Linear Algebra for Quantum Computing

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Why Do We Need Linear Algebra?

- Classical computing uses Boolean algebra.
- Quantum computing uses **Linear Algebra**.
- Quantum states are represented as **vectors**.
- Quantum gates are represented as **matrices**.

Goal: Learn vectors + matrices = understand quantum mechanics mathematically.

What is a Vector?

- A vector is simply a **list of numbers arranged vertically or Horizontally**.
- The dimension of vector is the number of element in the list.

In quantum mechanics, we represent vectors as **Bras** and **Kets**.

- **Ket (Column Matrix):** Represents a state vector $|\psi\rangle$.

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

- **Bra (Row Matrix):** Represents the conjugate transpose $\langle\psi|$.

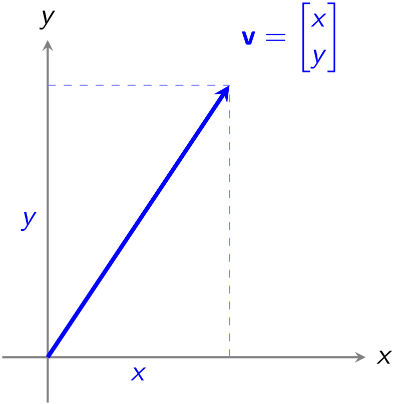
$$\langle\psi| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}$$

Think of it as:

- A point in Hilbert space
- A state of a quantum system (like a Qubit)
- A way to calculate probabilities via the inner product $\langle\phi|\psi\rangle$

Example: 2D Vector Visualization

A vector **v** represents both **direction** and **magnitude**.



The **magnitude** (length) of the vector is calculated using the Pythagorean theorem:

$$||\mathbf{v}|| = \sqrt{x^2 + y^2}$$

Vector Addition

To add two vectors, simply add their corresponding entries (**component-wise**):

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + 2 \\ 3 + 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Geometric Intuition:

- **Head-to-Tail Rule:** Place the tail of the second vector at the head of the first.
- **Resultant Vector:** The sum is the vector pointing from the start of the first to the end of the second.

Scalar Multiplication

To multiply a vector by a scalar, multiply each entry by that number:

$$k \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} k\alpha \\ k\beta \end{bmatrix}$$

Example:

$$2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2(1) \\ 2(3) \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Geometric Interpretation: Scalar multiplication changes the **magnitude** (length) and can flip the **direction**:

- If $|k| > 1$: The vector **stretches**.
- If $0 < |k| < 1$: The vector **shrinks**.
- If $k < 0$: The vector points in the **opposite direction**.

The Conjugate Transpose

To convert a **Ket** (column) into a **Bra** (row), we apply the conjugate transpose, denoted by the "dagger" symbol (†):

$$\langle\psi| = (|\psi\rangle)^\dagger$$

The Two-Step Process:

1. **Transpose:** Flip the column vector into a row vector.
2. **Conjugate:** Change the sign of the imaginary part ($i \rightarrow -i$).

Example: If $|\psi\rangle = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$, then:

$$\langle\psi| = \begin{bmatrix} 1-i & 2 \end{bmatrix}$$

- Note: Real numbers (like 2) remain unchanged.
- This operation is essential for calculating the **inner product** $\langle\psi|\psi\rangle$.

Example: Bra-Ket Transformation

Given the state vector:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

(a) What is $\langle a|$ in terms of $\langle 0|$ and $\langle 1|$?
(b) What is $\langle a|$ as a row vector?

- **Rule:** Ket $|v\rangle$ (Column) $\xrightarrow{\dagger}$ Bra $\langle v|$ (Row).

Inner Product

In Quantum Mechanics, we start with two **Ket** vectors, but we calculate the inner product by transforming the first into a **Bra**.

1. Start with two Kets:

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

2. Transform $|\psi\rangle$ into a Bra: Applying the conjugate transpose ($\dagger$):

$$\langle\psi| = \begin{bmatrix} \alpha_1^* & \alpha_2^* \end{bmatrix}$$

3. The Inner Product (Bra-Ket): The "bracket" notation $\langle\psi|\phi\rangle$ represents the matrix multiplication:

$$\langle\psi|\phi\rangle = \begin{bmatrix} \alpha_1^* & \alpha_2^* \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \alpha_1^*\beta_1 + \alpha_2^*\beta_2$$

- Result is always a **scalar**.
- In QM, $|\langle\psi|\phi\rangle|^2$ gives the **probability** of finding $|\phi\rangle$ in state $|\psi\rangle$.

Example: Inner Product of Bra and Ket

Let's calculate the overlap (inner product) between two quantum states, $|\psi\rangle$ and $|\phi\rangle$.

1. Define the Kets:

$$|\psi\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. Convert $|\psi\rangle$ to a Bra (Conjugate Transpose):

$$\langle\psi| = \begin{bmatrix} 1^* & i^* \end{bmatrix} = \begin{bmatrix} 1 & -i \end{bmatrix}$$

3. Multiply Bra $\times$ Ket:

$$\langle\psi|\phi\rangle = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\langle\psi|\phi\rangle = (1 \times 2) + (-i \times 1) = 2 - i$$

- **Observation:** The result $2 - i$ is a **complex scalar**. .

Example: Inner Product of $|a\rangle$ and $|+\rangle$

Let's find the inner product $\langle +|a\rangle$ for the following states:

$$|a\rangle = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

.

The Self Inner Product and Length

The inner product of a vector with itself, $\langle\psi|\psi\rangle$, is always a non-negative real number and represents the **squared length** of the vector.

1. **The Relationship** The magnitude (norm) of a vector $||\mathbf{v}||$ is defined as:

$$||\mathbf{v}|| = \sqrt{\langle\mathbf{v}|\mathbf{v}\rangle} \quad \Longrightarrow \quad ||\mathbf{v}||^2 = \langle\mathbf{v}|\mathbf{v}\rangle$$

2. **Example Calculation** Using a complex vector $|\psi\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix}$:

$$\langle\psi|\psi\rangle = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (1)(1) + (-i)(i) = 1 + 1 = 2$$

The length of this vector is $||\psi|| = \sqrt{2}$.

Why is this critical in Quantum Mechanics?

- If $\langle\psi|\psi\rangle = 1$, the vector is **Normalized**.
- This ensures the sum of all probabilities equals exactly 100%.
- A "Unit Vector" is simply a vector with a self-inner product of 1.

Orthogonal Vectors

Two vectors are **orthogonal** if their inner product is zero. Geometrically, this means they are perpendicular (at a 90° angle).

The Condition:

$$\langle \mathbf{a} | \mathbf{b} \rangle = 0$$

Example (Standard Basis): The most common orthogonal vectors in 2D are the unit vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\langle 0|1\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (1 \times 0) + (0 \times 1) = 0$$

Why it matters:

- In **Quantum Computing**, $|0\rangle$ and $|1\rangle$ are perfectly distinguishable states.

Example: Inner Product with Complex Coefficients

Given the two quantum states:

$$|a\rangle = \frac{3 + i\sqrt{3}}{4}|0\rangle + \frac{1}{2}|1\rangle, \quad |b\rangle = \frac{1}{4}|0\rangle + \frac{\sqrt{15}}{4}|1\rangle$$

(a) Find $\langle a|b\rangle$:
(b) Find $\langle b|a\rangle$:
(c) Relationship:

Orthonormal Vectors

A set of vectors is **orthonormal** if they satisfy two conditions simultaneously:

1. **Orthogonal:** Every vector is perpendicular to every other vector ($\langle i|j\rangle = 0$ for $i \neq j$).
2. **Normalized:** Each vector has a magnitude (length) of exactly 1 ($\langle i|i\rangle = 1$).

Example: The Computational Basis The states $|0\rangle$ and $|1\rangle$ are orthonormal:

- $\langle 0|1\rangle = 0$ (They are perfectly distinguishable)
- $\langle 0|0\rangle = 1$ and $\langle 1|1\rangle = 1$ (They are valid probability states)

Orthonormal bases simplify calculations because they eliminate "cross-talk" between dimensions.

Example

Consider the following vectors in $\mathbb{R}^2$:

$$|v_1\rangle = \begin{bmatrix} \cos(30^\circ) \\ \sin(30^\circ) \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} -\sin(30^\circ) \\ \cos(30^\circ) \end{bmatrix}$$

find orthonarmality of these vector

Polar Form of Complex Numbers

Any complex number $z = a + ib$ can be represented as a point in the **Complex Plane** (Argand Diagram).

1. **Polar Representation:**

$$z = re^{i\theta}$$

2. **Magnitude (r):**

$$r = |z| = \sqrt{a^2 + b^2}$$

3. **Phase (θ):**

$$\theta = \tan^{-1}(b/a)$$

Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This relates the **angle** θ to the real and imaginary components.

- **Quantum Context:** In a qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the amplitudes α and β are often written in polar form to highlight the

Example: Polar Form of $z = 1 + i$

Consider the complex number $z = 1 + i$. Let's convert it to polar form $re^{i\theta}$.

1. Calculate the Magnitude (r):

$$r = |z| = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

2. Calculate the Phase Angle (θ):

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ \text{ or } \frac{\pi}{4} \text{ radians}$$

3. Resulting Polar Form:

$$z = \sqrt{2}e^{i\pi/4}$$

Quantum States as Vectors

A **qubit** is a two-level quantum system represented as a column vector in a complex vector space ($\mathbb{C}^2$):

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Components:

- $\alpha, \beta \in \mathbb{C}$: Complex **probability amplitudes**.
- $|\alpha|^2$: Probability of measuring state $|0\rangle$.
- $|\beta|^2$: Probability of measuring state $|1\rangle$.

The Normalization Constraint: For $|\psi\rangle$ to be a valid physical state, the total probability must be 1:

$$|\alpha|^2 + |\beta|^2 = 1$$

Geometrically, this means the vector $|\psi\rangle$ always sits on the surface of a unit sphere (The Bloch Sphere).

Quantum Gates as Matrices

A quantum gate U is a transformation that maps basis states $|0\rangle$ and $|1\rangle$ to new superpositions. We can represent U as a 2×2 matrix by placing the resulting amplitudes side-by-side.

1. Define the action on Basis States:

$$U|0\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad U|1\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

2. Construct the Matrix:

$$U = \begin{bmatrix} | & | \\ U|0\rangle & U|1\rangle \\ | & | \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

- The first column is the "output" of $|0\rangle$.
- The second column is the "output" of $|1\rangle$.
- This allows us to use **Matrix-Vector Multiplication** for any state.

Example: Constructing an Operator from Mappings

Consider an operator U that performs the following mapping on the Z-basis:

$$U|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad U|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

(a) What is U as a matrix? The columns of a matrix are the results of applying the operator to the basis vectors $|0\rangle$ and $|1\rangle$.

$$U = \begin{bmatrix} U|0\rangle & U|1\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

(b) What is $U \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$? Applying the matrix to a general state:

$$U \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha - i\beta \\ -i\alpha + \beta \end{bmatrix}$$

Note: This operator is a specific type of rotation gate often used in quantum algorithms.

Example: The Hadamard Gate (H)

The Hadamard gate is the most common way to create **superposition**. It maps basis states to "diagonal" states.

1. Define the action on Basis States:

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad H|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

2. Construct the Matrix: Placing these columns side-by-side (and factoring out the constant):

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

This matrix is its own inverse ($H^2 = I$), a unique property of the Hadamard gate.

Special Linear Operators

In the space of qubits, two operators serve as the "one" and "zero" of matrix algebra.

1. The Identity Operator (I) The operator that maps every vector to itself ($I|\psi\rangle = |\psi\rangle$).

$$I = |0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Geometrically, this represents doing nothing to the state.

2. The Zero Operator ($\mathbf{0}$) The operator that maps every vector to the zero vector ($\mathbf{0}|\psi\rangle = \vec{0}$).

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note: The result is the zero vector, which is not a valid physical quantum state (length is 0).

Eigenvalues and Eigenvectors

For a linear operator (matrix) A, a non-zero vector $|v\rangle$ is an **eigenvector** if the action of A only scales the vector by a constant λ.

The Eigenvalue Equation:

$$A|v\rangle = \lambda|v\rangle$$

Key Components:

- $|v\rangle$: The **eigenvector** (the "characteristic direction").
- λ: The **eigenvalue** (a scalar representing the scaling factor).

Quantum Physical Meaning:

- Eigenvectors represent the **definite states** of an observable.
- Eigenvalues represent the **possible measurement results**.

Solving for Eigenvalues

To find the eigenvalues of a matrix A, we solve the **Characteristic Equation**:

$$\det(A - \lambda I) = 0$$

Step-by-Step Process:

1. Subtract λ from the diagonal elements of A.
2. Calculate the determinant of the resulting matrix.
3. Solve the resulting polynomial (characteristic polynomial) for λ.
4. For each λ, solve $(A - \lambda I)|v\rangle = 0$ to find the corresponding eigenvector.

For a 2×2 quantum gate, you will typically find two eigenvalues.

Example: The Pauli-X Operator

Consider the operator $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

1. Solve the Characteristic Equation:

$$\det(X - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = 0$$

The **eigenvalues** are $\lambda_1 = 1$ and $\lambda_2 = -1$.

2. Find the Eigenvector for $\lambda_1 = 1$: Solve $(X - I)|v_1\rangle = 0$:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies -a + b = 0 \implies a = b$$

Normalized eigenvector: $|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. Find the Eigenvector for $\lambda_2 = -1$: Solve $(X + I)|v_2\rangle = 0$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies a + b = 0 \implies a = -b$$

Normalized eigenvector: $|-\rangle = \frac{1}{\ldots} \begin{bmatrix} 1 \\ \ldots \end{bmatrix}$

The Adjoint Operator

For a linear operator A, its **adjoint** $A^\dagger$ is the unique operator that satisfies the following inner product relationship for all vectors $|u\rangle$ and $|v\rangle$:

$$\langle u|A|v\rangle = \langle A^\dagger u|v\rangle$$

Matrix Representation: If A is represented by a matrix, $A^\dagger$ is found by:

1. **Transposing** the matrix (rows become columns).
2. **Conjugating** each entry ($i \to -i$).

Properties:

- $(A^\dagger)^\dagger = A$
- $(cA)^\dagger = c^* A^\dagger$ (where c is a scalar)
- $(AB)^\dagger = B^\dagger A^\dagger$ (order reverses!)

Example: The Adjoint Operator

Consider the operator A:

$$A = \begin{bmatrix} 2 & 1-i \\ i & 3+2i \end{bmatrix}$$

Step 1: Take the Transpose (A^T) Swap the rows and columns:

$$A^T = \begin{bmatrix} 2 & i \\ 1-i & 3+2i \end{bmatrix}$$

Step 2: Complex Conjugation (*) Flip the sign of all imaginary components ($i \to -i$):

$$A^\dagger = (A^T)^* = \begin{bmatrix} 2 & -i \\ 1+i & 3-2i \end{bmatrix}$$

Key Observations:

- The real number 2 remains unchanged.
- The element $1-i$ moved position and became $1+i$.
- This new matrix $A^\dagger$ represents the "inverse" operation if A is unitary.

Hermitian Operators

An operator (matrix) H is called **Hermitian** if it is equal to its own conjugate transpose.

The Condition:

$$H^\dagger = H$$

Where $H^\dagger = (H^*)^T$.

Properties of Hermitian Matrices:

- The diagonal elements must be **real numbers**.
- The off-diagonal elements must be complex conjugates: $h_{ij} = h_{ji}^*$.
- **Example:** The Pauli matrices X, Y, and Z are all Hermitian.

In Quantum Computing, if you want to extract information from a qubit, you must use a Hermitian operator.

Example: The Pauli-Y Operator

Let's check if the Pauli-Y matrix is Hermitian:

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

1. Take the Transpose (Y^T):

$$Y^T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

2. Take the Complex Conjugate ($(Y^T)^*$): Flip the sign of i:

$$Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Conclusion: Since $Y^\dagger = Y$, the operator is **Hermitian**.
Key Property: Because it is Hermitian, its eigenvalues ($\lambda = \pm 1$) are **real numbers**, which is necessary for physical measurements.

Basis of a Vector Space

A **basis** is a set of vectors $\{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\}$ that satisfies two conditions:

1. **Linear Independence:** No vector in the set can be written as a combination of the others.
2. **Spanning the Space:** Any vector $|\psi\rangle$ in the space can be written as a **linear combination**:

$$|\psi\rangle = c_1|v_1\rangle + c_2|v_2\rangle + \cdots + c_n|v_n\rangle$$

Dimension: The number of vectors in the basis is the dimension of the space. For a single qubit, the dimension is 2.

The Z-Basis (Standard/Computational)

The Z-basis corresponds to the North and South poles of the Bloch Sphere. It is the default basis for measurement in most quantum algorithms.

Basis States:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characteristics:

- These are the eigenvectors of the **Pauli-Z** matrix.
- They represent classical bit states 0 and 1.
- $\langle 0|1\rangle = 0$ (Orthonormal).

The X-Basis (Superposition Basis)

The X-basis represents states on the equator of the Bloch Sphere along the x-axis. We reach these states from the Z-basis using a Hadamard gate.

Basis States:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Characteristics:

- These are the eigenvectors of the **Pauli-X** matrix.
- $|+\rangle$ has a relative phase of 0, while $|-\rangle$ has a phase of π.
- $\langle +|-\rangle = 0$ (Orthonormal).

The Y-Basis (Complex Basis)

The Y-basis involves complex numbers and represents points on the equator of the Bloch Sphere along the y-axis.

Basis States:

$$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \quad |-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

Characteristics:

- These are the eigenvectors of the **Pauli-Y** matrix.
- They incorporate a 90° phase shift.
- $\langle i|-i\rangle = 0$ (Orthonormal).

The Norm of a Vector

The **norm** (or length) of a vector $|\psi\rangle$ is denoted by $||\psi||$. In quantum mechanics, we use the L^2 norm, defined via the inner product.

Definition:

$$||\psi|| = \sqrt{\langle\psi|\psi\rangle}$$

For a vector $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$**:**

$$||\psi|| = \sqrt{|\alpha|^2 + |\beta|^2}$$

Properties:

- **Non-negativity:** $||\psi|| \geq 0$.
- **Definiteness:** $||\psi|| = 0$ if and only if $|\psi\rangle = \vec{0}$.
- **Homogeneity:** $||c \cdot \psi|| = |c| \cdot ||\psi||$ for any scalar c.

Example: Norm of a Complex State Vector

Consider the unnormalized quantum state:

$$|\psi\rangle = (1+i)|0\rangle + 2|1\rangle$$

1. Identify the Coefficients: $\alpha = 1+i$ and $\beta = 2$.
2. Calculate the Inner Product $\langle\psi|\psi\rangle$: Recall that $\langle\psi|$ is the conjugate transpose:

$$\begin{aligned}\langle\psi|\psi\rangle &= (1-i)(1+i) + (2)(2) \\ &= (1^2+1^2) + 4 = 2+4 = 6\end{aligned}$$

3. Find the Norm $||\psi||$:

$$||\psi|| = \sqrt{\langle\psi|\psi\rangle} = \sqrt{6}$$

Normalization: To make this a valid physical state, we divide by the norm:

$$|\psi_{normalized}\rangle = \frac{1}{\sqrt{6}}(1+i)|0\rangle + \frac{2}{\sqrt{6}}|1\rangle$$

The Tensor Product ($\otimes$)

When we have two independent qubits, $|\psi\rangle$ and $|\phi\rangle$, their combined state is represented by the **tensor product**:

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$$

The Kronecker Product (Matrix Form): If $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ and $|\phi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$, then:

$$|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} a \begin{bmatrix} c \\ d \end{bmatrix} \\ b \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

- A 2-qubit system is a vector in a **4-dimensional** space.
- For n qubits, the dimension is 2^n.

Tensor Product of Gates

If we apply gate A to the first qubit and gate B to the second, the total operation is $A \otimes B$.
Example:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

Applying an Identity: If we only apply a Hadamard gate H to the first qubit and do nothing to the second, the operator is:

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \cdot I & 1 \cdot I \\ 1 \cdot I & -1 \cdot I \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Unitary Operators

An operator U is **Unitary** if its adjoint (conjugate transpose) is equal to its inverse.

The Condition:

$$U^\dagger U = UU^\dagger = I$$

Key Properties:

- **Reversibility:** Every unitary operation can be "undone" by applying $U^\dagger$.
- **Norm Preservation:** They preserve the length of vectors: $||U\psi|| = ||\psi||$.
- **Inner Product Preservation:** They preserve the angle between vectors: $\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle$.

Physical interpretation: Unitary operators represent "rotations" in Hilbert space.

Example: Orthogonal Matrix Check

Consider the 2D rotation matrix A for an angle θ:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

1. Find the Transpose (A^T):

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

2. Compute the Product AA^T:

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\sin - \sin\cos \\ \sin\cos - \cos\sin & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

3. Apply Trigonometric Identity ($\cos^2\theta + \sin^2\theta = 1$):

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Matrix Reversibility (Invertibility)

A square matrix A is reversible if there exists another matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I$$

The Determinant Rule: A matrix A is reversible if and only if its determinant is non-zero:

$$\det(A) \neq 0$$

Why this matters: If $\det(A) = 0$, the matrix "collapses" a dimension (e.g., projects a 2D plane onto a 1D line). Once information is collapsed, it cannot be recovered, making the operation **irreversible**.

Example: The Inverse of the T Gate

The T gate (Phase gate) is defined as:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{bmatrix}$$

1. The Unitarity Rule: Since T is unitary, its inverse is its adjoint: $T^{-1} = T^\dagger$.

2. Calculate $T^\dagger$: Transpose the matrix and take the complex conjugate of the entries:

$$T^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{bmatrix}$$

3. Verification ($TT^\dagger = I$):

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i(\pi/4-\pi/4)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The inverse gate $T^\dagger$ (often called T-dg) performs a rotation in the opposite direction on the Bloch sphere.

References

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