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Maths Assignment - 2

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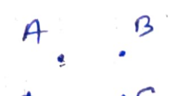
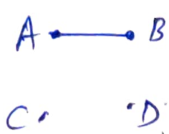
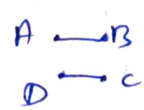
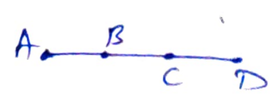
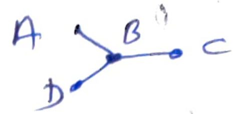
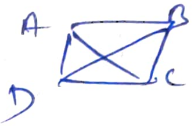
252 IS032.

Que 1) Find total no. of possible isomorphism of simple undirected graph ~~possible~~ with 4 vertices. Also calculate number of graphs possible in each such isomorphism classes.

Sol^n:- All edge possibilities with 4 vertices $^4C_2 = 6$

- So, total simple labelled graph {each edge have 2 option} $= 2^6 = 64$
- No. of unlabelled simple graph = 11 classes {non isomorphic}

Isomorphism Class (G)	Edges	No of Graphs	Graph
① Empty Graph ($\bar{K}_4$)	0	1	 $= {}^4C_0$
② One edge ($K_2 \cup 2K_1$)	1	6	 $= {}^4C_2$
③ Two disconnected	2	3	 ${}^4C_2/2$
④ Path of length 2 ($P_3 \cup P_1$)	2	$6 \times 2 = 12$	
⑤ Path of length 3 (P_4)	3	12	

(6) Triangle ($K_3 \cup K_1$)	3	4	
(7) Cycle of length 4 (C_4)	4	3	
(8) Three edge star ($K_{1,3}$)	3	4	
(9) Complete graph minus 1 edge	5	6	
(10) Complete graph minus No edges	5	12	
(11) Complete graph	6	1	

Q2)

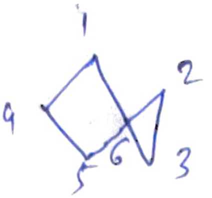
Solⁿ :- Let G be a eulerian Connected & every vertex have even degree. Let V be common endpoint of edges e & f.

(i) Follow a cycle starting with e.
Start with 'V' & traverse edge e and then continue. waiting along unused edges, always taking unused edges out at current edge.

(ii) If all edges are used in c; is an Eulerian circuit Check if edge immediately after e in C, is f. If yes then done. Else, you can rotate C, so that v-occurrence after e is the place when you splice in cycle containing f.

(iii) If unused edges remain, proceed as in pick any vertex that lies on current closed trail & that has unused incident edges

Statement is not true only when deleting those edge do not disconnect graph.

So it is false eg:-  $Deg(6) = 4$, Rest $= 2$

Pick edges that share vertex 1

$e = 6 \to 1$ $f = 6 \to 2$

$\therefore$ Euler circuit $[6-3-2-6-5-4-1]$

Que 3) Prove by induction number of vertices n, that maxm no of edges in a simple non-hamiltonian graph is ${}^{n-1}C_2 + 1$

Soln Let $n = 3$ $\Rightarrow {}^{2}C_2 + 1 = 2$

on 3 vertices, a non hamiltonian simple graph is a path with 2 edges (K_3 Δ is hamiltonian). So, maximum no. of edges is 2 for non hamiltonian graph.

Let G be non-hamiltonian graph

$|E(G)| \leq \binom{n-1}{2} + 1$ --- To prove

This is because G was chosen to be edge-maximal subject to non hamiltonian, adding any missing edge to G create Hamiltonian cycle

Let u, v be vertices in G, $\therefore G + v =$ hamiltonian

To find path P

$P: x = v_1 v_2 v_3 --- v_n = y$ n vertices.

& $x-y$ is path

Remove endpoint y & let $G' = G - y$

$G' =$ graph on $n-1$ vertices.

By induction hypothesis

$$|E(G')| \le {}^{n-2}C_2 + 1$$

Now since, y = non adjacent & x

$$\deg(y) \le n-2$$

$$|E(G)| = |E(G')| + \deg(y) \le \underbrace{{}^{n-2}C_2 + 1 + n-2}_{{}^{n-2}C_2}$$

Hence we found that for non Hamiltonian graph G, no of edges will be less than or equal to ${}^{n-1}C_2 + 1$

Hence proved.

let n = 5

no. of edges $= {}^{4}C_2 + 1 = 7$

Que 4) Prove that

~~BG~~ Bipartite graph (BG) $K_{n,n}$ has $\frac{(n-1)!\,n!}{2}$ Hamiltonian cycles.

Soln

let bipartite graph be $A = \{a_1, a_2 \cdots a_n\}$
$B = \{b_1, b_2 \cdots b_n\}$

any hamiltonian cycle (HC) in BG must alternate between A & B

HC is of form $a_{i_1} - b_{j_1} - a_{i_2} - b_{j_2} \cdots a_{i_n} - b_{j_n} - a_i \cdots$ visiting each a-vertex & b-vertex only once.

let a be fixed starting vertex.

(1) Now to choose n vertices of for filling all the possibilities $= n!$

(2) For A vertices choices are : $(n-1)!$ since a_1 is already fixed. Total ways $= n!\,(n-1)!$

But we also need to make sure undirected graph is not counted twice, so we divide it by ~~b~~ 2

Total ways $= \frac{n!\,(n-1)!}{2}$ Hence Proved

Que 5

Sol^n If $T =$ tree on n vertices

$|E(T)| = n-1$ --- By handshaking lemma.

$\therefore \sum_{i=1}^{n} d_i = 2|E(T)|$. where $E(T) =$ edges of tree

use the Prufer code sequence bijection on labelled between tree on $\{1, 2, \dots n\}$ & length $n-2$ sequence over $\{1 \dots n\}$ under the prufer correspondence a vertex 'i' appears exactly $d_i - 1$ times in prufer sequence where $d_i =$ degree of i in that force

$\therefore$ Multiset of multiplicities in prufer sequence $= \{d_i - 1 \dots d_n - 1\}$

length of sequence $= \sum_{i=1}^{n} d_i - 1 = \left(\sum_{i=1}^{n} d_i\right) - n$.

Now, if $\sum_{i=1}^{n} d = 2n-2$ then $\left[\sum_{i=1}^{n} d_i - 1 = n-2\right]$

Multiplicities sum exactly to $n-2$ length required length $\therefore$ The tree exists with degree $d_i = i + (d_i - 1)$

Hence Proved by prufer sequence

Que 6

① We use prufer code that says vertex V has degree = $1 +$ no of times V appears in prufer sequence

i) For a leaf i.e degree $= 1$:-
Exactly 2 labels must not appear in prufer sequence (for 2 leaves)

So, the prufer sequence will use exactly $n-2$ distinct labels. No of sequence of length $n-2$
$= n(n-1)(n-3) \cdots = 3 = \left\lfloor \frac{n!}{2} \right\rfloor$

ii) Exactly $n-2$ leaves
for exactly $n-2$ leaves only 2 nodes should appear in prufer sequence.
Choose 2 labels (i,j) that appear in prufer sequence for length $n-2$ sequence i,j appear $= 2^{n-2} - 2$ times.
Total choices $\left[{}^nC_2 \times (2^{n-2} - 2) \right]$

Que 7)

Sol^n Let us take a cycle $V_1 - V_2 - V_3 \cdots k \cdot V_n$ we break this cycle $(V_1 - V_2 - V_3 \cdots)$ into Δs that share the first vertex V_1

$\Delta_1 (V_1 V_2 V_3)$
$\Delta_2 (V_1 V_3 V_4)$
$\vdots$
$\Delta_K (V_1 V_3 V_K')$

Each Δ used are edge of main cycle & some edges that are not in cycle.

Not that each cycle edge appears once & non cycle edges appears twice among these triangles

Total parity = cycle edges + non cycle edge

$\therefore$ Total parity depends on cycle edges

Thus if total weight on every Δ is even then total weight on every edge will also be even. The converse is also true.

Que 8

Sol^n Let G = simple planar graph with n vertices & m edges. Eulers formula state for a simple planar graph with $n \geq 3$ $\boxed{m \leq 3n-6}$

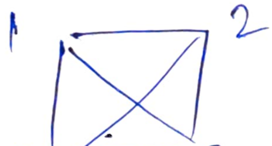
Euler : $n - m + f = 2$ $(2m \geq 3f)$

Avg degree $= \frac{1}{n} \sum \deg(v) = \frac{2m}{n} \leq \frac{2(3n-6)}{n} \leq 6$

Thus average degree is < 6.

Max degree of vertex can be atmost 5, because if every vertex had degree ≥ 6, average would be > 6 which is not possible for simple planar graph.

eg:-



K_4 All degree are 3 (≤ 5)

$\therefore$ valid graph