

Entanglement MidSem



Exercise 4.3. Calculate the following inner products:

- (a) $\langle 10 | 11 \rangle$.
- (b) $\langle + - | 01 \rangle$.
- (c) $\langle 1 + 0 | 1 - 0 \rangle$.

- (a) **0**
- (b) **-1/2**
- (c) **0**



solve

(a) $\langle 10 | 11 \rangle$

$$\begin{aligned}\langle 10 | 11 \rangle &= \langle 1 | 1 \rangle \langle 0 | 1 \rangle \\ &= (1)(0) = 0\end{aligned}$$

Answer: 0

(b) $\langle + - | 01 \rangle$

Recall:

$$\begin{aligned}|+\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ \langle + - | 01 \rangle &= \langle + | 0 \rangle \langle - | 1 \rangle \\ \langle + | 0 \rangle &= \frac{1}{\sqrt{2}}, \quad \langle - | 1 \rangle = -\frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \cdot \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\end{aligned}$$

Answer: -1/2

(c) $\langle 1+0 \mid 1-0 \rangle$

Interpret as tensor products:

$$= \langle 1|1 \rangle \langle +|- \rangle \langle 0|0 \rangle$$

We know:

$$\begin{aligned} \langle 1|1 \rangle &= 1, & \langle 0|0 \rangle &= 1, & \langle +|- \rangle &= 0 \\ &= (1)(0)(1) &= 0 \end{aligned}$$

Answer: 0



Exercise 4.4. Verify that

$$|1\rangle \otimes |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We verify step by step.

Step 1: Write basis vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Step 2: Compute $|1\rangle \otimes |1\rangle$

$$|1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 \\ 0 \cdot 1 \\ 1 \cdot 0 \\ 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

This is the 4-dimensional vector for $|11\rangle$.

Step 3: Tensor with $|0\rangle$

$$|11\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now expand:

$$= \begin{pmatrix} 0 \cdot 1 \\ 0 \cdot 0 \\ 0 \cdot 1 \\ 0 \cdot 0 \\ 0 \cdot 1 \\ 0 \cdot 0 \\ 1 \cdot 1 \\ 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Final Answer

$$|1\rangle \otimes |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

✓ Verified.



Exercise 4.5. Consider a two-qubit state

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{i}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}+i}{4}|11\rangle.$$

- (a) What is $|\psi\rangle$ as a (column) vector?
- (a) What is $\langle\psi|$ as a (row) vector?

Given

$$|\psi\rangle = \frac{1}{2}|00\rangle + \frac{i}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}+i}{4}|11\rangle$$

Step 1: Standard two-qubit basis order

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis order:

$$(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$$

(a) Column vector $|\psi\rangle$

There is **no $|01\rangle$ term**, so its coefficient is 0.

$$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{i}{\sqrt{2}} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix}$$

(b) Row vector $\langle\psi|$

Take complex conjugate transpose:

- $\frac{1}{2} \rightarrow \frac{1}{2}$
- $0 \rightarrow 0$
- $\frac{i}{\sqrt{2}} \rightarrow -\frac{i}{\sqrt{2}}$
- $\frac{\sqrt{3}+i}{4} \rightarrow \frac{\sqrt{3}-i}{4}$

$$\langle\psi| = \left(\frac{1}{2} \quad 0 \quad -\frac{i}{\sqrt{2}} \quad \frac{\sqrt{3}-i}{4} \right)$$

Final Answers

$$|\psi\rangle = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{i}{\sqrt{2}} \\ \frac{\sqrt{3}+i}{4} \end{pmatrix}$$

$$\langle\psi| = \left(\frac{1}{2} \quad 0 \quad -\frac{i}{\sqrt{2}} \quad \frac{\sqrt{3}-i}{4} \right)$$



Exercise 4.6. Show that $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is a complete orthonormal basis for the state of two qubits by showing that it satisfies the completeness relation

$$|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| = I,$$

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4 Multiple Quantum Bits

ere I is the 4×4 identity matrix:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We verify the completeness relation:

$$|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| = I$$

Step 1: Write basis vectors (column form)

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Step 2: Compute outer products

$$|00\rangle\langle 00| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|01\rangle\langle 01| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|10\rangle\langle 10| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|11\rangle\langle 11| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3: Add them

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Final Result

$$|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| = I_{4 \times 4}$$

Hence,

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

is a **complete orthonormal basis**. ✓



$|00\rangle\langle 00|$
how

You're asking **how exactly**

$$|00\rangle\langle 00|$$

becomes that matrix.

Let's do it slowly.

Step 1: Write $|00\rangle$ as a column vector

In standard two-qubit basis:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Step 2: Write $\langle 00|$ as a row vector

Take transpose (no conjugate needed here since entries are real):

$$\langle 00| = (1 \quad 0 \quad 0 \quad 0)$$

Step 3: Compute the outer product

Outer product = column \times row

$$|00\rangle\langle 00| = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (1 \quad 0 \quad 0 \quad 0)$$

Now multiply:

Row 1:

$$1 \times (1 \ 0 \ 0 \ 0) = (1 \ 0 \ 0 \ 0)$$

Row 2:

$$0 \times (1 \ 0 \ 0 \ 0) = (0 \ 0 \ 0 \ 0)$$

Row 3:

$$0 \times (1 \ 0 \ 0 \ 0) = (0 \ 0 \ 0 \ 0)$$

Row 4:

$$0 \times (1 \ 0 \ 0 \ 0) = (0 \ 0 \ 0 \ 0)$$

Final Matrix

$$|00\rangle\langle 00| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Intuition

This operator:

- Picks out the $|00\rangle$ component
- Zeros everything else

It acts like a **projector onto** $|00\rangle$.

If you want, I can also show why this projects states geometrically — that usually makes it very clear.



Exercise 4.7. Two qubits are in the state

$$\frac{i}{\sqrt{10}}|00\rangle + \frac{1-2i}{\sqrt{10}}|01\rangle + \frac{e^{i\pi/100}}{\sqrt{10}}|10\rangle + \frac{\sqrt{3}}{\sqrt{10}}|11\rangle.$$

If we measure the qubits in the Z-basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, what are the possible outcomes and with what probabilities?

Given state:

$$|\psi\rangle = \frac{i}{\sqrt{10}}|00\rangle + \frac{1-2i}{\sqrt{10}}|01\rangle + \frac{e^{i\pi/100}}{\sqrt{10}}|10\rangle + \frac{\sqrt{3}}{\sqrt{10}}|11\rangle$$

Measurement in Z-basis

In the computational basis, the probability of each outcome is:

$$P = |\text{amplitude}|^2$$

1 Outcome: $|00\rangle$

$$\left| \frac{i}{\sqrt{10}} \right|^2 = \frac{1}{10}$$

2 Outcome: $|01\rangle$

$$\left| \frac{1-2i}{\sqrt{10}} \right|^2 = \frac{|1-2i|^2}{10} = \frac{1^2 + (-2)^2}{10} = \frac{5}{10} = \frac{1}{2}$$

3 Outcome: $|10\rangle$

$$\left| \frac{e^{i\pi/100}}{\sqrt{10}} \right|^2 = \frac{|e^{i\theta}|^2}{10} = \frac{1}{10}$$

(since complex exponentials have magnitude 1)

4 Outcome: $|11\rangle$

$$\left| \frac{\sqrt{3}}{\sqrt{10}} \right|^2 = \frac{3}{10}$$

Final Probabilities

Outcome	Probability
(00\rangle)	
(01\rangle)	
(10\rangle)	
(11\rangle)	

Check sum:

$$\frac{1}{10} + \frac{1}{2} + \frac{1}{10} + \frac{3}{10} = \frac{1 + 5 + 1 + 3}{10} = 1$$

✓ Correctly normalized.



Exercise 4.8. Normalize the following quantum state:

$$A \left(\frac{1}{2} |00\rangle + i |01\rangle + \sqrt{2} |10\rangle - |11\rangle \right).$$

Given state:

$$A \left(\frac{1}{2} |00\rangle + i |01\rangle + \sqrt{2} |10\rangle - |11\rangle \right)$$

We find A such that the state is normalized.

Step 1: Compute squared magnitudes

Normalization condition:

$$|A|^2 \left(\sum |\text{coefficients}|^2 \right) = 1$$

Compute each:

$$\left| \frac{1}{2} \right|^2 = \frac{1}{4}$$

$$|i|^2 = 1$$

$$|\sqrt{2}|^2 = 2$$

$$|-1|^2 = 1$$

Step 2: Sum

$$\frac{1}{4} + 1 + 2 + 1 = \frac{1}{4} + 4 = \frac{17}{4}$$

Step 3: Solve for A

$$|A|^2 \cdot \frac{17}{4} = 1$$

$$|A|^2 = \frac{4}{17}$$

$$A = \frac{2}{\sqrt{17}}$$

(choosing positive real normalization constant)

Final Normalized State

$$\frac{2}{\sqrt{17}} \left(\frac{1}{2}|00\rangle + i|01\rangle + \sqrt{2}|10\rangle - |11\rangle \right)$$



$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle.$$

If we first measure the left qubit, we get

$|0\rangle$ with probability $\frac{3}{4}$, and the state collapses to $\sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle$,

$|1\rangle$ with probability $\frac{1}{4}$, and the state collapses to $\frac{\sqrt{3}}{2}|10\rangle + \frac{1}{2}|11\rangle$.

Now if we measure the right qubit after this, the possible outcomes for the sequence of measurements are $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. The probability of getting $|00\rangle$ is the probability of first getting $|0\rangle$ for the left qubit, which was $3/4$, times the probability of getting $|0\rangle$ for the right qubit, which is $2/3$ because the state collapsed after the first measurement. Multiplying these, the probability of getting $|00\rangle$ is $(3/4)(2/3) = 2/4 = 1/2$. We can perform this calculation for every possible outcome:

$$\text{Prob}(|00\rangle) = \text{Prob}(\text{first left } |0\rangle) \text{Prob}(\text{then right } |0\rangle) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2},$$

$$\text{Prob}(|01\rangle) = \text{Prob}(\text{first left } |0\rangle) \text{Prob}(\text{then right } |1\rangle) = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4},$$

$$\text{Prob}(|10\rangle) = \text{Prob}(\text{first left } |1\rangle) \text{Prob}(\text{then right } |0\rangle) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16},$$

$$\text{Prob}(|11\rangle) = \text{Prob}(\text{first left } |1\rangle) \text{Prob}(\text{then right } |1\rangle) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}.$$

how prob of right 0 became 2/3

Good question — this comes from the **collapsed (renormalized) state** after measuring the left qubit.

Step 1 After measuring the left qubit and getting $|0\rangle$

The state collapses to:

$$\sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle$$

Notice this state is already normalized.

Step 2 Now measure the right qubit

We only look at the coefficients of:

- $|00\rangle \rightarrow$ right qubit = 0
 - $|01\rangle \rightarrow$ right qubit = 1
-

Probability that right qubit = 0

Take magnitude squared of the coefficient of $|00\rangle$:

$$\left| \sqrt{\frac{2}{3}} \right|^2 = \frac{2}{3}$$

Probability that right qubit = 1

$$\left| \frac{1}{\sqrt{3}} \right|^2 = \frac{1}{3}$$

Why not use original coefficients?

Because **measurement changes the state**.

Originally:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle$$

But after measuring the left qubit as 0, we:

1. Throw away all terms with left = 1
2. Renormalize the remaining part

That renormalization is exactly what converts the probabilities into $2/3$ and $1/3$.

Intuition

After learning “left qubit = 0”, the system is **no longer the original state** — it becomes a smaller 2D system, and probabilities must sum to 1 again.

$$2/3 + 1/3 = 1$$

That's why the probability of right 0 became **2/3**.

If you want, I can show the renormalization calculation explicitly (one-line derivation).



Exercise 4.9. Consider the two-qubit state

$$\frac{1}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle.$$

If you measure only the left qubit, what are the resulting states, and with what probabilities?

Given state:

$$|\psi\rangle = \frac{1}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$$

Step 1 Probability of measuring left qubit = 0

Take all terms with left bit 0:

$$\frac{1}{4}|00\rangle + \frac{1}{2}|01\rangle$$

Probability:

$$\left|\frac{1}{4}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{16} + \frac{1}{4} = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$P(0_L) = \frac{5}{16}$$

Collapsed state (normalize)

Norm factor:

$$\sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}$$

Divide coefficients by this:

$$\frac{1/4}{\sqrt{5}/4} = \frac{1}{\sqrt{5}}$$

$$\frac{1/2}{\sqrt{5}/4} = \frac{2}{\sqrt{5}}$$

So collapsed state:

$$\frac{1}{\sqrt{5}}|00\rangle + \frac{2}{\sqrt{5}}|01\rangle$$

Step 2 Probability of measuring left qubit = 1

Terms with left bit 1:

$$\frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$$

Probability:

$$\left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{\sqrt{3}}{4} \right|^2 = \frac{1}{2} + \frac{3}{16} = \frac{8}{16} + \frac{3}{16} = \frac{11}{16}$$

$$P(1_L) = \frac{11}{16}$$

Collapsed state (normalize)

Norm factor:

$$\sqrt{\frac{11}{16}} = \frac{\sqrt{11}}{4}$$

New coefficients:

$$\frac{1/\sqrt{2}}{\sqrt{11}/4} = \frac{4}{\sqrt{22}}$$

$$\frac{\sqrt{3}/4}{\sqrt{11}/4} = \frac{\sqrt{3}}{\sqrt{11}} = \sqrt{\frac{3}{11}}$$

So collapsed state:

$$\frac{4}{\sqrt{22}}|10\rangle + \sqrt{\frac{3}{11}}|11\rangle$$

Final Answer

Outcome (left qubit)	Probability	Collapsed state
0	5/16	$(\frac{1}{\sqrt{5}})$
1	11/16	$(\frac{4}{\sqrt{22}})$

✓ Done.



Exercise 4.10. Consider the three-qubit state

$$\frac{1}{6}|000\rangle + \frac{1}{3\sqrt{2}}|001\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{1}{2}|011\rangle + \frac{1}{6}|100\rangle + \frac{1}{3}|101\rangle + \frac{1}{6}|110\rangle + \frac{1}{\sqrt{3}}|111\rangle.$$

4.3 Entanglement

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If you measure only the left and right qubits, but not the middle qubit, what are the resulting states, and with what probabilities?

Given the 3-qubit state:

$$\begin{aligned} |\psi\rangle = & \frac{1}{6}|000\rangle + \frac{1}{3\sqrt{2}}|001\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{1}{2}|011\rangle \\ & + \frac{1}{6}|100\rangle + \frac{1}{3}|101\rangle + \frac{1}{6}|110\rangle + \frac{1}{\sqrt{3}}|111\rangle \end{aligned}$$

We measure **left and right qubits only** (first and third qubits).

So outcomes are:

00, 01, 10, 11

We group states by (left, right).

1 Outcome: 00

States with pattern $|0?0\rangle$:

$|000\rangle, |010\rangle$

Probability:

$$\left|\frac{1}{6}\right|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 = \frac{1}{36} + \frac{1}{6} = \frac{1}{36} + \frac{6}{36} = \frac{7}{36}$$

Collapsed middle-qubit state:

$$\frac{1/6}{\sqrt{7/36}}|0\rangle + \frac{1/\sqrt{6}}{\sqrt{7/36}}|1\rangle = \frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$$

2 Outcome: 01

States:

$$|001\rangle, |011\rangle$$

Probability:

$$\left| \frac{1}{3\sqrt{2}} \right|^2 + \left| \frac{1}{2} \right|^2 = \frac{1}{18} + \frac{1}{4} = \frac{2}{36} + \frac{9}{36} = \frac{11}{36}$$

Collapsed state:

$$\frac{1/\sqrt{2}}{\sqrt{11}}|0\rangle + \frac{3}{\sqrt{11}}|1\rangle$$

3 Outcome: 10

States:

$$|100\rangle, |110\rangle$$

Probability:

$$\left| \frac{1}{6} \right|^2 + \left| \frac{1}{6} \right|^2 = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} = \frac{1}{18}$$

Collapsed state:

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

4 Outcome: 11

States:

$$|101\rangle, |111\rangle$$

Probability:

$$\left| \frac{1}{3} \right|^2 + \left| \frac{1}{\sqrt{3}} \right|^2 = \frac{1}{9} + \frac{1}{3} = \frac{1}{9} + \frac{3}{9} = \frac{4}{9} = \frac{16}{36}$$

Collapsed state:

$$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

✓ Final Answers

Outcome (L,R)	Probability	Middle qubit state
00	$7/36$	$(\frac{1}{\sqrt{7}})$
01	$11/36$	$(\frac{1}{\sqrt{22}})$
10	$1/18$	$(\frac{1}{\sqrt{2}})$
11	$4/9$	$(\frac{1}{2})$

Probabilities check:

$$\frac{7}{36} + \frac{11}{36} + \frac{2}{36} + \frac{16}{36} = 1$$

✓ Done.



Exercise 4.11. Are each of the following states a product state or entangled state? If it is a product state, give the factorization.

(a) $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

(b) $\frac{1}{\sqrt{2}}(|10\rangle + i|11\rangle)$.

We check whether each state can be written as

$$|\psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

If yes → **product state**

If no → **entangled**

(a) $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

This is

$$\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)$$

This is one of the **Bell states**.

Try factorization:

Suppose

$$= (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

Expanding gives:

$$ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

Matching coefficients:

- $ac = 0$
- $bd = 0$
- $ad = 1/\sqrt{2}$
- $bc = 1/\sqrt{2}$

From $ac = 0$ and $bd = 0$, either $a = 0$ or $c = 0$, and either $b = 0$ or $d = 0$.

But if $a = 0$, then $ad = 0$ (contradiction).

If $c = 0$, then $bc = 0$ (contradiction).

So impossible.

✓ **Conclusion:**

Entangled state

(b) $\frac{1}{\sqrt{2}}(|10\rangle + i|11\rangle)$

Factor out $|1\rangle$ (first qubit):

$$= |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

That is clearly separable.

✓ **Conclusion:**

Product state.

Factorization:

$$|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

Final Answers

(a) Entangled

(b) Product state:

$$|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$



Exercise 4.12. Are each of the following states a product state or entangled state? If it is a product state, give the factorization.

- (a) $\frac{1}{4} (3|00\rangle - \sqrt{3}|01\rangle + \sqrt{3}|10\rangle - |11\rangle)$.
 (b) $\frac{1}{\sqrt{3}}|0\rangle|+\rangle + \sqrt{\frac{2}{3}}|1\rangle|-\rangle$.

We test whether each state can be factorized as

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

(a)

$$\frac{1}{4} (3|00\rangle - \sqrt{3}|01\rangle + \sqrt{3}|10\rangle - |11\rangle)$$

Ignore overall factor $1/4$ (doesn't affect separability).

Coefficient matrix:

$$\begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

For a **product state**, determinant must be 0.

$$\det = (3)(-1) - (-\sqrt{3})(\sqrt{3}) = -3 - (-3) = -3 + 3 = 0$$

Determinant = 0 \rightarrow separable.

Now factor rows:

Row2 is $\frac{\sqrt{3}}{3}$ times Row1:

$$\sqrt{3} = 3 \cdot \frac{\sqrt{3}}{3}, \quad -1 = (-\sqrt{3}) \cdot \frac{\sqrt{3}}{3}$$

So factor:

$$(3|0\rangle + \sqrt{3}|1\rangle) \otimes (|0\rangle - \frac{1}{\sqrt{3}}|1\rangle)$$

After including normalization:

$$\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \right)$$

Product state.

(b)

$$\frac{1}{\sqrt{3}}|0\rangle|+\rangle + \sqrt{\frac{2}{3}}|1\rangle|-\rangle$$

This is already in Schmidt form.

Coefficients:

$$\frac{1}{\sqrt{3}}, \quad \sqrt{\frac{2}{3}}$$

Since **both Schmidt coefficients are nonzero**, Schmidt rank = 2.

Therefore state is **entangled**.

Final Answers

(a) **Product state**

$$\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$$

(b) **Entangled state**



Exercise 6.1. Using the techniques from Section 4.3, show that

$$\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle$$

cannot be factored into the tensor product of single-qubit states.

Given state:

$$|\psi\rangle = \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle$$

Step 1 Form coefficient matrix

Write coefficients in matrix form:

$$M = \begin{pmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

For a product state, determinant must be 0.

Step 2 Compute determinant

$$\det(M) = \left(\frac{\sqrt{3}}{2\sqrt{2}} \right) \left(\frac{1}{4} \right) - \left(\frac{\sqrt{3}}{2\sqrt{2}} \right) \left(\frac{\sqrt{3}}{4} \right)$$

First term:

$$= \frac{\sqrt{3}}{8\sqrt{2}}$$

Second term:

$$= \frac{3}{8\sqrt{2}}$$

So

$$\det = \frac{\sqrt{3}}{8\sqrt{2}} - \frac{3}{8\sqrt{2}} = \frac{\sqrt{3} - 3}{8\sqrt{2}}$$

Step 3 Is determinant zero?

Since

$$\sqrt{3} - 3 \neq 0$$

we have

$$\det(M) \neq 0$$

Final Conclusion

State cannot be factored \rightarrow Entangled state

Because determinant $\neq 0$, Schmidt rank = 2.



Exercise 6.2. Consider the following state of two qubits:

$$\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$$

If you measure the left qubit, what outcomes can you get, what are the corresponding probabilities of those outcomes, and what does the state collapse to for each outcome? Is this state a product state, partially entangled state, or maximally entangled state?

Given state:

$$|\psi\rangle = \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$$

1 Measure the left qubit

Probability of outcome 0

Take terms with left = 0:

$$\begin{aligned} & \frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle \\ P(0) &= \left| \frac{\sqrt{3}}{2\sqrt{2}} \right|^2 + \left| \frac{1}{2\sqrt{2}} \right|^2 = \frac{3}{8} + \frac{1}{8} = \frac{1}{2} \end{aligned}$$

Collapsed state (left = 0)

Normalize:

Since probability = 1/2, multiply by $\sqrt{2}$.

$$= \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|01\rangle$$

Factor out $|0\rangle$:

$$= |0\rangle \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \right)$$

2 Probability of outcome 1

Take terms:

$$\begin{aligned} & \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle \\ P(1) &= \frac{1}{8} + \frac{3}{8} = \frac{1}{2} \end{aligned}$$

Collapsed state (left = 1)

Multiply by $\sqrt{2}$:

$$= \frac{1}{2}|10\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

Factor:

$$= |1\rangle \otimes \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \right)$$

3 Entanglement type?

Coefficient matrix:

$$\frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

Determinant:

$$(\sqrt{3})(\sqrt{3}) - (1)(1) = 3 - 1 = 2 \neq 0$$

So state is entangled.

Check Schmidt coefficients.

Eigenvalues give unequal values \rightarrow not maximally entangled.

✓ Final Answer

Probabilities:

$$P(0) = \frac{1}{2}, \quad P(1) = \frac{1}{2}$$

Collapsed states:

$$0 : |0\rangle \otimes \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \right)$$

$$1 : |1\rangle \otimes \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \right)$$

Entanglement type:

Partially entangled state



Exercise 6.3. Consider the following state of two qubits:

$$\frac{1}{\sqrt{2}} (|01\rangle + |10\rangle).$$

If you measure the left qubit, what outcomes can you get, what are the corresponding probabilities of those outcomes, and what does the state collapse to for each outcome? Is this state a product state, partially entangled state, or maximally entangled state?

Given state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

This is a Bell state.

1 Measure the left qubit

Probability of outcome 0

Terms with left = 0:

$$\frac{1}{\sqrt{2}}|01\rangle$$

$$P(0) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$

Collapsed state

State becomes:

$$|01\rangle = |0\rangle \otimes |1\rangle$$

Probability of outcome 1

Terms with left = 1:

$$\frac{1}{\sqrt{2}}|10\rangle$$

$$P(1) = \frac{1}{2}$$

Collapsed state

$$|10\rangle = |1\rangle \otimes |0\rangle$$

2 Type of entanglement?

This state has equal Schmidt coefficients:

$$\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$$

So it is a Bell state.

Maximally entangled state

✓ Final Answer

Outcomes:

- 0 with probability 1/2 → state becomes $|01\rangle$
- 1 with probability 1/2 → state becomes $|10\rangle$

State type:

Maximally entangled



Exercise 6.19. You are Bob, and you and Alice are communicating using a one-time pad. You receive from Alice the ciphertext

```
1101010101001100000110010010110100110110011001100,
```

and you know the secret key

```
0101001011101111000101100000000010101100000101001.
```

What is the plaintext binary string, and what does it encode in ASCII?

In a one-time pad, decryption is done by bitwise XOR:

$$\text{Plaintext} = \text{Ciphertext} \oplus \text{Key}$$

Step 1: XOR the two binary strings

Ciphertext

```
1101010101001100000110010010110100110110011001100
```

Key

```
0101001011101111000101100000000010101100000101001
```

Bitwise XOR gives the plaintext binary:

```
1000011110100011000011110010110110011010011100101
```

Step 2: Group into 7-bit ASCII

```
1000011 1101000 1100001 1110010 1101100 1100101
```

Convert each to decimal:

- 1000011 → 67 → C
- 1101000 → 104 → h

- $1100001 \rightarrow 97 \rightarrow a$
- $1110010 \rightarrow 114 \rightarrow r$
- $1101100 \rightarrow 108 \rightarrow l$
- $1100101 \rightarrow 101 \rightarrow e$

✓ Final Answer

Plaintext binary:

1000011110100011000011110010110110011010011100101

ASCII message:

Charlie



Exercise 6.25. You are Alice, and you and Bob are establishing a secret key using BB84. You have the following random bits and random bases. What qubits do you send to Bob?

Alice's Bits	1 0 0 1 0 0 0 1 1
Alice's Bases	X X Z Z Z X X X X
Alice Sends	? ? ? ? ? ? ? ?

From the table:

Alice's Bits:

1 0 0 1 0 0 0 1 1

Alice's Bases:

X X Z Z Z X X X X

BB84 Encoding Rules

Z-basis

- $0 \rightarrow |0\rangle$
- $1 \rightarrow |1\rangle$

X-basis

- $0 \rightarrow |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$
- $1 \rightarrow |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

Encode each position

Bit	Basis	State Sent
1	X	(
0	X	(
0	Z	(
1	Z	(
0	Z	(
0	X	(
0	X	(
1	X	(
1	X	(

✓ Final Answer

Alice sends:

$$|-\rangle, |+\rangle, |0\rangle, |1\rangle, |0\rangle, |+\rangle, |+\rangle, |-\rangle, |-\rangle$$


Exercise 6.26. You are Bob, and you and Alice are establishing a secret key using BB84. You choose the following random bases to measure each qubit in, and you got the following results.

Bob's Bases	X	X	Z	X	Z	Z	X	X	Z
Bob's Measurement	$ +\rangle$	$ -\rangle$	$ 0\rangle$	$ -\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$ +\rangle$	$ 1\rangle$
Bob's Bits	0	1	0	1	0	1	1	0	1

Next, you call Alice and learn that she used the following bases:

Alice's Bases	Z	Z	Z	Z	Z	X	X	Z	Z
---------------	---	---	---	---	---	---	---	---	---

What is your shared secret key?

Step 1 Given

Bob's Bases:

X X Z X Z Z X X Z

Bob's Bits:

0 1 0 1 0 1 1 0 1

Alice's Bases:

Z Z Z Z Z X X Z Z

Step 2 Keep only positions where bases match

Compare position by position:

Position	Bob	Alice	Keep?	Bob's Bit
1	X	Z	X	–
2	X	Z	X	–
3	Z	Z	✓	0
4	X	Z	X	–
5	Z	Z	✓	0
6	Z	X	X	–
7	X	X	✓	1
8	X	Z	X	–
9	Z	Z	✓	1

Step 3 Extract bits from matching positions

Positions kept: **3, 5, 7, 9**

Bob's bits:

0 0 1 1

✓ Shared Secret Key

0011



Exercise 6.27. Alice and Bob want to catch a possible eavesdropper with a probability of 99%. How many bits of their shared secret key should they reveal?

In BB84, if Eve performs intercept-resend:

- Each revealed bit has a **1/4 probability of showing an error**.
- So probability a checked bit **shows no error** = $3/4$.

If Alice and Bob reveal n bits, probability Eve is **not detected** is:

$$\left(\frac{3}{4}\right)^n$$

We want detection probability $\geq 99\%$, i.e.,

$$1 - \left(\frac{3}{4}\right)^n \geq 0.99$$

So

$$\left(\frac{3}{4}\right)^n \leq 0.01$$

Take logarithm:

$$n \geq \frac{\ln(0.01)}{\ln(3/4)}$$

$$n \geq \frac{-4.605}{-0.2877}$$

$$n \geq 16$$

Final Answer

16 bits

They should reveal **at least 16 bits** to detect Eve with 99% probability.