

# One Quantum Bit MidSem

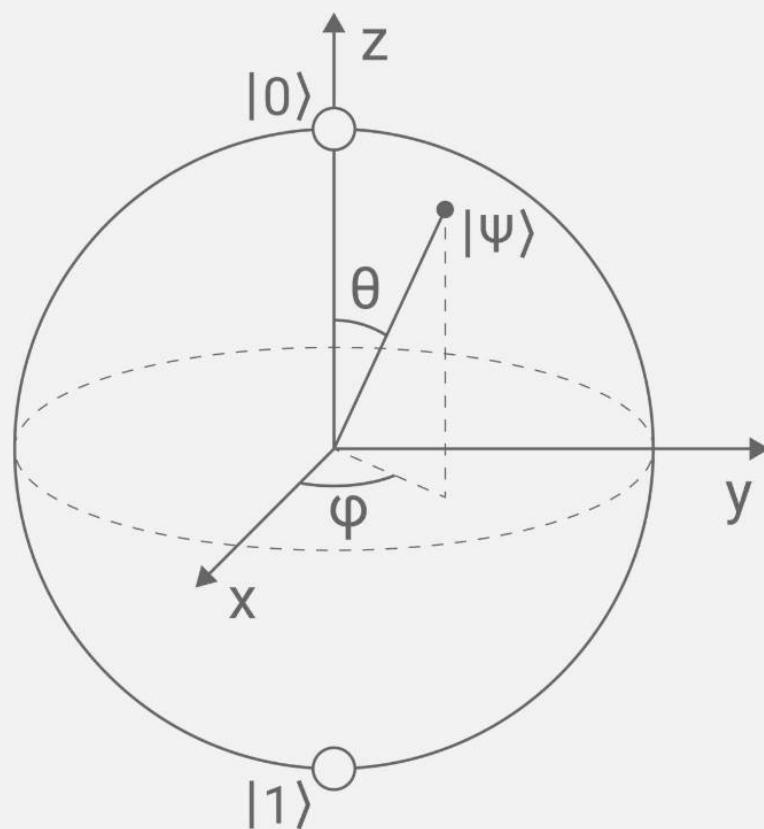


**Exercise 2.3.** Draw a Bloch sphere and label the following locations:

- (a) Where a qubit is exactly  $|0\rangle$ .
- (b) Where a qubit is exactly  $|1\rangle$ .
- (c) Where a qubit is half  $|0\rangle$  and half  $|1\rangle$ .
- (d) Where a qubit is more  $|0\rangle$  than  $|1\rangle$ .
- (e) Where a qubit is more  $|1\rangle$  than  $|0\rangle$ .

## Bloch Sphere Representation

# BLOCH SPHERE



In quantum mechanics and computing, the Bloch sphere is a geometrical representation of the pure state space of a two-level quantum mechanical system (qubit), named after the physicist Felix Bloch.



<https://www.researchgate.net/publication/335028508/figure/fig1/AS%3A789466423762944%401565234871365/The-Bloch-sphere-provides-a-useful-means-of-visualizing-the-state-of-a-single-qubit-and.ppm>



<https://www.researchgate.net/publication/381311252/figure/fig2/AS%3A11431281250988090%401718097792177/The-Bloch-sphere-The-North-Pole-corresponds-to-the-state-0i-the-South-Pole-to-the.ppm>

Now labeling the required locations:

## (a) Exactly $|0\rangle$

### • North Pole

- Coordinates:  $(0, 0, +1)$
- Angle:  $\theta = 0$

**(b) Exactly  $|1\rangle$** 📍 **South Pole**

- Coordinates:  $(0, 0, -1)$
- Angle:  $\theta = \pi$

**(c) Half  $|0\rangle$  and Half  $|1\rangle$** 

(Equal probability: 50% – 50%)

📍 **Equator of the sphere**

- Example:  $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$
- Here  $\theta = \pi/2$

(Any point on equator gives equal probabilities.)

**(d) More  $|0\rangle$  than  $|1\rangle$** 📍 **Upper hemisphere (above equator)**

- $0 < \theta < \pi/2$
- Closer to north pole

**(e) More  $|1\rangle$  than  $|0\rangle$** 📍 **Lower hemisphere (below equator)**

- $\pi/2 < \theta < \pi$
- Closer to south pole

If you want, I can also give a quick 2-line rule to immediately answer any Bloch sphere probability question in exams.



**Exercise 2.4.** Consider the complex number  $z = 1 + 2i$ .

- (a) Find  $\Re(z)$ .
- (b) Find  $\Im(z)$ .
- (c) Plot  $z$  as a point in the complex plane.
- (d) Write  $z$  in polar form  $re^{i\theta}$ .
- (e) Find  $z^*$ .
- (f) Find  $|z|$ .
- (g) Find  $|z|^2$ .

**Exercise 2.5.** Consider the complex number  $z = -3 - i$ .

- (a) Find  $\Re(z)$ .
- (b) Find  $\Im(z)$ .
- (c) Plot  $z$  as a point in the complex plane.
- (d) Write  $z$  in polar form  $re^{i\theta}$ . Hint: The angle should be between  $\pi$  and  $3\pi/2$  (i.e.,  $180^\circ$  and  $270^\circ$ ).
- (e) Find  $z^*$ .
- (f) Find  $|z|$ .
- (g) Find  $|z|^2$ .

## Exercise 2.4

Given  $z = 1 + 2i$

- (a)**  $\Re(z) = 1$
- (b)**  $\Im(z) = 2$
- (c)** Point in complex plane:  $(1, 2)$
- (d)** Polar form:

$$z = \sqrt{5} e^{i \tan^{-1}(2)}$$

(since  $r = \sqrt{5}$ ,  $\theta = \tan^{-1}(2)$ )

- (e)**  $z^* = 1 - 2i$
- (f)**  $|z| = \sqrt{5}$
- (g)**  $|z|^2 = 5$

## Exercise 2.5

Given  $z = -3 - i$

- (a)**  $\Re(z) = -3$
- (b)**  $\Im(z) = -1$
- (c)** Point in complex plane:  $(-3, -1)$
- (d)** Polar form:

$$z = \sqrt{10} e^{i(\pi + \tan^{-1}(\frac{1}{3}))}$$

(Angle is between  $\pi$  and  $3\pi/2$ , third quadrant.)

- (e)**  $z^* = -3 + i$
- (f)**  $|z| = \sqrt{10}$
- (g)**  $|z|^2 = 10$



\*\*\*(d)\*\* Polar form:  $z = 10 e^{i(\pi + \tan^{-1}(13))}$  (Angle is between  $\pi$  and  $3\pi/2$ , third quadrant.)  
explain

You're asking about this part:

$$z = -3 - i$$

and why its polar form angle is

$$\theta = \pi + \tan^{-1}\left(\frac{1}{3}\right)$$

Let's understand it clearly.

## Step 1 Magnitude $r$

$$r = \sqrt{(-3)^2 + (-1)^2} = \sqrt{9 + 1} = \sqrt{10}$$

So,

$$z = \sqrt{10}e^{i\theta}$$

Now we only need  $\theta$  (argument).

## Step 2 Find the Reference Angle

First compute:

$$\tan \theta = \frac{\text{Im}(z)}{\text{Re}(z)} = \frac{-1}{-3} = \frac{1}{3}$$

So the reference angle is:

$$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$$

This angle is acute (between 0 and  $\pi/2$ ).

## Step 3 Determine the Quadrant

Point:

$$(-3, -1)$$

Both real and imaginary parts are negative → **Third Quadrant**

In the third quadrant:

$$\theta = \pi + \alpha$$

Why?

Because:

- $\pi$  moves us to the negative x-axis.
- Then we go down by the reference angle.

## Final Angle

$$\theta = \pi + \tan^{-1}\left(\frac{1}{3}\right)$$

So the polar form is:

$$z = \sqrt{10} e^{i(\pi + \tan^{-1}(1/3))}$$

## Quick Exam Trick

If:

- $x > 0, y > 0 \rightarrow \theta = \tan^{-1}(y/x)$
- $x < 0 \rightarrow \theta = \pi + \tan^{-1}(y/x)$
- $x > 0, y < 0 \rightarrow \theta = 2\pi + \tan^{-1}(y/x)$

If you want, I can also show the exact numerical value of the angle in radians and degrees.



**Exercise 2.7.** A qubit is in the state

$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Say you measure the qubit and get  $|0\rangle$ . If you measure the qubit a *second time*, what is the probability of getting

- (a)  $|0\rangle$ ?  
(b)  $|1\rangle$ ?

State given:

$$|\psi\rangle = \frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle$$

## First Measurement

You are told the measurement result is  $|0\rangle$ .

When a measurement gives  $|0\rangle$ , the state **collapses** to:

$$|\psi_{\text{after}}\rangle = |0\rangle$$

All superposition is gone.

## Second Measurement

Now the system is exactly  $|0\rangle$ .

So:

**(a) Probability of  $|0\rangle$  again**

1

**(b) Probability of  $|1\rangle$**

0

## Important Concept 🔥

Measurement in the computational basis is **projective**.

Once you measure  $|0\rangle$ :

- The qubit becomes exactly  $|0\rangle$
- Repeated measurements in the same basis always give the same result with probability 1.

This is called **wavefunction collapse**.



**Exercise 2.8.** A qubit is in the state

$$\frac{e^{i\pi/8}}{\sqrt{5}}|0\rangle + \beta|1\rangle.$$

What is a possible value of  $\beta$ ?

Given state:

$$|\psi\rangle = \frac{e^{i\pi/8}}{\sqrt{5}}|0\rangle + \beta|1\rangle$$

A valid qubit must satisfy **normalization**:

$$|\alpha|^2 + |\beta|^2 = 1$$


---

### Step 1: Compute $|\alpha|^2$

$$\alpha = \frac{e^{i\pi/8}}{\sqrt{5}}$$

Magnitude of a complex exponential:

$$|e^{i\theta}| = 1$$

So,

$$|\alpha|^2 = \frac{1}{5}$$


---

### Step 2: Use normalization

$$\frac{1}{5} + |\beta|^2 = 1$$

$$|\beta|^2 = \frac{4}{5}$$

$$|\beta| = \frac{2}{\sqrt{5}}$$


---

## Final Answer

A possible value of  $\beta$  is:

$$\boxed{\beta = \frac{2}{\sqrt{5}}}$$

But remember:

Since only the magnitude is fixed, the most general form is

$$\boxed{\beta = \frac{2}{\sqrt{5}} e^{i\phi}}$$

for any real phase  $\phi$ .

---

If you want, I can also explain why the global phase  $e^{i\pi/8}$  doesn't physically matter.



**Exercise 2.9.** A qubit is in the state

$$A \left( 2e^{i\pi/6} |0\rangle - 3|1\rangle \right).$$

- (a) Normalize the state (i.e., find  $A$ ).
- (b) If you measure the qubit, what is the probability that you get  $|0\rangle$ ?
- (c) If you measure the qubit, what is the probability that you get  $|1\rangle$ ?

Given state:

$$|\psi\rangle = A \left( 2e^{i\pi/6} |0\rangle - 3|1\rangle \right)$$

## (a) Normalize the state (find $A$ )

Normalization condition:

$$|A|^2 \left( |2e^{i\pi/6}|^2 + |-3|^2 \right) = 1$$

Now,

$$|2e^{i\pi/6}|^2 = 4$$

(because  $|e^{i\theta}| = 1$ )

$$|-3|^2 = 9$$

So:

$$|A|^2(4 + 9) = 1$$

$$|A|^2 \cdot 13 = 1$$

$$|A|^2 = \frac{1}{13}$$

Taking positive normalization constant,

$$A = \frac{1}{\sqrt{13}}$$

## (b) Probability of getting $|0\rangle$

Coefficient of  $|0\rangle$ :

$$\frac{2e^{i\pi/6}}{\sqrt{13}}$$

Probability:

$$\frac{4}{13}$$

$\frac{4}{13}$
----------------

### (c) Probability of getting $|1\rangle$

Coefficient:

$$\frac{-3}{\sqrt{13}}$$

Probability:

$$\frac{9}{13}$$

$\frac{9}{13}$
----------------

Quick check:

$$\frac{4}{13} + \frac{9}{13} = 1$$

✓ Correct normalization.



**Exercise 2.10.** A qubit is in the state

$$\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle.$$

90

2 One Quantum Bit

- (a) If you measure it in the Z-basis  $\{|0\rangle, |1\rangle\}$ , what states can you get and with what probabilities?
- (b) Write the qubit's state in terms of  $|+\rangle$  and  $|-\rangle$ .
- (c) If you measure it in the basis  $\{|+\rangle, |-\rangle\}$ , what states can you get and with what probabilities?

Given state:

$$|\psi\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

(Already normalized since  $\frac{1}{4} + \frac{3}{4} = 1$ )

## (a) Measurement in Z-basis $\{|0\rangle, |1\rangle\}$

Probabilities are squares of coefficients:

$$P(0) = \left| \frac{1}{2} \right|^2 = \frac{1}{4}$$

$$P(1) = \left| \frac{\sqrt{3}}{2} \right|^2 = \frac{3}{4}$$

**Answer:**

- $|0\rangle$  with probability  $\boxed{\frac{1}{4}}$
- $|1\rangle$  with probability  $\boxed{\frac{3}{4}}$

## (b) Write state in $\{|+\rangle, |-\rangle\}$

Recall:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Substitute:

$$|\psi\rangle = \frac{1}{2} \frac{|+\rangle + |-\rangle}{\sqrt{2}} - \frac{\sqrt{3}}{2} \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Combine coefficients:

Coefficient of  $|+\rangle$ :

$$\frac{1 - \sqrt{3}}{2\sqrt{2}}$$

Coefficient of  $|-\rangle$ :

$$\frac{1 + \sqrt{3}}{2\sqrt{2}}$$

So,

$$|\psi\rangle = \frac{1 - \sqrt{3}}{2\sqrt{2}} |+\rangle + \frac{1 + \sqrt{3}}{2\sqrt{2}} |-\rangle$$

## (c) Measurement in $\{|+\rangle, |-\rangle\}$

Square the magnitudes:

$$P(+) = \left| \frac{1 - \sqrt{3}}{2\sqrt{2}} \right|^2 = \frac{2 - \sqrt{3}}{4}$$

$$P(-) = \left| \frac{1 + \sqrt{3}}{2\sqrt{2}} \right|^2 = \frac{2 + \sqrt{3}}{4}$$

## Final Answers

- $|+\rangle$  with probability

$$\frac{2 - \sqrt{3}}{4}$$

- $|-\rangle$  with probability

$$\frac{2 + \sqrt{3}}{4}$$

If you want, I can also show the Bloch sphere angle interpretation (this state actually corresponds to a specific polar angle).



**Exercise 2.11.** The following two states are opposite points on the Bloch sphere:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle,$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

So, we can measure relative to them. Now consider a qubit in the state

$$\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle.$$

- Write the qubit's state in terms of  $|a\rangle$  and  $|b\rangle$ .
- If you measure the qubit in the basis  $\{|a\rangle, |b\rangle\}$ , what states can you get and with what probabilities?

Given:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

and

$$|\psi\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

Since  $|a\rangle, |b\rangle$  form an orthonormal basis, coefficients are:

$$c_a = \langle a|\psi\rangle, \quad c_b = \langle b|\psi\rangle$$


---

## (a) Expand $|\psi\rangle$ in $\{|a\rangle, |b\rangle\}$

**Compute  $c_a$**

$$\begin{aligned} \langle a| &= \frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1| \\ c_a &= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \left(-\frac{i}{2}\right) \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{4} + \frac{i\sqrt{3}}{4} = \frac{\sqrt{3}}{4}(1+i) \end{aligned}$$


---

**Compute  $c_b$**

$$\begin{aligned} \langle b| &= -\frac{i}{2}\langle 0| + \frac{\sqrt{3}}{2}\langle 1| \\ c_b &= -\frac{i}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= -\frac{i}{4} - \frac{3}{4} = -\frac{3+i}{4} \end{aligned}$$


---

**So,**

$$|\psi\rangle = \frac{\sqrt{3}}{4}(1+i)|a\rangle - \frac{3+i}{4}|b\rangle$$


---

## (b) Measurement in basis $\{|a\rangle, |b\rangle\}$

Probabilities are squared magnitudes.

**Probability of  $|a\rangle$**

$$\begin{aligned} \left| \frac{\sqrt{3}}{4}(1+i) \right|^2 &= \frac{3}{16} \cdot |1+i|^2 \\ |1+i|^2 &= 2 \end{aligned}$$

$$P(a) = \frac{6}{16} = \frac{3}{8}$$

## Probability of $|b\rangle$

$$\left| -\frac{3+i}{4} \right|^2 = \frac{1}{16} |3+i|^2$$

$$|3+i|^2 = 10$$

$$P(b) = \frac{10}{16} = \frac{5}{8}$$

## Final Answers

$$|\psi\rangle = \frac{\sqrt{3}}{4}(1+i)|a\rangle - \frac{3+i}{4}|b\rangle$$

Measurement gives:

$$P(a) = \frac{3}{8}$$

$$P(b) = \frac{5}{8}$$

(Sum = 1 ✓)

If you want, I can also show the shortcut geometric Bloch sphere reasoning that avoids this algebra.



how are a and b orthonormal

Good question 👍 let's verify it properly.

We are given:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

To be **orthonormal**, two things must hold:

1. ✓ Each state has norm 1

2.  Their inner product is 0

## 1 Check Normalization

For  $|a\rangle$

$$\begin{aligned}\langle a|a \rangle &= \left| \frac{\sqrt{3}}{2} \right|^2 + \left| \frac{i}{2} \right|^2 \\ &= \frac{3}{4} + \frac{1}{4} = 1\end{aligned}$$

So  $|a\rangle$  is normalized.

For  $|b\rangle$

$$\langle b|b \rangle = \left| \frac{i}{2} \right|^2 + \left| \frac{\sqrt{3}}{2} \right|^2 = \frac{1}{4} + \frac{3}{4} = 1$$

So  $|b\rangle$  is also normalized.

## 2 Check Orthogonality

Now compute:

$$\langle a|b \rangle$$

First take conjugate of  $|a\rangle$ :

$$\langle a| = \frac{\sqrt{3}}{2} \langle 0| - \frac{i}{2} \langle 1|$$

Now multiply with  $|b\rangle$ :

$$\left( \frac{\sqrt{3}}{2} \langle 0| - \frac{i}{2} \langle 1| \right) \left( \frac{i}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle \right)$$

Multiply terms:

$$\begin{aligned}&= \frac{\sqrt{3}i}{4} - \frac{i\sqrt{3}}{4} \\ &= 0\end{aligned}$$

## ✓ Conclusion

$$\langle a|b\rangle = 0$$

So they are orthogonal.

Since:

- $\langle a|a\rangle = 1$
- $\langle b|b\rangle = 1$
- $\langle a|b\rangle = 0$

They are **orthonormal**.

Geometrically:

They correspond to opposite points on the Bloch sphere → that always means orthonormal basis states.



$|\psi\rangle = 43(1+i)|a\rangle - 43+i|b\rangle$   
how this

Ahhh you're asking:

How did

$$|\psi\rangle = \frac{\sqrt{3}}{4}(1 + i)|a\rangle - \frac{3 + i}{4}|b\rangle$$

come out?

Let's derive it cleanly.

## Step 1 Idea

Since  $|a\rangle, |b\rangle$  are orthonormal, any state can be written as:

$$|\psi\rangle = \langle a|\psi\rangle|a\rangle + \langle b|\psi\rangle|b\rangle$$

So we compute:

$$c_a = \langle a|\psi\rangle \quad c_b = \langle b|\psi\rangle$$

## Step 2 Write Everything

$$|\psi\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

### Step 3 Compute $c_a = \langle a|\psi\rangle$

Take conjugate of  $|a\rangle$ :

$$\langle a| = \frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|$$

Now multiply:

$$\begin{aligned} c_a &= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \left(-\frac{i}{2}\right) \left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{4} + \frac{i\sqrt{3}}{4} \\ &= \frac{\sqrt{3}}{4}(1+i) \end{aligned}$$

### Step 4 Compute $c_b = \langle b|\psi\rangle$

Conjugate of  $|b\rangle$ :

$$\langle b| = -\frac{i}{2}\langle 0| + \frac{\sqrt{3}}{2}\langle 1|$$

Now multiply:

$$\begin{aligned} c_b &= -\frac{i}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \left(-\frac{\sqrt{3}}{2}\right) \\ &= -\frac{i}{4} - \frac{3}{4} \\ &= -\frac{3+i}{4} \end{aligned}$$

### Step 5 Substitute

$$|\psi\rangle = \frac{\sqrt{3}}{4}(1+i)|a\rangle - \frac{3+i}{4}|b\rangle$$

That's it.

## Why This Works

Because in an orthonormal basis:

$$c_k = \langle k | \psi \rangle$$

always gives the expansion coefficient.

If you'd like, I can also show a **faster 2×2 matrix method** that avoids bra-ket multiplication.



**Exercise 2.12.** A qubit is in the state  $|0\rangle$ . If you measure it in the X-basis  $\{|+\rangle, |-\rangle\}$  and then measure it again in the Z-basis  $\{|0\rangle, |1\rangle\}$ , what is the probability of getting

- (a)  $|0\rangle$ ?
- (b)  $|1\rangle$ ?

Initial state:

$$|\psi\rangle = |0\rangle$$

### Step 1 Measure in X-basis $\{|+\rangle, |-\rangle\}$

Recall:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

So,

$$P(+) = \frac{1}{2}, \quad P(-) = \frac{1}{2}$$

After this measurement, the state collapses to:

- $|+\rangle$  with probability 1/2
- $|-\rangle$  with probability 1/2

### Step 2 Measure again in Z-basis $\{|0\rangle, |1\rangle\}$

Recall:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

For either case:

$$P(0|+) = \frac{1}{2}, \quad P(1|+) = \frac{1}{2}$$

$$P(0|-) = \frac{1}{2}, \quad P(1|-) = \frac{1}{2}$$

## Step 3 Total Probability

Use total probability rule.

$$P(0) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$$P(1) = \frac{1}{2}$$

## ✓ Final Answer

$$P(0) = \frac{1}{2}$$

$$P(1) = \frac{1}{2}$$

## 🔥 Key Idea

Measuring in the X-basis **destroys the original Z-basis information**.

After the first measurement, the system forgets it started in  $|0\rangle$ .



**Exercise 2.13.** Is there a measurement that can distinguish the following pairs of states? If yes, give a measurement. If no, explain your reasoning.

(a)  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $e^{i\pi/8}|+\rangle = \frac{e^{i\pi/8}}{\sqrt{2}}(|0\rangle + |1\rangle)$ .

(b)  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ .

(c)  $|0\rangle$  and  $e^{i\pi/4}|0\rangle$ .

The key rule:

👉 Two quantum states can be perfectly distinguished **iff they are orthogonal**.

If they differ only by a **global phase**, they are physically identical → impossible to distinguish.

**(a)  $|+\rangle$  and  $e^{i\pi/8}|+\rangle$**

$$e^{i\pi/8}|+\rangle$$

This is just a **global phase** multiplied to the same state.

Global phase has **no physical effect**.

These two states represent the **same point on the Bloch sphere**.

### ✓ Conclusion:

No measurement can distinguish them.

## (b) $|+\rangle$ and $|-\rangle$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Compute inner product:

$$\langle +|-\rangle = 0$$

They are orthogonal.

### ✓ Conclusion:

Yes, we can distinguish them perfectly.

### Measurement:

Measure in the **X-basis**  $\{|+\rangle, |-\rangle\}$ .

Result directly tells the state.

## (c) $|0\rangle$ and $e^{i\pi/4}|0\rangle$

Again:

$$e^{i\pi/4}|0\rangle$$

This is just global phase.

Same physical state.

### ✓ Conclusion:

No measurement can distinguish them.

# 🔥 Final Summary

Pair	Distinguishable?	Reason
(a)	✗ No	Global phase only
(b)	✓ Yes	Orthogonal states
(c)	✗ No	Global phase only

If you want, I can also show the Bloch sphere geometric intuition for this — it makes it instantly obvious.



**Exercise 2.14.** A qubit is in the state

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle).$$

- (a) Where on the Bloch sphere is this state? Give your answer in  $(\theta, \phi)$  coordinates.
- (b) Sketch the point on the Bloch sphere.

Given state:

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

## Step 1 Compare with General Bloch Form

A general qubit can be written as:

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$

Now compare.

We have:

$$\begin{aligned}\cos \frac{\theta}{2} &= \frac{1}{\sqrt{2}} \\ \sin \frac{\theta}{2} &= \frac{1}{\sqrt{2}}\end{aligned}$$

So,

$$\frac{\theta}{2} = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{2}$$

Now look at the phase:

$$e^{i\phi} = i = e^{i\pi/2}$$

So,

$$\phi = \frac{\pi}{2}$$

## ✓ Final Answer (a)

$$(\theta, \phi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

## (b) Where is it on Bloch Sphere?

- $\theta = \pi/2 \rightarrow$  on the **equator**
- $\phi = \pi/2 \rightarrow$  along **positive Y-axis**

So this is the **+Y direction** on the Bloch sphere.

It is usually called:

$$|+i\rangle$$

## Geometric Summary

- $|+\rangle \rightarrow +X$  direction
- $|-\rangle \rightarrow -X$  direction
- $|i\rangle \rightarrow +Y$  direction
- $| -i\rangle \rightarrow -Y$  direction

If you want, I can also show the quick shortcut trick to immediately read  $\theta$  and  $\varphi$  without comparing formulas.



no i prefer above method only

Perfect  that **is** the cleanest method.

Let me just restate it cleanly in final-exam style using only that method.

---

Given:

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

General Bloch form:

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$


---

## Compare coefficients

$$\cos \frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

$$\sin \frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

So,

$$\frac{\theta}{2} = \frac{\pi}{4}$$

$\theta = \frac{\pi}{2}$

Now compare the phase term:

$$e^{i\phi} = i$$

We know:

$$i = e^{i\pi/2}$$

So,

$\phi = \frac{\pi}{2}$

---

## Final Answer

$(\theta, \phi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

That's exactly the correct Bloch sphere coordinates using your preferred comparison method.



**Exercise 2.15.** A qubit is in the state

$$\frac{1-i}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

(a) Where on the Bloch sphere is this state? Give your answer in  $(\theta, \phi)$  coordinates.

Given state:

$$|\psi\rangle = \frac{1-i}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

General Bloch form:

$$|\psi\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2}|1\rangle$$

## Step 1 Find $\theta$

First compute magnitudes.

**Magnitude of  $|0\rangle$  coefficient:**

$$\left| \frac{1-i}{2\sqrt{2}} \right| = \frac{|1-i|}{2\sqrt{2}}$$

$$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

So,

$$\frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

Thus:

$$\cos \frac{\theta}{2} = \frac{1}{2}$$

$$\frac{\theta}{2} = \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

## Step 2 Find $\phi$

Rewrite the  $|0\rangle$  coefficient in polar form.

$$1 - i = \sqrt{2}e^{-i\pi/4}$$

So,

$$\frac{1-i}{2\sqrt{2}} = \frac{1}{2}e^{-i\pi/4}$$

Thus the state becomes:

$$|\psi\rangle = \frac{1}{2}e^{-i\pi/4}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Factor out global phase  $e^{-i\pi/4}$  (physically irrelevant):

$$|\psi\rangle = e^{-i\pi/4} \left( \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}e^{i\pi/4}|1\rangle \right)$$

Now compare with standard form.

So,

$$e^{i\phi} = e^{i\pi/4}$$

$$\boxed{\phi = \frac{\pi}{4}}$$

## ✓ Final Answer

$$\boxed{(\theta, \phi) = \left(\frac{2\pi}{3}, \frac{\pi}{4}\right)}$$



(b) Sketch the point on the Bloch sphere.

From part (a), we found:

$$(\theta, \phi) = \left(\frac{2\pi}{3}, \frac{\pi}{4}\right)$$

## How to Sketch It on the Bloch Sphere

**1** Polar angle  $\theta = \frac{2\pi}{3}$

- $\theta$  is measured from the **+Z axis**.

- $\frac{2\pi}{3} = 120^\circ$ .
- This is **below the equator** (since equator is  $\theta = \pi/2$ ).
- So the point is in the **southern hemisphere**.

## 2 Azimuth angle $\phi = \frac{\pi}{4}$

- Measured in the **XY-plane** from +X direction.
- $\frac{\pi}{4} = 45^\circ$ .
- So projection onto XY-plane lies halfway between +X and +Y.

## Final Position Description

- Southern hemisphere
- $120^\circ$  down from north pole
- $45^\circ$  counterclockwise from +X axis
- Between +X and +Y directions (but below equator)

If you want a quick Cartesian coordinate check for sketching:

$$x = \sin \theta \cos \phi = \frac{\sqrt{6}}{4}$$

$$y = \sin \theta \sin \phi = \frac{\sqrt{6}}{4}$$

$$z = \cos \theta = -\frac{1}{2}$$

So it's symmetric in x and y, with negative z.

That gives you the exact sketch location.



**Exercise 2.16.** Consider the following two states from Exercise 2.11:

$$\begin{aligned}|a\rangle &= \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle, \\ |b\rangle &= \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.\end{aligned}$$

Prove these are opposite points of the Bloch sphere by finding their points in spherical coordinates  $(\theta_a, \phi_a)$  and  $(\theta_b, \phi_b)$ . Verify that  $\theta_b = \pi - \theta_a$  and  $\phi_b = \phi_a + \pi$ , which means they lie on opposite points of the Bloch sphere.

We compare each state with the standard Bloch form:

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$

## 1 State $|a\rangle$

$$|a\rangle = \frac{\sqrt{3}}{2} |0\rangle + \frac{i}{2} |1\rangle$$

**Step A: Find  $\theta_a$**

$$\cos \frac{\theta_a}{2} = \frac{\sqrt{3}}{2}$$

$$\frac{\theta_a}{2} = \frac{\pi}{6}$$

$$\boxed{\theta_a = \frac{\pi}{3}}$$

**Step B: Find  $\phi_a$**

$$e^{i\phi_a} \sin \frac{\theta_a}{2} = \frac{i}{2}$$

Since

$$\sin \frac{\theta_a}{2} = \sin \frac{\pi}{6} = \frac{1}{2}$$

So

$$e^{i\phi_a} = i$$

$$\boxed{\phi_a = \frac{\pi}{2}}$$

Thus,

$$\boxed{(\theta_a, \phi_a) = \left(\frac{\pi}{3}, \frac{\pi}{2}\right)}$$

## 2 State $|b\rangle$

$$|b\rangle = \frac{i}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle$$

**Step A: Remove global phase**

Factor out  $i$ :

$$|b\rangle = i \left( \frac{1}{2}|0\rangle - \frac{i\sqrt{3}}{2}|1\rangle \right)$$

Global phase  $i$  does not affect Bloch coordinates.

So consider:

$$|b\rangle \sim \frac{1}{2}|0\rangle - \frac{i\sqrt{3}}{2}|1\rangle$$

### Step B: Find $\theta_b$

$$\cos \frac{\theta_b}{2} = \frac{1}{2}$$

$$\frac{\theta_b}{2} = \frac{\pi}{3}$$

$$\theta_b = \frac{2\pi}{3}$$

### Step C: Find $\phi_b$

Coefficient of  $|1\rangle$ :

$$e^{i\phi_b} \sin \frac{\theta_b}{2} = -\frac{i\sqrt{3}}{2}$$

Since

$$\sin \frac{\theta_b}{2} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

So

$$e^{i\phi_b} = -i$$

$$\phi_b = \frac{3\pi}{2}$$

## Final Coordinates

$$(\theta_a, \phi_a) = \left( \frac{\pi}{3}, \frac{\pi}{2} \right)$$

$$(\theta_b, \phi_b) = \left( \frac{2\pi}{3}, \frac{3\pi}{2} \right)$$

### 3 Verify Opposite Points Condition

Check:

$$\theta_b = \pi - \theta_a$$

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

 Correct

$$\phi_b = \phi_a + \pi$$

$$\frac{\pi}{2} + \pi = \frac{3\pi}{2}$$

 Correct

### ✓ Conclusion

$$\theta_b = \pi - \theta_a, \quad \phi_b = \phi_a + \pi$$

Hence they lie on **opposite points of the Bloch sphere**.



**Exercise 2.22.** Consider a map  $U$  that transforms the Z-basis states as follows:

$$\begin{aligned} U|0\rangle &= |0\rangle + |1\rangle, \\ U|1\rangle &= |0\rangle - |1\rangle. \end{aligned}$$

Say  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  is a normalized quantum state, i.e.,  $|\alpha|^2 + |\beta|^2 = 1$ .

- (a) Calculate  $U|\psi\rangle$ .
- (b) From your answer to (a), is  $U$  a valid quantum gate? Explain your reasoning.

Given:

$$U|0\rangle = |0\rangle + |1\rangle$$

$$U|1\rangle = |0\rangle - |1\rangle$$

and

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with  $|\alpha|^2 + |\beta|^2 = 1$ .

### (a) Calculate $U|\psi\rangle$

Using linearity:

$$\begin{aligned} U|\psi\rangle &= \alpha U|0\rangle + \beta U|1\rangle \\ &= \alpha(|0\rangle + |1\rangle) + \beta(|0\rangle - |1\rangle) \end{aligned}$$

Group terms:

Coefficient of  $|0\rangle$ :

$$\alpha + \beta$$

Coefficient of  $|1\rangle$ :

$$\alpha - \beta$$

So,

$$U|\psi\rangle = (\alpha + \beta)|0\rangle + (\alpha - \beta)|1\rangle$$

## (b) Is $U$ a valid quantum gate?

Quantum gates must be **unitary** → must preserve normalization.

Compute norm of the output:

$$\|U|\psi\rangle\|^2 = |\alpha + \beta|^2 + |\alpha - \beta|^2$$

Use identity:

$$|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$$

So,

$$= 2(|\alpha|^2 + |\beta|^2)$$

$$= 2$$

But input norm was 1.

Output norm is 2.

So normalization is **not preserved**.

## Conclusion

$$U \text{ is NOT unitary}$$

Hence,

$$U \text{ is NOT a valid quantum gate}$$

Note: If there were a factor  $\frac{1}{\sqrt{2}}$ , this would become the Hadamard gate and would be unitary.



**Exercise 2.24.** Consider each of the following classical logic gates with input  $A$ , output  $B$ , and truth table shown below. Is each gate a valid quantum gate? Why?

(a)	$A B$	(b)	$A B$
	$0 0$		$0 1$
	$1 1$		$1 1$

From the tables in the image:

**(a)**

Truth table:

$$0 \rightarrow 0$$

$$1 \rightarrow 1$$

This is the **Identity gate**.

**Is it a valid quantum gate?**

Yes.

Matrix form:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix is unitary since:

$$I^\dagger I = I$$

- ✓ Preserves normalization
- ✓ Reversible
- ✓ Unitary

Valid quantum gate

**(b)**

Truth table:

$$0 \rightarrow 1$$

$$1 \rightarrow 1$$

This maps **both inputs to 1**.

Matrix form would look like:

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

This gate:

- Is not reversible (both inputs give same output)
- Columns are not orthonormal
- Does not preserve normalization

So it is **not unitary**.

Not a valid quantum gate

## 🔥 Key Rule

A classical gate is valid quantum gate **iff it is reversible**, because quantum gates must be unitary (invertible).



**Exercise 2.25.** Consider each of the following classical logic gates with inputs  $A$  and  $B$ , outputs  $C$  and  $D$ , and truth table shown below. Is each gate a valid quantum gate? Why?

(a)	$A$	$B$	$C$	$D$	(b)	$A$	$B$	$C$	$D$
	0	0	0	1		0	0	0	0
	0	1	1	1		0	1	0	0
	1	0	0	0		1	0	1	0
	1	1	1	0		1	1	1	1

A classical gate is a valid quantum gate **iff it is reversible**, i.e., the mapping of basis states is one-to-one (a permutation). That ensures the matrix is unitary.

**(a)**

Truth table:

$$00 \rightarrow 01$$

$$01 \rightarrow 11$$

$$10 \rightarrow 00$$

$$11 \rightarrow 10$$

All four inputs map to **distinct outputs**.

So the mapping is one-to-one and onto (a permutation of the four states).

- ✓ Reversible
- ✓ Matrix is a permutation matrix
- ✓ Therefore unitary

(a) is a valid quantum gate

**(b)**

Truth table:

$$00 \rightarrow 00$$

$$01 \rightarrow 00$$

$$10 \rightarrow 10$$

$$11 \rightarrow 11$$

Here:

$$00 \text{ and } 01 \rightarrow 00$$

Two different inputs give the same output.

- ✗ Not reversible
- ✗ Not one-to-one
- ✗ Columns of matrix would not be orthonormal

(b) is NOT a valid quantum gate

## 🔥 Key Rule (Exam Line)

A classical logic gate is a valid quantum gate **iff it represents a reversible permutation of computational basis states.**