Mathematical Foundations of Computer Science

This Lecture: Graph Theory - Directed and Undirected Graphs (Introduction, Properties, Subgraphs, Isomorphism)

Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

MA714 (Odd Semester [2025-26])

Definition

A graph G is an ordered pair (V, E) where:

V is a finite set of vertices (nodes).

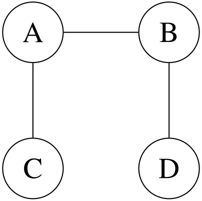
E is a set of edges.

Graphs model **networks, maps, relationships, dependencies**.

Undirected Graph:

Edge is an unordered pair $\{u, v\}$.

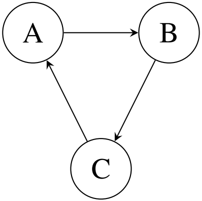
$\{u, v\} = \{v, u\}$.



Directed Graph (Digraph):

Edge is an ordered pair (u, v).

Represents a one-way connection.



Degree: Number of edges incident to a vertex.

Indegree (digraphs): Number of edges entering a vertex.

Outdegree (digraphs): Number of edges leaving a vertex.

Handshaking Lemma

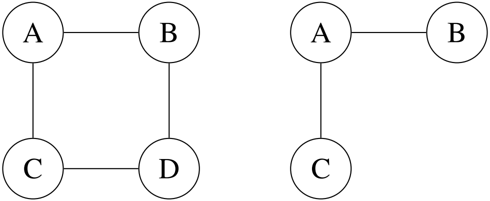
Undirected: $\sum_{v \in V} \deg(v) = 2|E|$.

Directed: $\sum \text{indeg}(v) = \sum \text{outdeg}(v) = |E|$.

A graph $H = (V_H, E_H)$ is a subgraph of $G = (V, E)$ if:

$V_H \subseteq V$,

$E_H \subseteq E$.



Graph G Subgraph H

Induced Subgraph

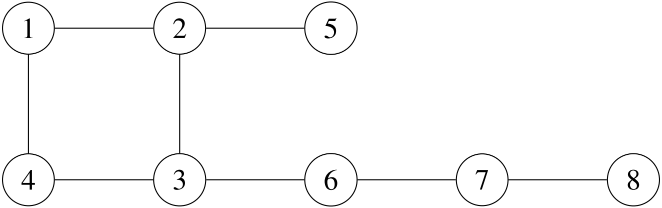
Given a graph $G = (V, E)$, let $V_H \subseteq V$. The induced subgraph (denoted by $< H >$) is a maximal subgraph of G with its vertex set V_H.

Spanning Subgraph

Given a graph $G = (V, E)$, a spanning subgraph H has the same vertex set V of G and $E_H \subseteq E$.

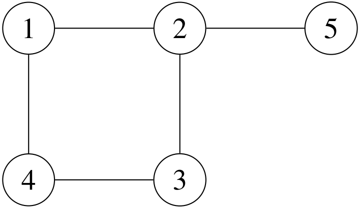
Vertices: $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
Edges:

$$E = \{(1, 2), (2, 3), (3, 4), (4, 1), (2, 5), (3, 6), (6, 7), (7, 8)\}$$



Chosen vertices: $V_H = \{1, 2, 3, 4, 5\}$
Edges included:

$$E_H = \{(1, 2), (2, 3), (3, 4), (4, 1), (2, 5)\}$$



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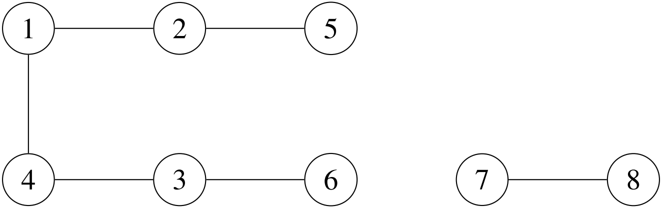
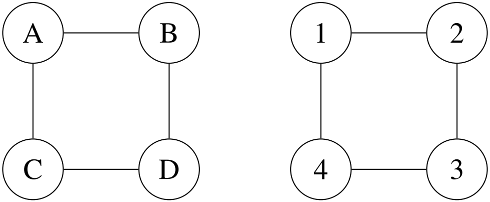


Definition

Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection $f : V_1 \to V_2$ such that:

$$\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2$$



Isomorphic via mapping: $A \to 1, B \to 2, C \to 4, D \to 3$.

1. Given a sequence of nonnegative integers $d_1, \ldots, d_n$,
2. Question: Is there a simple undirected graph with this degree sequence?
3. The Havel–Hakimi algorithm gives a step-by-step test.

Before applying Havel–Hakimi, check:

1. All entries ≥ 0.
2. No entry exceeds $n - 1$.
3. Sum of degrees is even (Handshaking Lemma).

If any fails $\Rightarrow$ not graphical.

Havel–Hakimi Iteration

1. Sort the sequence in nonincreasing order.
2. Remove the first entry r.
3. If $r >$ length of remaining sequence $\Rightarrow$ not graphical.
4. Subtract 1 from the next r entries.
5. If any entry becomes negative $\Rightarrow$ not graphical.
6. Repeat.

If the sequence reduces to all zeros $\Rightarrow$ graphical.

Sequence: $[3, 3, 2, 2, 2, 2]$

1. Sort: $[3, 3, 2, 2, 2, 2]$, remove 3, subtract from next 3 $\longrightarrow$ $[2, 1, 1, 2, 2]$.
2. Sort: $[2, 2, 2, 1, 1]$, remove 2, subtract from next 2 $\longrightarrow$ $[1, 1, 1, 1]$.
3. Remove 1, subtract $\longrightarrow$ $[1, 1, 0]$.
4. Remove 1, subtract $\longrightarrow$ $[0, 0]$.

All zeros $\Rightarrow$ Graphical.

Sequence: $[3, 3, 2, 2, 2, 2]$

1. Sort: $[3, 3, 2, 2, 2, 2]$, remove 3, subtract from next 3 $\rightarrow$ $[2, 1, 1, 2, 2]$.
2. Sort: $[2, 2, 2, 1, 1]$, remove 2, subtract from next 2 $\rightarrow$ $[1, 1, 1, 1]$.
3. Remove 1, subtract $\rightarrow$ $[1, 1, 0]$.
4. Remove 1, subtract $\rightarrow$ $[0, 0]$.

$$\text{All zeros} \Rightarrow \text{Graphical.}$$

Sequence: $[4, 3, 3, 1, 1]$

1. Sort: $[4, 3, 3, 1, 1]$, remove 4, subtract from next 4 $\longrightarrow [2, 2, 0, 0]$.
2. Sort: $[2, 2, 0, 0]$, remove 2, subtract from next 2 $\longrightarrow [1, -1, 0]$.

Negative entry appears $\Longrightarrow$ Not graphical.

Sequence: $[4, 3, 3, 1, 1]$

1. Sort: $[4, 3, 3, 1, 1]$, remove 4, subtract from next 4 $\longrightarrow [2, 2, 0, 0]$.
2. Sort: $[2, 2, 0, 0]$, remove 2, subtract from next 2 $\longrightarrow [1, -1, 0]$.

Negative entry appears $\Rightarrow$ Not graphical.

If a vertex has degree r, it must be connected to r others.

Removing it forces r other vertices to lose one degree.

Choosing the largest r is always safe (degree-swapping argument).

Induction: sequence is graphical iff reduced sequence is graphical.

Naive implementation: $O(n^2)$.

Sorting each iteration $\rightarrow O(n^2 \log n)$ worst case.

Using priority queues: $O(m \log n)$ where m is number of decrements.

Practical: efficient for small/medium n.

Summary of Havel - Hakimi Algorithm

Havel–Hakimi decides if a degree sequence is graphical.

Iterative: remove largest degree, reduce next r entries.

Stops at all zeros (graphical) or negative entry (not graphical).

Useful in graph theory, random graph generation, and proofs.

Question: Is there a simple graph with degree sequence [3,3,2,2,2,2]?

Solution:

Sum of degrees = 14, hence $|E| = 7$.

Apply Havel-Hakimi algorithm.

Reduction succeeds $\Rightarrow$ Such a graph exists.

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Solution:

Sum of degrees = 14, hence $|E| = 7$.

Apply Havel-Hakimi algorithm.

Reduction succeeds $\Rightarrow$ Such a graph exists.

Question: A directed graph has 8 vertices and 20 edges. Show that there exists a vertex with indegree ≥ 3 or outdegree ≥ 3.

Solution:

Total indegree = total outdegree = 20.

If all indegree ≤ 2, total ≤ 16, contradiction.

Hence some vertex must have indegree or outdegree ≥ 3.

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Hence some vertex must have indegree or outdegree ≥ 3.

Question: Is a 4-cycle with one diagonal isomorphic to a star graph with 4 vertices?

Solution:

Cycle + diagonal degrees: [3,2,3,2].

Star degrees: [3,1,1,1].

Degree sequences differ $\Rightarrow$ Not isomorphic.

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Solution:

Cycle + diagonal degrees: [3,2,3,2].

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Degree sequences differ $\Rightarrow$ Not isomorphic.

Graphs may be directed or undirected.

Key properties: degree, indegree, outdegree, Handshaking Lemma.

Subgraphs: spanning and induced.

Graph isomorphism preserves structure.

Degree sequences useful to test existence and isomorphism.