Entanglement

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Quantum Entanglement

- Quantum entanglement is a phenomenon where two or more qubits become linked such that the quantum state of one qubit cannot be described independently of the others.
- Measuring one qubit immediately determines the state of the other entangled qubit, no matter how far apart they are.

Mathematical Representation

To determine if a state is entangled, we look at whether the joint state vector $|\psi\rangle_{AB}$ can be written as a tensor product of individual states $|\phi\rangle_A \otimes |\chi\rangle_B$.

- **separable State:** $|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B$
- **Entangled State:** $|\psi\rangle \neq |\phi\rangle_A \otimes |\chi\rangle_B$

For a general two-qubit state:

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

The state is **entangled** if $\alpha\delta - \beta\gamma \neq 0$.

Bell States (Maximally Entangled States)

The **Bell states** form a complete orthonormal basis for two-qubit entangled systems.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

These states exhibit **maximum correlation** between qubits.

Example 1:

Q1) Are each of the following states a product state or entangled state ? If it is a product state,give the factorization.

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

Identify Coefficients: $\alpha = 0, \beta = \frac{1}{\sqrt{2}}, \gamma = \frac{1}{\sqrt{2}}, \delta = 0.$

Test: $\alpha\delta - \beta\gamma = (0 \cdot 0) - (\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}) = -\frac{1}{2}.$

Result: Since the result is not zero, this is an **entangled state.** Specifically, it is one of the Bell states ($|\Psi^+\rangle$).

Example 2:

Q2) Are each of the following states a product state or entangled state ? If it is a product state,give the factorization.

$\frac{1}{\sqrt{2}}(|10\rangle + i|11\rangle)$

Identify Coefficients: $\alpha = 0, \beta = 0, \gamma = \frac{1}{\sqrt{2}}, \delta = \frac{i}{\sqrt{2}}$.

Test: $\alpha\delta - \beta\gamma = (0 \cdot \frac{i}{\sqrt{2}}) - (0 \cdot \frac{1}{\sqrt{2}}) = 0.$

Result: This is a **product state**.

Factorization: We can factor out the first qubit $|1\rangle$:

$$\frac{1}{\sqrt{2}}|1\rangle \otimes (|0\rangle + i|1\rangle) = |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

Example 3:

Q.3) Are each of the following states a product state or entangled state ? If it is a product state, give the factorization.

(a) $\frac{1}{4}(3|00\rangle - \sqrt{3}|01\rangle + \sqrt{3}|10\rangle - |11\rangle)$

- **Identify Coefficients**: $\alpha = \frac{3}{4}, \beta = -\frac{\sqrt{3}}{4}, \gamma = \frac{\sqrt{3}}{4}, \delta = -\frac{1}{4}$.
- **Test**: $\alpha\delta - \beta\gamma = (\frac{3}{4} \cdot -\frac{1}{4}) - (-\frac{\sqrt{3}}{4} \cdot \frac{\sqrt{3}}{4}) = -\frac{3}{16} - (-\frac{3}{16}) = 0$.
- **Result**: This is a **product state**.

Now, we will factorize it.

State: $|\psi\rangle = \frac{3}{4}|00\rangle - \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle - \frac{1}{4}|11\rangle$

Step 1: Group by the first qubit

- **First Qubit is $|0\rangle$:** $\frac{3}{4}|00\rangle - \frac{\sqrt{3}}{4}|01\rangle = |0\rangle \otimes \left(\frac{3}{4}|0\rangle - \frac{\sqrt{3}}{4}|1\rangle\right)$
- **First Qubit is $|1\rangle$:** $\frac{\sqrt{3}}{4}|10\rangle - \frac{1}{4}|11\rangle = |1\rangle \otimes \left(\frac{\sqrt{3}}{4}|0\rangle - \frac{1}{4}|1\rangle\right)$

Step 2: Look for a common factor

Notice that $\frac{3}{4}|0\rangle - \frac{\sqrt{3}}{4}|1\rangle$ is just $\sqrt{3}$ times larger than $\frac{\sqrt{3}}{4}|0\rangle - \frac{1}{4}|1\rangle$.
We can pull $\sqrt{3}$ out of the first group:

- $|0\rangle \otimes \sqrt{3}\left(\frac{\sqrt{3}}{4}|0\rangle - \frac{1}{4}|1\rangle\right)$

Step 3: Factor out the shared second qubit

Now both groups share the state $\left(\frac{\sqrt{3}}{4}|0\rangle - \frac{1}{4}|1\rangle\right)$:

$$\boxed{(\sqrt{3}|0\rangle + |1\rangle) \otimes \left(\frac{\sqrt{3}}{4}|0\rangle - \frac{1}{4}|1\rangle\right)}$$

Example 4: Non-Equal Probabilities

Question: Consider a 2-qubit entangled system prepared in the following initial state:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

If Alice measures the first qubit and yields a value of **1**, determine the new collapsed state and the resulting probability of Bob measuring his qubit as **0**.

Solution:

Step 1: Apply the Measurement Filter * Alice measures the first qubit and observes **1**.

- We eliminate all terms in the superposition where the first qubit is **not 1** (the $|00\rangle$ and $|01\rangle$ terms disappear).
- The unnormalized remaining state is: $|\psi_{rem}\rangle = \frac{1}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$.

Step 2: Calculate the Total Remaining Probability * The probability of this specific outcome occurring is the sum of the squares of the coefficients of the surviving terms:

$$P(Alice = 1) = \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Step 3: Renormalize the State * To ensure the total probability of the new state equals 1, we divide the remaining terms by the square root of the total remaining probability ($\sqrt{1/3}$):

$$|\psi'\rangle = \frac{\frac{1}{\sqrt{6}}}{\sqrt{1/3}}|10\rangle + \frac{\frac{1}{\sqrt{6}}}{\sqrt{1/3}}|11\rangle = \frac{\sqrt{3}}{\sqrt{6}}|10\rangle + \frac{\sqrt{3}}{\sqrt{6}}|11\rangle$$

$$|\psi'\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

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Step 4: Determine Bob's Probability * In the collapsed state $|\psi'\rangle$, the coefficient for Bob measuring **0** (the state $|10\rangle$) is $\frac{1}{\sqrt{2}}$.

- Probability $P(Bob = 0) = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$.

Final Result: The state collapses to $|\psi'\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$. Bob now has a **50% chance** of measuring 0 and a **50% chance** of measuring 1.

E91 Protocol

- It is a Quantum Key Distribution (QKD) protocol developed by Artur Ekert in 1991.
- It uses the power of entanglement to ensure two people can share a secret key that is physically impossible to steal without detection.
- Here, the security is built on the **Bell Inequality Test**.
- This protocol is unique because Alice and Bob do not send information to each other; instead, they both receive qubits from a common source.
- **By the Monogamy of Entanglement:** If two qubits are perfectly entangled, they cannot be entangled with a third party and because these qubits are linked, measuring one instantly affects the other.

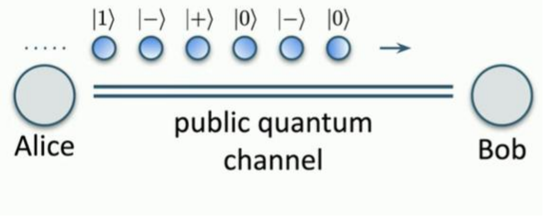
The Mechanism: Shared Correlation

1. **Entangled Source:** A central source generates a pair of entangled qubits in a specific state, such as the $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.
2. **Distribution:** One qubit is sent to Alice and the other to Bob.
3. **Measurement:** Alice and Bob each choose from three different measurement angles (bases) to measure their qubits.

Example 1: Without Eve (Perfect Secrecy)

In this scenario, the qubits arrive exactly as the source created them.

- **Key Generation:** When Alice and Bob happen to choose the same measurement angle, their results are perfectly correlated (in the singlet state, they will always get opposite results). They can turn these results into a secret key.
- **Security Verification:** They use the trials where they chose *different* angles to perform the **Bell Test**.
- **The Result:** Because the qubits are untouched, they satisfy the Bell Inequality with a high value (specifically, the CHSH inequality results in $|S| = 2\sqrt{2}$).
- **Conclusion:** This high value proves the qubits are "maximally entangled." Alice and Bob are certain that no one else has information about their results because entanglement is monogamous.



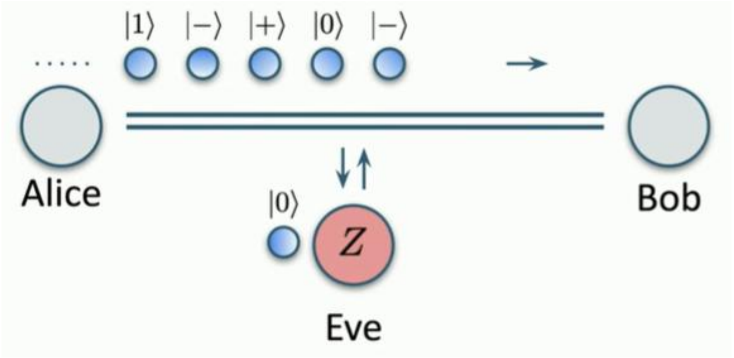
Pair	Alice's Basis	Bob's Basis	Alice's Bit	Bob's Bit	Outcome
1	**Z**	**Z**	0	1	**Keep for Key**
2	**X**	**X**	1	0	**Keep for Key**
3	**Z**	**X**	0	0	**Use for Bell Test**
4	**Y**	**Z**	1	1	**Use for Bell Test**

- **Key Logic:** Alice and Bob discard pairs 3 and 4 but keep the bits from 1 and 2. After Bob inverts his bits, they have the secret key: `01`.
- **Bell Result:** The math on the "Test" bits results in $|S| \approx 2.82$.
- **Conclusion:** The high S value proves the link is private and quantum.

Example 2: With Eve (Active Eavesdropping)

In this scenario, Eve attempts to intercept the qubits to learn the key.

- **The Interception:** Eve intercepts the qubits traveling to Bob and measures them to find their state.
- **State Collapse:** The act of measuring an entangled qubit causes the joint quantum state to "collapse". The qubits are no longer entangled; they become two separate, independent particles.
- **The Check:** Bob eventually receives the qubit and measures it, but it no longer shares that "spooky" connection with Alice's qubit.
- **The Result:** When Alice and Bob perform the Bell Test on their data, the value of $|S|$ drops from $2\sqrt{2}$ down to **2 or less**.
- **Conclusion:** A Bell value of 2 is the maximum possible for regular "local" physics. This drop is a massive red flag. Alice and Bob instantly know the connection was tampered with, discard the data, and do not use it for a key.



Pair	Alice's Basis	Bob's Basis	Alice's Bit	Bob's Bit	Outcome
1	**Z**	**Z**	0	0	**Error** (Correlation broken)
2	**X**	**X**	1	1	**Error** (Random results)
3	**Z**	**X**	0	1	**Bell Test Data**
4	**Y**	**Z**	0	0	**Bell Test Data**

- **The "Peek" Effect:** By measuring the qubits, Eve forces them into a definite classical state. They are no longer entangled.
- **Bell Result:** When Alice and Bob calculate the S parameter, it drops to $|S| \leq 2$.
- **Conclusion:** Alice and Bob see the score is 2 or lower, realize the "quantumness" is gone, and **abort the communication**.

Bell’s Inequality

The CHSH Formula: Used to calculate the "S-parameter" to detect eavesdropping.

- $S = E(a_1, b_1) - E(a_1, b_2) + E(a_2, b_1) + E(a_2, b_2)$

The Threshold:

- $S \approx 2.82$**:** Pure entanglement (Secure).
- $S \leq 2$**:** No entanglement/Classical behavior (Eve detected).

References

- “Introduction to Classical and Quantum Computing” - Thomas G. Wong
- “Quantum Computing for Everyone” – Chris Bernhardt

Thank You!