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Ratio test:

Let $\sum a_n$ be a series of positive terms and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = k$$

then,
if $k > 1$, the series diverges
if $k < 1$, the series converges
if $k = 1$, the test is inconclusive.

Questions:

(i): $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

Sol^n:

Here,

$a_n = \frac{3^n}{n!}$ and $a_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3^n \cdot 3}{(n+1)\times n!}$

Now,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{3^n} \cdot 3}{(n+1)\times \cancel{n!}} \times \frac{\cancel{n!}}{\cancel{3^n}} \right| = \frac{3}{n+1}$$

$$= \lim_{n\to\infty} \frac{3}{n+1}$$

$$= 0 < 1$$

Hence, the series converges.

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(ii): $\sum_{n=1}^{\infty} \frac{n}{4^n}$

Sol^n:

$a_n = \frac{n}{4^n}$ and $a_{n+1} = \frac{n+1}{4^{n+1}} = \frac{(n+1)}{4^n \cdot 4}$

Now,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{\cancel{4^n}\cdot 4} \times \frac{\cancel{4^n}}{n}$$

$$= \lim_{n\to\infty} \frac{1}{4} \times 1 + \frac{1}{n}$$

$$= \frac{1}{4} < 0$$

The series converges.

(iii) $\sum_{n=1}^{\infty} \frac{3^{n+2} \cdot n^2}{4^n}$

Sol^n:

$a_n = \frac{3^{n+2} \cdot n^2}{4^n} = \frac{3^n \cdot 3^2 \cdot n^2}{4^n}$

$a_{n+1} = \frac{3^{n+3} \cdot (n+1)^2}{4^{n+1}} = \frac{3^n \cdot 3^3 \cdot (n+1)^2}{4^n \cdot 4}$

Now,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\cancel{3^n} \cdot 3^{\cancel{3}} \cdot (n+1)^2}{\cancel{4^n} \cdot 4} \times \frac{\cancel{4^n}}{\cancel{3^n} \cdot \cancel{3^2} \cdot n^2}$$

$$= \frac{3}{4} \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 = \frac{3}{4} \lim_{n\to\infty} \left(\frac{1+1/n}{1} \right)^2$$

$= 3/4 < 1$. The series converges.

(iv): $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Soln:

Given,

$a_n = \frac{n^n}{n!}$ $\quad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)^n \cdot \cancel{(n+1)}}{\cancel{(n+1)}\, n!}$

Now,

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^n \cdot \cancel{(n+1)}}{\cancel{n!}} \times \frac{\cancel{n!}}{n^n}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

Let $y = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

$\ln y = \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)$

or $\ln y = \lim_{n\to\infty} \frac{\frac{1}{1+1/n} \times \cancel{\left(\frac{-1}{n^2}\right)}}{\cancel{\left(-1/n^2\right)}}$

or $\ln y = 1$

$\therefore y = e$

So,

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$

The series diverges.

(v): $\sum_{n=1}^{\infty} \frac{2^{4n+1}}{n^n}$

Soln:

$a_n = \frac{2^{4n+1}}{n^n}$ $\quad a_{n+1} = \frac{2^{4(n+1)+1}}{(n+1)^{n+1}} = \frac{2^{4n+1} \cdot 2^4}{(n+1)^n \cdot (n+1)}$

Now,

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\cancel{2^{4n+1}} \cdot 2^4}{(n+1)^n \cdot (n+1)} \times \frac{n^n}{\cancel{2^{4n+1}}}$

$= 16 \cdot \lim_{n\to\infty} \frac{1}{n+1} \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

$= 16 \cdot \lim_{n\to\infty} \frac{1}{n+1} \cdot \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$

Let $y = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$

or $-\ln y = \lim_{n\to\infty} n \cdot \ln\left(1+\frac{1}{n}\right)$

or, $-\ln y = \lim_{n\to\infty} \frac{\frac{1}{1+1/n} \times \cancel{\left(-1/n^2\right)}}{\cancel{\left(-1/n^2\right)}}$

or, $-\ln y = \lim_{n\to\infty} \frac{1}{1+1/n}$

or $-\ln y = 1$

or $y = e^{-1}$.

So,

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 16 \times 0 \times e^{-1} = 0 < 1$

The series converges.

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(vi): $\sum_{n=1}^{\infty} \frac{2n!}{n!\,n!}$

Sol^n.

$a_n = \frac{2n!}{n!\,n!}$ $\qquad a_{n+1} = \frac{(2(n+1))!}{(n+1)!\,(n+1)!} = \frac{(2n+2)(2n+1)\,2n!}{(n+1)(n+1)\times n!\,n!}$

Now,

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{2n!}}{(n+1)(n+1)\cancel{n!}\,\cancel{n!}} \times \frac{\cancel{n!}\,\cancel{n!}}{\cancel{2n!}}$

$= \lim_{n\to\infty} \frac{(2+2/n)}{(1+1/n)} \frac{(2+1/n)}{(1+1/n)}$

$= 4 > 1$

The series diverges.

(vii): $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{\sqrt{n}}$

Sol^n:

$a_n = \frac{(-1)^{n+1}(n+2)}{\sqrt{n}}$ $\qquad a_{n+1} = \frac{(-1)^{n+2}(n+3)}{\sqrt{n+1}}$

Now,

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+3)}{\sqrt{n+1}} \times \frac{\sqrt{n}}{(n+2)}$

$= \lim_{n\to\infty} \left\{ \frac{n+3}{n+2} \cdot \sqrt{\frac{n}{n+1}} \right.$

$= \lim_{n\to\infty} \frac{(1+3/n)}{(1+2/n)} \cdot \lim_{n\to\infty} \sqrt{\frac{1}{1+1/n}}$

$= 1$

The test is inconclusive.

Root test:

Let $\sum a_n$ be the series with $a_n > 0$ for $n \geq N$

and suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = k$

then,

if $k < 1$, the series converges

if $k > 1$, the series diverges

if $k = 1$, the test is inconclusive.

Question:

i) $\sum_{n=1}^{\infty} \frac{1}{4^n}$

Sol^n:

$a_n = \frac{1}{4^n}$ $\qquad (a_n)^{1/n} = \frac{1}{(4^n)^{1/n}} = \frac{1}{4}$

Now,

$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{4}$

$= \frac{1}{4} < 1$

The series converges.

(ii): $\sum_{n=1}^{\infty} \left[\frac{3+5n}{2+3n}\right]^n$

Soln:

$a_n = \left[\frac{3+5n}{2+3n}\right]^n$ $\therefore (a_n)^{1/n} = \frac{3+5n}{2+3n} = \frac{\frac{3}{n}+5}{\frac{2}{n}+3}$

Now,

$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{\frac{3}{n}+5}{\frac{2}{n}+3}\right)$

$= \frac{5}{3} > 1.$

The series diverges.

(iii) $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{n}\right]^n$

Soln:

$a_n = \left[\frac{1}{n^2} + \frac{1}{n}\right]^n$ $\therefore (a_n)^{1/n} = \frac{1}{n^2} + \frac{1}{n}$

Now,

$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{n^2} + \lim_{n\to\infty} \frac{1}{n}$

$= 0 < 1$

The series converges.

(iv) $\sum_{n=1}^{\infty} \frac{n^n}{2^{1+4n}}$

Soln:

$a_n = \frac{n^n}{2^{1+4n}}$ $\therefore (a_n)^{1/n} = \frac{n}{2^{1/n} \cdot 2^4}$

Now,

$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{2^4} \cdot \frac{n}{2^{1/n}} = \frac{1}{16} \times \infty$

$= \infty > 1$

The series diverges #

(v): $\sum_{n=1}^{\infty} \frac{n}{2^n}$

Soln:

$a_n = \frac{n}{2^n}$ $\therefore (a_n)^{1/n} = \frac{n^{\frac{1}{n}}}{2}$

Now,

$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{2} \cdot n^{1/n}$

$= \frac{1}{2} \lim_{n\to\infty} n^{1/n}$

Let $y = \lim_{n\to\infty} n^{1/n}$

or, $\ln y = \lim_{n\to\infty} \frac{1}{n} \ln n$

or, $\ln y = \lim_{n\to\infty} \frac{1}{n}$ or $\ln y = 0$

$\therefore y = 1.$

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So, $\lim_{n\to\infty} (a_n)^{1/n} = \frac{1}{2} \times 1 = \frac{1}{2} < 1$

The series converges.

(vi): $\sum_{n=1}^{\infty} \frac{(-1)^n \, 3^{n+2}}{(n-1)^n}$

Soln:

$$\sum_{n=1}^{\infty} |a_n| = \frac{3^{n+2}}{(n+1)^n}$$

$$\therefore (a_n)^{1/n} = \left(\frac{3^n \cdot 3^2}{(n+1)^n}\right)^{\frac{1}{n}} = \frac{3 \cdot 3^{\frac{2}{n}}}{n+1}$$

Now,

$$\lim_{n\to\infty} (a_n)^{1/n} = 3 \lim_{n\to\infty} \frac{3^{\frac{2}{n}}}{n+1}$$

$$= 3 \cdot \lim_{n\to\infty} \frac{1}{n+1} \cdot \lim_{n\to\infty} \cancel{3^{\frac{2}{n}}}\, 9^{\frac{1}{n}}$$

$$= 3 \times 0 \times 9^0$$

$$= 0 < 1$$

The series ~~a~~ converges.