Computing Torsion:

$$\tau = \frac{1}{|v \times a|^2} \times \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}$$

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}, \quad \dddot{x} = \frac{d^3x}{dt^3}$$

$$K = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

Q7: Find K and τ for the space curve

$$\vec{r}(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + (bt)\vec{k} \quad a, b > 0$$

$$a^2 + b^2 \neq 0.$$

Soln.

Given,

$$\vec{r}(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + (bt)\vec{k}$$

$$\vec{v}(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + b\vec{k}$$

$$|\vec{v}(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2+b^2}$$

$$\vec{a}(t) = \frac{d\,\vec{v}(t)}{dt} = -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0\vec{k}$$

$$\frac{d^3\vec{r}}{dt^3} = a\sin t\,\vec{i} - a\cos t\,\vec{j} + 0\vec{k}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= ab\sin t\,\vec{i} - ab\cos t\,\vec{j} + a^2\vec{k}$$

$$|\vec{v} \times \vec{a}| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2}$$

$$= a\sqrt{a^2+b^2}$$

$$K = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{a\cancel{\sqrt{a^2+b^2}}}{(\sqrt{a^2+b^2})^2(\cancel{\sqrt{a^2+b^2}})} = \frac{a}{a^2+b^2}$$

$$\tau = \frac{1}{|\vec{v} \times \vec{a}|} \begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$$= \frac{1}{(a\sqrt{a^2+b^2})^2} \times b \begin{vmatrix} -a\cos t & -a\sin t \\ a\sin t & -a\cos t \end{vmatrix}$$

$$= \frac{1}{(a\sqrt{a^2+b^2})^2} \times b \times a^2$$

$$\therefore \tau = \frac{b}{a^2+b^2}$$