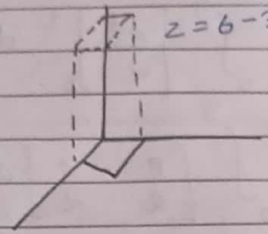
Q: Find the volume between xy plane and $z = 6-3x-2y$ above unit square
$R = \{(x,y): 0 \le x \le 1,\ 0 \le y \le 1\}$

Soln,



We know,

$$\text{Volume} = \int_0^1\int_0^1 (6-3x-2y)\,dy\,dx$$

$$= \int_0^1 \left[6y - 3xy - y^2\right]_0^1 dx$$

$$= \int_0^1 \left[6\times 1 - 3\times x\times 1 - 1^2\right] dx$$

$$= \int_0^1 [5-3x]\,dx = \left[5x - \frac{3x^2}{2}\right]_0^1$$

$$= 5\times 1 - \frac{3\times 1^2}{2} = \frac{7}{2} \text{ cubic units.}$$

Triple Integral in Rectangular Coordinates

$$\iiint_D F(x,y,z)\,dV \quad \text{or} \quad \iiint_D F(x,y,z)\,dx\,dy\,dz.$$

The volume of a closed bounded region D in space is $V = \iiint_D dV = \iiint_D dx\,dy\,dz$

Q: Evaluate:

(i) $\int_0^1\int_0^2\int_0^3 (x+y+z)\,dx\,dy\,dz$

Soln:

$$= \int_0^1\int_0^2\int_0^3 (x+y+z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^2 \left[\frac{x^2}{2} + xy + xz\right]_0^3 dy\,dz$$

$$= \int_0^1\int_0^2 \left(\frac{9}{2} + 3y + 3z\right) dy\,dz$$

$$= \int_0^1 \left[\frac{9y}{2} + \frac{3y^2}{2} + 3yz\right]_0^2 dz$$

$$= \int_0^1 (9 + 6 + 6z)\,dz$$

$$= \int_0^1 (15+6z)\,dz = \left[15z + \cancel{6}\,3z^2\right]_0^1$$

$$= 18$$

(ii): $\int_0^1\int_0^2\int_0^3 xyz\,dx\,dy\,dz$

Soln:

$$= \left(\int_0^1 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^3 z\,dz\right)$$

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^2 \left[\frac{z^2}{2}\right]_0^3$$

$$= \frac{1}{2} \times 2 \times \frac{9}{2} \qquad = \frac{9}{2}$$

(iii) $\int_1^e\int_1^e\int_1^e \frac{1}{xyz}\,dx\,dy\,dz$

$$= \left(\int_1^e \frac{1}{x}\,dx\right)\left(\int_1^e \frac{1}{y}\,dy\right)\left(\int_1^e \frac{1}{z}\,dz\right)$$

$$= [\ln x]_1^e\,[\ln y]_1^e\,[\ln z]_1^e$$

$$= [\ln e - \ln 1][\ln e - \ln 1][\ln e - \ln 1]$$

$$= 1$$

(iv): $\int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$

Soln:

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} [z]_{x^2+3y^2}^{8-x^2-y^2}\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - x^2 - \cancel{3}y^2 - x^2 - 3y^2)\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\,dy\,dx$$

$$= \int_{-2}^{2}\left[8y - 2x^2y - \frac{4}{3}y^3\right]_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\,dx$$

$$= \int_{-2}^{2} 8\sqrt{(4-x^2)/2} - 2x^2\left(\sqrt{(4-x^2)/2}\right) - \frac{4}{3}\left(\sqrt{(4-x^2)/2}\right) + 8\sqrt{(4-x^2)/2}$$
$$+ 2x^2\left(\sqrt{(4-x^2)/2}\right) + \frac{4}{3}\left(\sqrt{(4-x^2)/2}\right)\,dx$$

$$= \int_{-2}^{2} \frac{16}{\sqrt{2}}\sqrt{(4-x^2)}\,dx \qquad = \frac{16}{\sqrt{2}}\int_{-2}^{2}(4-x^2)^{1/2}\,dx$$

$$= \frac{16}{\sqrt{2}}\times\frac{2}{3}\left[(4-x^2)^{3/2}\right]_{-2}^{2} \qquad = \frac{32}{3\sqrt{2}}\left[(4-(2)^2)^{3/2} - (4-(-2)^2)^{3/2}\right]$$

$$= 0$$

$$= \int_{-2}^{2} \cancel{16\sqrt{(4-x^2)/2} \; 4x^2\left(\sqrt{(4-x^2)/2}\right) - \frac{8}{3}\left(4-x^2/2\right)^{3/2}}$$

~~solve~~ ... solve.

<Q>: Integrate $g(x,y,z) = xyz$ over the cube in the first octant bounded by coordinate planes and the planes $x=2$, $y=2$, $z=2$.

Soln:

Given,

$g(x,y,z) = xyz$.

Now,

$$= \int_0^2\int_0^2\int_0^2 xyz \, dx\,dy\,dz$$

$$= \left(\int_0^2 x\,dx\right)\left(\int_0^2 y\,dy\right)\left(\int_0^2 z\,dz\right)$$

$$= \left(\left[\frac{x^2}{2}\right]_0^2\right)\left(\left[\frac{y^2}{2}\right]_0^2\right)\left(\left[\frac{z^2}{2}\right]_0^2\right)$$

$$= \left(\frac{4}{2}\right)^3 = 8$$

$$\text{Volume} = \int_0^2\int_0^2\int_0^2 dx\,dy\,dz$$

$$= \left(\int_0^2 dx\right)\left(\int_0^2 dy\right)\left(\int_0^2 dz\right)$$

$$= \left([x]_0^2\right)\left([y]_0^2\right)\left([z]_0^2\right)$$

$$= (2-0)\times(2-0)\times(2-0) = 8$$

<Q>: Find the volume of the region D enclosed by the surface $z = x^2+3y^2$ and $z = 8-x^2-y^2$

Soln:

Now,

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} [z]_{x^2+3y^2}^{8-x^2-y^2} \, dy\,dx$$

Expanding ....., (skipping some steps)

$$= \frac{16}{\sqrt{2}}\int_{-2}^{2}\left(\sqrt{4-x^2}\right)dx - \frac{4}{\sqrt{2}}\int_{-2}^{2} x^2\left(\sqrt{4-x^2}\right)dx - \frac{8}{3\sqrt{2}}\int_{-2}^{2}(4-x^2)^{3/2}\,dx$$

$$= \cancel{\frac{16}{\sqrt{2}}\times\frac{2}{3}\left[\frac{(4-x^2)^{3/2}}{2}\right]_{-2}^{2}} -$$

Let $x = 2\sin\theta$

So, $\frac{dx}{d\theta} = \cos\theta$

or, $dx = \cos\theta\, d\theta$

When $x=-2,\ \theta=-\pi/2$ When $x=2,\ \theta=\pi/2$.

Sol

$$\frac{16}{\sqrt2}\int_{-\pi/2}^{\pi/2}\sqrt{4-4\sin^2\theta}\,\cos\theta\,d\theta-\frac{4}{\sqrt2}\int_{-\pi/2}^{\pi/2}4\sin^2\theta\sqrt{4-4\sin^2\theta}\,\cos\theta\,d\theta$$

$$-\frac{8}{3\sqrt2}\int_{-\pi/2}^{\pi/2}(4-4\sin^2\theta)^{3/2}\cos\theta\,d\theta$$

$$=\frac{16\times2}{\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta-\frac{4\times4\times2}{\sqrt2}\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$-\frac{8}{3\sqrt2}\times8\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$=\frac{32}{\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta-\frac{32}{\sqrt2}\int_{-\pi/2}^{\pi/2}(\cos^2\theta-\cos^4\theta)\,d\theta$$

$$-\frac{64}{3\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta.$$

$$=\frac{32}{\sqrt2\times2}\int_{-\pi/2}^{\pi/2}(1+\cos2\theta)\,d\theta-\frac{32}{\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta+\frac{32}{\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$-\frac{64}{3\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$\frac{32}{2\sqrt2}\int_{-\pi/2}^{\pi/2}1\,d\theta+\frac{32}{2\sqrt2}\int_{-\pi/2}^{\pi/2}\cos2\theta\,d\theta-\frac{32}{2\sqrt2}\int_{-\pi/2}^{\pi/2}1\cdot d\theta-\frac{32}{2\sqrt2}\int_{-\pi/2}^{\pi/2}\cos2\theta\,d\theta$$

$$\cancel{+\frac{32}{\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta}\ \cancel{-\frac{64}{3\sqrt2}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta}\quad\frac{16\sqrt2}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$=\frac{32}{2\sqrt2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]+\frac{32}{2\sqrt2\times2}\left[\sin\pi+\sin\pi\right]-\frac{32}{2\sqrt2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$-\frac{32}{2\sqrt2}\left[\sin\pi+\sin\pi\right]+\frac{16\sqrt2}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$=\frac{16\sqrt2}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta=\frac{16\sqrt2}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)^2d\theta$$

$$=\frac{16\sqrt2}{3\times4}\int_{-\pi/2}^{\pi/2}(1+2\cos2\theta+\cos^22\theta)\,d\theta$$

$$=\frac{4\sqrt2}{3}\left[\int_{-\pi/2}^{\pi/2}1\cdot d\theta+\int_{-\pi/2}^{\pi/2}2\cos2\theta+\int_{-\pi/2}^{\pi/2}\cos^22\theta\right]$$

$$=\frac{4\sqrt2}{3}\left[\pi+\cancel{2}\left[\sin\pi+\sin\pi\right]+\int_{-\pi/2}^{\pi/2}\cos^22\theta\right]$$

$$=\frac{4\sqrt2\pi}{3}+\frac{4\sqrt2}{3\times2}\int_{-\pi/2}^{\pi/2}(1+\cos4\theta)$$

$$=\frac{4\sqrt2\pi}{3}+\frac{2\sqrt2}{3}\left[\int_{\pi/2}^{-\pi/2}1\,d\theta+\int_{\pi/2}^{\pi/2}\cos4\theta\,d\theta\right]$$

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$$= \frac{4\sqrt{2}}{3}\pi + \frac{2\sqrt{2}}{3}\left[\pi + \frac{1}{4}\left\{\sin\frac{4\pi}{2} + \sin\frac{4\pi}{2}\right\}\right]$$

$$= \frac{6\sqrt{2}\,\pi}{3} = 2\pi\sqrt{2}$$

*) Average value of F over D: $= \dfrac{1}{\text{Volume of } D}\iiint_D F\cdot dv$

$$= \frac{\iiint_D F\cdot dv}{\iiint_D dv}$$

<Q7: Find the average value of $F(x,y,z) = xyz$ over the cube bounded by the coordinate planes and the planes $x = 2$, $y = 2$, $z = 2$ in the first octant.

Soln:

We know

A·value $= \dfrac{\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz}{\int_0^2\int_0^2\int_0^2 dx\,dy\,dz}$

$$= \frac{\left(\int_0^2 x\,dx\right)^3}{\left(\int_0^2 dx\right)^3} \qquad \frac{\left(\left|\frac{x^2}{2}\right|_0^2\right)^3}{\left(x\,|_0^2\right)^3}$$

$$= \frac{2^3}{2^3} = 1.$$

Date. No.

<Q>: Evaluate:

<i>: $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$

Soln:

$$= \int_0^1\int_0^1 \left(x^2z + y^2z + \frac{z^3}{3}\right)_0^1 dy\,dx$$

$$\int_0^1\int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx$$

$$= \int_0^1 \left(x^2y + \frac{y^3}{3} + \frac{1}{3}y\right)_0^1 dx$$

$$\int_0^1 x^2 + \frac{1}{3} + \frac{1}{3}\,dx$$

$$= \left(\frac{x^3}{3} + \frac{1}{3}x + \frac{1}{3}x\right)_0^1 = \frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$$

<ii>: $\int_0^1\int_0^1\int_0^1 x^2y^2z^2\,dz\,dy\,dx$

Soln:

$$\left(\int_0^1 x^2\,dx\right)^3 = \left[\left(\frac{x^3}{3}\right)_0^1\right]^3 = \frac{1}{27}$$

(X) Triple Integral in cylindrical coordinates:

$$dv = dz\, r\,dr\, d\theta$$

$$\iiint_D F(x,y,z)\, dv = \iiint_G F(r,\theta,z)\, dz\, r\, dr\, d\theta$$

<Q>: Evaluate: $\int_0^{2\pi}\int_0^{2}\int_0^{r^2} z\,dz\,r\,dr\,d\theta$

Sol^n.

$$= \int_0^{2\pi}\int_0^{2}\left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}\frac{r^5}{2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^{2} d\theta = \int_0^{2\pi}\frac{64}{12}\,d\theta = \int_0^{2\pi}\frac{16}{3}\,d\theta$$

$$= \frac{16}{3}\times 2\pi = \frac{32\pi}{3}$$

(X) Triple Integral in spherical coordinates:

$$dv = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\iiint_D F(x,y,z)\, dv = \iiint_G F(\rho,\phi,\theta)\, \rho^2\, d\rho\, \sin\phi\, d\phi\, d\theta$$

<Q>: Evaluate: $\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Solution.

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2\, d\rho\, \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^3}{3}\right]_0^{\frac{1-\cos\phi}{2}} \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\frac{1-\cos\phi}{2}\right)^3 \times \frac{1}{3}\,\sin\phi\, d\phi\, d\theta \qquad \left[\because \frac{d\cos\phi}{d\phi} = \sin\phi\, d\phi\right]$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{-1}{24}(1-\cos\phi)^3\, d\cos\phi\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{-1}{24}\,\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi} d\theta$$

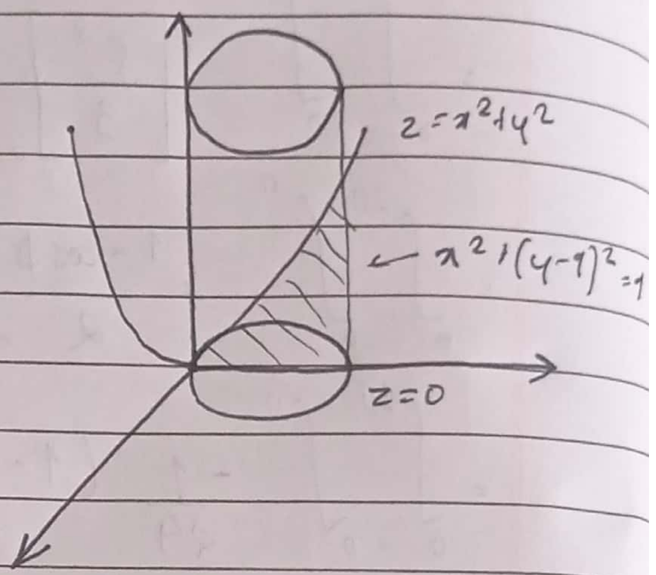
$$= \int_0^{2\pi} \frac{-1\,(1-\cos\pi)^4}{24\times 4}\, d\theta$$

$$= \int_0^{2\pi} \frac{-16}{24\times 4}\, d\theta = \int_0^{2\pi} -\frac{1}{6}\, d\theta$$

$$= -\frac{1}{6}\, 2\pi = \frac{-\pi}{3}$$

<2>: Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region D bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$ and above by the paraboloid $z=x^2+y^2$.

Soln:



Given,

$z=x^2+y^2$ or, $z=r^2$

and

$x^2+(y-1)^2=1$

or, $x^2+y^2-2y+1=1$

or, $r^2-2r\,\cancel{\cos}\sin\theta=0$

or, $r(r-2\sin\theta)=0$

$\therefore r=0,\ 2\sin\theta$

$$\text{Volume (V)} = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{\cancel{x^2+y^2}\ r^2} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$