Advanced Calculus
Multiple Integrals

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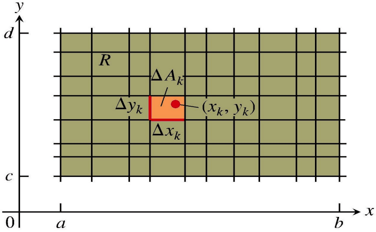
Double Integrals

September 28, 2023



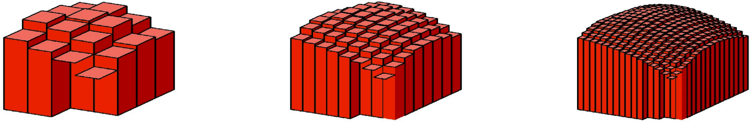
$$R: \quad a \le x \le b, \quad c \le y \le d.$$

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k .$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k ,$$

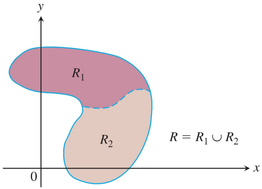
When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over R, written as

$$\iint_R f(x, y)\, dA \qquad \text{or} \qquad \iint_R f(x, y)\, dx\, dy.$$



- Find the values of the following integrals
  a. $\int_{-1}^{0}\int_{-1}^{1}(x+y+1)dxdy$
  b. $\int_{-1}^{1}\int_{-1}^{0}(x+y+1)dydx$

Properties of Double Integrals



$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

FIGURE 15.17 The Additivity Property for rectangular regions holds for regions bounded by smooth curves.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA \quad$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

3. *Domination:*

(a) $\displaystyle\iint_R f(x,y)\,dA \geq 0 \qquad$ if $\qquad f(x,y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA \qquad$ if $\qquad f(x,y) \geq g(x,y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

if R is the union of two nonoverlapping regions R_1 and R_2

Double Integrals

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R: \quad 0 \leq x \leq 2, \quad -1 \leq y \leq 1.$$

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

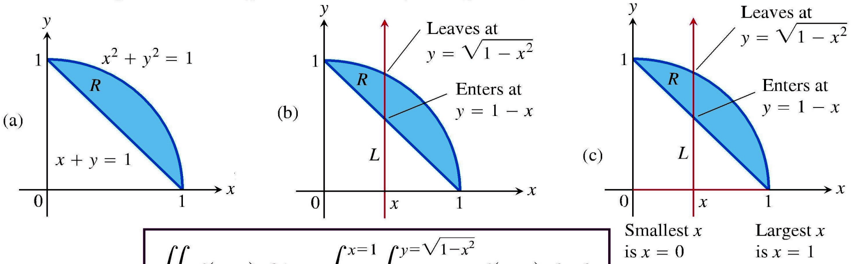
Example

Integrate $f(x, y) = x/y$ over the region in the first quadrant bounded by $y = x$, $y = 2x$, $x = 1$, $x = 2$. Ans $3/2 \ln 2$.

Double Integrals: Finding Limits of Integration

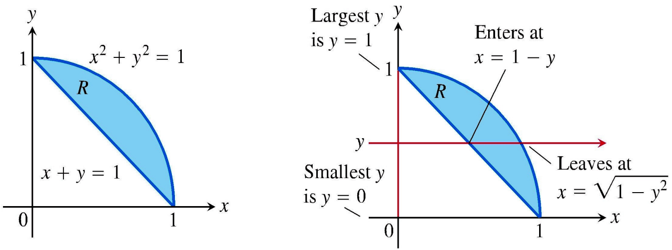
Using Vertical Cross-sections When faced with evaluating $\iint_R f(x,y)\,dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch*. Sketch the region of integration and label the bounding curves (Figure a).
2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants) (Figure b).
3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral shown here (see Figure c) is



$$\boxed{\iint_R f(x,y)\,dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx.}$$

Using Horizontal Cross-sections To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3 (see Figure 15.15). The integral is



$$\boxed{\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.}$$

- Sketch the region of integration and write an equivalent integral with the order of integration reversed.

1. $\int_0^2 \int_{x^2}^{2x} (4x+2)\,dydx$ Ans: $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dxdy$

2. $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\,dxdy$ Ans: $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y\,dydx$

3. $\int_0^1 \int_2^{4-x} dydx$ Ans: $\int_2^4 \int_0^{(4-y)/2} dxdy$

Area of Plane Region

Definition

The area of closed and bounded plane region R is

$$A = \iint_R dA$$

Example

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Ans: $1/6$ sq. units.

Average Value

Definition

Average value of $f(x,y)$ over $R = \dfrac{1}{\text{area of R}} \displaystyle\iint_R f\, dA$

Example

Find the average value of $f(x,y) = x\cos xy$ over the rectangle

$$R : 0 \leq x \leq \pi,\ 0 \leq y \leq 1.$$

Ans: $2/\pi$.

Double Integral in Polar Form

$$\iint_R f(r,\theta)dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r dr d\theta$$

Why $r\,drd\theta$ in place of dA ????(See in book)

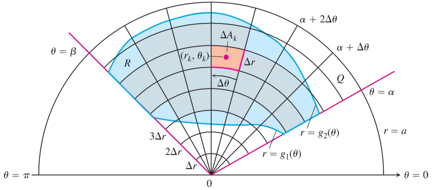


FIGURE 15.21 The region R: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, is contained in the fan-shaped region Q: $0 \le r \le a$, $\alpha \le \theta \le \beta$. The partition of Q by circular arcs and rays induces a partition of R.

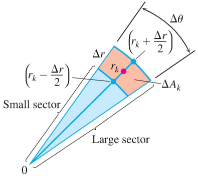


FIGURE 15.22 The observation that $\Delta A_k = \begin{pmatrix}\text{area of}\\ \text{large sector}\end{pmatrix} - \begin{pmatrix}\text{area of}\\ \text{small sector}\end{pmatrix}$ leads to the formula $\Delta A_k = r_k\,\Delta r\,\Delta\theta$.

Double Integrals: Finding Limits of Integration in Polar Form

EXAMPLE Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure 1).

2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.

3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

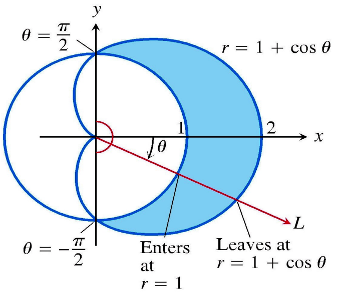


FIGURE 1 Finding the limits of integration in polar coordinates

Double Integral in Polar Form

Area in Polar Coordinates

The area of closed and bounded plane region R in polar cooridnate plane is

$$A = \iint_R r dr d\theta$$

Average value in Polar Coordinates

The area of closed and bounded plane region R in polar cooridnate plane is

$$A = \frac{1}{\text{Area of R}} \iint_R f(r,\theta) r dr d\theta$$

Problems

Area

Find the area of the region cut from the first quadrant by the cardiod $r = 1 + \sin\theta$.

Ans: $(3\pi/8 + 1)$ sq. units.

Average Value

Find the average distance from a point $P(x, y)$ in the disk $x^2 + y^2 \le a^2$ to the origin.

Hints: Average $= \dfrac{1}{\pi a^2} \displaystyle\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dydx.$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a r^2\, dr\, d\theta = \cdots = 2a/3$$

Cartesian and Polar Integrals

I. Change the Cartesian integral into an equivalent polar integral.

1. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dydx$

2. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dydx$

3. $\displaystyle\int_{0}^{6}\int_{0}^{y} xdxdy$

4. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)dydx$

II. Change into Cartesian Form:

1. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{1} r^3\cos\theta\sin\theta drd\theta$

2. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{2\sec\theta} r^5\sin^2\theta drd\theta$

Examples

1. Evaluate $\iint_R e^{x^2+y^2}dydx$, where R is the semi-circular region bounded by x - axis and the curve $y = \sqrt{1-x^2}$.

2. Find the average distance from a point $P(x, y)$ in the disk $x^2 + y^2 \leq a^2$ to the origin.