Signals

Euler's formula: $e^{i\theta} = \cos\theta + j\sin\theta$

$z = m \cdot e^{j\theta}$, $m \cdot \sin\theta$, $m \cdot \cos\theta$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2i}$$

SIGNALS

Delta (Kronecker & Dirac)

$x(t)\delta(t-a) = x(a)\delta(t-a)$

- D $\delta(t) = 0 \;\; \forall t \neq 0$; ∞ at $t=0$
- D $\int_A^B \delta(t)\,dt = 1,\ A < 0 < B,\ 0$ O.W
- D Sampling: $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
- D Sifting: $\int_{-\infty}^{\infty} y(\tau)\delta(x-\tau)\,d\tau = y(x)$
- D Convolution: $y(t) = a\,\delta(t-t_0) \Rightarrow (x*y)(t) = a\,x(t-t_0)$
- D Time-Scaling: $\delta(at) = \frac{1}{|a|}\delta(t)$

D: Other defs:

$$\delta(t) = \frac{du(t)}{dt} = \lim_{\epsilon\to 0} \frac{u(t) - u(t-\epsilon)}{\epsilon}$$

$$= \int_{-\infty}^{\infty} e^{j2\pi f t}\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\,d\omega$$

Heaviside Step

Rectangle

Ramp

Triangle

Sha/comb

SYSTEM Prop

- **Linearity:** $\alpha x_1[n] + \beta x_2[n] \to \boxed{Sys} \to \alpha y_1[n] + \beta y_2[n]$
  - or: zero input $\Rightarrow$ zero output; $\alpha, \beta = 0$
- **Time Invariance** $x[n] \to \boxed{Sys} \to y[n] \Rightarrow x[n-k] \to \boxed{Sys} \to y[n-k]$
- **Memoryless:** sys only depends on present (No future & past)
- **Causality** $\to$ System only depends on past & present (No future)
- **BiBo Stability** $\to \sum_{n=-\infty}^{\infty} |h(n)| < \infty$, $\int_{-\infty}^{\infty} h(n) < \infty$

$(f*g)*h = f*(g*h)$

$f*(g+h) = (f*g)+(f*h)$

$f*\delta = f$

$\to$ LTI

New freq. can't be created ✗

ZIZO prop. zero input = zero output

Convolution & Correlation

$\mathrm{Cov}(x,y) = (x*y)(t) = x(t)*y(t)$
$= \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau$

$r_{xy} = \mathrm{Cor}(x,y) = (x \circ y)(t) =$
$= \int_{-\infty}^{\infty} x(\tau)\,y(t+\tau)\,d\tau$

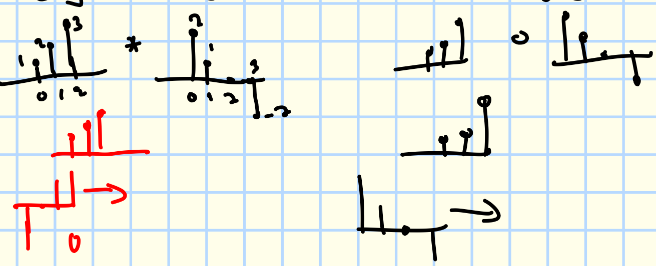
$\mathrm{Cov}(x,y) = (x*y)[n] = x[n]*y[n]$; $r_{xy}[n] = \mathrm{Cor}(x,y) = (x \circ y)[n]$

$= \sum_{k=-\infty}^{\infty} x[k]\,y[n-k]$ $\qquad$ $= \sum_{k=-\infty}^{\infty} x[k]\,y[n+k]$

Flips, Drags

Flip & drag

$x[n] * y[n]$ $\qquad$ $x[n] \circ y[n]$



CTFT

$r_{yx}(t) = r_{xy}(-t)$;
$r_{xx}(t) = r_{xx}(-t)$

$x(t) \Leftrightarrow X(f)$

$x(t)$	$X(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$
$\sum_{k=-\infty}^{\infty}\delta(t-k)$	$\sum_{n=-\infty}^{\infty}\delta(f-n)$
$\Pi(t)$	$\mathrm{Sinc}(f) = \frac{\sin(\pi f)}{\pi f}$
$x_1(t)*x_2(t)$	$X_1(f)X_2(f)$

LTI Systems & Representations

CLCCDE: $a_k \frac{d^k}{dt^k}y(t) + \dots + a_0 y(t) = b_m \frac{d^m}{dt^m}x(t) + \dots + b_0 x(t)$

D $a_k y[n-k] + \dots + a_0 y[n] = b_m x[n-k] + \dots + b_0 x[n]$

General Sol $\Rightarrow y(t) = y_p(t) + y_h(t)$

$$y_h(t) = \sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0$$

$$\left\{ \int \frac{d\,y_h(t)}{dt} = \ln(y_h(t)) \qquad e^{\ln(y_h(t))} = y_h(t) \right\}$$

particular Sol $\Rightarrow \sum_{k=0}^{N} a_k \frac{d^k y_p(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$

gen $\to A e^{s_1 t} + A_2 e^{s_2 t} + \dots$
$\to$ find A at end w/ initial cond.

Impulse response: $\boxed{LTI} = H$

$\delta[n] \to \boxed{H} \to h[n]$

delay: $\delta[n-N] \to \boxed{H} \to h[n-N]$

$\times\, x[n]$: $x[n]\delta[n-N] \to \boxed{H} \to x[n]h[n-N]$

Summation: $\sum_{n=-\infty}^{\infty} x[N]\delta[n-N] \to \boxed{H} \to \sum_{n=-\infty}^{\infty} x[N]h[n-N]$

sifting: $x[n] \to \boxed{H} \to y[n] = \sum_{n=-\infty}^{\infty} x[N]h[n-N]$

Frequency Response: $\to e^{i\omega n}$

$$e^{i\omega n} * h(n) = \sum_{m=-\infty}^{\infty} h(m)e^{i\omega(n-m)} = \underbrace{\sum_{m=-\infty}^{\infty} h(m)e^{-i\omega m}}_{H(\omega)} e^{i\omega n}$$

$e^{i\omega n} \to \boxed{H} \to H(\omega)\,e^{i\omega n}$

$$H(\omega) = \sum_{m=-\infty}^{\infty} h(m)e^{-i\omega m}$$

C $H(\omega) = \dfrac{b_m(i\omega)^m + \dots + b_1(i\omega) + b_0}{a_k(i\omega)^k + \dots + a_1(i\omega) + a_0}$

$h(n) = \delta(n-k) \Leftrightarrow H(\omega) = e^{-i\omega k}$

$h(n) = a^n u(n) \Leftrightarrow H(\omega) = \dfrac{1}{1-a e^{-i\omega}}$

D $H(\omega) = \dfrac{b_m e^{-i\omega m} + \dots + b_1 e^{-i\omega} + b_0}{a_k e^{-i\omega k} + \dots + a_1 e^{-i\omega} + a_0}$

CTFS

(Periodic function $\Leftrightarrow$ Discrete frequencies)

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Exp. ∫

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi n f_0 t}\,dt$$

sin/cos

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n\cos(2\pi n f_0 t) - b_n\sin(2\pi n f_0 t)\right]$$

$a_0 = X_0,\ a_n = X_n + X_{-n},\ b_n = j(X_n - X_{-n})$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(2\pi n f_0 t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(2\pi n f_0 t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt$$

amp/phase rep.

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n\cos(2\pi n f_0 t + \phi_n)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n \begin{cases} -\tan^{-1}\left(\frac{b_n}{a_n}\right), & a_n > 0 \\ \pi - \tan^{-1}\left(\frac{b_n}{a_n}\right), & a_n < 0 \end{cases}$$

DTFT

$x[n]$	$\Leftrightarrow X(e^{j\omega})$
$\delta(n-N)$	$e^{j\omega N}$
$a^n u(n)$ $\lvert a\rvert<1$	$\frac{1}{1-ae^{-j\omega}}$
1	$2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$
$e^{j\omega_0 n}$	$2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0-2\pi k)$
$\cos(\omega_0 n)$	$\pi\left[\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)\right]$
$\sin(\omega_0 n)$	$-j\pi\left[\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0-2\pi k)+\delta(\omega+\omega_0-2\pi k)\right]$
$\Pi[n]$	$2\cos(\omega)+1$
$\Pi[\frac{n}{N}]$	$\frac{\sin[\omega(N+1/2)]}{\sin(\omega/2)}$

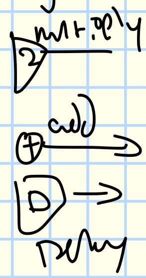
$y(n)=(h*x)(n)$ $\quad Y(\omega)=H(\omega)X(\omega)$

$y(n)=x(n)p(n)$ $\quad Y(\omega)=\frac{1}{2\pi}\int_0^{2\pi}X(\lambda)P(\omega-\lambda)\,d\lambda$

$y(n)=x(n-N)$ $\quad Y(\omega)=e^{-j\omega N}X(\omega)$

$y(n)=e^{j\omega_1 n}x(n)$ $\quad Y(\omega)=X(\omega-\omega_1)$

1) Ay



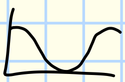
DTFS if $x[n]$ is periodic

$$X_k=\frac{1}{N_0}\sum_{n=0}^{N_0-1}x[n]e^{-jk\omega_0 n}\quad k=0,1,\dots N_0-1$$

$$x[n]=\sum_{k=0}^{N_0-1}X_k e^{jk\omega_0 n}$$

$\sin(\pi n)\Rightarrow$ period of 1

$\cos^2(x)=$



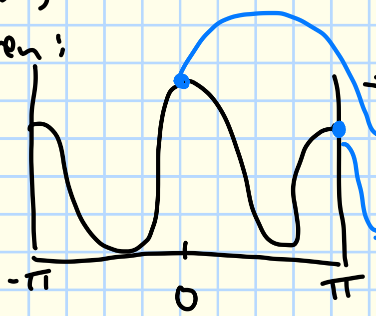
$\sin(x)=\cos(x-\frac{\pi}{2})$

Guess the function

Causal filter: nothing b4 0.

$\int_{\langle 2\pi\rangle}A(\omega)d\omega=a$ | use $\frac{1}{2\pi}\int_{\langle 2\pi\rangle}X(e^{j\omega})e^{j\omega n}d\omega$, $n=0$

when:



→ use

$A(0)=\sum_{n\in\langle n\rangle}e^{-jk\omega_0 n}$ [$k=0$]

$A(\pi)=\sum_{n\in\langle n\rangle}e^{-j\pi n}$

$\frac{2\pi}{2\pi}=1$

when given: $1+(-1)^n$

Convert to e: $e^{j0n}+e^{j\pi n}$

Sum of Geo

$a+ar+ar^2\dots$

$\frac{a}{1-r}$

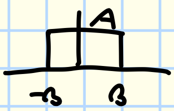
LTI iff no new freqs are generated

Common Fourier Pairs

$1\longrightarrow 2\pi\,\delta(\omega)$

$e^{j\omega_0 n}\longrightarrow 2\pi\,\delta(\omega-\omega_0)$

$h=\frac{A}{\pi n}\sin(Bn)\longrightarrow$



$f(t)$	$F(\omega)$
$f(t-t_0)$	$F(\omega)e^{-j\omega t_0}$
$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
$f(at)$	$\frac{1}{\lvert a\rvert}F(\omega/a)$
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\text{rect}(\frac{t}{\tau})$	$\text{sinc}(\tau/\pi)$
$\cos(\alpha t)$	$\frac{1}{2}(\delta(\omega-\alpha)+\delta(\omega+\alpha))$
$\text{sinc}(\alpha t)=\frac{\sin(\alpha t\pi)}{(\alpha t\pi)}$	$\frac{1}{\alpha}\text{rect}(\frac{\omega}{2\alpha\pi})$
triangle(t)	

Purely Real in $x[n]$ time domain

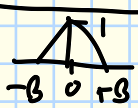
$X[\omega]=X^*[\omega]\Rightarrow x_e[n]=\frac{x[n]+x[-n]}{2}$

TF:

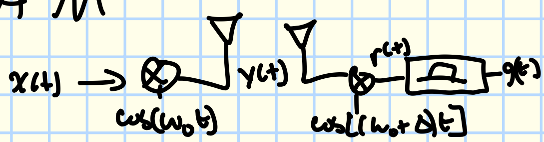
$H(\omega)=|H(\omega)|\cdot e^{j\angle H(\omega)}$

AM

$x=$



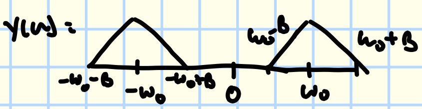
use fourier props



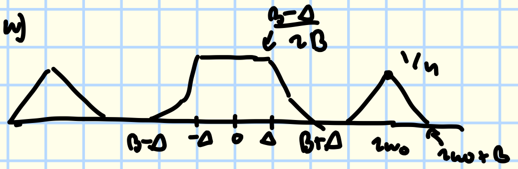
$e^{j\angle H(\omega)}$ is freq. shift.

$e^{j\angle H(\omega)}\cdot\delta(\omega-\omega_0)=\delta(\omega-\omega_0+\angle H(\omega))$

$Y(\omega)=$



$R(\omega)$

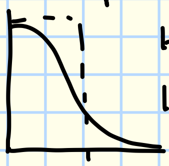


Filtering

$H(\omega) = \frac{Y(\omega)}{X(\omega)} = |H(\omega)| e^{j\angle H(\omega)}$ where $\angle H(\omega) = \tan^{-1}\left(\frac{im|H(\omega)|}{re|H(\omega)|}\right)$

TF

Low pass

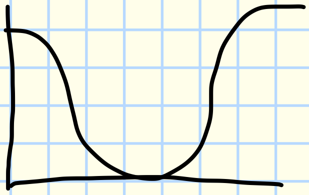


$H_c(\omega) = \frac{V_c}{V_s} = \frac{1}{1+j\omega RC}$

$|H_c(\omega)| = \frac{1}{\sqrt{1^2+(\omega rc)^2}}$

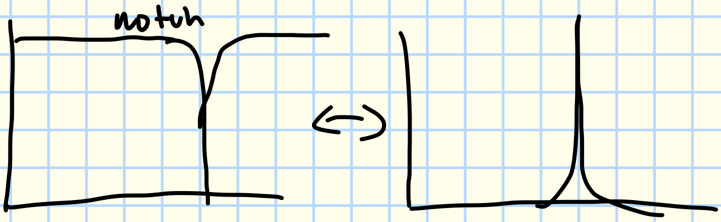
$\phi = \tan^{-1}(\phi_{top} - \phi_{bottom})$

$= 0 - (0 + \phi\omega rc)$

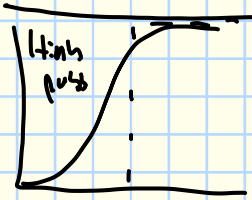


band stop

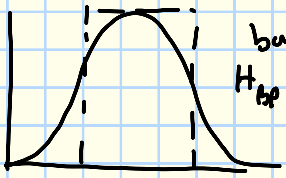
notch



High pass



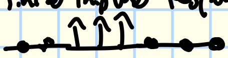
$H_R(\omega) = \frac{V_R}{V_S} = \frac{j\omega RC}{1+j\omega RC}$



band pass

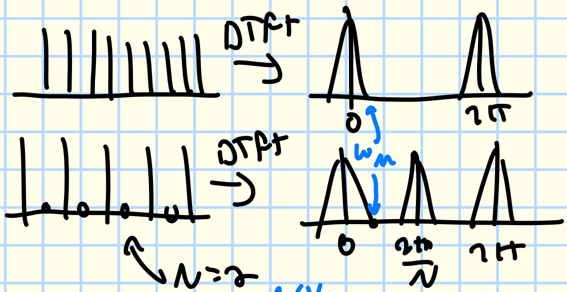
$H_{BP} = \frac{V_R}{V_S} = \frac{I R}{V_S} = \frac{j\omega RC}{j\omega RC + (1-\omega^2 LC)}$

Finite impulse response $\leftrightarrow$ Infinite impulse response



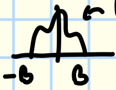
Sampling CT / DT

$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$



Nyquist: $\omega_s > 2\omega_m$

Nyquist Theorem

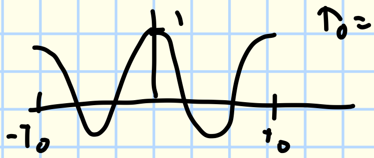


B Hz only reconstructed

$\omega \quad f \geq 2B$

$\frac{2\pi}{f}$

Sampling ex:

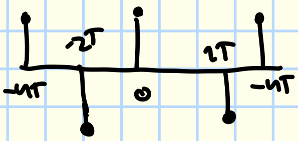


$T_0 = \frac{2\pi}{\omega_0}$

Note: When reconstructing original sig, ω has to be to be same or Alias happens

Sampling p: $\frac{4}{3}\omega_0$

$x_p =$



expression for $X_p(\omega)$

Sampling

$X_p(\omega) = \frac{1}{2\pi}(X(\omega) * P(\omega))$

$X(\omega) = \mathcal{F}(\frac{1}{2}(e^{-j\omega_0 t} + e^{j\omega_0 t})) = \pi(\delta(\omega-\omega_0) + \delta(\omega-\omega_0))$

$= \pi(\delta(\omega - \frac{3}{4}\omega_s) + \delta(\omega + \frac{3}{4}\omega_s))$

$P(\omega) = \mathcal{F}\left(\frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{j\omega_s k T}\right) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$

$\overset{t}{Ш} = \overset{\omega}{Ш}$

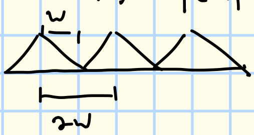
2D Stuff

$\mathbb{A}$: When Sampling remember Scaling (multiply by Shan Prod?)

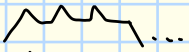
$\int_{-\infty}^{\infty} x(t)\,dt = X(0)$

$\rightarrow$ if Sampling at ω_s, Sampling period $= T = \frac{2\pi}{\omega_s}$

if sampling freq $\left(\frac{2\pi}{T}\right)$ is = Nyquist rate.
$\searrow$ 2 + highest frequency



If smaller than sampling freq. Aliasing occurs.



and we can't make signal with $X(\text{rect})$

Has to be band limited

Ex:

Nyquist:	funct.	
4π	$\cos(2\pi t)$	
N/A	rect(t)	not band lim in f domain
2π	sinc(t)	
2π	sinc(t) * sinc(t)	
4π	sinc$^2(t)$	convolve 2 rects : 4π band
N/A	e^{-t^2}	no band

Laplace puzzle.

1. real even.
2. $X(s)$ 4 poles no zeros
3. $X(s)$ has pole $s = \frac{e^{j\pi/4}}{2} = \alpha$
4. $\int_{-\infty}^{\infty} x(t)\,dt = 4$

$1 \Rightarrow$ Symmetry & Conjugate symmetry

$$(3 \,\&\, 2) \Rightarrow X(s) = \frac{A}{(s-\alpha)(s+\alpha)(s-\alpha^*)(s+\alpha^*)}$$

$$4 \Rightarrow \int_{-\infty}^{\infty} x(t)\,dt = X(0) = \frac{A}{|\alpha|^4} = 16A = 4$$

$A = 1/4$

since $X(0)$ is defined, ROC $= \frac{-1}{2\sqrt{2}} < \mathrm{Re}(s) < \frac{1}{2\sqrt{2}}$

Z

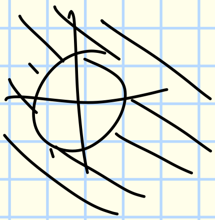
Causality / Stability,

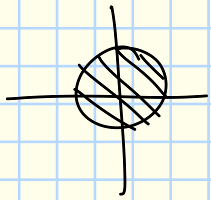
if causal roc $|z| \to \infty$,
roc has ∞.

if ROC contains unit circle } Stability

1 is in ROC: Z.

Causal / Anti Causal





$\rightarrow$ Causal LTI Sys Stable
Iff poles lie in unit circle.

$\rightarrow$ Sys function $H(z)$ if It includes unit circle in ROC
~ non causal $\Rightarrow$ J

L $\rightarrow$

Causality
roc extends to ∞

Stability
if ROC contains $j\omega$ axis

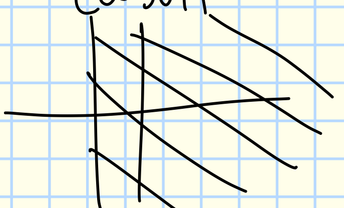
$\int_0^t = \frac{1}{s}$

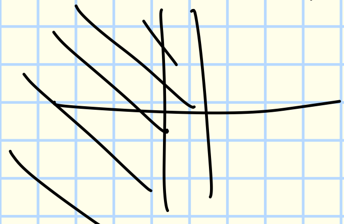
$\frac{d}{dt} = s$

Partial fract:

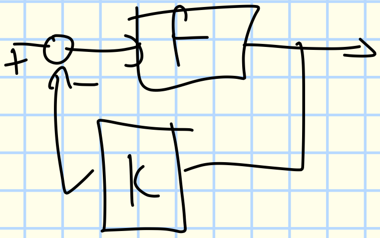
$$\frac{A}{(s+B)^2} = \frac{A}{(s+B)} + \frac{B}{(s+B)^2}$$

Casual / anti Casual





feedback



$$\frac{F}{1+FK} = T_F$$