VECTOR SPACE AND LINEAR TRANSFORMATIONS

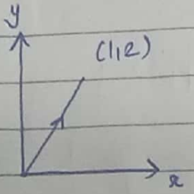
Vector equations:

$\mathbb{R}^2$: vectors in 2-dimension.

$\mathbb{R}^2 = (x, y)$ $\quad \vec{u_1}$

Representation in column form.

$\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \therefore \vec{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

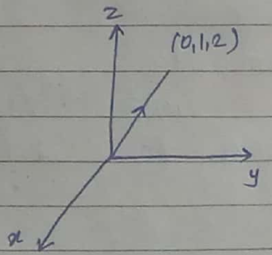


$\mathbb{R}^3$: vectors in 3-dimension

$\mathbb{R}^3 = (x, y, z)$

Representation: in column form.

$\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \therefore \vec{u_1} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$



So, vector in n dimension

$\mathbb{R}^n = (x_1, x_2, \ldots\ldots, x_n)$

Linear Combinations of Vectors:

If $\vec{u_1}, \vec{u_2}, \ldots\ldots, \vec{u_k}$ are vectors on $\mathbb{R}^n$ and if $c_1, c_2, \ldots\ldots, c_k$ are scalars, the vector

$\vec{V} = c_1\vec{u_1} + c_2\vec{u_2} + \ldots\ldots + c_k\vec{u_k}$

is ~~called~~ the linear combination of the given vectors.

Q: Let $\vec{u_1} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

Determine if $\vec{u}$ is the linear combination of $\vec{u_1}$ and $\vec{u_2}$

Sol^n:

Let the linear combinations of vector be

$\vec{u} = c_1\vec{u_1} + c_2\vec{u_2}$

where, c_1 & c_2 are scalars.

So,

$$\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

This implies that,

$7 = 1 \times c_1 + 2 \times c_2$

or $7 = c_1 + 2c_2$ —— (i)

$4 = -2 \times c_1 + 5 \times c_2$

or $4 = -2c_1 + 5c_2$ —— (ii)

$-3 = -5c_1 + 6c_2$ —— (iii)

Writing eq^n's (i), (ii), (iii) in augmented form,

$$\begin{bmatrix} 1 & 2 & : & 7 \\ -2 & 5 & : & 4 \\ -5 & 6 & : & -3 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 + 2R_1$ and $R_3 \rightarrow R_3 + 5R_1$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right]$$

Applying $R_2 \to \frac{1}{9} R_2$ and $R_3 \to \frac{1}{16} R_3$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

Applying $R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

From R_2; $c_2 = 2$

and

From R_1; $c_1 + 2c_2 = 7$

$\therefore c_1 = 3$

Thus,

$$\begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

i.e.,

$\vec{u}$ can be represented as linear combination of $\vec{u_1}$ and $\vec{u_2}$.

Q2): Determine if $\vec{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ can be written as linear combination of,

$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ and $\vec{v_3} = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

Sol^n:

Let the linear combination of vectors be.

$$\vec{u} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

where, c_1, c_2, c_3 are scalars.

So,

$$\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$$

This implies that,

$c_1 + 3c_2 + 4c_3 = 1$ —— (i)

$2c_1 + 5c_2 + 7c_3 = -3$ —— (ii)

$-c_1 + 2c_2 + c_3 = 2$ —— (iii)

Writing eq^n (i), (ii), (iii) in augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 2 & 5 & 7 & -3 \\ -1 & 2 & 1 & 2 \end{array}\right]$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 5 & 5 & 3 \end{array}\right]$$

Applying $R_2 \to -R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 5 & 3 \end{array}\right]$$

Applying; $R_3 \to R_3 - 5R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 4 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

Here,
rank of coefficient matrix = 2
rank of augmented matrix = 3
Since, the system of linear equations is inconsistent, we cannot determine the scalars c_1, c_2, c_3.
Hence, the linear combination is not possible.

Theorem:

The linear system $A\vec{x} = \vec{b}$ has a solution iff, $\vec{b}$ can be expressed as linear combination of the column vectors of A.

<Q>: Find the solution of $A\vec{x} = \vec{b}$ where,

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 10 & 5 \\ 2 & 7 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

Soln:

The augmented matrix is,

$$\left[\begin{array}{ccc:c} 1 & 3 & 4 & 6 \\ 3 & 10 & 5 & 4 \\ 2 & 7 & 1 & -2 \end{array}\right]$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$.

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 4 & 6 \\ 0 & 1 & -7 & -14 \\ 0 & 1 & -7 & -14 \end{array}\right]$$

Applying $R_3 \to R_3 - R_2$,

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 4 & 6 \\ 0 & 1 & -7 & -14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

So,
From R_1, $x + 3y + 4z = 6$
From R_2, $y - 7z = -14$
From R_1 $0 \times z = 0$.

Thus, 2 is a free variable.

Let $z = t$.

Now,

$$y = -14 + 7z$$

and

$$x = 6 - 4z - 3y$$

$$\therefore x = 6 - 4z + 42 - 21z$$

$$= 48 - 25z$$

If $t = 1$,

$z = 1$, $y = -7$, $x = 23$

Thus,

$$\begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix} = 23 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - 7 \begin{bmatrix} 3 \\ 10 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

This is the linear combination when $z = 1$.

Similarly, other solutions and linear combinations can be found.