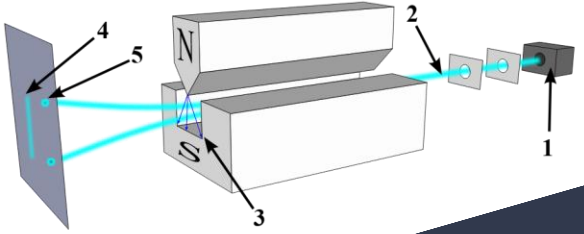
Quantum Spin



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What is Spin?

Applications:

MRI works because:

- Hydrogen nuclei have spin. Body images are created by measuring the magnetic response of hydrogen nuclei aligned within a strong field.

MRAM: More energy-efficient

Quantum Cryptography

Spin is fundamental Quantum property of particle. It behaves mathematically like angular momentum of a particle.

Electrons behave like tiny magnets. When measured, their magnetic orientation can only have two outcomes(if we measure on a vertical basis):

- Spin Up (N)
- Spin Down (S)

Nothing in between.

This property is **quantized**, meaning it does not vary continuously.

Measurements of Spin

Goal: Stop Eavesdropping

Measurement changes it

You cannot measure without disturbing it

$\langle\uparrow| = [1 \quad 0]$ can be also called as **North/Up**

$\langle\downarrow| = [0 \quad 1]$ can be also called as **South/Down**

$\langle\rightarrow| = \frac{1}{\sqrt{2}}\ [1 \quad 1]$ can be also called as **East/Right**

$\langle\leftarrow| = \frac{1}{\sqrt{2}}\ [1 \quad -1]$ can be also called as **West/Left**

Quantum Measurement:

$|\psi\rangle = c_1\,|\uparrow\rangle + c_2\,|\downarrow\rangle$

State collapses to either $|\uparrow\rangle$ or $|\downarrow\rangle$

Probability of $|\uparrow\rangle$ is $|c_1|^2$

Probability of $|\downarrow\rangle$ is $|c_2|^2$

Measuring Spin in Different Directions

Given a spin let's say: $|\psi\rangle = 1\ \langle\uparrow| + 0\ \langle\downarrow|$

Probability of $\langle\uparrow|$ is 1, Probability of $\langle\downarrow|$ is 0

If we measure this vector in the horizontal direction, we will get ?

Given : $|\psi\rangle = 1|\uparrow\rangle + 0|\downarrow\rangle$

means it is UP : $|\psi\rangle = |\uparrow\rangle$

Vertical Basis:
$|\uparrow\rangle = [1\ 0]^T$ and $|\downarrow\rangle = [0\ 1]^T$

Horizontal basis:
$|\rightarrow\rangle = 1/\sqrt{2}\,[1\ \ 1]^T$ and $|\leftarrow\rangle = 1/\sqrt{2}\,[1\ \ -1]^T$

Step1:
Express Up State in Horizontal Basis
Lets say $|\uparrow\rangle = x1\,|\rightarrow\rangle + x2\,|\leftarrow\rangle$

step 2:
Now substitute the vectors.
$[1\ \ 0]^T = x_1\,[1/\sqrt{2}\ \ 1\sqrt{2}]^T + x_2[1/\sqrt{2}\ \ -1/\sqrt{2}]^T$

$[1\ \ 0]^T = 1/\sqrt{2}\,[x_1 + x_2\ \ x_1 - x_2]^T$

Step3 :

Equate Components:

$[1\ \ 0]^T = [(x_1+x_2)/\sqrt{2}\quad (x_1-x_2)/\sqrt{2}]^T$

$x_1 = 1/\sqrt{2},\ \ x_2 = 1/\sqrt{2}$

Final Expression:

$|\uparrow\rangle = 1/\sqrt{2}\,|\rightarrow\rangle + 1/\sqrt{2}\,|\leftarrow\rangle$

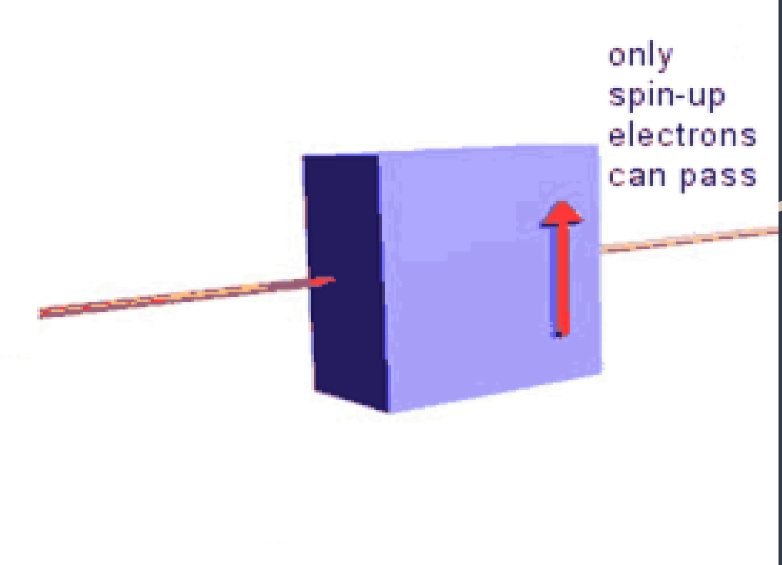
$P(\rightarrow) = 1/2$ and for left also 1/2

Hence,If we measure spin in horizontal direction, starting from Up state:

50% chance of Right

50% chance of Left

Measuring Spin in Same Direction



Given a spin lets say:

|ψ⟩ = 1|↑⟩ + 0|↓⟩

Probability of |↑⟩ is **1** (100%) and for |↓⟩ is **0**

If we measure this same vector again in the same direction, the result will be the same only.

Eg: Vertical → Vertical → Vertical

Result: Always identical (in this case always 1 for |↑⟩)

Three Measurements in a Row:

Vertical → Horizontal → Vertical
We consider the sequence:
Assume the first result is Up.

Sol^n:
Initially $|\psi\rangle = |\uparrow\rangle$

Since if we measures this vertically i.e. same direction and the state collapses to:

$$|\psi\rangle = |\uparrow\rangle$$

Now we measure in horizontal basis.

$$|\uparrow\rangle = 1/\sqrt{2}\ |\rightarrow\rangle + 1/\sqrt{2}\ |\leftarrow\rangle$$

Suppose the second measurement gives(**collapse to a single state**):

$|\psi\rangle = |\leftarrow\rangle$ $\{ P(\rightarrow) = ½ \quad P(\leftarrow) = ½ \}$

If we measure for the third time this vector but now in vertical direction, we will get result as:

$$|\psi\rangle = 1/\sqrt{2}\ |\uparrow\rangle + 1/\sqrt{2}\ |\downarrow\rangle \quad \text{..............................(1)}$$

Thus, we see that the result is not same as the previous state.

First vertical gives **Up**.Second horizontal gives **Left** or **Right(with 50-50%)**. Now, if we measure this vector in vertical direction, we will get up(with 50 -50 probability).

Means total P(up) =50% $\{ |1/\sqrt{2}|^2 = ½$ (50%) from eq. 1$\}$

Measurement Collapse

Before measurement → **superposition**

After measurement → single state

State vector jumps to basis vector

This is called wavefunction collapse.

Question 1

Given quantum state representation of Vector V in horizontal basis, express the vector in vertical bases.

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

$$|V\rangle = \frac{1}{\sqrt{2}}|\rightarrow\rangle + \frac{1}{\sqrt{2}}|\leftarrow\rangle$$

Solution:

Substitute the vertical basis equivalents into the equation:

$$|V\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)\right] + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)\right]$$

$$|V\rangle = \frac{1}{2}|\uparrow\rangle + \frac{1}{2}|\downarrow\rangle + \frac{1}{2}|\uparrow\rangle - \frac{1}{2}|\downarrow\rangle$$

$$|V\rangle = (\frac{1}{2} + \frac{1}{2})|\uparrow\rangle = 1|\uparrow\rangle$$

$|V\rangle = 1\langle\uparrow| + 0\langle\downarrow|$

So ,in vertical basis it is purely Spin Up.

Equivalent State vectors in quantum spin

Two state vectors are equivalent if no measurement can distinguish them.

Consider two states $\langle\uparrow|$ and $-\langle\uparrow|$ & let their measurement basis be $\langle b_1|$ and $\langle b_2|$

$\langle\uparrow| = x_1\langle b_1| + x_2\langle b_2|$

$-\langle\uparrow| =- x_1\langle b_1| - x_2\langle b_2|$

Probability for $\langle b_1| = |x_1|^2$ (in case of $\langle\uparrow|$)

Probability for $\langle b_2| = |x_2|^2$ (in case of $\langle\uparrow|$)

Probability for $\langle b_1| = |-x_1|^2$ (in case of $-\langle\uparrow|$)

Probability for $\langle b_2| = |-x_2|^2$ (in case of $-\langle\uparrow|$)

Thus, we say that $\langle\uparrow|$ and $- \langle\uparrow|$ are equivalent state vectors.

Equivalent State vectors in quantum spin

The concept in equivalent state vectors in quantum spin is that for 2 vectors to be equivalent their probabilities should be same. But this is not the main concept.

Main concept for 2 vectors to be equivalent is that their measurements in horizontal and vertical direction should be the same.

That is:

Say vector $|V1|$ & $|V2|$ should have same horizontal measurement (lets say x_1 and x_2) and same vertical measurement (lets say x_3 and x_4)

Question 2

State 1: $1/\sqrt{2}\ \langle\uparrow| + 1/\sqrt{2}\ \langle\downarrow|$
State 2: $-1/\sqrt{2}\ \langle\uparrow| + 1/\sqrt{2}\ \langle\downarrow|$
State 3: $1/\sqrt{2}\ \langle\uparrow| - 1/\sqrt{2}\ \langle\downarrow|$
State 4: $-1/\sqrt{2}\ \langle\uparrow| - 1/\sqrt{2}\ \langle\downarrow|$

Check if they are equivalent or not?

If we go by the probability logic, all 4 vectors have same probability i.e (50%-50%) for $\langle\uparrow|$ & $\langle\downarrow|$. So all 4 should be equivalent but that's not the case.

Solution:

States 1 & 4 are equivalent (same measurement in both horizontal & vertical direction)

Also States 2 & 3 are equivalent (same measurement in both horizontal & vertical direction)

Question 2

State 1: $1/\sqrt{2}\ \langle\uparrow| + 1/\sqrt{2}\ \langle\downarrow|$
State 2: $-1/\sqrt{2}\ \langle\uparrow| + 1/\sqrt{2}\ \langle\downarrow|$
State 3: $1/\sqrt{2}\ \langle\uparrow| - 1/\sqrt{2}\ \langle\downarrow|$
State 4: $-1/\sqrt{2}\ \langle\uparrow| - 1/\sqrt{2}\ \langle\downarrow|$

Check if they are equivalent or not?

$$\langle\rightarrow| = \frac{1}{\sqrt{2}}\ [1 \quad 1]$$
$$\langle\leftarrow| = \frac{1}{\sqrt{2}}\ [1 \quad -1]$$

Solution(Continue):

Now, let's check for State 1 & 3:

State 1 in horizontal direction:

$1/\sqrt{2}[1 \quad 0] + 1/\sqrt{2}[0 \quad 1] = x_1\,[1/\sqrt{2} \quad 1/\sqrt{2}] + x_2[1/\sqrt{2} \quad -1/\sqrt{2}]$

After solving, we find that $\mathbf{x_1 = 1, x_2 = 0}$

State 3 in horizontal direction:

$1/\sqrt{2}[1 \quad 0] - 1/\sqrt{2}[0 \quad 1] = x_1[1/\sqrt{2} \quad 1/\sqrt{2}] + x_2[1/\sqrt{2} \quad -1/\sqrt{2}]$

After solving, we find that $\mathbf{x_1 = 0, x_2 = 1}$

Question 2

State 1: $1/\sqrt{2}\ \langle\uparrow| + 1/\sqrt{2}\ \langle\downarrow|$
State 2: $-1/\sqrt{2}\ \langle\uparrow| + 1/\sqrt{2}\ \langle\downarrow|$
State 3: $1/\sqrt{2}\ \langle\uparrow| - 1/\sqrt{2}\ \langle\downarrow|$
State 4: $-1/\sqrt{2}\ \langle\uparrow| - 1/\sqrt{2}\ \langle\downarrow|$

Check if they are equivalent or not?

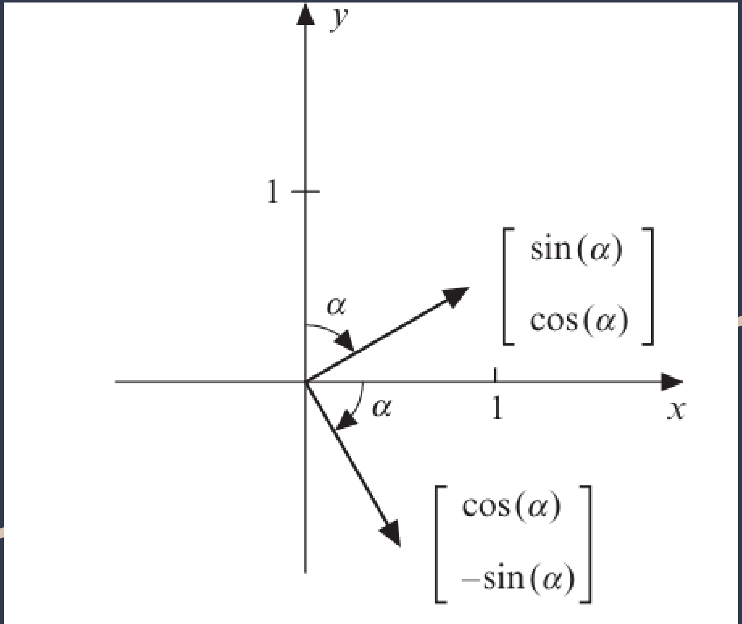
Solution(continue):

Thus, we find that State 1 and State 3 have different measurements in horizontal direction and hence, they are not equivalent.

Similarly, State 2 and State 4 are also not equivalent.

Hence, State 1 and State 4 are equivalent.
State 2 and State 3 are equivalent.

Rotation of Measurement



Angle **α** is the Basis angle - The angle the **state vectors / basis vectors** rotate in Hilbert space (math space)

Angle **θ** is the Apparatus angle - The angle you rotate the measuring device (magnets / detector)

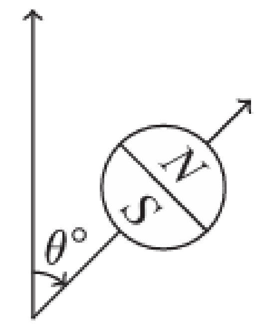
Quantum states can be rotated using a rotation matrix:

R(**α**) = $\left(\begin{bmatrix} \cos(\alpha) \\ -\sin(\alpha) \end{bmatrix}, \begin{bmatrix} \sin(\alpha) \\ \cos(\alpha) \end{bmatrix} \right).$

If **α** = 90°, then R(90) 0 = [0 -1] &

R(90) 1 = [1 0]

Relation b/w apparatus & basis angle



) Measurement angle

The relation between apparatus angle and basis angle is that **θ = 2α**

$$R(\theta) = \left(\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \end{bmatrix}, \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \right)$$

Question 3

What will be the measurement after we rotate the apparatus through 60°

Rotating the Apparatus through 60° **(θ = 60)**

Solution:

We know:

$$R(\theta) = \left(\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \end{bmatrix}, \begin{bmatrix} \sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \right)$$

$$\therefore \left(\begin{bmatrix} \cos(30^\circ) \\ -\sin(30^\circ) \end{bmatrix}, \begin{bmatrix} \sin(30^\circ) \\ \cos(30^\circ) \end{bmatrix} \right)$$

$$\therefore \left(\begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \right).$$

Question 3

What will be the measurement after we rotate the apparatus through 60°

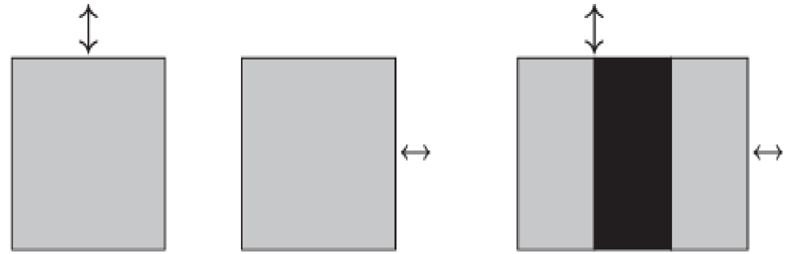
Solution(continue):

If we measure this vector in N direction:

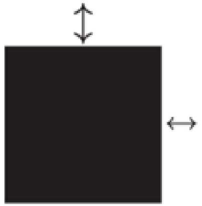
$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix},$$

Probability of N = $|\sqrt{3}/2|^2 = \frac{3}{4}$ (75%)

Photons & Polarization



(a) Two polarized sheets (b) Slightly overlapping



(c) Fully overlapping

Photon polarization behaves exactly like spin

Consider 2 filters:

1st filter is 0°(vertical) = [1 0]

2nd filter is 90°(horizontal) = [0 1]

Projection = $\langle H | V \rangle = 0$

No photon(light) will pass through both the filters

Three Polarizer Experiment

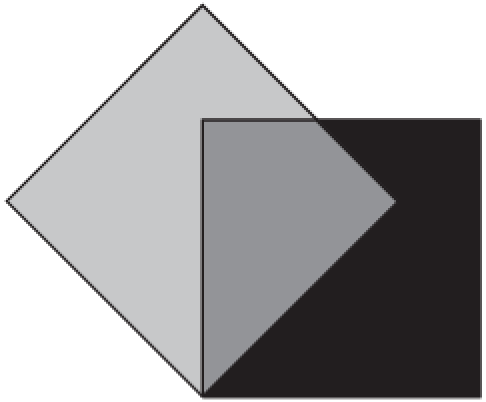


Figure 3.5
Three polarized squares.

Consider 3 filters:

1st filter is at 0°(vertical) = [1 0]

2nd filter is at 45°:

$\langle b_1| = \frac{1}{\sqrt{2}}$ [1 -1] , $\langle b_2| = \frac{1}{\sqrt{2}}$ [1 1]

3rd filter is at 90°(horizontal) = [0 1]

Photons that pass through the first filter will be in state = [1 0]

Probability of passing through 2nd filter is 50% (1/2)

After 2nd filter, state is: $\langle b_1| = \frac{1}{\sqrt{2}}$[1 -1]

Three Polarizer Experiment

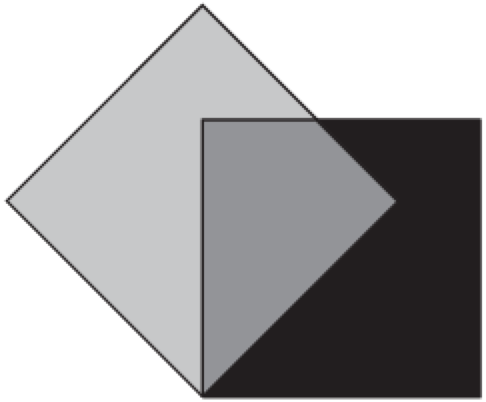


Figure 3.5
Three polarized squares.

Third filter basis is at 90°:

$\langle H| = [0 \;\; 1]$

Projection onto the 3rd filter pass direction:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{-1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\langle H \mid b_1 \rangle = -\frac{1}{\sqrt{2}}$ as

Thus, probability of passing through 3rd filter is 50%.

Overall probability = ½ * ½ * ½ = ⅛

Only 12.5% of photons will pass through the setup

Thank You

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Reference

- Quantum Computing for Everyone (Book by Chris Bernhardt)