Lattice study of the chiral condensate with staggered and overlap fermions in the multi-flavoured Schwinger model

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Motivation

In QCD:

Big energy/momenta: sufficiently small coupling $\rightarrow$ can be treated perturbatively

Small energy/momenta: coupling too strong $\rightarrow$ perturbative approach not applicable

Non-perturbative approach: Lattice QCD [Wilson (1974)]

- Concept can be used for other QFT's as well
- QFT of choice: Schwinger model with $N_f \leq 2$
- Fermions: Naïve, Wilson, Staggered and Overlap
- Observable: Chiral condensate for a range of masses
- Evaluation via: Monte Carlo methods and implementation of topological charge

[Kenneth G. Wilson]

Basics: Schwinger model

Two-dimensional QED:

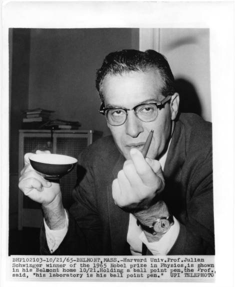
Free Dirac equation ($\hbar = 1 = c$):

$(i\partial\!\!\!/ - m)\psi = 0,$

γ -matrices in 2d: represented by Pauli-matrices,
ψ: Two-component vector $\rightarrow$ represents fermionic fields

Interaction of fields: mediated by gauge fields
$\Rightarrow$ replace ∂_μ with covariant derivative:

$$D_\mu = \partial_\mu + ieA_\mu,$$



[Julian Schwinger]

Euclidean action:

$S_E = \int d^2x \left(\overline{\psi}(D\!\!\!\!/ + m)\psi + \frac{1}{4}(F_{\mu\nu})^2\right),$

Field strength tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, Dirac adjoint: $\overline{\psi}(x)$,

Locally gauge invariant under U(1) transformations.

Schwinger model quantization via:

$Z = \int \mathcal{D}A_\mu \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-S_F - S_G}$,

Exactly solvable for zero fermion mass [Schwinger (1962)]

Exhibits features of QCD:

- Confinement $\rightarrow$ existence of bound states [Coleman (1975)]
- Coupling not energy dependent $\rightarrow$ $\beta = 1/(ae)^2$.
- In comparison to QCD: SM computationally more efficient.

$\Rightarrow$ Good test laboratory for QCD

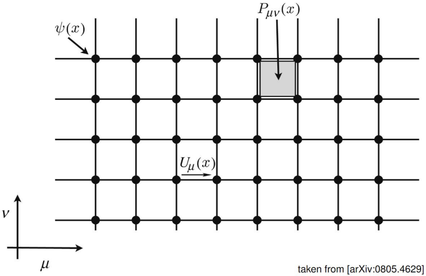
Additionally: simulations can be done on a regular laptop.

Discretization on a 2-dimensional lattice:

Replacement of: Euclidean space-time by discretized space-time grid
$\overline{\psi}(n)$, $\psi(n)$: located on each lattice site

Implementation of:

- Parallel transport $\rightarrow$ translation mechanism between lattice sites: $U_\mu(n) \in U(1) \rightarrow$ gauge links.
- Plaquette $\rightarrow$ most local gauge invariant object: $P_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n)$



taken from [arXiv:0805.4629]

Integration:

Fermionic action can be integrated out [Matthews and Salam (1954)] :

$$Z_F = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{-\sum\limits_{i,j} \overline{\psi}_i D_{ij} \psi_j} = \det(D), \quad D: \text{square matrix.}$$

Only pure euclidean Wilson gauge action left:

$$S_E^G = \beta \sum_{n\in\Lambda} \sum_{\mu<\nu} \text{Re}\left(1 - P_{\mu\nu}(n)\right),$$

$\rightarrow$ sum over plaquettes!

Λ: complete set of discrete lattice space-time points.

Full partition function:

$$Z = \int \mathcal{D}U_\mu \left[\det(D)\right]^{N_f} e^{-S_G}, \quad N_f: \text{number of flavours.}$$

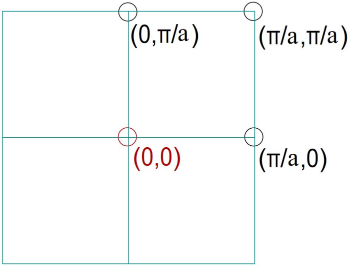
Fermion doubling

Naïve lattice fermions:

$\rightarrow$ fermion doubler problem.

$\Rightarrow$ additional poles, which do not appear in the continuum theory.

Poles in 2d at positions: $(\pi/a, 0)$, $(0, \pi/a)$, $(\pi/a, \pi/a)$.



[Lang: LQCD for pedestrians]

Wilson fermions

Solution: Wilson lattice fermion

Approach of removing the doubler modes by adding Wilson term [Wilson (1974)].

Wilson Dirac operator:

$$D_W = \frac{1}{2}\gamma_\mu \left(\nabla_\mu^f + \nabla_\mu^b\right) + m\mathbb{1} - \frac{ar}{2}\Delta,$$

Derivatives with respect to gauge links:

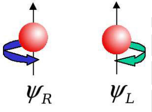
$$\nabla_\mu^f \psi(n) = \frac{U_\mu(n)\psi(n+\hat{\mu}) - \psi(n)}{a}, \quad \nabla_\mu^b \psi(n) = \frac{\psi(n) - U_\mu^\dagger(n-\hat{\mu})\psi(n-\hat{\mu})}{a}.$$

- Wilson parameter: $r \in (0,1] \rightarrow$ vanishes for $a \rightarrow 0$,
- Laplace operator: $\Delta\psi(n) = \nabla_\mu^b \nabla_\mu^f \psi(n)$.

Problem:
Breaks chiral symmetry explicitly.

Solution:
Replacement by non-vanishing anti-commutator

$$\{D, \gamma_5\} = aD\gamma_5 D, \quad \text{Chirality matrix:} \gamma_5 = \text{diag}(-1.1)$$



- r.h.s: lattice artefact $\rightarrow$ vanishes for $a \rightarrow 0$.
- Now possible to define set of transformations, that gives invariance.
- GWC provides crucial security, that lattice chiral symmetry is preserved.

Task: Find an operator, that satisfies these transformations!

Overlap fermions

Solution: Overlap fermions [Narayanan and Neuberger (1993)]

Projection of D_W with negative mass parameter:

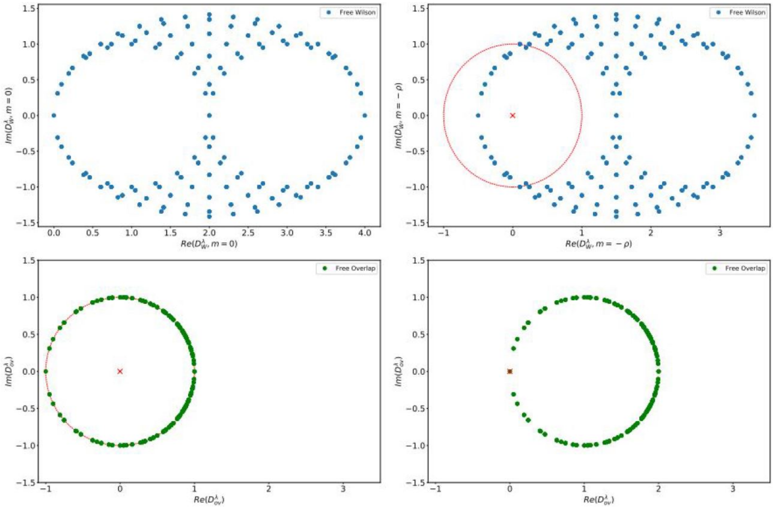
$$\mathrm{D}_{ov} = \frac{\rho}{a}\left(\mathbb{1} + \frac{D_W(-\rho/a)}{\sqrt{D_W^\dagger(-\rho/a)D_W(-\rho/a)}}\right).$$

Fulfills γ_5-hermiticity: $D_{ov}^\dagger = \gamma_5 D_{ov} \gamma_5$

- Eigenmodes of opposite chirality $\Rightarrow$ Non-real eigenvalues of D_{ov} split into conjugated pairs.
- $\det(D_{ov}) \in \mathbb{R}$.
- D_{ov} is normal: $\left[D_{ov}^\dagger, D_{ov}\right] = 0 \Rightarrow \lambda + \lambda^\star = a\lambda\lambda^\star$.

  Explicitly: $\left(\mathrm{Re}\lambda - \frac{1}{a}\right)^2 + (\mathrm{Im}\lambda)^2 = \frac{1}{a^2}$.
- Doubler modes are lying at $2/a$.
- Circle touches origin $\rightarrow$ D_{ov} is able to have exact zero modes!

From Wilson to overlap



$N = 20$, both operators are free

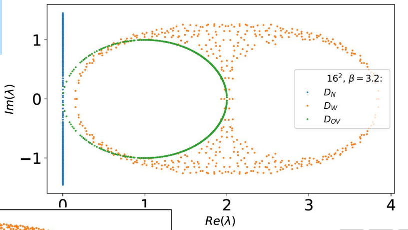
Staggered fermions

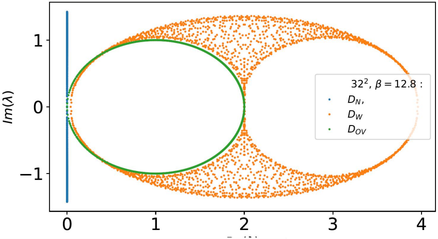
Non-GW type [Kogut and Susskind (1975)]
Constructed from Naïve lattice fermions:

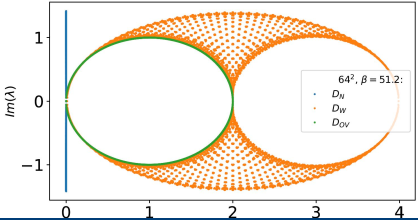
$\rightarrow$ "spreading" the two components and discard all but one component on each site.

- Staggered operator: $D_{st} = \eta_\mu \nabla_\mu$.
- Staggered sign function: $\eta_\mu \rightarrow$ replacement for γ_μ.
  $\eta_1 = 1, \eta_2 = (-1)^{n_1}, \ldots$
- Reduction from 4 down to 2 Dirac fermions $\rightarrow$ one doubler left.
- Remnant $U(1)$ symmetry.
- Simpler implementation, but harder to interpret.

Eigenvalue spectra







Eigenmodes and topological charge

Atiyah-Singer index theorem: [Atiyah and Singer (1971)]

Counts the zero modes via fermionic index:

$n_+ - n_-,$

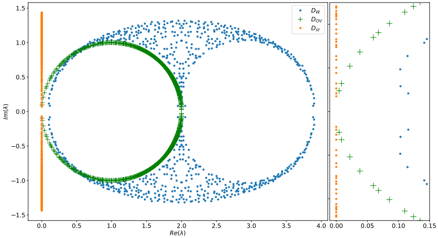
n_+: numbers of positive chirality zero modes,
n_-: numbers of negative chirality zero modes.

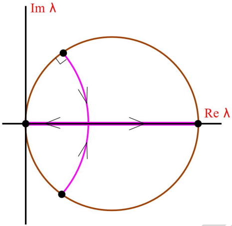
Topological charge:

$Q(U) = a^2 \sum_{n \in \Lambda} \frac{1}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu},$

Theorem connects property of gauge fields to fermionic observable:

$$\boxed{Q(U) = \text{index}(U) = n_+ - n_-}$$





[Creutz: arXiv:0708.1295v2]

Overlap formulation: Transition from $Q = 0$ to $Q = 1$ possible
$\Rightarrow$ pair of complex eigenvalues disappears $\rightarrow$ replaced by exact zero mode PLUS compensating mode on opposite side.

$\rightarrow$ exact zero modes are chiral and define the topological sector.

Staggered formulation: No clear distinction between different topological sectors [Dürr and Hoelbling (2003)] .

Chiral condensate

Expectation value of quark-antiquark pairs $\rightarrow$ condensate.

Chiral limit

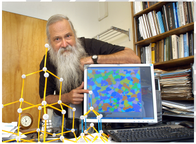
$N_f = 1$: SM in overlap formalism shows non-vanishing chiral condensate

$\rightarrow$ Schwinger value: $\frac{\exp(\gamma)}{2\pi^{3/2}} = 0.1599\ldots$ [Schwinger (1962)]

$\rightarrow$ Selection theorem: chiral condensate exclusively formed by zero modes with $|Q| = 1$ in an overlap formalism [Dürr, Hoelbling (2003)]

$N_f = 2$: Non-zero value would violate Mermin-Wagner theorem [Mermin and Wagner (1966)] .

[Michael Creutz]



Overlap formulation: [Dürr and Hoelbling (2006)]

Chiral condensate:

$$\frac{\chi^{ov}_{scal}}{e} = \frac{\sqrt{\beta}}{N^2}\frac{\langle \det(D^{ov}_m)^{N_f} \sum' \frac{1}{\tilde{\lambda}+m}\rangle}{\langle \det(D^{ov}_m)^{N_f}\rangle}, \quad \det(D^m_{ov}) = \prod_\lambda \left[\left(1-\frac{m}{2}\right)\lambda + m\right].$$

$$\tilde{\lambda} = \frac{1}{\frac{1}{\lambda}-\frac{1}{2}} = \frac{2\lambda}{2-\lambda} \Rightarrow \sum{}' \frac{1}{\tilde{\lambda}+m} = \sum{}' \frac{1-\lambda/2}{\lambda\left(1-\frac{m}{2}\right)+m}.$$

Staggered formulation: [Dürr and Hoelbling (2006)]

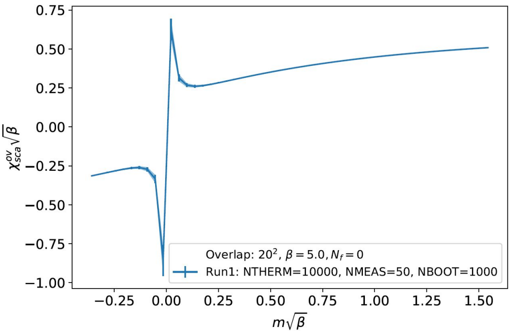
Chiral condensate:

$$\frac{\chi^{st}_{scal}}{e} = \frac{\sqrt{\beta}}{2N^2}\frac{\langle \det(D^{st}_m)^{N_f/2} \sum \frac{1}{\lambda+m}\rangle}{\langle \det(D^{st}_m)^{N_f/2}\rangle}, \quad \det(D^{st}_m) = \prod\left[\lambda + m\right].$$

Evaluation of the chiral condensate via:

- Generating up to 1000 gauge configurations with non-trivial topological charge on different lattice-sizes: $N^2 = 8 \times 8$ up to $N^2 = 50 \times 50$
  $\rightarrow$ Monte Carlo-Metropolis,
- Using: Wilson gauge action at different β,
- For each configuration: Determination of complete eigenvalue spectrum: Overlap formalism and staggered formalism.
- Computation of chiral condensate and fermion determinant for a range of positive and negative masses.
  Errors $\rightarrow$ Bootstrap method.

Quenched simulation



Chiral limit behaviour for $N_f = 0$

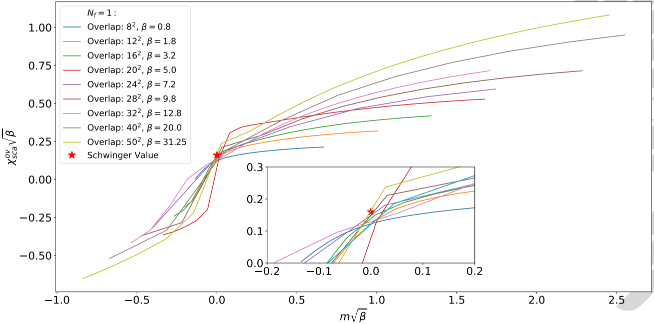
$\frac{\chi_{scal}}{e} \sim \langle \sum \frac{1}{\lambda_i + m} \rangle \Rightarrow 1/m$ -behaviour, $\exists\ \lambda \to 0$.

Behaviour matches previous works: [Kiskis and Narayanan (2000), Dürr and Hoelbling (2003)]

Lattice: Appearing of discontinuity.

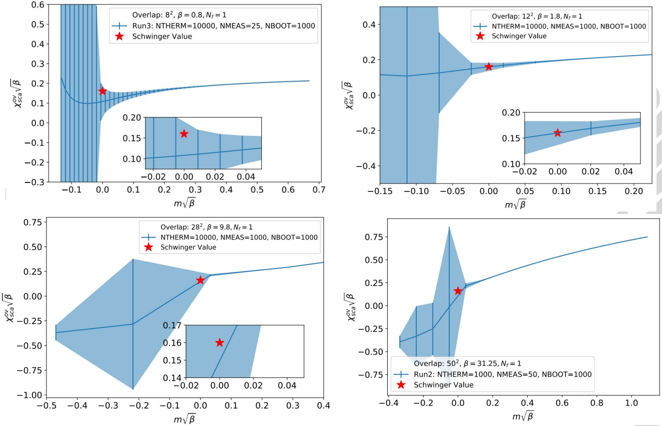
Single flavour, overlap formalism

NTHERM: 10000, NMEAS: 1000, resp. 50 for $N = 40, 50$.



Behaviour for $N_f = 1$. Schwinger value: $\frac{\exp(\gamma)}{2\pi^{3/2}} = 0.1599\ldots$ [Schwinger (1962)]

$$\frac{\chi_{scal}^{ov}}{e} = \frac{\sqrt{\beta}}{N^2} \frac{\langle \det(D_m^{ov})^{N_f} \sum' \frac{1}{\tilde{\lambda}+m} \rangle}{\langle \det(D_m^{ov})^{N_f} \rangle}.$$



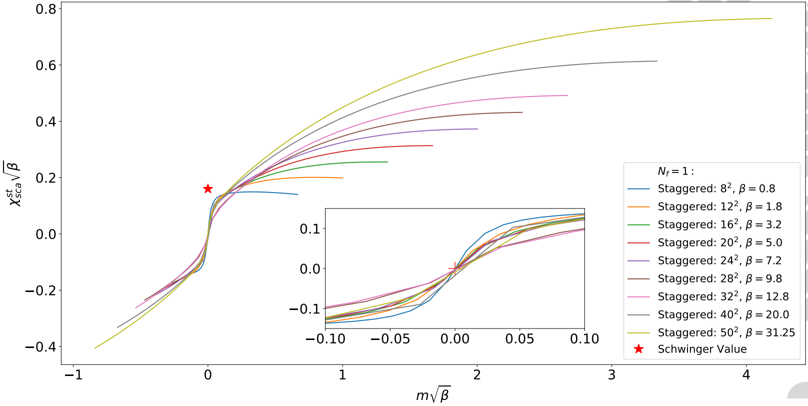
Behaviour for $N_f = 1$. Schwinger value: $\frac{\exp(\gamma)}{2\pi^{3/2}} = 0.1599\ldots$

Negative mass region: high fluctuations $\Rightarrow$ errors increase significantly.
Higher statistics: Visible improvement.

Overlap chiral condensate agrees with Schwinger value!

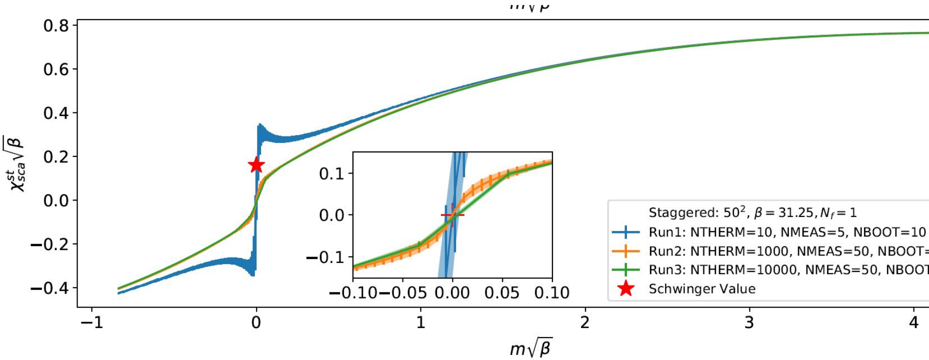
Single flavour, staggered formalism

NTHERM: 10000, NMEAS: 1000, resp. 50 for $N = 40, 50$.



Behaviour for $N_f = 1$. Schwinger value: $\frac{\exp(\gamma)}{2\pi^{3/2}} = 0.1599\ldots$

$$\frac{\chi^{st}_{scal}}{e} = \frac{\sqrt{\beta}}{2N^2} \frac{\langle \det(D^{st}_m)^{N_f/2} \sum \frac{1}{\lambda + m} \rangle}{\langle \det(D^{st}_m)^{N_f/2} \rangle}.$$



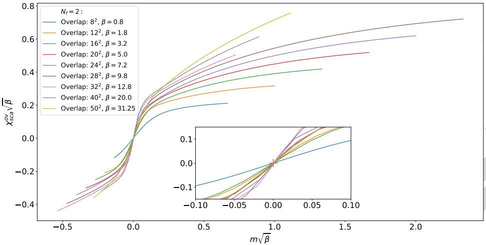
Behaviour for $N_f = 1$. Schwinger value: $\frac{\exp(\gamma)}{2\pi^{3/2}} = 0.1599\ldots$

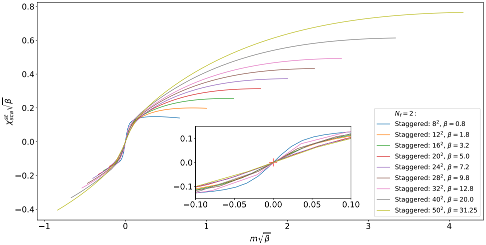
Too little statistic: $1/m$ outweighs $\rightarrow$ high statistics: significant improvement.

Staggered chiral condensate fails to reproduce the Schwinger value!

Two flavours

NTHERM: 10000, NMEAS: 25, resp. 50 for $N = 40, 50$.





Behaviour for $N_f = 2$.

Summary & Outlook

Schwinger model: 2d QFT with essential features of 4d QCD.
$\Rightarrow$ great test laboratory

Naïve lattice fermions: leads to fermion doubling

Solution: Wilson lattice fermions $\rightarrow$ removes doublers, but breaks chiral symmetry.

Solutions:

- Operator which satisfies Ginsparg-Wilson-condition $\rightarrow$ Overlap formalism.
- Staggered formalism $\rightarrow$ removes fermion doublers.

Chiral condensate for $N_f = 1$:

- Overlap formalism: Sensible to selection theorem:
- Staggered formalism: Vanishes in chiral limit.

Chiral condensate for $N_f = 2$: Both vanish in the chiral limit.

How to improve results?

- Higher statistics
- APE-smearing

Next steps?

- Calculation on higher lattice-sizes
  $\rightarrow$ Approaching the continuum limit.
- Evaluation of meson propagators (Master-thesis Niklas Pielmeier)
- Exponential fine-tuning (Unparticle physics $\rightarrow$ H. Georgi)

In conclusion: Schwinger model provides an indispensable lattice QFT

$\Rightarrow$ its journey is far from over!

Thank you for listening!

BU SVD

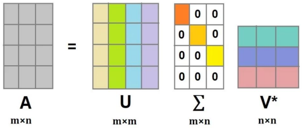
Implementation via SVD
Singular value decomposition:

- Valid for any $m \times n$ matrix over $\mathbb{C} \to A = U\Sigma V^\dagger$.
- $A \in \mathbb{M}_{m,n}$, unitary $U \in \mathbb{M}_m$ and $V \in \mathbb{M}_n$.
- Diagonal rectangular matrix $\Sigma \leftarrow$ singular values.

Result: Overlap operator in simpler form:

massless: $D_{ov} = \rho\left(\mathbb{1} + UV^\dagger\right) \to$ massive: $D_{ov}^m = \left(\mathbb{1} - \frac{m}{2\rho}\right) D_{ov} + m.$

askpython.com/python/examples/singular-value-decomposition



BU Markov chain

- Produces stationary distribution.
- Ensures transition from current state to a new state.

In order to get a (possible) new configuration U' with a starting configuration U, define transition probability:

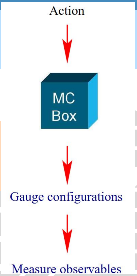
$$P(U_n = U' | U_{n-1} = U) = T(U'|U),$$

obeys:

- Range of a probability $0 \leq T(U'|U) \leq 1$,
- Total probability $\sum_{U'} T(U'|U) = 1$.

Ergodicity: Different topological sectors can be reached!

BU Monte Carlo of $U(1)$



[physik.uni-graz.at/ cbl]

Computation of expectation values:

$$\langle \mathcal{O}(U) \rangle = \frac{1}{Z} \int \mathcal{D}U \, \mathcal{O}(U) e^{-S_G(U)}, \quad \text{with} \quad Z = \int \mathcal{D}U e^{-S_G(U)}$$

Metropolis algorithm: [Metropolis et al. (1953)]

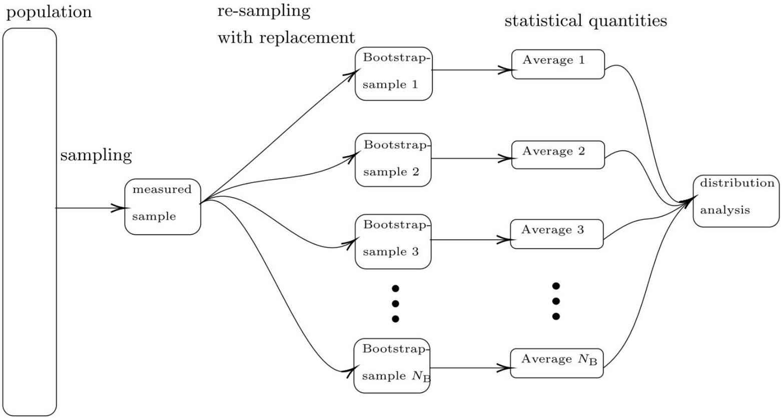
Receive new gauge configuration via:

- Initialize lattice: Trivial configuration $\rightarrow$ all gauge links are set to unity: $U_\mu(n) = \mathbb{1}, \forall n$.
- Choose possible new configuration $U'_\mu(n)$ and
- accept, if: $\Delta S_G(U_\mu(n)) \leq 0$,

  else: acceptance according to $\exp\left[-\Delta S_G(U_\mu(n))\right] > r$.
- Add topological configuration: multiply accepted U' with field per plaquette $F \sim Q \rightarrow$ receive final configuration U''.
- Repeat.

BU Bootstrap

Errors calculated via Bootstrap method: [Efron (1979)]



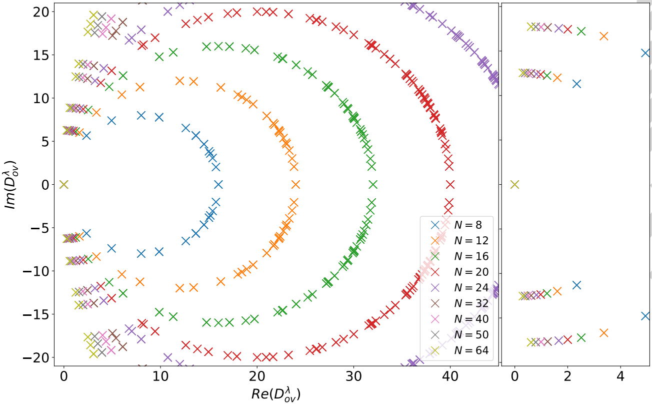
BU Determinant-Implementation

- Product over all $NX \times NT \times 2$ elements $\rightarrow$ Eigenvalues
- Re$(\lambda) \in [0, 2]$
- Im$(\lambda) \sim 0$
- $\det$ can get VERY big

$\Rightarrow$ Divide each eigenvalue by a constant c,
$\Rightarrow$ product is reduced by $1/c^{NX \times NT \times 2}$
In the end: determinant in the condensate in numerator and in denominator
$\Rightarrow$ reduction is eliminated!

BU Overlap for increasing N:

- $a \sim 1/N \to$ approaching continuum limit.
- Eigenvalues decouple from the theory



BU Formulas

$\Delta S_G(U_\mu(n)) = -\beta \text{Re}\left[\left(U'_\mu(n) - U_\mu(n)\right) A_\mu(n)\right]$,
Eigenvalues D_{ov} explicitly: $\text{Re}\lambda + i\ \text{Im}\lambda + \text{Re}\lambda - i\ \text{Im}\lambda = a\left(|\lambda|^2\right)$,
Naïve Dirac operator in momentum space:

$$\tilde{D}(p)^{-1} = \frac{-i\sum_\mu \gamma_\mu \sin(p_\mu a)/a + m\mathbb{1}}{\sum_\mu \sin^2(p_\mu a)/a^2 + m}.$$