Adaptive generation of quasi-optimal tetrahedral meshes

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Adaptive generation of quasi-optimal tetrahedral meshes

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Abstract — A method for adaptive 3D mesh generation based on approximate Hessian recovery is proposed and analyzed. Several numerical results illustrate basic properties of the method.

The techniques of adaptive mesh generation are of particular interest to mathematicians and engineers. Large attention is paid to both the implementation issues and theoretical aspects of adaptivity. Significant improvement of the accuracy of approximation through the adaptive nodes distribution rather than increasing the number of mesh nodes, allows to solve large problems arised in applications.

During the past decade the most popular were the regular conformal triangulations whose mesh size is governed by some *a posteriori* error estimators [18, 23]. The error estimators take into account both the discrete solution and the type of the problem to be solved. The set of conformal triangulations (possibly anisotropic) is much wider than that of regular-shaped meshes, then, it provides bigger opportunities in mesh adaptivity. Until recent times the wide usage of adaptive anisotropic meshes was retarded by two barriers. First, the presence of acute angles in the triangulation causes a deterioration in approximation of piece-wise polynomial functions with an isotropic second derivative [5]. Second, there was not a robust method for adaptive generation of unstructured anisotropic meshes. However, as it follows from [2-4, 20] the simplexes with acute angles stretched along the direction of minimal second derivative of a solution, are the best elements for minimizing the interpolation error. This being so, anisotropic elements might be and should be applied in the regions of the anisotropic solution Hessian. Besides, in recent years a set of robust methods for adaptive anisotropic mesh generation was proposed [8-10, 12, 16, 17, 19, 22]. The basic idea is that a quasi-uniform (in a metric based on the solution Hessian) mesh is to be generated. Since the solution and its Hessian are unknown, the metric is generated by the discrete Hessian recovered from the discrete solution. Thus, the mesh generator takes on input a mesh and respective discrete solution and outputs a new mesh which is more adapted to the solution. The output is obtained by a sequence of local modifications of the current mesh in such a way that the modified mesh be more quasi-uniform in the given metric. Besides the problem independence, adaptive methods based on a Hessian recovery produce meshes with a prescribed number of elements distributing them in an almost optimal way. Furthermore, the methods may be implemented as a "black box" regardless of the discrete problem generation and solver. The appeal of such an approach in engineering is

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difficult to overestimate.

In the paper we extend the 2D analysis of the method based on the Hessian recovery [22] to the 3D case. The rest of the paper is organized as follows. In Section 1 we give basic definitions and pose the minimization problem. In Section 2 we introduce the adaptive algorithm whose analysis is presented in Section 3. Numerical experiments illustrating the basic features of the algorithm constitute Section 4. The proofs of statements from Section 3 are contained in the Appendix.

1. OPTIMAL AND QUASI-OPTIMAL TRIANGULATIONS

Let $\Omega \in \mathbb{R}^3$ be a polyhedral domain, Ω^h be a conformal simplicial triangulation of Ω where any two tetrahedra have either a common face, or an edge, or a vertex, or have no common points. We denote by Tr the set of tetrahedra of Ω^h, and by $\#Tr$ the number of elements in Tr. We shall refer to $W(\Omega^h)$ as the space of continuous in Ω and linear in each tetrahedron $\Delta \in Tr$ functions. The symbol $C^k(\Omega)$ will denote the space of all functions which have continuous partial derivatives up to the order k in Ω. The space $C^k(\bar{\Omega})$ is formed by all functions from $C^k(\Omega)$ whose all derivatives up to the order k can be continuously extended onto $\bar{\Omega}$. In contrast to the standard assumption, we admit the norm of a continuous function to be arbitrary large but finite. Thus, functions with essential anisotropic properites are included in the set we are dealing with.

Let $u \in C(\bar{\Omega})$ and $P^h u \in W(\Omega^h)$ be the solutions of a boundary value problem and its discrete counterpart on Ω^h, respectively.

Definition 1.1. Given the type of approximation P^h, we refer to Ω^h as the optimal triangulation consisting of N_T tetrahedra, for approximation of the solution u by $P^h u$, if

$$\Omega^h = \text{arg} \min_{\Omega^h : \#Tr = N_T} \|u - P^h u\|_{L_\infty(\Omega)} . \tag{1.1}$$

Due to $\|u - P^h u\|_{L_\infty(\Omega)} = \max\limits_{\Delta \in Tr} \|u - P^h u\|_{L_\infty(\Delta)}$, the relation (1.1) may be rewritten as:

$$\Omega^h = \text{ arg } \min_{\Omega^h : \#Tr = N_T} \max_{\Delta \in Tr} \|u - P^h u\|_{L_\infty(\Delta)} .$$

The Definition 1.1 is quite natural but the existence of the optimal triangulation is not guaranteed in general case. However, this does not pose a problem since we will deal with meshes which approximate the optimal one only in the sense of minimization of the error norm. One may evaluate such meshes by comparing the error norm with $\inf\limits_{\Omega^h : \#Tr = N_T} \|u - P^h u\|_{L_\infty(\Omega)}$ which exists. On the other hand, the following statement is valid.

Lemma 1.1. *Let P^h be such a mapping that $\|u - P^h u\|_{L_\infty(\Omega)}$ is a continuous functional of the nodes coordinates. Then the optimal triangulation consisting of N_T tetrahedra exists.*

Proof. Any conformal triangulation may be defined in terms of a connectivity table Tb and a set X of the nodes of the mesh. Since $\|u - P^h u\|_{L_\infty(\Omega)}$ is continuous with respect to the nodes coordinates and the domain Ω is a bounded set, there exists arg $\min\limits_{X} \|u - P^h u\|_{L_\infty(\Omega)}$ for any fixed connectivity table Tb. Further, since the number N_T of tetrahedra is fixed, a set of all possible connectivity tables is finite, and there exists arg $\min\limits_{Tb, \#Tr = N_T} \min\limits_{X} \|u - P^h u\|_{L_\infty(\Omega)}$. $\square$

Corollary 1.1. *Let $u \in C^2(\bar{\Omega})$ and P^h be the operator of piece-wise linear nodal interpolation. Then the optimal triangulation consisting of N_T tetrahedra exists.*

The main objects of the research are the so-called quasi-optimal meshes which are defined in terms of a triangulation quality. The next definitions specify the respective terms.

Definition 1.2. [24] Let Δ be a tetrahedron, $G = \{g_{ij}\}_{i,j=1}^{3}$ be a constant metric defined on Δ, and $h^\star$ be a real positive number. The quality of Δ with reference to $h^\star$ in metric G is referred to as the number

$$Q_{G,h^\star}(\Delta) = 6^4\sqrt{2}\frac{|\Delta|_G}{|\partial\partial\Delta|_G^3}F\left(\frac{|\partial\partial\Delta|_G}{6h^\star}\right), \quad F(x) = \left(\min\left\{x, \frac{1}{x}\right\}\left(2 - \min\left\{x, \frac{1}{x}\right\}\right)\right)^3 .$$

Here $|\Delta|_G$ and $|\partial\partial\Delta|_G$ are the volume of the tetrahedron and the sum of lengths of its edges, respectively, measured in metric G.

One can easily show that $0 < Q_{G,h^\star}(\Delta) \leq 1$ and $Q_{G,h^\star}(\Delta) = 1$ if and only if the tetrahedron Δ is equilateral in the metric G, with the edge length $h^\star$. The first factor in $Q_{G,h^\star}(\Delta)$ controls the shape of the tetrahedron and the second one (function F) its size. In general, the function $F(x)$ may be defined in many ways. We remind that the G-measured volume of the tetrahedron and the G-measured length of a vector $\boldsymbol{l} \in \mathbb{R}^3$ are expressed via the respective values in the Euclidean space:

$$|\Delta|_G = |\Delta|_E \cdot (\det G)^{1/2}, \qquad |\boldsymbol{l}|_G = (G\boldsymbol{l}, \boldsymbol{l})^{1/2}.$$

Definition 1.3. The quality of triangulation Ω^h consisting of N_T tetrahedra in continuous metric $G(x)$ is defined as the worst quality of the tetrahedra which constitute Ω^h:

$$Q_{G,N_T}(\Omega^h) = \min_{\Delta\in Tr} Q_{G_\Delta,h^\star}(\Delta), \quad G_\Delta = G(\text{arg}\ \max_{x\in\Delta} \det G(x)), \quad h^\star = \sqrt[3]{\frac{12|\Omega|_G}{\sqrt{2}N_T}}.$$

Thus, for evaluating the quality of a triangulation consisting of N_T tetrahedra it is necessary to find the element the most distinguished in the metric G from the equilateral tetrahedron with the volume $|\Omega|_G/N_T$.

Let $u \in C^2(\bar{\Omega})$ and the Hessian $H = \{H_{ps}\}_{p,s=1}^{3}$ of u is nonsingular i.e. $\det H(x) \neq 0$. Due to $H = H^T$ at any point of Ω the spectral decomposition of H is possible,

$$H = W^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} W$$

so that the following metric may be defined:

$$|H| = W^T \begin{pmatrix} |\lambda_1| & 0 & 0 \\ 0 & |\lambda_2| & 0 \\ 0 & 0 & |\lambda_3| \end{pmatrix} W .$$

Definition 1.4. Let G be a continuous metric. Triangulation Ω^h consisting of N_T tetrahedra, is referred to as G-quasi-optimal, if $Q_{G,N_T}(\Omega^h) \geq q_0$, with $q_0 > 0$ being a fixed constant. Triangulation Ω^h consisting of N_T tetrahedra, is referred to as quasi-optimal (with reference to u), if $Q_{|H|,N_T}(\Omega^h) \geq Q_0$, with $Q_0 > 0$ being a fixed constant.

At the first glance, it is not clear what is common between the optimal and the quasi-optimal meshes. It turns out that at least in certain cases the quasi-optimal triangulation is an approximate solution of the optimization problem (1.1). Namely, a quasi-optimal triangulation Ω^h

consisting of N_T tetrahedra may satisfy the following estimate with a constant $C(Q_0)$ depending only on Q_0:

$$\|u - P^h u\|_{L_\infty(\Omega^h)} \le C(Q_0) \min_{\tilde{\Omega}^h : \#Tr = N_T} \|u - P^h u\|_{L_\infty(\tilde{\Omega}^h)} \,. \tag{1.2}$$

It follows from Definition 1.4 that quasi-optimal triangulations consisting of N_T tetrahedra constitute certain set which enlarges with the diminishing of Q_0. Next, the quasi-optimal triangulation satisfying $Q_0 = 1$ rather does not exist, since it is impossible to cover the space by the equilateral tetrahedra (measured in constant metric) without holes. Moreover, for a given u, Ω and Q_0 sufficiently close to 1, the existence of a quasi-optimal triangulation is unlikely since the boundary $\partial\Omega$ poses strong restrictions on the position of the tetrahedra faces. Nevertheless, smaller values of Q_0 relax the above limitations due to larger freedom in choosing a tetrahedra covering.

2. ALGORITHM FOR GENERATION OF QUASI-OPTIMAL MESHES

Let u be a solution of a given boundary value problem in a domain Ω, Ω_k^h be a triangulation of Ω consisting of N_T tetrahedra ($\#Tr_k = N_T$), $P_k^h u \in W(\Omega_k^h)$ be an approximation of u on Ω_k^h, and H_k be the discrete Hessian of $P_k^h u$ [9, 16], which is defined via its values in the interior grid nodes a_i, $a_i \notin \partial\Omega$:

$$H_k = \{H_{k,ps}\}_{p,s=1}^{3} \,, \qquad H_{k,ps} \in W(\Omega_k^h) \tag{2.1}$$

$$\int\limits_{\sigma_i} H_{k,ps}(a_i) v_i \,\mathrm{d}x = -\int\limits_{\sigma_i} \frac{\partial P_k^h u}{\partial x_p} \frac{\partial v_i}{\partial x_s} \,\mathrm{d}x \quad \forall v_i \in W(\sigma_i),\ v_i = 0 \ \text{ on } \ \partial\sigma_i, \quad p,s = 1,2,3 \tag{2.2}$$

where the superelement σ_i is a set of tetrahedra $\Delta_{ij} \in Tr_k$ sharing the node a_i. At the boundary nodes a_i the values of the Hessian H_k are the weighted extrapolations from the neighboring interior values [1]:

$$H_{k,ps}(a_i) = \frac{(\varphi(a_i), \overset{\circ}{H}_{k,ps})}{\sum\limits_{a_j \in \sigma_i, a_j \notin \partial\Omega} (\varphi(a_i), \varphi(a_j))} \,. \tag{2.3}$$

Here $\varphi(a_i)$ denotes the nodal basis function, $\overset{\circ}{H}_{k,ps}$ stands for the finite element functions defined by (2.2) and vanishing on $\partial\Omega$. Alternative extrapolation was proposed in [16].

The algorithm for the construction of the quasi-optimal meshes reads as follows.

Algorithm.

Generate an initial triangulation Ω_0^h. Set $k = 0$.

Do while $k \le k_{\max}$:

1. Find an approximation $P_k^h u$ associated with Ω_k^h;
2. Compute the discrete Hessian H_k of $P_k^h u$, $H_{k,ps} \in W(\Omega_k^h)$; if $Q_{|H_k|,N_T}(\Omega_k^h) > Q_0$ then stop;
3. Generate the next triangulation Ω_{k+1}^h such that $Q_{|H_k|,N_T}(\Omega_{k+1}^h) \ge q_0$;
4. $k = k + 1$;

End Do

The structure of the algorithm implies that four completely independent steps have to be performed.

- Initialization. An initial triangulation Ω_0^h has to be generated. As we shall show further, in theory it should be fine enough to provide an accurate approximation $P_0^h u$ to u. We do not consider methods of generation of the initial triangulation leaving the initial meshing of Ω to the user.

- Discrete solution. The user has to find the solution $P_k^h u$ of the discrete problem which approximates the given continuous one on the mesh Ω_k^h. The user is free in the choice of the algebraic solver and the type of discretization. The only requirement is that the solution should be a scalar piece-wise linear continuous function. In fact, there exists an extension of the method for the vector-valued functions, see [12].

- Hessian recovery. Computation of the discrete Hessian H_k of the discrete solution $P_k^h u$ on the basis of the rules (2.2)-(2.3); generation of the current metric $|H_k|$. The step is very fast since the computations are local, their arithmetical cost being proportional to the number of elements N_T.

- Generation of $|H_k|$-quasi-optimal triangulation Ω_{k+1}^h consisting of N_T tetrahedra. This is the most important and technical step in the overall technique. There are several heuristic methods for generation of $|H_k|$-quasi-optimal triangulation [8, 9, 12, 16, 17]. We follow the approach proposed in [9] for the construction of 2D $|H_k|$-quasi-optimal triangulation where the quality of a triangle is measured in respect to the equilateral (in metric $|H_k|$) triangle with an edge length $h^\star$. The value $h^\star$ was proposed to be equal to the edge length (in metric $|H_k|$) of the equilateral triangle with the square $|\Omega|_{|H_k|}/N_T$. We extend the method to tetrahedral meshes. We try to construct the mesh consisting of N_T elements, each of them being as much as possible close to the equilateral (in metric $|H_k|$) tetrahedron with the edge length $h^\star$. Of course, the conformal mesh may not consist of equilateral tetrahedra measured in a constant metric. But the space may be covered by almost similar tetrahedra which are close to a reference equilateral one. The admissibility of tetrahedra with the quality less than 1 relaxes the topological barrier. Thus, the edge length of the reference equilateral (in metric $|H_k|$) tetrahedron is that of the element with the volume $|\Omega|_{|H_k|}/N_T$: $h^\star = \sqrt[3]{\frac{12|\Omega|_{|H_k|}}{\sqrt{2}N_T}}$. Assuming that $|H_k| \approx |H|$ we have $h^\star \approx \sqrt[3]{\frac{12|\Omega|_{|H|}}{\sqrt{2}N_T}}$, i.e. $h^\star$ is approximately equal to a fixed value depending on $|\Omega|_{|H|}$, N_T. The proposed algorithm inherits the main features of the 2D prototype and uses several rules for local transformation of the spatial mesh [18, 24]. The method for producing the $|H_k|$-quasi-optimal triangulation is based on generation of a grid sequence $\Omega_k^h, \Omega_{k+\frac{1}{l_{\max}}}^h, \ldots, \Omega_{k+\frac{l}{l_{\max}}}^h, \ldots, \Omega_{k+1}^h$ such that

$$Q_{|H_k|,N_T}(\Omega_k^h) \le Q_{|H_k|,N_T}(\Omega_{k+\frac{1}{l_{\max}}}^h) \le \ldots \le Q_{|H_k|,N_T}(\Omega_{k+\frac{l}{l_{\max}}}^h) \le \ldots \le Q_{|H_k|,N_T}(\Omega_{k+1}^h)\,. \tag{2.4}$$

Each term of the sequence $\Omega_{k+\frac{l}{l_{\max}}}^h$ is a local modification of the previous term which rises the quality $Q_{|H_k|,N_T}$. It turns out there exists such set of local operations over the current mesh that the application of at least one of them usually improves the mesh quality. Due to the heuristic origination of the method we are not able to evaluate “a rate of the improvement". In practice, we take $l_{\max}$ large enough to have the value $Q_{|H_k|,N_T}(\Omega^h_{k+1})$ as much as a few tenths. Given a set of local operations, the worst tetrahedron with the minimal quality is considered, and a set of admissible mesh modifications in a vicinity of the tetrahedron is virtually tested until the proper operation will be found. Then, the detected operation is applied to change the mesh. In the case when all admissible local operations do not improve the mesh quality, we remove for a while the "bad" tetrahedron from the list of the elements admitted to testing. Such relaxation allows us to avoid intractable situations and provides the monotonicity of (2.4). Below we present the set of local operations in the priority of testing.

– **Add a new node.** Consider all the edges of the given tetrahedron. Try to insert a new point in the middle of the current edge and split all the tetrahedra sharing the edge by connecting the new point and the vertices of tetrahedra located outside the edge.

– **Swap face to edge.** Consider all the faces of the given tetrahedron. In the couple of tetrahedra sharing the face try to remove the common face and connect the opposite vertices of the couple by the edge.

– **Swap edge to face.** Consider all the edges of the given tetrahedron. If the edge is shared by three tetrahedra, try to remove the edge and introduce a new face which contains the vertices of the element triple outside the considered edge. The operation is inverse to **Swap face to edge.**

– **Swap edges in a superelement.** This operation is a generalization of the previous one. Consider all the edges of the given tetrahedron. If the edge is shared by more than three tetrahedra, try to remove the edge and split thus appearing polyhedron into a new set of tetrahedra.

– **Delete a node.** Consider all the vertices of the given tetrahedron. Form a superelement consisting of the tetrahedra sharing the current vertex. Try to delete the common node and split the resulted polyhedron into tetrahedra by the edges emanated from a superelement boundary node.

– **Move a node.** Consider all the vertices of the given tetrahedron. Form a superelement consisting of the tetrahedra sharing the current vertex. If this vertex is interior both for the superelement and the whole triangulation, compute the pseudo-gradient (with respect to the current node coordinates) of the mesh quality functional in terms of finite differences. Try to maximize the mesh quality functional by shifting the current node along the pseudo-gradient (line search). If the node is on the boundary of the whole triangulation, the pseudo-gradient is to be projected on the boundary of the domain Ω.

We note that in real practice the mesh Ω^h_{k+1} obtained at the Step 3 of the **Algorithm** contains the number of tetrahedra which just approximately equals N_T (deviation of several percents is possible). The main reasons are the approximate evaluation of $h^\star$, topological restrictions posed by $\partial\Omega$, and the difference between q_0 and the unity. Furthermore, we have noted that the number of elements in the the sequence of triangulations (2.4) may vary, but with $l_{\max}$ large enough and $l \approx l_{\max}$ the number of tetrahedra in the mesh $\Omega^h_{k+\frac{l}{l_{\max}}}$ returns to a vicinity of N_T.

The above algorithm seems to be appealing in applications since the Step 3 of the **Algorithm** is performed as a "black box" independent of a problem to be solved. The user has to possess an initial triangulation and a discrete solver for the problem posed on a tetrahedral mesh. By this means one may improve approximation of the problem by adapting to the solution a mesh consisting of a fixed number of tetrahedra. The alternative way which is an improvement of approximation by increasing the number of elements or degrees of freedom, often faces the problem of computer memory restrictions.

The Step 3 of the **Algorithm** is itself of practical significance. By its use the meshes with prescribed number of elements and desirable properties may be generated. To this end, it is enough to substitute a certain analytical metric G for the metric $|H_k|$. For example, the metric $G = \text{diag}\,\{1,\,1,\,1\}$ results in a quasi-uniform mesh, the metric $G = \text{diag}\,\{g(x),\,g(x),\,g(x)\}$ with $g(x) > 0$ attaining a large maximum on set M yields a regular mesh [15] refined to M. The metric $G(x) = W^T(x)\text{diag}\,\{\lambda_1,\,\lambda_2,\,\lambda_3\}W(x)$, where $0 < \lambda_1 \ll \lambda_2 \approx \lambda_3$, $W^T(x) = W^{-1}(x)$, $W(x) = (\boldsymbol{w}_1(x),\,\boldsymbol{w}_2(x),\,\boldsymbol{w}_3(x))$, provides a mesh stretched along the direction specified by $\boldsymbol{w}_1(x)$. Furthermore, one may pose the following problem: find the mesh adapted to a particular solution and possessing prescribed properties. The substitution in the Step 3 of the **Algorithm** $|H_k| \to G^T|H_k|G$ with a filter $G = \text{diag}\,\{g(x),\,g(x),\,g(x)\}$ causes a solution adapted mesh with an *a priori* weighted distribution of mesh nodes. User control of the mesh properties is possible through the above filtering.

3. ANALYSIS OF THE ALGORITHM

We suppose that the Step 3 of the **Algorithm** may be performed. The heuristic method used in the construction of the $|H_k|$-quasi-optimal triangulation is discussed above. The analysis of the **Algorithm** reduces to the following questions:

1. Why a triangulation satisfying $Q_{|H|,N_T}(\Omega^h) \geq Q_0$ is referred to as quasi-optimal?
2. Why the stopping criterion for the **Algorithm** reads as $Q_{|H_k|,N_T}(\Omega_k^h) \geq Q_0$?
3. Why the **Algorithm** converges?

The first question is a primal one. Indeed, even if the **Algorithm** is supposed to produce the meshes satisfying $Q_{|H|,N_T}(\Omega^h) \geq Q_0$, it is not clear yet what is the sense of such meshes. The answer reported in Theorem 3.1 is that these meshes are the approximate solutions of the optimization problem (1.1) in the sense (1.2), at least for the simplest interpolation problem. The second question is related to the problem of identification of the quasi-optimality. Since the Hessian of the differential solution u is not known, one can not compute the value $Q_{|H|,N_T}(\Omega^h)$. However, under certain conditions the inequality $Q_{|H_k|,N_T}(\Omega_k^h) \geq Q_0$ implies the quasi-optimality. This result is formulated in Theorem 3.2 and Corollary 3.2. The third question follows from the second one, namely, why the iterates of the **Algorithm** eventually satisfy the conditions of Theorem 3.2. Theorem 3.3 gives the answer for the case of the simplest interpolation problem. Though the statements are formulated for tetrahedral meshes, they are readily rewritten in the 2D case. Early the 2D version of the analysis was reported in [22].

The proofs of the theorems are presented in Appendix. Hereinafter we denote by I the identity matrix and by $C(z)$ a positive constant depending on the parameter z and independent of other parameters. We recall that by $C^2(\bar{\Omega})$ we define a set of functions possessing finite but arbitrary large C^2-norm which includes functions with essentially anisotropic Hessian.

Theorem 3.1. *Let $u \in C^2(\bar{\Omega})$, its Hessian H be nonsingular:* $\det H(x) \neq 0\ \forall x \in \Omega$*; P_{Ω^h} be the piece-wise linear interpolation operator. Let Ω^h and $\hat{\Omega}^h$ be a quasi-optimal and the optimal triangulations consisting of N_T tetrahedra, respectively, and for both any tetrahedron*

Δ from $\hat{\Omega}^h$ and the tetrahedron from Ω^h where $||u - P_{\Omega^h} u||_{L_\infty(\Omega)}$ is attained the following estimate holds true:

$$||H_{ps} - H_{\Delta,ps}||_{L_\infty(\Delta)} < q|\lambda_1(H_\Delta)|, \quad 0 < q < 1, \quad p, s = 1, 2, 3\,. \tag{3.1}$$

Here $H_\Delta := H(\arg \max_{x\in\Delta} |\det H(x)|)$, $\lambda_1(H_\Delta)$ is the closest to zero eigenvalue of H_Δ.

Then

$$||u - P_{\Omega^h} u||_{L_\infty(\Omega)} \le C(Q_0, q)||u - P_{\hat{\Omega}^h} u||_{L_\infty(\Omega)}\,. \tag{3.2}$$

We remind that the constant Q_0 characterizes the quasi-optimality of the mesh Ω^h. Condition (3.1) may be treated as the requirement of small relative variations for the Hessian H of the function u on any tetrahedron of the optimal mesh and the worst element of the quasi-optimal one. Small relative variations for matrices are naturally expressed in terms of the minimal eigenvalue. Condition (3.1) is satisfied at least for sufficiently refined meshes and functions such that $|\lambda_1(H_\Delta)| \ge C$. It is pertinent to note that the last inequality holds for a wide set of anisotropic functions.

An important consequence of the proof of Theorem 3.1 is the following corollary.

Corollary 3.1. *Let $u \in C^2(\bar{\Omega})$, the Hessian of u be nonsingular, $\hat{\Omega}^h$ be the optimal triangulation consisting of N_T elements, and for any tetrahedron Δ from $\hat{\Omega}^h$ the estimate* (3.1) *be valid. Then*

$$C_1(q)\left(\frac{|\Omega|_{|H|}}{N_T}\right)^{\frac{2}{3}} \le ||u - P_{\hat{\Omega}^h} u||_{L_\infty(\Omega)} \le C_2(q)\left(\frac{|\Omega|_{|H|}}{N_T}\right)^{\frac{2}{3}}$$

where $0 < C_1(q) \le C_2(q)$.

The next theorem is valid for any approximation $P_k^h u$ of the function u such that a recovered discrete Hessian H_k approaches H at the nodes of Ω_k^h.

Theorem 3.2. *Let $P_k^h u \in W(\Omega_k^h)$ be an approximation of the function $u \in C^2(\bar{\Omega})$, H be the nonsingular Hessian of u, H_k be a discrete Hessian of $P_k^h u$, $\det H_k(x) \neq 0$, and Ω_k^h be a $|H_k|$-quasi-optimal triangulation consisting of N_T tetrahedra, $Q_{|H_k|,N_T}(\Omega_k^h) \ge q_0$. Let for any superelement $\sigma \in \Omega_k^h$ associated with the nodal basis function the following estimates hold:*

$$||H_{ps} - H_{\sigma,ps}||_{L_\infty(\sigma)} \ < \ \delta \tag{3.3}$$

$$|H_{k,ps}(a) - H_{\sigma,ps}| \ < \ \varepsilon \tag{3.4}$$

where a is a common point of all tetrahedra constituting σ, $H_\sigma = H(\arg \max_{x\in\sigma} |\det H(x)|)$ and $\delta > 0$, $\varepsilon > 0$ are small with respect to the minimal eigenvalue of $|H_k|$. Then the mesh Ω_k^h is quasi-optimal:

$$Q_{|H|,N_T}(\Omega_k^h) \ge C q_0\,. \tag{3.5}$$

Condition (3.3) implies small relative variations of the Hessian H on any superelement σ, while (3.4) is the requirement of a nodal approximation for the Hessian. The requirement analogous to (3.4) was announced recently in [11]. In the case of discrete Hessian recovered via (2.1)-(2.2), the following corollary gives sufficient conditions for the quasi-optimality of Ω_k^h.

Corollary 3.2. *Let $P_k^h u \in W(\Omega_k^h)$ be an approximation of the function $u \in C^2(\bar{\Omega})$, H be the nonsingular Hessian of u, H_k be the discrete Hessian recovered from $P_k^h u$ via* (2.1)-(2.2), *$\det H_k(x) \neq 0$, and Ω_k^h be a $|H_k|$-quasi-optimal triangulation consisting of N_T tetrahedra,*

$Q_{|H_k|,N_T}(\Omega_k^h) \geq q_0$. *Let for any superelement* $\sigma \in \Omega_k^h$ *associated with the nodal basis function the following estimates hold:*

$$||H_{ps} - H_{\sigma,ps}||_{L_\infty(\sigma)} < \delta \tag{3.6}$$

$$||\nabla(u - P_k^h u)||_{L_2(\sigma)} < \varepsilon|\sigma|^{1/2}\rho \tag{3.7}$$

where ρ *is the radius of the largest sphere inscribed in* σ, $H_\sigma = H(\arg \max_{x\in\sigma} |\det H(x)|)$ *and* $\delta > 0$, $\varepsilon > 0$ *are small with respect to the minimal eigenvalue of* $|H_k|$.
Then the mesh Ω_k^h *is quasi-optimal:*

$$Q_{|H|,N_T}(\Omega_k^h) \geq Cq_0\,. \tag{3.8}$$

The condition (3.6) implies small relative variations of the Hessian H on any superelement σ, while (3.7) is indicative of higher accuracy for the gradient approximations. Indeed, in the case of regular-shaped tetrahedra it may be rewritten as $\|\nabla(u - P_k^h u)\|_{L_2(\sigma)} < \varepsilon\rho^{5/2}$. The discrete Hessian H_k (2.1)-(2.2) approaches the differential one H when the conditions (3.6), (3.7) are satisfied. It is pertinent to note here that condition (3.7) is rather restrictive. One way of relaxing it might be the use of better discrete Hessian recovering compared to (2.1)-(2.2) [11]. However, in some cases, no discrete Hessian recovering satisfies (3.4). Then the stopping criterion $Q_{|H_k|,N_T}(\Omega_k^h) \geq Q_0$ of the **Algorithm** is not adequate anymore. In such cases a fixed number of the **Algorithm** iterations is performed regardless the value of $Q_{|H_k|,N_T}(\Omega_k^h)$.

Another simple but useful consequence of Theorem 3.2 is as follows.

Corollary 3.3. *Let* $P_k^h u \in W(\Omega_k^h)$ *be an approximation of* $u \in C^2(\bar{\Omega})$ *on the mesh* Ω_k^h, *consisting of* N_T *elements and satisfying* (3.6), (3.7) *and* Ω_k^h *be not quasi-optimal. Assume that* Ω_{k+1}^h *is the* $|H_k|$*-quasi-optimal triangulation consisting of* N_T *elements:*

$$Q_{|H_k|,N_T}(\Omega_{k+1}^h) \geq q_0$$

where H_k *is the discrete Hessian of* $P_k^h u$ *recovered by* (2.1)-(2.2). *Then the mesh* Ω_{k+1}^h *is quasi-optimal:*

$$Q_{|H|,N_T}(\Omega_{k+1}^h) \geq Cq_0\,.$$

By the convergence of the **Algorithm** we understand a gradual increase of value $Q_{|H_k|,N_T}(\Omega_k^h)$. This value, however, is not always indicative of the mesh quasi-optimality. We restrict ourselves by the case when $Q_{|H_k|,N_T}(\Omega_k^h)$ is an adequate measure of the mesh quality $Q_{|H|,N_T}(\Omega_k^h)$. As it follows from Corollary 3.2, quasi-optimality of the mesh may be conditioned by a small value of the gradient error. With the help of the next theorem we show that under certain conditions the gradient error will be reduced. The analysis here is restricted by the simplest interpolation problem.

Theorem 3.3. *Let* $u \in C^2(\bar{\Omega})$, H *be nonsingular Hessian of* u, *and for any superelement (and tetrahedron)* σ *from the current mesh* Ω_k^h *consisting of* N_T *elements the following estimate holds:*

$$||H_{ps} - H_{\sigma,ps}||_{L_\infty(\sigma)} \leq \delta, \quad H_\sigma = H(\arg \max_{x\in\sigma} |\det H(x)|)\,. \tag{3.9}$$

Besides, let the piece-wise linear interpolant $P_{\Omega_k^h} u \in W(\Omega^h)$ *be such that for any superelement* σ *from* Ω_k^h

$$||\nabla(u - P_{\Omega_k^h} u)||_{L_2(\sigma)} \leq \varepsilon|\sigma|^{1/2}\rho \tag{3.10}$$

where ρ is the radius of the largest sphere inscribed in σ. Further, let Ω^h_{k+1} be a $|H_k|$-quasi-optimal triangulation consisting of N_T elements:

$$Q_{|H_k|,N_T}(\Omega^h_{k+1}) \geq q_0 \tag{3.11}$$

where H_k is the discrete Hessian of $P_{\Omega^h_k} u$. Then for the interpolant $P_{\Omega^h_{k+1}} u$ and any tetrahedron Δ from Ω^h_{k+1} the following estimation is valid:

$$\|\nabla(u - P_{\Omega^h_{k+1}} u)\|_{L_2(\Delta)} \leq C(q_0, \delta, \varepsilon) \left(\sqrt{\lambda_{3,\Delta} \cdot \min_{\Omega^h : \# Tr = N_T} \|u - P_{\Omega^h} u\|_{L_\infty(\Omega)}} + \delta \right). \tag{3.12}$$

Here $\lambda_{3,\Delta}$ is the maximal eigenvalue of $|H|$ on Δ.

The above assertion states that, if the right hand side of (3.12) is less than $\|\nabla(u - P_{\Omega^h_k} u)\|_{L_2(\Delta)}$, then the gradient error is reduced and consequently, the mesh quality $Q_{|H|,N_T}(\Omega^h_{k+1})$ increases. Otherwise the gradient error may increase and the quality may deteriorate. Indeed, in practice, at the first steps of the **Algorithm** the current mesh quality $Q_{|H|,N_T}(\Omega^h_k)$ increases. Then it falls into certain stagnation zone where inequality (3.12) is violated. The mesh quality is specified there by the values of q_0, δ and $\|\nabla(u - P_{\Omega^h_k} u)\|_{L_2(\Delta)}$.

We note that the restrictions on the solution function u and the discrete operator P_{Ω^h} used in the analysis are not severe both from the practical and some theoretical view points.

Due to Theorems 3.1 and 3.3 quasi-optimal meshes satisfy (1.2) and the **Algorithm** "converges" when P^h is the interpolation operator. In those methods where the approximation error $u - P^h u$ is bounded by the interpolation error $u - P_{\Omega^h} u$ for any tetrahedral mesh,

$$\|u - P^h u\|_{L_\infty(\Omega^h)} \leq C \|u - P_{\Omega^h} u\|_{L_\infty(\Omega^h)} \tag{3.13}$$

estimate (1.2) is transformed to

$$\|u - P^h u\|_{L_\infty(\Omega^h)} \leq C C(Q_0) \min_{\tilde{\Omega}^h : \# Tr = N_T} \|u - P_{\Omega^h} u\|_{L_\infty(\tilde{\Omega}^h)} \tag{3.14}$$

and according to Corollary 3.1 the approximation error on a quasi-optimal mesh Ω^h is estimated as follows:

$$\|u - P^h u\|_{L_\infty(\Omega^h)} \leq C \left(\frac{|\Omega|_{|H|}}{N_T} \right)^{\frac{2}{3}}.$$

On the other hand, due to Tichomirov's result [21], for any discrete spaces V_h and $\Omega \in \mathbb{R}^3$

$$\inf_{\# V_h \leq N_T} \sup_{\|u\|_{C^2(\bar{\Omega})} = 1} \inf_{v_h \in V_h} \|u - v_h\|_{L_\infty(\Omega)} \simeq N_T^{-2/3}. \tag{3.15}$$

Therefore,

$$c \left(\frac{1}{N_T} \right)^{\frac{2}{3}} \leq \|u - P^h u\|_{L_\infty(\Omega^h)} \leq C \left(\frac{|\Omega|_{|H|}}{N_T} \right)^{\frac{2}{3}}.$$

It means that for functions satisfying Corollary 3.1 and approximations satisfying (3.13) quasi-optimal meshes are approximate solutions of problem (1.1).

The assumption of nonsingularity of the Hessian H is used in the proofs. In actual practice, the **Algorithm** works even in the case of locally singular Hessian. The diameter of an optimal tetrahedron will be infinite in this case. Due to the boundness of the domain Ω the tetrahedra diameters are bounded and the mesh may be not quasi-optimal in the sense of Definitions 1.2–1.4. The meshes generated by the **Algorithm** correspond to a perturbed Hessian where some small values are substituted for zero eigenvalues of the Hessian H. The last observation points the way to cope with the theoretical difficulties. Instead of dealing with a singular Hessian at a tetrahedron Δ, we introduce there a constant metric G and a symmetric invertible matrix $\hat{G}$ which are perturbations of the Hessian. Then, for any $u \in C^2(\bar{\Delta})$ and its interpolant $P_{\Omega^h} u$:

$$\|u - P_{\Omega^h} u\|_{L_\infty(\Delta)} \leq 2\,\frac{\left(6\sqrt[4]{2}\right)^{2/3}}{Q_{l_G,G}} \left(1 + \frac{\||H|_{ps} - G_{ps}\|_{L_\infty(\Delta)}}{\lambda_G}\right) |\Delta|_G^{2/3} \tag{3.16}$$

$$\|u - P_{\Omega^h} u\|_{L_\infty(\Delta)} \geq C \max_{\hat{G}:\|H_{ps} - \hat{G}_{ps}\|_{L_\infty(\Delta)} < 2\lambda_{|\hat{G}|}} \left(1 - \frac{\|H_{ps} - \hat{G}_{ps}\|_{L_\infty(\Delta)}}{2\lambda_{|\hat{G}|}}\right) |\Delta|_G^{2/3} \tag{3.17}$$

where l_G stands for the diameter of Δ measured in metric G. The above estimates, in conjunction with Tichomirov's result (3.15), give the basis for the theory of the method in the case of locally singular Hessian.

We note, however, that theoretical difficulties in deriving estimate (1.2) for general types of approximations do not affect the practical performance of the method.

4. NUMERICAL EXPERIMENTS

The numerical properties of the method are illustrated by an example of two types of approximation of a given function, the piece-wise linear interpolation and the finite element energy projector, respectively. Although the respective optimal triangulations are unknown, we compare the meshes obtained in the iterative process with the quasi-optimal triangulation generated by the Step 3 of the **Algorithm** on the basis of analytically known Hessian. The usage of the quasi-optimal mesh as a reference one makes sense since it is the best mesh the **Algorithm** may produce. This being so, Theorems 3.2 and 3.3 admit the numerical illustration while Theorem 3.1 seems to be impossible to confirm numerically due to the unattainability of the optimal triangulation in contrast to the 2D case [22].

We consider the function

$$u(x) = 0.06/\sqrt{(x_1 - 0.5)^2 + 0.3(x_2 - 0.5)^2 + 0.3^2(x_3 + 0.2)^2}, \;\; x \in \Omega = (0,1)^3$$

which is the solution of the boundary value problem

$$\begin{aligned} -\sum_{i=1}^{3} a_{ii}\,\frac{\partial^2 u}{\partial x_i^2} + u \;&=\; f \;\text{ in }\; \Omega \\ \sum_{i=1}^{3} a_{ii}\,\frac{\partial u}{\partial x_i}\cos(\boldsymbol{n}, x_i) \;&=\; g \;\text{ on }\; \partial\Omega \end{aligned} \tag{4.1}$$

where $a_{11} = 1,\; a_{22} = 0.3,\; a_{33} = 0.3^2,\; f(x) = u(x)$,

$$g(x) = \frac{0.06 \sum_{i=1}^{3} \cos(\boldsymbol{n}, x_i)}{\left((x_1 - 0.5)^2 + 0.3(x_2 - 0.5)^2 + 0.3^2(x_3 + 0.2)^2\right)^{3/2}}$$

$\cos(\boldsymbol{n}, x_i)$ is the cosine of the angle between the outer unit normal $\boldsymbol{n}$ to $\partial\Omega$ and the coordinate axis x_i. The function $P^h u \in W(\Omega^h)$ stands for the solution of the finite element counterpart of the problem (4.1), and $P_{\Omega^h} u \in W(\Omega^h)$ stands for the piece-wise linear interpolation of $u(x)$ on a mesh Ω^h. We note that $0 < u(x) \leq 1$, $x \in \Omega$, $u(x)$ has anisotropic behavior and the point

$(0.5, 0.5, 0)$ is the closest one to its singularity. The Hessian H of function u does not have elliptic properties and an $|H|$-equilateral tetrahedron has likely non-optimal shape for minimization of interpolation error, cf. [2]. Therefore, even the quasi-optimal triangulation with the quality close to 1 does not approach the optimal mesh, though the estimate for interpolation error (1.2) is true.

To produce the initial mesh Ω_0^h, we split the unit cube Ω into 6 tetrahedra and refine the obtained triangulation by chopping each tetrahedron into 8 smaller ones. Three (respectively, four) levels of such refinement result in the hierarchical grid consisting of 3072 (respectively, 24576) elements. The first grid will be referred to as the coarse mesh and the second one as the fine mesh. By Ω_∞^h we denote the quasi-optimal mesh generated by the Step 3 of the **Algorithm** when the Hessian H of function u is specified analytically instead of its recovering by (2.1)-(2.2). It worths to mention that in order to keep the number of elements about the same in our computations we used the modified formula for $h^\star$, $h^\star = \sqrt[3]{\frac{12|\Omega|_G}{\sqrt{2}N_T} \cdot 1.5}$, instead of $\sqrt[3]{\frac{12|\Omega|_G}{\sqrt{2}N_T}}$. The heuristic factor 1.5 is caused probably by topological conditions.

Table 1.
Uniform hierarchical grids versus quasi-optimal meshes.

mesh	coarse Ω_0^h	coarse Ω_∞^h	fine Ω_0^h	fine Ω_∞^h
$\#Tr_k$	3072	3591	24576	27161
$\|\|u - P_{\Omega^h} u\|\|_{L_\infty(\Omega)}$	0.111	0.045	0.087	0.013
$Q_{\|H\|,\#Tr_0}(\Omega_k^h)$	0.009	0.13	0.002	0.1

In Table 1 we compare the uniform hierarchical grids Ω_0^h and the quasi-optimal triangulations Ω_∞^h with approximately the same number of elements. We see that the quality $Q_{|H|,\#Tr_0}(\Omega_\infty^h)$ of the triangulation Ω_∞^h in the given metric $|H|$ is much higher than that of Ω_0^h, and the mesh Ω_∞^h provides the interpolation error significantly smaller. Although the meshes Ω_∞^h are not optimal, they are quasi-optimal with qualities being about the same. The ratio of the interpolation error for the coarse and the fine meshes equals to the power $2/3$ of the ratio for the respective numbers of elements, $\#Tr_\infty$. The latter observation may be treated as an illustration of the important Corollary 3.1.

Table 2.
Application of the **Algorithm** to the interpolation problem. Coarse meshes.

mesh	Ω_0^h	Ω_1^h	Ω_2^h	Ω_3^h	Ω_4^h	Ω_5^h	Ω_6^h	Ω_7^h	Ω_8^h	Ω_9^h	Ω_∞^h
$Q_{\|H_{k-1}\|,N_T}(\Omega_k^h)$		0.24	0.12	0.15	0.18	0.16	0.15	0.14	0.13	0.14	
$Q_{\|H_k\|,N_T}(\Omega_k^h)$	0.02	0.01	0.05	0.06	0.06	0.05	0.05	0.04	0.05	0.06	
$\|\|u - P_{\Omega_k^h} u\|\|_{L_\infty(\Omega)}$	0.111	0.068	0.065	0.053	0.053	0.07	0.07	0.07	0.07	0.07	0.045
$\#Tr_k$	3072	3379	3497	3511	3563	3619	3678	3674	3699	3692	3591

In Table 2 we present the behavior of the **Algorithm** applied to the interpolation problem. The ordered number of elements, N_T, is chosen to be the same as in the coarse hierarchical grid, $N_T = 3072$. The difference between N_T and $\#Tr_k$ is conditioned probably by nonsmoothness of the recovered Hessian. However, the number of elements in the meshes Ω_k^h remains more or less stable. At each iteration k of the **Algorithm** the Step 3 produces $|H_k|$-quasi-optimal mesh with a good quality, $Q_{|H_k|,N_T}(\Omega^h_{k+1}) \sim 0.1 - 0.2$. On the other hand, the quality $Q_{|H_k|,N_T}(\Omega^h_k)$ increases in the course of iterations. Relatively small values of $Q_{|H_k|,N_T}(\Omega^h_k)$ are dictated possibly by inadequate values of ε and δ in (3.3)–(3.4). As for the interpolation error, its reduction has a nonmonotone character. There exist many reasons for that. First, the minimization of the interpolation error by the **Algorithm** is merely a side effect of generating a quasi-optimal mesh which is an approximation of the optimal mesh in the sense (1.2). Second, the convergence of the **Algorithm** does not imply that the qualities $Q_{|H_k|,N_T}(\Omega^h_k)$ increase monotonically, cf. Theorem 3.3 and comments done after it. This means that the **Algorithm** does not increase the quality of the mesh $Q_{|H_k|,N_T}(\Omega^h_k)$ at each iteration, except the first ones. Nevertheless, the comparison of the interpolation error on the hierarchical grid Ω^h_0, the quasi-optimal mesh Ω^h_∞ and the intermediate grids Ω^h_k exhibits the usefulness of the technique, especially in the case of the fine meshes with a large number of elements (cf. Table 4, upper rows).

Table 3.
Application of the **Algorithm** to the finite element problem. Coarse meshes.

mesh	Ω^h_0	Ω^h_1	Ω^h_2	Ω^h_3	Ω^h_4	Ω^h_5	Ω^h_6	Ω^h_7	Ω^h_8	Ω^h_9	Ω^h_{10}
$Q_{|H_{k-1}|,N_T}(\Omega^h_k)$		0.21	0.27	0.24	0.23	0.24	0.22	0.21	0.21	0.21	0.24
$Q_{|H_k|,N_T}(\Omega^h_k)$	0.02	0.05	0.05	0.07	0.11	0.08	0.11	0.08	0.08	0.08	0.09
$\|u - P^h_k u\|_{L_\infty(\Omega)}$	0.49	0.27	0.20	0.17	0.14	0.12	0.11	0.11	0.11	0.12	0.11
$\#Tr_k$	3072	3389	3451	3545	3578	3650	3608	3553	3537	3540	3468

Table 4.
Application of the **Algorithm** to the interpolation (upper rows) and finite element (bottom rows) problems. Fine meshes.

mesh	Ω^h_0	Ω^h_1	Ω^h_2	Ω^h_3	Ω^h_4	Ω^h_5	Ω^h_6	Ω^h_7	Ω^h_8	Ω^h_9	Ω^h_∞
$\|u - P_{\Omega^h_k} u\|_{L_\infty(\Omega)}$	0.087	0.045	0.042	0.022	0.021	0.021	0.021	0.021	0.021	0.021	0.013
$\#Tr_k$	24576	24707	25236	25519	25375	25448	25462	25305	25133	25169	27161
$\|u - P^h_k u\|_{L_\infty(\Omega)}$	0.18	0.091	0.085	0.078	0.076	0.073	0.075	0.068	0.067	0.067	
$\#Tr_k$	24576	26351	25938	25611	25081	24945	24640	24599	24380	24237	

Table 3 contains the analogous data for the **Algorithm** applied to the adaptive finite element solution of the problem (4.1). The **Algorithm** demonstrates the same properties that in the case of the simplest interpolation. The only difference here is larger values of the approximation errors $\|u - P^h_k u\|_{L_\infty(\Omega)}$ as compared with the interpolation errors from Table 2. In the Table 4 we present the history of the **Algorithm** in the case of fine meshes, for the problem of interpolation (upper rows) and the finite element counterpart of (4.1) (bottom rows). The ordered number of elements N_T is the number of tetrahedra in the fine hierarchical grid, $N_T = 24576$.

The above numerical results imply the following. The **Algorithm** produces quasi-optimal meshes with the fixed number of elements. It does converge at a few initial steps. The convergence here does not imply a monotone behavior of the mesh quality $Q_{|H_k|,N_T}(\Omega^h_k)$. Nevertheless, it is increased in average. A side effect of the convergence is the minimization of the errors. The minimal error is attained on the meshes produced by the **Algorithm** on the basis of the explicitly known Hessian. The application of the **Algorithm** yields significant reduction of the error in comparison with the initial uniform grid.

5. CONCLUSIONS

The algorithm for adaptive tetrahedral mesh generation based on the Hessian recovery is presented and analyzed. Under certain assumptions, the quasi-optimal meshes are shown to be approximations to optimal meshes (1.1). The stopping criterion of the **Algorithm** is proven to be adequate, when the recovered discrete Hessian approaches the differential one at the nodes of the mesh. Under the assumption of small gradient error, the convergence of the **Algorithm** is analyzed. Several numerical examples illustrate the basic properties of the method. The above technique possesses both drawbacks and advantages. Among the drawbacks the following ones are the most important: to provide the convergence towards a quasi-optimal mesh, one has to start the iterations from rather fine initial grid Ω_0^h, so that the recovered discrete Hessian H_0 approaches the differential Hessian H (cf. Theorems 3.2 and 3.3). This restriction is not typical of techniques which take into account the differential equations to be solved, cf. [23]. On the other hand, in practice, the initial mesh is not obliged to be very fine, even “traces” of singularities in the discrete solution are sufficient to catch problem characteristics. The second drawback is a trivial consequence of the quasi-optimality concept. The algorithm produces the mesh which consists of almost equilateral tetrahedra measured in $|H|$-metric. However, such elements may be very stretched in the Euclidean metric that may cause troubles in some approximations. Alternatively, quasi-optimal meshes seem to be the best ones fitted to approximate a particular solution and application of non-robust approximations in the case of highly anisotropic solutions probably makes no sense. The most important advantage of the approach is that it is independent of the particular type of problems or equations. It may be considered as a “black box” having on input the mesh and discrete solution and generating the grid with a prescribed number of elements which is more adapted to the solution. The only restriction of the mesh conformity enables to perform the maximal optimization in the sense of (1.1). The appeal of the technique in applications is difficult to overestimate.

APPENDIX

Before proving the theorems announced in Section 3, we present two lemmas which play an important role in the analysis.

The following lemma is a 3D counterpart of the result proved in [2].

Lemma 5.1. *Let Δ be a tetrahedron with vertices a_1, a_2, a_3, a_4, and $u \in P_2(\Delta)$ be a quadratic function with the nonsingular Hessian:* $\det H \neq 0$*, and $P_\Delta u$ be the linear interpolant of u on Δ. Then*

$$C|\tilde{\Delta}|^{\frac{2}{3}} \leq \|u - P_\Delta u\|_{L_\infty(\Delta)} \leq \tilde{r}^2/2 \tag{5.1}$$

where $\tilde{r}$ is the radius of the circumscribed sphere for the tetrahedron $\tilde{\Delta}$ which is the image of Δ under the transformation $\tilde{x} = R(x)$ reducing H to canonical form

$$\tilde{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

Proof. We shall use the following identity which is multipoint Taylor formula [13, 14] for quadratic functions:

$$u(x) - P_\Delta u(x) = -\frac{1}{2}\sum_{i=1}^{4}(H(x - a_i), (x - a_i))p_i(x) \tag{5.2}$$

where $p_i(x)$ is the linear function on Δ attaining 1 at a_i and vanishing at other vertices of Δ.

Due to the symmetry of the Hessian the spectral decomposition is possible, $H = W^T DW$, $D = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3\}$. The module of H is defined as $|H| = W^T|D|W$ where $|D| = \operatorname{diag}\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}$. It is clear that

$$|(Hy, y)| \leq (|H|y, y) \quad \forall y \in \mathbb{R}^3 . \tag{5.3}$$

Due to $p_i(x) \geq 0,\ x \in \Delta$, (5.2) and (5.3) we get

$$\max_{x \in \Delta} |u(x) - P_\Delta u(x)| \leq \max_{x \in \Delta} |\hat{u}(x) - P_\Delta \hat{u}(x)| \tag{5.4}$$

where $\hat{u}(x)$ is defined via the multipoint Taylor formula,

$$\hat{u}(x) = P_\Delta u(x) - \frac{1}{2} \sum_{i=1}^{4} (|H|(x - a_i), (x - a_i)) p_i(x) .$$

Let $R(x) = \sqrt{|D|}W$ and $\tilde{\Delta} = R\Delta$. It follows from [2] that

$$\max_{x \in \Delta} |\hat{u}(x) - P_\Delta \hat{u}(x)| = \tilde{r}^2/2$$

where $\tilde{r}$ is the radius of the sphere circumscribed about $\tilde{\Delta}$. Hence,

$$\max_{x \in \Delta} |u(x) - P_\Delta u(x)| \leq \tilde{r}^2/2$$

and the right inequality of (5.1) is shown provided that (5.2) is valid.

Let us show the left inequality of (5.1). To this end, we use the result [2]:

$$\max_{x \in f_i} |u(x) - P_\Delta u(x)| \geq C \max_{j=1,2,3} |l_{ij}|^2_{|H|} \tag{5.5}$$

where f_i, $i = 1, \ldots, 4$, stand for the faces of Δ and l_{ij}, $j = 1, 2, 3$, stand for the edges of f_i. We have:

$$\begin{aligned} \max_{x \in \Delta} |u(x) - P_\Delta u(x)| &\geq \max_{i=1,\ldots,4} \max_{x \in f_i} |u(x) - P_\Delta u(x)| \geq C \max_{i=1,\ldots,4} \max_{j=1,2,3} |l_{ij}|^2_{|H|} \\ &\geq C|\Delta|^{\frac{2}{3}}_{|H|} = C|\tilde{\Delta}|^{\frac{2}{3}} . \end{aligned}$$

The left inequality of (5.1) is proven. □

Lemma 5.2. *Let G^1, G^2 be two constant metric defined on a tetrahedron Δ and for a small enough $\varepsilon > 0$*

$$|G^1_{ps} - G^2_{ps}| \leq \varepsilon, \quad p, s = 1, 2, 3 .$$

If

$$Q_{G^1,h^\star}(\Delta) \geq Q_0$$

then

$$Q_{G^2,h^\star}(\Delta) \geq Q_0 \cdot (1 - C\varepsilon/\lambda_1(G^1))^5$$

where $\lambda_1(G^1)$ is the minimal eigenvalue of the matrix G^1.

Proof. By the definition,

$$Q_{G^2,h^\star}(\Delta) = 6^4\sqrt{2}\frac{|\Delta|_{G^2}}{|\partial\partial\Delta|^3_{G^2}} F\left(\frac{|\partial\partial\Delta|_{G^2}}{6h^\star}\right), \quad F(x) = \left(\min\left\{x, \frac{1}{x}\right\}\left(2 - \min\left\{x, \frac{1}{x}\right\}\right)\right)^3 .$$

We denote by $\lambda_k(G^j)$, $k = 1,2,3$, the eigenvalues of the matrix G^j, $j = 1,2$, and by $\boldsymbol{l}_i$, $i = 1,\dots,6$, the oriented edges of Δ. Then

$$|\Delta|_{G^2} = |\Delta|_{G^1} \frac{\det G^2}{\det G^1} = |\Delta|_{G^1} \frac{\prod\limits_{k=1}^{3} \lambda_k(G^2)}{\prod\limits_{k=1}^{3} \lambda_k(G^1)} \geq |\Delta|_{G^1} \frac{\prod\limits_{k=1}^{3} (\lambda_k(G^1) - C\varepsilon)}{\prod\limits_{k=1}^{3} \lambda_k(G^1)} \geq |\Delta|_{G^1} \left(1 - \frac{C\varepsilon}{\lambda_1(G^1)}\right)^3 . \tag{5.6}$$

$$(G^2 \boldsymbol{l}_i, \boldsymbol{l}_i) \leq (G^1 \boldsymbol{l}_i, \boldsymbol{l}_i) + |((G^2 - G^1)\boldsymbol{l}_i, \boldsymbol{l}_i)|$$
$$|((G^2 - G^1)\boldsymbol{l}_i, \boldsymbol{l}_i)| \leq 3\,\varepsilon(\boldsymbol{l}_i, \boldsymbol{l}_i) .$$

Since

$$\frac{|\partial\partial\Delta|^2_{G^1}}{|\partial\partial\Delta|^2_{G^2}} = \left(\frac{\sum\limits_{i=1}^{6} (G^1 \boldsymbol{l}_i, \boldsymbol{l}_i)^{1/2}}{\sum\limits_{i=1}^{6} (G^2 \boldsymbol{l}_i, \boldsymbol{l}_i)^{1/2}} \right)^2 \geq \min_{i=1,\dots,6} \frac{(G^1 \boldsymbol{l}_i, \boldsymbol{l}_i)}{(G^2 \boldsymbol{l}_i, \boldsymbol{l}_i)} \geq \frac{1}{1 + 3\varepsilon/\lambda_1(G^1)}$$

$$F\left(\frac{|\partial\partial\Delta|_{G^2}}{6h^\star}\right) / F\left(\frac{|\partial\partial\Delta|_{G^1}}{6h^\star}\right) \geq 1 - C\varepsilon/\lambda_1(G^1)$$

we have

$$Q_{G^2,h^\star}(\Delta) \geq 6^4\sqrt{2} \frac{|\Delta|_{G^1}}{|\partial\partial\Delta|^3_{G^1}} F\left(\frac{|\partial\partial\Delta|_{G^1}}{6h^\star}\right) \frac{(1 - C\varepsilon/\lambda_1(G^1))^4}{1 + 3\varepsilon/\lambda_1(G^1)} \geq Q_0 (1 - C\varepsilon/\lambda_1(G^1))^5 .$$

□

Proof of Theorem 3.1. To prove (3.2), we shall show the following estimates:

$$\|u - P_{\Omega^h} u\|_{L_\infty(\Omega)} \leq A(Q_0, q) \left(\frac{|\Omega|_{|H|}}{N_T}\right)^{\frac{2}{3}} \tag{5.7}$$

$$\|u - P_{\hat{\Omega}^h} u\|_{L_\infty(\Omega)} \geq B(q) \left(\frac{|\Omega|_{|H|}}{N_T}\right)^{\frac{2}{3}} \tag{5.8}$$

where constant $A(Q_0, q)$ depends only on Q_0 and q, while $B(q) > 0$ depends on q, and

$$|\Omega|_{|H|} = \int_\Omega (\det |H|)^{1/2} \, \mathrm{d}x .$$

Let us assume that for any $u \in C^2(\bar{\Delta})$ and any tetrahedron Δ satisfying (3.1) there exists quadratic on Δ function

$$u_2(x) = \frac{1}{2} (H_2 \boldsymbol{x}, \boldsymbol{x}) - (\boldsymbol{b}, \boldsymbol{x}) + d + P_\Delta u(x) \tag{5.9}$$

with a Hessian H_2 such that

$$P_\Delta u = P_\Delta u_2 \tag{5.10}$$

$$C_1 \|u_2 - P_\Delta u_2\|_{L_\infty(\Delta)} \leq \|u - P_\Delta u\|_{L_\infty(\Delta)} \leq C_2 \|u_2 - P_\Delta u_2\|_{L_\infty(\Delta)} \tag{5.11}$$

$$|H_{2,ps} - H_{\Delta,ps}| \leq \delta, \quad \delta = q|\lambda_1(H_\Delta)|/2 . \tag{5.12}$$

The proof of the assumption is postponed to the end of the theorem proof.

We proceed to verification of (5.7). Let Δ be a tetrahedron from the quasi-optimal triangulation Ω^h, where $||u - P_{\Omega^h} u||_{L_\infty(\Omega)}$ is attained. Due to (3.1) there exists $u_2 \in P_2(\Delta)$ such that it satisfies Lemma 5.1 and (5.10)-(5.12) are valid.

Let us estimate the radius $\tilde{r}$ of the circumsphere defined in Lemma 5.1. Let l_j be the edge lengths for $\tilde{\Delta}$ which is the image of Δ under the transformation $\tilde{x} = R(x)$ reducing H_2 to the canonical form. The coordinates $\tilde{x}_i$ of the circumsphere center $\tilde{a}_c$ satisfy the following equations [7]:

$$12|\tilde{\Delta}|\tilde{x}_i(\tilde{a}_c) + (-1)^i \begin{vmatrix} 1 & 1 & 1 & 1 \\ \tilde{x}_{k_1(i)}(a_1) & \tilde{x}_{k_1(i)}(a_2) & \tilde{x}_{k_1(i)}(a_3) & \tilde{x}_{k_1(i)}(a_4) \\ \tilde{x}_{k_2(i)}(a_1) & \tilde{x}_{k_2(i)}(a_2) & \tilde{x}_{k_2(i)}(a_3) & \tilde{x}_{k_2(i)}(a_4) \\ w^2(a_1) & w^2(a_2) & w^2(a_3) & w^2(a_4) \end{vmatrix} = 0$$

where $\tilde{x}_i(a_j)$ is the ith coordinate of the vertex $R(a_j)$, $w^2(a_j) = \sum\limits_{i=1}^{3} \tilde{x}_i^2(a_j)$, $k_1(1) = 2$, $k_1(2) = 1$, $k_1(3) = 1$, $k_2(1) = 3$, $k_2(2) = 3$, $k_2(3) = 2$. Hence, the following estimate is straightforward:

$$\tilde{r} \le C\frac{\left(\sum\limits_{j=1}^{6} l_j\right)^4}{|\tilde{\Delta}|} \le C\frac{|\partial\partial\tilde{\Delta}|^4}{|\tilde{\Delta}|} = C\frac{|\partial\partial\Delta|^4_{|H_2|}}{|\Delta|_{|H_2|}} = Ch^\star\frac{|\partial\partial\Delta|_{|H_2|}}{6h^\star}\frac{|\partial\partial\Delta|^3_{|H_2|}}{6^4\sqrt{2}|\Delta|_{|H_2|}}\,. \tag{5.13}$$

By virtue of Lemma 5.2, (5.12) and (3.1):

$$Q_{|H_2|,h^\star}(\Delta) \ge Q_0 \cdot C(q)\,.$$

From Definition 1.2 we get

$$6^4\sqrt{2}\frac{|\Delta|_{|H_2|}}{|\partial\partial\Delta|^3_{|H_2|}} \ge Q_0C(q), \quad F\left(\frac{|\partial\partial\Delta|_{|H_2|}}{6h^\star}\right) \ge Q_0C(q), \quad \frac{|\partial\partial\Delta|_{|H_2|}}{6h^\star} \le \tilde{C}(Q_0,q) \tag{5.14}$$

and hence due to (5.13) there exists a constant $\tilde{A}(Q_0,q) > 0$ such that

$$\tilde{r}^3 \le \tilde{A}(Q_0,q)(h^\star)^3 = \frac{12\tilde{A}(Q_0,q)}{\sqrt{2}}\frac{|\Omega|_{|H|}}{N_T}\,. \tag{5.15}$$

Thus,

$$||u - P_{\Omega^h}u||_{L_\infty(\Omega)} = ||u - P_\Delta u||_{L_\infty(\Delta)} \le C_2||u_2 - P_\Delta u_2||_{L_\infty(\Delta)} \le A(Q_0,q)\left(\frac{|\Omega|_{|H|}}{N_T}\right)^{\frac{2}{3}}$$

that completes the proof of (5.7).

Now we shall show (5.8). Let $\hat{\Delta}$ be an arbitrary tetrahedron from the optimal triangulation $\hat{\Omega}^h$. By virtue of (3.1) there exists $u_2 \in P_2(\hat{\Delta})$ satisfying (5.10)-(5.12) and Lemma 5.1. Hence,

$$\max_{x\in\hat{\Delta}} |u(x) - P_{\hat{\Delta}}u(x)| \ge C_1 \max_{x\in\hat{\Delta}} |u_2(x) - P_{\hat{\Delta}}u_2(x)| \ge C|\hat{\Delta}|^{\frac{2}{3}}_{|H_2|}\,.$$

Due to (5.12),(3.1),(5.6) there exists a constant $\tilde{B}(q)$ such that

$$|\hat{\Delta}|_{|H_2|} \ge \tilde{B}(q)|\hat{\Delta}|_{|H_\Delta|} \ge \tilde{B}(q)|\hat{\Delta}|_{|H|}.$$

The following estimate holds for the element with the maximal volume $|\hat{\Delta}|_{|H|}$:

$$\max_{x\in\hat{\Delta}} |u(x) - P_{\hat{\Delta}}u(x)| \geq C\left[\tilde{B}(q)\right]^{\frac{2}{3}} |\hat{\Delta}|_{|H|}^{\frac{2}{3}}.$$

Therefore,

$$\|u - P_{\hat{\Omega}^h}u\|_{L_\infty(\Omega)} \geq C\left[\tilde{B}(q)\right]^{\frac{2}{3}} \max_{\hat{\Delta}\subset\hat{\Omega}^h} |\hat{\Delta}|_{|H|}^{\frac{2}{3}} \geq B(q)\left(\frac{|\Omega|_{|H|}}{N_T}\right)^{\frac{2}{3}}$$

that proves (5.8).

Now we return to the assumption on the existence of the quadratic function satisfying (5.10)-(5.12). Let

$$u_2(x) = \frac{1}{2}\left(H_\Delta \boldsymbol{x}, \boldsymbol{x}\right) - (\boldsymbol{b}, \boldsymbol{x}) + d + P_\Delta u(x) \tag{5.16}$$

where H_Δ is defined in the statement of the theorem. Since (5.10) is equivalent to

$$u_2(a_i) = u(a_i),\ i = 1, 2, 3, 4 \tag{5.17}$$

equations (5.16)-(5.17) constitute a linear nonsingular system for unknown values of $\boldsymbol{b}$ and d. Thus, we constructed the quadratic function which satisfies (5.10),(5.12). In order to verify (5.11) we take advantage of the multipoint Taylor formula [13, 14]:

$$E(x) := u(x) - P_\Delta u(x) = -\frac{1}{2}\sum_{i=1}^{4}(H(\tilde{x}_i)(x - a_i), (x - a_i))p_i(x)$$

$$E_2(x) := u_2(x) - P_\Delta u_2(x) = -\frac{1}{2}\sum_{i=1}^{4}(H_\Delta(x - a_i), (x - a_i))p_i(x)$$

$$E_{2,\text{mod}}(x) := -\frac{1}{2}\sum_{i=1}^{4}(|H_\Delta|(x - a_i), (x - a_i))p_i(x)$$

where $\tilde{x}_i = \tilde{x}_i(x, a_i) \in \Delta$. Due to (3.1)

$$|E(x) - E_2(x)| \leq \frac{q\sqrt{3}}{2}|\lambda_1(H_\Delta)|\sum_{i=1}^{4}((x - a_i), (x - a_i))p_i(x)$$

$$\leq \sqrt{3}\,q\frac{1}{2}\sum_{i=1}^{4}(|H_\Delta|(x - a_i), (x - a_i))p_i(x) = \sqrt{3}\,q|E_{2,\text{mod}}(x)| \leq \sqrt{3}\,q\|E_{2,\text{mod}}\|_{L_\infty(\Delta)}.$$

Transformations of Δ reducing H_Δ and $|H_\Delta|$ to canonical forms, are identical. Taking into account (5.5) and the geometry of interpolation errors [2], we have

$$\|E_{2,\text{mod}}\|_{L_\infty(\Delta)} \leq C_0\|E_2\|_{L_\infty(\Delta)}$$

where $C_0 > 1$ is certain constant independent of Δ. Hence,

$$|E(x) - E_2(x)| \leq C_0\sqrt{3}q\|E_2\|_{L_\infty(\Delta)}$$

$$\|E\|_{L_\infty(\Delta)} \leq \left(1 + C_0\sqrt{3}q\right)\|E_2\|_{L_\infty(\Delta)}$$

and for $C_0\sqrt{3}q < 1$

$$\left(1 - C_0\sqrt{3}q\right)\|E_2\|_{L_\infty(\Delta)} \leq \|E\|_{L_\infty(\Delta)}. \qquad \square$$

Proof of Theorem 3.2. We consider an arbitrary tetrahedron $\Delta \subset \sigma$. Since $H_{k,ps} \in W(\Omega^h)$, then $H_{k,\Delta} = H_k(\arg \max_{x\in\Delta} |\det H_k(x)|)$ is attained in one of the vertices of Δ, where (3.4) is fulfilled. Therefore, for any element Δ there exists a superelement σ containing Δ such that

$$|H_{\sigma,ps} - H_{k,\Delta,ps}| \le \varepsilon\,.$$

Let denote $H_\Delta := H(\arg \max_{x\in\Delta} |\det H(x)|)$. Then (3.3) yields

$$|H_{\Delta,ps} - H_{k,\Delta,ps}| \le \delta + \varepsilon\,. \tag{5.18}$$

Lemma 5.2, (5.18) and the assumption that δ, ε are sufficiently small give

$$Q_{|H|,N_T}(\Omega_k^h) \ge q_0 \min_{\Delta\in\Omega_k^h} (1 - C(\delta+\varepsilon)/\lambda_1(|H_{k,\Delta}|))^5 = Cq_0\,.$$

$\square$

Proof of Corollary 3.2. We shall show that for any superelement σ from Ω_k^h the following estimate is valid

$$|H_{\sigma,ps} - H_{k,ps}(a)| \le C(\delta+\varepsilon) \tag{5.19}$$

where the node a is a common point for all tetrahedra constituting σ.

Let $u_k := P_k^h u$. From (2.2) for any interior node a we get

$$\int\limits_\sigma (H_{ps} - H_{k,ps}(a)) v \,\mathrm{d}x = -\int\limits_\sigma \frac{\partial(u-u_k)}{\partial x_p}\frac{\partial v}{\partial x_s}\,\mathrm{d}x \quad \forall v \in W(\sigma),\ v=0 \ \text{ on } \ \partial\sigma. \tag{5.20}$$

Cauchy inequality and the triangle inequality yield:

$$\int\limits_\sigma |H_{\sigma,ps} - H_{k,ps}(a)||v|\,\mathrm{d}x \le \left\|\frac{\partial(u-u_k)}{\partial x_p}\right\|_{L_2(\sigma)} \left\|\frac{\partial v}{\partial x_s}\right\|_{L_2(\sigma)} + \int\limits_\sigma |H_{\sigma,ps} - H_{ps}||v|\,\mathrm{d}x\,. \tag{5.21}$$

Let v be such that $v(a) = \dfrac{\rho}{\sqrt{|\sigma|}}$. We evaluate all the terms in (5.21):

$$\int\limits_\sigma |H_{\sigma,ps} - H_{k,ps}(a)|v\,\mathrm{d}x \ge C_1|H_{\sigma,ps} - H_{k,ps}(a)|\rho\sqrt{|\sigma|}$$

$$\left\|\frac{\partial v}{\partial x_s}\right\|_{L_2(\sigma)} \le C_2$$

$$\int\limits_\sigma |H_{\sigma,ps} - H_{ps}|v\,\mathrm{d}x \le C_3\delta\rho\sqrt{|\sigma|}\,.$$

Hence, for any interior node a

$$|H_{\sigma,ps} - H_{k,ps}(a)| \le \frac{C_2}{C_1\rho\sqrt{|\sigma|}}\varepsilon\rho\sqrt{|\sigma|} + \frac{C_3}{C_1}\delta \le \frac{C_2}{C_1}\varepsilon + \frac{C_3}{C_1}\delta \le C(\varepsilon+\delta)\,. \tag{5.22}$$

For the boundary node the inequality (5.19) stems from (5.22), (3.6) and (2.3). $\square$

Proof of the Theorem 3.3. When proving Theorem 3.1 we showed that given a tetrahedron Δ and a function $u \in C^2(\bar\Delta)$ the quadratic function u_2 exists such that

$$\left\{\begin{array}{c} P_\Delta u_2 = P_\Delta u \\ \|u_2 - P_\Delta u_2\|_{L_\infty(\Delta)} \simeq \|u - P_\Delta u\|_{L_\infty(\Delta)} \\ |H_{2,ps} - H_{\Delta,ps}| \le \delta\,. \end{array}\right. \tag{5.23}$$

Moreover, the following estimate holds true:

$$\|\nabla(u - P_\Delta u)\|_{L_2(\Delta)} \leq C(\|\nabla(u_2 - P_\Delta u_2)\|_{L_2(\Delta)} + \delta)\,. \tag{5.24}$$

The inequality (5.24) implies that to evaluate the gradient error for u it suffices to estimate the gradient error for u_2. To prove (5.24) we note that the Hessian $H(x)$ of $u - P_\Delta u$ is nonsingular, hence the function $(\nabla(u - P_\Delta u))^2$ attains its maximum only at one of the vertices of Δ a_i. Thus,

$$\|\nabla(u - P_\Delta u)\|_{L_2(\Delta)} \leq \|\nabla(u - P_\Delta u)(a_i)\|_{L_2(\Delta)}\,.$$

For any point $a \in \Delta$ Taylor formula yields:

$$u(a) = u(a_i) + \nabla u(a_i)(a - a_i) + \frac{1}{2}(H(\tilde{a}_i)(a - a_i), (a - a_i))$$

$$u_2(a) = u_2(a_i) + \nabla u_2(a_i)(a - a_i) + \frac{1}{2}(H_\Delta(a - a_i), (a - a_i))$$

$$P_\Delta u(a) = P_\Delta u(a_i) + \nabla P_\Delta u(a_i)(a - a_i)$$

where $\tilde{a}_i \in \Delta$ depends on a, a_i. We take $a = a_j$, $j \neq i$, then

$$(\nabla(u - P_\Delta u)(a_i) - \nabla(u_2 - P_\Delta u_2)(a_i), a_j - a_i) = -\frac{1}{2}\left((H(\tilde{a}_i) - H_\Delta)(a_j - a_i), a_j - a_i\right)$$

and from (3.9) we get

$$\|\nabla(u - P_\Delta u)(a_i) - \nabla(u_2 - P_\Delta u_2)(a_i)\|_2 \leq C\delta\,.$$

Since $\nabla(u_2 - P_\Delta u_2)$ is a linear vector-function,

$$\|\nabla(u_2 - P_\Delta u_2)(a_i)\|_{L_2(\Delta)} \leq C\|\nabla(u_2 - P_\Delta u_2)\|_{L_2(\Delta)}$$

which implies (5.24).

Let $\tilde{\Delta}$ be the image of Δ under the transformation specified in Lemma 5.1, $0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3$ be the eigenvalues of the matrix $|H_2|$,

$$\tilde{R} = \begin{pmatrix} \sqrt{\bar{\lambda}_1} & 0 & 0 \\ 0 & \sqrt{\bar{\lambda}_2} & 0 \\ 0 & 0 & \sqrt{\bar{\lambda}_3} \end{pmatrix}$$

$\tilde{\tilde{\Delta}} = \tilde{R}(\tilde{\Delta})$. As it is shown in [3], $\|\nabla(u_2 - P_\Delta u_2)\|_{L_2(\Delta)}$ is the radius $\tilde{\tilde{r}}$ of the minimal sphere with the center $\tilde{a}_c$ and containing $\tilde{\tilde{\Delta}}$. Hence,

$$\|\nabla(u_2 - P_\Delta u_2)\|_{L_2(\Delta)} = \tilde{\tilde{r}} \leq \sqrt{\bar{\lambda}_3}\tilde{r} \tag{5.25}$$

where $\tilde{r}$ is defined in Lemma 5.1.

Let Q_0 be the quality of the mesh Ω^h_{k+1} in metric $|H|$. Applying the technique presented in the proof of Theorem 3.1 we have

$$\tilde{r} \leq C(Q_0, \delta)\sqrt{\min_{\Omega^h : \#Tr = N_T} \|u - P_{\Omega^h} u\|_{L_\infty(\Omega)}}\,. \tag{5.26}$$

By virtue of Corollary 3.3

$$Q_0 \geq C(\varepsilon, \delta) q_0 \, .$$

Therefore due to (5.24)-(5.26) we get

$$\begin{aligned}\|\nabla(u - P_{\Omega^h_{k+1}} u)\|_{L_2(\Delta)} &\leq C(q_0, \delta, \varepsilon) \left(\sqrt{\bar{\lambda}_3 \min_{\Omega^h : \#Tr = N_T} \|u - P_{\Omega^h} u\|_{L_\infty(\Omega)}} + \delta \right) \\ &\leq C(q_0, \delta, \varepsilon) \left(\sqrt{\lambda_{3,\Delta} \min_{\Omega^h : \#Tr = N_T} \|u - P_{\Omega^h} u\|_{L_\infty(\Omega)}} + \delta \right) \, .\end{aligned}$$

□

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