Qubit and Quantum Computing

1. Qubit: Definition and Mathematical Representation

Definition:
A qubit is the fundamental unit of quantum information.

Mathematical Representation:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1$$

Measurement Probabilities:

$$P(0) = |\alpha|^2, \quad P(1) = |\beta|^2$$

Measurement collapses the state to either $|0\rangle$ or $|1\rangle$.

2. Qubit vs Electron Spin

Electron spin behaves like a qubit:

- Spin-up $\uparrow \equiv |0\rangle$
- Spin-down $\downarrow \equiv |1\rangle$
- Exists in superposition before measurement
- Collapses to one state after measurement

3. Why Qubits Are Better Than Classical Bits

Classical bit: 0 or 1 only

Qubit:

$$\alpha|0\rangle + \beta|1\rangle$$

Key Properties of Quantum Computing:

- Superposition
- Entanglement
- Interference
- Exponential state space
- Quantum parallelism
- Potential speedup for certain problems

4. Superposition

A qubit can exist in multiple states simultaneously.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Example equal superposition:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$P(0) = P(1) = \frac{1}{2}$$

Superposition enables quantum parallel computation.

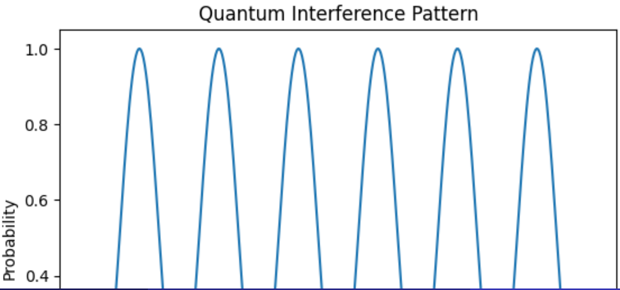
Entanglement Example (Bell State):

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Measurement of one qubit determines the other instantly.

Quantum Interference:

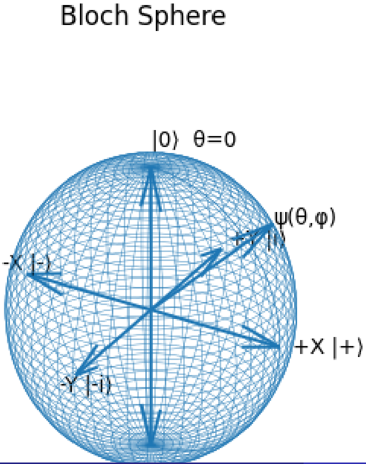
$$|a+b|^2 = |a|^2 + |b|^2 + 2\mathrm{Re}(ab^*)$$



6. Geometrical Representation: Bloch Sphere

Any single qubit corresponds to a point on a unit sphere.

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$



7. Global vs Relative Phase

Global Phase:

$$e^{i\gamma}|\psi\rangle$$

No physical effect on measurement.

Relative Phase:

$$|\psi\rangle = \alpha|0\rangle + e^{i\phi}\beta|1\rangle$$

Affects interference outcomes.

Bloch Sphere Form Derivation:
Using normalization and removing global phase:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

8. Basis States on Bloch Sphere

Z-basis (Computational):

$$|0\rangle \rightarrow (0, 0, +1)$$

$$|1\rangle \rightarrow (0, 0, -1)$$

X-basis:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad (\theta = \pi/2,\ \phi = 0)$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad (\theta = \pi/2,\ \phi = \pi)$$

Y-basis:

$$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \quad (\theta = \pi/2,\ \phi = \pi/2)$$

$$|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}} \quad (\theta = \pi/2,\ \phi = 3\pi/2)$$

Global Phase and Relative Phase

Global Phase

If a state is:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

Multiplying whole state by phase:

$$|\psi'\rangle = e^{i\theta} |\psi\rangle$$

$$|\psi'\rangle = e^{i\theta}(\alpha |0\rangle + \beta |1\rangle) = e^{i\theta}\alpha |0\rangle + e^{i\theta}\beta |1\rangle$$

Now check probabilities:

$$|e^{i\theta}\alpha|^2 = |e^{i\theta}|^2|\alpha|^2 = |\alpha|^2$$

$$|e^{i\theta}\beta|^2 = |\beta|^2$$

Since $|e^{i\theta}| = 1$, probabilities are unchanged.

$$|\psi'\rangle \equiv |\psi\rangle$$

Global phase can be ignored.

Relative Phase

$$|\psi\rangle = \alpha|0\rangle + e^{i\phi}\beta|1\rangle$$

Phase difference between components is called:
Relative Phase

- Cannot be removed completely
- Affects interference
- Physically observable

Question 1

Given:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/3}|1\rangle)$$

Find:

- α
- β
- $|\alpha|^2$
- $|\beta|^2$

Solution

$$\alpha = \frac{1}{\sqrt{2}}$$

$$\beta = \frac{e^{i\pi/3}}{\sqrt{2}}$$

Since:

$$|e^{i\theta}| = 1$$

$$|\alpha|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\beta|^2 = \left|\frac{e^{i\pi/3}}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$|\alpha|^2 + |\beta|^2 = 1$$

State is normalized.

Problem 2 (a)

Simplify:

$$e^{i\pi/4}|0\rangle + e^{i\pi/4}|1\rangle$$

Solution (a)

Factor common term:

$$= e^{i\pi/4}(|0\rangle + |1\rangle)$$

Since entire state multiplied by $e^{i\pi/4}$,
This is Global Phase.
Physically equivalent to:

$$|0\rangle + |1\rangle$$

Problem 2 (b)

Simplify:

$$e^{i\pi/6}\left|0\right\rangle + e^{i\pi/3}\left|1\right\rangle$$

Solution (b)

Factor smallest phase

$$= e^{i\pi/6} \left(|0\rangle + e^{i(\pi/3 - \pi/6)} |1\rangle \right)$$

$$\pi/3 - \pi/6 = \pi/6$$

$$= e^{i\pi/6} \left(|0\rangle + e^{i\pi/6} |1\rangle \right)$$

Final form:

$$|0\rangle + e^{i\pi/6} |1\rangle$$

Problem (c)

Simplify:

$$e^{i\theta}\alpha\,|0\rangle - \beta\,|1\rangle$$

Solution (c)

$$e^{i\theta}\alpha|0\rangle - \beta|1\rangle$$

$$= e^{i\theta}\left(\alpha|0\rangle - e^{-i\theta}\beta|1\rangle\right)$$

- $e^{i\theta} \rightarrow$ Global phase
- $e^{-i\theta}$ inside $\rightarrow$ Relative phase

$$= \alpha|0\rangle - e^{-i\theta}\beta|1\rangle$$

Problem (d)

Simplify:

$$-e^{i\pi/4}|0\rangle + e^{i\pi/4}|1\rangle$$

Solution (d)

Given:

$$-e^{i\pi/4}|0\rangle + e^{i\pi/4}|1\rangle$$

$$= e^{i\pi/4}(-|0\rangle + |1\rangle)$$

$$= e^{i\pi/4}\left[-(|0\rangle + |1\rangle)\right]$$

$$= -e^{i\pi/4}(|0\rangle - |1\rangle)$$

Since $-1 = e^{i\pi}$, this is just a global phase.
Global phase can be ignored.
Final state:

$$|0\rangle - |1\rangle$$

Thank You