Lecture 14

Electromagnetic Induction

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Keshav Raj Sigdel

Assistant Professor

Department of Physics

School of Science

Kathmandu University

Outline

1. Motional *emf*
   - Flux rule for rectangular loop moving in an uniform magnetic field
   - Flux rule for arbitrary loop moving in a non-uniform magnetic field
2. Faraday's laws of electromagnetic induction
3. Lenz's law
4. Self induction
5. Problems

Flux rule for rectangular loop moving in an uniform magnetic field

The emf produced by the relative motion between magnetic field and conducting loop (or rod) is called **motional *emf***.

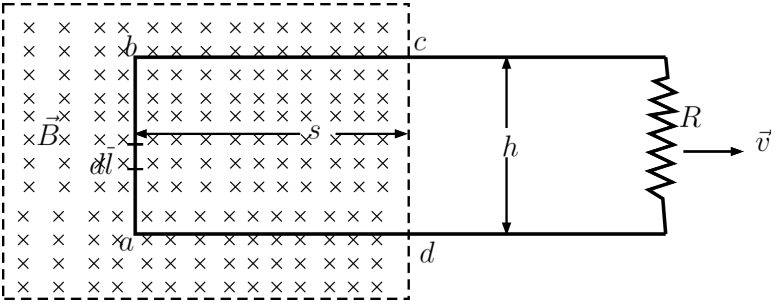


Figure 1

Motional *emf*

Flux rule for rectangular loop moving in an uniform magnetic field (contd.)

Suppose a portion *abcd* of a conducting loop is inside of a magnetic field $\vec{B}$ as shown in figure 1. Let $\vec{v}$ be the velocity of loop, h the width ab, s is length of loop inside the field at a time t and R is resistance connected with loop. When loop moves, the charge particles inside ab experiences a Lorentz force $\vec{F} = q(\vec{v} \times \vec{B})$ due to magnetic field. Therefore, the magnetic force experience by the unit charge is $f_{\text{mag}} = \vec{v} \times \vec{B}$. Along side ab, a unit charge experiences the magnetic force $\left(f_{\text{mag}}\right)_{ab} = vB$ along the direction from a to b. The unit charge on side ad or bc experiences the magnetic force is $\left(f_{\text{mag}}\right)_{ad} = \left(f_{\text{mag}}\right)_{bc} = vB$ along the direction normal to the side. The charge in the remaining portion does not experience magnetic force because of the

Motional *emf*

Flux rule for rectangular loop moving in an uniform magnetic field (contd.)

absence of magnetic field. The amount of work by the magnetic force done to drive a unit charge in a closed circuit is called induced *emf* i.e. the induced emf.

$$\begin{aligned}
\mathscr{E} &= \oint \vec{f}_{\text{mag}} \cdot d\vec{l} \\
&= \underbrace{\int_a^b \vec{f}_{\text{mag}} \cdot d\vec{l}}_{\vec{f}_{\text{mag}} \parallel d\vec{l}} + \underbrace{\cancelto{0}{\int_b^c \vec{f}_{\text{mag}} \cdot d\vec{l}}}_{\vec{f}_{\text{mag}} \perp d\vec{l}} + \underbrace{\cancelto{0}{\int_c^d \vec{f}_{\text{mag}} \cdot d\vec{l}}}_{\vec{B}=0} + \underbrace{\cancelto{0}{\int_d^a \vec{f}_{\text{mag}} \cdot d\vec{l}}}_{\vec{f}_{\text{mag}} \perp d\vec{l}} \\
\therefore \mathscr{E} &= vBh \qquad (1)
\end{aligned}$$

Flux rule for rectangular loop moving in an uniform magnetic field (contd.)

Again the magnetic flux crossing the whole circuital loop is,

$$\Phi = \oint \vec{B} \cdot d\vec{A} = BA = Bsh$$

As loop moves, the flux decreases i.e

$$\frac{d\Phi}{dt} = -B\frac{ds}{dt}h = -vBh \tag{2}$$

From equations (1) and (2),

$$\mathscr{E} = -\frac{d\Phi}{dt}. \tag{3}$$

This is called the **flux rule** for motional *emf*.

Motional *emf*

Flux rule for arbitrary loop moving in a non-uniform magnetic field

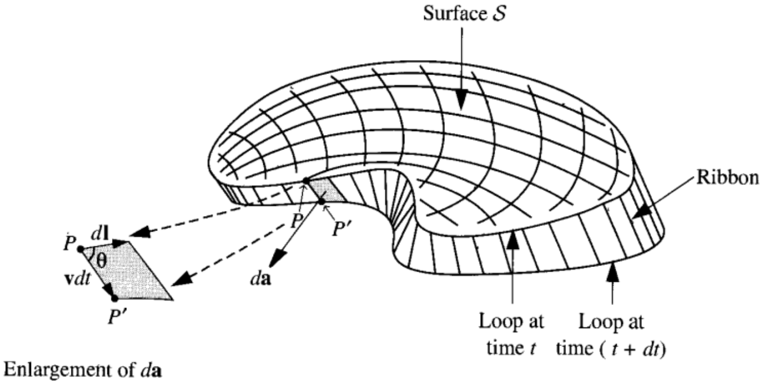


Figure 2

Flux rule for arbitrary loop moving in a non-uniform magnetic field (contd.)

Suppose a loop of arbitrary shape is moving with velocity $\vec{v}$ in a static magnetic field $\vec{B}$. The position of loop in time t and $t+dt$ are shown in figure 2. The change in flux due to motion of loop is then

$$d\Phi = \Phi(t+dt) - \Phi(t) = \Phi_{ribbon} = \oint_{ribbon} \vec{B}\cdot d\vec{a}$$

Here $d\vec{a}$ is small area generated by $d\vec{l}$ and $\vec{v}\,dt$. So,

$d\vec{a} = \vec{v}\,dt \times d\vec{l} = (\vec{v}\times d\vec{l})\,dt$

$$\therefore\ d\Phi = \oint \vec{B}\cdot(\vec{v}\times d\vec{l})dt$$

Motional *emf*

Flux rule for arbitrary loop moving in a non-uniform magnetic field (contd.)

$$\implies \frac{d\Phi}{dt} = \oint (\vec{B} \times \vec{v}) \cdot d\vec{l} = -\oint (\vec{v} \times \vec{B}) \cdot d\vec{l} \tag{4}$$

The quantity $\vec{v} \times \vec{B}$ is the magnetic force per unit charge. Therefore,

$$\frac{d\Phi}{dt} = -\oint \vec{f}_{mag} \cdot d\vec{l} \tag{5}$$

The quantity on right hand side is called induced *emf*.

Therefore, from equation (4) and (5)

$$\boxed{\mathscr{E} = -\frac{d\Phi}{dt}.}$$

Faraday's laws of electromagnetic induction

The phenomenon of generation of *emf* in a conducting loop due to the relative motion between a magnet and the loop is called electromagnetic induction and the *emf* thus produced is called induced *emf*. Faraday's laws of electromagnetic induction are as follows:

1. The change in magnetic flux produces an induced *emf* in the circuit,
2. The induced *emf* lasts as long as change in flux continues, and
3. The induced *emf* is directly proportional to the rate of change of magnetic flux i.e.

$$\mathscr{E} = -\frac{d\Phi}{dt}.$$

Faraday's laws of electromagnetic induction (contd.)

Suppose a surface S bounded by curve C in a time varying magnetic field $\vec{B}$. Then the magnetic flux through the surface is given by

$$\Phi = \int_S \vec{B} \cdot d\vec{a}$$

Suppose $\vec{E}$ be the electric field produced by induced *emf* then

$$\mathscr{E} = \oint_C \vec{E} \cdot d\vec{l}, \qquad \therefore \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

Applying the Stoke's theorem in the left hand side

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{a} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a}$$

$$\implies \int_S (\nabla \times \vec{E}) \cdot d\vec{a} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

Faraday's laws of electromagnetic induction (contd.)

$$\implies \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6}$$

This is differential form of Faraday's law. Also, $\vec{B} = \nabla \times \vec{A}$, so that

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) = \nabla \times \left(-\frac{\partial \vec{A}}{\partial t}\right)$$

$$\implies \vec{E} = -\frac{\partial \vec{A}}{\partial t} \tag{7}$$

This gives the electric field due to induced *emf*.

Lenz's law

When a magnet is brought towards a coil, a current is induced in it due to the change in magnetic flux. The induced current itself produces a magnetic field and hence the magnetic flux. The direction of induced current is given by the Lenz's law. It states that the induced current is such that it opposes the change of flux which produced it. That means flux due to induced current always opposes the original flux which produced the current.

Lenz's law is in accordance with the law of conservation of energy according to which energy is always conserved. Here, the induced *emf* is produced at the cost of mechanical energy.

From Faraday's law, $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

Lenz's law (contd.)

and from Ampere's law, $\nabla \times \vec{B} = \mu_0 \vec{J}$.

As $\vec{E}$ is purely Faraday induced electric field, $\nabla \cdot \vec{E} = 0$

and also for magnetic field, $\nabla \cdot \vec{B} = 0$.

Thus, Faraday's induced electric field can be determined by $\left(-\dfrac{\partial \vec{B}}{\partial t}\right)$

exactly in the same way as magnetic field is determined by $\mu_0 \vec{J}$.

Thus from Biot-Savart law, we have

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \vec{\mathscr{r}}}{\mathscr{r}^3} d\tau$$

The analog to Biot-Savart law is

$$\vec{E} = -\frac{1}{4\pi} \int \frac{\left(\frac{\partial \vec{B}}{\partial t}\right) \times \vec{\mathscr{r}}}{\mathscr{r}^3} d\tau$$

Lenz's law (contd.)

$$\Longrightarrow \vec{E} = -\frac{d}{dt}\left\{\frac{1}{4\pi}\int \frac{\vec{B}\times\vec{\mathscr{r}}}{\mathscr{r}^3}d\tau\right\} \tag{8}$$

Again we know that

$$\nabla\cdot\vec{A} = 0,\ \nabla\times\vec{A} = \vec{B}$$

$$\text{and} \qquad \nabla\cdot\vec{B} = 0,\ \nabla\times\vec{B} = \mu_0\vec{J}$$

So $\vec{A}$ depends on $\vec{B}$ exactly in the same way as $\vec{B}$ depends on $\mu_0\vec{J}$.

Since

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\vec{\mathscr{r}}}{\mathscr{r}^3}d\tau$$

it must be the case that

$$\vec{A} = \frac{1}{4\pi}\int \frac{\vec{B}\times\vec{\mathscr{r}}}{\mathscr{r}^3}d\tau$$

Thus, eq (8) can be written as

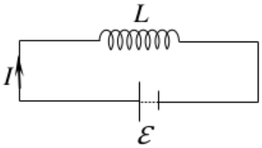
$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

This is the expression for electric field due to Faraday induced *emf*.

The result can be checked by applying the curl i.e.

$$\nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) = -\frac{\partial \vec{B}}{\partial t}.$$

Self induction



Suppose a current I is flowing through a coil thereby producing a magnetic field. If the current through the coil changes, the magnetic flux linked with coil also changes and this changing flux produces an *emf* called induced *emf* or back *emf* in the coil. The direction of current due to induced emf is opposite to the direction of original current. The phenomenon by which an oppositely induced *emf* is produced is called self induction.

Self induction (contd.)

It is found that magnetic flux linked with coil is proportional to the current flowing through it. Let Φ denote the flux then

$$\Phi \propto I \Rightarrow \Phi = LI$$

Here, L is a constant called the coefficient of self induction of the coil or self inductance.

Also, from Faraday's law, the induced *emf* $\mathscr{E}$ is

$$\mathscr{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt} \qquad \therefore L = \frac{\mathscr{E}}{\left(-\frac{dI}{dt}\right)}.$$

Thus, coefficient of self induction (self inductance) is defined as the ratio of induced *emf* to time rate of change of current through the coil. Its unit is Henry (H)

Problems

1. **A battery of constant emf $\mathscr{E}_0$ is connected to a circuit of resistance R and inductance L as shown in figure 3. What is the current flowing in the circuit at any time t?**

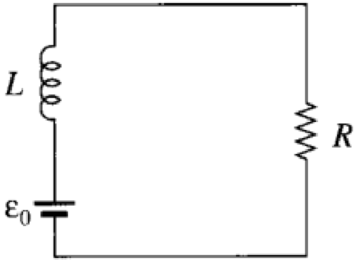


Figure 3

Hint:

The total *emf* (i.e. the *emf* of battery plus the induced *emf*) on the circuit is equal voltage dropped on the resistance, i.e.

$$\begin{aligned}
\mathscr{E}_0 - L\frac{dI}{dt} &= IR \\
\implies L\frac{dI}{dt} &= \mathscr{E}_0 - IR \\
\implies \frac{dI}{dt} &= \frac{R}{L}\left(\frac{\mathscr{E}_0}{R} - I\right) \\
\implies \frac{dI}{\left(\frac{\mathscr{E}_0}{R} - I\right)} &= \frac{R}{L}dt
\end{aligned}$$

Problems (contd.)

$$\implies -\frac{d\left(\frac{\mathscr{E}_0}{R} - I\right)}{\left(\frac{\mathscr{E}_0}{R} - I\right)} = \frac{R}{L}dt$$

upon integration

$$\ln\left(\frac{\mathscr{E}_0}{R} - I\right) = -\frac{R}{L}t + C_0 \qquad (9)$$

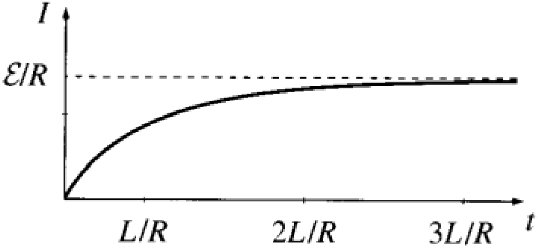
C_0 is the constant of integration and can be determined using initial condition. Just before the switch is on i.e. at $t = 0$ there is no current $I = 0$, which gives $C_0 = \ln\left(\frac{\mathscr{E}_0}{R}\right)$. Therefore, equation (9) becomes

$$\ln\left(\frac{\mathscr{E}_0}{R} - I\right) = -\frac{R}{L}t + \ln\left(\frac{\mathscr{E}_0}{R}\right)$$

Problems (contd.)

$$\implies \frac{\mathscr{E}_0}{R} - I = \frac{\mathscr{E}_0}{R} e^{-(R/L)t}$$

$$\therefore I(t) = \frac{\mathscr{E}_0}{R}\left(1 - e^{-(R/L)t}\right) \tag{10}$$



2. **Find the self-inductance per unit length of a long solenoid, of radius R, carrying n turns per unit length.**

   Hint:

   Consider a portion of length l on the solenoid. The number of turns on the portion is $N = nl$.

   The magnetic field inside the solenoid is $B = \mu_0 nI$ parallel to the axis and constant.

   The magnetic flux linked to the portion

$$\Phi = (\mu_0 nI)(nl)(\pi R^2) = \left(\mu_0 \pi n^2 R^2 l\right) I$$

Problems (contd.)

Comparing this with $\Phi = LI$, the self inductance is

$$L = \mu_0 \pi n^2 R^2 l$$

and the self inductance per unit length

$$\boxed{L/l = \mu_0 \pi n^2 R^2}$$

End of Lecture 14

Thank you