Additional Examples Chap 12. (old book)

1.) Vector valued function for curve of intersection.

12.1 # 62 $\quad 4x^2 + 4y^2 + z^2 = 16 \qquad x = z^2 \qquad \underline{z = t}$

(Ellipsoid) $\qquad$ (parabolic cylinder)

$$z = t \Rightarrow \quad x = t^2 \Rightarrow \quad 4(t^2)^2 + 4y^2 + t^2 = 16$$

$$4t^4 + 4y^2 + t^2 = 16$$

$$4y^2 = 16 - 4t^4 - t^2$$

$$y^2 = 4 - t^4 - \tfrac{1}{4}t^2 \Rightarrow y = \pm\sqrt{4 - t^4 - \tfrac{1}{4}t^2}$$

$$\underline{\vec{r}(t) = t^2\,\hat{i} \pm \sqrt{4 - t^4 - \tfrac{1}{4}t^2}\,\hat{j} + t\,\hat{k}}$$

12.1 # 64 $\quad x^2 + y^2 + z^2 = 10 \qquad x + y = 4 \qquad x = 2 + \sin t$

(sphere) $\qquad$ (plane)

$$\overset{x}{y + (2 + \sin t)} = 4 \qquad y = 4 - (2 + \sin t) = 2 - \sin t$$

$$x^2 = (2 + \sin t)^2 = 4 + 4\sin t + \sin^2 t$$

$$y^2 = (2 - \sin t)^2 = 4 - 4\sin t + \sin^2 t$$

$$x^2 + y^2 + z^2 = 10 \rightarrow (4 + 4\sin t + \sin^2 t) + (4 - 4\sin t + \sin^2 t) + z^2$$

$$10 = 8 + 2\sin^2 t + z^2 \rightarrow z^2 = 2 - 2\sin^2 t = 2(1 - \sin^2 t)$$

12.1 #64 (Continued)

$$z^2 = 2(1-\sin^2 t) = 2\cos^2 t \Rightarrow z = \sqrt{2}\cos t$$

Curve of intersection:

$$\vec{r}(t) = (2+\sin t)\,\hat{i} + (2-\sin t)\,\hat{j} + \sqrt{2}\cos t\,\hat{k}$$

12.2 #36 $\vec{r}(\theta) = (\theta + \sin\theta)\,\hat{i} + (1-\cos\theta)\,\hat{j}$

"Smooth" $\vec{r}\,'(\theta)$ exists and $\vec{r}\,'(\theta) \neq \vec{0} = 0\hat{i} + 0\hat{j}$

$\vec{r}\,'(\theta) = (1+\cos\theta)\,\hat{i} + \sin\theta\,\hat{j}$

$\vec{r}\,'(\theta)$ exist for all θ, $1+\cos\theta = 0$ at $\theta = \pi, 3\pi, \ldots$

$\sin\theta = 0$ at $\theta = 0, \pi, 2\pi, \ldots$

So $\vec{r}\,'(\theta) = 0\hat{i} + 0\hat{j}$ at θ = any odd multiple of π.

Not smooth if $\theta = \pm(2n+1)\pi$ $\quad n = 0, 1, 2, \ldots$

12.2 #58 $\int \left(e^t\,\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}\right) dt$

$$= \left(e^t\,\hat{i} - \cos t\,\hat{j} + \sin t\,\hat{k}\right) + \vec{c}$$

$$\vec{c} = \langle c_1, c_2, c_3 \rangle$$

12.2 # 72) $\vec{r}\,'(t) = \frac{1}{1+t^2}\hat{i} + \frac{1}{t^2}\hat{j} + \frac{1}{t}\hat{k}$

$\vec{r}(1) = 2\hat{i}$

$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \left(\tan^{-1}(t) + c_1\right)\hat{i} + \left(-\frac{1}{t} + c_2\right)\hat{j} + \left(\ln|t| + c_3\right)$

$\vec{r}(1) = \left(\tan^{-1}(1) + c_1\right)\hat{i} + \left(\frac{-1}{1} + c_2\right)\hat{j} + \left(\cancel{\ln 1}^{\,0} + c_3\right)\hat{k}$

$= \left(\pi/4 + c_1\right)\hat{i} + \left(c_2 - 1\right)\hat{j} + \left(c_3\right)\hat{k} = 2\hat{i}$

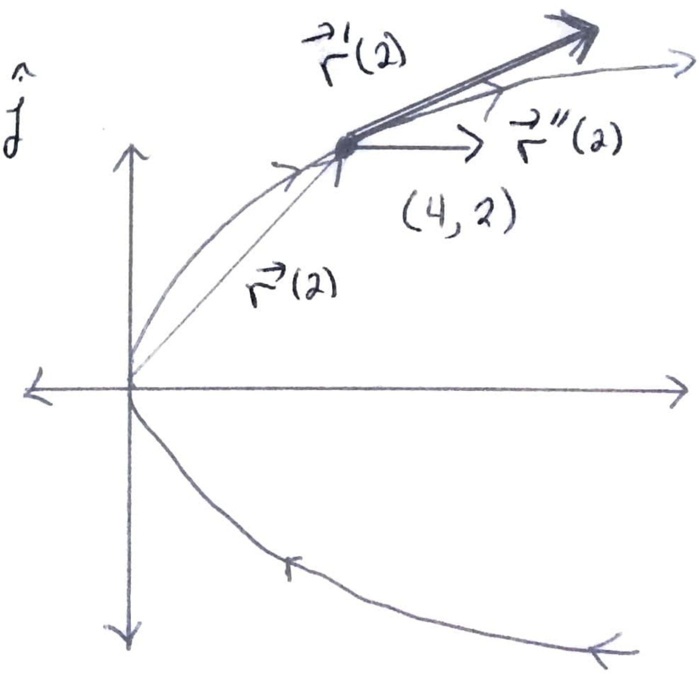
$\Rightarrow \frac{\pi}{4} + c_1 = 2 \quad c_1 = 2 - \frac{\pi}{4} \quad c_2 - 1 = 0 \quad c_2 = 1 \quad c_3 = 0$

$\vec{r}(t) = \left(\tan^{-1}(t) + 2 - \pi/4\right)\hat{i} + \left(1 - \frac{1}{t}\right)\hat{j} + \ln|t|\,\hat{k}$

12.3 #3 $\vec{r}(t) = t^2\hat{i} + t\hat{j}$

$\vec{r}\,'(t) = 2t\hat{i} + \hat{j}$

$\vec{r}\,''(t) = 2\hat{i}$



at (4, 2) → t = 2

$\vec{r}\,'(2) = \langle 4, 1 \rangle$

$\vec{r}\,''(2) = \langle 2, 0 \rangle$

12.3

#28 $\vec{a}(t) = e^t \hat{i} - 8\hat{k}$ $\vec{v}(0) = \langle 2, 3, 1 \rangle$

$\vec{r}(0) = \vec{0} = \langle 0, 0, 0 \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt = (e^t + c_1)\hat{i} + c_2\hat{j} + (-8t + c_3)\hat{k}$

$\vec{v}(0) = (1 + c_1)\hat{i} + c_2\hat{j} + c_3\hat{k} = \langle 2, 3, 1 \rangle$

$1 + c_1 = 2$ $c_1 = 1$ $c_2 = 3$ $c_3 = 1$

$\underline{\vec{v}(t) = (e^t + 1)\hat{i} + 3\hat{j} + (-8t + 1)\hat{k}}$

$\vec{r}(t) = \int \vec{v}(t)\,dt = (e^t + t + c_1)\hat{i} + (3t + c_2)\hat{j} + (-4t^2 + t + c_3)\hat{k}$

$\vec{r}(0) = (1 + c_1)\hat{i} + c_2\hat{j} + c_3\hat{k} = \langle 0, 0, 0 \rangle$

$1 + c_1 = 0$ $c_1 = -1$ $c_2 = 0$ $c_3 = 0$

$\underline{\vec{r}(t) = (e^t + t - 1)\hat{i} + (3t)\hat{j} + (-4t^2 + t)\hat{k}}$

12.3 #46 projectile fired from ground level $\rightarrow h_0 = 0$

at angle of $8^\circ \rightarrow \theta = 8^\circ$ Needs range of 50 meters

$g = -9.8\ m/s^2$

Find V_0.

12.3 46.) (continued)

$$\vec{r}(t) = (V_0 \cos\theta) t \,\hat{i} + \left(-\tfrac{1}{2} g t^2 + (V_0 \sin\theta) t + h_0\right) \hat{k}$$

$$\vec{r}(t) = (V_0 \cos(8^\circ)) t \,\hat{i} + \left(-4.9 t^2 + V_0 (\sin 8^\circ) t\right) \hat{k}$$

Need $(V_0 \cos 8^\circ) t = 50$ when $(-4.9 t^2 + (V_0 \sin 8^\circ) t) = 0$

$$t(-4.9t + (V_0 \sin 8^\circ)) = 0$$

$$\Rightarrow t = 0 \quad \text{or} \quad t = \frac{V_0 \sin 8^\circ}{4.9}$$

$$(V_0 \cos 8^\circ)\left[\frac{V_0 \sin 8^\circ}{4.9}\right] = 50$$

$$V_0^2 = \frac{50(4.9)}{(\sin 8^\circ)(\cos 8^\circ)} \approx 1777.6981$$

$V_0 = 42.16$ m/s

12.4

57 + extra K = curvature

at $t = \frac{\pi}{3}$

$$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 2t\,\hat{k}$$

$$\vec{v}(t) = \vec{r}\,'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + 2\,\hat{k}$$

$$\vec{a}(t) = \vec{r}\,''(t) = -\cos t\,\hat{i} - \sin t\,\hat{j} + 0\,\hat{k}$$

$$\vec{v}(\pi/3) = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 2 \right\rangle$$

$$\vec{a}(\pi/3) = \left\langle -1/2, -\sqrt{3}/2, 0 \right\rangle$$

$$\|\vec{v}\| = \sqrt{\frac{3}{4} + \frac{1}{4} + 4} = \sqrt{5} \qquad \vec{v} \cdot \vec{a} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} + 0 = 0$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{\sqrt{3}}{2} & 1/2 & 2 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{vmatrix} = \left\langle 0 + \sqrt{3}, -(0+1), \frac{3}{4} + \frac{1}{4} \right\rangle = \langle \sqrt{3}, -1, 1 \rangle$$

$$\|\vec{v} \times \vec{a}\| = \sqrt{3+1+1} = \sqrt{5}$$

$$a_T = \frac{0}{\sqrt{5}} = 0 \qquad a_N = \frac{\sqrt{5}}{\sqrt{5}} = 1 \qquad \vec{T} = \frac{1}{\sqrt{5}} \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 2 \right\rangle$$

$$\vec{a} = a_T \vec{T} + a_N \vec{N} \Rightarrow \vec{N} = \frac{1}{a_N}\left(\vec{a} - a_T \vec{T}\right)$$

$$\vec{N} = 1\left(\left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0 \right\rangle - 0 \left\langle -\frac{\sqrt{3}}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle\right)$$

$$\vec{N} = \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0 \right\rangle$$

curvature $K = \frac{\sqrt{5}}{(\sqrt{5})^3} = \frac{1}{5}$

12.5

#14 $\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j} + t^2\hat{k}$

on $0 \leq t \leq \pi/2$. Find arc length

$\vec{r}\,'(t) = (-\cancel{\sin t} + \cancel{\sin t} + t\cos t)\hat{i} + (\cancel{\cos t} - \cancel{\cos t} + t\sin t)\hat{j} + 2t\,\hat{k}$

$\|\vec{r}\,'(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t + 4t^2} = \sqrt{5t^2} = \sqrt{5}\,t$

$$S = \int_0^{\frac{\pi}{2}} \sqrt{5}\,t\,dt = \frac{\sqrt{5}\,t^2}{2}\Bigg|_0^{\pi/2} = \frac{\sqrt{5}}{2}\left(\frac{\pi^2}{4} - 0\right) = \frac{\sqrt{5}\,\pi^2}{8}$$