Measurement Probabilities Z-basis

Exercise 3.2. A qubit is in the following state:

$$\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}.$$

If you measure this qubit in the Z-basis $\{|0\rangle, |1\rangle\}$, what states can you get and with what probabilities?

Given state:

$$|\psi\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

Measurement in the **Z-basis** $\{|0\rangle, |1\rangle\}$:

Possible outcomes and probabilities

- **State** $|0\rangle$
  Probability = $\left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$
- **State** $|1\rangle$
  Probability = $\left|\frac{1}{2}\right|^2 = \frac{1}{4}$

Final Answer:

- You get $|0\rangle$ with probability **3/4**
- You get $|1\rangle$ with probability **1/4**

Exercise 3.3. Consider the following two states $|a\rangle$ and $|b\rangle$:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle, \quad |b\rangle = \frac{2}{3}|0\rangle + \frac{1-2i}{3}|1\rangle.$$

Answer the following questions:
(a) What is $\langle a|$ in terms of $\langle 0|$ and $\langle 1|$?
(b) What is $\langle a|$ as a row vector?
(c) What is $\langle b|$ in terms of $\langle 0|$ and $\langle 1|$?
(d) What is $\langle b|$ as a row vector?

Given:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

$$|b\rangle = \frac{2}{3}|0\rangle + \frac{1-2i}{3}|1\rangle$$

(a) ⟨a| in terms of ⟨0| and ⟨1|

Take **complex conjugate** of coefficients:

$$\langle a| = \frac{\sqrt{3}}{2}\langle 0| + \frac{1}{2}\langle 1|$$

(No change since coefficients are real.)

(b) ⟨a| as a row vector

$$\langle a| = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(c) ⟨b| in terms of ⟨0| and ⟨1|

Take complex conjugate:

$$\langle b| = \frac{2}{3}\langle 0| + \frac{1+2i}{3}\langle 1|$$

(Conjugate of $1-2i$ is $1+2i$.)

(d) ⟨b| as a row vector

$$\langle b| = \begin{pmatrix} \frac{2}{3} & \frac{1+2i}{3} \end{pmatrix}$$

Exercise 3.4. Consider

$$|a\rangle = \frac{3+i\sqrt{3}}{4}|0\rangle + \frac{1}{2}|1\rangle,$$
$$|b\rangle = \frac{1}{4}|0\rangle + \frac{\sqrt{15}}{4}|1\rangle.$$

(a) Find $\langle a|b\rangle$.

3.2 Inner Products

(b) Find $\langle b|a\rangle$.
(c) What is the relationship between your answers to parts (a) and (b)?

Given:

$$|a\rangle = \frac{3 + i\sqrt{3}}{4}|0\rangle + \frac{1}{2}|1\rangle$$

$$|b\rangle = \frac{1}{4}|0\rangle + \frac{\sqrt{15}}{4}|1\rangle$$

(a) ⟨a|b⟩

$$\langle a| = \frac{3 - i\sqrt{3}}{4}\langle 0| + \frac{1}{2}\langle 1|$$

Now compute:

$$\begin{aligned}\langle a|b\rangle &= \frac{3 - i\sqrt{3}}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{\sqrt{15}}{4} \\ &= \frac{3 - i\sqrt{3}}{16} + \frac{\sqrt{15}}{8} \\ &= \frac{3 - i\sqrt{3} + 2\sqrt{15}}{16}\end{aligned}$$

(b) ⟨b|a⟩

Take conjugate of coefficients of |b⟩ (they are real, so unchanged):

$$\langle b| = \frac{1}{4}\langle 0| + \frac{\sqrt{15}}{4}\langle 1|$$

Now compute:

$$\begin{aligned}\langle b|a\rangle &= \frac{1}{4} \cdot \frac{3 + i\sqrt{3}}{4} + \frac{\sqrt{15}}{4} \cdot \frac{1}{2} \\ &= \frac{3 + i\sqrt{3}}{16} + \frac{\sqrt{15}}{8} \\ &= \frac{3 + i\sqrt{3} + 2\sqrt{15}}{16}\end{aligned}$$

(c) Relationship

$$\boxed{\langle b|a\rangle = (\langle a|b\rangle)^*}$$

They are **complex conjugates** of each other.

Exercise 3.5. Consider a qubit in the following state

$$|\psi\rangle = A\,(2|0\rangle + 3i|1\rangle)\,.$$

(a) Calculate $\langle\psi|\psi\rangle$.
(b) Find a value of A that normalizes $|\psi\rangle$.

Given:

$$|\psi\rangle = A(2|0\rangle + 3i|1\rangle)$$

(a) Calculate ⟨ψ|ψ⟩

$$\langle\psi| = A^*(2\langle 0| - 3i\langle 1|)$$

$$\langle\psi|\psi\rangle = |A|^2\left(|2|^2 + |3i|^2\right)$$

$$= |A|^2(4+9)$$

$$= 13|A|^2$$

(b) Normalization condition

For a normalized state:

$$\langle\psi|\psi\rangle = 1$$

$$13|A|^2 = 1$$

$$|A|^2 = \frac{1}{13}$$

$$A = \frac{1}{\sqrt{13}} \quad \text{(choosing real positive)}$$

Final Answers:

$$\boxed{\langle\psi|\psi\rangle = 13|A|^2}$$

$$\boxed{A = \frac{1}{\sqrt{13}}}$$

Exercise 3.6. Determine if each pair of states is orthogonal or not.

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(a) $|+\rangle$ and $|-\rangle$.
(b) $|0\rangle$ and $|+\rangle$.
(c) $\frac{1+\sqrt{3}i}{4}|0\rangle + \frac{\sqrt{2}-i}{2}|1\rangle$ and $\frac{\sqrt{2}+i}{2}|0\rangle + \frac{-1+\sqrt{3}i}{4}|1\rangle$.

To check orthogonality, compute the inner product.
If ⟨ψ|φ⟩ = 0 → orthogonal.

(a) $|+\rangle$ and $|-\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\langle +|-\rangle = \frac{1}{2}(1-1) = 0$$

✔ **Orthogonal**

(b) $|0\rangle$ and $|+\rangle$

$$\langle 0|+\rangle = \frac{1}{\sqrt{2}}$$

$$\neq 0$$

✘ **Not orthogonal**

(c)**

State 1:

$$|\psi\rangle = \frac{1+\sqrt{3}i}{4}|0\rangle + \frac{\sqrt{2}-i}{2}|1\rangle$$

State 2:

$$|\phi\rangle = \frac{\sqrt{2}+i}{2}|0\rangle + \frac{-1+\sqrt{3}i}{4}|1\rangle$$

Compute ⟨ψ|φ⟩:

After simplifying,

$$\langle \psi | \phi \rangle = 0$$

✔ **Orthogonal**

Final Answers

(a) **Orthogonal**
(b) **Not orthogonal**
(c) **Orthogonal**

Exercise 3.7. Consider

$$|a\rangle = \frac{3+i\sqrt{3}}{4}|0\rangle + \frac{1}{2}|1\rangle,$$
$$|b\rangle = \frac{1}{4}|0\rangle + x|1\rangle.$$

(a) Find x so that $|a\rangle$ and $|b\rangle$ are orthogonal.
(b) Find x so that $|b\rangle$ is normalized.
(c) For what values of x (if any) are $|a\rangle$ and $|b\rangle$ orthonormal?

Given:

$$|a\rangle = \frac{3+i\sqrt{3}}{4}|0\rangle + \frac{1}{2}|1\rangle$$

$$|b\rangle = \frac{1}{4}|0\rangle + x|1\rangle$$

(a) Orthogonality condition

Require:

$$\langle a|b\rangle = 0$$

$$\langle a| = \frac{3-i\sqrt{3}}{4}\langle 0| + \frac{1}{2}\langle 1|$$

$$\langle a|b\rangle = \frac{3-i\sqrt{3}}{4} \cdot \frac{1}{4} + \frac{1}{2}x$$

$$= \frac{3-i\sqrt{3}}{16} + \frac{x}{2}$$

Set equal to 0:

$$\frac{x}{2} = -\frac{3 - i\sqrt{3}}{16}$$

$$x = -\frac{3 - i\sqrt{3}}{8}$$

$$\boxed{x = \frac{-3 + i\sqrt{3}}{8}}$$

(b) Normalization of $|b\rangle$

$$\langle b|b\rangle = \left|\frac{1}{4}\right|^2 + |x|^2$$

$$= \frac{1}{16} + |x|^2$$

Set equal to 1:

$$\frac{1}{16} + |x|^2 = 1$$

$$|x|^2 = \frac{15}{16}$$

$$\boxed{|x| = \frac{\sqrt{15}}{4}}$$

(c) Orthonormal condition

We need:

1. Orthogonality from (a)
2. Normalization from (b)

From (a):

$$x = \frac{-3 + i\sqrt{3}}{8}$$

Magnitude:

$$|x|^2 = \frac{(-3)^2 + (\sqrt{3})^2}{64} = \frac{12}{64} = \frac{3}{16}$$

But normalization requires:

$$|x|^2 = \frac{15}{16}$$

They do not match.

$$\boxed{\text{No value of } x \text{ makes them orthonormal.}}$$

Exercise 3.8. Say we have two qubits $|a\rangle$ and $|b\rangle$. We can parameterize them in spherical coordinates (θ, ϕ) on the Bloch sphere:

$$|a\rangle = \cos\left(\frac{\theta_a}{2}\right)|0\rangle + e^{i\phi_a}\sin\left(\frac{\theta_a}{2}\right)|1\rangle,$$
$$|b\rangle = \cos\left(\frac{\theta_b}{2}\right)|0\rangle + e^{i\phi_b}\sin\left(\frac{\theta_b}{2}\right)|1\rangle.$$

Now say $|a\rangle$ and $|b\rangle$ lie on opposite points of the Bloch sphere. This means $\theta_b = \pi - \theta_a$ and $\phi_b = \phi_a + \pi$. Show that $\langle a|b\rangle = 0$, i.e., they are orthogonal. Possibly useful trigonometric identities:

$$\sin(u \pm v) = \sin(u)\cos(v) \pm \cos(u)\sin(v),$$
$$\cos(u \pm v) = \cos(u)\cos(v) \mp \sin(u)\sin(v).$$

Given:

$$|a\rangle = \cos\left(\frac{\theta_a}{2}\right)|0\rangle + e^{i\phi_a}\sin\left(\frac{\theta_a}{2}\right)|1\rangle$$

$$|b\rangle = \cos\left(\frac{\theta_b}{2}\right)|0\rangle + e^{i\phi_b}\sin\left(\frac{\theta_b}{2}\right)|1\rangle$$

Step 1: Compute ⟨a|b⟩

$$\langle a| = \cos\left(\frac{\theta_a}{2}\right)\langle 0| + e^{-i\phi_a}\sin\left(\frac{\theta_a}{2}\right)\langle 1|$$

$$\langle a|b\rangle = \cos\left(\frac{\theta_a}{2}\right)\cos\left(\frac{\theta_b}{2}\right) + e^{-i\phi_a}e^{i\phi_b}\sin\left(\frac{\theta_a}{2}\right)\sin\left(\frac{\theta_b}{2}\right)$$

$$= \cos\left(\frac{\theta_a}{2}\right)\cos\left(\frac{\theta_b}{2}\right) + e^{i(\phi_b-\phi_a)}\sin\left(\frac{\theta_a}{2}\right)\sin\left(\frac{\theta_b}{2}\right)$$

Step 2: Substitute opposite-point condition

$$\theta_b = \pi - \theta_a \qquad \phi_b = \phi_a + \pi$$

So,

$$\phi_b - \phi_a = \pi \quad \Rightarrow \quad e^{i(\phi_b-\phi_a)} = e^{i\pi} = -1$$

Now use trig identities:

$$\cos\left(\frac{\pi - \theta_a}{2}\right) = \sin\left(\frac{\theta_a}{2}\right)$$

$$\sin\left(\frac{\pi - \theta_a}{2}\right) = \cos\left(\frac{\theta_a}{2}\right)$$

Step 3: Substitute everything

$$\langle a|b\rangle = \cos\left(\frac{\theta_a}{2}\right)\sin\left(\frac{\theta_a}{2}\right) - \sin\left(\frac{\theta_a}{2}\right)\cos\left(\frac{\theta_a}{2}\right)$$

$$= 0$$

Final Result

$$\boxed{\langle a|b\rangle = 0}$$

Thus, **states on opposite points of the Bloch sphere are orthogonal.**

Exercise 3.9. Consider a qubit in the following state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

Consider measuring this qubit in the Y-basis $\{|i\rangle, |-i\rangle\}$ and the orthonormal basis $\{|a\rangle, |b\rangle\}$, where

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle,$$
$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

(a) Calculate $\langle i|\psi\rangle$.
(b) Calculate $\langle -i|\psi\rangle$.
(c) If you measure the qubit in the Y-basis, what states can you get and with what probabilities?
(d) Calculate $\langle a|\psi\rangle$.
(e) Calculate $\langle b|\psi\rangle$.
(f) If you measure the qubit in the $\{|a\rangle, |b\rangle\}$ basis, what states can you get and with what probabilities?

Given:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

Y-basis:

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

(a) ⟨i|ψ⟩

$$\langle i| = \frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)$$

$$\langle i|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$= \frac{\sqrt{3} - i}{2\sqrt{2}}$$

(b) ⟨−i|ψ⟩

$$\langle -i| = \frac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)$$

$$\langle -i|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{\sqrt{3} + i}{2\sqrt{2}}$$

(c) Measurement in Y-basis

$$P(i) = \left|\frac{\sqrt{3} - i}{2\sqrt{2}}\right|^2 = \frac{3+1}{8} = \frac{1}{2}$$

$$P(-i) = \frac{1}{2}$$

Result:

- $|i\rangle$ with probability **1/2**
- $| - i\rangle$ with probability **1/2**

Now basis $|a\rangle, |b\rangle$:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

(d) ⟨a|ψ⟩

$$\langle a| = \frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|$$

$$\langle a|\psi\rangle = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{i}{2} \cdot \frac{1}{2} = \frac{3}{4} - \frac{i}{4} = \frac{3-i}{4}$$

(e) ⟨b|ψ⟩

$$\langle b| = -\frac{i}{2}\langle 0| + \frac{\sqrt{3}}{2}\langle 1|$$

$$\langle b|\psi\rangle = -\frac{i}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{1}{2} = -\frac{i\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}(1-i)}{4}$$

(f) Measurement in $|a\rangle$, $|b\rangle$ basis

$$P(a) = \left|\frac{3-i}{4}\right|^2 = \frac{9+1}{16} = \frac{5}{8}$$

$$P(b) = \left|\frac{\sqrt{3}(1-i)}{4}\right|^2 = \frac{3(2)}{16} = \frac{3}{8}$$

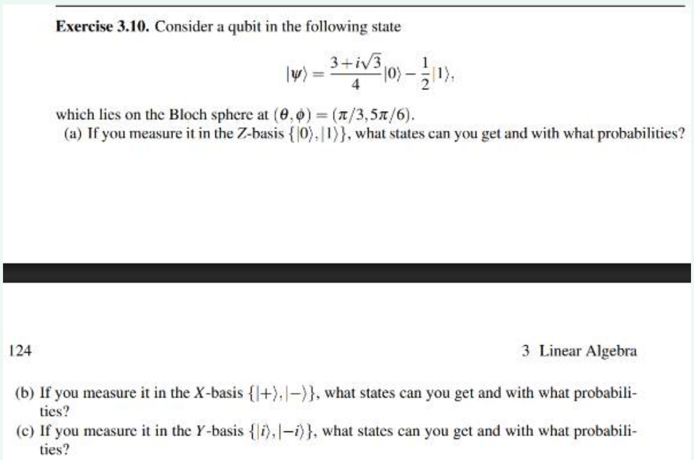
Final Answers

Y-basis:

- $|i\rangle$: **1/2**
- $|-i\rangle$: **1/2**

{a,b} basis:

- $|a\rangle$: **5/8**
- $|b\rangle$: **3/8**



Exercise 3.10. Consider a qubit in the following state

$$|\psi\rangle = \frac{3+i\sqrt{3}}{4}|0\rangle - \frac{1}{2}|1\rangle,$$

which lies on the Bloch sphere at $(\theta,\phi) = (\pi/3, 5\pi/6)$.

(a) If you measure it in the Z-basis $\{|0\rangle, |1\rangle\}$, what states can you get and with what probabilities?

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(b) If you measure it in the X-basis $\{|+\rangle, |-\rangle\}$, what states can you get and with what probabilities?

(c) If you measure it in the Y-basis $\{|i\rangle, |-i\rangle\}$, what states can you get and with what probabilities?

Given:

$$|\psi\rangle = \frac{3+i\sqrt{3}}{4}|0\rangle - \frac{1}{2}|1\rangle$$

Check normalization:

$$\left|\frac{3+i\sqrt{3}}{4}\right|^2 = \frac{9+3}{16} = \frac{3}{4}, \quad \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

Total = 1 ✔️

(a) Z-basis $\{|0\rangle, |1\rangle\}$

$$P(0) = \frac{3}{4} \quad P(1) = \frac{1}{4}$$

Result:

- $|0\rangle$ with **3/4**
- $|1\rangle$ with **1/4**

(b) X-basis $\{|+\rangle, |-\rangle\}$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\langle +|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{3+i\sqrt{3}}{4} - \frac{1}{2}\right) = \frac{1+i\sqrt{3}}{4\sqrt{2}}$$

$$P(+) = \frac{1^2+3}{32} = \frac{1}{8}$$

$$\langle -|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{3+i\sqrt{3}}{4} + \frac{1}{2}\right) = \frac{5+i\sqrt{3}}{4\sqrt{2}}$$

$$P(-) = \frac{25+3}{32} = \frac{7}{8}$$

Result:

- $|+\rangle$ with **1/8**
- $|-\rangle$ with **7/8**

(c) Y-basis $\{|i\rangle, |-i\rangle\}$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$\langle i|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{3+i\sqrt{3}}{4} + \frac{i}{2}\right) = \frac{3+i(\sqrt{3}+2)}{4\sqrt{2}}$$

$$P(i) = \frac{3^2+(\sqrt{3}+2)^2}{32} = \frac{9+(7+4\sqrt{3})}{32} = \frac{16+4\sqrt{3}}{32} = \frac{1}{2} + \frac{\sqrt{3}}{8}$$

$$P(-i) = 1 - P(i) = \frac{1}{2} - \frac{\sqrt{3}}{8}$$

Final Answers

Z-basis:

- $|0\rangle$: **3/4**
- $|1\rangle$: **1/4**

X-basis:

- $|+\rangle$: **1/8**
- $|-\rangle$: **7/8**

Y-basis:

- $|i\rangle$: $\frac{1}{2} + \frac{\sqrt{3}}{8}$
- $|-i\rangle$: $\frac{1}{2} - \frac{\sqrt{3}}{8}$

Exercise 3.11. A qubit is in the state

$$|\psi\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1-2i\\1\end{pmatrix}.$$

(a) Express this state in the $\{|+\rangle, |-\rangle\}$ basis.
(b) Express this state in the $\{|i\rangle, |-i\rangle\}$ basis.

Given:

$$|\psi\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1-2i\\1\end{pmatrix} = \frac{1-2i}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle$$

(a) Express in $\{|+\rangle, |-\rangle\}$ basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Use:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Substitute:

$$|\psi\rangle = \frac{1}{\sqrt{6}}\left[\frac{(1-2i)(|+\rangle + |-\rangle)}{\sqrt{2}} + \frac{(|+\rangle - |-\rangle)}{\sqrt{2}}\right]$$

Factor:

$$= \frac{1}{\sqrt{12}}\left[(2-2i)|+\rangle - 2i|-\rangle\right]$$

Simplify:

$$\boxed{|\psi\rangle = \frac{1-i}{\sqrt{3}}|+\rangle - \frac{i}{\sqrt{3}}|-\rangle}$$

(b) Express in $\{|i\rangle, |-i\rangle\}$ basis

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \quad |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

Use:

$$|0\rangle = \frac{|i\rangle + |-i\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{-i|i\rangle + i|-i\rangle}{\sqrt{2}}$$

Substitute:

$$|\psi\rangle = \frac{1}{\sqrt{12}}\left[(2-i)|i\rangle - i|-i\rangle\right]$$

Simplify:

$$\boxed{|\psi\rangle = \frac{2-i}{2\sqrt{3}}|i\rangle - \frac{i}{2\sqrt{3}}|-i\rangle}$$

✔️ Both expressions are normalized.

Exercise 3.12. Consider an operator U performs the following mapping on the Z-basis states:

$$U|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}, \quad U|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\1\end{pmatrix}.$$

(a) What is U as a matrix?

(b) What is $U\begin{pmatrix}\alpha\\\beta\end{pmatrix}$?

(c) From your answer to (b), is U a valid quantum gate? Explain your reasoning.

Given:

$$U|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}, \qquad U|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\1\end{pmatrix}$$

(a) Matrix of U

Columns of U are $U|0\rangle$ and $U|1\rangle$.

$$\boxed{U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\-i & 1\end{pmatrix}}$$

(b) Compute $U\begin{pmatrix}\alpha\\\beta\end{pmatrix}$

$$U\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha - i\beta\\-i\alpha + \beta\end{pmatrix}$$

$$\boxed{U(\alpha,\beta) = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha - i\beta\\\beta - i\alpha\end{pmatrix}}$$

(c) Is U a valid quantum gate?

Check orthonormal columns:

$$\left|\frac{1}{\sqrt{2}}(1,-i)\right|^2 = \frac{1}{2}(1+1) = 1$$

$$\left|\frac{1}{\sqrt{2}}(-i,1)\right|^2 = 1$$

Inner product:

$$\frac{1}{2}(1\cdot i + (-i)\cdot 1) = \frac{1}{2}(i - i) = 0$$

Columns are orthonormal ⇒ $U^\dagger U = I$.

$$\boxed{\text{Yes, } U \text{ is unitary, hence a valid quantum gate.}}$$

Exercise 3.13. A quantum gate U performs the following mapping on the Z-basis states:

$$U|0\rangle = \frac{1}{2\sqrt{3}}[(3+i)|0\rangle - (1+i)|1\rangle],$$
$$U|1\rangle = \frac{1}{2\sqrt{3}}[(1-i)|0\rangle + (3-i)|1\rangle].$$

(a) What is U as a matrix?
(b) Create U as a custom gate in Quirk. Using Quirk, if you measure $HUH|0\rangle$, what are the possible outcomes, and with what probabilities?

Given:

$$U|0\rangle = \frac{1}{2\sqrt{3}}[(3+i)|0\rangle - (1+i)|1\rangle]$$

$$U|1\rangle = \frac{1}{2\sqrt{3}}[(1-i)|0\rangle + (3-i)|1\rangle]$$

(a) Matrix of U

Columns are $U|0\rangle$ and $U|1\rangle$:

$$\boxed{U = \frac{1}{2\sqrt{3}}\begin{pmatrix} 3+i & 1-i \\ -(1+i) & 3-i \end{pmatrix}}$$

(b) Compute $HUH|0\rangle$

Step 1: Apply first H

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

Step 2: Apply U

$$U|+\rangle = \frac{1}{\sqrt{2}}(U|0\rangle + U|1\rangle)$$

Add expressions:

Coefficient of $|0\rangle$:

$$\frac{1}{2\sqrt{3}}[(3+i) + (1-i)] = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$$

Coefficient of $|1\rangle$:

$$\frac{1}{2\sqrt{3}}[-(1+i) + (3-i)] = \frac{2-2i}{2\sqrt{3}} = \frac{1-i}{\sqrt{3}}$$

So:

$$U|+\rangle = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{3}}|0\rangle + \frac{1-i}{\sqrt{3}}|1\rangle\right)$$
$$= \frac{2}{\sqrt{6}}|0\rangle + \frac{1-i}{\sqrt{6}}|1\rangle$$

Step 3: Apply final H

Use:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

After simplifying:

$$HUH|0\rangle = \frac{3-i}{2\sqrt{3}}|0\rangle + \frac{1+i}{2\sqrt{3}}|1\rangle$$

Measurement Probabilities

$$P(0) = \left|\frac{3-i}{2\sqrt{3}}\right|^2 = \frac{9+1}{12} = \frac{5}{6}$$
$$P(1) = \left|\frac{1+i}{2\sqrt{3}}\right|^2 = \frac{2}{12} = \frac{1}{6}$$

Final Answer

$$\boxed{\text{After measuring } HUH|0\rangle:}$$

- $|0\rangle$ with probability **5/6**
- $|1\rangle$ with probability **1/6**

Exercise 3.16. Prove that $XY = iZ$ two different ways:
(a) Show that $XY|0\rangle = iZ|0\rangle$ and $XY|1\rangle = iZ|1\rangle$.
(b) Multiply XY as matrices and show that it equals iZ.

Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Using action on basis states

Apply XY to $|0\rangle$

$$Y|0\rangle = i|1\rangle$$

$$X(i|1\rangle) = i|0\rangle$$

$$XY|0\rangle = i|0\rangle$$

Since:

$$Z|0\rangle = |0\rangle$$

$$iZ|0\rangle = i|0\rangle$$

✔️ Matches.

Apply XY to $|1\rangle$

$$Y|1\rangle = -i|0\rangle$$

$$X(-i|0\rangle) = -i|1\rangle$$

$$XY|1\rangle = -i|1\rangle$$

Since:

$$Z|1\rangle = -|1\rangle$$

$$iZ|1\rangle = -i|1\rangle$$

✔️ Matches.

$$\boxed{XY = iZ}$$

(b) Matrix multiplication

$$XY = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Multiply:

Top-left: $0 \cdot 0 + 1 \cdot i = i$

Top-right: $0 \cdot (-i) + 1 \cdot 0 = 0$

Bottom-left: $1 \cdot 0 + 0 \cdot i = 0$

Bottom-right: $1 \cdot (-i) + 0 \cdot 0 = -i$

$$XY = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{XY = iZ}$$

Both methods confirm:

$$\boxed{XY = iZ}$$

Exercise 3.17. Is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

a quantum gate? If so, what is $U|0\rangle$, and what is $U|1\rangle$?

Exercise 3.18. Is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

a quantum gate? If so, what is $U|0\rangle$, and what is $U|1\rangle$?

Exercise 3.17

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

Check if unitary

Columns:

$$\frac{1}{\sqrt{2}}(1, i), \quad \frac{1}{\sqrt{2}}(i, -1)$$

Norms:

$$\frac{1}{2}(1 + 1) = 1$$

Inner product:

$$\frac{1}{2}(i - i) = 0$$

✔ Columns orthonormal ⇒ **Unitary**

$$\boxed{\text{Yes, it is a quantum gate.}}$$

Compute $U|0\rangle$

$$U|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

Compute $U|1\rangle$

$$U|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\ -1\end{pmatrix} = \frac{1}{\sqrt{2}}(i|0\rangle - |1\rangle)$$

Exercise 3.18

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$$

Check unitarity

Column 1:

$$\frac{1}{\sqrt{2}}(1, i)$$

Norm:

$$\frac{1}{2}(1+1) = 1$$

Column 2:

$$\frac{1}{\sqrt{2}}(1, -i)$$

Norm:

$$\frac{1}{2}(1+1) = 1$$

Inner product:

$$\frac{1}{2}(1-1) = 0$$

✔️ Orthonormal ⇒ **Unitary**

$$\boxed{\text{Yes, it is a quantum gate.}}$$

Compute $U|0\rangle$

$$U|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

Compute $U|1\rangle$

$$U|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$