Module ID

1 Laplace Transform Review - 5 pts

A system can be modeled using the following differential equation:

$$17\frac{d^4w}{dt^4}+3\frac{d^3w}{dt^3}+5\frac{d^2w}{dt^2}+3\frac{dw}{dt}+7w=\frac{d^3r}{dt^3}+\frac{d^2r}{dt^2}+4\frac{dr}{dt}+5r$$

Determine the transfer function from $R(s)$ to $W(s)$.

Solution

Taking the Laplace transform of the given equation:

$$[17s^4+3s^3+5s^2+3s+7]W(s)=[s^3+s^2+4s+5]R(s)$$

$$\frac{W(s)}{R(s)}=\frac{s^3+s^2+4s+5}{17s^4+3s^3+5s^2+3s+7}$$

5 Linearization - 20 pts

A system is described by:

$$f(t)=m\ddot{x}(t)+b\dot{x}(t)+f_s(x,t)$$

where f_s is the force from a non-linear spring. The spring force is defined by:

$$f_s(x,t)=k(1-e^{-x})$$

where $x(t)$ is the spring displacement and k is some constant. Find the transfer function, $X(s)/F(s)$, for small excursions around $x=0$.

Solution

First linearize $f_s(x,t)$ around $x=0$.

$$f_s(x,t)=k(1-e^{-x})\approx k((-1)(-e^{-0}))(x-0)=kx$$

Then the linearized system is

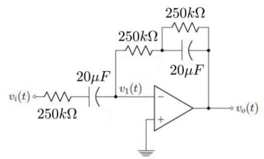
$$f(t)=m\ddot{x}+b\dot{x}+kx$$

Laplace transform on both sides

$$F(s)=(ms^2+bs+k)X(s)$$

Transfer function $\frac{X(s)}{F(s)}=\frac{1}{ms^2+bs+k}$

Find the transfer function, $G(S)=\frac{V_o(S)}{V_i(S)}$, for the following operational ampli fier:



Solution

Using equation 2.97 from the textbook for a non-inverting op-amp $\frac{V_o(s)}{V_i(s)}=-\frac{Z_2(s)}{Z_1(s)}$. In this case:

$$Z_1(s)=R+\frac{1}{Cs}=\frac{RCs+1}{Cs}$$

$$Z_2(s)=R+\frac{1}{\frac{1}{R}+Cs}=\frac{2R+R^2Cs}{1+RCs}$$

Substituting the values of R and C

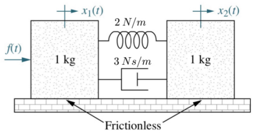
$$\frac{V_o(s)}{V_i(s)}=-\frac{\frac{2R+R^2Cs}{1+RCs}}{\frac{RCs+1}{Cs}}=-\frac{R(RC^2s^2+2Cs)}{(1+RCs)^2}$$

$$=-\frac{(250*10^3)((250*10^3)(20*10^{-6})^2s^2+2(20*10^{-6})s)}{(1+(250*10^3)(20*10^{-6})s)^2}$$

$$=-\frac{5s(5s+2)}{(5s+1)^2}=-\frac{25s^2+10s}{25s^2+10s+1}$$

4 Mechanical Systems - 20 pts

Find the transfer function, $G(s)=X_2(s)/F(s)$, for the translational mechanical network below.



Solution

Writing the equations of motion,

$$(s^2+3s+2)X_1(s)-(3s+2)X_2(s)=F(s)$$

$$-(3s+2)X_1(s)+(s^2+3s+2)X_2(s)=0$$

Solving for $X_1(s)$ in each equation,

$$X_1(s)=\frac{(3s+2)X_2(s)+F(s)}{s^2+3s+2}=\frac{(s^2+3s+2)X_2(s)}{3s+2}$$

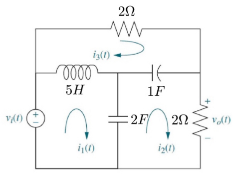
Reordering terms and solving for $G(s)$,

$$((s^2+3s+2)^2-(3s+2)^2)X_2=(3s+2)F$$

$$G(s)=X_2(s)/F(s)=\frac{3s+2}{s^2(s^2+6s+4)}$$

State space representation - 20 pts

Find the state-space representation of the network shown in the follow figure if the output is $v_o(t)$.



Solution

Let C_1 be the grounded capacitor and C_2 be the other. The equation for the energy storage components yields

$$5\frac{di_L}{dt}=v_i-v_{C_1}$$

$$2\frac{dv_{C_1}}{dt}=i_1-i_2$$

$$1\frac{dv_{C_2}}{dt}=i_2-i_3$$

KVL around loop 2 yields

$$v_{C_1}=v_{C_2}+2i_2\Rightarrow i_2=\frac{v_{C_1}-v_{C_2}}{2}$$

KVL around outer loop yields

$$v_i=2i_3+2i_2\Rightarrow i_3=\frac{v_i-v_{C_1}+v_{C_2}}{2}$$

And $i_L=i_1-i_3$, substituting the current yields

$$\frac{d}{dt}\begin{bmatrix}i_L\\v_{C_1}\\v_{C_2}\end{bmatrix}=\begin{bmatrix}0&-\frac{1}{5}&0\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&1&-1\end{bmatrix}\begin{bmatrix}i_L\\v_{C_1}\\v_{C_2}\end{bmatrix}+\begin{bmatrix}\frac{1}{5}\\\frac{1}{4}\\-\frac{1}{2}\end{bmatrix}v_i$$

$$v_o=2i_2=\begin{bmatrix}0&1&-1\end{bmatrix}\begin{bmatrix}i_L\\v_{C_1}\\v_{C_2}\end{bmatrix}+0v_i$$

2 Inverse Laplace Transform Review - 15 pts

For each transfer function below determine $h(t)$

(a) $H_1(s)=\frac{s}{s^2+8s+41}$

(b) $H_2(s)=\frac{7s+18}{s^2+8s+41}$

(c) $H_3(s)=\frac{s^2}{s^2+8s+41}$

Solution

(a)

$$H_1(s)=\frac{s}{s^2+8s+41}=\frac{s}{(s^2+8s+16)+25}$$

$$=\frac{s+4}{(s+4)^2+5^2}-\frac{4}{(s+4)^2+5^2}$$

$$\mathcal{L}^{-1}[H_1(s)]=e^{-4t}[\cos 5t-\frac{4}{5}\sin 5t]$$

(b)

$$H_2(s)=\frac{7s+18}{s^2+8s+41}=\frac{7s+18}{(s+4)^2+5^2}$$

$$\mathcal{L}^{-1}[H_2(s)]=e^{-4t}[7\cos 5t-2\sin 5t]$$

(c)

$$H_3(s)=\frac{s^2}{s^2+8s+41}=1-\frac{8s+41}{(s^2+8s+16)+25}$$

$$=1-8\frac{s}{(s+4)^2+5^2}-41\frac{1}{(s+4)^2+5^2}$$

$$\mathcal{L}^{-1}[H_3(s)]=\delta(t)-e^{-4t}[8\cos 5t+\frac{9}{5}\sin 5t]$$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	
1.	$\mathscr{L}[f(t)]=F(s)$	$=\int_{0-}^{\infty}f(t)e^{-st}dt$
2.	$\mathscr{L}[kf(t)]$	$=kF(s)$
3.	$\mathscr{L}[f_1(t)+f_2(t)]$	$=F_1(s)+F_2(s)$
4.	$\mathscr{L}[e^{-at}f(t)]$	$=F(s+a)$
5.	$\mathscr{L}[f(t-T)]$	$=e^{-sT}F(s)$
6.	$\mathscr{L}[f(at)]$	$=\frac{1}{a}F\left(\frac{s}{a}\right)$
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$=sF(s)-f(0-)$
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$=s^2F(s)-sf(0-)-f'(0-)$
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$=s^nF(s)-\sum_{k=1}^{n}s^{n-k}f^{k-1}(0-)$
10.	$\mathscr{L}\left[\int_{0-}^{t}f(\tau)d\tau\right]$	$=\frac{F(s)}{s}$
11.	$f(\infty)$	$=\lim_{s\to 0}sF(s)$
12.	$f(0+)$	$=\lim_{s\to\infty}sF(s)$

[1]For this theorem to yield correct finite results, all roots of the denomi parts, and no more than one can be at the origin.

[2]For this theorem to be valid, $f(t)$ must be continuous or have a st impulses or their derivatives at $t=0$).

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^nu(t)$	$\frac{n!}{s^n+1}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin\omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos\omega t u(t)$	$\frac{s}{s^2+\omega^2}$

formulas:

: $V(s)=R\cdot I(s)\Rightarrow$ T.F. $\frac{V(s)}{I(s)}=R$

: $v(t)=\frac{1}{C}\int_0^t i(\tau)d\tau$

$\hookrightarrow V(s)=\frac{1}{C(s)}\cdot\frac{1}{s}\cdot I(s)\Rightarrow$ T.F. $=\frac{1}{C(s)}$

: $v(t)=L\frac{d}{dt}i(t)$

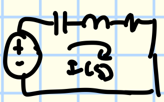
$\hookrightarrow V(s)=Ls\,I(s)$

$k\to$ $f(t)=k\int_0^t v(\tau)d\tau=kx(t)$

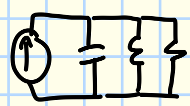
f_v $\to f(t)$; $f(t)=f_v\cdot v(t)=f_v\cdot\frac{dx(t)}{dt}$

$\boxed{M}\to f(t)$; $f(t)=M\frac{dv(t)}{dt}=M\frac{d^2x(t)}{dt}$

$(\alpha+\beta+\gamma)I(s)=E(s)$



$(\alpha+\beta+\gamma)E(s)=I(s)$



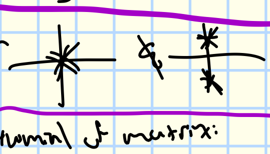
TF: $\frac{Y(s)}{U(s)}=\frac{b_ms^m+b_{m-1}s^{m-1}+\dots+b_0}{s^n+a_{n-1}s^{n-1}+\dots+a_0}$

$$A=\begin{bmatrix}1&&&\\0&&I&\\1&&&\\-a_0&-a_1&\dots&-a_{n-1}\end{bmatrix}\quad B=\begin{bmatrix}0\\0\\\dots\\0\\1\end{bmatrix}$$

$C=[b_0\ b_1\ b_2\dots b_m\ 0\dots]$ $D=[0]$

$$H(s)=\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B+D$$

remember

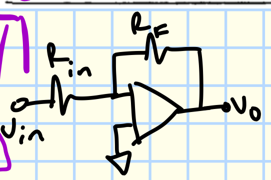


unstable

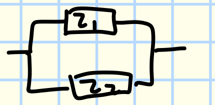
polynomial of matrix:

$\det(sI-\hat{A})=S$

$\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=ad-cd$



$V_o=-\frac{R_F}{R_{in}}V_{in}$



$\frac{Z_1\cdot Z_2}{Z_1+Z_2}$

1 **Phase-variable representation - 10 points**

For the transfer function below, write the state equations and the output equation for the phase-variable representation.

$$\frac{s^2+3s+3}{s^4+5s^3+7s^2+5s+13}$$

Solution

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -13 & -5 & -7 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 3 & 1 & 0 \end{bmatrix}, D = 0$$

Find the transfer function $G(s) = Y(s)/U(s)$ for the following system represented in state space $\dot{x} = Ax + Bu, y = Cx$.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$$

Solution

Note that this form is close to our phase-variable representation, but not quite in B. One way to solve this quickly is to perform a variable substitution for u:

$$w \triangleq 2u$$

Substituting into above, our state space equations become:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$
$$y = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix} x$$

From here we can write out the transfer function between w and y and then re-substitute to get $G(s)$:

$$H(s) \triangleq \frac{Y(s)}{W(s)} = \frac{3}{s^3+4s^2+s+2}$$
$$G(s) = \frac{Y(s)}{U(s)} = 2\frac{Y(s)}{W(s)} = \frac{6}{s^3+4s^2+s+2}$$

Alternatively, this can be done through matrix manipulations:

$$G(s) = C(sI-A)^{-1}B + D$$

Sys resp

For each of the transfer functions shown below, find the locations of the poles and zeros, plot them on the s-plane, and then write an expression for the general form of the step response without solving for the inverse Laplace transform, i.e. fill out $f_i(t)$ in the form of

$$c(t) = A_0 + A_1 f_1(t) + A_2 f_2(t) + \ldots$$

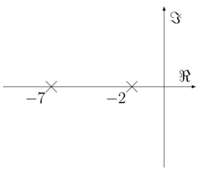
Additionally, state the nature of each response (overdamped, underdamped, and so on).

(a)
$$G(s) = \frac{1}{(s+2)(s+7)}$$

(b)
$$G(s) = \frac{3}{s^2+6s+13}$$

Solution

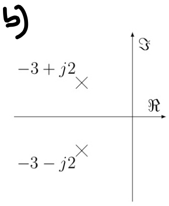
(a)



$$c(t) = A_0 + A_1 e^{-2t} + A_2 e^{-7t}$$

Overdamped

b)



$$c(t) = A_0 + A_1 e^{-3t}\cos(2t) + A_2 e^{-3t}\sin(2t)$$

Underdamped

For each pair of second-order system specifications that follow, find the location of the second-order pair of poles.

(a) $\%OS = 20\%$; $T_s = 0.2$ second

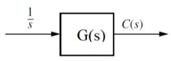
(b) $T_s = 8$ seconds; $T_p = 3$ seconds

Solution

(a) $\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}} \Rightarrow \zeta = 0.456$
$T_s = \frac{4}{\zeta\omega_n} \Rightarrow \omega_n = 43.86$
$-\sigma_d = -\zeta\omega_n = -20$
$\pm j\omega_d = \pm j\omega_n\sqrt{1-\zeta^2} = \pm j39.04$
$s = -\sigma_d \pm j\omega_d = -20 \pm j39.04$

(b) $T_p = \frac{\pi}{\omega_d} \Rightarrow \omega_d = \frac{\pi}{3}$
$T_s = \frac{4}{\sigma_d} \Rightarrow \sigma_d = \frac{4}{8} = \frac{1}{2}$
$s = -\sigma_d \pm j\omega_d = -\frac{1}{2} \pm j\frac{\pi}{3}$

Find the output response, $c(t)$, for a step input to each of the systems. Also find the time constant, rise time, and 2% settling time for each case. Assume initial condition $c(0) = 0$.



(a)
$$G(s) = \frac{4}{s+2}$$

(b)
$$G(s) = \frac{1}{s+6}$$

Solution

(a)

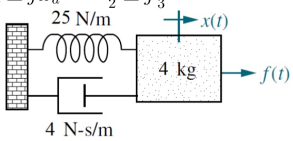
$$C(s) = \frac{4}{s(s+2)} = \frac{2}{s} - \frac{2}{s+2} \Rightarrow c(t) = 2\left(1-e^{-2t}\right)$$

time constant $\tau = 1/2 = 0.5$, rise time $T_r = 2.2/2 = 1.1$, settling time $T_s = 4/2 = 2$.

(b)

$$C(s) = \frac{1}{s(s+6)} = \frac{1}{6s} - \frac{1}{6(s+6)} \Rightarrow c(t) = \frac{1}{6}\left(1-e^{-6t}\right)$$

time constant $\tau = \frac{1}{6} = .167$, rise time $T_r = \frac{2.2}{6} = 0.367$, settling time $T_s = \frac{4}{6} = 0.667$.



For the system, do the following:

(a) Find the transfer function $G(s) = X(s)/F(s)$

(b) Find $\zeta, \omega_n, \%OS, T_s, T_p$, and T_r.

Solution

(a) Transfer function

$$G(s) = X(s)/F(s) = \frac{1}{ms^2+bs+k} = \frac{1}{4s^2+4s+25} = \frac{1/4}{s^2+s+25/4}$$

(b) Natural frequency $\omega_n = \sqrt{\frac{25}{4}} = \frac{5}{2}$; Damping ratio $\zeta = \frac{1}{2\omega_n} = \frac{1}{5}$;

Percent overshoot $\%OS = \exp(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}) \times 100 = 52.66\%$;

Peak time $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.2825$; Settling time $T_s = \frac{4}{\zeta\omega_n} = 8$;

Rise time $T_r = \frac{1.76\zeta^3-0.417\zeta^2+1.039\zeta+1}{\omega_n} = 0.4821$.

gen 2nd order sys:

$$\boxed{H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

poles @ $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

ζ?			
$\zeta = 0$ $s = \pm\omega_n j$ marg. stable	$0 < \zeta < 1$ underdamped	$\zeta = 1$ $s = -\omega_n$ crit. damp.	$\zeta > 1$ over damped

under damped sys:

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta}}}$$

$$\boxed{\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}} \quad \zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2+\ln^2(\frac{\%OS}{100})}}$$

$$\boxed{T_s = \frac{4}{\zeta\omega_n}}$$

if poles A, B, C, & $re\{C\} \geq 5 \cdot Re\{a\}$ approx 2nd order

State trans. & Stability

Given the following state-space representation of a system, solve for

(a) the state-transition matrix $\Phi(t)$

(b) the state vector $x(t)$

(c) the output of the system $y(t)$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} x(t), x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $u(t)$ is the unit step.

Solution

(a)

$$\Phi(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

(b)

$$X(s) = (sI - A)^{-1}(x(0) + BU(s))$$

$$= \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{1}{s(s^2+1)} \\ \frac{1}{s^2+1} \end{bmatrix}$$

$$\Rightarrow x(t) = \begin{bmatrix} 1 - \cos(t) \\ \sin(t) \end{bmatrix}$$

(c)

$$y(t) = 1x_1(t) + 2x_2(t)$$
$$= (1 - \cos(t)) + 2\sin(t)$$
$$= 1 - \cos(t) + 2\sin(t)$$

```
syms s t
Phi = ilaplace(inv(s*eye(2)-A))
Phi = simplify(expm(A*t)) % Alternative method
x = ilaplace(inv(s*eye(2)-A)*B*1/s)
y = C*x
```

$$\boxed{\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \\ e^{At} &= \phi = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) \end{aligned}}$$

Routh-Hurwitz table

↳ Signs must all be the same: $s^4 + 3s^3 \boxed{-} 5s^2 + s + 2$

unstable

reg case. $As^5 + Bs^4 + Cs^3 + Ds^2 + Es^1 +$

s^5	A	C	E	0
s^4	B	D	F	0
s^3	☐ (red)	☐ (green)	☐ (orange)	
s^2		…		
s^1				
s^0				

☐ (red) = $\begin{matrix} A & C \\ B & D \end{matrix} \Rightarrow \dfrac{B \cdot C - A \cdot D}{B}$

☐ (green) = $\begin{matrix} A & C & E \\ B & D & F \end{matrix} \Rightarrow \dfrac{B \cdot E - A \cdot F}{B}$

☐ (orange) = $\begin{matrix} A & C & E & 0 \\ B & D & F & 0 \end{matrix}$

Special Cases: row w/ zeros

1	2	5
2	4	0
0	5	

→

1	2	5
2	4	
ϵ	5	

take lim $\epsilon \to 0$ $\dfrac{\epsilon \cdot 4 - 2 \cdot 5}{\epsilon}$

row of zeros

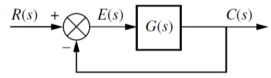
s^5	1	6	8
s^4	2	10	12
s^3	1	2	0
s^2	6	12	0
s^1	0	0	0
s^0			

→ aux. poly $6s^2 + 12s^{(2-2)} = 0$

$\frac{d}{dt}(s^2 + 2s) = 0$

$2s$

$s^1 \mid 2$



In the system, let

$$G(s) = \frac{K(s+8)}{s(s-2)(s+4)}$$

Find the range of K for closed-loop stability.

Solution The closed-loop system has transfer function as

$$G_{closed} = \frac{G(s)}{1 + G(s)} = \frac{K(s+8)}{s(s-2)(s+4) + K(s+8)}$$
$$= \frac{K(s+8)}{s^3 + 2s^2 + (K-8)s + 8K}$$

Using Routh-Hurwitz criterion, the condition for closed-loop system t be (asymptotically) stable is

$$2(K - 8) > 8K$$
$$2K - 16 > 8K$$
$$K < -8/3$$

Also

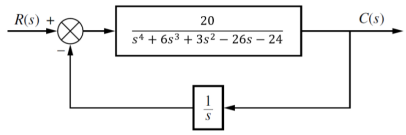
$$8K > 0 \Rightarrow K > 0$$

There is no valid K for the system.

Note: the condition for polynomial $a_3s^3 + a_2s^2 + a_1s + a_0$ to hav negative real-part roots is $a_2 > 0, a_0 > 0, a_2a_1 > a_3a_0$ assuming $a_3 > 0$.

3 Routh-Hurwitz stability - 25 pts

Using the Routh-Hurwitz criterion, tell how many closed-loop poles of the system shown below lie in the left half-plane, in the right half-plane, and or the $j\omega-$axis.



Solution The closed-loop transfer function is

$$G(s) = \frac{C(s)}{R(s)} = \frac{20s}{s^5 + 6s^4 + 3s^3 - 26s^2 - 24s + 20}$$

The Routh table is

s^5	1	3	-24
s^4	6	-26	20
s^3	44/6	-164/6	0
s^2	-40/11	20	0
s^1	13	0	0
s^0	20	0	0

There is 2 sign changes in the first column, therefore there are 2 RHP poles and 3 LHP poles.

4 Routh criterion - 25 pts

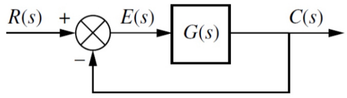
Consider the following Routh table. Notice that the s^5 row was originally all zero. Tell how many roots of the original polynomial were in the right half-plane, in the left half-plane, and on the $j\omega-$axis.

s^7	1	2	−1	−2
s^6	1	2	−1	−2
s^5	3	4	−1	0
s^4	1	−1	−3	0
s^3	7	8	0	0
s^2	−15	−21	0	0
s^1	−9	0	0	0
s^0	−21	0	0	0

Solution All entries from the even polynomial at the s^6 row down to the s^0 row are a test of the even polynomial. One sign change exists from the s^6 row down to the s^0 row. Thus, the even polynomial has 1 right-half-plane pole, which also leads to 1 left-half-plane pole because of the requirement for symmetry. Hence, the even polynomial must have all other four poles on the $j\omega$-axis.

In summary, the original polynomial has 1 RHP pole, 2 LHP poles, and 4 poles on the $j\omega-$axis.

Steady State Error



For the unity feedback system shown above, where

$$G(s) = \frac{25(s+4)(s+6)(s+9)}{s(s+8)(s^2+4s+10)}$$

find the steady-state errors for the following test inputs: $15u(t), 10tu(t), 20t^2u(t)$

Solution The closed-loop transfer function is

$$\begin{aligned} G_{closed}(s) &= \frac{G(s)}{1+G(s)} \\ &= \frac{25(s+4)(s+6)(s+9)}{s(s+8)(s^2+4s+10)+25(s+4)(s+6)(s+9)} \\ &= \frac{25s^3+475s^2+2850s+5400}{s^4+37s^3+517s^2+2930s+5400} \end{aligned}$$

Checking either via MATLAB or the Routh-Hurwitz criteria, the transfer function has 4 LHP poles, thus the system is stable.

Using Final Value Theorem,

$$\begin{aligned} e_{ss} = \lim_{t\to+\infty} e(t) &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} sR(s)(1-G_{closed}(s)) \\ &= \lim_{s\to 0} sR(s)\frac{1}{1+G(s)} \end{aligned}$$

(a) $15u(t) \Leftrightarrow R(s) = \frac{15}{s}$

$$\begin{aligned} e(\infty) &= 15\lim_{s\to 0}\frac{1}{1+G(s)} \\ &= 0 \end{aligned}$$

since we can see $G(s)\to\infty$ as $s\to 0$, thus $\frac{1}{1+G(s)}\to 0$ as $s\to 0$.

(b) $10tu(t) \Leftrightarrow R(s) = \frac{10}{s^2}$

$$\begin{aligned} e(\infty) &= 10\lim_{s\to 0}\frac{1}{s+sG(s)} \\ &= \lim_{s\to 0}\frac{10(s+8)(s^2+4s+10)}{s(s+8)(s^2+4s+10)+25(s+4)(s+6)(s+9)} \\ &= \frac{10\cdot 8\cdot 10}{25\cdot 4\cdot 6\cdot 9} \\ &= \frac{4}{27} \end{aligned}$$

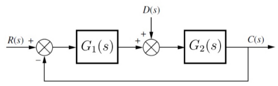
(c) $20t^2u(t) \Leftrightarrow R(s) = \frac{40}{s^3}$

$$\begin{aligned} e(\infty) &= 40\lim_{s\to 0}\frac{1}{s^2+s^2G(s)} \\ &= \frac{40(s+8)(s^2+4s+10)}{3s^2(s+8)(s^2+4s+10)+25s(s+4)(s+6)(s+9)} \\ &= \infty \end{aligned}$$

as the denominator is a polynomial of s^4 and numerator s^3, hence it approaches ∞ as $s\to 0$.

We can only apply Final Value Theorem when the final value exists.

Find the total steady-state error due to a unit step input $R(s)$ and a unit step disturbance $D(s)$ in the system of the following figure.



where

$$G_1(s) = \frac{4}{s+3} \qquad G_2(s) = \frac{30}{s+2}$$

Solution The system is stable. Taking $E(s) = C(s) - R(s)$, we can compute the steady-state error $e(\infty)$ as the sum of the steady-state errors due to reference and disturbances:

$$e(\infty) = e_R(\infty) + e_D(\infty)$$

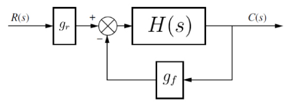
Then letting $R(s) = D(s) = 1/s$:

$$\begin{aligned} e_R(\infty) &= \lim_{s\to 0}\frac{s}{1+G_1(s)G_2(s)}\frac{1}{s} \\ &= \lim_{s\to 0}\frac{(s+3)(s+2)}{(s+3)(s+2)+120} \\ &= \frac{6}{6+120} = \frac{1}{21} \end{aligned}$$

$$\begin{aligned} e_D(\infty) &= -\lim_{s\to 0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}\frac{1}{s} \\ &= -\lim_{s\to 0}\frac{30(s+3)}{(s+3)(s+2)+120} \\ &= -\frac{90}{126} = -\frac{5}{7} \end{aligned}$$

$$\implies e(\infty) = \frac{1}{21} - \frac{15}{21} = -\frac{14}{21}$$

For the system shown below, find the steady-state error for unit step, ramp, and parabolic inputs. g_r and g_f are gains.



where

$$g_r = 1, \qquad g_f = 2, \qquad H(s) = \frac{s+2}{(s+6)(s+1)}$$

Solution The system is stable. And the equivalent transfer function is

$$\begin{aligned} H_{cl}(s) &= \frac{g_r H(s)}{1+g_f H(s)} \\ &= \frac{s+2}{(s+6)(s+1)+2(s+2)} \\ &= \frac{s+2}{s^2+9s+10} \end{aligned}$$

For a unit step input:

$$\begin{aligned} e(\infty) &= \lim_{s\to 0} s\frac{1}{s}(1-H_{cl}(s)) \\ &= 1 - \lim_{s\to 0}\frac{s+2}{s^2+9s+10} \\ &= 1 - \frac{2}{10} = \frac{4}{5} \end{aligned}$$

For a ramp input $tu(t)$:

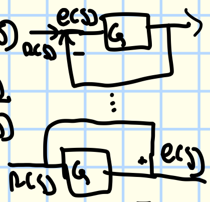
$$\begin{aligned} e(\infty) &= \lim_{s\to 0} s\frac{1}{s^2}(1-H_{cl}(s)) \\ &= \lim_{s\to 0}\left(\frac{1}{s} - \frac{s+2}{s(s^2+9s+10)}\right) \\ &= \lim_{s\to 0}\left(\frac{s^2+8s+8}{s(s^2+9s+10)}\right) \\ &= \infty \end{aligned}$$

For a parabolic input $\frac{1}{2}t^2u(t)$:

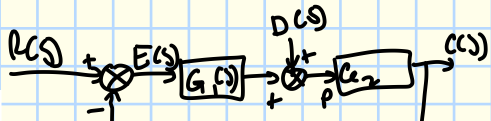
$$\begin{aligned} e(\infty) &= \lim_{s\to 0} s\frac{1}{s^3}(1-H_{cl}(s)) \\ &= \lim_{s\to 0}\left(\frac{1}{s^2} - \frac{s+2}{s^2(s^2+9s+10)}\right) \\ &= \lim_{s\to 0}\left(\frac{s^2+8s+8}{s^2(s^2+9s+10)}\right) \\ &= \infty \end{aligned}$$

Formulas

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1+G(s)}$$



Disturbance system



$$C = \frac{G_2}{1+G_1G_2}\cdot D + \frac{G_1G_2}{1+G_1G_2}\cdot R$$

$$E(s) = R(s) - C(s) = \frac{-G_2}{1+G_1G_2}D + \frac{1}{1+G_1G_2}\cdot R$$

$e_D(\infty)$ $e_R(\infty)$

$E(\infty) = \lim_{s\to 0} sE(s)$

given A is stable:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)(1-C(sI-A)^{-1}B)$$

proportional constant: K_p

$$K_p = \lim_{s\to 0} G(s)$$

Velocity Constant K_v

$$K_v = \lim_{s\to 0} sG(s)$$

accelleration constant

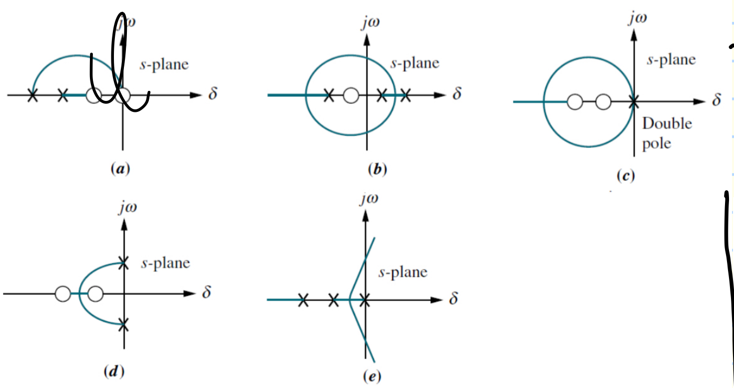
$$K_a = \lim_{s\to 0} s^2G(s)$$

SSE for:

type = # of poles @ origin in open loop TF G

type	Step	Ramp	Parabolic
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$

For each of the root loci shown below, tell whether or not the sketch can be a root locus. If the sketch cannot be a root locus, explain why. Give all reasons.



Solution

(a) No. Not symmetric about the real axis; On real axis to left of an even number of poles and zeros.

(b) No. On real axis to left of an even number of poles and zeros.

(c) No. On real axis to left of an even number of poles and zeros.

(d) Yes.

(e) Yes.

4 Steady-state error - 20 pts

For the following closed-loop system, find the steady-state error for unit step and unit ramp inputs.

$$\dot{x} = \begin{bmatrix} 0 & 2 & 8 \\ -3 & -5 & -4 \\ 1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Solution

$$e(\infty) = \lim_{s\to 0} sR(s)(1-T(s)) = \lim_{s\to 0} sR(s)(1-C(sI-A)^{-1}B) =$$

$$= \lim_{s\to 0} sR(s)\left(1 - \frac{4s+16}{s^3+5s^2+6s+16}\right) = \lim_{s\to 0} sR(s)\left(\frac{s^3+5s^2+2s}{s^3+5s^2+6s+16}\right)$$

For a unit step, $R(s) = 1/s$, and $e(\infty) = 0$.

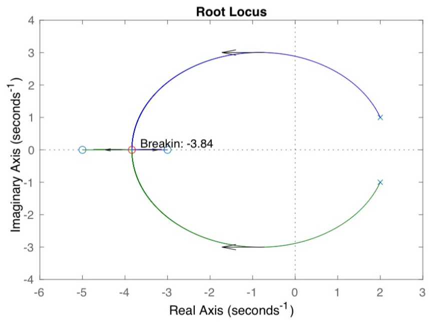
For a unit ramp, $R(s) = 1/s^2$, and $e(\infty) = \frac{2}{16} = 0.125$.

For a system with open-loop poles and zeros given below, sketch the root locus. Label the direction of the root locus for increasing K with arrows. Derive any breakaway and break-in points, and annotate them on your plot.

Zeros: $z_1 = -3$ $\quad z_2 = -5$

Poles: $p_1 = 2 + j1$ $\quad p_2 = 2 - j1$

Solution A final sketch is shown below:



First, we know the root locus on the real axis will be between the two zeros, i.e. between -3 and -5, since that would be to the left of an odd number of poles and/or zeros. Second, since there's an equal number of finite zeros and finite poles, we won't have any asymptotes that go to ∞ as K increases. Finally, combined with the fact that the root locus starts at the poles and ends at the zeros, we know we'll just have a single break-in point between -3 and -5.

To compute the break-in point, we can use a couple methods:

Differential calculus method The break-in point can be found by

$$
\begin{aligned}
KH(s) &= -1 \\
K &= -\frac{(\sigma - (2+j1))(\sigma - (2-j1))}{(\sigma+3)(\sigma+5)} \\
K &= -\frac{\sigma^2 - 4\sigma + 5}{\sigma^2 + 8\sigma + 15} \\
0 &= \frac{dK}{d\sigma} \\
&= -\frac{(\sigma^2+8\sigma+15)(2\sigma-4) - (2\sigma+8)(\sigma^2-4\sigma+5)}{(\sigma^2+8\sigma+15)^2} \\
&= -\frac{12s^2 + 20s - 100}{(\sigma^2+8\sigma+15)^2}
\end{aligned}
$$

Using the quadratic formula, this yields roots at:

$$
\begin{aligned}
\sigma &= \frac{-20 \pm \sqrt{400 + 400 \cdot 12}}{24} \\
&= \frac{-5 \pm 5\sqrt{13}}{6}
\end{aligned}
$$

In this case, we just care about the break-in point between -3 and -5, thus

$$\sigma = \frac{-5 - 5\sqrt{13}}{6} \approx -3.83$$

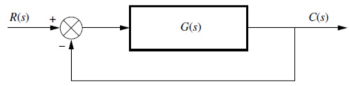
Transition method

$$
\begin{aligned}
\frac{1}{\sigma+3} + \frac{1}{\sigma+5} &= \frac{1}{\sigma - (2+j1)} + \frac{1}{\sigma - (2-j1)} \\
\frac{2\sigma+8}{\sigma^2+8\sigma+15} &= \frac{2\sigma-4}{\sigma^2-4\sigma+5} \\
(2\sigma+8)(\sigma^2-4\sigma+5) - (2\sigma-4)&(\sigma^2+8\sigma+15) = 0 \\
12s^2 + 20s - 100 &= 0
\end{aligned}
$$

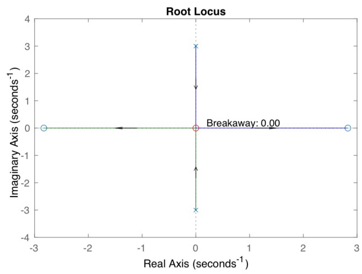
And thus we arrive at the same root equation and solution as above.

Root Locus

sketch the root locus in a similar way as the previous question. Additionally, tell for what exact values of $K \geq 0$ the closed-loop system is asymptotically stable, marginally stable, and unstable.



Solution A completed root locus sketch:



We have zeros at $\pm 2\sqrt{2}$, and poles at $\pm 3j$. First, we know the root locus on the real axis will be between the two zeros, i.e. from $-2\sqrt{2}$ to

$2\sqrt{2}$. Second, there are 2 finite poles and 2 finite zeros, so no asympt
We go from poles to zeros, so we'll have a break-in point.

We can then compute the break-in point, using the transition met

$$
\begin{aligned}
\frac{1}{\sigma+2\sqrt{2}} + \frac{1}{\sigma-2\sqrt{2}} &= \frac{1}{\sigma-3j} + \frac{1}{\sigma+3j} \\
\frac{2\sigma}{\sigma^2-8} &= \frac{2\sigma}{\sigma^2+9} \\
2\sigma(\sigma^2+9) &= 2\sigma(\sigma^2-8) \\
\implies \sigma &= 0
\end{aligned}
$$

Thus the root locus hits the origin and then goes straight up and d

To compute the values of $K \geq 0$ for stability analysis, we compute closed-loop transfer function:

$$
\begin{aligned}
G_{cl}(s) &= \frac{G(s)}{1+G(s)} \\
&= \frac{K(s^2-8)}{s^2+9+K(s^2-8)} \\
&= \frac{K(s^2-8)}{(1+K)s^2+(9-8K)}
\end{aligned}
$$

Inspecting the denominator, the poles will be at

$$s = \pm\sqrt{-\frac{9-8K}{1+K}}$$

Inspecting the denominator, the poles will be at

$$s = \pm\sqrt{-\frac{9-8K}{1+K}}$$

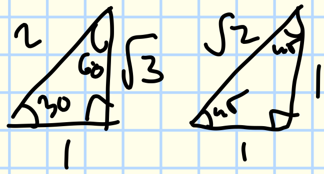
If the term under the square root term is negative, then we'll have purely imaginary (distinct) poles, so:

$$
\begin{aligned}
-\frac{9-8K}{1+K} &< 0 \\
8K - 9 &< 0 \\
K &< 9/8
\end{aligned}
$$

Thus the system is marginally stable for $0 \leq K < 9/8$. If the term under the square root is either zero or positive, then we'll be unstable, as we'll either have poles on the jw axis with multiplicity > 1 or in the RHP. Thus the system is unstable for $K \geq 9/8$, and the system is never asymptotically stable.

4) Breakaway / Breakin points:

$$\sum_{i=1}^{m} \frac{1}{\sigma - z_i} = \sum_{j=1}^{n} \frac{1}{\sigma - p_j}$$



5) angle of departure

$$\theta_{depart, p_j} = 180 + \sum_{i=1}^{m} \angle(p_j - z_i) - \sum_{i=1, i\neq j}^{n} \angle(p_j - p_n)$$

$$\theta_{arrival, z_j} = 180 - \sum_{i=1, i\neq j}^{m} \angle(z_j - z_i) + \sum_{i=1}^{n} \angle(z_j - p_i)$$

6) jw intercept:

→ Make routh table with gain k.

→ make k to make first possible row of zeros.

n = order of denominator

m = order of numerator

P = poles $\quad$ Z = zeros

1) real axis segments (# poles & zeros on right side of $\bar{s}$: odd)

2) Branches going to ∞ $(n - m)$

3) Branches going to ∞ $\left(\text{angle} = \frac{(2r+1)180}{n-m}\right)$ $\{r = 0, 1, \ldots n-m-1\}$

$\left(\text{centroid} = \frac{\sum \text{Poles} - \sum \text{zeros}}{n-m}\right)$ $\left\{\begin{matrix}\text{pole loc}(-2,-3\ldots) \\ \text{zero loc}(-2,-3\ldots)\end{matrix}\right.$

angle rule $\quad \bar{s}$ = point on RL

$$\sum_{i=1}^{m} \angle(\bar{s} - z_j)_{\text{zeros}} - \sum_{j=1}^{n} \angle(\bar{s} - p_j)_{\text{poles}} = (2r-1)180$$

3 Controller design - 25 pts

$\dot{x} = Ax + Bu$
$u = K(r - x)$
$\dot{x} = Ax + B(K(r-x))$
$Ax + BKr - BKx$
$(A - BK)x$
$[k_1, k_2, k_3]$

Given the following open-loop plant,

$$G(s) = \frac{1}{(s+2)(s+3)(s+10)}$$

design a controller to yield a 10% overshoot and a settling time of 1 second. Place the third pole 5 times as far from the imaginary axis as the dominant pole pair. Use the phase variables for state-variable feedback.

$\frac{1}{s^3 + 15s^2 + 56s + 60}$

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -56 & -15 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C = [1\ 0\ 0\ 0]$

$\dot{x} = (A + BK)x(t)$ $3\times1 \ \ 1\times3$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -56 & -15 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1, k_2, k_3]$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60-k_1 & -56-k_2 & -15-k_3 \end{bmatrix}$

$s^3 + (15+k_3)s^2 + (56+k_2)s + (60+k_1)$

$K = [k_1, k_2, k_3] = \begin{bmatrix} 13 & 149.8 & 856 \end{bmatrix}$

desired poles:
10% OS
ST: 1s $\Rightarrow 1 = \frac{4}{\zeta \cdot \omega_n}$ $\omega_n = \frac{4}{.591} = 6.77$
$e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = .1$
$\zeta = .591$
poles: $s^2 + 2 \cdot .591 \cdot 6.77 s + (6.77)^2$
$s^2 + 8x + 45.8$
$\frac{-8 \pm \sqrt{64 - 4 \cdot 45.8}}{2}$
$\frac{-4 \pm \sqrt{119}}{2}$ 64−
$s_1, s_2 = -4 \pm 5.46j$
$s_3 = -20$
$(s+20)(s^2 + 8s + 45.8)$
$s^3 + 8s^2 + 45.8s + 20s^2 + 160s + 916$
$s^3 + 28s^2 + (45.8 + 160)s + 916$
$s^3 + 28s^2 + 205.8s + 916$

Page 3

4 Observer design - 25 pts

Controller design

(a)

$$\frac{s+5}{s+7}$$

(b)

$$\left(\frac{s+0.2}{s+0.02}\right)\left(\frac{s+2}{s+20}\right)$$

Solution Refer to Table 9.11 in textbook

(a) Lead compensator

$$\frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_1 C} + \frac{1}{R_2 C}}$$

$\Rightarrow C = 2\mu F, R_1 = 100k\Omega, R_2 = 250k\Omega.$

(b) Lag-Lead compensator

$$\frac{\left(s + \frac{1}{R_1C_1}\right)\left(s + \frac{1}{R_2C_2}\right)}{s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}\right)s + \frac{1}{R_1R_2C_1C_2}}$$

$$\frac{1}{R_1C_1} = 0.2$$

$$\frac{1}{R_2C_2} = 2$$

$$\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1} = 10.02 \Rightarrow \frac{1}{R_2C_1} = 7.82$$

$\Rightarrow$ Choose $C_1 = 10\mu F$

$\Rightarrow R_1 = 2M\Omega, R_2 = 13k\Omega, C_2 = 38\mu F.$

Given the following open-loop plant,

$$G(s) = \frac{1}{(s+1)(s+4)(s+8)}$$

design a controller to yield a 10% overshoot and a settling time of 1 second. Place the third pole 5 times as far from the imaginary axis as the dominant pole pair. Use the phase variables for state-variable feedback.

Solution

The plant is given by

$$G(s) = \frac{1}{(s+2)(s+3)(s+10)} = \frac{1}{s^3 + 15s^2 + 56s + 60}$$

The characteristic polynomial for the plant with phase-variable state feedback, i.e. $u = -Kx$ where $K = [k_1, k_2, k_3]$, is

$$s^3 + (k_3 + 15)s^2 + (k_2 + 56)s + (k_1 + 60)$$

The constraints of 10% overshoot ($M_p = 0.1$) and settling time 1s determine our dominant 2nd order closed loop poles:

$$\zeta = \frac{-\ln(M_p)}{\sqrt{\pi^2 + (\ln(M_p))^2}} = 0.5912$$

$$T_s = \frac{4}{\zeta\omega_n}$$

$$\omega_n = 6.7659$$

Knowing damping ratio corresponds to the cosine of the angle with the negative real axis and ω_n is the magnitude, our desired closed-loop poles are thus:

$$\theta = (\pi - \cos^{-1}(\zeta))$$
$$p_1, p_2 = \omega_n \cos(\theta) \pm 1j \cdot \omega_n \sin(\theta) = -4 \pm 5.4570j$$

And then we can place our third pole at -20. Thus, the desired characteristic equation is

$$\chi(s) = (s+20)(s - (-4 + 5.457j))(s - (-4 - 5.457j)) = s^3 + 28s^2 + 205.8s + 916$$

Comparing the two characteristic equations,

$$k_1 = 856, k_2 = 149.8, k_3 = 13$$

Note that, by design, our second order approximation is valid, since there are no zeros and the third pole is intentionally placed far from the other poles.

Consider the plant

$$G(s) = \frac{1.5}{s(s+3)(s+5)}$$

whose state variables are not available. Design an observer for the observer canonical variables to yield a transient response described by $\xi = 0.866$ and $\omega_n = 15$. Place the third pole 5 times farther from the imaginary axis than the dominant poles.

Solution

The plant is given by

$$G(s) = \frac{1.5}{s(s+3)(s+5)} = \frac{1.5}{s^3 + 8s^2 + 15s}$$

The characteristic polynomial for the plant with phase-variable state feedback is

$$s^3 + (l_3 + 8)s^2 + (l_2 + 15)s + l_1$$

According to our desired transient response, our dominant 2nd order closed-loop poles should be:

$$\theta = (\pi - \cos^{-1}(\zeta))$$
$$p_1, p_2 = \omega_n \cos(\theta) \pm 1j \cdot \omega_n \sin(\theta) = -13 \pm 7.5j$$

And hence our third pole is placed at -65. The desired characteristic equation is then

$$(s+65)(s^2 + 2(0.866)(15)s + 15^2) = s^3 + 90.98s^2 + 1914s + 14625$$

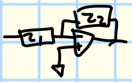
based upon $\xi = 0.866$ and $\omega_n = 15$, and a third pole ten times further from the imaginary axis than the dominant poles. Comparing the two characteristic equations,

$$l_1 = 14625, l_2 = 1899, l_3 = 82.98$$

Note that, by design, our second order approximation is valid, since there are no zeros and the third pole is intentionally placed far from the other poles.

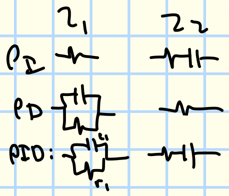
PID:

op amp



PD → PID   PI

PD



1) get system reqs, find ζ, ω_n
2) make polynomial
$s^2 + \omega_n \zeta s + \omega_n^2$
3) find poles
4. find new zeros
↳ $\Sigma\theta_z - \Sigma\theta_p = 180$

$K_{overall} = \frac{\Pi(\text{dist to all poles})}{\Pi(\text{dist to all zeros})}$

Controlability

controlability iff $C = [B\ AB \ldots A^{n-1}B]$ is full column rank   n = rank of sys
↳ $\det C \neq 0$

$\dot{x} = Ax + B(-Kx(t))$

gen form:
$\dot{x} = Ax + bu$
$y = Cx + Du$
$u = -Kx$

Phase variable

let $\dot{x} = Ax + Bu$ $C_x = [B\ AB\ AB \ldots A^{n-1}B]$
$y = C$

any plant can be transformed into phase var. form $x = Pz$

If not in OCF form:

1. get characteristic poly for sys → $\det(sI - A)$
2. make cont. matrix $C_x = [B\ AB \ldots]$
3. make new obs. matrix in OCF form $O_z = [\tilde{B}\ \ \tilde{A}\tilde{B} \ldots]$ $\tilde{A}, \tilde{B}$
4. design $\tilde{K}$ s.t. $\tilde{A} - \tilde{B}\tilde{K}$ meets reqs.
↳ $\det(sI - (\tilde{A} - \tilde{B}\tilde{K}))$ = poly that meets reqs.
5. transform back. $P = C_x C_z^{-1}$
$K = \tilde{K} P^{-1}$

Observability

observable iff $O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n} \end{bmatrix}$ is row space full space
↳ $\det O \neq 0$

$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))$
$e = \dot{x} - \dot{\hat{x}}$
$\dot{e} = (A - LC)e$
design L s.t. $e \Rightarrow 0$

observable canonicle form

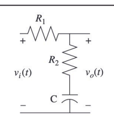
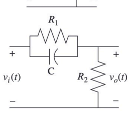
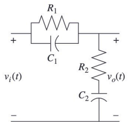
$\tilde{A} = \begin{bmatrix} -a_{n-1} & [\ I\ ] \\ \vdots & \\ -a_0 & \text{—}0\text{—} \end{bmatrix} = A - LC$

$\tilde{C} = [1\ 0 \ldots 0]$

1. get characteristic poly for sys
2. make obs. matrix $O_x = \begin{bmatrix} C \\ CA \end{bmatrix}$
3. make new obs. matrix in OCF form $O_z = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \ldots \end{bmatrix}$ $\tilde{A}, \tilde{C}$
4. design $\tilde{L}$ s.t. $\tilde{A} - \tilde{L}\tilde{C}$ meets reqs.
↳ $\det(sI - (\tilde{A} - \tilde{L}\tilde{C}))$ = poly that meets reqs.
5. transform back. $P = O_x^{-1} O_z$
$L = P\tilde{L}$

Lead / Lag:

$$G_{lead}\,G_{lag} = \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\gamma T_2}}\right)$$

Function	Network	Transfer function, $\frac{V_o(s)}{V_i(s)}$
Lag compensation		$\frac{R_2}{R_1 + R_2} \frac{s + \frac{1}{R_2 C}}{s + \frac{1}{(R_1 + R_2)C}}$
Lead compensation		$\frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_1 C} + \frac{1}{R_2 C}}$
Lag-lead compensation		$\frac{\left(s + \frac{1}{R_1C_1}\right)\left(s + \frac{1}{R_2C_2}\right)}{s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}\right)s + \frac{1}{R_1R_2C_1C_2}}$