8Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 9: 7/21-7/26
Sections 14.5 – 14.8, Surface Area, Triple Integrals, Cylindrical and Spherical Integrals, Jacobians.

Due this week:

Tuesday 7/21/2020	**Thursday 7/23/2020** WebAssign 14.7 – 15.2	**Sunday 7/19/2020** Weekly Assignment 9, 15.1 – 15.4

Lecture Notes 15.1: Vector Fields

I hope everyone managed to push through triple integrals in cylindrical and spherical coordinates. That is, in my opinion, the most difficult part of calculus III. And now we begin what I generally consider to be one of the easier parts of the course. That being said, this is not the time to start coasting.

Section 15.1: Vector Fields
Continuing our evolution of the concept of function we look at a set of functions whose domain is the *xy* plane (or *xyz* space) and whose range is the set of vectors in that space.

$$\boldsymbol{F}(x,y) = M(x,y)\,\boldsymbol{i} + N(x,y)\boldsymbol{j} \quad \text{or} \quad \boldsymbol{F}(x,y,z) = M(x,y,z)\,\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$$

Functions of this form which associate a vector with each point in their domain generate what is known as a Vector Field. These are vectors so when you write them you must use vector notation, an arrow over the function letter and arrows or carrots over the *i, j,* k. For example we can plot vectors for the rather simple field $\boldsymbol{F}(x,y) = 2xy\,\boldsymbol{i} + x^2\,\boldsymbol{j}$ Lets begin with a table

(*x*,*y*)	$\boldsymbol{F}(x,y) = 2xy\,\boldsymbol{i} + x^2\,\boldsymbol{j}$
(0,0)	$< 0,0 >$
(1,0)	$< 0,1 >$
(0,1)	$< 0,0 >$
(1,1)	$< 2,1 >$
(1,2)	$< 4,1 >$
(2,1)	$< 4,4 >$
(2,2)	$< 8,4 >$

With a lot of patience we can imagine plotting each of these vectors and more on the *xy* plane. For example the vector $\boldsymbol{F}(1,2) = < 4,1 >$ would be plotted with the tail at (1,2) and the head (arrow) at (1+4, 2+1) = (5,3). What a field of vectors like this represents are the 2 dimensional forces acting on a particle as the particle moves through the plane. Take a look at some of the diagrams in section 15.1 of your book for better and more artistic renderings of vector fields.

We want to define a few metrics on these vector fields.

First we define a vector operator called del which is the same as our gradient operator.

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle \quad \text{or in 3 space} \quad \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

Define the divergence of vector field, *div*(***F***), to be $div(\mathbf{F}) = \nabla \bullet \mathbf{F} = M_x + N_y$ in 2 space and $div(\mathbf{F}) = \nabla \bullet \mathbf{F} = M_x + N_y + P_z$ for a 3 dimensional vector field. Notice that as a dot product the divergence is not a vector, it is a scalar which in a general translation measures the tendency of a field to try to separate particles as they move through the field. For example light, in a particle form point of view, has a high diffusivity (light spreads) so its vector field would have a positive divergence. This is just meant to be an example, not a physics lecture. And recall from above that we are defining a vector field in three dimensions as $\mathbf{F}(x,y,z) = M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j} + P(x,y,z)\,\mathbf{k}$ or in 2 dimensions as the same without the **k** term.

Now we define the **curl** of a vector field as Curl(***F***) to be the cross product of ∇ and the field, ***F***

$$curl(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x,y,z) & N(x,y,z) & P(x,y,z) \end{vmatrix} = \left(P_y - N_z\right)\mathbf{i} - \left(P_x - M_z\right)\mathbf{j} + \left(N_x - M_y\right)\mathbf{k}$$

Note: *curl*(**F**) is not defined for a 2-dimensional vector field.

Recall the method for finding cross products we employed when finding vectors perpendicular to a given pair of non-parallel vectors at the beginning of the semester. This is the same process except the middle row of the matrix is the ∇ operator vector and the bottom row is the vector field.

You may have noticed that a vector field $\mathbf{F}(x,y,z) = M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j} + P(x,y,z)\,\mathbf{k}$ or $\mathbf{F}(x,y,z) = M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j}$ looks just like a gradient vector in that it is a collection of traditional two dimensional functions serving as coefficients of basis vectors, **i**, **j**, and **k**. In fact a gradient vector is a vector field, we just didn't know to call it that at the time. Every gradient vector is a vector field but not every vector field is a gradient vector. A vector field which is also a gradient vector is said to be a **conservative** vector field and conservative fields have several nice features which we will be exploring over the next few weeks.

Recall how we built gradient vectors. Given a higher dimensional function, $z = f(x,y)$ or $w = f(x,y,z)$ we calculated the gradient vector $\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{j}$ using partial derivatives. If a given vector field $\mathbf{F}(x,y,z)$ is **Conservative** then $\mathbf{F}(x,y,z)$ is a gradient of some function $f(x,y,z)$ and we call $f(x,y,z)$ the **Potential Function** of the field. Yes this is related to potential energy etc. but again this is not a physics class, thank goodness. So how do we tell if a vector field is conservative or not? Any three dimensional field, $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$, is conservative (has a potential function) if the *curl*(**F**) is the zero vector. In the case of a 2-dimensional vector field like $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j}$ the field is conservative if $M_y = N_x$ that is the partial derivative of the **i** coefficient with respect to *y* is equal to the partial derivative of the **j** coefficient.

So now the question becomes, if we know that a vector field $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j}$ is conservative then we know that he field has a potential function $f(x,y)$ can we find the potential function? First remember what it means to be conservative. The gradient of *f* must be equal to **F**.

That is to say $\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j} = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$.

Clearly then $f_x(x,y) = M(x,y)$ and $f_y(x,y) = N(x,y)$. Then since *M* and *N* are equal to derivatives we must surely use integration to the potential function. So $M(x,y) = \int f_x(x,y)\, dx = g(x,y) + c$ but since the derivative was a partial derivative and we don't have such a thing as an anti-partial derivative we have only been able to recover the *x* pieces of the potential function. Integrating with respect to *x* treats all *y* terms as constants and so the +*c* at the end is really a +*c*(*y*) meaning there could well be some *y* terms in the constant. So we turn to

$N(x,y) = f_y(x,y)$ and we find the partial derivative of our new found $g(x,y)+c(y)$ with respect to y and set the result equal to $N(x,y)$. $\frac{\partial}{\partial y}\big(g(x,y)+c(y)\big) = g_y(x,y)+c'(y) = N_y(x,y)$. We can use the prime notation on the c term because as the constant of integration from an x integration it cannot have any variables other than y. So then since we know our potential function is $g(x,y)+c(y)$ and the only piece we are missing is $c(y)$ we just have to solve $g_y(x,y)+c'(y) = N_y(x,y)$ for $c'(y) = N_y(x,y) - g_y(x,y)$ which should be a function of y only, provided we did our integration correctly, and finish with $c(y) = \int c'(y)\,dy = \int N_y(x,y) - g_y(x,y)\,dy$

And we have our potential function $g(x,y)+c(y) = g(x,y) + \int N_y(x,y) - g_y(x,y)\,dy$ or in other words the potential function is $g(x,y) = \int f_x(x,y)\;dx + \int \big(N_y(x,y) - g_y(x,y)\big)dy$. AND if the vector field was three dimensional we would need to repeat this process again to find the z components of the potential function as the original constant of integration would really be a function both y and z.

That is a whole lot of work!!! This process of integrating, differentiating, equating and then integrating again is also used in differential equations when solving a certain type of equation, so most of you will see it again. But for now I have a more efficient approach.

Given a conservative field such as $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ rather than integrate and differentiate we can just integrate each component and then assemble the pieces as long as we know what we are doing.

$\int M(x,y,z)\,dx = f_1(x,y,z) + c_1(y,z)$ this integral will identify every term of the potential function that contains an x and may turn up several and may turn up several y's and z's as well if they are being multiplied by an x expression.

$\int N(x,y,z)\,dy = f_2(x,y,z) + c_2(x,z)$ identifies every term containing a y. In much the same way as above.

$\int P(x,y,z)\,dx = f_3(x,y,z) + c_3(x,y)$ identifies every term containing a z. Again, same as above. So then the one and only potential function needs to be the compilation of these 3 results. **NOT THE SUM!**

The potential function will consist of each unique term appearing in any of the three integrals plus a constant. So if the first integral has a term, x^2y^3 and the second integral has the same x^2y^3 term then the potential function has the same x^2y^3 term **not** $2x^2y^3$. A term does not have to be part of more than one integral to be a part of the potential function.

Example 1: Given the vector field $\mathbf{F}(x,y) = xy^2\,\mathbf{i} + x^2y\,\mathbf{j} = M\,\mathbf{i} + N\,\mathbf{j}$

A) Find the $div(\mathbf{F})$
B) Is the field conservative? Why?
C) If the field is conservative then find the potential function.

Solution: A) $div(\mathbf{F}) = M_x + N_y = y^2 + x^2 \neq 0$ so the field is not divergence free.

B) To test a 2 dimensional field for conservativeness check to see if $M_y = N_x$. $M_y = 2xy$, $N_x = 2xy$ since $M_y = N_x$ we know the field is conservative and therefore a potential function exists.

C) Find the potential function: $\int M\,dx = \int xy^2\,dx = \dfrac{x^2y^2}{2} + c(y)$, $\int N\,dy = \int x^2 y\,dx = \dfrac{x^2y^2}{2} + c(x)$

and so the potential function is $f(x,y) = \dfrac{x^2y^2}{2} + c$.

Example 2: Given the vector field $\mathbf{F}(x,y,z) = x^2z\,\mathbf{i} - 2xz\,\mathbf{j} + yz\,\mathbf{k} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$

A) Find the $div(\mathbf{F})$
B) Find the curl(F). Is the field conservative? Why?
C) If the field is conservative then find the potential function.

Solution: A) $div(\mathbf{F}) = M_x + N_y + P_z = 2xz - 0 + y = 2xz + y \neq 0$ so the field is not divergence free.

B) curl($\mathbf{F}$)

$$= \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = \left(z - (-2x)\right)\mathbf{i} - \left(0 - x^2\right)\mathbf{j} + \left((-2x) - 0\right)\mathbf{k} = (z + 2x)\mathbf{i} + x^2\,\mathbf{j} - 2x\,\mathbf{k}$$

Since the curl($\mathbf{F}$) is not the zero vector then the field is **not conservative**, and therefore we cannot find a potential function.

Example 3: Given the vector field $\mathbf{F}(x,y,z) = y^2z^3\,\mathbf{i} + \left(2xyz^3 - 4y\right)\mathbf{j} + \left(3xy^2z^2 + 6z^2\right)\mathbf{k} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$

A) Find the $div(\mathbf{F})$
B) Find the curl(F). Is the field conservative? Why?
C) If the field is conservative then find the potential function.

Solution: A) $div(\mathbf{F}) = M_x + N_y + P_z = 0 + \left(2xz^3 - 4\right) + \left(6xy^2z + 12z\right) = 2xz^3 - 46xy^2z + 12z \neq 0$ so the field is not divergence free.

B) curl($\mathbf{F}$) $= \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(y^2z^3\right) & \left(2xyz^3 - 4y\right) & \left(3xy^2z^2 + 6z^2\right) \end{vmatrix}$

$$= \left(\frac{\partial}{\partial y}\left(3xy^2z^2 + 6z^2\right) - \frac{\partial}{\partial z}\left(2xyz^3 - 4y\right)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}\left(3xy^2z^2 + 6z^2\right) - \frac{\partial}{\partial z}\left(y^2z^3\right)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\left(2xyz^3 - 4y\right) - \frac{\partial}{\partial y}\left(y^2z^3\right)\right)\mathbf{k}$$

$$= \left(\left(6xyz^2\right) - \left(6xyz^2\right)\right)\mathbf{i} - \left(\left(3y^2z^2\right) - \left(3y^2z^2\right)\right)\mathbf{j} + \left(\left(2yz^3\right) - \left(2yz^3\right)\right)\mathbf{k} = \langle 0,0,0\rangle$$

Since the curl($\mathbf{F}$) is the zero vector then the field is **conservative**, and therefore we can find a potential function.

$$\int M\,dx = \int \left(y^2z^3\right)dx = xy^2z^3 + C(y,z)$$

$$\int N\,dy = \int \left(2xyz^3 - 4y\right) dy = xy^2z^3 - 2y^2 + C(x,z)$$

$$\int P\,dz = \int \left(3xy^2z^2 + 6z^2\right) dz = xy^2z^3 + 2z^3 + C(x,y)$$

After compiling the results we see the potential function is $f(x,y,z) = xy^2z^3 - 2y^2 + 2z^3 + C$

Lecture Notes 15.2: Line Integrals

A line integral comes in several forms although there are really only two distinct styles that we work with in calculus III.

$\int_C f(x,y)\,ds$ represents the integral over the curve C up to a height of $z = f(x,y)$ with respect to arclength, s.

Hopefully everyone can picture a surface like a paraboloid lying above the xy plane. Now draw a curve in the xy plane and imagine a vertical sheet of fabric that has its bottom edge lying on the curve and the top edge butting up against the paraboloid. Like a curtain with a flat bottom, multiple curves, and a variable height. The above integral being height, $z = f(x,y)$, times length, ds, along this curve would give us the area of the curtain if it were laid flat. Pretty Cool

So how do we define the curve? Either parametrically or as a vector valued function. Same Thing Really.
Parametric form $C:\ x = x(t),\quad y = y(t)\quad a \le t \le b$ Vector form $C:\ \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}\quad a \le t \le b$

And what is ds? Recall from calculus II that the length of a curve defined parametrically or as a vector valued function is $s = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$.

Therefore $\int_C f(x,y)\,ds = \int_a^b f\left(x(t), y(t)\right)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$ for $C:\ x = x(t),\quad y = y(t)\quad a \le t \le b$
The above integral would be interpreted as area.

This process extends naturally to higher dimensions

$$\int_C f(x,y,z)\,ds = \int_a^b f\left(x(t), y(t), z(t)\right)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\,dt$$

For $C:\ x = x(t),\quad y = y(t),\quad z = z(t)\quad a \le t \le b$

Although the concept of curtain area is lost in this space.

The second general form of a line integral is $\int_C \mathbf{F}(x,y) \bullet d\mathbf{r}$. This is the integral along a curve, $\mathbf{r}(t)$, through a vector field, $\mathbf{F}$. Because we are looking at the dot product of the Field and the derivative of $\mathbf{r}(t)$ over infinitely many infinitely small intervals we are really measuring force times distance also known as work. This type of line integral then is known as a work integral for obvious reasons.

Computationally then we have:

$\int_C \mathbf{F}(x,y)\bullet d\mathbf{r}$ for the curve C: $\mathbf{r}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}\quad a\le t\le b$, and the vector field $\mathbf{F}(x,y)$.

$\mathbf{r}'(t)=x'(t)\,\mathbf{i}+y,(t)\,\mathbf{j}\quad a\le t\le b$ and $\mathbf{F}(x,y)=M\,\mathbf{i}+N\,\mathbf{j}$ so $\mathbf{F}\bullet d\mathbf{r}=M\,x'(t)+N\,y'(t)$

$x=x(t)$ and $y=y(t)$ so $\frac{dx}{dt}=x'(t)\rightarrow dx=x'(t)dt$ and $\frac{dy}{dt}=y'(t)\rightarrow dy=y'(t)dt$

Then $\int_C \mathbf{F}(x,y)\bullet d\mathbf{r}=\int_C M\,dx+N\,dy$ and $\int_C M\,dx+N\,dy=\int_a^b\left(\left[M\left(x(t),y(t)\right)\right]x'(t)+\left[N\left(x(t),y(t)\right)\right]y'(t)\right)dt$

Hopefully the attached examples will clarify this, it really isn't as bad as it looks.

(See hand written examples from section 15.2)

Lecture Notes 15.3: Conservative Vector Fields and Independence of Path

Recall from section 15.2 the definition of the Work Integral

$$W = \int_C \mathbf{F}\bullet d\mathbf{r} = \int_C M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$$

Where $\mathbf{F}(x,y,z) = M(x,y,z)\,\mathbf{i} + N(x,y,z)\,\mathbf{j} + P(x,y,z)\,\mathbf{k}$
And C is defined by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \quad a \le t \le b$

Now recall from section 15.1 that a vector field is conservative iff $curl(\mathbf{F}) = \nabla \times \mathbf{F} = \vec{0}$.
Conservative vector fields are said to be Independent of Path. That is to say that the work integral over any path with the same initial point and terminal point will be the same.

$\int_{C_1} \mathbf{F}\bullet d\mathbf{r} = \int_{C_2} \mathbf{F}\bullet d\mathbf{r} = \int_{C_3} \mathbf{F}\bullet d\mathbf{r}$... provided curves 1,2, and 3 all start from the same point and end at the same point in the domain of **F**.

Example 1: Given the vector field $\mathbf{F}(x,y) = y\,\mathbf{i} + x\,\mathbf{j} = M\,\mathbf{i} + N\,\mathbf{j}$

A) Show that **F** is a conservative vector field.

B) Show that the work integral, $\int_C \mathbf{F}(x,y)\bullet d\mathbf{r}$ is the same for both of the curves below

$C_1 : \mathbf{r}_1(t) = (2+t)\,\mathbf{i} + (3-t)\,\mathbf{j},\ 0 \le t \le 1$ and $C_2 : \mathbf{r}(w) = (2+\ln w)\,\mathbf{i} + (3-\ln w)\,\mathbf{j},\ 1 \le w \le e$

Solution: A) $M_y = 1 = N_x$ and so the field is conservative.

B) On C_1 $\mathbf{r}_1(0) = (2+0)\,\mathbf{i} + (3-0)\,\mathbf{j} = \langle 2,3\rangle$ and $\mathbf{r}_1(1) = (2+1)\,\mathbf{i} + (3-1)\,\mathbf{j} = \langle 3,2\rangle$ so we will be finding the work done in moving from the point (2,3) to (3,2) along the line segment given by $\mathbf{r}(t)$ through the field **F** and on C_2, $\mathbf{r}_2(1) = (2+\ln 1)\,\mathbf{i} + (3-\ln 1)\,\mathbf{j} = \langle 2,3\rangle$ and $\mathbf{r}_2(e) = (2+\ln e)\,\mathbf{i} + (3-\ln e)\,\mathbf{j} = \langle 3,2\rangle$ so we will be moving through the same field from the same initial point to the same terminal point along the logarithmic curve given by $\mathbf{r}_2(w)$.

$$\int_C \mathbf{F}(x,y)\bullet d\mathbf{r} = \int_C M\,dx + N\,dy = \int_C y\,dx + x\,dy$$

For $C_1 : \mathbf{r}_1(t) = (2+t)\,\mathbf{i} + (3-t)\,\mathbf{j},\ 0 \le t \le 1$ we have $x = 2+t,\ \ dx = dt$ and $y = 3-t,\ \ dy = -dt$ so

$$\int_C y\,dx + x\,dy$$

For $C_2 : \mathbf{r}(w) = (2+\ln w)\,\mathbf{i} + (3-\ln w)\,\mathbf{j},\ 1 \le w \le e$ we have $x = 2+\ln w,\ \ dx = \frac{1}{w}dw$ and

$y = 3-\ln w,\ \ dy = \frac{-1}{w}dw$ so

$$\int_C y\,dx + x\,dy = \int_1^e (3-\ln w)\frac{1}{w}dw + (2+\ln w)\frac{-1}{w}dw = \int_1^e (3-2-\ln w-\ln w)\frac{1}{w}dw = \int_1^e \left(\frac{1}{w}-\frac{2\ln w}{w}\right)dw$$

$= \left[\ln|w| - (\ln w)^2\right]_1^e = \left(0-0^2\right)-\left(1-1^2\right) = 0$ and we see that both paths lead to the same result.

Example 2: Given the three dimensional vector field $\mathbf{F}(x,y,z) = 2yz\,\mathbf{i} + 2xz\,\mathbf{j} + 2xy\,\mathbf{k} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$

A) Show that $\mathbf{F}$ is a conservative vector field.

B) Show that the work integral, $\int_C \mathbf{F}(x,y,z)\bullet d\mathbf{r}$ is the same for both of the curves below

$$C_1 : \mathbf{r}_1(t) = t\,\mathbf{i} - 4t\,\mathbf{j} + t^2\,\mathbf{k},\ 0 \le t \le 3 \text{ and } C_2 : \mathbf{r}_2(s) = s^2\,\mathbf{i} - \frac{4s^4}{3}\mathbf{j} + s^4\,\mathbf{k},\ 0 \le s \le \sqrt{3}$$

Solution: A) To see if the field is conservative we should find the curl($\mathbf{F}$).

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2yz & 2xz & 2xy \end{vmatrix} = (2x-2x)\mathbf{i} + (2y-2y)\mathbf{j} + (2z-2z)\mathbf{k} = \vec{0}$$

and so the field is conservative.

Short Cut – test for conservative three dimensional fields calculate the "off" partial derivatives, then the field is conservative if the top left pair, bottom right pair, and opposite corners are all pairwise equal as illustrated below.

$$\begin{array}{lll} M = 2yz & N = 2xz & P = 2xy \\ M_y = 2z & N_x = 2z & \boxed{P_x = 2y} \\ \boxed{M_z = 2y} & N_z = 2x & P_y = 2x \end{array}$$

$M_y = N_x,\ N_z = P_y,\ M_z = P_x$ implies conservative.

B) On $C_1 : \mathbf{r}_1(0) = 0\,\mathbf{i} - 0\,\mathbf{j} + 0\,\mathbf{k}$ and $\mathbf{r}_1(3) = 3\,\mathbf{i} - 12\,\mathbf{j} + 9\,\mathbf{k}$ on $C_2 : \mathbf{r}_2(0) = 0\,\mathbf{i} - 0\,\mathbf{j} + 0\,\mathbf{k}$ and $\mathbf{r}_2(\sqrt{3}) = 3\,\mathbf{i} - 12\,\mathbf{j} + 9\,\mathbf{k}$ so both curves have the same initial and terminal points.

$$\int_C \mathbf{F}(x,y,z)\bullet d\mathbf{r} = \int_C M\,dx + N\,dy + P\,dz = \int_C 2yz\,dx + 2xz\,dy + 2xy\,dz$$

on $C_1 : \mathbf{r}_1(t) = t\,\mathbf{i} - 4t\,\mathbf{j} + t^2\,\mathbf{k},\ 0 \le t \le 3 \quad x = t,\ dx = dt, \quad y = -4t,\ dy = -4\,dt, \quad z = t^2,\ dz = 2t\,dt$

$$\int_C 2yz\,dx + 2xz\,dy + 2xy\,dz = \int_0^3 2(-4t)\left(t^2\right)dt + 2(t)\left(t^2\right)(-4)\,dt + 2(t)(-4t)(2t)\,dt$$

$$= \int_0^3 \left(-8t^3 - 8t^3 - 16t^3\right) dt = \int_0^3 \left(-32t^3\right) dt = \left[-8t^4\right]_0^3 = -648$$ Remember a negative work value indicates that motion along this curve is generally in the same direction as the field.

on $C_2 : \mathbf{r}_2(s) = s^2\,\mathbf{i} - \dfrac{4s^4}{3}\,\mathbf{j} + s^4\,\mathbf{k},\ 0 \le s \le \sqrt{3}$

$x = s^2,\ dx = 2s\,ds, \quad y = -\dfrac{4s^4}{3},\ dy = -\dfrac{16s^3}{3}ds, \quad z = s^4,\ dz = 4s^3\,ds$

$$\int_C 2yz\,dx + 2xz\,dy + 2xy\,dz = \int_0^{\sqrt{3}} 2\left(-\frac{4s^4}{3}\right)\left(s^4\right)(2s)\,ds + 2\left(s^2\right)\left(s^4\right)\left(-\frac{16s^3}{3}\right)ds + 2\left(s^2\right)\left(-\frac{4s^4}{3}\right)\left(4s^3\right)ds$$

$$= \int_0^{\sqrt{3}} \left(-\frac{16s^9}{3}\right)ds + \left(-\frac{32s^9}{3}\right)ds + \left(-\frac{32s^9}{3}\right)ds = \int_0^{\sqrt{3}} \left(-\frac{80s^9}{3}\right)ds = \left[-\frac{80s^{10}}{30}\right]_0^{\sqrt{3}} = \left[-\frac{8\left(3^5\right)}{3} - 0\right] = -648$$

And so again we see that for a conservative vector field the work integral is independent of path.

Example 3: Given the three dimensional vector field $\mathbf{F}(x,y,z) = x^2y\,\mathbf{i} + 2xyz\,\mathbf{j} + xz^2\,\mathbf{k} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ Evaluate the work line integral $\int_C \mathbf{F}(x,y,z) \bullet d\mathbf{r}$ where C is the curve made up of the line segments

C_1 from $(0,2,1)$ to $(2,2,1)$ and C_2 from $(2,2,1)$ to $(2,6,1)$ and C_3 from $(2,6,1)$ to $(2,6,3)$

Solution: Check to see if the field is independent of path (conservative):

$$\begin{array}{lll} M = x^2y & N = 2xyz & P = xz^2 \\ M_y = x^2 & N_x = 2yz & P_x = z^2 \\ M_z = 0 & N_z = 2xy & P_y = 0 \end{array}$$

since $M_y \ne N_x$ the field is not conservative and therefore is not independent of path so we must follow the given path.

Because this collection of line segments is always parallel to a coordinate axis (only one variable is changing per segment) we can evaluate the integral as follows.

On C_1

$x: 0 \to 2$

$y = 2$ so $dy = 0$

$z = 1$ so $dz = 0$

$$\int_0^2 x^2(2)\,dx = \left[\frac{2x^3}{3}\right]_0^2$$

$$= \frac{16}{3} - 0 = \frac{16}{3}$$

On C_2

$x = 2$ so $dx = 0$

$y: 2 \to 6$

$z = 1$ so $dz = 0$

$$\int_2^6 2(2)y(1)\,dy = \left[\frac{4y^2}{2}\right]_2^6$$

$$= 72 - 8 = 64$$

On C_3

$x = 2$ so $dx = 0$

$y = 6$ so $dy = 0$

$z: 1 \to 3$

$$\int_1^3 (2)z^2\,dz = \left[\frac{2z^3}{3}\right]_1^3$$

$$= 18 - \frac{2}{3} = \frac{52}{3}$$

And so $\displaystyle\int_C \mathbf{F}(x, y, z) \bullet d\mathbf{r} = \frac{16}{3} + 64 + \frac{52}{3} = \frac{260}{3}$

The Fundamental Theorem of Line Integrals

Further, recall that conservative vector fields all have potential functions. Review your notes on how to find the potential function for a conservative field.

If $\mathbf{F}(x, y, z)$ is a conservative field with potential function $f(x, y, z)$ then $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$

where C is any curve with initial point (x_0, y_0, z_0) and terminal point (x_1, y_1, z_1). This is known as the Fundamental Theorem of Line Integrals. Another way to express this is

$$\int_{(x_0, y_0, z_0)}^{(x_1, y_1, z_1)} \mathbf{F} \bullet d\mathbf{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

It is critical to understand that the Fundamental Theorem of Line Integrals **only** applies to **conservative** vector fields. Much like the original Fundamental Theorem of Calculus only applies to continuous integrable functions.

Example 4: Given the vector field $\mathbf{F}(x, y) = e^x \sin(y)\,\mathbf{i} + e^x \cos(y)\,\mathbf{j} = M\,\mathbf{i} + N\,\mathbf{j}$

Evaluate the work line integral $\displaystyle\int_C \mathbf{F}(x, y) \bullet d\mathbf{r}$ where C is the cycloid $x = \theta - \sin(\theta),\quad y = 1 - \cos(\theta)$

from $(0, 0)$ to $(2\pi, 0)$ by showing that **F** is conservative and applying the fundamental theorem of line integrals.

Solution: Check to see if the field is independent of path (conservative):

$M_y = e^x \cos(y) = N_x = e^x \cos(y)$ so the field is conservative.

Find the potential function:

$$\int M\,dx = \int e^x \sin(y)\,dx = e^x \sin(y) + c(y)$$
$$\int N\,dy = \int e^x \cos(y)\,dy = e^x \sin(y) + c(x)$$

So compiling the results of the antiderivatives we have $f(x,y) = e^x \sin(y) + c$

Then by the fundamental theorem of line integrals:

$$\int_C \mathbf{F}(x,y)\bullet d\mathbf{r} = f(2\pi,0) - f(0,0) = e^{2\pi}\sin(0) - e^0 \sin(0) = 0 - 0 = 0$$

Example 5: Given the three dimensional vector field

$\mathbf{F}(x,y,z) = 8x^3\,\mathbf{i} + z^2\cos(2y)\,\mathbf{j} + z\sin(2y)\,\mathbf{k} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$

Evaluate the work line integral $\int_C \mathbf{F}(x,y,z)\bullet d\mathbf{r}$ where C is any curve from $\left(0,\frac{\pi}{4},1\right)$ to $(-2,0,-1)$ using the fundamental theorem of line integrals.

Solution: Check to see if the field is independent of path (conservative):

$$M = 8x^3 \quad N = z^2\cos(2y) \quad P = z\sin(2y)$$
$$M_y = 0 \quad N_x = 0 \quad P_x = 0$$
$$M_z = 0 \quad N_z = 2z\cos(2y) \quad P_y = 2z\cos(2y)$$

$M_y = N_x,\ N_z = P_y,\ M_z = P_x$ implies the field is conservative.

Now find the potential function:

$$\int M\,dx = \int 8x^3\,dx = 2x^4 + c(y,z)$$
$$\int N\,dy = \int z^2\cos(2y)\,dy = \tfrac{1}{2}z^2\sin(2y) + c(x,z)$$
$$\int P\,dz = \int z\sin(2y)\,dz = \tfrac{1}{2}z^2\sin(2y) + c(x,y)$$

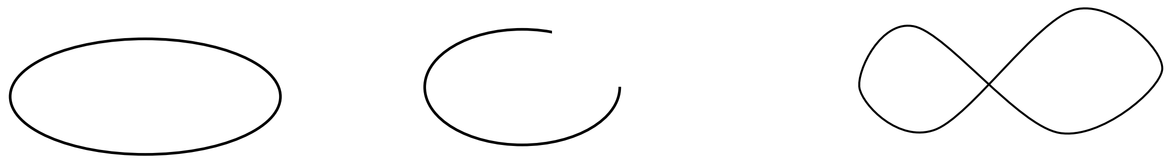
So compiling the results of the antiderivatives we have $f(x,y,z) = 2x^4 + \frac{1}{2}z^2\sin(2y) + c$

Then by the fundamental theorem of line integrals:

$$\int_C \mathbf{F}(x,y,z)\bullet d\mathbf{r} = f(-2,0,-1) - f\left(0,\frac{\pi}{4},1\right) = \left(2(-2)^4 + \tfrac{1}{2}(-1)^2\sin(0)\right) - \left(2(0)^4 + \tfrac{1}{2}(1)^2\sin\left(\tfrac{\pi}{4}\right)\right)$$
$$= (32+0) - \left(0 + \frac{\sqrt{2}}{4}\right) = 32 - \frac{\sqrt{2}}{4}$$

Lecture Notes 15.4: Green's Theorem

A curve in the plane is considered "**simple"** if the curve does not intersect itself at any points. A curve is considered "**closed**" if the initial point and terminal point are the same.



Simple and Closed Simple but not Closed Closed but not Simple

A simple closed curve does not need to be smooth, a curve traversing the vertices of a triangle would be a simple closed curve assuming it included each side exactly once. Retracing a side would be considered an intersection at the infinite number of points along the side.

First let us state a logical result. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ if **F** is conservative and C is any simple closed curve. If **F** is conservative then it satisfies the requirements of the Fundament Theorem of Line Integrals and so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(x_1, y_1) - f(x_0, y_0)$ but if C is a closed curve then $(x_1, y_1) = (x_0, y_0)$ and so $\int_C \mathbf{F} \cdot d\mathbf{r} = f(x_1, y_1) - f(x_1, y_1) = 0$. We know that gravity is a conservative field and that the work done in going from one location to another depends only on those locations and that if we start and end at the same location the total work is always zero.

Now, a not so logical result. The following is known as Green's Theorem.

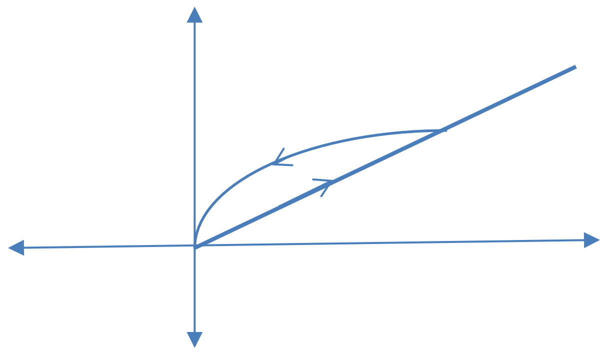
$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$ Where C is any simple closed curve, oriented counter-clockwise, R is the region bounded by the curve, and $\mathbf{F}(x, y) = M(x, y)\,\mathbf{i} + N(x, y)\,\mathbf{j}$. Green's Theorem applies only to 2-dimensional vector fields and curves in the plane.

From a physics point of view, Green's theorem is used to relate the Flux across a region in a field to the Flow around the region. The form of Green's theorem we study here is often called the Tangential form and relates to the work done in going around a region in field.

As an interesting note, George Green was a self-educated son of a miller and first published this theorem in an essay on Electricity and Magnetism in 1828. Over 1000 pages of calculus and we are just now getting to something that was first published within the last 200 years, barely.

See Example 3.

Example 1: Use Green's theorem to evaluate $\int_C y^2\,dx + x^2\,dy$ where C is the boundary of the region bounded by the graphs of $y = \sqrt{x}$ and $y = x$ oriented counter clockwise.

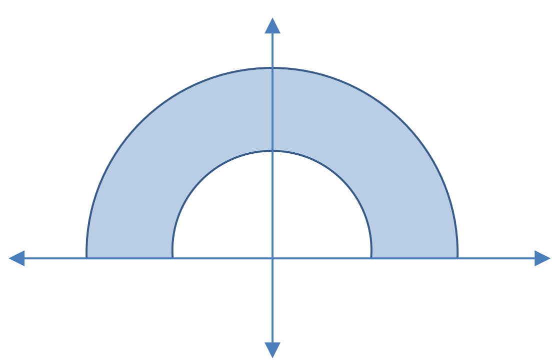


Solution: $M = y^2$, $\dfrac{\partial M}{\partial y} = 2y$ and $N = x^2$, $\dfrac{\partial N}{\partial x} = 2x$ so by Green's Theorem

$$\int_C y^2\,dx + x^2\,dy = \int_0^1 \int_x^{\sqrt{x}} (2x - 2y)\,dy\,dx = \int_0^1 \left(2xy - y^2\right)_x^{\sqrt{x}} dx = \int_0^1 \left(2x^{3/2} - x\right) - \left(2x^2 - x^2\right) dx$$

$$= \int_0^1 \left(2x^{3/2} - x - x^2\right) dx = \left[2\left(\frac{2}{5}\right)x^{5/2} - \frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{4}{5} - \frac{1}{2} - \frac{1}{3} - 0 = \frac{-1}{30}$$

Example 2: Use Green's theorem to evaluate $\int_C (y - x)\,dx + (2x - y)\,dy$ where C is the boundary of the region inside the semicircle $y = \sqrt{25 - x^2}$ and above the semicircle $y = \sqrt{9 - x^2}$ for $y \geq 0$ oriented counter clockwise.



Solution: $M = y - x$, $\dfrac{\partial M}{\partial y} = 1$ and $N = 2x - y$, $\dfrac{\partial N}{\partial x} = 2$ so by Green's Theorem

$$\int_C (y-x)\,dx + (2x-y)\,dy = \iint_R (2-1)\,dA = \iint_R (1)\,dA$$

This region looks like it would be a lot nicer in polar coordinates.

$$\iint_R (1)\,dA = \int_0^{\pi}\int_3^5 1r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_3^5 d\theta = \int_0^{\pi}\left(\frac{25}{2}-\frac{9}{2}\right)d\theta = \int_0^{\pi}(8)\,d\theta = 8\pi$$