Que) Prove that any grp of order 5 or less than 5 is abelian

(Q) (14) Let G be group with $|G| \le 5$

case ①:- $|G| = 1$

$\Rightarrow$ only one element $\rightarrow$ trivial grp $\rightarrow$ ∴ abelian

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case ②:- $|G| = p$ $\rightarrow$ Grp of prime order, Then there exist an element a such that

$\langle a \rangle = G = \{e, a, a^2, \ldots a^{p-1}\}$ $\Rightarrow$ This is always abelian

for any element $a^i, a^j \in G$ $\Rightarrow a^i a^j = a^{i+j} = a^j a^i$

∴ G is abelian

Case ③ $|G| = 4$ ← (Example: Klein four grp)

By Lagrange theorem, order of element divides order of $|G|$ i.e 4 ∴ possible orders = 1, 2, 4

If $o(G) = 4$ and $\Rightarrow$ If there exist ele. of order 4, $\Rightarrow$ G is cyclic $\Rightarrow$ abelian

$\langle a \rangle = \{e, a, a^2, a^3\}$

• If no element has order 4, then all non-identity elements have order 2

Let $G = \{e, a, b, c\}$ (V_4)

$a^2 = b^2 = c^2 = e$

$ab = c = ba$

∴ all elements commute ∴ G is abelian

Case ④ $|G| = 5$

∵ 5 is prime, it is case②, any group of prime order is cyclic and therefore abelian

Note :- In a group G :-

1. Identity element is unique
2. Every element has a inverse
3. $ab = ac \Rightarrow b = c$
4. $(a^{-1})^{-1} = a$ 5. $(ab)^{-1} = b^{-1}a^{-1}$

5. Order of a Group $\Rightarrow$ Total no. of elements present in Group.

6. Order of element $\Rightarrow$ smallest number n such that
   If $\boxed{a^n = e} \Rightarrow n$ is order of a
   identity element

7. Any grp of order 5 or less than 5 is abelian

8. Order of element Divides order of grp

9. Order of identity element $e = 1$

Subgroups. $(G;*)$ be a group, a non empty subset of G is $(H,*)$ is a subgrp if it is a group.

Lemma 2 :- A non-empty subset H of a group G is a Subgrp of G iff ~~for every a, b ∈ H~~

(i) For every $a, b \in H$, $\boxed{a \cdot b \in H}$

(ii) For every $a \in H$, $\boxed{a^{-1} \in H}$

Lemma :- (iii) For every $a, b \in H$, $\boxed{ab^{-1} \in H}$

or

Note :- (1) Order of Subgroup divides order of Group.
(Lagrange's Theo.)

Prove that every cyclic grp is abelian
(7)

Cyclic grp :- If A group G contains an element a such that a^k forms every element of group G (k is some integer) we ~~abc~~ call ~~$G = \langle a \rangle$~~ G a cyclic group and a is generator. $\therefore G = \langle a \rangle$.

↑ a cyclic grp — ↑ generator
$\because a^k$ generates all element of G

Let $G = \langle a \rangle$

Consider two elements x, y. Clearly $x = a^m$ and $y = a^n$ for some integers m, n.

Now,
$$xy = a^m a^n = a^{m+n} = a^{n+m} = a^n \cdot a^m = yx$$
$\therefore$ G is abelian

Prove that order of an element 'a' in a grp is same as order of a^{-1}
(9)

Let $a \in G$ $\Rightarrow$ suppose order of a is n, $\Rightarrow \boxed{a^n = e}$

Now, compute $(a^{-1})^n = a^{-n} = (a^n)^{-1} = e^{-1} = e$

so a^{-1} also satisfies $\boxed{(a^{-1})^n = e}$

$\therefore$ order (a) = order (a^{-1})

If G is a grp and $a, x \in G$. Show that a and

(10) $x^{-1}ax$ have same order.

Let order $(a) = n \Rightarrow \boxed{a^n = e}$

Let order $(x^{-1}ax) = m \Rightarrow (x^{-1}ax)^m = e$ — (1)

Now,

$(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax) = x^{-1}a(xx^{-1})ax$

$= x^{-1}a^2x$.

Similarly $(x^{-1}ax)^n = x^{-1}a^n x$

$= x^{-1}ex = x^{-1}x$

$\boxed{(x^{-1}ax)^n = e}$. — (2)

By (1) & (2) $\boxed{m = n}$,

Let G be a group, If $a \in G$ and $a^n = e$, then

(11) Prove that $O(a)$ divides n.

Solⁿ Let $O(a) = m \Rightarrow a^m = e$ i.e m is least +ve integer such that $a^m = e$

We know, by divisⁿ algo $n = qm + r$ $0 \le r < m$

Now, given $a^n = e \Rightarrow$ Calculate $a^n = a^{qm+r}$

$= (a^m)^q \cdot a^r$

$= e^q \cdot a^r = a^r$

$\therefore a^n = a^r$

but $a^n = e \Rightarrow a^r = e$

$\Rightarrow r = 0 \Rightarrow n = mq$

$\therefore m$ divides n

$O(a)$ divides n

Note :- $G = \{e, a, a^2, a^3 --- a^{p-1}\}$ where p is prime.

↳ This group is of prime order.
And every grp of prime order is cyclic
And every cyclic group is abelian.

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(12) # Show that every grp of prime order is cyclic

Let $O(G) = p$ ← no of elements in a grp is prime

So, G must have atleast 2 elements. and one of these elements must be e.

choose one of the element $a \in G$ where $a \neq e$.

- Now, consider a subgroup generated by a;

$$\langle a \rangle = \{e, a, a^2, a^3 ---\}$$

- By lagrange's Theorem, order of subgrp divides order of group.

$$\therefore \frac{p}{|\langle a \rangle|}$$

Since p is prime only divisors of p are 1 and p itself

$\therefore |\langle a \rangle|$ cannot be 1 $(\because a \neq e)$

$\therefore |\langle a \rangle| = p$ ← means entire group of p elements is generated by a. This means group is cyclic.

Hence $\boxed{\langle a \rangle = G}$

Cyclic Group. Properties

If you can get every element of the group by repeatedly applying grp operation to one element then group is cyclic.

$a \in G$ such that $G = \langle a \rangle = a^n \quad \forall n \in \mathbb{Z}$

then G is cyclic grp — a is generator

* Every cyclic grp is abelian $\Leftrightarrow$ A grp is cyclic if it has a generator
* Subgrp of cyclic grp is also cyclic

(**) How to check if grp is cyclic?

- A grp is cyclic if some element has order equal to order of group.

Eg:- If $o(G) = n$ and $a^n = e$

$\therefore$ order of grp $= n$, order of ele $= n$ $\Big\}$ $\therefore G$ is cyclic

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(13) # Give an example of a group G in which every proper subgrp is cyclic but G is not cyclic

Soln Take a Klein four group (V_4)

It is a special grp of order 4 where every non identity element has order 2

denoted by V_4 or K_4 $\Rightarrow$ $V_4 = \{e, a, b, c\}$ such that

- e is identity ele
- $a^2 = b^2 = c^2 = e$
- $ab = c$, $bc = a$, $ca = b$.

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	
c	c	b	a	e

⇒ This grp is abelian

⇒ This grp is Not cyclic

(∵ no single ele generate whole grp)

comparison

cyclic grp of order 4	Klein four grp.
• $O(G) = 4$	$O(V) = 4$
• Has element of order 4	• All elements have order 2
• Generated by one element	• Cannot be generated by one element

⇒ Now to prove proper subgrp of G is cyclic

By lagrange theo proper subgrp's order divides order of group V_4.

∴ If $V_4 = \{e, a, b, c\}$

⇒ Proper subgrp cannot have order 3, ∴ it can have order 2.

∴ Subgroups ~~are~~ can be ⇒ $\{e, a\}$ $\{e, b\}$ $\{e, c\}$

- Each subgrp has 2 elements ⇒ ∴ $O(G) = 2$

Also order of each element $= 2$

$(\because a^2 = b^2 = c^2 = e)$

∴ Each subgrp is cyclic

Note :- If grp G is cyclic ⇒ (means G is abelian)

① ⇒ It must have a generator

② ⇒ $O(G) = O(\text{some element})$

#Note :- Show that every grp of prime order is abelian.

Let G be of prime order. ∴ Min^m no of elements that can be present in G = 2

Let $G = \{e, a\}$

⇓

we will show that G contains an element which is a generator. and if G contains a generator, G is cyclic. And every cyclic grp is abelian.

Now :- To show $a \in G$ is generator

Consider a subgrp generated by a

$\therefore \langle a \rangle = \{e, a, a^2, a^3 \dots\}$

Now $o(a)$ divides $O(G)$ $\therefore \frac{\text{Prime } p}{O(a)}$

∵ $O(G)$ is prime, $O(a)$ can only be 1 or Prime itself.

$O(a)$ cannot be 1 $(\because a \neq e)$ $\therefore O(a) = \text{Prime } p$

$\therefore O(a) = O(G) = p$ ← means entire grp of p element generated by a ∴ a is a generator.

∴ G has an element 'a' which is a generator ⇒ G is cyclic.

Now To show every cyclic grp is abelian.

cyclic grp G with generator a.

$G = \langle a \rangle$ ∴ a can generate all elements of G

let two such ele. be x, y

$\therefore x = a^m$ and $y = a^n$

$\Rightarrow xy = a^m \cdot a^n = a^{m+n} = a^n \cdot a^m = yx \Rightarrow$ ∴ G is abelian

Recap

① Every grp of prime order ⟹ is cyclic ⟹ Every cyclic grp is abelian

∴ Every grp of prime order is abelian

we can show this by showing group G (of prime order) has a generator.

If grp is cyclic, it has a generator 'a'
$x = a^m$, $y = a^n$
$xy = a^{mn} = a^{nm} = yx$.
∴ abelian

Que 4

② Every group of order 4 is abelian.

⟹ we know order of element divides order of group.

∴ possible order of element = 1, 2, 4

Let Group of order 4 has 4 elements {- - - -}.

If one of these elements has order 4

then $O(a) = 4$ } ∴ cyclic
⇓
abelian

If none of these elements has order 4,

Then every element has order 2
⇓
Klein 4 group
⇓
Abelian

∴ Every group with 4 elements is abelian.

- why is it true?

Because there are only 2 possible grp of order 4

① cyclic grp (generated by one element of order 4)
⟹ cyclic ⟶ Abelian

② Klein's four group ($V_4 = \{e, a, b, c\}$ ⟹ $a^2 = b^2 = c^2 = e$
⟹ $ab = c = ba$)
Not cyclic But Abelian

Since these are the only two possible grp of order 4 & both are abelian
∴ Every order 4 grp is abelian

(*) **Note** :- Every cyclic group is abelian ✓

But we can not say every abelian is cyclic

→ eg:- Klein 4 is abelian but not cyclic

- Some of the abelian can be cyclic
- But not all abelian are cyclic

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(5) # If H and K are subgroups of a group G then show that $H \cap K$ is also a subgrp of G.

By lemma of subgroups, we know

If H is a subgroup of G

then (1) For every $a, b \in H \Rightarrow \boxed{a \cdot b \in H}$

(2) For every ~~a b~~ $a \in H \Rightarrow \boxed{a^{-1} \in H}$

lemma 2 ⇒ (3) For every $a, b \in H \Rightarrow \boxed{ab^{-1} \in H}$

Sol^n we know if H is a subgrp of G

then $\boxed{\forall\, a, b \in H \Rightarrow ab^{-1} \in H}$

⇓

$H \cap K$ cannot be empty
($\because e \in H \cap K$)

⇒ ∴ To show $H \cap K$ is a subgrp of G

we take $a, b \in H \cap K$

⇓

$\therefore a, b \in H$ and $a, b \in K$

$\therefore ab^{-1} \in H$ and $ab^{-1} \in K$

$\therefore ab^{-1} \in H \cap K$

$\therefore H \cap K$ is a subgrp of G

Cosets in a group.

Let G is a group and let H is a subgrp of G

- Left coset :- Multiply all elements of subgrp H by a fixed element of the group G

  $g \in G \Rightarrow gH \Rightarrow \{gh : h \in H\}$

- Right coset :- $Hg \Rightarrow \{hg : h \in H\}$

(*) It's Important to note that $gH \neq Hg$ always.

When are they equal ?

→ If grp is abelian $\Rightarrow gh = hg \Rightarrow gH = Hg$

→ If subgrp H is normal

$\hookrightarrow gH = Hg \quad \forall g \in G$

(*) Example when they are not equal

$H = \{e, (12)\} \qquad g = (13)$

$gH = \{(13), (13)(12)\} = \{(13), (132)\}$

$Hg = \{(13), (12)(13)\} \Rightarrow \{(13), (123)\}$

$\Rightarrow gH \neq Hg$

Normal Subgroups.

Let G be a group and H be a subgroup of G

H is called Normal subgroup if it behaves equal from both sides.

$\therefore$ A subgrp $H \leq G$ is Normal if for every $g \in G$

$\boxed{gH = Hg}$ ⟸ means left & right cosets are same

* Equivalent easier condⁿ.

If H is normal in G $\Rightarrow$ ~~hg~~ $gh = hg$

$\Rightarrow \boxed{ghg^{-1} = h \in H}$ for all $g \in G$, $h \in H$.

\# Examples

(1) Every sub grp of abelian grp is Normal

$\Rightarrow$ $\because$ Abelian grp has $ab = ba$

A sub grp of abelian grp is also abelian

This is because every ele. of sub grp is also ele. of abelian grp.

$\therefore ab = ba$ also holds here

Now, $\because$ This subgrp is abelian

$ab = ba \Rightarrow$ or $gh = hg$

$\boxed{= ghg^{-1} = h \in H}$

$\therefore$ Every subgrp of abelian is normal also

(2) Klein 4 group.

All its subgrps are ~~abelian~~ Normal

because All are abelian

Summary ;- A subgrp H is normal if

- It is unchanged when sandwitched by any element of the group

$$\boxed{gHg^{-1} = H \quad \text{for all } g \in G}$$

(15) # Show that every subgroup of an Abelian grp is normal.

Sol^n Let G be abelian grp. Let H be subgrp of G

we must show that

H is normal i.e $gHg^{-1} = H$ for all $g \in G$

Since G is abelian

$$gh = hg$$

so ⇒ $ghg^{-1} = hgg^{-1}$

$ghg^{-1} = he = h$

$\boxed{\therefore ghg^{-1} = h \in H}$ ← This is the cond^n of normal subgroup.

∴ H is normal subgrp.