# Mathematical Foundations of Computer Science

## This Lecture: Combinatorics - Countable and Uncountable sets

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MA714 (Odd Semester [2025-26])

**Introduction**

These notes introduce the concepts of countable and uncountable sets, with definitions, key examples, and applications in Computer Science.

### Definitions

A set is said to be ***countable*** if its elements can be put into a one-to-one correspondence with the set of natural numbers $\mathbb{N} = \{0, 1, 2, \cdots\}$.

A set is ***uncountable*** if it is not countable, i.e., there does not exist a one-to-one correspondence with $\mathbb{N}$.

## Examples of Countable Sets

- The set of natural numbers $\mathbb{N}$
- The set of integers $\mathbb{Z}$
- The set of rational numbers $\mathbb{Q}$.

Each of these can be listed in a sequence (even if it is an infinite one), so they are countable.

To show a set is countable, we demonstrate a bijection with $\mathbb{N}$.

**Example:**

Mapping $\mathbb{Z}$ to $\mathbb{N}$: $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$

This function maps all integers to distinct natural numbers.

**Theorem 1**

*If $S$ is both countable and infinite, then there is a bijection between $S$ and $\mathbb{N}$ itself.*

*Proof*: For any $s \in S$, we let $f(s)$ denote the value of $k$ such that $s$ is the $k^{th}$ smallest element of $S$. This map is well defined for any $s$, because there are only finitely many natural numbers between 1 and $s$. It is impossible for two different elements of $S$ to both be the $k^{th}$ smallest element of $S$. Hence $f$ is one-to-one. Also, since $S$ is infinite, $f$ is onto.

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**Theorem 2**

*If $S$ is countable and $S' \subset S$, then $S'$ is also countable.*

*Proof*: Since $S$ is countable, there is a bijection $f : S \longrightarrow \mathbb{N}$. But then $f(S') = \mathbb{N}'$ is a subset of $\mathbb{N}$, and $f$ is a bijection between $S'$ and $\mathbb{N}'$.

**Theorem 3**

*If $S' \subset S$ and $S'$ is uncountable, then so is $S$.*

This theorem is actually a kind of restatement of Theorem 2.

A real number $x$ is called *algebraic* if $x$ is the root of a polynomial equation $c_0 + c_1x + \cdots + c_nx^n$ where all the $c$'s are integers. For instance, $\sqrt{2}$ is an algebraic integer because it is a root of the equation $x^2 - 2 = 0$.

**Theorem 4**

*The set of algebraic numbers is countable.*

*Proof*: Let $L_k$ denote the set of algebraic numbers that satisfy polynomials of the form $c_0+c_1x+\cdots+c_nx^n$, where $n < k$ and $\max(|c_j|) < k$. Note that there are at most $k^k$ polynomials of this form, and each one has at most $k$ roots. Hence $L_k$ is a finite set having at most $k^{k+1}$ elements. We make our list $L_1, L_2, L_3, \cdots$ of all algebraic numbers and weed out repeaters.

**Theorem 5**

*Suppose that $S_1, S_2, \cdots \subset T$ are disjoint countable sets. Then $S = \bigcup_i S_i$ is a countable set. (Exercise)*

### Two simple examples of uncountable sets

- The set of real numbers $\mathbb{R}$
- The set of irrational numbers.

Cantor's diagonalization argument proves that $\mathbb{R}$ is uncountable.

### Cantor's Diagonalization Argument

Suppose the real numbers in the interval $[0, 1]$ were countable and could be listed as $r_1, r_2, r_3, \cdots$. We can construct a number that differs in the $n^{th}$ decimal place from $r_n$. This number is not in the list, contradicting the assumption.

### The Set of Binary Sequences

Let $S$ denote the set of infinite binary sequences. Suppose that $f : S \longrightarrow \mathbb{N}$ is a bijection. We form a new binary sequence $A$ by declaring that the $n^{th}$ digit of $A$ is the opposite of the $n^{th}$ digit of $f^{-1}(n)$. The idea here is that $f^{-1}(n)$ is some binary sequence and we can look at its $n^{th}$ digit and reverse it.
Supposedly, there is some $N$ such that $f(A) = N$. But then the $N^{th}$ digit of $A = f^{-1}(N)$ is the opposite of the $N^{th}$ digit of $A$, and this is a contradiction.

## The Transcendental Numbers

A real number $x$ is called *transcendental* if $x$ is not an algebraic number.

Let $A$ denote the set of algebraic numbers and let $T$ denote the set of tran- scendental numbers. Note that $\mathbb{R} = A \bigcup T$, and $A$ is countable. If $T$ were countable then $\mathbb{R}$ would be the union of two countable sets. Since $\mathbb{R}$ is uncountable, $\mathbb{R}$ is not the union of two countable sets. Hence $T$ is uncountable.

The consequence of this argument is that there are many more transcendental numbers than algebraic numbers.

1. **Decidability and Computability**: The set of all Turing machines is countable, while the set of all languages over an alphabet is uncountable. Therefore, there exist languages that are not decidable by any Turing machine.
2. **Data Representation**: All data structures in a computer are countable as they are represented using a finite alphabet.
3. **Compression Algorithms**: Understanding countability helps in understanding the limits of data compression and Kolmogorov complexity.

- ✓ Countable sets can be listed in a sequence; uncountable sets can not.
- ✓ $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable; $\mathbb{R}$ and the power set of $\mathbb{N}$ are uncountable.
- ✓ This concept is essential in theoretical computer science, especially in automata theory, complexity, and information theory.

# Suggested References


1. Kenneth Rosen, *Discrete Mathematics and its Applications*, $5^{th}$ edition, McGraw Hill, NY, 2003.
2. C.L. Liu, Elements of Discrete Mathematics, $2^{nd}$ edition, McGraw Hill.