Null Space:

The null space of $A_{m\times n}$ is the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$.

ie, $\text{Null}(A)$ or $N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$.

<Q>: Find null space of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$.

Soln:

Let $\vec{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $A\vec{x} = \vec{0}$.

or,
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cancel{0} \end{bmatrix}$$

By matrix multiplication,

$$\begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 + 2x_2 + 3x_3 + 4x_4 \\ 4x_1 + 3x_2 + 2x_3 + x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From R_1, $x_1 + x_2 + x_3 + x_4 = 0$ —— (i)

From R_2, $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ —— (ii)

From R_3, $4x_1 + 3x_2 + 2x_3 + x_4 = 0$ —— (iii)

Writing in augmented matrix form,

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$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 4R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & -3 & 0 \end{array}\right]$$

Applying $R_3 \to R_3 + R_2$ and $R_1 \to R_1 - R_2$.

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From R_1; $x_1 - x_3 - 2x_4 = 0$ $\therefore x_1 = x_3 + 2x_4$
$x_2 + 2x_3 + 3x_4 = 0$ $\therefore x_2 = -2x_3 - 3x_4$.

~~Here~~, So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{x} \in \mathbb{R}^4$ satisfies $A\vec{x} = \vec{0}$ and is linear combination of $\vec{a} = (1, -2, 1, 0)$ and $\vec{b} = (2, -3, 0, 1)$

So, nullspace $(A) = N(A) = x_3 \vec{a} + x_4 \vec{b}$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

or, $N(A) = \text{span}\{\vec{a}, \vec{b}\} = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

<Q>: Find null space for $A = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 3 & -7 \end{bmatrix}$

Soln:

Let $\vec{x} = (x_1, x_2, x_3, \cancel{x_4})$ such that $A\vec{x} = \vec{0}$

or, $\begin{bmatrix} 1 & -2 & 4 \\ -1 & 3 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cancel{0} \end{bmatrix}$

By matrix multiplication,

$$\begin{bmatrix} x_1 - 2x_2 + 4x_3 \\ -x_1 + 3x_2 - 7x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From R_1, $x_1 - 2x_2 + 4x_3 = 0$ —(i)
$-x_1 + 3x_2 - 7x_3 = 0$ —(ii)

Writing in augmented matrix form,

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ -1 & 3 & -7 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 + R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

Applying $R_1 \to R_1 + 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

Here,

From R_1, $x_1 - 2x_3 = 0$ $\quad \therefore x_1 = 2x_3$

From R_2, $x_2 - 3x_3 = 0$ $\quad \therefore x_2 = 3x_3$

~~Here~~, So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Here,

$\vec{x} \in \mathbb{R}^3$ satisfies $A\vec{x} = \vec{0}$ and is linear combination of $\vec{a} = (2, 3, 1)$

Thus, nullspace $(A) = N(A) = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = x_3 \vec{a}$

or,

$N(A) = \text{span}\{\vec{a}\} = \text{span}\left\{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right\}$

<Q>: Find null space for $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$

Soln.

Let $\vec{x} = (x_1, x_2, x_3)$ such that $A\vec{x} = \vec{0}$.

or,

$$\begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By matrix multiplication, we get

$$\begin{bmatrix} x_1 - 3x_2 - 2x_3 \\ -5x_1 + 9x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From R_1, $x_1 - 3x_2 - 2x_3 = 0$ —(i)

from R_2, $-5x_1 + 9x_2 + x_3 = 0$ —(ii)

Writing in augmented matrix form,

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -5 & 9 & 1 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 + 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -6 & -9 & 0 \end{array}\right]$$

Applying $R_2 \to -\frac{1}{6} R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 3/2 & 0 \end{array}\right]$$

Applying $R_1 \to R_1 + 3R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 5/2 & 0 \\ 0 & 1 & 3/2 & 0 \end{array}\right]$$

From R_1, $x_1 + 5/2\, x_3$ $\quad \therefore x_1 = -5/2\, x_3$

From R_2, $x_2 + 3/2\, x_3$ $\quad \therefore x_2 = -3/2\, x_3$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -3/2 \\ 1 \end{bmatrix}$$

Here, $\vec{x} \in \mathbb{R}^3$ satisfies $A\vec{x} = \vec{0}$ and is linear combination of $\vec{a} = (-5/2, -3/2, 1)$

Thus,

$$\text{Nullspace}(A) = N(A) = x_3\, \vec{a} = x_3 \begin{bmatrix} -5/2 \\ -3/2 \\ 1 \end{bmatrix}$$

or, $\text{span}\{\vec{a}\} = \text{span}\left\{ \begin{bmatrix} -5/2 \\ -3/2 \\ 1 \end{bmatrix} \right\}$