UIT2201 Programming and Data Structures
AVL Trees

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Order among the objects

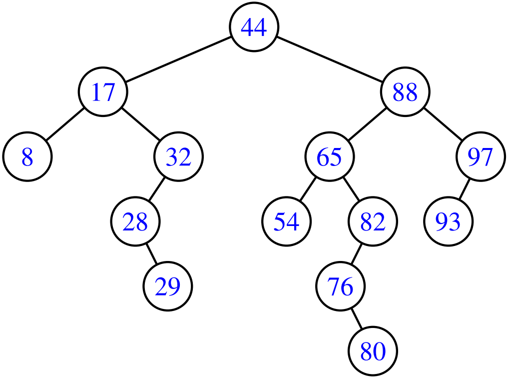
- Searching for an object in a tree collection is as complex as searching for an object in a linear collection
- There are several applications where we need operations such as
  - find()
  - findMin()
  - findMax()
  - deleteMin()
  - deleteMax()
  - insert()
  - delete()
- It is safe to assume that a partial order $<$ can be defined on set of objects and this order can be imposed on the tree as well!

Binary Search Tree

Binary Search Tree

Binary Search Tree (BST) is a binary tree that stores objects in its nodes in such a way that all the objects in the left subtree of any node with object x are $< x$ and all the objects in the right subtree are $x <$, for some partial order $<$ among the objects.

Binary Search Tree: Example



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- Is it possible to keep the height under control (close to $\log N$)?
- Of course, the amount of additional work we do to keep the height under control should preferably be $O(1)$ or should not exceed $O(\log N)$

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- Ideally we may expect both the left and right sub-trees to have the same height
- But, that may not be always possible!

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For every node in the tree, the heights of left and right sub-trees can differ by at most 1

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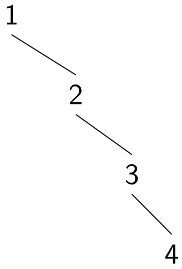
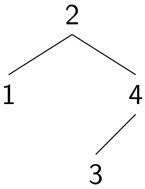
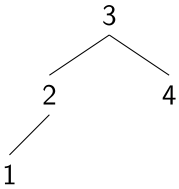
Note

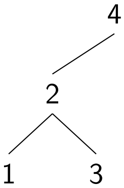
Note that the height of an empty tree is considered as -1

Which of these are AVL Trees?









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- This is closely related to Fibonacci numbers!

AVL Analysis

AVL Tree Property

Height of an AVL tree is approximately equal to

$$1.44 \log (N + 2) - 1.328$$

Preserving the structural condition

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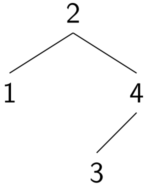
- AVL Tree approach: Insertion and deletion operations are carried out obeying first the ordering conditions — that is, perform BST insertion or deletion
- If the structural property is violated, perform certain operations (in constant time?) to restore the balance

Example

- Insert 5 into the following AVL Tree



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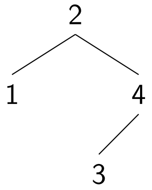

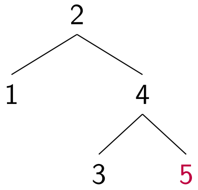

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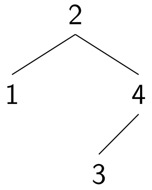
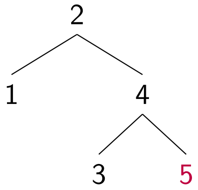
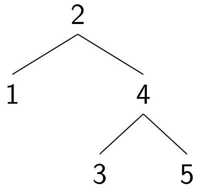




- Is the structural condition violated after the insertion?

Example

- Let us now continue to insert 6



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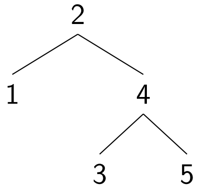

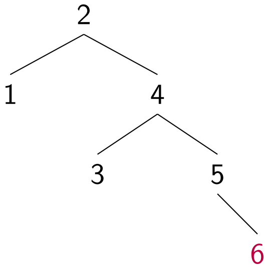

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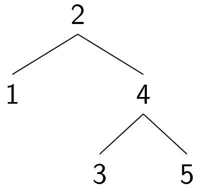

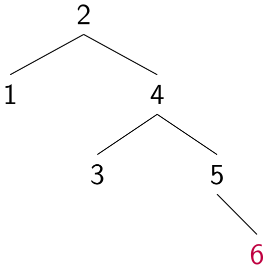

- Now, violation happens at the root node

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- There is no need to look up further in the path!

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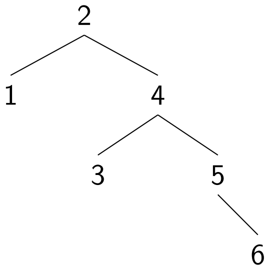

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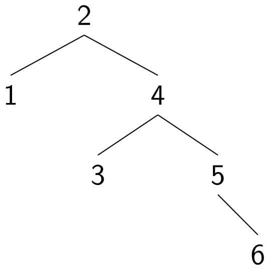

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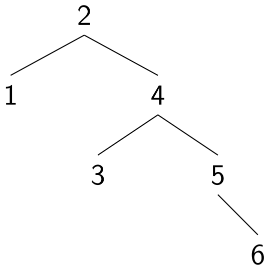

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- This can be corrected by performing a 'rotation' operation at α

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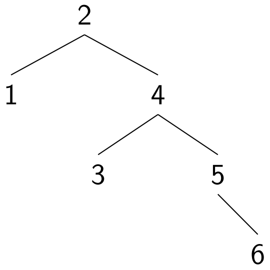
- Anti-clockwise rotation can be performed at α to restore the structural balance of the tree



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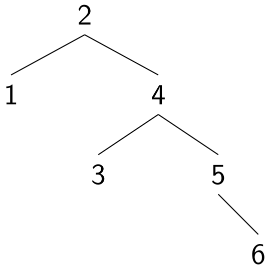

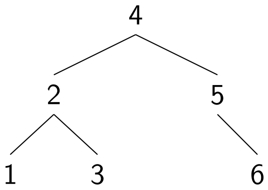

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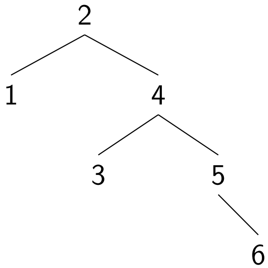
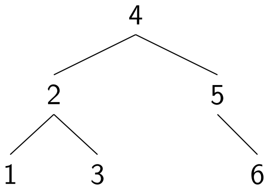
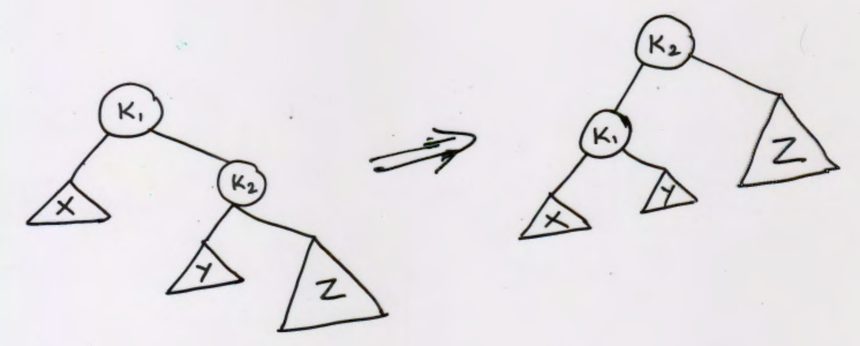




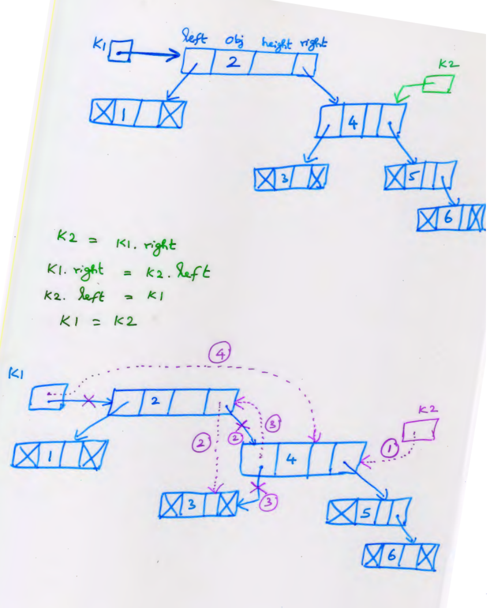
- Since only one path is involved, the complexity remains as $O(\log N)$

Rotation Operation

- As we have seen, Case 2 violation is corrected by performing an anti-clockwise rotation at α



Data Strucure view of rotation



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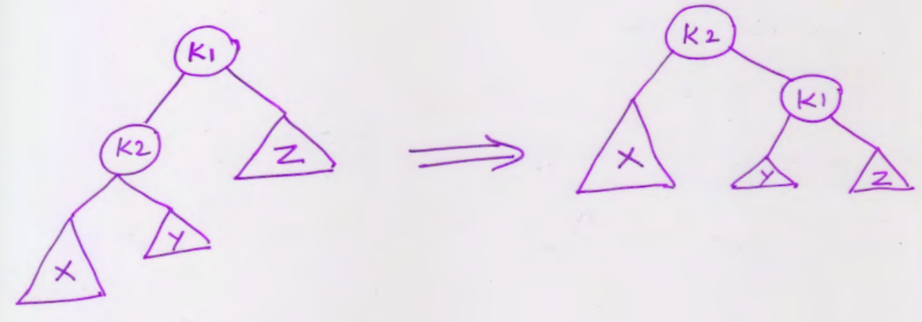

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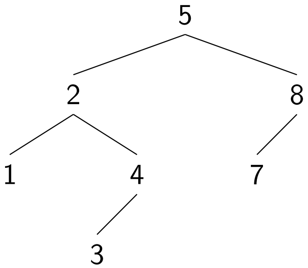

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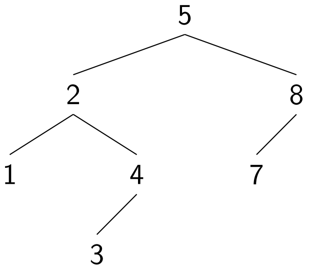

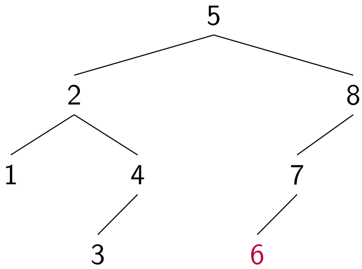

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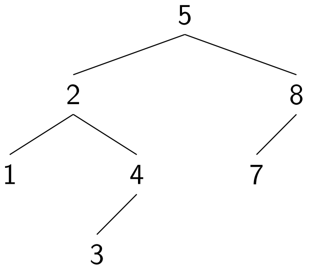
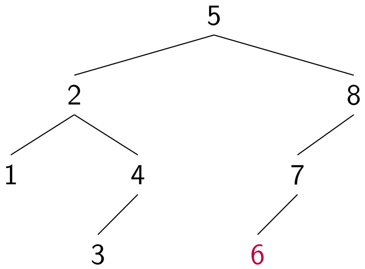
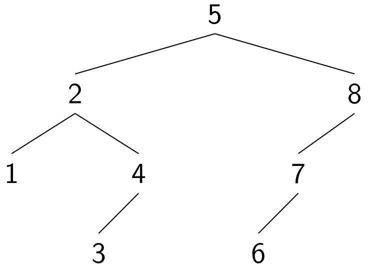




- Is there any violation after the insertion? If so, which node α is the first violating node?

Example

- Violation occurs at the node containing the key 8
- This is Case 1 violation and can be corrected by a clockwise rotation



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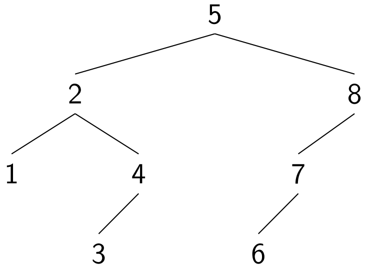

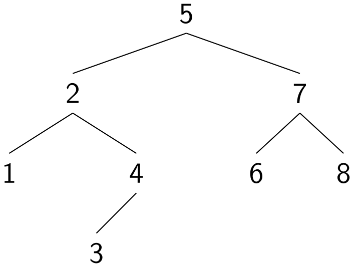

Inside Insertions

- Solution similar to Case 1 and Case 2 does not work for Case 3 and Case 4 inside insertions (left-right and right-left insertions)

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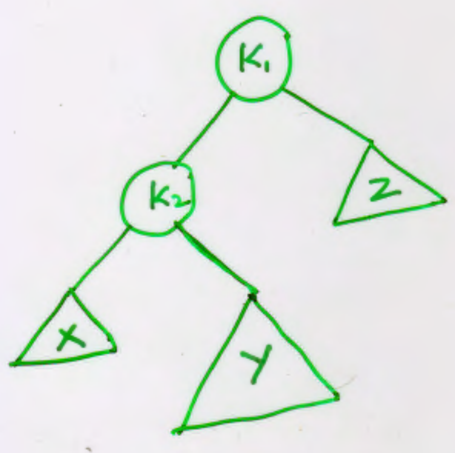

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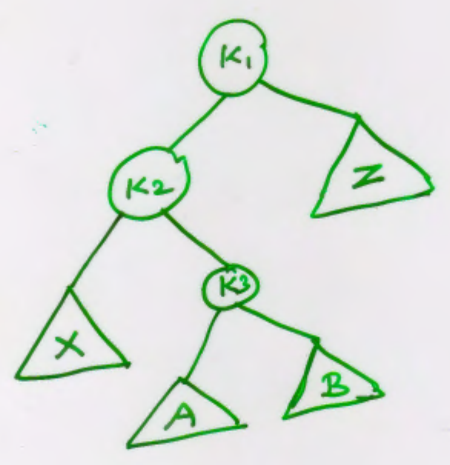

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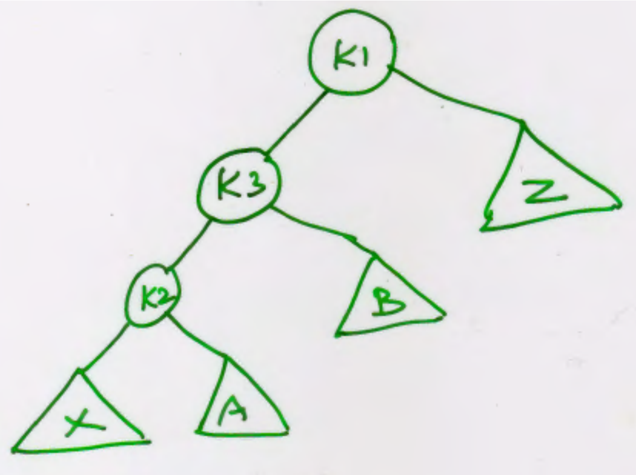

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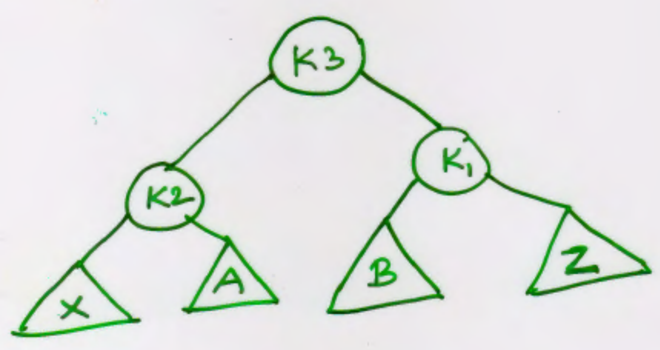

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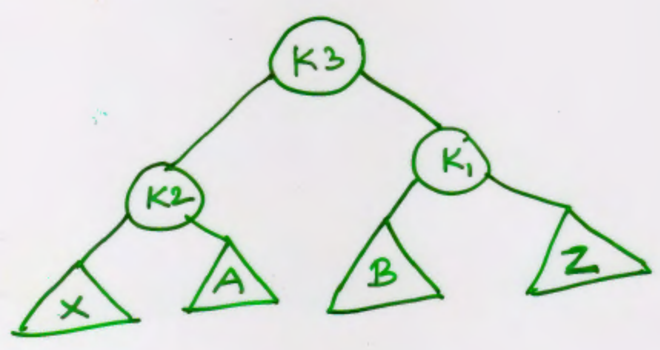

- Thus, a "double rotation" operation restores balance in Case 3 violation

Case 4 Violation

- A similar "first clockwise then anti-clockwise" "double rotation" can be performed to restore the balance in Case 4 violation

Exercise

- Insert the following keys in the given order into an initially empty AVL tree: $2, 1, 4, 5, 9, 3, 6, 7, 8$