

One Quantum Bit MidSem

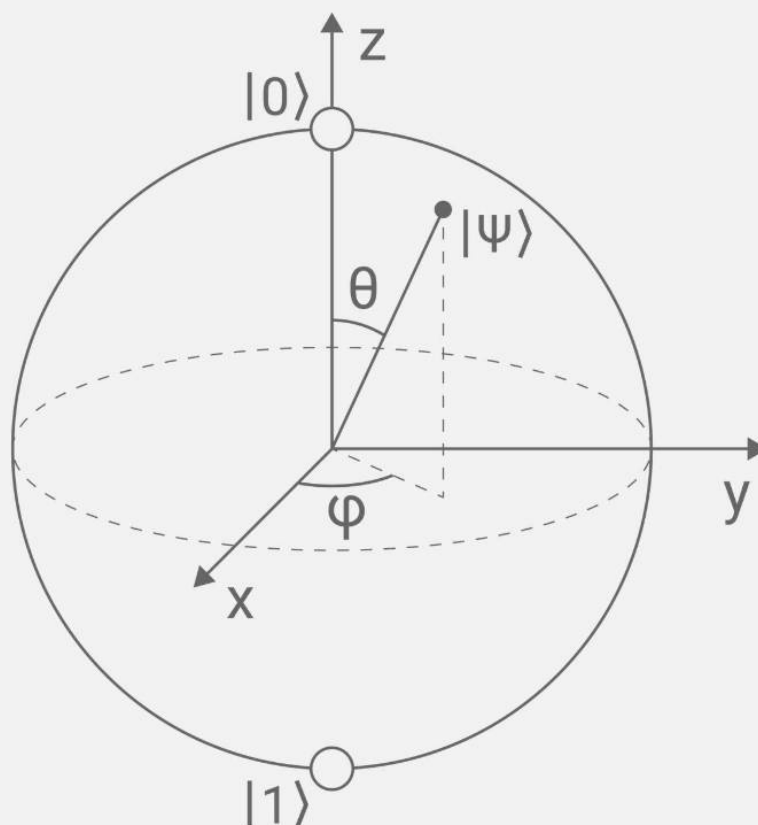


Exercise 2.3. Draw a Bloch sphere and label the following locations:

- (a) Where a qubit is exactly $|0\rangle$.
- (b) Where a qubit is exactly $|1\rangle$.
- (c) Where a qubit is half $|0\rangle$ and half $|1\rangle$.
- (d) Where a qubit is more $|0\rangle$ than $|1\rangle$.
- (e) Where a qubit is more $|1\rangle$ than $|0\rangle$.

Bloch Sphere Representation

BLOCH SPHERE



In quantum mechanics and computing, the Bloch sphere is a geometrical representation of the pure state space of a two-level quantum mechanical system (qubit), named after the physicist Felix Bloch.



<https://www.researchgate.net/publication/335028508/figure/fig1/AS%3A789466423762944%401565234871365/The-Bloch-sphere-provides-a-useful-means-of-visualizing-the-state-of-a-single-qubit-and.ppm>



<https://www.researchgate.net/publication/381311252/figure/fig2/AS%3A11431281250988090%401718097792177/The-Bloch-sphere-The-North-Pole-corresponds-to-the-state-0i-the-South-Pole-to-the.ppm>

Now labeling the required locations:

(a) Exactly $|0\rangle$

• North Pole

- Coordinates: $(0, 0, +1)$
- Angle: $\theta = 0$

(b) Exactly $|1\rangle$

• South Pole

- Coordinates: $(0, 0, -1)$
- Angle: $\theta = \pi$

(c) Half $|0\rangle$ and Half $|1\rangle$

(Equal probability: 50% – 50%)

• Equator of the sphere

- Example: $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$
- Here $\theta = \pi/2$

(Any point on equator gives equal probabilities.)

(d) More $|0\rangle$ than $|1\rangle$

• Upper hemisphere (above equator)

- $0 < \theta < \pi/2$
- Closer to north pole

(e) More $|1\rangle$ than $|0\rangle$

• Lower hemisphere (below equator)

- $\pi/2 < \theta < \pi$
- Closer to south pole

If you want, I can also give a quick 2-line rule to immediately answer any Bloch sphere probability question in exams.



Exercise 2.4. Consider the complex number $z = 1 + 2i$.

- (a) Find $\Re(z)$.
- (b) Find $\Im(z)$.
- (c) Plot z as a point in the complex plane.
- (d) Write z in polar form $re^{i\theta}$.
- (e) Find z^* .
- (f) Find $|z|$.
- (g) Find $|z|^2$.

Exercise 2.5. Consider the complex number $z = -3 - i$.

- (a) Find $\Re(z)$.
- (b) Find $\Im(z)$.
- (c) Plot z as a point in the complex plane.
- (d) Write z in polar form $re^{i\theta}$. Hint: The angle should be between π and $3\pi/2$ (i.e., 180° and 270°).
- (e) Find z^* .
- (f) Find $|z|$.
- (g) Find $|z|^2$.

Exercise 2.4

Given $z = 1 + 2i$

- (a)** $\Re(z) = 1$
- (b)** $\Im(z) = 2$
- (c)** Point in complex plane: **(1, 2)**
- (d)** Polar form:

$$z = \sqrt{5} e^{i \tan^{-1}(2)}$$

(since $r = \sqrt{5}$, $\theta = \tan^{-1}(2)$)

- (e)** $z^* = 1 - 2i$
- (f)** $|z| = \sqrt{5}$
- (g)** $|z|^2 = 5$

Exercise 2.5

Given $z = -3 - i$

- (a)** $\Re(z) = -3$
- (b)** $\Im(z) = -1$
- (c)** Point in complex plane: **(-3, -1)**
- (d)** Polar form:

$$z = \sqrt{10} e^{i(\pi + \tan^{-1}(\frac{1}{3}))}$$

(Angle is between π and $3\pi/2$, third quadrant.)

- (e)** $z^* = -3 + i$
- (f)** $|z| = \sqrt{10}$
- (g)** $|z|^2 = 10$



(d) Polar form: $z = 10 e^{i(\pi + \tan^{-1}(1/3))}$ ($z = \sqrt{10} e^{i(\pi + \tan^{-1}(1/3))}$) (Angle is between π and $3\pi/2$, third quadrant.)
explain

You're asking about this part:

$$z = -3 - i$$

and why its polar form angle is

$$\theta = \pi + \tan^{-1}\left(\frac{1}{3}\right)$$

Let's understand it clearly.

Step 1 Magnitude r

$$r = \sqrt{(-3)^2 + (-1)^2} = \sqrt{9 + 1} = \sqrt{10}$$

So,

$$z = \sqrt{10} e^{i\theta}$$

Now we only need θ (argument).

Step 2 Find the Reference Angle

First compute:

$$\tan \theta = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} = \frac{-1}{-3} = \frac{1}{3}$$

So the **reference angle** is:

$$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$$

This angle is acute (between 0 and $\pi/2$).

Step 3 Determine the Quadrant

Point:

$$(-3, -1)$$

Both real and imaginary parts are negative → **Third Quadrant**

In the third quadrant:

$$\theta = \pi + \alpha$$

Why?

Because:

- π moves us to the negative x-axis.
- Then we go down by the reference angle.

Final Angle

$$\theta = \pi + \tan^{-1}\left(\frac{1}{3}\right)$$

So the polar form is:

$$z = \sqrt{10} e^{i(\pi + \tan^{-1}(1/3))}$$



Quick Exam Trick

If:

- $x > 0, y > 0 \rightarrow \theta = \tan^{-1}(y/x)$
- $x < 0 \rightarrow \theta = \pi + \tan^{-1}(y/x)$
- $x > 0, y < 0 \rightarrow \theta = 2\pi + \tan^{-1}(y/x)$

If you want, I can also show the exact numerical value of the angle in radians and degrees.



Exercise 2.7. A qubit is in the state

$$\frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle.$$

Say you measure the qubit and get $|0\rangle$. If you measure the qubit a *second time*, what is the probability of getting

- $|0\rangle$?
- $|1\rangle$?

State given:

$$|\psi\rangle = \frac{2}{3}|0\rangle + \frac{1+2i}{3}|1\rangle$$

First Measurement

You are told the measurement result is $|0\rangle$.

When a measurement gives $|0\rangle$, the state **collapses** to:

$$|\psi_{\text{after}}\rangle = |0\rangle$$

All superposition is gone.

Second Measurement

Now the system is exactly $|0\rangle$.

So:

(a) Probability of $|0\rangle$ again

1

(b) Probability of $|1\rangle$

0

Important Concept 🔥

Measurement in the computational basis is **projective**.

Once you measure $|0\rangle$:

- The qubit becomes exactly $|0\rangle$
 - Repeated measurements in the same basis always give the same result with probability 1.
-

This is called **wavefunction collapse**.



Exercise 2.8. A qubit is in the state

$$\frac{e^{i\pi/8}}{\sqrt{5}}|0\rangle + \beta|1\rangle.$$

What is a possible value of β ?

Given state:

$$|\psi\rangle = \frac{e^{i\pi/8}}{\sqrt{5}}|0\rangle + \beta|1\rangle$$

A valid qubit must satisfy **normalization**:

$$|\alpha|^2 + |\beta|^2 = 1$$

Step 1: Compute $|\alpha|^2$

$$\alpha = \frac{e^{i\pi/8}}{\sqrt{5}}$$

Magnitude of a complex exponential:

$$|e^{i\theta}| = 1$$

So,

$$|\alpha|^2 = \frac{1}{5}$$

Step 2: Use normalization

$$\frac{1}{5} + |\beta|^2 = 1$$

$$|\beta|^2 = \frac{4}{5}$$

$$|\beta| = \frac{2}{\sqrt{5}}$$

Final Answer

A possible value of β is:

$$\boxed{\beta = \frac{2}{\sqrt{5}}}$$

But remember:

Since only the magnitude is fixed, the most general form is

$$\boxed{\beta = \frac{2}{\sqrt{5}}e^{i\phi}}$$

for any real phase ϕ .

If you want, I can also explain why the global phase $e^{i\pi/8}$ doesn't physically matter.



Exercise 2.9. A qubit is in the state

$$A \left(2e^{i\pi/6} |0\rangle - 3|1\rangle \right).$$

- (a) Normalize the state (i.e., find A).
- (b) If you measure the qubit, what is the probability that you get $|0\rangle$?
- (c) If you measure the qubit, what is the probability that you get $|1\rangle$?

Given state:

$$|\psi\rangle = A \left(2e^{i\pi/6} |0\rangle - 3|1\rangle \right)$$

(a) Normalize the state (find A)

Normalization condition:

$$|A|^2 \left(|2e^{i\pi/6}|^2 + |-3|^2 \right) = 1$$

Now,

$$|2e^{i\pi/6}|^2 = 4$$

(because $|e^{i\theta}| = 1$)

$$|-3|^2 = 9$$

So:

$$|A|^2 (4 + 9) = 1$$

$$|A|^2 \cdot 13 = 1$$

$$|A|^2 = \frac{1}{13}$$

Taking positive normalization constant,

$$A = \frac{1}{\sqrt{13}}$$

(b) Probability of getting $|0\rangle$

Coefficient of $|0\rangle$:

$$\frac{2e^{i\pi/6}}{\sqrt{13}}$$

Probability:

$$\frac{4}{13}$$

$$\boxed{\frac{4}{13}}$$

(c) Probability of getting $|1\rangle$

Coefficient:

$$\frac{-3}{\sqrt{13}}$$

Probability:

$$\frac{9}{13}$$

$$\boxed{\frac{9}{13}}$$

Quick check:

$$\frac{4}{13} + \frac{9}{13} = 1$$

✓ Correct normalization.



Exercise 2.10. A qubit is in the state

$$\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle.$$

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2 One Quantum Bit

- If you measure it in the Z-basis $\{|0\rangle, |1\rangle\}$, what states can you get and with what probabilities?
- Write the qubit's state in terms of $|+\rangle$ and $|-\rangle$.
- If you measure it in the basis $\{|+\rangle, |-\rangle\}$, what states can you get and with what probabilities?

Given state:

$$|\psi\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

(Already normalized since $\frac{1}{4} + \frac{3}{4} = 1$)

(a) Measurement in Z-basis $\{|0\rangle, |1\rangle\}$

Probabilities are squares of coefficients:

$$P(0) = \left| \frac{1}{2} \right|^2 = \frac{1}{4}$$

$$P(1) = \left| \frac{\sqrt{3}}{2} \right|^2 = \frac{3}{4}$$

Answer:

- $|0\rangle$ with probability $\boxed{\frac{1}{4}}$
- $|1\rangle$ with probability $\boxed{\frac{3}{4}}$

(b) Write state in $\{|+\rangle, |-\rangle\}$

Recall:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Substitute:

$$|\psi\rangle = \frac{1}{2} \frac{|+\rangle + |-\rangle}{\sqrt{2}} - \frac{\sqrt{3}}{2} \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

Combine coefficients:

Coefficient of $|+\rangle$:

$$\frac{1 - \sqrt{3}}{2\sqrt{2}}$$

Coefficient of $|-\rangle$:

$$\frac{1 + \sqrt{3}}{2\sqrt{2}}$$

So,

$$|\psi\rangle = \frac{1 - \sqrt{3}}{2\sqrt{2}} |+\rangle + \frac{1 + \sqrt{3}}{2\sqrt{2}} |-\rangle$$

(c) Measurement in $\{|+\rangle, |-\rangle\}$

Square the magnitudes:

$$P(+)=\left|\frac{1-\sqrt{3}}{2\sqrt{2}}\right|^2=\frac{2-\sqrt{3}}{4}$$

$$P(-)=\left|\frac{1+\sqrt{3}}{2\sqrt{2}}\right|^2=\frac{2+\sqrt{3}}{4}$$

Final Answers

- $|+\rangle$ with probability

$$\frac{2-\sqrt{3}}{4}$$

- $|-\rangle$ with probability

$$\frac{2+\sqrt{3}}{4}$$

If you want, I can also show the Bloch sphere angle interpretation (this state actually corresponds to a specific polar angle).



Exercise 2.11. The following two states are opposite points on the Bloch sphere:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle,$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

So, we can measure relative to them. Now consider a qubit in the state

$$\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle.$$

- Write the qubit's state in terms of $|a\rangle$ and $|b\rangle$.
- If you measure the qubit in the basis $\{|a\rangle, |b\rangle\}$, what states can you get and with what probabilities?

Given:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

and

$$|\psi\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

Since $|a\rangle, |b\rangle$ form an orthonormal basis, coefficients are:

$$c_a = \langle a|\psi\rangle, \quad c_b = \langle b|\psi\rangle$$

(a) Expand $|\psi\rangle$ in $\{|a\rangle, |b\rangle\}$

Compute c_a

$$\begin{aligned} \langle a| &= \frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1| \\ c_a &= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \left(-\frac{i}{2}\right) \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{4} + \frac{i\sqrt{3}}{4} = \frac{\sqrt{3}}{4}(1+i) \end{aligned}$$

Compute c_b

$$\begin{aligned} \langle b| &= -\frac{i}{2}\langle 0| + \frac{\sqrt{3}}{2}\langle 1| \\ c_b &= -\frac{i}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= -\frac{i}{4} - \frac{3}{4} = -\frac{3+i}{4} \end{aligned}$$

So,

$$|\psi\rangle = \frac{\sqrt{3}}{4}(1+i)|a\rangle - \frac{3+i}{4}|b\rangle$$

(b) Measurement in basis $\{|a\rangle, |b\rangle\}$

Probabilities are squared magnitudes.

Probability of $|a\rangle$

$$\begin{aligned} \left| \frac{\sqrt{3}}{4}(1+i) \right|^2 &= \frac{3}{16} \cdot |1+i|^2 \\ |1+i|^2 &= 2 \end{aligned}$$

$$P(a) = \frac{6}{16} = \frac{3}{8}$$

Probability of $|b\rangle$

$$\left| -\frac{3+i}{4} \right|^2 = \frac{1}{16} |3+i|^2$$

$$|3+i|^2 = 10$$

$$P(b) = \frac{10}{16} = \frac{5}{8}$$

Final Answers

$$|\psi\rangle = \frac{\sqrt{3}}{4}(1+i)|a\rangle - \frac{3+i}{4}|b\rangle$$

Measurement gives:

$$P(a) = \frac{3}{8}$$

$$P(b) = \frac{5}{8}$$

(Sum = 1 ✓)

If you want, I can also show the shortcut geometric Bloch sphere reasoning that avoids this algebra.



how are a and b orthonormal

Good question 👍 let's verify it properly.


We are given:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

To be **orthonormal**, two things must hold:

1. ✓ Each state has norm 1

2.  Their inner product is 0

Check Normalization

For $|a\rangle$

$$\begin{aligned}\langle a|a\rangle &= \left|\frac{\sqrt{3}}{2}\right|^2 + \left|\frac{i}{2}\right|^2 \\ &= \frac{3}{4} + \frac{1}{4} = 1\end{aligned}$$

So $|a\rangle$ is normalized.

For $|b\rangle$

$$\langle b|b\rangle = \left|\frac{i}{2}\right|^2 + \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{1}{4} + \frac{3}{4} = 1$$

So $|b\rangle$ is also normalized.

Check Orthogonality

Now compute:

$$\langle a|b\rangle$$

First take conjugate of $|a\rangle$:

$$\langle a| = \frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|$$

Now multiply with $|b\rangle$:

$$\left(\frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|\right) \left(\frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)$$

Multiply terms:

$$\begin{aligned}&= \frac{\sqrt{3}i}{4} - \frac{i\sqrt{3}}{4} \\ &= 0\end{aligned}$$

Conclusion

$$\langle a|b\rangle = 0$$

So they are orthogonal.

Since:

- $\langle a|a\rangle = 1$
- $\langle b|b\rangle = 1$
- $\langle a|b\rangle = 0$

They are **orthonormal**.

Geometrically:

They correspond to opposite points on the Bloch sphere → that always means orthonormal basis states.



$|\psi\rangle = \frac{1}{2}(1+i)|a\rangle - \frac{1}{2}(1-i)|b\rangle$
how this

Ahhh 👍 you're asking:

How did

$$|\psi\rangle = \frac{\sqrt{3}}{4}(1+i)|a\rangle - \frac{3+i}{4}|b\rangle$$

come out?

Let's derive it cleanly.

Step 1 Idea

Since $|a\rangle, |b\rangle$ are orthonormal, any state can be written as:

$$|\psi\rangle = \langle a|\psi\rangle|a\rangle + \langle b|\psi\rangle|b\rangle$$

So we compute:

$$c_a = \langle a|\psi\rangle \quad c_b = \langle b|\psi\rangle$$

Step 2 Write Everything

$$|\psi\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Step 3 Compute $c_a = \langle a|\psi\rangle$

Take conjugate of $|a\rangle$:

$$\langle a| = \frac{\sqrt{3}}{2}\langle 0| - \frac{i}{2}\langle 1|$$

Now multiply:

$$\begin{aligned} c_a &= \frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \left(-\frac{i}{2}\right) \left(-\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{4} + \frac{i\sqrt{3}}{4} \\ &= \frac{\sqrt{3}}{4}(1+i) \end{aligned}$$

Step 4 Compute $c_b = \langle b|\psi\rangle$

Conjugate of $|b\rangle$:

$$\langle b| = -\frac{i}{2}\langle 0| + \frac{\sqrt{3}}{2}\langle 1|$$

Now multiply:

$$\begin{aligned} c_b &= -\frac{i}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \left(-\frac{\sqrt{3}}{2}\right) \\ &= -\frac{i}{4} - \frac{3}{4} \\ &= -\frac{3+i}{4} \end{aligned}$$

Step 5 Substitute

$$|\psi\rangle = \frac{\sqrt{3}}{4}(1+i)|a\rangle - \frac{3+i}{4}|b\rangle$$

That's it.

Why This Works

Because in an orthonormal basis:

$$c_k = \langle k | \psi \rangle$$

always gives the expansion coefficient.

If you'd like, I can also show a **faster 2×2 matrix method** that avoids bra-ket multiplication.



Exercise 2.12. A qubit is in the state $|0\rangle$. If you measure it in the X -basis $\{|+\rangle, |-\rangle\}$ and then measure it again in the Z -basis $\{|0\rangle, |1\rangle\}$, what is the probability of getting

- (a) $|0\rangle$?
- (b) $|1\rangle$?

Initial state:

$$|\psi\rangle = |0\rangle$$

Step 1 Measure in X-basis $\{|+\rangle, |-\rangle\}$

Recall:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

So,

$$P(+)=\frac{1}{2}, \quad P(-)=\frac{1}{2}$$

After this measurement, the state collapses to:

- $|+\rangle$ with probability $1/2$
- $|-\rangle$ with probability $1/2$

Step 2 Measure again in Z-basis $\{|0\rangle, |1\rangle\}$

Recall:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

For either case:

$$P(0|+) = \frac{1}{2}, \quad P(1|+) = \frac{1}{2}$$

$$P(0|-) = \frac{1}{2}, \quad P(1|-) = \frac{1}{2}$$

Step 3 Total Probability

Use total probability rule.

$$P(0) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$$P(1) = \frac{1}{2}$$

✓ Final Answer

$$P(0) = \frac{1}{2}$$

$$P(1) = \frac{1}{2}$$

🔥 Key Idea

Measuring in the X-basis **destroys the original Z-basis information**.

After the first measurement, the system forgets it started in $|0\rangle$.



Exercise 2.13. Is there a measurement that can distinguish the following pairs of states? If yes, give a measurement. If no, explain your reasoning.

(a) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $e^{i\pi/8}|+\rangle = \frac{e^{i\pi/8}}{\sqrt{2}}(|0\rangle + |1\rangle)$.

(b) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

(c) $|0\rangle$ and $e^{i\pi/4}|0\rangle$.

The key rule:

👉 Two quantum states can be perfectly distinguished **iff they are orthogonal**.

If they differ only by a **global phase**, they are physically identical → impossible to distinguish.

(a) $|+\rangle$ and $e^{i\pi/8}|+\rangle$

$$e^{i\pi/8}|+\rangle$$

This is just a **global phase** multiplied to the same state.

Global phase has **no physical effect**.

These two states represent the **same point on the Bloch sphere**.

✓ **Conclusion:**

No measurement can distinguish them.

(b) $|+\rangle$ and $|-\rangle$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Compute inner product:

$$\langle + | - \rangle = 0$$

They are orthogonal.

✓ **Conclusion:**

Yes, we can distinguish them perfectly.

Measurement:

Measure in the **X-basis** $\{|+\rangle, |-\rangle\}$.

Result directly tells the state.

(c) $|0\rangle$ and $e^{i\pi/4}|0\rangle$

Again:

$$e^{i\pi/4}|0\rangle$$

This is just global phase.

Same physical state.

✓ **Conclusion:**

No measurement can distinguish them.

🔥 Final Summary

Pair	Distinguishable?	Reason
(a)	✗ No	Global phase only
(b)	✓ Yes	Orthogonal states
(c)	✗ No	Global phase only

If you want, I can also show the Bloch sphere geometric intuition for this — it makes it instantly obvious.



Exercise 2.14. A qubit is in the state

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle).$$

- (a) Where on the Bloch sphere is this state? Give your answer in (θ, ϕ) coordinates.
 (b) Sketch the point on the Bloch sphere.

Given state:

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

Step 1 Compare with General Bloch Form

A general qubit can be written as:

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$

Now compare.

We have:

$$\cos \frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

$$\sin \frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

So,

$$\frac{\theta}{2} = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{2}$$

Now look at the phase:

$$e^{i\phi} = i = e^{i\pi/2}$$

So,

$$\phi = \frac{\pi}{2}$$

✓ Final Answer (a)

$$(\theta, \phi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

(b) Where is it on Bloch Sphere?

- $\theta = \pi/2 \rightarrow$ on the **equator**
- $\phi = \pi/2 \rightarrow$ along **positive Y-axis**

So this is the **+Y direction** on the Bloch sphere.

It is usually called:

$$|+i\rangle$$

Geometric Summary

- $|+\rangle \rightarrow$ +X direction
- $|-\rangle \rightarrow$ -X direction
- $|i\rangle \rightarrow$ +Y direction
- $|-i\rangle \rightarrow$ -Y direction

If you want, I can also show the quick shortcut trick to immediately read θ and ϕ without comparing formulas.



no i prefer above method only

Perfect 👍 that **is** the cleanest method.

Let me just restate it cleanly in final-exam style using only that method.

Given:

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

General Bloch form:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

Compare coefficients

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

$$\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$$

So,

$$\frac{\theta}{2} = \frac{\pi}{4}$$

$$\boxed{\theta = \frac{\pi}{2}}$$

Now compare the phase term:

$$e^{i\phi} = i$$

We know:

$$i = e^{i\pi/2}$$

So,

$$\boxed{\phi = \frac{\pi}{2}}$$

Final Answer

$$\boxed{(\theta, \phi) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)}$$

That's exactly the correct Bloch sphere coordinates using your preferred comparison method.



Exercise 2.15. A qubit is in the state

$$\frac{1-i}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

(a) Where on the Bloch sphere is this state? Give your answer in (θ, ϕ) coordinates.

Given state:

$$|\psi\rangle = \frac{1-i}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

General Bloch form:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

Step 1 Find θ

First compute magnitudes.

Magnitude of $|0\rangle$ coefficient:

$$\left| \frac{1-i}{2\sqrt{2}} \right| = \frac{|1-i|}{2\sqrt{2}}$$

$$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

So,

$$\frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

Thus:

$$\cos\frac{\theta}{2} = \frac{1}{2}$$

$$\frac{\theta}{2} = \frac{\pi}{3}$$

$$\boxed{\theta = \frac{2\pi}{3}}$$

Step 2 Find ϕ

Rewrite the $|0\rangle$ coefficient in polar form.

$$1 - i = \sqrt{2}e^{-i\pi/4}$$

So,

$$\frac{1 - i}{2\sqrt{2}} = \frac{1}{2}e^{-i\pi/4}$$

Thus the state becomes:

$$|\psi\rangle = \frac{1}{2}e^{-i\pi/4}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Factor out global phase $e^{-i\pi/4}$ (physically irrelevant):

$$|\psi\rangle = e^{-i\pi/4} \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}e^{i\pi/4}|1\rangle \right)$$

Now compare with standard form.

So,

$$e^{i\phi} = e^{i\pi/4}$$

$$\boxed{\phi = \frac{\pi}{4}}$$

✓ Final Answer

$$\boxed{(\theta, \phi) = \left(\frac{2\pi}{3}, \frac{\pi}{4} \right)}$$



(b) Sketch the point on the Bloch sphere.

From part (a), we found:

$$(\theta, \phi) = \left(\frac{2\pi}{3}, \frac{\pi}{4} \right)$$

How to Sketch It on the Bloch Sphere

1 Polar angle $\theta = \frac{2\pi}{3}$

- θ is measured from the **+Z axis**.

- $\frac{2\pi}{3} = 120^\circ$.
- This is **below the equator** (since equator is $\theta = \pi/2$).
- So the point is in the **southern hemisphere**.

2 Azimuth angle $\phi = \frac{\pi}{4}$

- Measured in the **XY-plane** from +X direction.
- $\frac{\pi}{4} = 45^\circ$.
- So projection onto XY-plane lies halfway between +X and +Y.

Final Position Description

- Southern hemisphere
- 120° down from north pole
- 45° counterclockwise from +X axis
- Between +X and +Y directions (but below equator)

If you want a quick Cartesian coordinate check for sketching:

$$x = \sin \theta \cos \phi = \frac{\sqrt{6}}{4}$$

$$y = \sin \theta \sin \phi = \frac{\sqrt{6}}{4}$$

$$z = \cos \theta = -\frac{1}{2}$$

So it's symmetric in x and y, with negative z.

That gives you the exact sketch location.



Exercise 2.16. Consider the following two states from Exercise 2.11:

$$|a\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle,$$

$$|b\rangle = \frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

Prove these are opposite points of the Bloch sphere by finding their points in spherical coordinates (θ_a, ϕ_a) and (θ_b, ϕ_b) . Verify that $\theta_b = \pi - \theta_a$ and $\phi_b = \phi_a + \pi$, which means they lie on opposite points of the Bloch sphere.

We compare each state with the standard Bloch form:

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$

1 State $|a\rangle$

$$|a\rangle = \frac{\sqrt{3}}{2} |0\rangle + \frac{i}{2} |1\rangle$$

Step A: Find θ_a

$$\cos \frac{\theta_a}{2} = \frac{\sqrt{3}}{2}$$

$$\frac{\theta_a}{2} = \frac{\pi}{6}$$

$$\boxed{\theta_a = \frac{\pi}{3}}$$

Step B: Find ϕ_a

$$e^{i\phi_a} \sin \frac{\theta_a}{2} = \frac{i}{2}$$

Since

$$\sin \frac{\theta_a}{2} = \sin \frac{\pi}{6} = \frac{1}{2}$$

So

$$e^{i\phi_a} = i$$

$$\boxed{\phi_a = \frac{\pi}{2}}$$

Thus,

$$\boxed{(\theta_a, \phi_a) = \left(\frac{\pi}{3}, \frac{\pi}{2}\right)}$$

2 State $|b\rangle$

$$|b\rangle = \frac{i}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle$$

Step A: Remove global phase

Factor out i :

$$|b\rangle = i \left(\frac{1}{2}|0\rangle - \frac{i\sqrt{3}}{2}|1\rangle \right)$$

Global phase i does not affect Bloch coordinates.

So consider:

$$|b\rangle \sim \frac{1}{2}|0\rangle - \frac{i\sqrt{3}}{2}|1\rangle$$

Step B: Find θ_b

$$\cos \frac{\theta_b}{2} = \frac{1}{2}$$

$$\frac{\theta_b}{2} = \frac{\pi}{3}$$

$$\boxed{\theta_b = \frac{2\pi}{3}}$$

Step C: Find ϕ_b

Coefficient of $|1\rangle$:

$$e^{i\phi_b} \sin \frac{\theta_b}{2} = -\frac{i\sqrt{3}}{2}$$

Since

$$\sin \frac{\theta_b}{2} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

So

$$e^{i\phi_b} = -i$$

$$\boxed{\phi_b = \frac{3\pi}{2}}$$

Final Coordinates

$$(\theta_a, \phi_a) = \left(\frac{\pi}{3}, \frac{\pi}{2} \right)$$

$$(\theta_b, \phi_b) = \left(\frac{2\pi}{3}, \frac{3\pi}{2} \right)$$

3 Verify Opposite Points Condition

Check:

$$\theta_b = \pi - \theta_a$$

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

✓ Correct

$$\phi_b = \phi_a + \pi$$

$$\frac{\pi}{2} + \pi = \frac{3\pi}{2}$$

✓ Correct

✓ Conclusion

$$\theta_b = \pi - \theta_a, \quad \phi_b = \phi_a + \pi$$

Hence they lie on **opposite points of the Bloch sphere**.



Exercise 2.22. Consider a map U that transforms the Z-basis states as follows:

$$U|0\rangle = |0\rangle + |1\rangle,$$

$$U|1\rangle = |0\rangle - |1\rangle.$$

Say $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a normalized quantum state, i.e., $|\alpha|^2 + |\beta|^2 = 1$.

(a) Calculate $U|\psi\rangle$.

(b) From your answer to (a), is U a valid quantum gate? Explain your reasoning.

Given:

$$U|0\rangle = |0\rangle + |1\rangle$$

$$U|1\rangle = |0\rangle - |1\rangle$$

and

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with $|\alpha|^2 + |\beta|^2 = 1$.

(a) Calculate $U|\psi\rangle$

Using linearity:

$$\begin{aligned} U|\psi\rangle &= \alpha U|0\rangle + \beta U|1\rangle \\ &= \alpha(|0\rangle + |1\rangle) + \beta(|0\rangle - |1\rangle) \end{aligned}$$

Group terms:

Coefficient of $|0\rangle$:

$$\alpha + \beta$$

Coefficient of $|1\rangle$:

$$\alpha - \beta$$

So,

$$U|\psi\rangle = (\alpha + \beta)|0\rangle + (\alpha - \beta)|1\rangle$$

(b) Is U a valid quantum gate?

Quantum gates must be **unitary** → must preserve normalization.

Compute norm of the output:

$$\|U|\psi\rangle\|^2 = |\alpha + \beta|^2 + |\alpha - \beta|^2$$

Use identity:

$$|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$$

So,

$$\begin{aligned} &= 2(|\alpha|^2 + |\beta|^2) \\ &= 2 \end{aligned}$$

But input norm was 1.

Output norm is 2.

So normalization is **not preserved**.

Conclusion

$$U \text{ is NOT unitary}$$

Hence,

$$U \text{ is NOT a valid quantum gate}$$

Note: If there were a factor $\frac{1}{\sqrt{2}}$, this would become the Hadamard gate and would be unitary.



Exercise 2.24. Consider each of the following classical logic gates with input A , output B , and truth table shown below. Is each gate a valid quantum gate? Why?

(a)	$A B$	(b)	$A B$
	$0 0$		$0 1$
	$1 1$		$1 1$

From the tables in the image:

(a)

Truth table:

$$0 \rightarrow 0$$

$$1 \rightarrow 1$$

This is the **Identity gate**.

Is it a valid quantum gate?

Yes.

Matrix form:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix is unitary since:

$$I^\dagger I = I$$

✓ Preserves normalization

✓ Reversible

✓ Unitary

Valid quantum gate

(b)

Truth table:

$$0 \rightarrow 1$$

$$1 \rightarrow 1$$

This maps **both inputs to 1**.

Matrix form would look like:

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

This gate:

- Is not reversible (both inputs give same output)
- Columns are not orthonormal
- Does not preserve normalization

So it is **not unitary**.

Not a valid quantum gate

Key Rule

A classical gate is valid quantum gate **iff it is reversible**, because quantum gates must be unitary (invertible).



Exercise 2.25. Consider each of the following classical logic gates with inputs A and B , outputs C and D , and truth table shown below. Is each gate a valid quantum gate? Why?

(a)				(b)			
A	B	C	D	A	B	C	D
0	0	0	1	0	0	0	0
0	1	1	1	0	1	0	0
1	0	0	0	1	0	1	0
1	1	1	0	1	1	1	1

A classical gate is a valid quantum gate **iff it is reversible**, i.e., the mapping of basis states is one-to-one (a permutation). That ensures the matrix is unitary.

(a)

Truth table:

$00 \rightarrow 01$

$01 \rightarrow 11$

$10 \rightarrow 00$

$11 \rightarrow 10$

All four inputs map to **distinct outputs**.

So the mapping is one-to-one and onto (a permutation of the four states).

- ✓ Reversible
- ✓ Matrix is a permutation matrix
- ✓ Therefore unitary

(a) is a valid quantum gate

(b)

Truth table:

00 → 00

01 → 00

10 → 10

11 → 11

Here:

00 and 01 → 00

Two different inputs give the same output.

- ✗ Not reversible
- ✗ Not one-to-one
- ✗ Columns of matrix would not be orthonormal

(b) is NOT a valid quantum gate

Key Rule (Exam Line)

A classical logic gate is a valid quantum gate **iff it represents a reversible permutation of computational basis states.**