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Eigenvalues of and Eigenvectors.

Let A be $n \times n$ matrix. Then, a non-zero vector $\vec{u}$ is an eigenvector of A, if there exists a scalar λ such that

$$A\vec{u} = \lambda\vec{u} \quad \text{— (i)}.$$

The scalar λ is said to be eigenvalue of corresponding eigenvector $\vec{u}$.

From (i), we can write,

$$A\vec{u} - \lambda\vec{u} = 0$$

or, $$(A - \lambda I)\vec{u} = 0$$

Since $\vec{u} \neq 0$. So,

$$|A - \lambda I| = 0 \quad \text{— (ii)}$$

This is characteristic equation of matrix A.

Eigenspace:

If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors together with the zero vector is subspace of $\mathbb{R}^n$. This subspace is called eigenspace of λ.

(Q): Find the eigenvalue and corresponding eigenspace of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

Soln:

Given,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

The characteristic equation of matrix A is,

$$|A - \lambda I| = 0$$

or, $$\left| \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

or, $$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & -2 & -3-\lambda \end{vmatrix} = 0$$

We know,

$$\lambda^3 - \left\{ \begin{matrix} \text{Sum of} \\ \text{diagonal elements} \end{matrix} \right\} \lambda^2 + \left\{ \begin{matrix} \text{Sum of diagonal} \\ \text{minors} \end{matrix} \right\} \lambda - |A| = 0$$

or $$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$\therefore \lambda = 1, 2$$

For $\lambda = 1$, the eigenspace,

$$(A - \lambda I)\vec{x} = 0$$

or $$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The augmented matrix,

$$\left[\begin{array}{ccc:c} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$$

Applying $R_2 \to -1 \times R_2$

$$\sim \left[\begin{array}{ccc:c} 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 2R_2$

$$\sim \left[\begin{array}{ccc:c} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

Applying $R_3 \to -1/6\, R_3$.

$$\sim \left[\begin{array}{ccc:c} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 + R_3$
and $R_1 \to R_1 - R_3$

$$\sim \left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Here, x_1 is free variable:
$x_2 = 0$
$x_3 = 0$

~~Let $x_1 = s$. $x_2 = 0$. $x_3 = 0$.~~

$$\vec{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \cancel{= s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$\vec{x}$ is the eigenvector for $\lambda = 1$.

Let $x_1 = s$, $x_2 = 0$, $x_3 = 0$.
Then,

$$\vec{x} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is eigenspace at $\lambda = 1$.

For $\lambda = 2$,

$$(A - \lambda I)\vec{x} = 0$$

or,
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

The augmented matrix is.

$$\left[\begin{array}{ccc:c} -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & -5 & 0 \end{array}\right]$$

Applying $R_1 \to -1 \times R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & -5 & 0 \end{array}\right]$$

Applying $R_2 \to -1/2 \times R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -2 & -5 & 0 \end{array}\right]$$

Applying $R_1 \to R_1 + R_2$ $\quad R_3 \to R_3 + 2R_1$

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$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

Applying $R_3 \to R_3 \times (-1/6)$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1 + 1/2 R_3$ $R_2 \to R_2 + 1/2 R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 = 0$
$x_2 = 0$
$x_3 = 0$

The eigenvector is $\vec{x} = [0, 0, 0]$

The eigenspace is $\vec{x} = s \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.