GRAM: An efficient (k, l) graph anonymization method (**Supplementary Material**)

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1. Illustrative example of the GRAM execution

1.1. The case of $l = 1$

In the following, the steps of the GRAM and the corresponding *costs* matrix are shown. A small graph with $n = 11$ vertices is anonymized for $k = 3$ and $l = 1$. The original graph G and the initial *costs* matrix are shown at first. Phase I of the GRAM adds some edges to the graph to increase the degree of *problematic* vertices whose degrees are $< k$ (see Proposition 1). In the first iteration, $dist = 1$ and each problematic vertex v_i is connected to all of its neighbors at distance $dist + 1$, i.e., v_i is linked to $N_G(v_i, 2)$. All added edges are shown in blue dashed lines in the following figures. In the first step, $e_{1,3}$ and $e_{1,11}$ are added to the graph. Correspondingly, $costs(1, 3) = costs(3, 1)$ and $costs(1, 11) = costs(11, 1)$ are reduced to 10. In the second step, v_3 is connected to both v_1 and v_{11}. Then related *costs* entries are decreased by 1. Please note that the values $costs(1, 3)$ and $costs(3, 1)$ are now equal to 9. In the third step $e_{4,11}$ and $e_{4,6}$ are added and *costs* entries are updated. In step 4, v_6 is connected to v_4 and v_{11}. In the next step, v_7 is linked to v_2, v_5, v_8, and v_{10}. In the final step of the first phase, $e_{9,11}$ is added to the graph. The GRAM in Phase II removes some redundant added edges to decrease information loss. These edges are shown in red dotted lines in steps 7 and 8 in which $e_{2,7}$ and $e_{5,7}$ are removed, respectively. Finally, the protected graph is returned in step 9.

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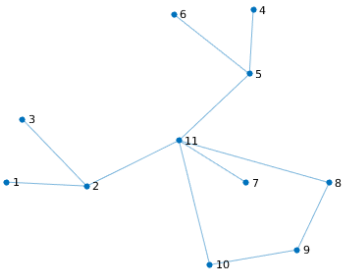
The original graph and initial *costs* matrix



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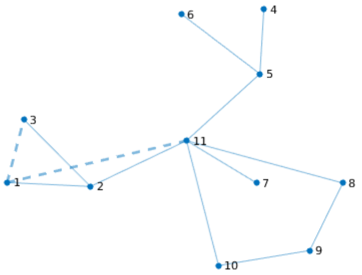
Phase I. Adding edges

Step Graph The *costs* matrix

1



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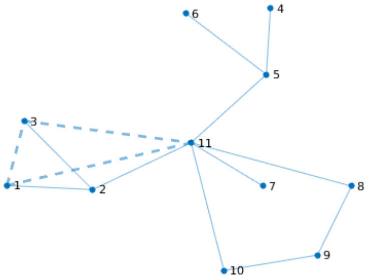

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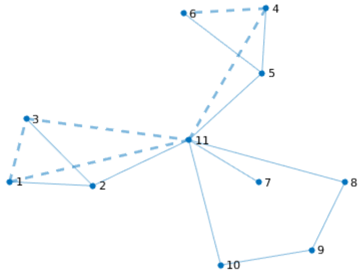

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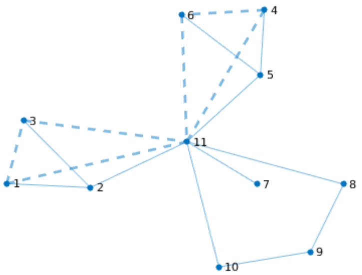

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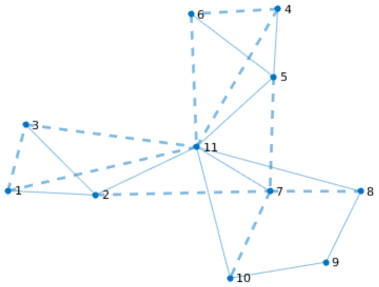

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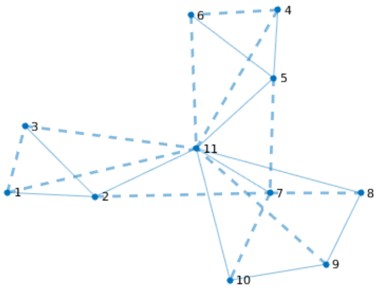
6



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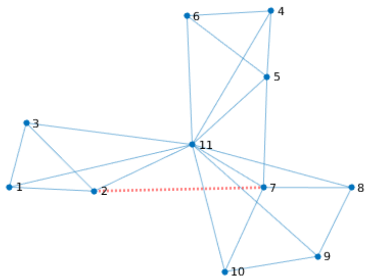
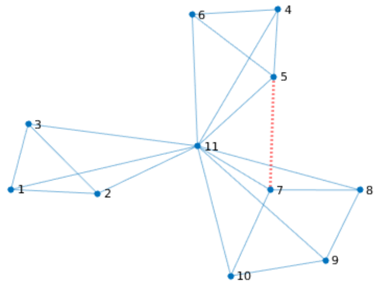
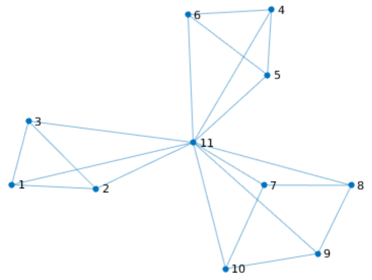
Phase II. Removing edges

Step	Graph	Step	Graph
7		8	

The final $(3,1)$-anonymous graph

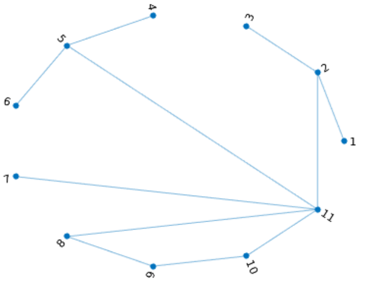
9



1.2. The case of $l > 1$

In the following, the same graph G of the previous section is anonymized for $k = 2$ and $l = 2$. In the original graph, v_2 may be uniquely re-identified as the only common neighbor of v_1 and v_3, or v_3 and v_{11}. Therefore, these neighbors are linked in step 1 by adding $e_{1,3}$, $e_{1,11}$, and $e_{3,11}$. In this graph, v_5 is the only common neighbor of all of its neighbors, i.e., v_4, v_6, and v_{11}. The GRAM adds $e_{4,6}$, $e_{4,11}$, and $e_{6,11}$ to the graph in step 2. Similarly, step 3 adds $e_{2,5}$, $e_{2,7}$, $e_{2,8}$, $e_{2,10}$, $e_{5,7}$, $e_{5,8}$, $e_{5,10}$, $e_{7,8}$, $e_{7,10}$, and $e_{8,10}$ to protect v_{11}. Again, the number of common neighbors of *all* neighbors of v_2 in the current graph G is less than k. Therefore, in step 4, *dist* gets 2, and $e_{1,5}$ and $e_{3,5}$ are added to the graph. Note that $e_{5,11}$ exists in the original graph. Correspondingly, the *costs* values are updated. In step 5, $e_{2,4}$, $e_{2,6}$, and $e_{2,11}$ are used to protect v_5 (v_2 is at distance 2 from v_5). Step 6 considers again v_{11}, and connects v_1 to v_7, v_8, and v_{10} ($e_{1,2}$ and $e_{1,5}$ exist). Note that $costs(1,5)$ and $costs(5,1)$ are decreased again in this step. In the second phase of the GRAM, all added edges are processed in decreasing order of their associated *costs* values and, some of them are removed if the privacy condition is not violated. More precisely, $e_{1,3}$, $e_{2,4}$, $e_{2,5}$, $e_{3,5}$, $e_{2,6}$, $e_{4,6}$, $e_{1,7}$, $e_{2,7}$, $e_{5,7}$, $e_{1,8}$, $e_{2,8}$, $e_{5,8}$, $e_{7,8}$, $e_{1,10}$, $e_{2,10}$, $e_{5,10}$, $e_{7,10}$, $e_{8,10}$, $e_{1,11}$, $e_{3,11}$, $e_{4,11}$, $e_{6,11}$, and $e_{1,5}$ are checked for removal in order, but only $e_{1,3}$, $e_{2,4}$, $e_{2,5}$, $e_{2,6}$, $e_{1,7}$, $e_{2,7}$, $e_{5,7}$, $e_{1,8}$, $e_{2,8}$, $e_{5,8}$, $e_{1,11}$, and $e_{3,11}$ are removed in phase II (shown in red dotted lines). The last step returns the final $(2, 2)$-anonymous graph.

The original graph and initial *costs* matrix



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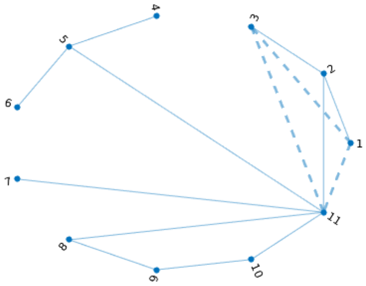
Phase I. Adding edges

Step Graph The *costs* matrix

1



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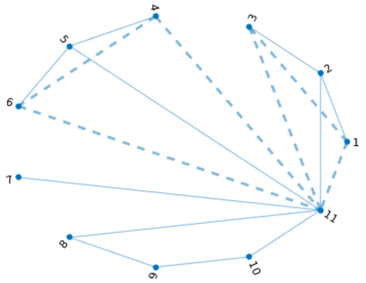

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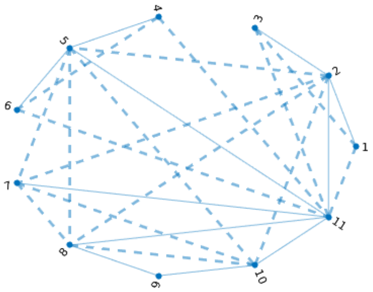

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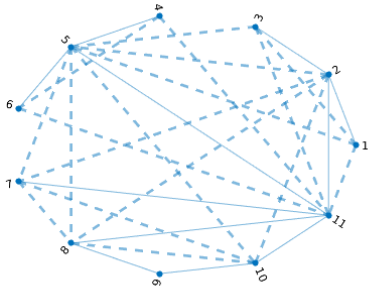

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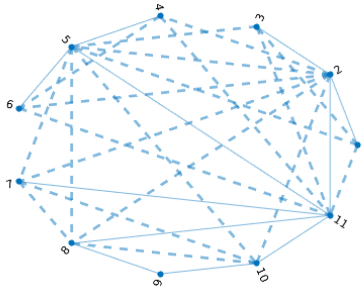

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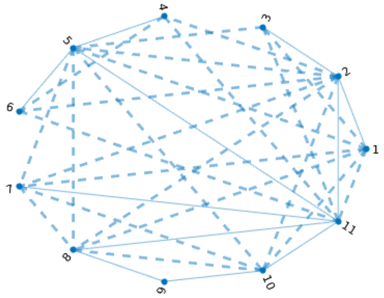

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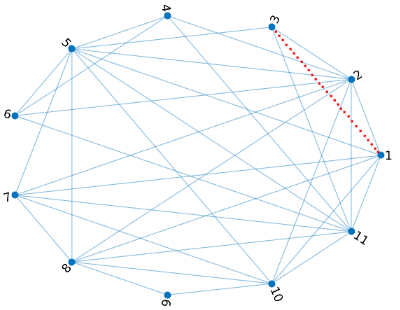
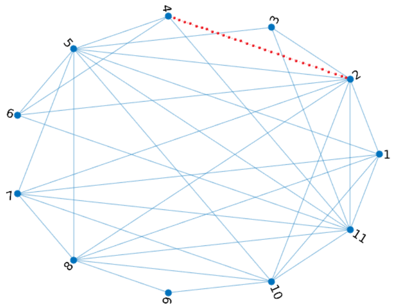
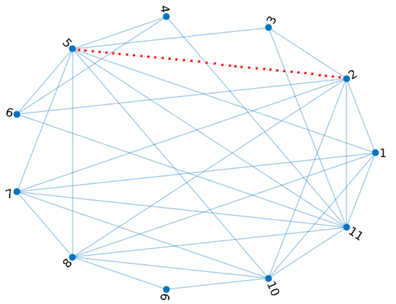
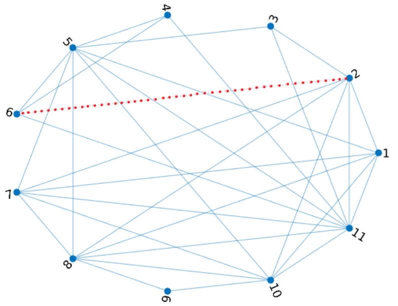
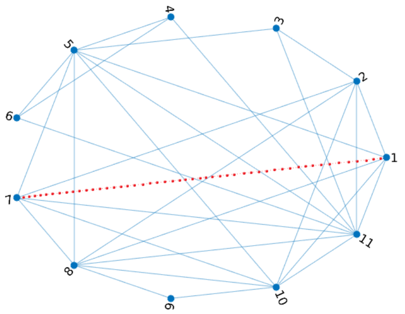
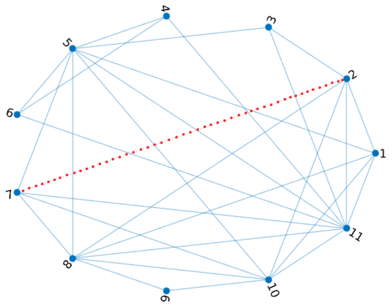
Phase II. Removing edges

Step	Graph	Step	Graph
8		9	
10		11	
12		13	

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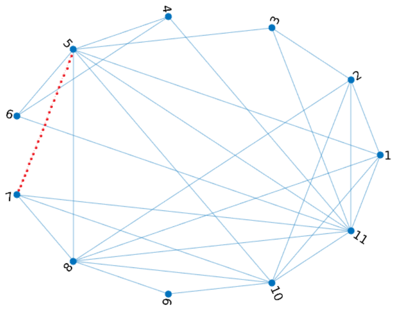

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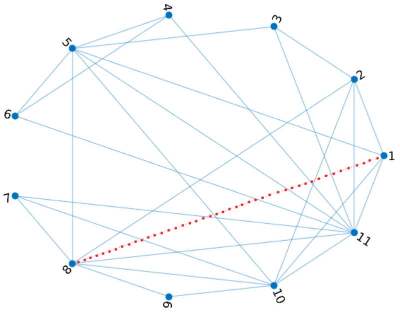

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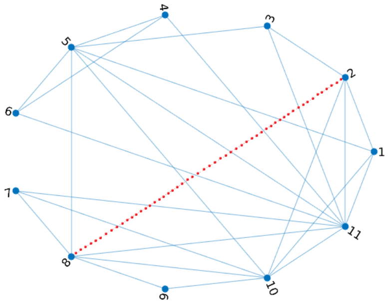

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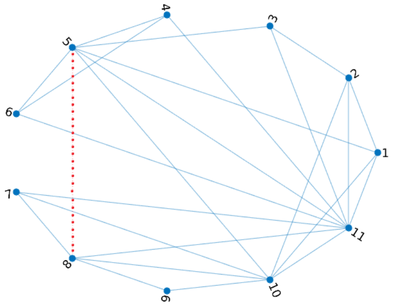

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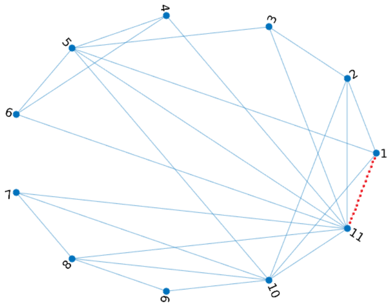

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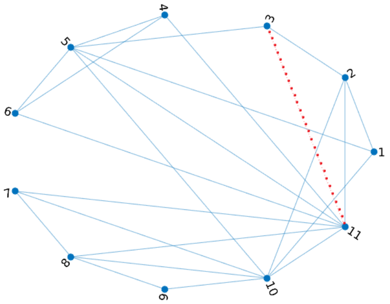

The final $(2, 2)$-anonymous graph