Embedded Control Laboratory

Report of lab Exercise 1

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Group 10

1762859	**Gopal Vaid**
1767672	**Samyak Indurkar**
1781238	**Darren Melroy Menezes**

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Lab 1: Computational Fluid Dynamics Modelling Flow in pipes.

1.1 Introduction

Getting a grip on how fluid behaves is crucial in various branches of engineering, whether it's designing airplanes or optimizing processes in chemical plants. Creating accurate computer simulations that depict the movement of these fluids is a key step in designing systems that perform efficiently, anticipating how these systems will behave in different scenarios, and troubleshooting any operational hiccups.

In our lab session, we are going to take a closer look at the mathematical and computational aspects of building models that represent how liquids flow through pipes. Think of it as creating a virtual version of what happens when water or air moves through a system of pipes. This virtual representation helps us understand the dynamics and intricacies of fluid flow, allowing us to predict and control the behavior of fluids in various engineering applications.

The tool we will be using as the foundation for these simulations is something called the Navier-Stokes equations. So, in a nutshell, this lab is about exploring the mathematical and computational tools we use to create digital models of fluid flow in pipelines, using the Navier-Stokes equations as our guide. It is a hands-on way of understanding and predicting how liquids move in various engineering scenarios.

1.1.1 Introduction to Python:

The selection of Python as the programming language for modeling fluid flow in pipes is underpinned by its unique combination of versatility, community support, and a robust scientific computing ecosystem. Python's syntax is renowned for its readability and simplicity, rendering it accessible to engineers and scientists with varying levels of programming expertise. The extensive community surrounding Python, along with its comprehensive suite of scientific libraries, including NumPy, and Matplotlib, endows it with powerful capabilities for numerical analysis, scientific computation, and data visualization.

Python's adaptability is particularly advantageous in the context of fluid dynamics modeling. Its ease of use facilitates rapid prototyping and experimentation, enabling the swift development and testing of computational models, such as those based on the Navier-Stokes equations. The language's cross-platform compatibility ensures the seamless deployment of models across diverse computing environments. Furthermore, Python's capacity for seamless integration with other languages, such as C and Fortran, allows for the incorporation of high-performance modules, addressing computational efficiency requirements.

In essence, the adoption of Python for fluid flow modeling is grounded in its accessibility, widespread acceptance, and computational robustness, making it an ideal choice for engineering applications demanding precision and versatility.

1.1.2 Introduction to Navier–Stokes equations:

The Navier-Stokes equations represent a sophisticated framework elucidating the temporal evolution of a fluid's velocity field. They constitute an advanced adaptation of Newton's second law, precisely calibrated for the intricate domain of fluid dynamics, encapsulating the fundamental principles of mass, momentum, and energy conservation. These equations, denoted as Equation 1 and Equation 2 in their generalized form, delineate the intricate dynamics inherent in fluid motion.

The Mass Conservation Equation (Eq. 1) eloquently articulates the dynamic variations in the fluid's density (ρ) concerning both temporal and spatial dimensions:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \; - \; eq.\ 1$$

Conceptualize it as a meticulous inventory, systematically documenting the distribution of the fluid across diverse spatial coordinates as it gracefully traverses its trajectory.

Conversely, the Momentum Conservation Equation (Eq. 2), a more intricate facet of this exposition, scrutinizes the dynamic interplay of velocity (u), pressure (p), stress (τ), and external forces (F), such as gravity:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{F} \; - \; \mathbf{eq.\ 2}$$

This equation unravels the intricacies of how the fluid's speed transforms over time, providing a comprehensive depiction of its trajectory under the influence of various forces.

In these equations: (ρ) denotes fluid density, (u) signifies velocity, (p) represents pressure, (τ) is stress, and (F) characterizes external forces, typically gravitational. The ∇ symbol functions as a gradient operator for three-dimensional space, guiding our understanding of spatial changes. When confronted with compressible fluids capable of contraction or expansion, the conservation of energy assumes an additional facet in this captivating symphony of fluid dynamics.

Navier–Stokes equations in cylindrical coordinates:

In cylindrical coordinates, the Navier-Stokes equations get a bit specialized, but offer a neat way to understand fluid flow in systems with circular symmetry. The different equations in cylindrical coordinates are

Mass Conservation Equation:

In simple terms, equation 3 says that, when accounting for radial (u_r), angular (u_θ), and axial (u_z) velocities, it is balanced. It ensures that the amount of fluid entering or leaving any given point is in harmony.

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0\ -\ eq.\ 3$$

Radial Momentum Conservation Equation:

Equation 4 dives into the radial direction. It considers how the radial velocity (u_r) changes over time and space, factoring in pressure (p), viscosity (μ), and gravitational forces (g). Essentially, it explains how the fluid moves outward or inward in the radial direction.

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_\theta\frac{1}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left[\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]+F_r\ -\ eq.\ 4$$

Angular Momentum Conservation Equation:

Here, we are looking at how the angular velocity (u_θ) changes, considering pressure gradients, viscosity effects, and gravitational forces in the angular direction. Equation 5 helps us understand the swirling motion of the fluid.

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+u_\theta\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\nabla^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]+F_\theta\ -\ eq.\ 5$$

Axial Momentum Conservation Equation:

In the axial direction, equation 6 tells us how the axial velocity (u_z) changes due to pressure, viscosity, and gravitational forces in the axial direction. It describes how the fluid moves along the length of the cylinder.

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_\theta\frac{1}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\nabla^2 u_z+F_z\ -\ eq.\ 6$$

In equation 7, equation 8, equation 9, ∇^2 is the Laplace operator in cylindrical coordinates, and are the body force components in the respective directions. ∇^2 for each component in cylindrical coordinates will be:

For u_r:

$$\nabla^2 u_r=\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}\ -\ eq.\ 7$$

For u_θ:

$$\nabla^2 u_\theta=\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}\ -\ eq.\ 8$$

For u_z:

$$\nabla^2 u_z=\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\ -\ eq.\ 9$$

These are Navier-Stokes equations for an incompressible fluid in cylindrical coordinates. These are general equations, and they include time dependence, radial, azimuthal, and axial velocities, and external body forces.

Task 1:

Model a steady-state and fully-developed axial flow of incompressible Newtonian fluid in a cylindrical pipe, assuming axisymmetric flow with no radial velocity.

Task 1.1:

Formulate the Partial differential equations by simplifying the Navier-Stokes equations for cylindrical coordinates.

Solution:

To obtain the partial differential equations for a steady-state and fully-developed axial flow of incompressible Newtonian fluid in the cylindrical pipe, we assume that the flow of fluid has the following properties.

1. Incompressible Flow (Constant Density):

- Explanation: The fluid's density (ρ) remains constant. It does not get compressed or spread out.

2. Steady-State:

- Explanation: The flow does not change with time ($\frac{\partial}{\partial t}$ = 0). It is like hitting pause; everything stays constant.

3. Axisymmetric Flow:

- Explanation: The flow is symmetrical around the pipe's axis. There is no swirling motion ($\frac{\partial}{\partial \theta}$ = 0) and no angular velocity (u_θ=0).

4. **Fully Developed:**

- Explanation: The fluid's velocity in the axial direction (u_z) does not change along the pipe's length ($\partial u_z/\partial z = 0$). However, there can be changes in pressure along the pipe ($\frac{\partial p}{\partial z} \neq 0$).

5. No Radial Velocity:

- Explanation: There is no movement in or out from the center of the pipe (u_r= 0). The fluid only moves along the length and around the axis.

6. Newtonian Fluid (Constant Viscosity):

- Explanation: The fluid behaves like a typical Newtonian fluid. The viscosity (μ) remains constant.

After applying the above assumptions in the Navier-Stokes equation, we get the final expression of the Partial differential equation in axial component (u_z):

As u_r= 0 (No radial velocity), $\frac{\partial}{\partial \theta} = 0$ (Axisymmetric flow), and $\frac{\partial u_z}{\partial z} = 0$,

So, the equation becomes 0.

Now consider the **Radial momentum conservation equation:**

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_\theta\frac{1}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] + F_r \;-\; eq.\ 10$$

As u_r = 0 (No radial velocity), the whole equation will become 0.

Similarly, consider the **Angular momentum conservation equation:**

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+u_\theta\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\nabla^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]+F_\theta\ -eq.\ 11$$

As u_θ=0 (Flow is axisymmetric), so this equation is also 0.

We are left with only the **Axial momentum conservation equation:**

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_\theta\frac{1}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\nabla^2 u_z+F_z\ -\ eq.\ 12$$

As flow is steady state, fully developed, with no radial velocity, and axisymmetric. The equation 12 reduces to the following expression:

$$0=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right]\ -\ eq.\ 13$$

Task 1.2:

In this, we will convert the Partial differential equation into a finite difference equation.

$$0=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right]\ -\ eq.\ 14$$

Substitute from PDE to make it a finite difference equation:-

$$\frac{\partial^2 u_z}{\partial r^2}\approx\frac{u_{z,i+1}-2u_{z,i}+u_{z,i-1}}{\Delta r^2}\ -\ eq.\ 15$$

$$\frac{\partial u_z}{\partial r}\approx\frac{u_{z,i+1}-u_{z,i}}{\Delta r}\ -\ eq.\ 16$$

After substituting the values, the finite difference equation becomes:

$$0=-\frac{\partial p}{\partial z}+\mu\left[\frac{u_{z,i+1}-2u_{z,i}+u_{z,i-1}}{\Delta r^2}+\frac{1}{r}\frac{u_{z,i}-uz,i-1}{\Delta r}\right]\ -\ eq.\ 17$$

$$0 = -\frac{\partial P}{\partial z} + \frac{\mu}{\Delta r^2}\left[u_{z,i+1} - 2u_{z,i} + u_{z,i-1}\right] + \frac{1}{r\Delta r}\left[u_{z,i} - u_{z,i-1}\right] \; - \; eq.\; 18$$

After Simplifying the equation in the form of linear equation Au = B, we get equation 19

$$\frac{\partial p}{\partial z}\frac{1}{\mu} = \left[\frac{1}{\Delta r^2} - \frac{1}{r_i \Delta r}\right]u_{z,i-1} - \left[\frac{2}{\Delta r^2} + \frac{1}{r_i \Delta r}\right]u_{z,i,} + \frac{1}{\Delta r^2}\cdot u_{z,i+1} \; - \; eq.\; 19$$

<u>Task 2:</u>

In Task 2, we write a Python program to solve the system of linear equations derived from the finite difference equations obtained from the previous task. The difference equation for the axial velocity u_z in cylindrical coordinates is given by:

$$\frac{\partial p}{\partial z}\frac{1}{\mu} = \left[\frac{1}{\Delta r^2} - \frac{1}{r_i \Delta r}\right]u_{z,i-1} - \left[\frac{2}{\Delta r^2} + \frac{1}{r_i \Delta r}\right]u_{z,i,} + \frac{1}{\Delta r^2}\cdot u_{z,i+1} \; - \; eq.\; 20$$

This equation is used to solve for u_z as a function of r. The following boundary conditions and parameters were considered.

Boundary Conditions:

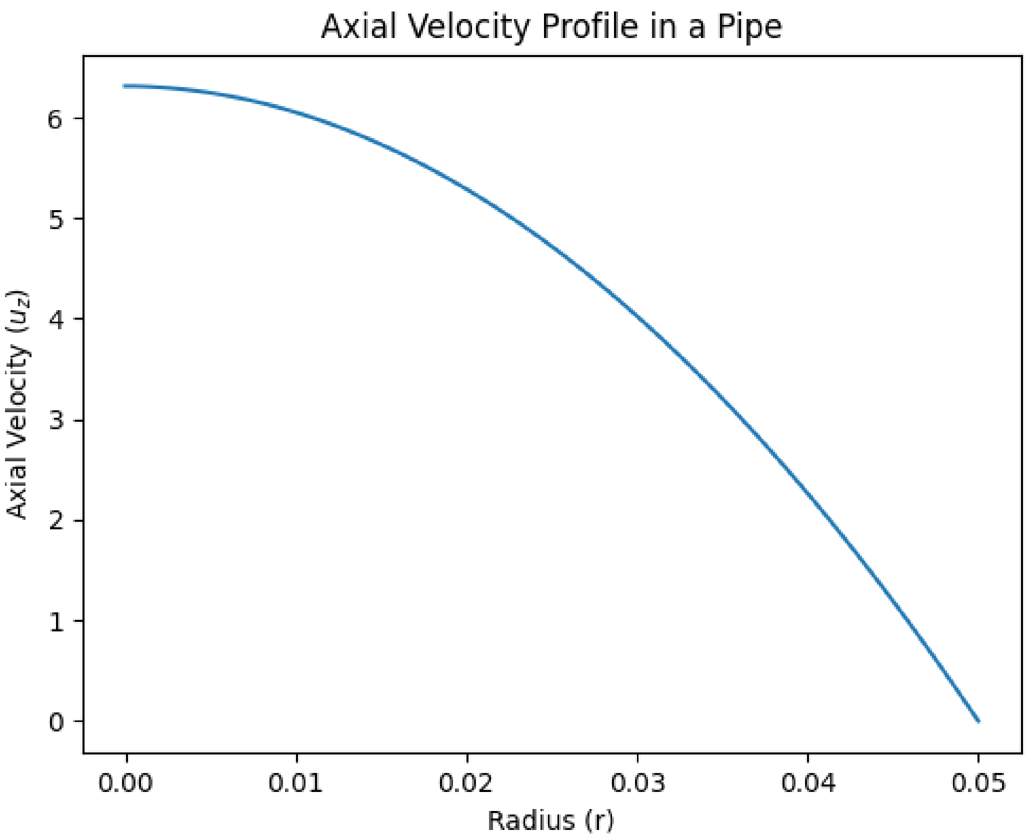
- at $r = 0$, $\frac{\partial u_z}{\partial r} = 0$
- at r = R, $u_z = 0$

Parameters:

- Pipe Diameter D = 0.1 m
- Viscosity, = 0.01 kg/m.s
- Constant pressure gradient = -100 Pa/m
- N =100 grid points

Observations / Result

The axial velocity profile u_z was visualized as a function of radial position r. The figure below illustrates the key observations:



From the figure, it is evident that the axial velocity at the u_z direction at r = 0 is 6.25, which represents the maximum velocity. As r decreases to R, the velocity follows a parabolic curve until r = 0, where u_z becomes 0.

Task 3:

In Task 3, we implemented a Python program to simulate fluid flow within a pipe, incorporating the dependence of fluid viscosity on temperature, denoted as μ(T). This viscosity-temperature relationship is described by the equation:

We used parameters and code from task 2 and replaced static viscosity with dynamic viscosity obtained from equation 21 and equation 22.

$$\mu(T) = \mu_0(1 + \alpha(T - T_0)) \; - \; eq.\ 21$$

Additionally, the fluid temperature within the pipe varies radially according to the expression:

$$T(r) = T_0 + \beta r^2 \;\; - eq.\ 22$$

Additional Parameters for task 3:

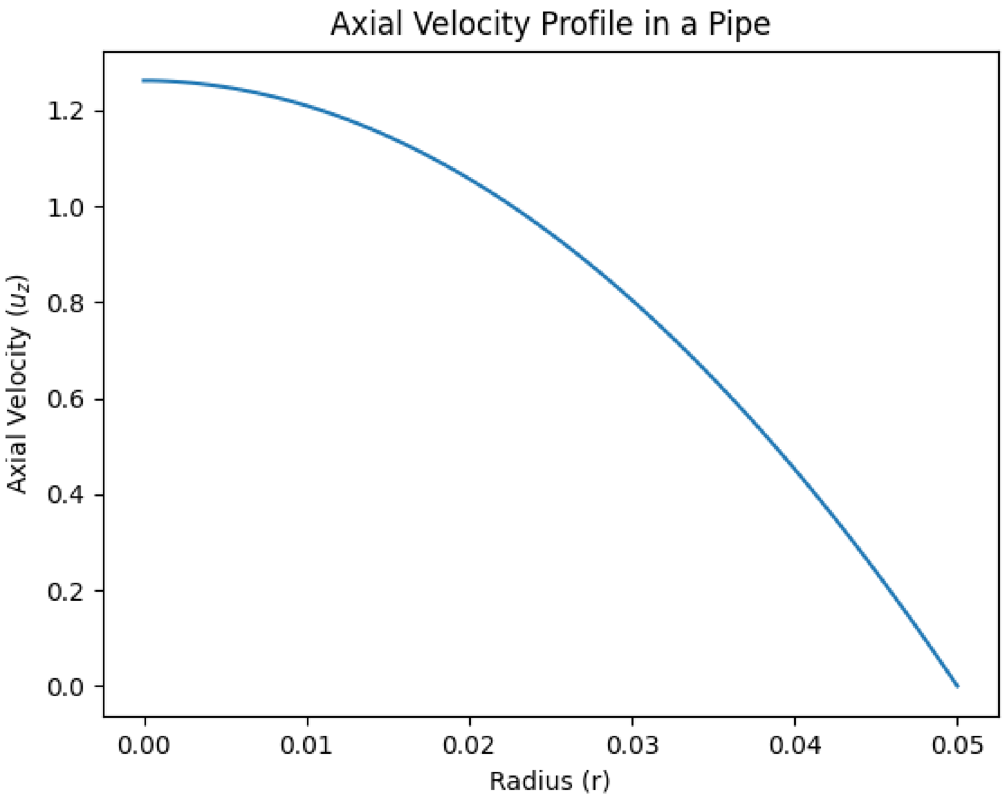
β = 5 K/m^2

Reference temperature T_0 = 273 K

Base viscosity μ_0 = 0.05 kg/m.s

Observations / Result

The simulation, with a base viscosity μ_0 set to 0.05 kg/(m·s), reveals that the resulting velocity at r = 0 is 1.2623 m/s. As we progress radially toward R, the velocity profile follows a parabolic trend, indicating a reduction in velocity with increasing radial distance.



Task 4:

Consider the model for the flow of incompressible Newtonian fluid in a pipe where the velocity profile u_z varies with both radial (r) and axial (z) coordinates, according to the given equation:

$$\mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + \gamma\frac{\partial^2 u_z}{\partial z^2} = \frac{\partial p}{\partial z} \quad - \ eq.\ 23$$

Solution:

In this task, change the given partial differential equation into finite difference equation and simplify further into linear form to solve it in given boundary conditions and parameters.

Substitute equation 15 and equation 16 in equation 23:

$$\mu\left[\frac{u_{z,i+1}-2u_{z,i}+u_{z,i-1}}{\Delta r^2}+\frac{u_{z,i+1}-u_{z,i}}{r\Delta r}\right]+\gamma\left[\frac{u_{z,i+1}-2u_{z,i}+u_{z,i-1}}{\Delta z^2}\right]=\frac{\partial p}{\partial z}\quad -\ eq.\ 24$$

$$\frac{\mu u_{z,i+1}}{\Delta r^2}+\frac{\mu u_{z,i+1}}{r\Delta r}+\frac{\gamma u_{z,i+1}}{\Delta z^2}-\frac{\mu 2u_{z,i}}{\Delta r^2}-\frac{\mu u_{z,i}}{r\Delta r}-\frac{\gamma 2u_{z,i}}{\Delta z^2}+\frac{\mu u_{z,i-1}}{\Delta r^2}+\frac{\gamma u_{z,i-1}}{\Delta z^2}=\frac{\partial p}{\partial z}\quad -\ eq.\ 25$$

After simplifying the above equation into linear form Au = B, we get

$$\left[\frac{\mu}{\Delta r^2}+\frac{\mu}{r\Delta r}+\frac{\gamma}{\Delta z^2}\right]u_{z,i+1}-\left[\frac{2\mu}{\Delta r^2}+\frac{\mu}{r\Delta r}+\frac{2\gamma}{\Delta z^2}\right]u_{z,i}+\left[\frac{\mu}{\Delta r^2}+\frac{\gamma}{\Delta z^2}\right]u_{z,i-1}=\frac{\partial p}{\partial z}\quad -\ eq.\ 26$$

Additional Parameters for task 4:

$\gamma = 0.25$

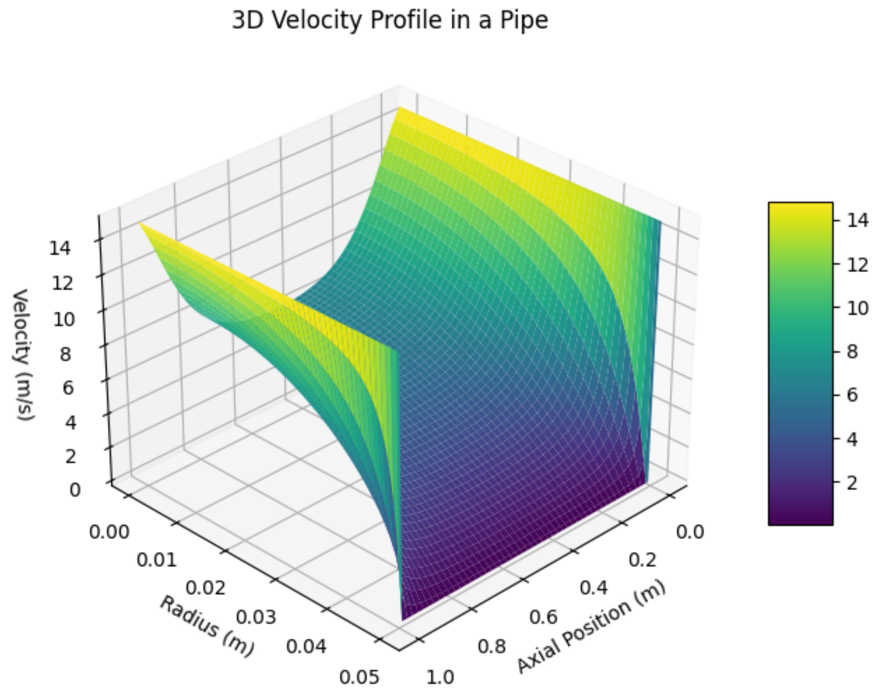
Boundary conditions for task 4:

- at $r = 0,\ \frac{\partial u_z}{\partial r} = 0$
- at, $r = R,\ u_z = 0$
- at $z = 0,\ u_z = 15$
- at, $r = L,\ u_z = 15$

Observations / Result

At the center of the pipe (r=0), there is a maximum velocity, consistent with the axial symmetry of the flow. At the pipe wall (r=R), the velocity is zero, adhering to the no-slip boundary condition. Along the axial direction, the velocity is initially maximum at both ends of the pipe but decreases towards the middle, forming a parabolic curve. This parabolic distribution is a characteristic feature of fully developed laminar flow. The viscosity of the fluid (μ) and the applied pressure gradient (∂P/∂z) influence the velocity distribution, introducing terms that contribute to the curvature of the profile. The imposed boundary conditions at the pipe ends (z=0 and z=L) and at the pipe axis (r=0) ensure a physically realistic flow pattern, with conditions at the pipe wall (r=R) enforcing the no-slip condition.

The 3D surface plot visually represents the velocity profile across axial and radial coordinates, providing a clear depiction of how the velocity varies within the pipe. The model parameters, including diameter (D), length (L), pressure gradient (dpdz), viscosity (μ), and a constant (γ), influence the flow, allowing for the exploration of different flow scenarios.



1	**Cover sheet**	
1	Group information on cover sheet	yes
2	Date and version specified	yes
2	**Fonts**	
1	Font 11-12pt, black	yes
2	Headings bold, main headings larger than subheadings	yes
3	Font: Times New Roman, CMU, Calibri, or Helvetica	yes
3	**Report structure**	
1	Page numbers in footer	yes
2	Proper headings for different sections (e.g., when answering the lab questions, number and repeat it beforehand)	yes
4	**Figures/plots**	
1	Figures/plots meaningfully planned: properly decided which information is shown/highlighted/demonstrated	yes
2	Figure horizontally centered and upright	yes
3	Figure labeled (caption/title)	yes
4	Axes in plots labeled (description and unit)	yes
5	Grid used (if it supports readability)	yes
6	Multiple colors or types for different lines within a plot	yes
7	Legend (if necessary)	N/A
8	Proper font size for labels and captions (10-12 pt)	yes
9	Plot/figure referenced and described in the text (especially with focus on important characteristics (e.g., behavior of signals))	yes
10	Reasons and conclusions from plots described in the text	yes
11	Appropriate file formats for figures (e.g., latex: eps, pdf, svg; word: svg, emf, (large scaled png), no screenshots)	yes

5	**Equations/formulas**	
1	Centered horizontally and numbered (if standalone)	yes
2	All symbols explained in the text after the equation (Symbols in the formula must have the same font and should not look different. For instance, compare: and β)	yes
6	**Submission**	
1	Grammar and spelling checked	yes
2	Complete report in a single document in pdf format	yes
3	Commented programming code/modeling files provided along with the report (scilab, uppal-files, etc.)	yes
4	Submitted programs run without errors	yes
5	All submission files packed in a zipped archive (.zip, .rar, etc.)	yes
6	Archive named as: group<*your group ID*>_lab<*exercise numbers*> (e.g., group01_lab1_2)	yes
7	Subject of the e-mail as in point 6.6 (e.g., group01_lab1_2)	yes