Potential Energy

The energy stored in a body or system by virtue its position or configuration is called potential energy.

Eg: a stretched catapult, water collected in dam

Depending on the nature of forces operating in the system, the potential energy of the system can be different types:

i) Gravitational Potential Energy:

→ P.E. associated with a system consisting of Earth and a nearby particle is called GPE.

$$\Delta U(y) = mg(y_f - y_i)$$

ii) Elastic Potential Energy:

→ P.E. associated with the state of compression or extension of elastic object.

$$\Delta U(x) = \frac{1}{2} k x^2$$

The potential energy is a function of position whose negative derivatives gives the force.

$$F(x) = -\frac{dU(x)}{dx}$$

Date. No.

When work is done by conservative force, the configuration of its parts change and so the PE from initial value 'U_i' and find value 'U_f'.

The change in potential energy due to change in configuration $(\Delta U) = U_f - U_i$

In an isolated system in which conservative force acts

$$K + U = \text{constant}$$

ie, $$\Delta K + \Delta U = 0$$

$$\therefore \Delta U = -\Delta K \quad \text{—— (1)}$$

According to Work-Energy theorem,

$$\Delta K = W \quad \text{—— (2)}$$

From eq^n (1) and (2), we get.

$$\cancel{\Delta U} \; \Delta U = -W. \quad \text{—— (3)}$$

Expression of PE at a point

We have, $\Delta K = -\Delta U$

Workdone by resultant force $(W) = \int f(x) \cdot dx$

$= -\Delta U$ [From eq^n (3)]

So,

$$\Delta U = -\int f(x)\, dx$$

So, workdone against conservative force is equal to the gain in potential energy.

Eg: While stretching the spring, workdone against the elastic force is equal to change in P.E.

ie, $\Delta U = U(b) - U(a)$

$$= -\int_a^b f(x)\,dx$$

$$\therefore U(b) = -\int_a^b f(x)\cdot dx + U(a)$$

Assuming potential energy at initial position $x = a$, $U(a) = 0$

Then,

$$U(b) = -\int_a^b f(x)\cdot dx$$

So, potential energy of a body (system) at a point is the amount of workdone against the conservative force in taking the body (system) from position where potential energy is taken to be zero $\{x = a, U(a) = 0\}$ to the present position of the point $x = b$.

Conservative Forces As Negative Gradient of P.E.

We know,

Under the action of conservative force, mechanical energy remains constant.

$$K + U = \text{constant}$$

or, $\Delta K + \Delta U = 0$

or, $\Delta U = -\Delta K$ ie, workdone against conservative force.

$$\therefore \Delta U = -\int_{x_0}^{x} F_x\,dx - \int_{y_0}^{y} F_y\,dy - \int_{z_0}^{z} F_z\,dz$$

In vector form,

$$-\int_{r_0}^{r} dU = \int_{r_0}^{r} \vec{F}\cdot\vec{dr}$$

Here, r_0 = position of zero P.E.

or, $-dU = \vec{F}\cdot\vec{dr}$

or, $-dU = (\hat{i}F_x + \hat{j}F_y + \hat{k}F_z)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

or, $-dU = F_x dx + F_y dy + F_z dz$

~~It~~ Undergoing partial differentiation,

We get,

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

Hence, the force $\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$

$$= -\left(\hat{i}\frac{\partial U}{dx} + \hat{j}\frac{\partial U}{dy} + \hat{k}\frac{\partial U}{\partial z}\right)$$

The quantity $\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right)$ is gradient of U or grad U or ∇U or (del U).

So,

$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$ is a vector operator that converts a scalar function to vector function.

Then,

$$\vec{F} = -\text{grad } U$$

$$\therefore \vec{F} = -\nabla U$$

ie, conservative force is equal to negative gradient of potential energy.

Conservation of Energy

(*) Conservation of Mechanical Energy:

Consider a particle is acted by both conservative and non-conservative forces.

Let 'W_c' = workdone by conservative force

'W_n' = workdone by non-conservative force.

Then, the total workdone by ~~#~~ both forces

$$W = W_c + W_n \qquad \text{——} (1)$$

From work-energy theorem,

$$W = \Delta K \quad \text{—— (2)}$$

From the definition of potential energy

$$W_c = -\Delta U \quad \text{—— (3)}$$

Using eqn (1), (2), (3), we get.

$$\Delta K = -\Delta U + W_n$$

or $W_n = \Delta K + \Delta U$

$\therefore W_n = \Delta E_0 \quad$ —— (4)

Here,

$E_0 = K + U$ is mechanical energy. and

ΔE_0 is the change in mechanical energy.

Hence, workdone by non-conservative forces is equal to the change in mechanical energy.

If non-conservative forces do not act or contribute to the workdone ie, $W_n = 0$ then,

$$\Delta E_0 = 0$$

$\therefore$ E_0 is constant.

That means, if a body is not acted by non-conservative forces, the total mechanical energy of body remains ~~constant~~ conserved.

(*) Conservation of Total Energy:

Workdone against non-conservative forces results to change the other form of energy except mechanical energy, ie,

$$-\Delta W_n = \Delta Q \quad -(5)$$

where, Q is other form of energy.

From eqⁿ (5) and (4),

$$-\Delta Q = \Delta E_0$$

$$\therefore \Delta E_0 + \Delta Q = 0$$

or, $\Delta K + \Delta U + \Delta Q = 0$

$$\therefore \Delta K + \Delta U = -\Delta Q$$

Change in mechanical energy = $-\Delta Q$
or, $K + U + Q$ = Constant.

Thus, under the action of non-conservative forces, mechanical energy changes into other forms of energy such as heat or some other form of energy. But the total energy of system remains constant.

Hence, the total energy of isolated system remains conserved.

This is the principle of conservation of energy.

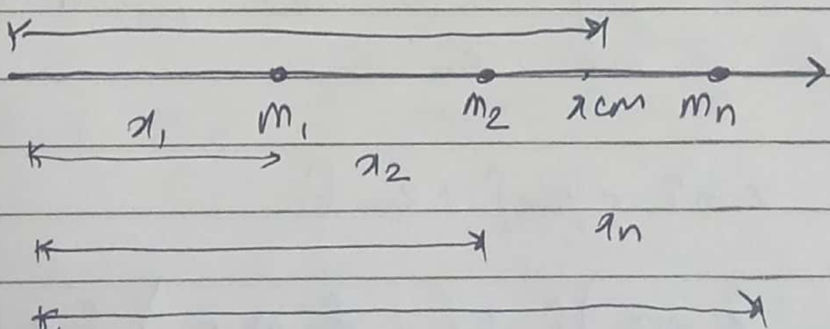
Center of Mass

The center of mass of a body is a point at which ~~tot~~ ~~mote~~ whole mass of the body is supposed to be concentrated and where the line of action of force passes through the point then the body accelerates without rotation.

The centre of mass of a body represents purely the translation motion of the body even though the body rotates during motion.

*) Center of mass for point mass distribution.

Consider a system of n-point masses $m_1, m_2, \ldots, m_n$ are located on x-axis with position $x_1, x_2, \ldots, x_n$ respectively.



The center of mass of the system.

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \quad \text{(i)}$$

or,

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

Here, $M = \sum_{i=1}^{n} m_i$ = total mass of system

If the point mass is distributed in space, the position of center of mass is given by

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \quad , \quad y_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i$$

$$z_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

In vector notation, each particle can be described by position vector as.

$\vec{r_i} = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$ and position vector of center of mass $\vec{r}_{cm}$ is

$$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

$$= \left(\frac{1}{M}\sum_{i=1}^{n} m_i x_i\right)\hat{i} + \left(\frac{1}{M}\sum_{i=1}^{n} m_i y_i\right)\hat{j} + \left(\frac{1}{M}\sum_{i=1}^{n} m_i z_i\right)\hat{k}$$

$$= \frac{1}{M}\sum_{i=1}^{n} m_i(x_i\hat{i} + y_i\hat{j} + z_i\hat{k}) = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r_i}$$

Therefore, in terms of position vector, the center of mass.

$$\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i \quad \text{—— (ii)}$$

For a continuous body, the center of mass can be obtained by subdividing the body in n-numbers of small elements by mass Δm_i located approximately at point (x_i, y_i, z_i).

$$x_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i x_i}{\sum_{i=1}^{n} \Delta m_i}, \quad y_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i y_i}{\sum_{i=1}^{n} \Delta m_i}, \quad z_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i z_i}{\sum_{i=1}^{n} \Delta m_i}$$

If the mass element Δm_i tends to zero, then number of elements 'n' tends to infinity,

$$x_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum_{i=1}^{n} \Delta m_i x_i}{\sum_{i=1}^{n} \Delta m_i}, \quad y_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum_{i=1}^{n} \Delta m_i y_i}{\sum_{i=1}^{n} \Delta m_i}, \quad z_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum_{i=1}^{n} \Delta m_i z_i}{\sum_{i=1}^{n} \Delta m_i}$$

Thus,

$$x_{cm} = \frac{\int x\, dm}{\int dm} = \frac{1}{M}\int x\, dm, \quad y_{cm} = \frac{1}{M}\int y\, dm, \quad z_{cm} = \frac{1}{M}\int z\, dm$$

Here, dm = differential element of mass.

In vector form,

$$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

$$= \frac{1}{M}\int x\,dm\,\hat{i} + \frac{1}{M}\int y\,dm\,\hat{j} + \frac{1}{M}\int z\,dm\,\hat{k}$$

$$= \frac{1}{M}\int (x\hat{i} + y\hat{j} + z\hat{k})\,dm = \frac{1}{M}\int \vec{r}\cdot dm.$$

Motion of Center of Mass

Consider the motion of group of particles whose masses are $m_1, m_2, \dots, m_n$ and whose total mass is M.

Now,
equation of center can be written as.

$$M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_n\vec{r}_n \quad \text{— (i)}$$

where,
$\vec{r}_{cm}$ is position vector of center of mass.

Differentiating (i) with respect to time 't',

$$M\frac{d\vec{r}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \dots + m_n\frac{d\vec{r}_n}{dt}$$

or, $$M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \dots + m_n\vec{v}_n \quad \text{— (ii)}$$

Date No.

Diffrentiating (ii) w.r.t t, we get.

$$M \cdot \frac{d\vec{v}_{cm}}{dt} = m_1 \frac{d\vec{v_1}}{dt} + m_2 \frac{d\vec{v_2}}{dt} + \dots + m_n \frac{d\vec{v_n}}{dt}$$

$$\Rightarrow M \vec{a}_{cm} = m_1 \vec{a_1} + m_2 \vec{a_2} + \dots + m_n \vec{a_n} \quad \text{— (iii)}$$

where,
$\vec{a}_{cm}$ is the acceleration of center of mass.

From Newton's second law $\vec{F} = m\vec{a}$, equation (iii)

$$\Rightarrow M \vec{a}_{cm} = \vec{F_1} + \vec{F_2} + \dots + \vec{F_n} \quad \text{— (iv)}$$

$\vec{F_1}, \vec{F_2}, \dots, \vec{F_n}$ are the forces acting on the individual particles.

ie, the total mass of group of particle times the acceleration of center of mass is equal to the vector sum of all the forces acting on the group of velocity.

From eq^n (iv), $M \vec{a}_{cm} = \vec{F}_{ext}$

This states that the center of mass of a system of particles moves if all the mass of the system were concentrated at the center of mass and the external forces were applied at that point.

If $\vec{F}_{ext} = 0$, then, $M \vec{a}_{cm} = 0$

or, $\vec{a}_{cm} = 0$ $\qquad \vec{v}_{cm} = \text{constant}.$

Under such condition the center of mass either remains at rest or moves at constant speed.