General Physics I (PHYS 101)

Lecture 07

Rotational Dynamics

Keshav Raj Sigdel

Assistant Professor

Department of Physics

Kathmandu University

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Outline

1. Rigid Body

2. Torque and Angular momentum
   - Relation between torque and angular momentum
   - Torque and Angular momentum of system of particles

3. Rotational Kinetic Energy and Moment of Inertia (Rotational Inertia)
   - Radius of Gyration
   - Physical significance of moment of inertia

Rigid Body

A body in which distance between two particles is independent of the motion of the body is called a rigid body. Every solid is a rigid body.

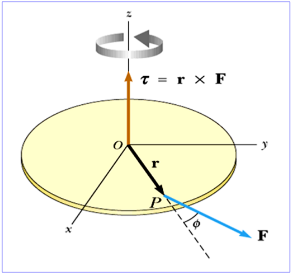
A rigid body is defined as a solid body in which the particle are completely arranged so that the inter-particle distance is small and fixed, and their positions are not distributed by any external force applied on it. A rigid body can undergo both translational and rotational motion.

Torque and Angular momentum

Consider a particle of mass m and linear velocity $\vec{v}$ rotating about a fixed point 0. The particle is located at position vector $\vec{r}$ relative to its axis of rotation. If the particle's linear momentum is $\vec{p}$, then the angular momentum $\vec{L}$ or $\vec{J}$ of the particle with respect to the point 0 is defined as

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = rpsin\phi\hat{n} \quad (1)$$

Where ϕ is the angle between $\vec{r}$ and $\vec{p}$.



Torque and Angular momentum (contd.)

Angular momentum is a vector quantity, whose magnitude is $L = rp\sin\phi\hat{n}$ and the direction is perpendicular to the plane of $\vec{r}$ and $\vec{p}$ and specified by right hand rule.

If $\phi = 90^0$ then L=pr

$\implies$ L = linear momentum $\times$ $\perp$ distance from the axis

$\implies$ L = moment of linear momentum = Angular momentum

When a force $\vec{F}$ is applied to the particle, then

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{d\vec{p}}{dt}$$

Torque and Angular momentum (contd.)

taking cross product with $\vec{r}$ on both sides, we get

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Now we define torque is the product of force and the distance from the axis of rotation (i.e. moment of force)

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} \tag{2}$$

Torque is a vector quantity whose magnitude is $\vec{\tau} = rF\sin\phi$, where ϕ is the angle between $\vec{F}$ and $\vec{r}$, and the direction is perpendicular to the plane of $\vec{F}$ and $\vec{r}$, and specified by right hand rule.

Differentiating equation (1) with respect to time, we get

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \frac{d\vec{r}}{dt}\times\vec{p}+\vec{r}\times\frac{d\vec{p}}{dt} = \vec{v}\times m\vec{v}+\vec{r}\times\vec{F}$$

Since $\vec{v}\times\vec{v}=0$

$$\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F} = \vec{\tau} \tag{3}$$

Hence, the rate of change of angular momentum is equal to the torque. This is analogy of Newton's law for translational motion.

Consider the system of n-particles of point masses $m_1, m_2, \ldots, m_n$ with position vectors $\vec{r}_1$, $\vec{r}_2, \ldots, \vec{r}_n$ respectively, have the angular momenta $\vec{L}_1 = \vec{r}_1 \times \vec{p}_1, \vec{L}_2 = \vec{r}_2 \times \vec{p}_2$ and so on. The total angular momentum of the system is equal to the vector sum of all angular momenta of the particles of the system i.e.

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots + \vec{L}_n = \sum_{i=1}^{n} \vec{L}_i \qquad (4)$$

Taking the derivative of equation (4) with respect to time, we get

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \ldots + \frac{d\vec{L}_n}{dt}$$

$$\implies \frac{d\vec{L}}{dt} = \vec{\tau}_1 + \vec{\tau}_2 + \ldots + \vec{\tau}_n \tag{5}$$

In the sum of torques in equation (5), there are two types of torques acting on the particles: ones due to the internal forces and the others due to the external forces. So the equation (5) is better to be written as

$$\frac{d\vec{L}}{dt} = (\vec{\tau}_{1int} + \vec{\tau}_{2int} + \ldots + \vec{\tau}_{nint}) + (\vec{\tau}_{1ext} + \vec{\tau}_{2ext} + \ldots + \vec{\tau}_{next}) \tag{6}$$

But the internal forces are in pair with equal in magnitude and opposite in direction. So the torques produced by each pair cancel each other and hence the net internal torques due to all internal forces is equal to zero. Therefore

$$\frac{d\vec{L}}{dt} = (\vec{\tau}_{1ext} + \vec{\tau}_{2ext} + \ldots + \vec{\tau}_{next}) = \vec{\tau}_{ext} \quad (7)$$

where $\vec{\tau}_{ext}$ is the resultant of all external torques acting on the particles of the system.

If $\vec{\tau}_{ext} = 0$, then $\frac{d\vec{L}}{dt} = 0$ implies $\vec{L} =$ constant. That mean total angular momentum remain conserved when the resultant of all the torques is zero. The individual particles may experience the external torques. This is called the principle of conservation of total angular momentum.

Rotational Kinetic Energy and Moment of Inertia (Rotational Inertia)

Consider a system of point mass particles with masses m_1, m_2, …, m_n. r_1, r_2, …, r_n are respectively the perpendicular distance of the particles from the axis of rotation AB. Considering all particles rotate about the axis with same angular velocity ω, the velocities of the particles are $v_1 = \omega r_1$, $v_2 = \omega r_2$, and so on. Hence the total kinetic energy of the system of rotating particles is

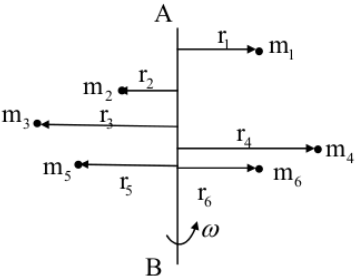


Figure 1: Rotional kinetic energy

Rotational Kinetic Energy and Moment of Inertia (Rotational Inertia) (contd.)

$$\begin{aligned}
K_{rot} &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \ldots + \frac{1}{2}m_n v_n^2 \\
&= \frac{1}{2}m_1 \omega^2 r_1^2 + \frac{1}{2}m_2 \omega^2 r_2^2 + \ldots + \frac{1}{2}m_n \omega^2 r_n^2 \\
&= \frac{1}{2}\left(m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2\right)\omega^2 \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2
\end{aligned}$$

$$\text{i.e. } K_{rot} = \frac{1}{2}I\omega^2 \tag{8}$$

Rotational Kinetic Energy and Moment of Inertia (Rotational Inertia) (contd.)

where,

$$I = \sum_{i=1}^{n} m_i r_i^2 \tag{9}$$

is called the moment of inertia or rotational inertia of the system of particle. For the continuous mass distribution (rigid body) system, the m_i of equation (9) must be replaced by infinitesimally small mass dm and the summation sign by integral as

$$I = \int r^2 dm \tag{10}$$

Here r is the perpendicular distance of any dm from the axis of rotation.

It is defined as the distance from the axis of rotation to the point where total mass of the body is supposed to be concentrated such that the moment of inertia about the axis remains same. it is denoted by K. In terms of radius of gyration the moment of inertia of a body of mass M is given by

$$I = MK^2 \implies K = \sqrt{\frac{I}{M}} \tag{11}$$

Hence the radius of gyration can also be defined as a distance, the square of which when multiplied by the total mass of the body gives

Radius of Gyration (contd.)

its moment of inertia about the given axis. This radius of gyration depends upon the shape and size of the body as well as upon the axis of rotation.

Physical significance of moment of inertia

Moment of inertia plays the same role in rotational motion as mass does in translational motion. This is the physical significance of moment of inertia.