

# Mathematical Foundations of Computer Science

This Lecture: Graph Theory: Line Graphs and Traversability

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MA714 (Odd Semester [2025-26])



A **Line Graph** represents adjacency relationships between the edges of a graph.

It transforms edge-based information of a graph into vertex-based information.

Traversability of these graphs (Eulerian and Hamiltonian properties) provides insights into network structure and flow.



Let  $G = (V, E)$  be a simple graph.

### Definition

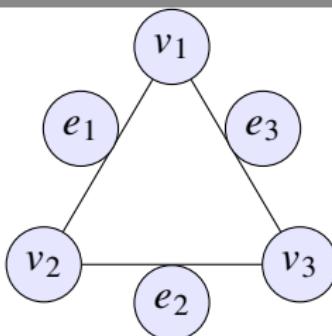
The **Line Graph** of  $G$ , denoted by  $L(G)$ , is the graph whose:

- vertices correspond to the edges of  $G$ ;
- two vertices of  $L(G)$  are adjacent if and only if their corresponding edges in  $G$  share a common vertex.

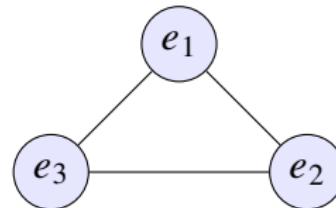
$$V(L(G)) = E(G), \quad e_i \sim e_j \text{ in } L(G) \iff e_i, e_j \text{ are incident in } G.$$



# Example 1: Line Graph of a Triangle $K_3$



Graph  $G = K_3$

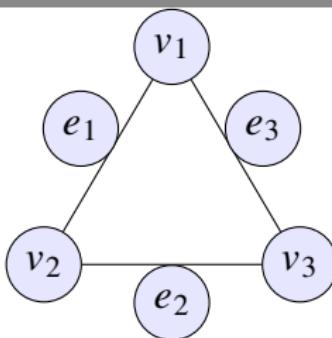


Line Graph  $L(G)$

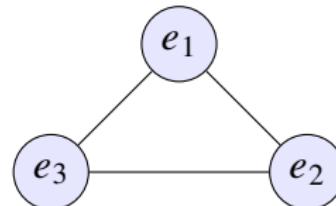
$$L(K_3) \cong K_3$$



# Example 1: Line Graph of a Triangle $K_3$



Graph  $G = K_3$

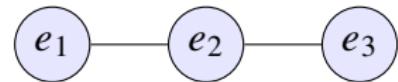
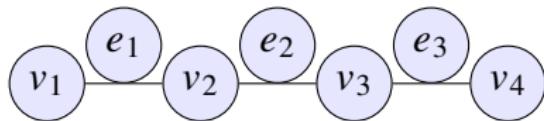


Line Graph  $L(G)$

$$L(K_3) \cong K_3$$



## Example 2: Line Graph of a Path $P_4$

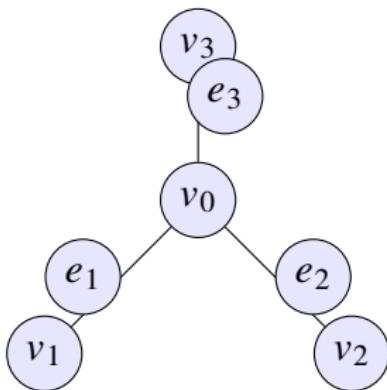


Graph  $G = P_4$

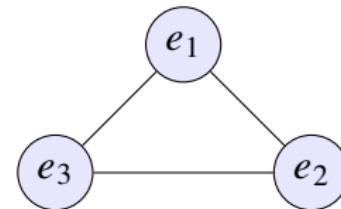
Line Graph  $L(G) = P_3$



# Example 3: Line Graph of a Star $K_{1,3}$



Graph  $G = K_{1,3}$



Line Graph  $L(G) = K_3$



If  $G$  has  $m$  edges,  $L(G)$  has  $m$  vertices.

Number of edges in  $L(G)$ :

$$|E(L(G))| = \sum_{v \in V(G)} \binom{d_G(v)}{2}$$

For edge  $e = uv$  in  $G$ :

$$d_{L(G)}(e) = d_G(u) + d_G(v) - 2$$

$L(G)$  is connected iff  $G$  is connected and has no isolated edges.

$$L(P_n) = P_{n-1}, \quad L(C_n) = C_n, \quad L(K_n) = K_{\binom{n}{2}}$$



## Theorem 1

If  $G$  is Eulerian, then  $L(G)$  is both Eulerian and Hamiltonian.

Traversing each edge of  $G$  once corresponds to visiting each vertex of  $L(G)$  once.

Hence, an Eulerian circuit in  $G \Rightarrow$  a Hamiltonian cycle in  $L(G)$ .

## Theorem 2

$L(G)$  is Eulerian iff every vertex of  $G$  has even degree  $\geq 2$ .



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## Theorem 2

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1. **Network Topology Design:** Models link-to-link connectivity in communication or optical networks.
2. **Resource Allocation:** Used in frequency assignment and edge coloring.
3. **Compiler Design:** Register interference graphs can be viewed as line graphs of flow graphs.
4. **Transportation and Circuits:** Models traffic flow between routes or current between connections.
5. **Social Networks:** Captures relationship among relationships (shared participants).



$L(G)$ : edges of  $G \rightarrow$  vertices.

Eulerian–Hamiltonian correspondence:

$$G \text{ Eulerian} \Rightarrow L(G) \text{ Hamiltonian}$$

$L(G)$  highlights edge interactions and traversal properties.

Applications span communication networks, compiler optimization, and transportation systems.



1. Construct  $L(G)$  for:

$P_5, C_5, K_{1,4}$

2. Prove: If  $G$  is Eulerian, then  $L(G)$  is Hamiltonian.
3. Find a graph  $G$  such that  $L(G)$  is not Hamiltonian.
4. Identify real-world systems modeled effectively by  $L(G)$ .

