Mathematical Foundations of Computer Science

This Lecture: Combinatorics - Generating Functions

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Introduction

Generating functions are formal power series in one or more variables whose coefficients encode information about a sequence of numbers. They are widely used in combinatorics, discrete mathematics, and computer science for counting, solving recurrence relations, and analyzing algorithms.

The power of generating functions concept rests upon its ability to not only solve the kinds of problems we have solved so far, but also acts as a support to new situations which involve additional restrictions.

Example 1

How many integer solutions does the equation $c_1 + c_2 + c_3 + c_4 = 25$ *have if* $c_i \geq 0$ *for* $1 \leq i \leq 4$*?*
This problem can also be looked at as distributing 25 identical chocolates among 4 children. For each child, the possibilities can be described by the polynomial $1 + x + x^2 + \cdots + x^{25}$*. Then the answer to the problem is the coefficient of* x^{25} *in the function* $f(x) = (1 + x + x^2 + \cdots + x^{25})^4$.
The answer can also be obtained as the coefficient of x^{25} *in the function* $g(x) = (1 + x + x^2 + \cdots + x^{25} + x^{26} + \cdots)^4$*, if we rephrase the question as distributing from a large number of identical chocolates, 25 chocolates among 4 children.*
Now which one is easier?!!
Although it appears that dealing with a finite sum is easier, often a power series is easier to deal with!

Example 2

There are red, green, white and black pearls, each 24 in number, in how many ways can I select 24 of these pearls so that I have an even number of white pearls and at least 6 black ones?
The polynomials associated with the different colours of the pearls are as follows:

1. *Red or Green:* $1 + x + x^2 + \cdots + x^{24}$.
2. *White:* $1 + x^2 + x^4 + \cdots + x^{24}$.
3. *Black:* $x^6 + x^7 + \cdots + x^{24}$

So, the answer to the problem is the coeffient of x^{24} *in the function*
$f(x) = (1+x+x^2+\cdots+x^{24})^2(1+x^2+x^4+\cdots+x^{24})(x^6+x^7+\cdots+x^{24})$.
One such selection, say 5 red, 2 green, 8 white and 9 black pearls, arises from x^5 *in the first factor,* x^2 *in the second factor,* x^8 *in the third factor and* x^9 *in the last factor.*

Definition 1

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The function $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$ is called the ***generating function*** *of the sequence.*

From where this idea could have come?

Example 3

For any $n \in \mathbb{Z}^+$, $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$. Thus, clearly, $(1+x)^n$ is generating the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}, 0, 0, 0, \cdots.$$

Example 4

For any $n \in \mathbb{Z}^+$, $(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + \cdots + x^n)$. *So,*

$$\frac{(1-x^{n+1})}{(1-x)} = 1 + x + x^2 + \cdots + x^n.$$

This means that $\frac{(1-x^{n+1})}{(1-x)}$ *is a generating function for the sequence* $1, 1, \cdots (n + 1 \textit{ times}) \cdots, 1, 0, 0, \cdots$.
Also, we know that $1 = (1 - x)(1 + x + x^2 + \cdots)$; *hence* $\frac{1}{1-x}$ *is a generating function of* $1, 1, 1, \cdots$ *because* $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ *is valid for all* x *with* $|x| < 1$.

Example 5

With $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{i=0}^{\infty} x^i$, *the differentiation of the LHS yields*

$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

Also, the differentiation of the RHS yields

$\frac{d}{dx}(1 + x + x^2 + \cdots) = 1 + 2x + 3x^2 + \cdots$.

Thus, $\frac{1}{(1-x)^2}$ *is a generating function of the sequence* $1, 2, 3, 4, \cdots$, *while* $\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + \cdots$ *is a generating function of the sequence* $0, 1, 2, 3, \cdots$.

Example 6

Taking hint from Example 5,
$\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{d}{dx}(0 + 1x + 2x^2 + 3x^3 + \cdots)$ *which means* $\frac{x+1}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + \cdots$. *Thus,* $\frac{x+1}{(1-x)^3}$ *is a generating sequence for* $1^2, 2^2, 3^2, \cdots$. *Again,* $\frac{x(x+1)}{(1-x)^3}$ *generates* $0^2, 1^2, 2^2, 3^2, \cdots$.

Let us now relook at the Examples 4,5,6 with a slightly differenct approach:

$f_0(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$

$f_1(x) = x\frac{d}{dx}f_0(x) = \frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + \cdots$

$f_2(x) = x\frac{d}{dx}f_1(x) = \frac{x(x+1)}{(1-x)^3} = 0^2 + 1^2x + 2^2x^2 + 3^2x^2 + \cdots$

$f_3(x) = x\frac{d}{dx}f_2(x) = \frac{x^3+4x^2+x}{(1-x)^4=0^3+1^3x+2^3x^2+3^3x^3+\cdots}$

$f_4(x) = x\frac{d}{dx}f_3(x) = \frac{x^4+11x^3+11x^2+x}{(1-x)^5} = 0^4 + 1^4x + 2^4x^2 + 3^4x^3 + \cdots.$

Exercise 1

Using the above relook, write a code to generate such generating functions and demonstrate the implementation.

Example 7

For $n \in \mathbb{Z}^+$, *the Maclaurin Series expansion of* $(1+x)^{-n}$ *is given by*
$(1+x)^{-n} = 1 + (-n)x + (-n)(-n-1)\frac{x^2}{2!} + (-n)(-n-1)(-n-2)\frac{x^3}{3!} = 1 + \sum_{r=1}^{\infty} \frac{(-n)(-n-1)(-n-2)\cdots(-n-r+1)}{r!}x^r = \sum_{r=0}^{\infty}(-1)^r\binom{n+r-1}{r}x^r$

Hence $(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots = \sum_{r=0}^{\infty} = \binom{-n}{r}x^r$. *Thus,* $(1+x)^{-n}$ *is a generating function for* $\binom{-n}{0}, \binom{-n}{1}, \binom{-n}{2}, \binom{-n}{3}, \cdots$.

Example 8

Determine the coefficient of x^{15} *in* $f(x) = (x^2 + x^3 + x^4 + \cdots)^4$.
Solution: $(x^2 + x^3 + x^4 + \cdots) = x^2(1 + x + x^2 + \cdots) = \frac{x^2}{(1-x)}$. *So, coefficient of* x^{15} *in* $f(x)$ *is the coefficient of* x^{15} *in* $\left(\frac{x^2}{(1-x)}\right)^4 = \frac{x^8}{(1-x)^4}$.
This is equivalent to finding the coefficient of $x^{15-8} = x^7$ *in* $(1-x)^{-4}$, *which is namely,* $\binom{-4}{7}(-1)^7 = (-1)^7\binom{4+7-1}{7}(-1)^7 = \binom{10}{7} = 120.$

Example 9

In how many ways can we select, with repetitions allowed, r objects from n distinct objects?
Solution: For each of the n distinct objects, the geometric series $1 + x + x^2 + \cdots$ *represents the possible choices for that object (namely none, one, two,. ..). Considering all of the n distinct objects, the generating function is* $f(x) = (1 + x + x^2 + \cdots)^n$*. Now the required answer is the coefficient of* x^r *in* $f(x)$*.*
We have $(1 + x + x^2 + \cdots)^n = \left(\frac{1}{(1-x)}\right)^n = \frac{1}{(1-x)^n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} x^i$*. So, the coefficient of* x^r *is* $\binom{n+r-1}{r}$*.*

Introduction

These concepts are historically noted to have originated around 1669 (as some correspondence between Leibnitz and Bernoulli). But it was Euler who used these in Combinatorics.

It is clearly possible to represent any positive integer n as a sum of one or more positive integers a_i as

$$n = a_1 + a_2 + \cdots + a_m \tag{1}$$

Interesting problems arise in the enumeration and efficient generation of such representations. Divisions of a positive integer as in (1) are of two types:

Definition 2

Compositions An ordered division of a positive integer n into m parts is known as a ***Composition****.*
If m is fixed, then it is called a ***restricted composition****; otherwise, it is* ***unrestricted****.*

Definition 3

Partitions An unordered division of a positive integer n into m parts is known as a ***Partition****.*
If m is fixed, then it is called a ***restricted partition****; otherwise, it is* ***unrestricted****.*

Remark 1

The distinction between the partitions and the compositions is similar to that made between the permutations and the combinations.

There are 7 unrestricted partitions of 5, namely,

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1$$

while there are 16 unrestricted compositions of 5, namely,

$$5, 4+1, 1+4, 3+2, 2+3, 3+1+1, 1+3+1, 1+1+3, 2+2+1, 2+1+2,$$
$$1+2+2, 2+1+1+1, 1+2+1+1, 1+1+2+1,$$
$$1+1+1+2, 1+1+1+1+1$$

One of the easiest ways of enumerating the unrestricted compositions of n is to consider n number of 1's in a row. Since there is no restriction on the number of parts, we may or may not put a marker in any of the $(n-1)$ spaces between the 1's in order to form groups. Clearly, this may be done in 2^{n-1} ways.
We can make same type of argument when we deal with a restriction on the composition to have exactly m parts. We need exactly $(m-1)$ markers to form m groups and number of ways of placing $(m-1)$ markers in the $(n-1)$ spaces between the 1's is $\binom{n-1}{m-1}$.

Now, how do we use generating functions in this context?

Let $C_m(x)$ be the enumerator for compositions of n with exactly m parts. Then

$$C_m(x) = \sum_n C_{m_n} x^n \tag{2}$$

where C_{m_n} is the coefficient of x^n representing the number of compositions of n into exactl;y m parts. Each part of any composition can be $1, 2, 3$ or any greater number so that the factor in the enumerator must contain each of these powers of x. It is so in $x + x^2 + x^3 + \cdots + x^k + \cdots = x(1-x)^{-1}$.

Since there are exactly m parts, the generating function must be the product of m such factors, namely,

$$C_m(x) = (x + x^2 + x^3 + \cdots + x^k + \cdots)^m \tag{3}$$

which can be written as

$$C_m(x) = x^m(1-x)^{-m} = x^m \sum_{i=0}^{\infty} \binom{m+i-1}{i} x^i$$

Replacing $m+i$ by r in the summation, we can write

$$C_m(x) = \sum_{r=m}^{\infty} \binom{r-1}{r-m} x^r = \sum_{r=m}^{\infty} \binom{r-1}{m-1} x^r$$

Thus, the coefficient of x^n in this enumerator is $\binom{n-1}{m-1}$, as we derived earlier.

The enumerating generating function for the compositions with no restriction on the number of parts $C(x)$ can be obtained as $C(x) = \sum_{m=1}^{\infty} C_m(x) = \sum_{m=1}^{\infty} x^m(1-x)^{-m}$.

Substituting $t = \frac{x}{1-x}$ in the series, we obtain

$$C(x) = t + t^2 + t^3 + \cdots = \frac{t}{1-t} = \frac{x}{1-2x} = \sum_{n=1}^{\infty} 2^{n-1} x^n \tag{4}$$

The coefficient of x^n in the enumerator is 2^{n-1}, which is the number of unrestricted compositions of n.

From $r!C(n,r) = P(n,r)$, we can easily obtain the required permutations. Unfortunately, we don't have so much simple relationship between the number of partitions and the number of compositions, because each partition, in general, will give rise to a different number of compositions.

For example, the two partitions of 10, namely, 811 and 4321, give respectively 3 and 24 compositions. Thus, it is impossible to use the results obtained for the compositions.

Let p_n be the number of unrestricted partitions of n so that the generating function is

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n + \cdots \tag{5}$$

Consider the polynomial

$$1 + x + x^2 + x^3 + \cdots + x^k + \cdots + x^n$$

The appearance of x^k can be interpreted as the existence of just k number of 1's in a partition of the integer n. Similarly, the polynomial

$$1 + x^2 + x^4 + \cdots + x^{2k} + \cdots$$

is about the 2's in the partition, and in particular, the coefficient of $x^{2k} = (x^2)^k$ represents the case of just k number of 2's in the partition. In general, the polynomial

$$1 + x^r + x^{2r} + \cdots + x^{kr} + \cdots$$

can represent the r's in the partition.

The generating function should have one factor for the 1's, one for the 2's and so on. Collecting together all these polynomials, we get the generating function for the partitions of n as

$$p(x) = (1+x+x^2+x^3+\cdots+x^k+\cdots)(1+x^2+x^4+\cdots+x^{2k}+\cdots)\cdots(1+x^r+x^{2r}+\cdots \tag{6}$$

$$= (1-x)^{-1}(1-x^2)^{-1}\cdots(1-x^r)^{-1} \tag{7}$$

The number of unrestricted partitions of n is, therefore, the coefficient of the term x^n in equation (7).
Unfortunately we can not obtain a simple formula for these coefficients. But this doesn't mean that the generating function approach to partitions serves no useful purpose.

In fact, many of standard results in partitions can be solved using them. For example, proving that the partitions with unequal parts are equinumerous with the partitions in which all parts are odd can be done in a very simple way.

Example 10

The enumerator for the partitions with unequal parts:
Each factor can only contain two terms - one which indicates that the corresponding number doesn't appear in the partition, and the other that the number appears once. So, we get the generating function as

$$u(x) = (1+x)(1+x^2)\cdots(1+x^k)\cdots \tag{8}$$

Exercise 2

Obtain an enumerator for the partitions with every part odd.

Exercise 3

Obtain an enumerator for the partitions in which no part is bigger than s.

1. **Algorithm Analysis**: Used to analyze recursive algorithms (e.g., divide and conquer strategies).
2. **Automata Theory**: Used in counting paths and transitions.
3. **Formal Languages**: Used in parsing and analyzing grammars.
4. **Symbolic Computation**: Used to simplify and solve counting problems.
5. **Complexity Analysis**: Helps in determining growth rates of functions in recurrence relations.

Generating functions are essential tools for solving problems involving sequences, recurrence relations, and combinatorics. They transform problems into algebraic manipulations, often revealing elegant solutions. Applications to computer science include algorithm analysis, combinatorial counting, and symbolic manipulation.

1. Kenneth Rosen, *Discrete Mathematics and its Applications*, 5^{th} edition, McGraw Hill, NY, 2003.
2. C.L. Liu, *Elements of Discrete Mathematics*, 2^{nd} edition, McGraw Hill.
3. E.S. Page and L.B. Wilson, *An introduction to Computational Combinatorics*, Cambridge University Press, 1979.