UIT2201 Programming and Data Structures

Introduction to Recursion

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Introduction

- We saw that divide and conquer strategy naturally results in a recursive implementation
- Recursive algorithms can be easily analyzed using recurrence equations
- Properties of recursive structures can be proved using structural induction
- In this lecture, we will discuss about recursive algorithms and their implementations

- Consider a classic problem to compute the factorial of a number n

Iteration Vs Recursion

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- $n! = 1 \times 2 \times \cdots \times (n-1) \times n$

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def fact(n):
    ans = 1
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    return ans
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```

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- $c(n-1) + d$

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def fact_rec(n):
    if (n < 2):
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    else:
        return n * fact_rec(n-1)
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$$T(1) = d$$

$$T(n) = T(n-1) + c$$

What happens in recursion?

fact_rec(4)

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$$= \quad 4 * \text{fact\_rec}(3)$$

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\begin{aligned}
&= 4 * \text{fact\_rec}(3) \\
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\end{aligned}
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\begin{aligned}
&= 4 * \text{fact\_rec}(3)\\
&= 4 * (\ 3 * \text{fact\_rec}(2)\ )\\
&= 4 * (\ 3 * (\ 2 * \text{fact\_rec}(1)\ )
\end{aligned}
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[base case reached; recursive calls start returning]

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&= 4 * (\ 3 * (\ 2 * \text{fact\_rec}(1)\ )\ ) \\
&\quad \text{[base case reached; recursive calls start returning]} \\
&= 4 * (\ 3 * (\ 2 * 1\ )\ )
\end{aligned}
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&\ \text{[base case reached; recursive calls start returning]} \\
&= 4 * (\ 3 * (\ 2 * 1\ )\ ) \\
&= 4 * (\ 3 * 2\ ) \\
&= 4 * 6
\end{aligned}
$$

What happens in recursion?

fact_rec(4)

= 4 * fact_rec(3)
= 4 * (3 * fact_rec(2))
= 4 * (3 * (2 * fact_rec(1))
[base case reached; recursive calls start returning]
= 4 * (3 * (2 * 1))
= 4 * (3 * 2)
= 4 * 6
= 24

What happens in recursion?

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&\ \text{[base case reached; recursive calls start returning]} \\
&= 4 * (\ 3 * (\ 2 * 1\ )\ ) \\
&= 4 * (\ 3 * 2\ ) \\
&= 4 * 6 \\
&= 24
\end{aligned}
$$

- Function calls require context information to be stored in a stack
- Context gets popped out when the call returns
- Depth of recursion is limited by the size of the system stack!

Tail Recursion

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- Tail recursion is a special case of recursion, where a recursive call appears as the last individual statement and not in an expression
- A good compiler may optimize a tail recursive function by not creating a new stack context!
- Tail recursion can be achieved by using "accumulator" pattern!

```
def fact_tail(n):
    def fact_tail_rec(n, acc):
        if (n < 2):
            return acc
        else:
            return fact_tail_rec(n-1, n*acc)

    return fact_tail_rec(n, 1)
```

What happens in tail recursion?

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\begin{aligned}
&= \text{fact\_tail\_rec}(4,\ 1)\\
&= \text{fact\_tail\_rec}(3,\ 4*1)
\end{aligned}
$$

What happens in tail recursion?

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\begin{aligned}
&= \text{fact\_tail\_rec}(4,\ 1)\\
&= \text{fact\_tail\_rec}(3,\ 4*1)\\
&= \text{fact\_tail\_rec}(2,\ 3 * 4)
\end{aligned}
$$

What happens in tail recursion?

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\begin{aligned}
&= \text{fact\_tail\_rec}(4,\ 1)\\
&= \text{fact\_tail\_rec}(3,\ 4*1)\\
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&= \text{fact\_tail\_rec}(1,\ 2 * 12)
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[base case reached; recursive calls start returning]

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&= \text{fact\_tail\_rec}(4,\ 1)\\
&= \text{fact\_tail\_rec}(3,\ 4{*}1)\\
&= \text{fact\_tail\_rec}(2,\ 3 * 4)\\
&= \text{fact\_tail\_rec}(1,\ 2 * 12)\\
&\ \text{[base case reached; recursive calls start returning]}\\
&= 24
\end{aligned}
$$

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\end{aligned}
$$

- Since recursive call is the last independent statement, no more work is needed when the call returns
- So, a compiler may be able to optimize this and execute the function in the same stack frame (like an iteration)
- However, a compiler may not perform this optimization (which is the case with most of the non-functional languages) and depth of recursion may be limited

Another Example: Fibonacci Series

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

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$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

```
def fib(n):
    current = 1; old = 0
    if (n == 0):
        return old
    elif (n == 1):
        return current
    else:
        for i in range(2, n+1):
            current, old = current+old, current
        return current
```

Recursive version

```
def fib_rec(n):
    if (n == 0):
        return 0
    elif (n < 3):
        return 1
    else:
        return fib_rec(n-1) + fib_rec(n-2)
```

Recursive version

```
def fib_rec(n):
    if (n == 0):
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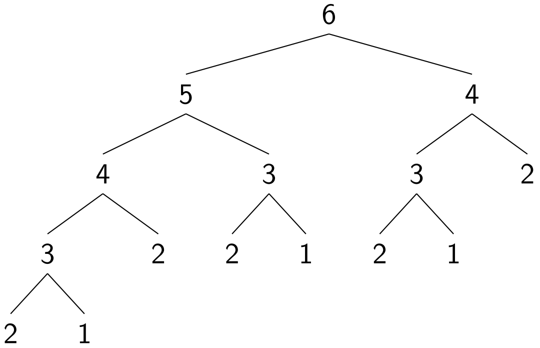
```

$$T(1) = T(2) = d$$

$$T(n) = T(n-1) + T(n-2) + c$$

Is recursion computationally expensive?



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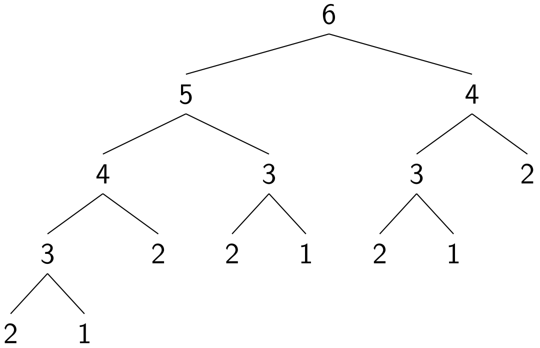

Is it possible to rewrite this function with tail recursion?

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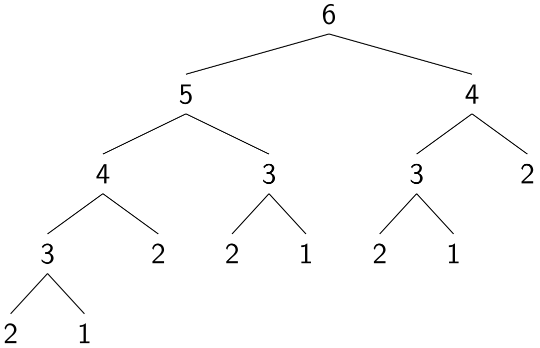

Is it possible to rewrite this function with tail recursion?

Hint: Use two accumulators!

Tail recursive version

```
def fib_tail(n):
    def fib_tail_rec(n, acc1, acc2):
        if (n < 3):
            return acc1
        else:
            return fib_tail_rec(n-1, acc1+acc2, acc1)

    if (n == 0):
        return 0
    elif (n < 3):
        return 1
    else:
        return fib_tail_rec(n, 1, 1)
```

Tail recursive version

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\begin{aligned}
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- Note that tail recursion has completely changed the complexity class!

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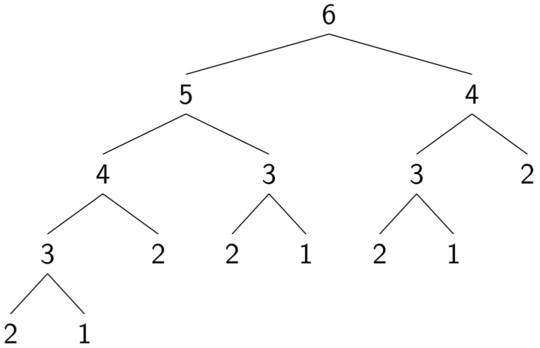
- Note that tail recursion has completely changed the complexity class!
- So, even if a compiler does not perform any stack optimization, tail recursion still helps!

Dynamic Programming

- Let us again look at what happens with standard recursion



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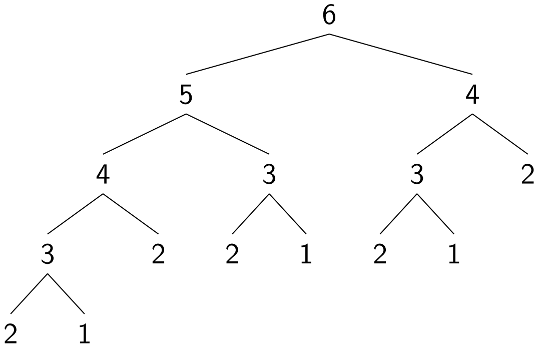

There is a lot of repetitive work!

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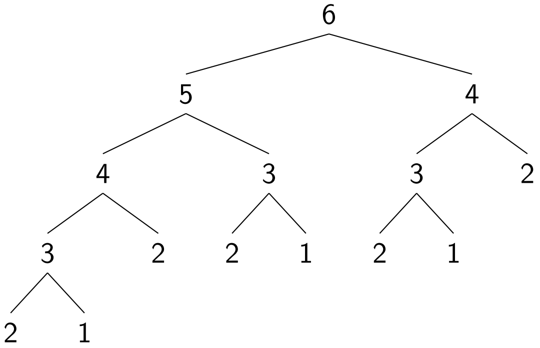

There is a lot of repetitive work!

Is it possible to store intermediate results to avoid repetitive computation?

Memoization

```python
known = {0:0, 1:1, 2:1}

def fib_dict(n):
    if n in known:
        return known[n]
    res = fib_dict(n-1) + fib_dict(n-2)
    known[n] = res
    return res
```

Memoization

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```

Point to note: Time Vs Space Trade-off

Memoization

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def fib_dict(n):
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    known[n] = res
    return res
```
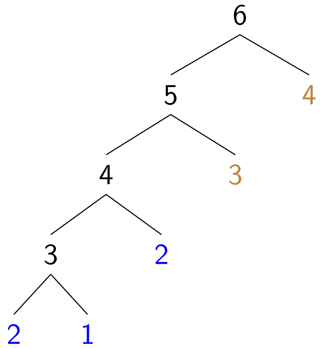

Point to note: Time Vs Space Trade-off
Another point to note: How much time is required to check for an entry in the look-up table?

Dynamic Programming

- What happens with dynamic programming?



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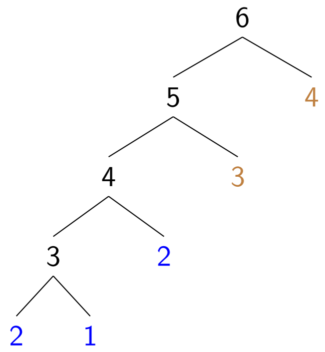

- Exponential time execution may be turned in to linear time!
- But space requirement grows with n (in this case it is linear growth)
- When we use dictionary like structure, it may be possible to dynamically manage which results need to be persisted with

Formulate as matrix multiplication problem

$$\begin{pmatrix} F_2 \\ F_1 \end{pmatrix} = M \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

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$$\vdots$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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$$\vdots$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Fibonacci through matrix multiplication

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import numpy as np

def fib_mat(n):
    if (n == 0):
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    elif (n < 3):
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    else:
        return ((np.matrix('1 1; 1 0', dtype='object')
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What is the complexity of computing $M^{(n-1)}$?

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- Look at the binary bit pattern of n!!!

Right to left evaluation

1	1	0	0	1	
M^{16}	M^8	M^4	M^2	M	term
$*$	$*$			$*$	result

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- Initialize term to M
- Initialize result to M if $b_0 = 1$, otherwise initialize to 1
- Iterate over the bits b_1 to b_k:
  - term = term * term
  - If $b_i = 1$, then result = result * term

Exercise

- It is also possible to do left to right evaluation of the bit pattern of n.
- Read about Horner's rule for evaluating a polynomial
- Read about how to use that idea for left to right evaluation of bit pattern to compute M^n
- Implement (in Python, without using Numpy or any such packages) both the left-to-right and right-to-left evaluations for computing M^n
- Re-implement the computation of n^{th} Fibonacci number using your exponentiation functions