General Physics II

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Course Outline

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- Problem Solving
- MCQ
- Fill in thee Blanks

SUMMARY

PRODUCT OF VECTORS

Scalar Product: $\vec{A}\bullet\vec{B} \equiv \mathrm{AB}\cos\theta$

Vector Product: $\vec{A}\times\vec{B} \equiv \mathrm{AB}\sin\theta\ \hat{n}$

Scalar Triple Product: $\vec{A}\bullet(\vec{B}\times\vec{C}) = \vec{B}\bullet(\vec{C}\times\vec{A}) = \vec{C}\bullet(\vec{A}\times\vec{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

Vector Triple Product: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\bullet\vec{C}) - \vec{C}(\vec{A}\bullet\vec{B})$

SUMMARY

VECTOR DERIVATIVES

Cartesian

The Infinitesimal displacement vector: $d\vec{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Gradient: $$\nabla T \equiv \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$$

Divergence: $$\nabla \cdot \vec{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Laplacian: $$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\nabla^2 \vec{v} \equiv \left(\nabla^2 v_x\right)\hat{i} + \left(\nabla^2 v_y\right)\hat{j} + \left(\nabla^2 v_z\right)\hat{k}$$

SUMMARY

VECTOR DERIVATIVES

Spherical

The Infinitesimal displacement vector: $d\vec{l} = dr\,\hat{r} + r\,d\theta\;\hat{\theta} + r\sin\theta\;d\phi\;\hat{\phi}$

An element of surface area on the sphere of radius R: $da = R^2 \sin\theta\; d\theta\; d\phi$

Gradient: $$\nabla f \equiv \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

SUMMARY

VECTOR IDENTITIES

Product Rules

1. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
2. $\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + (\nabla f) \cdot \vec{A}$
3. $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
4. $\nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) + (\nabla f) \times \vec{A} = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f)$

Second Derivatives

1. $\nabla \cdot (\nabla \times \vec{A}) = 0$
2. $\nabla \times (\nabla f) = 0$
3. $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

SUMMARY

FUNDAMENTAL THEOREMS

Gradinet Theorem: $\int_a^b (\nabla f)\cdot d\vec{l} = f(b) - f(a)$

Divergence Theorem: $\int_V (\nabla\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$ [Gauss's Theorem]

Curl Theorem: $\int_S (\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$ [Stoke's Theorem]

NOTES:

- Divergence of gradient: $\nabla\cdot(\nabla T) \equiv \nabla^2 T \rightarrow$ a scalar
- Curl of gradient: $\nabla\times(\nabla T) \rightarrow$ a vector
- Gradient of divergence: $\nabla(\nabla\cdot\vec{v}) \rightarrow$ a vector
- Divergence of curl: $\nabla\cdot(\nabla\times\vec{v}) \rightarrow$ a scalar
- Curl of curl: $\nabla\times(\nabla\times\vec{v}) \rightarrow$ a vector

SUMMARY

Notes:

- $$\nabla\cdot\vec{F}=0 \qquad \Rightarrow \vec{F} \text{ is solenoidal}$$
  $$\nabla\cdot\vec{F}=0 \ \ \& \ \nabla\cdot\nabla\times\vec{G} \qquad \Rightarrow \vec{F} \text{ can be written as curl of a vector } \left[\vec{F}=\nabla\times\vec{G}\right]$$

- $$\nabla\times\vec{F}=0 \qquad \Rightarrow \vec{F} \text{ is irrotational}$$
  $$\nabla\times\vec{F}=0 \ \ \& \ \nabla\times\left(\nabla T\right)=0 \qquad \Rightarrow \vec{F} \text{ can be written as gradient of a scalar function } \left[\vec{F}=\nabla T\right]$$

- Which one of the following properties of a vector $\vec{F}$ does not allow us to get the identity, $\nabla\times\left(\nabla\times\vec{F}\right)=-\nabla^2\vec{F}$?

  [a] $\vec{F}$ is solenoidal .

  [b] $\nabla\cdot\vec{F}$ is a non-zero constant.

  [c] $\vec{F}$ is expressible to the curl of another vector.

  [d] $\vec{F}$ is expressible to the gradient of some scalar function.

  Hints: [d] $\left[\because \nabla\times\left(\nabla\times\vec{F}\right)=\nabla\left(\nabla\cdot\vec{F}\right)-\nabla^2\vec{F}\right]$

SUMMARY

Notes:

- The vector function $\vec{v} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is

  solenoidal as well as irrotational.

  Hints: $\nabla \cdot \vec{v} = 0$ and $\nabla \times \vec{v} = 0$.

- In two vectors $\vec{V}_1 = \nabla \times \vec{F}$ and $\vec{V}_2 = \nabla \phi$,

  $\vec{V}_1$ **is solenoidal and** $\vec{V}_2$ **is irrotational.**

- $\nabla \times (\frac{\hat{r}}{r^2}) = \nabla \times (\frac{\vec{r}}{r^3}) = 0$

- A vector field $\vec{A}$ is conservative if $\vec{A} = \nabla \phi$

- $\nabla \cdot \hat{r} = \frac{2}{r}$ $\quad \left[\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \right]$

SUMMARY

Gradients:

- Grad $\rightarrow$ ∇ ,

  $$\nabla T \equiv \hat{i}\frac{\partial T}{\partial x} + \hat{j}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z} \qquad [\mathrm{T} = \text{a scalar function}]$$

- ∇T is a vector quantity.
- The gradient ∇T points in the direction of maximum increase of the function T .

- $\nabla r = \nabla\left(\sqrt{x^2 + y^2 + z^2}\right) = \hat{r}$
- $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3}$
- $\nabla(r^n) = nr^{n-1}\hat{r} = nr^{n-2}\vec{r}$

SUMMARY

Divergence:

- div $\rightarrow$ $\nabla\bullet$,

  $$\nabla \cdot \vec{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

- $\nabla \cdot \vec{v}$ is a scalar.
- $\nabla \cdot \vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out from the point in question.

  - $\nabla \cdot \hat{r} = \frac{2}{r}$.

SUMMARY

Curl:

- Curl $\rightarrow$ $\nabla\times$,

$$\nabla\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}=\hat{i}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-\hat{j}\left(\frac{\partial v_z}{\partial x}-\frac{\partial v_x}{\partial z}\right)+\hat{k}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

- $\nabla\times\vec{v}$ is a measure of how much the vector $\vec{v}$ "curls around" the point in question.

  - $\nabla\times\frac{\hat{r}}{r^2}=\nabla\times\frac{\vec{r}}{r^3}=0$.

Laplacian:

- The **Laplacian** of T : $\nabla^2T=\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}+\frac{\partial^2T}{\partial z^2}$
- The **Laplacian** of $\vec{v}$: $\nabla^2\vec{v}\equiv\left(\nabla^2v_x\right)\hat{i}+\left(\nabla^2v_y\right)\hat{j}+\left(\nabla^2v_z\right)\hat{k}$

Problems

1. **If** $\phi(x,y,z) = 3x^2y - y^3z^2$ **, find** $\nabla\Phi$ at $(1,-2,-1)$ **.**

2. **Calculate the divergence of vector function**

$$\vec{v} = xyz(x\hat{i} + y\hat{j} + z\hat{k})$$

3. **If** $\vec{r}$ **is the position vector, then show that** $\nabla \cdot \hat{r} = \dfrac{2}{r}$ **.**

4. **Calculate the divergence and curl of the vector function**

$$\vec{v} = y^2\hat{i} + \left(2xy + z^2\right)\hat{j} + 2yz\hat{k}.$$

5. **Calculate the Laplacian of the following function**

$$\vec{v} = x^2y\hat{i} + (x^2 - y)\hat{k} \text{ .}$$

Solutions

1. Solution:

$$\nabla\Phi = \hat{i}\frac{\partial}{\partial x}\left(3x^2y - y^3z^2\right) + \hat{j}\frac{\partial}{\partial y}\left(3x^2y - y^3z^2\right) + \hat{k}\frac{\partial}{\partial z}\left(3x^2y - y^3z^2\right)$$

$$= 6xy\,\hat{i} + \left(3x^2 - 3y^2z^2\right)\hat{j} - 2y^3z\,\hat{k}$$

$$= 6(1)(-2)\,\hat{i} + \left[3(1)^2 - 3(-2)^2(-1)^2\right]\hat{j} - 2(-2)^3(-1)\hat{k} \qquad \text{at } (1,-2,-1)$$

$$= -12\,\hat{i} - 9\,\hat{j} - 16\,\hat{k}$$

2. Solution:

$$\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(x^2yz\right) + \frac{\partial}{\partial y}\left(xy^2z\right) + \frac{\partial}{\partial z}\left(xyz^2\right)$$

$$= 2xyz + 2xyz + 2xyz$$

$$= 6xyz$$

Solutions

3. Solution:

$$\nabla\cdot\hat{r} = \nabla\cdot\frac{\vec{r}}{r} = \frac{1}{r}\left(\nabla\cdot\vec{r}\right)+\nabla\left(\frac{1}{r}\right)\cdot\vec{r}$$

$$= \frac{1}{r}(3)+\left[(-1)r^{-1-2}\vec{r}\right]\cdot\vec{r} \qquad \left[\because \nabla\cdot\vec{r}=3; \quad \nabla r^{n}=nr^{n-2}\vec{r}\right]$$

$$= \frac{3}{r}-\frac{1}{r} \qquad \left[\because \vec{r}\cdot\vec{r}=r^{2}\right]$$

$$= \frac{2}{r}$$

4. Solution:

$$* \quad \nabla\cdot\vec{\mathrm{v}} = \frac{\partial \mathrm{v}_x}{\partial x}+\frac{\partial \mathrm{v}_y}{\partial y}+\frac{\partial \mathrm{v}_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\mathrm{y}^2\right)+\frac{\partial}{\partial y}\left(2\,\mathrm{xy}+\mathrm{z}^2\right)+\frac{\partial}{\partial z}\left(2\,\mathrm{yz}\right)$$

$$= 0+2\mathrm{x}+2\mathrm{y}$$

$$= 2(\mathrm{x}+\mathrm{y})$$

Solutions

4. Solution:

$$* \ \nabla\times\vec{v} = \nabla\times\left[y^2\hat{i} + \left(2xy+z^2\right)\hat{j} + 2yz\hat{k}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}\left(2xy+z^2\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}\left(y^2\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(2xy+z^2\right) - \frac{\partial}{\partial y}\left(y^2\right)\right]$$

$$= \hat{i}\left[2z-2z\right] - \hat{j}\left[0-0\right] + \hat{k}\left[2y-2y\right]$$

$$= 0$$

5. Solution:

$$\nabla^2\vec{v} = \left(\nabla^2 v_x\right)\hat{i} + \left(\nabla^2 v_y\right)\hat{j} + \left(\nabla^2 v_z\right)\hat{k}$$

$$= \left[\nabla^2\left(x^2y\right)\right]\hat{i} + \left[\nabla^2(0)\right]\hat{j} + \left[\nabla^2\left(x^2-y\right)\right]\hat{k}$$

$$= \left[\frac{\partial^2}{\partial x^2}\left(x^2y\right) + \frac{\partial^2}{\partial y^2}\left(x^2y\right) + \frac{\partial^2}{\partial z^2}\left(x^2y\right)\right]\hat{i} + 0$$

$$+ \left[\frac{\partial^2}{\partial x^2}\left(x^2-y\right) + \frac{\partial^2}{\partial y^2}\left(x^2-y\right) + \frac{\partial^2}{\partial z^2}\left(x^2-y\right)\right]\hat{k}$$

$$= 2y\hat{i} + 2\hat{k}$$

Multiple Choice Questions

- If the curl of a vector function is zero, then the vector function can be expressed as

  [a] the gradient of a scalar function.

  [b] the curl of some other vector function.

  [c]the divergence of some other vector field.

  [d] the gradient of another vector function.

- The vector function $\vec{v} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is

  [a] solenoidal but not irrotational. [b] irrotational but not solenoidal.

  [c] solenoidal as well as irrotational. [d] neither solenoidal nor irrotational.

- Which of the following statements is NOT CORRECT?

  [a]The gradient points in the direction of maximum increase of the function .

  [b]The curl of a gradient is always zero.

  [c]The divergence of a curl is always zero.

  [d]The curl of a curl is always zero.

Fill in the Blanks

- **The gradient of the function** $t = x^2 y + e^z$ at the point $p(1,5,-2)$ **is ………**

- **The divergence of the vector function** $\vec{F} = xe^{-x}\hat{i} + y\hat{j} - xz\hat{k}$ **is ……….**

- **If** $\vec{A} = \frac{1}{2}\mu_0 nI\left(-z\hat{j} + y\hat{k}\right)$, then $\nabla \times \vec{A} =$ ……….

- **The curl of curl of a vector function** $\vec{\mathrm{V}} = -\mathrm{x}^2\hat{k}$ **is** ……….

- **If the divergence of a vector function is zero, then the vector function can be expressed as………..**

Text Books & References

1. **David J. Griffith**, Introduction to Electrodynamics
2. **R. A. Serway and J.W. Jewett**, Physics for Scientist and Engineers with Modern Physics
3. **Halliday and Resnick**, Fundamental of Physics


Thank you