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Unit: 6
SEQUENCE AND INFINITE SERIES

Sequence:

The list of numbers in which the difference / ratio between a term and its preceeding term is equal throughout the list is called sequence.

Examples:

(i): 1, 2, 3, 4, 5, ... Arithmetic sequence.

$a_n = 1 + (n-1) \times 1$

$\therefore a_n = n$

(ii): $\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, ...$ ⇒ Geometric sequence which is also alternating sequence.

$\therefore a_n = (-1)^{n+1} \frac{(n+2)}{5^n}$

If $a_n = (-1)^{n+1}$ ..... then alternating.

(iii) 1, 1, 2, 3, 5, 8, ... ⇒ Fibonacci series

$a_n = a_{n-1} + a_{n-2}$

$\lim_{n \to \infty} a_n =$ finite number. Sequence is convergent.

$\lim_{n \to \infty} a_n = \infty$ Sequence is ~~co~~ divergent.

Convergence and Divergence of Sequence

A sequence $\{a_n\}$ is convergent to L, if for every $\epsilon > 0$, there exists a two number N such that

$$|a_n - L| < \varepsilon \iff n > N.$$

i.e, $\lim_{n \to \infty} a_n = L$

otherwise, it is divergent.

Eg: $\lim_{n \to \infty} \frac{n-1}{n}$

Soln.

$$= \lim_{n \to \infty} \frac{n}{n} - \frac{1}{n} = \lim_{n \to \infty} 1 - \frac{1}{n}$$

$$= 1$$

It is convergent.

Sandwich Theorem of Sequence

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be a sequence of real numbers.

If $a_n < b_n < c_n$ holds for all n and if

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L$$

then,

$$\lim_{n \to \infty} b_n = L$$

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<Q7: If $\lim_{n\to\infty} \frac{1}{n} = 0$ is convergent and show that $\lim_{n\to\infty} \frac{\cos n}{n}$ is convergent.

Soln:

Here,

$$-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$$

Taking $\lim_{n\to\infty}$,

$$\lim_{n\to\infty} -\frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} = 0$$

So,

$$\cancel{\lim_{n\to\infty} -\frac{1}{n}} \cancel{\leq} \cancel{\lim_{n\to\infty} \frac{\cos n}{n}} \cancel{\leq} \cancel{\lim_{n\to\infty} \frac{1}{n}}$$

Since $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$

So,

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0$$

This is convergent.

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Bounded Sequence:

i) Bounded lower:

If $a_n \leq M$.

Eg: Sequence bounded lower 1,

$$1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}$$

(ii) Bounded above:

If $a_n \geq M$

Eg: Sequence bounded ~~to~~ above 1,

1, 2, 3, 4.

Monotonic Sequence

If a sequence is non-decreasing or non-increasing then, it is called montonic sequence.

for non-decreasing; $a_n < a_{n+1}$

for non-increasing; $a_n > a_{n+1}$

<Q>: Check convergence and divergence of a function.

(a): $a_n = \dfrac{1-2n}{2+2n}$

Soln:

Given,

$$a_n = \frac{1-2n}{2+2n}$$

Taking $\lim_{n\to\infty} a_n$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1-2n}{2+2n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(\frac{1}{n}-2\right)}{2\cancel{n}\left(\frac{1}{n}+1\right)}$$

$$= \frac{0-2}{2(0+1)} = \frac{-2}{2} \cancel{= \frac{-1}{1}} = -1$$

Hence,

$\lim_{n\to\infty} a_n = -1$ ie, convergent.

(b) : $a_n = \frac{\sin^2 n}{2^n}$

Soln:

Given,

$a_n = \frac{\sin^2 n}{2^n}$

Taking $\lim_{n\to\infty} a_n$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin^2 n}{2^n}$$

We know,

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(b): $a_n = \frac{\sin^2 n}{2n}$

Soln:

Here,

$0 \cancel{\frac{1}{2n}} \leq \frac{\sin^2 n}{2n} \leq \frac{1}{2n}$

Taking $\lim_{n\to\infty} \cancel{\frac{-1}{2n}}^{0} = 0$ and $\lim_{n\to\infty} \frac{1}{2n} = 0$

ie, $\lim_{n\to\infty} \cancel{\frac{-1}{2n}} 0 = \lim_{n\to\infty} \frac{1}{2n}$

So, using Sandwich theorem,

$\lim_{n\to\infty} \frac{\sin^2 n}{2n} = 0$

it is convergence.

(c): $a_n = \left(1 - \frac{1}{n}\right)^n$

Soln:

Given,

$a_n = \left(1 - \frac{1}{n}\right)^n$

Let $y = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$

or, $\ln y = \cancel{\ln} \lim_{n\to\infty} n \ln\left(1 - \frac{1}{n}\right)$

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$$= \lim_{n\to\infty} \frac{\ln(1-1/n)}{1/n}$$

$$: \lim_{n\to\infty} \frac{\frac{-1/n^2}{1-1/n} \times (-n_2)}{-1/n^2 \quad \times(-n_2)}$$

or, $\ln y = \lim_{n\to\infty} \frac{1}{1-1/n}$

or $\ln y = 1$

$y = e^1$

$\therefore \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e$

It converges.

(d): $a_n = \frac{n}{2^n}$

Soln:

Given,

$a_n = \frac{n}{2^n}$

Let $y = \lim_{n\to\infty} \frac{n}{2^n}$ $= \frac{1}{2} \lim_{n\to\infty} \frac{1}{2^n}$

$= \lim_{n\to\infty} \frac{1}{2^n \ln 2}$ $= \frac{1}{\ln(2)} \times 0 = 0$

$\therefore$ It is convergent.

Infinite Series

The expression in the form $a_1 + a_2 + \cdots\cdots$

$\sum_{1}^{\infty} a_n$ is an infinite series.

Partial Sums:

A ~~practice~~ partial sum of an infinite series is the sum of a finite numbers of consecutive terms beginning with the first term.

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2$

$\vdots$

$S_i = a_1 + a_2 + \cdots + a_i$

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n$

If the sequence of partial sums for an infinite series converges to a limit L, then the sum of the series is said to be L and series is convergent.

Otherwise, ie, if limit reaches to infinity and the series doesn't have a sum, it is divergent.

i.e, If $\lim_{n\to\infty} S_n = L$ then, Σa_n is converged.

So, $\lim_{n\to\infty} S_{n-1} = L$ is also convergent.

which means the series is also convergent.

<Q>: $\sum_{n=1}^{\infty}\left(\frac{1}{(n+1)(n+2)}\right)$

~~Ho~~ Sol^n:

Here,

$$a_n = \frac{1}{(n+1)(n+2)}$$

$$\therefore a_n = \frac{1}{(n+1)} - \frac{1}{(n+2)}$$

So,

$$S_n = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$$

$$\cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$\therefore S_n = \frac{1}{2} - \frac{1}{n+2}$$

So,

$$\lim_{n\to\infty} S_n = \frac{1}{2} - \frac{1}{\infty+2} = \frac{1}{2}$$

Hence, the given series is convergent.

nth Root Test

If infinite series $\sum a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

Proof:

If $\sum a_n$ is convergent, then the partial sum of infinite series is also convergent.

ie, $\lim_{n \to \infty} S_n = \lim_{n \to \infty} S_{n-1} = L$

We know,

$$a_n = S_n - S_{n-1}$$

Taking $\lim_{n \to \infty}$ on both sides,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1}$$

$$= L - L = 0$$

$$\therefore \lim_{n \to \infty} a_n = 0$$

Hence, proved.

But converse may not be true.

Cases for converse not being true

<Q>: $\sum_{1}^{\infty} \frac{1}{\sqrt{n}}$

Sol^n:

Given,

$$a_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$$

Now,

$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}$$

$$= \cancel{\frac{\sum}{\sqrt{n}}} \quad n \times \frac{1}{\sqrt{n}} \quad = \sqrt{n} \times \cancel{\sqrt{n}} \times \frac{1}{\cancel{\sqrt{n}}}$$

$$\therefore S_n = \sqrt{n}$$

Now,

$$\cancel{S_n} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sqrt{n} = \infty$$

It is divergent.

Which means converse of n^{th} root test is not true.

<Q>: $\sum_{1}^{n} 1/n$

Sol^n:

Given,

$$a_n = \frac{1}{n}$$

~~lim~~ Taking $\lim_{n\to\infty}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Now,

$$S_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\cdots + \frac{1}{16} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$\therefore S_n = n \times \frac{1}{2}$$

Now,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n \times \frac{1}{2} = \infty$$

It is divergent.

Hence, the converse of n^{th} root test is not true.

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Infinite Geometric Series

$$a + ar + ar^2 + \cdots\cdots + ar^{n-1} + \cdots\cdots$$

$\sum_{1}^{\infty} ar^{n-1}$ is the infinite geometric series.

We know,

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{for} \quad r \neq 1.$$

Case I:

If $|r| = 1$, $S_n = \infty$ ie, divergent.

The series becomes,

$S_n = a + a + a + \cdots\cdots + a_n$

$S_n = na$

so,

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} na = \infty$$

This is ~~co~~ divergent.

Case II:

if $|r| < 1$,

$\lim_{n \to \infty} r^n = 0$

$$S_n = \frac{a}{1-r}$$

so,

$$\lim_{n \to \infty} S_n = \lim_{r \to \infty} \frac{a}{1-r} = 0$$

This is convergent.

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(*) Case III :

If $|r| > 1$,

$$\lim_{n\to\infty} r^n = \infty$$

So,

$$\lim_{n\to\infty} S_n = \lim_{ar^n\to\infty} \frac{a(r^n - 1)}{r-1} = \infty$$

This is divergent.

<Q7: Check for : $\sum_{0}^{\infty} (-1)^{n+1} \frac{5}{4^{n-1}}$

Soln

$$= 5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \frac{5}{4^4} - \cdots$$

$$= 5\left(1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \frac{1}{4^4} - - \right)$$

Here,

$$|r| = \frac{1/4}{1} = \frac{1}{4} < 1$$

This series is convergence.

Integral Test

Let $\{a_n\}$ be the sequence of positive terms.

Suppose that $a_n = f(n)$ where f is continuous positive decreasing function of x for all $x > N$ (N is positive integer).

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Then, the series $\sum_{n=N}^{\infty} a_n$ and integral $\int_n^{\infty} f(x)\, dx$.

both converges or diverges.

(Q) Check: $\sum_{n=1}^{\infty} 1/(n^2+1)$

Soln:

Here,

$$a_n = \frac{1}{n^2+1}$$

writing $\sum_{n=1}^{\infty} a_n$ in integral form,

$$\sum_{1}^{\infty} \frac{1}{n^2+1} = \int_1^{\infty} \frac{dn}{1+n^2}$$

$$= \lim_{b \to \infty} \int_1^{b} \frac{dn}{1+n^2}$$

$$= \lim_{b \to \infty} \tan^{-1} n \Big|_1^b$$

$$= \lim_{b \to \infty} (\tan^{-1} b - \tan^{-1} 1)$$

$$= \pi/4 = \text{finite.}$$

Here, $\sum_{1}^{\infty} \frac{1}{n^2+1}$ is convergent.

<Q>: $\sum_{1}^{\infty} \ln(n)/n$.

Soln.

Here,

$$a_n = \frac{\ln(n)}{n}$$

Writing $\sum_{1}^{\infty} \ln(n)/n$ in integral form,

$$\sum_{1}^{\infty} a_n = \int_{1}^{\infty} \frac{\ln x}{x} \, dx$$

$$= \lim_{b \to \infty} \int_{1}^{b} \ln x \, d\ln x$$

$$= \lim_{b \to \infty} \left. \frac{(\ln x)^2}{2} \right|_{1}^{b}$$

$$= \lim_{b \to \infty} \left[\frac{(\ln \infty)^2}{2} - \frac{(\ln 1)}{2} \right]$$

$= \infty$ = divergent.

Hence, $\sum_{1}^{\infty} \ln(n)/n$ is divergent.

<Q>: $\sum_{1}^{\infty} \frac{1}{n^p}$ Show that $p \leq 1$ is ~~convergent~~ divergent, $p > 1$ is convergent.

Soln.

Given,

$$a_n = \frac{1}{n^p}$$

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Writing $\sum_{1}^{\infty} 1/n^p$ in integral form,

$$\sum_{1}^{\infty} 1/n^p = \int_{1}^{\infty} \frac{1}{n^p}\, dn$$

$$= \lim_{b\to\infty} \int_{1}^{b} n^{-p}\, dn$$

$$= \lim_{b\to\infty} \left. \frac{n^{-p+1}}{1-p} \right|_{1}^{b}$$

$$= \lim_{b\to\infty} \left. \frac{n^{1-p}}{1-p} \right|_{1}^{b}$$

$$= \frac{1}{(1-p)} \times \lim_{b\to\infty} \left. n^{1-p} \right|_{1}^{b}$$

When $p = 1$,

$$= \frac{1}{(1-1)} \times \lim_{b\to\infty} \left. n^{1-1} \right|_{1}^{b} = \infty$$

diverges.

When $p < 1$, let $1-p = a$ for $p<1$

$$= \frac{1}{(1-p)} \times \lim_{b\to\infty} \left. n^{a} \right|_{1}^{b}$$

$$= \frac{1}{(1-p)} \times [\infty^a - 1^a] = \infty$$

. diverges.

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When, let $1-p = -a$ for $p > 1$.

$p > 1$,

$$= \frac{1}{(1-p)} \times \lim_{b\to\infty} n^{-a} \Big|_1^b$$

$$= \frac{1}{(1-p)} \times \lim_{b\to\infty} \frac{1}{n^a} \Big|_1^b$$

$$= \frac{1}{(1-p)} \times \left(\frac{1}{\infty} - \frac{1}{1}\right)$$

$$= \frac{1}{(1-p)} = \text{finite}$$

$= \text{converges.}$

Hence, proved.

The ~~Root~~ Ratio Test

~~Let Σa_n be a sum with $a_n \geq 0$ for $n \geq N$ and suppose that~~

Let Σa_n be a series with positive terms, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = k.$$

then,

the series converges if $k < 1$.

the series diverges if $k > 1$ or infinity

test fail if $k = 1$.

The Root Test

Let $\sum a_n$ be a series with $a_n > 0$ for $n \geq N$ and suppose that.

$$\lim_{n \to \infty} \sqrt[n]{a_n} = k.$$

then,

the series converges if $k < 1$.

the series diverges if $k > 1$. or infinity

test fail if $k = 1$.

<Q7: Use ratio test:

a) $\sum 1/n$ (diverges in reality)

Soln:

$$a_{n+1} = \frac{1}{n+1}$$

$$a_n = \frac{1}{n}$$

Now,

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1}$$

$$= \frac{\cancel{n}}{\cancel{n}\left(1 + \frac{1}{n}\right)}$$

$$\lim_{n \to \infty} \frac{1}{(1 + 1/n)}$$

$$= 1$$

(test fail)

b) $\sum 1/n^2$ (converges in relation)

$$a_{n+1} = \frac{1}{(n+1)^2}$$

$$a_n = \frac{1}{n^2}$$

Now,

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2}$$

$$= \frac{\cancel{n^2}}{\cancel{n^2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)}$$

$$= \lim_{n \to \infty} \frac{1}{(1 + 2/n + 1/n^2)}$$

$$= 1$$

(test fail).

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<Q>: $\sum_{1}^{\infty} 2n! / n!n!$

Sol^n.

Given,

$$a_n = \frac{2n!}{n!n!}$$

and

$$a_{n+1} = \frac{2(n+1)!}{(n+1)!(n+1)!}$$

$$= \frac{(2n+2)!}{(n+1)!(n+1)!} \quad = \cancel{(2n+2)}(2n$$

Now, conducting ratio test,

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!(n+1)!} \times \frac{n!n!}{2n!}$$

$$= \frac{(2n+2)\times(2n+1)\times\cancel{2n!}}{(n+1)^2 \cancel{\times n!} \cancel{\times n!}} \times \frac{\cancel{2}\cancel{n!}\cancel{n!}}{\cancel{2n!}}$$

$$= \frac{2\cancel{(n+1)^2}(2n+1)}{(n+1)^{\cancel{2}}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2(2n+1)}{(n+1)} = \frac{2\cancel{n}(2+2/n)}{\cancel{n}(1+1/n)}$$

$$= 4 > 1$$

The series diverges.

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<Q>: $\sum_{1}^{\infty} \frac{4^n}{(3n)^n}$

Soln:

Given,

$a_n = \frac{4^n}{(3n)^n}$ or, $a_n = \left(\frac{4}{3n}\right)^n$

Conducting root test;

$(a_n)^{1/n} = \left(\frac{4}{3n}\right)^{\cancel{n} \times 1/\cancel{n}}$

or, $\lim_{n \to \infty} (a_n)^{1/n} = \lim_{n \to \infty} \frac{4}{3n} = 0 < 1$

The series converges.

<Q>: $a_n = \begin{cases} \frac{n}{2^n} & ; n \text{ odd} \\ \frac{1}{2^n} & ; n \text{ even} \end{cases}$

Soln:

(* Don't know if this is correct)

Here, $a_n = \frac{n}{2^n}$,

$a_{n+1} = \frac{n+1}{2^{n+1}}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{(n+1)}{2^{n+1}} \times \frac{2^n}{n} = \frac{\cancel{2^n} \cdot 2 \times n}{2(n+1) \times \cancel{2^n}} = \frac{2}{\frac{\cancel{n}(1 + 1/n)}{2\cancel{n}}}$

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} < 1$, ~~diverges~~. converges.

for $a_n = \frac{1}{2^n}$,

$a_{n+1} = \frac{1}{2^{n+1}}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{2^n} = \frac{\cancel{2^n} \times 2}{\cancel{2^n}} = 2 > 1$ diverge.

Here, ~~sequence~~ series for odd terms converges
whereas ~~sequence~~ series for even terms.

ie, $\sum a_n$ odd and $\sum a_n$ even don't tend to a specific number.

We know,

$\sum a_n = \sum a_n \ (n = \text{odd}) + \sum a_n \ (n = \text{even})$

So,

$\sum a_n$ doesn't convergence.