Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 1: 5/25-5/31
Sections 11.1 – 11.4, Vectors in the Plane, Vectors in Space, Dot Product, Cross Product

Due this week: Thursday – WebAssign 11.1-11.3 Sunday – Weekly Assignment 1 11.1-11.4

Lecture Notes 11.1, 11.2: Vectors in the Plane and in Space

Everyone should have a fundamental understanding of vectors from either a physics class, general science, or college algebra or precalculus. As a refresher, a vector is a mathematical object with a magnitude (length) and direction most commonly represented by an arrow where the length of the arrow represents magnitude and the point of the arrow naturally shows direction.

Mathematically a vector is generally represented in either component form $\mathbf{a} = <2,5>$ or as a linear combination of standard basis vectors, $\mathbf{a} = 2\,\mathbf{i} + 5\,\mathbf{j}$ in the plane or $\mathbf{b} = <2,5,1>$ and $\mathbf{b} = 2\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k}$ in 3-space. In component form, $\mathbf{a} = <2,5>$, the first value represents the horizontal (*x*) displacement and the second value represents the vertical (*y*) displacement. In 3-space the third component or **k** term represents the 3-dimensional height or *z* component. Similarly in basis form the **i** represents a vector of magnitude one in the positive *x* direction and **j** represents a vector of magnitude one in the positive *y* direction and if necessary the **k** represents a vector of magnitude one in the positive *z* direction. It is important to always use one of these forms when representing a vector. Usually I prefer to use component form when the vector is constant and basis form if I am working a vector valued function, but there will be more on that later.

Since vectors are only magnitude and direction they can be moved anywhere in the plane as long as the direction and magnitude remain the same and they are considered to the be the same vector. All of the vectors below are considered to be the same because the have a common magnitude and direction.

Perhaps the best way to think of a vector in a Euclidean space (2 or 3 dimensions) is as a set of directions on how to get from one point to another. Thus our vector from above, $\mathbf{a} = <2,5>$, would indicate that from wherever we are we should move 2 units in the positive *x* direction and 5 units in the positive direction and 5 units in the positive *y* direction to find the head (arrow head) of the vector. The most common application of vectors in science and engineering is the representation of forces where the vector's magnitude represents the amount of force and the direction is, well, the direction of the force.

Now that we have a fundamental concept of a vector let's review or define some operations on the set of vectors. One additional comment at this point, the vector $\vec{0} = <0,0,0>$ is known as the zero vector. This is often not considered to be a vector at all since it has a magnitude of 0 and hence no direction. It would be like trying to define the length of a point. Sometimes we will make reference to the zero vector but will not consider it to be a true vector.

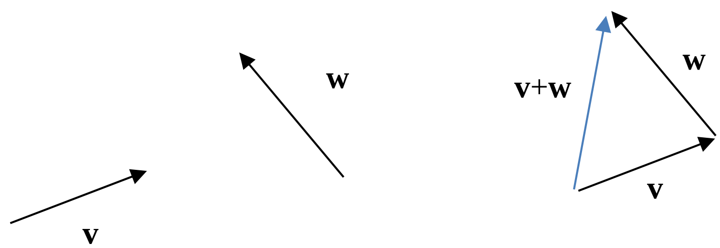
For the purpose of definition consider vectors $\mathbf{v} = <v_1, v_2, v_3>$ and $\mathbf{w} = <w_1, w_2, w_3>$

1. Creating a vector from one point to another. The vector from point $P = (p_1, p_2, p_3)$ to point $Q = (q_1, q_2, q_3)$ denoted as $\overrightarrow{PQ} = <q_1 - p_1, q_2 - p_2, q_3 - p_3>$. It is important to note the order of subtraction, since $\overrightarrow{PQ}$ is the vector from *P* to *Q* we need to subtract *P*'s coordinates from *Q*'s.

2. Magnitude of a vector is represented by a double absolute value symbol (some texts use just a single)

   Magnitude of vector **v** is $\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ and this formula can be extended to any number of dimensions.

3. Vector addition/subtraction $\mathbf{v} \pm \mathbf{w} = <v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3>$. This is sometimes thought of geometrically as placing the tail of **w** at the head of **v** then drawing the vector $\mathbf{v} + \mathbf{w}$ by connecting the tail of **w** to the head of **v**.



4. Scalar Products: Multiplying a vector by a constant multiplies the magnitude of the vector without changing its direction, it **scales** the length up or down.
   $a\mathbf{w} = <aw_1, aw_2, aw_3>$ and so
   $\| a\mathbf{w} \| = \sqrt{(aw_1)^2 + (aw_2)^2 + (aw_3)^2} = \sqrt{a^2 w_1^2 + a^2 w_2^2 + a^2 w_3^2} = a\sqrt{w_1^2 + w_2^2 + w_3^2} = a\| \mathbf{w} \|$.

5. Unit vector. A unit vector is a vector of magnitude one. Unit vectors are important to us when the direction of the vector is what we are interested in and we don't need to concern ourselves with direction. A unit vector can always be constructed by multiplying a vector by the reciprocal of the magnitude of the vector.

   Unit vector in the direction of **v** would be $\mathbf{u} = \dfrac{1}{\| \mathbf{v} \|}\mathbf{v} = \dfrac{\mathbf{v}}{\| \mathbf{v} \|}$

6. Parallel vectors are any pair of vectors such that $\mathbf{v} = c\mathbf{w}$. That is to say one vector is a scale multiple of the other. Vectors are only equal if each component is **exactly** the same.

11.3: Dot Product

We looked at scalar products in the previous notes now let us consider another type of multiplication known as **dot product**. The dot product between two vectors can be calculated in two different ways and it is this feature that we can take advantage of.

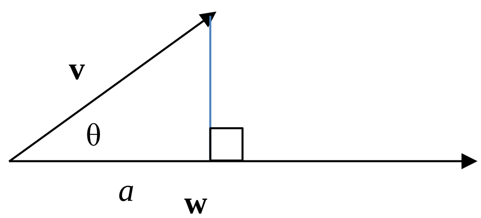
The dot product is denoted by a dot, surprise. **IMPORTANT** the results of a dot product is **NOT** a vector, it is a scalar (a real number).

$\mathbf{v} \bullet \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$ where θ represents the angle between the vectors, $0° \le \theta \le 180°$

Solving the dot product formula for θ allows us to determine the angle between the vectors, $\theta = \cos^{-1} \dfrac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$.

Since the zero vector is not truly a vector we know that both the magnitude of **v** and **w** cannot be zero so if $\mathbf{v} \bullet \mathbf{w} = 0$ and $\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$ then $\cos\theta = 0$ and so $\theta = 90°$. This indicates the vectors form a right angle so we can say they are perpendicular. In actuality the terms perpendicular, right angle and even the degree measurement of $\theta = 90°$ are only defined in a Euclidean space like the plane or our 3-dimensional world but since vectors can be of any dimension and the dot product operation is defined for any two vectors of the same dimension we need a more general term for such situations. In general if $\mathbf{v} \bullet \mathbf{w} = 0$ we say the vectors are orthogonal. Orthogonality has applications in most mathematics courses at or above calculus III including differential equations where we like to study sets of functions that have orthogonal trajectories, but that is for another time and course. For now, be aware that when we mention orthogonal vectors we are talking about perpendicular vectors in a more general fashion.

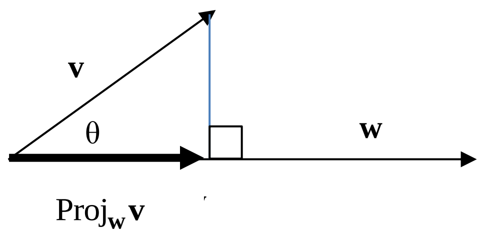
Consider the diagram below, vectors **v** and **w** could have any orientation but are presented like this for simplicity and ease of understanding. Clearly θ is the angle between the vectors **v** and **w**. We will use this diagram and the included right triangle to develop the idea of the projection of one vector onto another. In this case we are looking at the projection of **v** onto **w**.



The projection of **v** onto **w** is a vector parallel to **w** of magnitude equal to length of the horizontal side in the right triangle above, *a*. From trigonometry we know that $\cos\theta = \dfrac{a}{\|\mathbf{v}\|}$ and from dot product we know

$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$ so $\cos\theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} = \dfrac{a}{\|\mathbf{v}\|}$ now solving for *a* we get $a = \dfrac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|}$. We call this the magnitude of the projection of **v** onto **w.** Now to construct the vector parallel to **w** of magnitude *a* all we need to do is scale **w** to unit size (magnitude one) then multiply that vector by *a*.

$$\text{Proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|} \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|^2} \mathbf{w}$$



It is important to note that the projection of **v** onto **w** is a vector and the magnitude of the projection of **v** onto **w** is a scalar, $a = \dfrac{\mathbf{v} \bullet \mathbf{w}}{\| \mathbf{w} \|}$. We will be using projections to determine distances between objects in 2 and 3 dimensions.

11.4: Cross Product

Finally we are going to examine yet another method of multiplying vectors known as the cross product and believe it or not denoted using a cross. It is important to note that the result of a cross product is a vector which is orthogonal to both **v** and **w**. Computationally

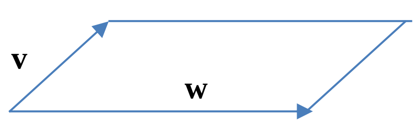
$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - w_2 v_3)\mathbf{i} - (v_1 w_3 - w_1 v_3)\mathbf{j} + (v_1 w_2 - w_1 v_2)\mathbf{k}$$

For those familiar with linear algebra this process is known as cofactor expansion by minors for finding the determinant of a 3 by 3 matrix. For those not familiar with linear algebra this is process you will need to become familiar and comfortable with. In words we set up a 3 by 3 matrix using the basis vectors **i**, **j**, and **k**¸ as the top row, the components of **v** as the middle row and components of **w** for the bottom row. Then to find the **i** coefficient ignore the entries under **i** and multiply the down diagonal by the up diagonal (from left to right). Repeat this process for **j** and **k** just note that outside the parentheses of the **j** term there is a negative. This is not a typo. The cross product is not defined for vectors in 2 dimensions although we can assign a 0 value to the **k** component if we really wanted to find the cross product of a pair of 2 dimensional vectors.

Properties of the cross product:

a. The result is a vector that is mutually orthogonal to the vectors of the product.
$(\mathbf{v} \times \mathbf{w}) \bullet \mathbf{u} = (\mathbf{v} \times \mathbf{w}) \bullet \mathbf{v} = \mathbf{0}$

b. The magnitude of the cross product is equal to the area of the parallelogram with non-parallel sides equal to the vectors of the product.
$\|\mathbf{v} \times \mathbf{w}\| =$ area of



c. $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$ and so $\|\mathbf{v} \times \mathbf{w}\| = 0$ iff **v** and **w** are parallel. This is not the best way to test for parallel vectors!