Mathematical Foundations of Computer Science

This Lecture: Combinatorics - Fundamental principles of counting

Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

MA714 (Odd Semester [2025-26])

Introduction

The Fundamental Principles of Counting are foundational concepts in combinatorics, a branch of mathematics concerned with counting, arranging, and analyzing discrete structures. These principles are essential in fields such as probability, statistics, computer science, and operations research. This lecture focuses on the two basic principles—the Addition Principle and the Multiplication Principle — and illustrates their relevance in solving problems in computer science.

The Addition Principle (a.k.a. The Rule of Sum)

The Addition Principle (or Rule of Sum) states: If there are m ways to do one task and n ways to do another task, and the two tasks are mutually exclusive (can not be done at the same time), then there are $m + n$ ways to choose one of these tasks.

Example 1

Suppose a user can log into a system using 2 types of authentication methods: password-based (5 methods) or biometric-based (3 methods). The total number of login options is $5 + 3 = 8$.

Example 2

A computer science instructor who has 7 different introductory books each on C++, Java and Perl can recommend any one of these 21 books to a student who is interested in learning a first programming language.

Example 3

In trying to reach a decision on plant expansion, an administrator assigns 12 of his employees to two committees. Committee A consists of 5 members and is to investigate possible favourable results from such an expansion. The other 7 employees who are in Committee B, will scrutinize possible unfavourable repurcussions. Should the administrator decide to speak to one committee member before making his decision, then by the Addition Principle, there are 12 employees he can call upon for input. However, to be a bit more unbiased, he decides to speak with a member of the Committee A on Monday, and then with a member of the Committee B on Tuesday, before reaching a decision. This means, he can select two such employees to speak with in $5 \times 7 = 35$ ways.

The Example 3 introduces the second principle of counting, namely,

The Multiplication Principle (a.k.a. The Rule of Product)
The Multiplication Principle (or Rule of Product) states: If a task can be broken into two successive independent subtasks, with m ways to do the first subtask and n ways to do the second subtask, then there are $m \times n$ ways to perform the entire task.

Example 4

If a computer system allows you to select a file (10 options) and then choose an operation (4 options: open, edit, delete, copy), there are 10 × 4 = 40 possible file-operation combinations.

Example 5

The Drama Club of NITK is holding practice sessions for the upcoming festival. With 6 men and 8 women auditioning for the lead roles of male and female, by the rule of product, the director can cast his leading couple in $6 \times 8 = 48$ ways.

Example 6

A license plate consists of two letters followed by 4 digits.

1. If no letter or digit can be repeated, there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3276000$ different possible license plates.
2. With repetitions of letters and digits allowed, $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6760000$ different possible license plates.

Example 7

In order to store data, a computer's main memory contains a large collection of circuits, each of which is capable of storing a *bit* - that is, one of the binary digits 0 or 1. These storage circuits are arranged in units called cells. To identify the cells in a computer's main memory, each is assigned a unique name called its *address*. For some computers, such as embedded microcontrollers (for example, found in the ignition system of an automobile), an address is represented by an ordered list of eight bits, collectively referred to as a *byte*. Using the rule of product, there are $2^8 = 256$ such bytes. So, we have 256 addresses that may be used for cells where certain information may be stored.

Example 7 (*contd.*)

A kitchen appliance, such as a microwave oven, incorporates an embedded microcontroller. These "small computers" contain thousands of memory cells and use two-byte addresses to identify these cells in their main memory. Such addresses are made up of two consecutive bytes, or 16 consecutive bits. Thus, there are $256 \times 256 = 2^8 \times 2^8 = 65536$ available addresses that could be used to identify cells in the main memory.

In continuation of the applications of the rule of product, we now turn to counting linear arrangements of objects. These arrangements are often called *permutations* when the objects are distinct.

Definition

Given a collection of n distinct objects, any (linear) arrangement of these objects is called a *permutation* of the collection.

If there are n objects and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects is

$P(n,r) = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$ because the 1^{st} position can be filled in 2^{nd} in $(n-1)$ ways, $\cdots$, r^{th} position in $(n-r+1)$ ways.

Thus, $P(n,r)$ can be simplified as

$P(n,r) = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) \times \frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)} = \frac{n!}{(n-r)!}$.

Remember that $P(n,r)$ counts the linear arrangements in which the objects are *not repeated*. However, if the repetitions are allowed, then by the rule of product, there are n^r possible arrangements, with $r \geq 0$.

Example 8

The number of permutations of the letters in the word COMPUTER is $8!$. If only 5 of the letters are used, then it is $P(8,5) = \frac{8!}{(8-5)!} = 6720$. If repetitions of letters are allowed, the number of possible 12-letter sequences is 8^{12}.

Example 9

Unlike Example 8, the number of linear arrangements of the 4 letters in BALL is 12, not $4! = 24$. The reason is that we do not have 4 distinct letters to arrange.

Example 10

The number of permutations of the letters in DATABASES: Note that A has the repeated occurrence of 3 times, while S occurs 2 times. When these repeated letters A and S are distinguishable, then we have 9! permutations. Of them, A's are permuted 3! times and S's are permuted 2! times. Obviously, when only such repeated letters are permuted, the resulting permutations are indistinguishable and they are $3! \times 2!$ in number. Hence the number of permutations of the letters in DATABASES=$\frac{9!}{2!3!} = 30240$.

The examples 9 and 10 can be generalized as follows:
If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable of a second type, $\cdots$, and n_r indistinguishable objects of an r^{th} type, where $n = n_1 + n_2 + \cdots + n_r$, then there are $\frac{n!}{n_1!n_2!\cdots n_r!}$ permutations of the given n objects.

Example 11

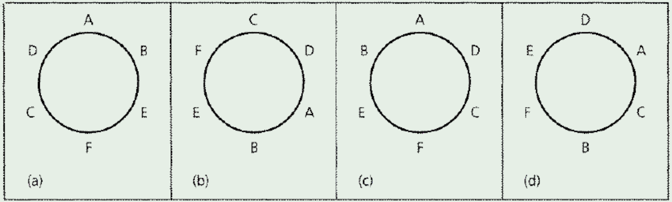
If n and k are two positive integers such that $n = 2k$, then show that $\frac{n!}{2^k}$ is an integer.

Solution: Consider two copies of $x_1, x_2, \cdots, x_k$ which will be totally n symbols (because $n = 2k$). The number of permutations of these n symbols is $\frac{n!}{2!2!\cdots(k \text{ times})\cdots 2!} = \frac{n!}{2^k}$, which is an integer.

Exercise 1

If 6 people, designated as A, B, C, D, E, F are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation? (*Ans.: 120*)

(In the following figure, the arrangements (a) and (b) are considered identical, whereas (b),(c) and (d) are three distinct arrangements.)



Exercise 2

In a slight modification to Exercise 1, the six people are three married couples and that A, B and C are females. The six people are to be arranged around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.) What is the total number of such arrangements? (*Ans.: 12*)

The standard deck of playing cards has 52 cards comprising four suits: clubs, diamonds, hearts and spades. Each suit has 13 cards: ace, $2, 3, \cdots, 9, 10$, jack, queen and king. If we are asked to draw 3 cards from such a standard deck, in succession, and without replacement, then the rule of product, there are $52 \times 51 \times 50 = \frac{52!}{49!} = P(52, 3)$ possibilities, one of which is AH (ace of hearts), $9C$ (9 0f clubs), KD (king of diamonds). If, instead, we simply select 3 cards at one time from the deck so the order of selection is no longer important, then the 6 permutations of $AH - 9C - KD$ will not be distinguishable. Consequently, each selection, or combination, of 3 cards, *with no reference to order*, corresponds to 3! permutations of 3 cards. That is,

$$(3!) \times (\text{No. of selections of size 3 from a deck of 52}) = P(52, 3)$$

This leads us to define:

Definition

If we start with n distinct objects, each selection or combination, of r of these objects, with no reference to the order, corresponds to $r!$ permutations of size r from the n objects. Thus, the no. of combinations of size r from a collection of size n is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} \text{ for } 0 \leq r \leq n.$$

Note

When dealing with any counting problem, we have to note the importance of order in the problem. When order is relevant, we consider permutations. When order is not relevant, combinations play a key role.

Example 12

The Sports faculty in NITK has to select 9 girls and the junior and senior classes of M.Tech. for a Volleyball team. If there are 28 juniors and 25 seniors, he can make the selection in $C(53, 9) = 4431613550$ ways.
If two juniors and one senior are the best spikers and must be on the team, then the rest of the team can chosen in $C(50, 6) = 15890700$ ways.
For a certain tournament, the team must comprise of 4 juniors and 5 seniors. Then the faculty can select 4 juniors in $C(28, 4)$ ways. For each of these selections, the faculty has $C(25, 5)$ ways to choose 5 seniors. So, by the rule of product, such a team can be selected in $C(28, 4)C(25, 5)$ ways.

Some problems can be treated as either permutations or combinations, depending on how one analyzes the situation. The next example illustrates this:

Example 13

The faculty in Example 12 has to make 4 volleyball teams of 9 girls each from the 36 freshers. In how many ways can he select the 4 teams? Let us call the teams A, B, C and D.
Solution: To form Team A, he can select any 9 girls from the 36 in $\binom{36}{9}$ ways. For Team B, the selection process has $\binom{27}{9}$ possibilities. This leaves $\binom{18}{9}$ and $\binom{9}{9}$ possible ways to select Teams C and D, respectively. So, by the rule of product, the four teams can be chosen in

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \left(\frac{36!}{9!27!}\right)\left(\frac{27!}{9!18!}\right)\left(\frac{18!}{9!9!}\right)\left(\frac{9!}{9!0!}\right) = \frac{36!}{9!9!9!9!} \text{ ways.}$$

Next example illustrates how some problems require the concepts of both permutations and combinations for their solutions:

Example 14

The number of arrangements of the letters in $TALLAHASSEE$ is $\frac{11!}{3!2!2!2!1!1!} = 831600$. How many of these arrangements have no adjacent A's?

Solution: When we disregard A's, there are $\frac{8!}{2!2!2!1!1!} = 5040$ ways to arrange the remaining letters. One of these 5040 ways is shown below, where the arrows indicate 9 possible locations for the 3 A's.

Three of these locations can be selected in $\binom{9}{3} = 84$ ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are $5040 \times 84 = 423360$ permutations/ arrangements of the letters $TALLAHASSEE$ with no consecutive A's.

We have seen earlier that when repetitions are allowed, for n distinct objects, a permutation of size r of these objects can be made in n^r ways, where $r \geq 0$. Let us now look at a comparable problem for the combinations.

Definition

When we wish to select with repetition, r of n distinct objects, it can be done in $\binom{n+r-1}{r}$ ways. We also denote this as $C(n+r-1, r)$.

Exercise 3

Explain the above with a suitable example and use the logic behind the explanation to exhibit the definition.

Example 15

A caterer offers 20 types of dosas. Assuming that he has batter and the other ingredients for making a dozen of each kind of the dosas, we can select the dosas in $C(20 + 12 - 1, 12) = C(31, 12) = 141120525$ ways of making the dosas!

Example 16

A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least 3 spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

Solution: There are 12! ways to arrange the 12 different symbols, and for each of these arrangements there are 11 positions between the 12 symbols. Because there must be at least 3 spaces between successive symbols, we use up 33 of the 45 spaces and must now locate the remaining 12 spaces. This is now a selection, with repetition, of size 12 (the spaces) from a collection of size 11 (the locations), and this can be accomplished in $C(11 + 12 - 1, 12) = 646646$ ways. Consequently, be the rule of product, the transmitter can send such messages with the required spacing in $(12!)\binom{22}{12} \approx 3.097 \times 10^{14}$ ways.

Example 17

Determine all integer solutions of $x_1 + x_2 + x_3 + x_4 = 7$ where $x_i \geq 0$ for all $1 \leq i \leq 4$.

Solution (outline only): One solution is $x_1 = 3$, $x_2 = 3$, $x_3 = 0$ and $x_4 = 1$. This is viewed as a different solution from $x_1 = 1$, $x_2 = 0$, $x_3 = 3$ and $x_4 = 3$ because a possible interpretation for the first solution can be given as distributing 7 identical chocolates among 4 children as 3 chocolates for the first two children, nothing to the third and one to the fourth. Continuing with this interpretation, we note that each non-negative integer solution of the equation corresponds to a selection, with repetition, of size 7 (identical chocolates) from a collection of size 4 (distinct children); so, there are $C(4 + 7 - 1, 7) = 120$ solutions.

Example 18

Each term of the binomial expansion of $(x+y)^n$ is of the form $\binom{n}{k}x^k y^{n-k}$; so the total number of terms in the expansion is the number of non-negative integer solutions of $n_1 + n_2 = n$ (n_1 is the exponent for x and n_2 is the exponent for y). This number is $C(2+n-1, n) = n+1$.

How many terms are there in the expansion of $(w+x+y+z)^{10}$?
Solution: Each distinct term here is of the form $\binom{10}{n_1, n_2, n_3, n_4} w^{n_1} x^{n_2} y^{n_3} z^{n_4}$, where $0 \le n_i$ for $1 \le i \le 4$ and $n_1 + n_2 + n_3 + n_4 = 10$. This last equation can be solved in $C(4+10-1, 10) = 286$ ways; so there are 286 terms in the expansion of $(w+x+y+z)^{10}$.

Example 19

Consider the following program segment, where i, j and k are integer variables.

```
for i := 1 to 20 do
    for j := 1 to i do
        for k := 1 to j do
            print (i * j + k)
```

How many times is the **print** statement executed in this program segment?

Solution: Among the choices for i, j and k(in the order i-first, j-second, k-third) that will lead to the execution of the **print** statement, we note that any selection a, b, c ($a \leq b \leq c$) of size 3, with repetitions allowed, from the list $1, 2, 3, \cdots, 20$ results in one of the correct selections: here, $k = a$, $j = b$, $i = c$. Consequently, the **print** statement is executed $\binom{20+3-1}{3} = \binom{22}{3} = 1540$ times.

If there were $r(\geq 1)$ **for** loops, then the **print** statement would have been executed $\binom{20+r-1}{r}$ times.

Algorithm Design: Estimating the number of operations in recursive and iterative algorithms.

Database Query Optimization: Determining the number of possible query plans.

Cryptography: Counting the number of keys or passwords possible in a given security system.

Network Design: Calculating paths in routing algorithms and communication models.

Automata Theory: Counting possible states and transitions in finite automata.

Programming: Loop combinations, nested iteration counts, and branching structure analysis.

Often, complex problems require the combined use of both principles. For instance, when different types of components are selected in combination (product rule) and different categories of selection are considered (sum rule).

Example

Consider a system where a user can either register (3 steps) or reset a password (2 steps). Each step has 2 methods. The total number of user interactions is $(3 \times 2) + (2 \times 2) = 10$.

1. How many unique IP addresses can be formed with 3-digit subnets where each digit is from 0–9?
2. In a web application, if there are 4 pages and each page has 5 navigational options, how many unique paths exist from the homepage to another page using one click?
3. A secure code requires selecting 2 lowercase letters followed by 3 digits. How many such codes are possible?

Understanding the Fundamental Principles of Counting is critical for effectively solving real-world problems in computer science. These principles support logical reasoning in algorithm analysis, data structure manipulation, and systems design. Mastery of these ideas lays the groundwork for advanced topics in discrete mathematics and theoretical computer science.

1. Allan Tucker, *Applied Combinatorics*, 4^{th} edition, Wiley, NY, 2002.
2. Kenneth Rosen, *Discrete Mathematics and its Applications*, 5^{th} edition, McGraw Hill, NY, 2003.