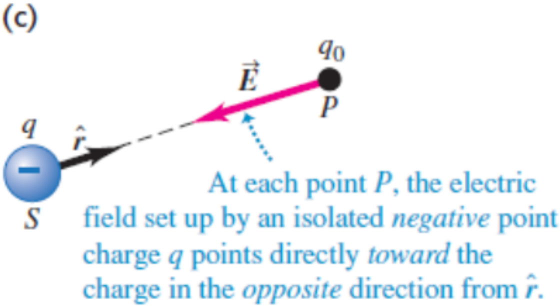
Electric Field of a Point Charge (contd.)



Electric Field of a Point Charge (contd.)



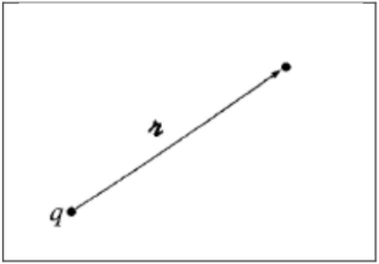
Electric Field of a Point Charge (contd.)



$$\boxed{\vec{E} \text{ is produced by } q \text{ but acts on the charge } q_0 \text{ at point } P.}$$

- The electric field $\vec{E}$ produced at field point P by an isolated point charge q at the source point S is given by

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^3}\vec{\mathscr{r}}}$$



The Line Integral of Electric Field

- The electric field at point P with position vector $\vec{r}$ due to a point charge q located at the origin is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

- The line integral of electric field $\vec{E}$ along a curve path from $a \to b$ as shown in figure 1

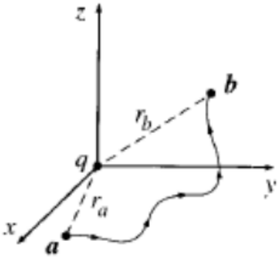


Figure 1

$$
\begin{aligned}
\int_a^b \vec{E}\cdot d\vec{l} &= \int_a^b \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\right)\cdot\left(dr\hat{r}+rd\theta\,\hat{\theta}+r\sin\theta d\phi\hat{\phi}\right)\\
&= \frac{q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}dr\\
&= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_a}^{r_b}\\
&= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_a}-\frac{1}{r_b}\right]
\end{aligned}
$$

where r_a is the distance from the origin to the point a and r_b is the distance from the origin to the point b.

The Line Integral of Electric Field (contd.)

- The line integral of electric field around a closed path is zero: $\boxed{\oint \vec{E}\cdot d\vec{l} = 0}$.
- The amount of work done by the electric field $\vec{E}$ when a unit positive charge moves from point a to point b is given by the line integral of electric field. i.e.

$$\boxed{W_E = \int_a^b \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right]}$$

The work done by the *electric field* $\vec{E}$ when a unit positive charge moves from some point a to some other point b depends only on these points and not on the path followed. Therefore the electric field due to stationary charges is **conservative field**.

- The line integral of electric field around a closed path is zero, i.e.

$$\oint_c \vec{E}\cdot d\vec{l} = 0$$

$$or \int_S (\nabla\times\vec{E})\cdot d\vec{a} = 0 \qquad \text{[Using Stoke's Theorem]}$$

$$\therefore \boxed{\nabla\times\vec{E} = 0}$$

The electric field is a very special kind of vector function whose curl is always zero.

OR

The Curl of Electric Field (contd.)

- The electric field at a point with position vector $\vec{r}$ due to a point charge q located at the origin is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}$$

So,

$$\begin{aligned}\nabla\times\vec{E} &= \nabla\times\left[\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\vec{r}\right]\\ &= \frac{q}{4\pi\varepsilon_0}\left[\nabla\times\left(\frac{\vec{r}}{r^3}\right)\right]\\ &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r^3}(\nabla\times\vec{r})+\nabla\left(\frac{1}{r^3}\right)\times\vec{r}\right]\end{aligned}$$

The Curl of Electric Field (contd.)

$$= \frac{q}{4\pi\varepsilon_0}\left[0 + (-3r^{-3-2})\vec{r} \times \vec{r}\right]$$

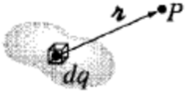
$$\therefore \nabla \times \vec{E} = 0$$

[Here, we have used $\nabla r^n = nr^{n-1}\hat{r} = nr^{n-2}\vec{r}$]

The curl of electric field $\vec{E}$ is zero.

Electric Field due to a Continuous Charge Distribution

If the source charges are continuously distributed in the space, then the charge can be divided into infinitely large number of elemental charge dq and the net electric field is given by



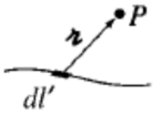
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{r}$$

where $\vec{r}$ is the separation vector of field point P from the elemental charge dq in the charge distribution

Electric Field due to a Continuous Charge Distribution

Line Charge Density

- If the charges are continuously distributed along the line, L, then the elemental charge dq can be written as



Line charge, λ

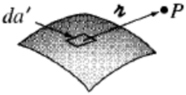
$$dq = \lambda dl'$$

where dl' is the elemental length on the line and λ is the charge per unit length, also called line charge density. The corresponding electric field is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(r')dl'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

Surface Charge Density

- If the charges are continuously distributed over the surface, S, then the elemental charge dq can be written as



Surface charge, σ

$$dq = \sigma da'$$

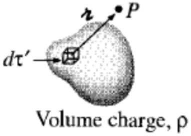
where da' is the elemental area on the surface and σ is the charge per unit area, also called surface charge density. The corresponding electric field is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(r')da'}{\mathscr{r}^2} \hat{\mathscr{r}}$$

Electric Field due to a Continuous Charge Distribution

Volume Charge Density

- If the charges are continuously distributed in the space, V, then the elemental charge dq can be written as



$$dq = \rho d\tau'$$

where $d\tau'$ is the elemental volume in the space and ρ is the charge per unit volume, also called volume charge density. The corresponding electric field is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')d\tau'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

1. **Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges, q, a distance d apart (Figure Ep-1). Check that your result is consistent with what you'd expect when $z \gg d$.**

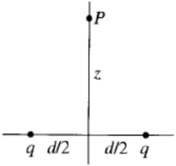
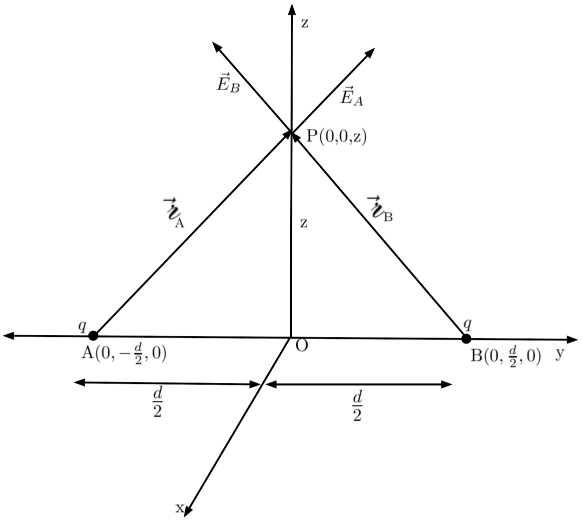


Figure Ep-1

Solution:-

Let the point P be a point at the distance z above the midpoint O between two equal charges q a distance d apart.


z
$\vec{E}_B$
$\vec{E}_A$
P(0,0,z)
$\vec{r}_A$
$\vec{r}_B$
z
q
q
A$(0,-\frac{d}{2},0)$
O
B$(0,\frac{d}{2},0)$
y
$\frac{d}{2}$
$\frac{d}{2}$
x

From Figure

$$\vec{r}_A = (0-0)\,\hat{i} + \left(0+\frac{d}{2}\right)\hat{j} + (z-0)\,\hat{k} = \frac{d}{2}\hat{j} + z\hat{k}$$

$$\Rightarrow r_A = \left(\frac{d^2}{4} + z^2\right)^{\frac{1}{2}}$$

and

$$\vec{r}_B = (0-0)\,\hat{i} + \left(0-\frac{d}{2}\right)\hat{j} + (z-0)\,\hat{k} = -\frac{d}{2}\hat{j} + z\hat{k}$$

$$\Rightarrow r_B = \left(\frac{d^2}{4} + z^2\right)^{\frac{1}{2}}$$

Electric Field at P due to the charge at A is

$$\begin{aligned}\vec{E}_A &= \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}_A^3}\vec{\mathscr{r}}_A \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}\left(\frac{d}{2}\hat{j}+z\hat{k}\right)\end{aligned} \tag{5}$$

Electric Field at P due to the charge at B is

$$\begin{aligned}\vec{E}_B &= \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}_B{}^3}\vec{\mathscr{r}}_B \\ &= \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}\left(-\frac{d}{2}\hat{j}+z\hat{k}\right)\end{aligned} \tag{6}$$

Therefore, total electric filed at P is

$$\begin{aligned}
\vec{E} &= \vec{E}_A + \vec{E}_B \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}\left(\frac{d}{2}\hat{j}+z\hat{k}\right) + \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}\left(-\frac{d}{2}\hat{j}+z\hat{k}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{\left(\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}\left[\left(\frac{d}{2}\hat{j}+z\hat{k}\right)+\left(-\frac{d}{2}\hat{j}+z\hat{k}\right)\right]
\end{aligned}$$

$$\therefore \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}q\frac{2z}{\left(\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}\hat{k}}$$

when $z \gg d$, the Electric field reduces to

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} q \frac{2z}{\left(z^2\right)^{\frac{3}{2}}} \hat{k}$$

$$\therefore \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{2q}{z^2} \hat{k}}$$

and same as that of due to point charge of magnitude $2q$

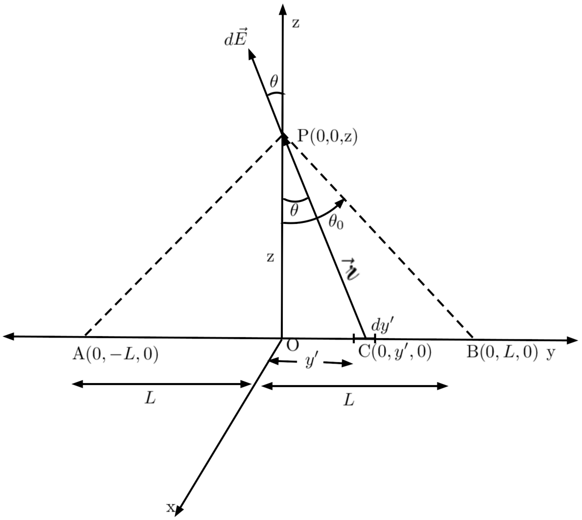
2. **Find the electric field a distance z above the midpoint of a straight line segment of length $2L$, which carries a uniform charge λ.**

   Solution:

   Consider a straight line segment AB of length $2L$ along y-axis, which carries a uniform charge λ.

   The origin O of the coordinate system is at the center of the straight line segment so that portion of the straight line segment AB along + y-axis $+L$ is and along – y-axis is $-L$. [Figure 2]

   Let the point P be at a distance z above the midpoint of the straight line segment AB.



z
$d\vec{E}$
θ
P(0,0,z)
θ
θ_0
z
$\vec{r}$
dy'
A$(0,-L,0)$
O
y'
C$(0,y',0)$
B$(0,L,0)$
y
L
L
x

Figure 2

From Figure

$$\vec{\imath} = (0-0)\,\hat{i} + \left(0-y'\right)\hat{j} + (z-0)\,\hat{k} = -y'\hat{j} + z\hat{k}$$

$$\Rightarrow \imath = \left(y'^2 + z^2\right)^{\frac{1}{2}}$$

The charge on an element of length dy' at C along the line is $dq = \lambda dy'$. The electric field at P due to the charge $dq\left(= \lambda dy'\right)$ is given by

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\imath^3}\vec{\imath}$$

$$\therefore \boxed{d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy'}{\left(y'^2+z^2\right)^{\frac{3}{2}}}\left(-y'\hat{j}+z\hat{k}\right)} \tag{7}$$

Hence total electric field at P due to the whole line segment AB extending from $-L$ to $+L$ is

$$\vec{E} = \int_{-L}^{+L} d\vec{E}$$

$$= \frac{1}{4\pi\varepsilon_0}\lambda\left[\int_{-L}^{L}\frac{\left(-y'\hat{j}+z\hat{k}\right)dy'}{\left(y'^2+z^2\right)^{\frac{3}{2}}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\lambda\left[\int_{-L}^{L}\frac{-y'\hat{j}dy'}{\left(y'^2+z^2\right)^{\frac{3}{2}}}+\int_{-L}^{L}\frac{z\hat{k}dy'}{\left(y'^2+z^2\right)^{\frac{3}{2}}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\lambda\left[2\int_0^L \frac{z\,dy'}{\left(y'^2+z^2\right)^{\frac{3}{2}}}\right]\hat{k} \tag{8}$$

Because,

$$\boxed{\because \int_{-\alpha}^{\alpha} f(x)dx = \begin{cases} 2\int_0^{\alpha} f(x)dx & \text{for even function } f(x) \\ = 0 & \text{for odd function } f(x) \end{cases}}$$

Letting $y' = z\tan\theta$

$\Rightarrow dy' = z\sec^2\theta d\theta$

when $y' = 0$, then $\theta = 0$

when $y' = L$, then $\theta = \tan^{-1}\left(\frac{L}{z}\right) = \theta_0$ (say)

Now,

$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0}\lambda \left[2z\int_0^{\theta_0}\frac{zsec^2\theta d\theta}{z^3\sec^3\theta}\right]\hat{k}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}\left[\int_0^{\theta_0}\cos\theta d\theta\right]\hat{k}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}\left[\sin\theta_0\right]\hat{k}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}\left[\frac{L}{\sqrt{z^2+L^2}}\right]\hat{k}$$

$$\therefore \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,(2L)}{z\sqrt{z^2+L^2}}\hat{k}} \tag{9}$$

For points far from the line $(z \gg L)$:

$$\vec{E} \cong \frac{1}{4\pi\varepsilon_0} \frac{\lambda\,(2L)}{z^2} \hat{k}$$

$$\therefore \boxed{\vec{E} \cong \frac{1}{4\pi\varepsilon_0} \frac{q}{z^2} \hat{k}} \qquad (10)$$

For far away the line ”looks” like a point charge $q = \lambda\,(2L)$

As $L \to \infty$:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\,(2L)}{z\sqrt{z^2 + L^2}} \hat{k}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{z\sqrt{\frac{z^2}{L^2}+1}}\hat{k}$$

$$\therefore \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{z}\hat{k}} \tag{11}$$

Therefore, the field of an infinite straight wire: $\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{z}\hat{k}}$.

where z is the distance from the wire.

3. **Find the electric field a distance z above the one end of a straight line segment of length L, which carries a uniform charge λ.**

Solution:

The line segment is placed on y-axis with one end at origin O and the other end at B.

Up to Equation (7) is the same and then, the total electric field at P due to the whole line segment OB extending from 0 to $+L$ is

$$\vec{E} = \int_{0}^{+L} d\vec{E}$$

$$= \frac{1}{4\pi\varepsilon_0}\lambda\left[\int_0^L \frac{\left(-y'\hat{j}+z\hat{k}\right)dy'}{\left(y'^2+z^2\right)^{\frac{3}{2}}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\lambda\left[\int_0^L \frac{-y'dy'}{\left(y'^2+z^2\right)^{\frac{3}{2}}}\hat{j}+\int_0^L \frac{zdy'}{\left(y'^2+z^2\right)^{\frac{3}{2}}}\hat{k}\right] \tag{12}$$

Letting $y' = z\tan\theta$

$\Rightarrow dy' = z\sec^2\theta d\theta$

when $y' = 0$, then $\theta = 0$

$when y' = L, then \theta = \tan^{-1}\left(\frac{L}{z}\right) = \theta_0$ (say)

Now,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{z}\left[-\int_0^{\theta_0}\sin\theta d\theta\hat{j}+\int_0^{\theta_0}\cos\theta d\theta\hat{k}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{z}\left[(\cos\theta_0 - 1)\hat{j} + \sin\theta_0\hat{k}\right] \tag{13}$$

$$\tag{14}$$

The magnitude of electric field is

$$\begin{aligned} E &= \frac{\lambda}{4\pi\varepsilon_0 z}\sqrt{(\cos\theta_0 - 1)^2 + \sin^2\theta_0} \\ &= \frac{\lambda}{4\pi\varepsilon_0 z}\sqrt{2 - 2\cos\theta_0} \\ &= \frac{\lambda}{4\pi\varepsilon_0 z}\sqrt{2 - 2\frac{z}{\sqrt{L^2 + z^2}}} \end{aligned}$$

For $z \gg L$

$$\begin{aligned} E &= \frac{\lambda}{4\pi\varepsilon_0 z}\sqrt{2-2\left(1+\frac{L^2}{z^2}\right)^{-\frac{1}{2}}} \\ &\approx \frac{\lambda}{4\pi\varepsilon_0 z}\sqrt{2-2\left(1-\frac{L^2}{2z^2}\right)} \\ &= \frac{\lambda L}{4\pi\varepsilon_0 z^2} \end{aligned}$$

The result converges as for the point charge of magnitude λL

4. **Find the electric field a distance z above the center of a circular loop of radius r, which carries a uniform line charge λ.**

   Solution:

   Consider a circular loop of radius r, which carries a uniform line charge λ.

   The origin O of the coordinate system is at the center of the circular loop and the circular loop is in the x-y plane [Figure 3].

   Let the point P be at a distance z above the center O of the circular loop.

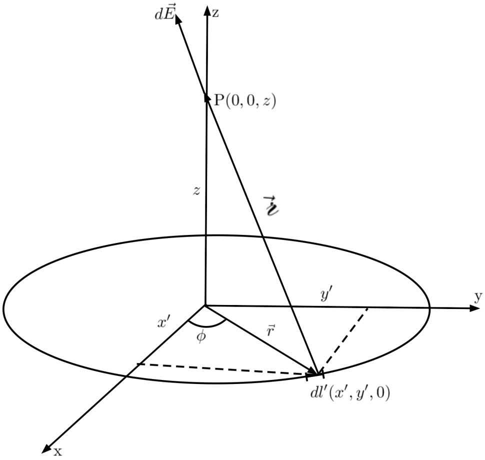


Figure 3

Let's take an elemental length dl' on the ring with coordinate (x', y', 0) which subtend an elemental angle $d\phi$ at the center. So that $dl' = rd\phi$. We can have, $x' = r\cos\phi$ and $y' = r\sin\phi$ assuming $\vec{r}$ as the position vector of the elemental length dl'. Since, the coordinate of field point P is (0, 0, z), the separation distance between P and dl' is

$$\vec{\mathscr{r}} = -x'\hat{i} - y'\hat{j} + z\hat{k} = -r\cos\phi\hat{i} - r\sin\phi\hat{j} + z\hat{k}$$

and

$$\mathscr{r} = \left(r^2 + z^2\right)^{\frac{1}{2}}$$

The electric field at P due to the charge on dl' is

$$d\vec{E} = \frac{\lambda dl'}{4\pi\varepsilon_0 \mathscr{r}^3}\vec{\mathscr{r}}$$
$$= \frac{\lambda r d\phi}{4\pi\varepsilon_0}\left[\frac{-r\cos\phi\hat{i} - r\sin\phi\hat{j} + z\hat{k}}{(r^2+z^2)^{\frac{3}{2}}}\right]$$

Therefore, the net electric field at P due to the charge on whole loop is

$$\vec{E} = \frac{\lambda r}{4\pi\varepsilon_0 (r^2+z^2)^{\frac{3}{2}}}\left[-\left(r\int_0^{2\pi}\cos\phi\, d\phi\right)\hat{i} - \left(r\int_0^{2\pi}\sin\phi\, d\phi\right)\hat{j} + \left(z\int_0^{2\pi}d\phi\right)\hat{k}\right]$$
$$= \frac{\lambda r}{4\pi\varepsilon_0 (r^2+z^2)^{\frac{3}{2}}}\left[-0\times\hat{i} - 0\times\hat{j} + z2\pi\hat{k}\right]$$
$$= \frac{2\pi r\lambda z}{4\pi\varepsilon_0 (r^2+z^2)^{\frac{3}{2}}}\hat{k}$$

Hence, the electric field at point P due to the charge on the ring is $\boxed{\vec{E} = \frac{2\pi r\lambda z}{4\pi\varepsilon_0\left(r^2+z^2\right)^{\frac{3}{2}}}\hat{k}}$ and directed along z-axis. Here $q = 2\pi r\lambda$ is the total charge on the ring. So the electric field at P can also be written as

$$\boxed{\vec{E} = \frac{qz}{4\pi\varepsilon_0\left(r^2+z^2\right)^{\frac{3}{2}}}\hat{k}}$$

For $z \gg r$, the above expression reduces to $\vec{E} = \frac{q}{4\pi\varepsilon_0 z^2}\hat{k}$ as for a point charge q

5. **Find the electric field a distance z above the center of a flat circular disk of radius R , which carries a uniform surface charge σ.What does your formula give in the limit $R \to \infty$? Also, check the case $z \gg R$.**

Solution:

Consider a flat circular disk of radius R, which carries a uniform surface charge σ.

The origin O of the coordinate system is at the center of the flat circular disk and the flat circular disk is in the x-y plane [Figure 4]. The disk can be considered as the combination of an infinite number of infinitesimally thin rings.

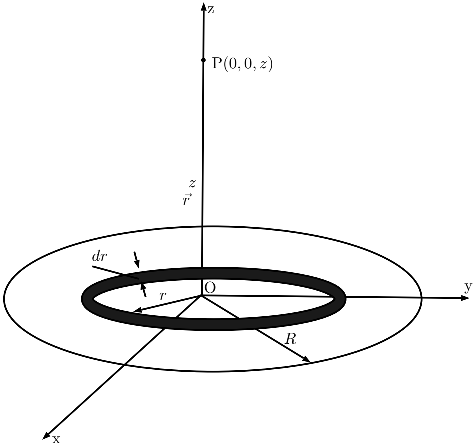


Figure 4

Consider a ring of radius r and thickness dr of this disk. The charge on this ring is

$$dq = \sigma\left(2\pi r dr\right) \tag{15}$$

Let the point P be at a distance z above the center O of the flat circular disk.

The electric field at P due to the charge $dq\left(=\sigma 2\pi r dr\right)$ on the ring is given by

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0} dq \frac{z}{\left(r^2+z^2\right)^{\frac{3}{2}}} \hat{k}$$

Using Equation (15)

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\sigma 2\pi r dr\right)\frac{z}{\left(r^2+z^2\right)^{\frac{3}{2}}}\hat{k}$$

$$\therefore \boxed{d\vec{E} = \frac{\sigma z}{2\varepsilon_0}\frac{rdr}{\left(r^2+z^2\right)^{\frac{3}{2}}}\hat{k}} \tag{16}$$

Hence the electric field at P due to the charge on the whole flat circular disk is given by

$$\vec{E}_{disk} = \int d\vec{E} = \frac{\sigma z}{2\varepsilon_0}\left[\int_0^R \frac{rdr}{\left(r^2+z^2\right)^{\frac{3}{2}}}\right]\hat{k} \tag{17}$$

put $r^2+z^2=t^2$

$\Rightarrow rdr = tdt$

when $r = 0$, then $t = z$

when $r = R$, then $t = \sqrt{R^2 + z^2}$

$$\because \vec{E} = \frac{\sigma z}{2\varepsilon_0}\left[\int_z^{\sqrt{R^2+z^2}} \frac{tdt}{t^3}\right]\hat{k} = \frac{\sigma z}{2\varepsilon_0}\left[\int_z^{\sqrt{R^2+z^2}} \frac{1}{t^2}dt\right]\hat{k}$$

$$= \frac{\sigma z}{2\varepsilon_0}\left[-\frac{1}{t}\right]_z^{\sqrt{R^2+z^2}}\hat{k} = \frac{\sigma z}{2\varepsilon_0}\left[\left(-\frac{1}{\sqrt{R^2+z^2}}\right) - \left(-\frac{1}{z}\right)\right]\hat{k}$$

$$\therefore \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{k}}$$

As $L \to \infty$:

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left[1-0\right]\hat{k} = \frac{\sigma}{2\varepsilon_0}\hat{k}$$

Therefore, the electric field due to the infinite sheet of charge is $\boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{k}}$. For points far from the disk $(z \gg R)$:

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{k} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\left(\frac{R^2+z^2}{z^2}\right)^{\frac{1}{2}}}\right]\hat{k}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[1 - \left(1 + \frac{R^2}{z^2}\right)^{-\frac{1}{2}}\right]\hat{k}$$

using binomial expansion

$$\vec{E} \cong \frac{\sigma}{2\varepsilon_0}\left[1-\left\{1-\frac{1}{2}\frac{R^2}{z^2}+\ldots\right\}\right]\hat{k}$$

$$\cong \frac{\sigma}{2\varepsilon_0}\frac{1}{2}\frac{R^2}{z^2}\hat{k} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\left(\pi R^2\right)}{z^2}\hat{k}$$

$$\therefore \vec{E} \cong \frac{1}{4\pi\varepsilon_0}\frac{q}{z^2}\hat{k}$$

where $q = \sigma\left(\pi R^2\right)$ is the total charge on the disk For far away the disk "looks" like a point charge $q = \sigma\left(4\pi R^2\right)$.

End of Lecture 04

Thank you