

Mathematical Foundations of Computer Science

This Lecture: Graph Theory: Line Graphs and Traversability

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MA714 (Odd Semester [2025-26])



A **Line Graph** represents adjacency relationships between the edges of a graph.

It transforms edge-based information of a graph into vertex-based information.

Traversability of these graphs (Eulerian and Hamiltonian properties) provides insights into network structure and flow.



Let $G = (V, E)$ be a simple graph.

Definition

The **Line Graph** of G , denoted by $L(G)$, is the graph whose:

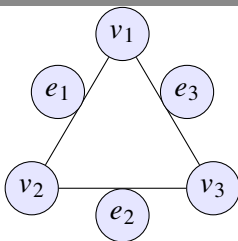
- vertices correspond to the edges of G ;

- two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G share a common vertex.

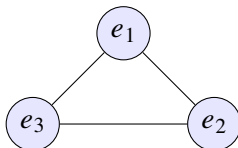
$$V(L(G)) = E(G), \quad e_i \sim e_j \text{ in } L(G) \iff e_i, e_j \text{ are incident in } G.$$



Example 1: Line Graph of a Triangle K_3



Graph $G = K_3$

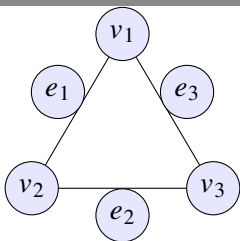


Line Graph $L(G)$

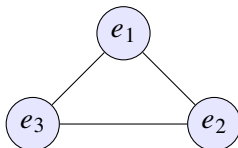
$$L(K_3) \cong K_3$$



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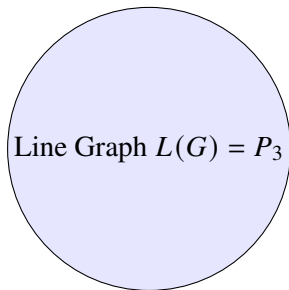
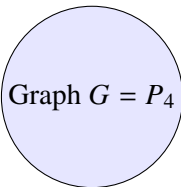
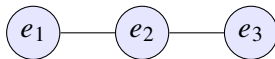
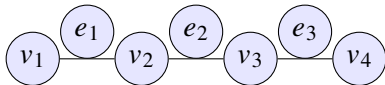


Line Graph $L(G)$

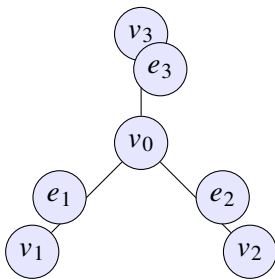
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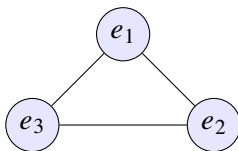
Example 2: Line Graph of a Path P_4



Example 3: Line Graph of a Star $K_{1,3}$



Graph $G = K_{1,3}$



Line Graph $L(G) = K_3$



If G has m edges, $L(G)$ has m vertices.

Number of edges in $L(G)$:

$$|E(L(G))| = \sum_{v \in V(G)} \binom{d_G(v)}{2}$$

For edge $e = uv$ in G :

$$d_{L(G)}(e) = d_G(u) + d_G(v) - 2$$

$L(G)$ is connected iff G is connected and has no isolated edges.

$$L(P_n) = P_{n-1}, \quad L(C_n) = C_n, \quad L(K_n) = K_{\binom{n}{2}}$$



Theorem 1

If G is Eulerian, then $L(G)$ is both Eulerian and Hamiltonian.

Traversing each edge of G once corresponds to visiting each vertex of $L(G)$ once.

Hence, an Eulerian circuit in $G \Rightarrow$ a Hamiltonian cycle in $L(G)$.

Theorem 2

$L(G)$ is Eulerian iff every vertex of G has even degree ≥ 2 .



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Theorem 2

$L(G)$ is Eulerian iff every vertex of G has even degree ≥ 2 .



1. **Network Topology Design:** Models link-to-link connectivity in communication or optical networks.
2. **Resource Allocation:** Used in frequency assignment and edge coloring.
3. **Compiler Design:** Register interference graphs can be viewed as line graphs of flow graphs.
4. **Transportation and Circuits:** Models traffic flow between routes or current between connections.
5. **Social Networks:** Captures relationship among relationships (shared participants).



$L(G)$: edges of $G \rightarrow$ vertices.

Eulerian–Hamiltonian correspondence:

$$G \text{ Eulerian} \Rightarrow L(G) \text{ Hamiltonian}$$

$L(G)$ highlights edge interactions and traversal properties.

Applications span communication networks, compiler optimization, and transportation systems.



1. Construct $L(G)$ for:

$$P_5, C_5, K_{1,4}$$

2. Prove: If G is Eulerian, then $L(G)$ is Hamiltonian.
3. Find a graph G such that $L(G)$ is not Hamiltonian.
4. Identify real-world systems modeled effectively by $L(G)$.

