Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 2: 6/1-6/7
Sections 11.5 – 11.7, Lines and Planes, Surfaces in Space, Cylindrical and Spherical Coordinates
Section 12.1 Vector Valued Functions

Due this week: Thursday – WebAssign 11.4-11.7 Sunday – Weekly Assignment 2, 11.5-11.7, 12.1

Lecture Notes 11.5: Lines and Planes in Space

Remember back to early in your algebra days. One of the first things you studied was the equation of a line. Recall, any 2 points uniquely determine a line, but how did we formulate the equation of a line through two points (x_1, y_1) and (x_2, y_2). The first thing we needed to find was the slope of the line, $m = \frac{y_2 - y_1}{x_2 - x_1}$ then depending on how you were taught or what style you prefer we could build either the point-slope form of the equation, $y - y_0 = m(x - x_0)$ where *m* is the slope and (x_0, y_0) is either of the given points. If we then distributed the slope and solved for *y*. $y - y_0 = mx - mx_0$ leading to $y = mx - mx_0 + y_0$ and combining the constants we would have $y = mx + b$ where $b = -mx_0 + y_0$ is the *y* intercept of the line. This is known as slope-intercept form. Many of you are still what I call b solvers, by that I mean once you find the slope you go directly to the slope intercept form although you don't know the *y* intercept, but if you substitute either of the given points for *x* and *y* then *b* is the only remaining unknown so we can solve for it.

__**Example 1**__: Find the equation of the line through points $(2,7)$ and $(5,-4)$.
Solution: Find slope, $m = \frac{7 - -4}{2 - 5} = \frac{11}{-3} = \frac{-11}{3}$. Remember it doesn't matter which is point #1 and which is point #2 as long as we are consistent between the numerator and denominator. Now with the slope select either point and formulate the point-slope form, $y - 7 = \frac{-11}{3}(x - 2)$. Distributing and simplifying we arrive at the equation, $y = \frac{-11}{3}x + \frac{43}{3}$.

Now, what if the points are in 3-space? Slope isn't defined any longer since slope measure rise and run in only 2 directions. Slope tells us how to move from one point on a line in the plane to another point on the line, from our example above we can locate another point by moving down 11 and right 3 units from any point on the line. Directions on how to move from one point to another is what a vector gives so we replace the slope concept with a vector. Just like point-slope form, to find the equation of a line in 3-space we need to have a point on the line (x_0, y_0, z_0) and a vector parallel to the line, $\langle a, b, c \rangle$. The next problem arises when we try to formulate an equation. A linear or first degree equation always defines an object of one dimension less than the space it is defined in. Therefore a linear equation of 3 variables defines a plane, a linear equation of 2 variables defines a line and a linear equation of one variable defines a point. So we can only formulate the equation of a line in 3-space by using parametric equations.

The line containing the point (x_0, y_0, z_0) and parallel to the vector $\langle a, b, c \rangle$ has the parametric equations $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$. This indicates the line consists of all points that are some multiple, *t*, of the vector $\langle a, b, c \rangle$ from the fixed point (x_0, y_0, z_0).

__**Example 2**__: Find the parametric equations of the line through points $P = (2,7,4)$ and $Q = (5,-4,1)$.

Solution: First we construct the vector through the points, $\overline{PQ} = \langle 5-2, -4-7, 1-4 \rangle = \langle 3, -11, -3 \rangle$. Then using either point we can build the parametric equations, $x = 2+3t, \quad y = 7-11t, \quad z = 4-3t$. Note that when $t=0$ the parametric equations give us point *P* and that when $t=1$ the parametric equations generate point *Q*.

Now lets take a look at the equation of a plane. From the above discussion we might think that to build the equation of a plane we would need a point in the plane and a vector in the plane as well but that won't work. Imagine a plane as an infinite sheet of paper, if we take a piece of paper and draw a line (vector) from corner to corner then we can rotate that piece of paper by holding on to the corners connected by the vector and each rotation would represent a different plane. We need to change our thinking. Now if we imagine a vector projecting out of the piece of paper at a right angle then any rotation around that vector would still be a subset of the same plane and any movement that would change the plane would also change the vector. I like to think of the vector as a handle sticking straight out of the plane that I can use to hold and change the orientation of the plane. So the short of it is that to construct the equation of a plane we need a point in the plane (x_0, y_0, z_0) and a vector perpendicular (also known as normal to the plane) $\langle a, b, c \rangle$. Then the equation of the plane is $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ which we can simplify to $ax+by+cz=d$ where $d = -(ax_0+by_0+cz_0)$.

<u>**Example 3**</u>: Find the equation of the plane containing the point $(3,-4,11)$ with normal vector $\mathbf{n} = \langle -2, 5, -4 \rangle$.
Solution: Going directly to the formula above we have $-2(x-3)+5(y-(-4))-4(z-11)=0$ then we can simplify that to be $-2x+5y-4z=-6-20-44=-70$ or $2x-5y+4z=72$. You can often check for arithmetic errors by substituting the given point into the final equation to make the equation is valid, $2(3)-5(-4)+4(11)=70$.

<u>**Example 4**</u>: Find the equation of the plane that contains the non-colinear points $P=(2,7,4)$, $Q=(5,-4,1)$, and $R=(-1,3,5)$.
Solution: In this case we are given a normal vector but we know that the cross product of two vectors is normal (perpendicular) to both vectors. We just need to find 2 vectors in the plane. From example 2 we know $\overline{PQ} = \langle 5-2, -4-7, 1-4 \rangle = \langle 3, -11, -3 \rangle$ now we need to involve our new point *R* either $\overline{RQ} = \langle -1-2, 3-7, 5-4 \rangle = \langle -3, -4, 1 \rangle$ or $\overline{PR} = \langle 5-(-1), -4-3, 1-5 \rangle = \langle 6, -7, -4 \rangle$ would suffice. Just to keep the numbers smaller lets find $\overline{PQ} \times \overline{RQ}$. Recall:

$$\overline{PQ} \times \overline{RQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -11 & -3 \\ -3 & -4 & 1 \end{vmatrix} = \left(-11(1)-(-4)(-3)\right)\mathbf{i} - \left(3(1)-(-3)(-3)\right)\mathbf{j} + \left(3(-4)-(-3)(-11)\right)\mathbf{k} = \langle -23, 6, -45 \rangle$$

Now we don't need to deal with the negatives either because if this vector is normal to the plane then so is it's opposite so let $\mathbf{n} = \langle 23, -6, 45 \rangle$. We also need a point in the plane and we have 3 to choose from, any point will do for the equation so lets use $P=(2,7,4)$. Now we just need to put the pieces together to get the equation of the plane,' $23(x-2)-6(y-7)+45(z-4)=0$ and simplify that to $23x-46-6y+42+45z-180=0$ or $23x-6y+45z=184$. Now just to check it lets try substituting one of the other points (you can substitute both but if you are unlucky enough for one to work out but not the other then you are one of the unluckiest people I know). Using point $Q=(5,-4,1)$ we get $23(5)-6(-4)+45(1)=184$ and so our equation checks.

In summary to form the equation of a line in space we need a point on the line, (x_0, y_0, z_0), and a vector **parallel** to the line, $\langle a,b,c \rangle$. Then the parametric equations of the line are: $x = x_0 + at, \quad y = y_0 + bt, \ z = z_0 + ct$ for all real numbers, *t*.
To form the equation of a plane we need a point in the plane, (x_0, y_0, z_0), and a vector **normal (perpendicular)** to the plane, $\langle a,b,c \rangle$. Then the equation of the plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or $ax + by + cz = d$ where $d = -(ax_0 + by_0 + cz_0)$.

Just imagine the things we can do now and the questions that an be asked of us. Find the angle between intersecting planes (equals the angle between normal vectors), find the equation of the line of intersection between 2 planes (find 2 points on the line to form a vector parallel to the line of intersection or find a point in common to both planes and use the cross product of the normals as a vector parallel to the line of intersection) find the distance from a point in space to a plane (form a vector from any point in the plane to the given point not in the plane and find the magnitude of the projection of that vector onto the normal) just to name a few possibilities.

Lecture Notes 11.6: Surfaces in Space

We will not be doing very much in this section but I would like everyone to become familiar with cylinders and the quadric surfaces. The cylinders in this context are different from the pop cans that we all picture when we hear the word. In general 3-dimensional space any time we have a cylinder any time we have a curve in the plane and a set of parallel lines not in the plane that intersect the curve which then form the cylinder. To relate this to our classic right circular cylinder (pop can) the curve in the plane is the circle that forms the base then all of the vertical lines intersecting the circle form the cylinder walls which we will call "the cylinder". Another example of such a cylinder can be fairly easily visualized by picturing a parabola in the xy plane. Now consider lines perpendicular (perpendicular is common but not required) to the xy plane coming out of the points of the parabola forming a 2-dimensional parabolic shape. That is known as a parabolic cylinder. The curve can take any form but we are not going to get too fancy. A quadric surface is any 3-dimensional surface whose generating equation contains variables of positive degrees less than or equal to 2. A list of such surfaces along with their graphical renditions can be found on pages 800 and 801 of the text book or pages 814-815 of the older edition pdf book I posted to Blackboard. You should look over these surfaces and get generally acquainted with their equation forms. The basics we should be familiar with are summarized below.

Ellipsoid – 3 squared terms each with positive coefficients. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ If all denominators are equal then this would be a sphere.

Hyperboloid – 3 squared terms but one or two negative coefficients. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, hyperboloid of one sheet. $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, hyperboloid of 2 sheets.

Cone – 3 squared terms with one negative term equal to 0. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ Often considered a double cone while $z=\pm\sqrt{\frac{c^2x^2}{a^2}+\frac{c^2y^2}{b^2}}$ would be just one cone above or below the xy plane depending on sign.

Elliptic Paraboloid – one linear term and 2 positive second degree terms. $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$. Each vertical cross section (for a fixed value of x or y) would be a parabola while each horizontal cross section (for a fixed value of z would be an ellipse.

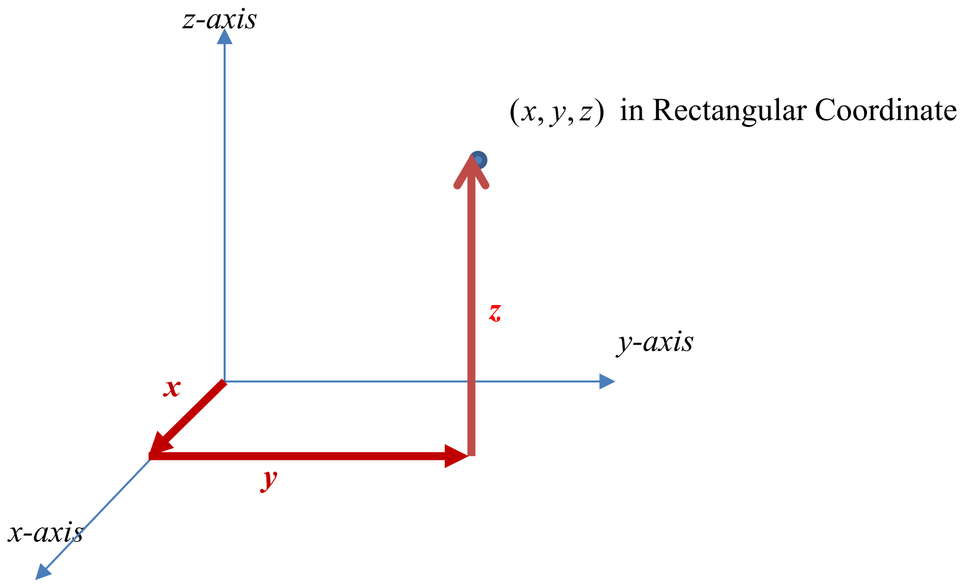
Hyperbolic Paraboloid (Saddle Function) – one linear term and one positive and one negative second degree terms. $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$. Again think of holding one variable constant thus creating cross sections. For fixed z values we have hyperbolas (difference of squares). For fixed y values in this case we have upward opening parabolas and for fixed x values we have downward opening parabolas. Looking at the diagram the "saddle" title should become obvious.

Each of the examples above are just for reference, they are all centered at the origin but using transformations studied in college algebra it would not be difficult to move them all over 3-space. We will for the most part limit ourselves to these general forms throughout the course if a visualization is required.

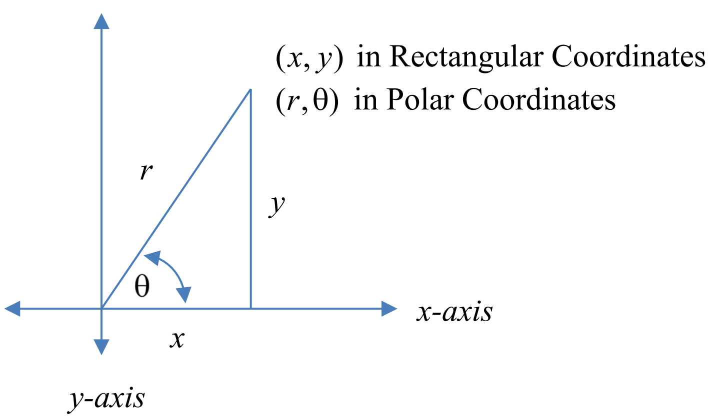
Omit the discussion of surfaces of revolution since we studied these fairly hard in calculus II.

Lecture Notes 11.7: Cylindrical and Spherical Coordinates

In this section we are going to talk about describing points and graphs in 3-dimensions using different coordinate systems. Let us begin with what is known as Rectangular Coordinates (also sometimes called Cartesian Coordinates). In a rectangular system we build the 3-dimensional world out of unit cubes (cubes with all sides of length 1 unit). To locate a point in space then we move out the *x*-axis some distance, *x*, then from that point we move in a direction parallel to the *y*-axis a distance, *y*, then from that point move in a direction parallel to the *z*-axis a distance, *z*. At the end of this "out, over, up" journey then we have the point (x, y, z). See the diagram below.



Now recall polar coordinates from Calculus II and your precalculus courses. Polar coordinates are limited to the plane but we will be adding the third dimension.

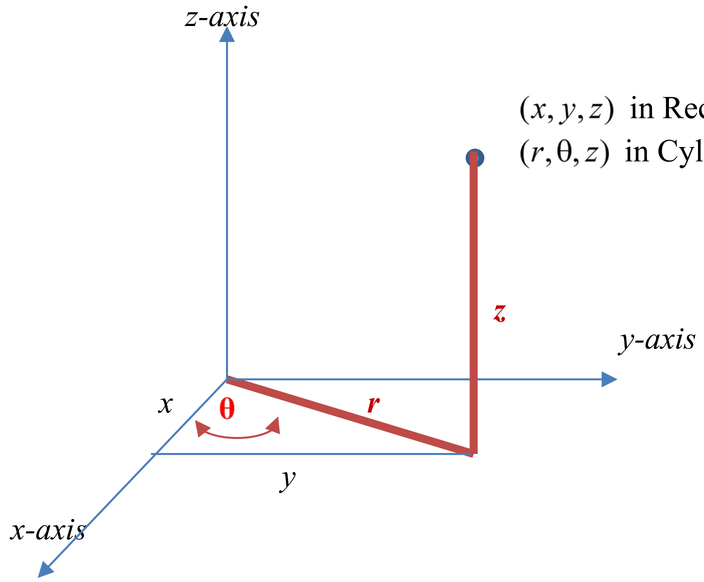


Polar Coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$\tan\theta = \frac{y}{x}$$

If we use polar coordinates to describe points in the *xy* plane and then add a vertical distance, *z*, we create what is known as Cylindrical Coordinates. Cylindrical coordinates are a 3-dimensional coordinate system which is based on right circular cylinders where each horizontal cross section in the *xy* plane has the origin as it's center. Each point is located on the side of such cylinders centered on the *z* axis with polar coordinates providing the location of the image of the point in the *xy* plane and the *z* value giving the distance above or below the plane. See the diagram below.

(x, y, z) in Rectangular Coordinates
(r, θ, z) in Cylindrical Coordinates



Cylindrical Coordinates: (r, θ, z)

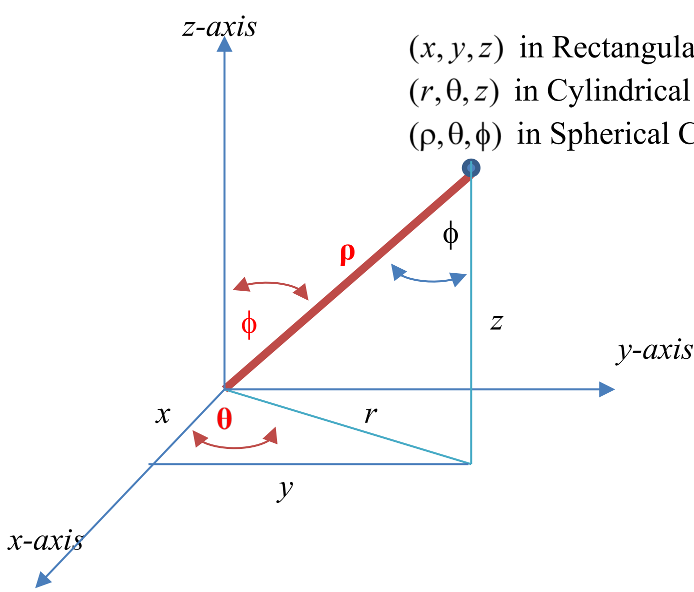
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$\tan\theta = \frac{y}{x}$$
$$z = z$$

At this point I want to emphasize that in an ordered triplet like (x, y, z) in rectangular coordinates or (r, θ, z) in cylindrical coordinates the order matters and the parentheses are important. A list of 3 numbers is not a coordinate without them.

Now, rather than locating points on a cylinder centered on the z axis we will look at locating points on concentric spheres with the origin at their center. This is known as the **spherical coordinate system.** Each point is located based on the distance from the origin (the radius of the sphere) denoted by the Greek letter rho, ρ, and the rotational angle as measured from the positive x axis, θ, (this is the same θ we use in polar or cylindrical coordinates), the final coordinate is the angle denoted by the Greek letter phi, (which can be pronounced like fee or fie but never foe or fum), and is measured down from the positive z axis. By convention we consider $0 \le \theta < 2\pi$ but $0 \le \phi \le \pi$, thus theta revolves around the z axis but phi only ranges from straight up at $\phi = 0$ to straight down when $\phi = \pi$. The reason for this is that any point with $0 \le \theta < \pi$ and $\pi < \phi < 2\pi$ can also be located using $\pi \le \theta < 2\pi$ and $0 \le \phi \le \pi$. If you want to point to something behind you, you can either face forward and swing your arm down past vertical until it points at the object $\pi < \phi < 2\pi$ or turn around and swing your arm down just until you are pointing at it, $0 \le \phi \le \pi$.

Consider the diagram below for a visual description of spherical coordinates.

(x, y, z) in Rectangular Coordinates
(r, θ, z) in Cylindrical Coordinates
(ρ, θ, ϕ) in Spherical Coordinates



Spherical Coordinates: (ρ, θ, ϕ)

$$r = \rho\sin\phi$$
$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$r^2 + z^2 = \rho^2$$
$$\tan\phi = \frac{r}{z}$$

So now what are we going to be expected to do with these coordinate systems? For now we will be primarily limited to translations between them.

Example 1: Translate the point $(6,\frac{-3\pi}{4},5)$ from cylindrical to rectangular and spherical coordinates.

Solution: $r=6$, $\theta=\frac{-3\pi}{4}$, and $z=5$ so $x=6\cos\frac{-3\pi}{4}=6(\frac{-\sqrt{2}}{2})=-3\sqrt{2}$ and $y=6\sin\frac{-3\pi}{4}=6(\frac{-\sqrt{2}}{2})=-3\sqrt{2}$ so $(6,\frac{-3\pi}{4},5)$ in cylindrical coordinates is the same as $(-3\sqrt{2},-3\sqrt{2},5)$ in rectangular coordinates.

$\rho^2=r^2+z^2=36+25=61$ so $\rho=\sqrt{61}$, and $\tan\phi=\dfrac{r}{z}=\dfrac{6}{5}$ so $\phi=\tan^{-1}\left(\frac{6}{5}\right)$ and so the same point in spherical coordinates would be $\left(\sqrt{61},\frac{-3\pi}{4},\tan^{-1}\left(\frac{6}{5}\right)\right)$.

Cylindrical equations should be solved for z if that is practical and does not lead to a plus or minus situation, or r if necessary. Spherical equations are generally expressed in $\rho=\ldots$ form, again if solving for ρ is practical and doesn't lead to a plus or minus situation We don't want to take the square root of both sides of an equation just to isolate a variable.

Example 2: Translate the rectangular equation $x^2+y^2+z^2=16-4z$ into cylindrical and spherical equations.

Solution: Cylindrical - $x^2+y^2=r^2$ so $r^2+z^2=16-4z$ and I would stop here since solving for z would require completing the square and solving for r would require the square root of both sides of the equation. Spherical - $x^2+y^2+z^2=\rho^2$ and $z=\rho\sin\phi$ so by direct substitution we see $\rho^2=16-4\rho\sin\phi$ or $\rho^2+4\rho\sin\phi=16$. Again we stop here since trying to isolate ρ would require completing the square.

Example 3: Translate the spherical equation $\rho=\sin\phi+3\cos\phi$ to both rectangular and cylindrical forms.

Solution: It is tempting to square both sides of the equation to produce a ρ^2 but although ρ^2 is a convenient form to have on the left hand side, the right hand side of the equation only gets worse. In fact it is always a good idea to look for the trig functions and try to do whatever it takes to produce either $\rho\sin\phi=r$, or $\rho\cos\phi=z$ in spherical coordinates and $r\cos\theta=x$ or $r\sin\theta=y$ in cylindrical coordinates and these manipulations should have a higher priority than building either ρ^2 or r^2. Multiply both sides of the equation by ρ to put a ρ with the sine and cosine. $\rho(\rho)=\rho(\sin\phi+3\cos\phi)$ so now $\rho^2=\rho\sin\phi+3\rho\cos\phi$ and so for cylindrical coordinates $r^2+z^2=r+3z$ or $r^2-r=3z-z^2$ again this is not a form we would want to try to isolate either variable in. For rectangular coordinates from $\rho^2=\rho\sin\phi+3\rho\cos\phi$ we move to $x^2+y^2+z^2=r+3z$ and we are closer but not yet all the way into rectangular form. Remember which variables exist in each system and never stop until those are the only variables remaining. Here we need to get rid of the remaining r. We could isolate it and square both sides of the equation to produce an r^2 or since r represents a radius we can assume $r\geq 0$ and so $r=\sqrt{x^2+y^2}$ is an acceptable substitution. Leading to $x^2+y^2+z^2=\sqrt{x^2+y^2}+3z$ and this would be an acceptable rectangular version of $\rho=\sin\phi+3\cos\phi$.

Sometimes it is easier to isolate a variable and in those cases we should continue.

Example 4: Translate the rectangular equation $x^2+y^2=z^2,\quad z\geq 0$ into both cylindrical and spherical coordinates.

Solution: Cylindrical - $r^2 = z^2, \quad z \geq 0$ here we can take the square root of both sides of the equation because we know $z \geq 0$ and so $r = z, \quad z \geq 0$, now continuing to spherical $\rho \sin \phi = \rho \cos \phi$ and since ρ is a radius generally though of as being greater than zero we divide both sides by ρ and arrive at $\sin \phi = \cos \phi$ which of course is only true if $\phi = \frac{\pi}{4}$ or $\phi = \frac{5\pi}{4}$ but we have agreed that $0 \leq \phi \leq \pi$ so we are left with just $\phi = \frac{\pi}{4}$

Example 5: Translate the rectangular equation $x^2 + y^2 = 9$ into both cylindrical and spherical coordinates.
Solution: Clearly $r^2 = 9$ and since this is a radius we can safely say $r = 3$ in cylindrical form. In spherical form then $\rho \sin \phi = 3$ or $\rho = \dfrac{3}{\sin \phi} = 3 \csc \phi$.

Finally we need to consider what the graphs of some of these equations might look like.

From Example 2: $x^2 + y^2 + z^2 = 16 - 4z$ is clearly a sphere which has been shifted vertically on the z axis. We could locate the exact center and radius if we completed the square on the z terms, $x^2 + y^2 + (z+2)^2 = 20$ so the sphere has a center at $(0,0,-2)$ and a radius of $\sqrt{20} = 2\sqrt{5}$. You can review the completing of the square process but it will not be required this semester, I just wanted to bring this to closure here.

I wouldn't want to even think of the what the graph of the equation in example 3 looks like.

From Example 4: $x^2 + y^2 = z^2, \quad z \geq 0$. In simplified rectangular form $z = \sqrt{x^2 + y^2}$ or cylindrical $r = z, \quad z \geq 0$ or spherical $\phi = \frac{\pi}{4}$. We will come to recognize the rectangular form but for now consider the cylindrical form. If $r = z, \quad z \geq 0$ we can think of this like $y = x$ where the vertical and horizontal values are always equal and in this case positive. In cylindrical thinking then the radius and the height above the xy plane are always equal and positive. Now, since θ is not involved in the equation it DOES NOT mean that $\theta = 0$, in fact just the opposite, it means that θ values are arbitrary and θ takes on all possible values. So we have the set of points whose distance from the z axis and height above the xy plane are equal for all possible rotational angles around the z axis. In other words a cone. From spherical coordinates it is even easier to see, $\phi = \frac{\pi}{4}$, with both ρ and θ arbitrarily taking on all possible values we have all points all distances from the origin on all possible rotational angles but all on the ray forming an angle of $\phi = \frac{\pi}{4}$ radians or $\phi = 45°$ down from vertical.

From Example 5: $x^2 + y^2 = 9$, or $r = 3$, or $\rho = \dfrac{3}{\sin \phi} = 3 \csc \phi$. In rectangular form we see the equation of a circle of radius 3, which is repeated in cylindrical form as well. But again the fact that z is not part of the equation means z is free to be anything so we have all circles of radius 3 located any distance above or below the xy plane. A right circular cylinder. The spherical form of a cylinder is not the easiest to see.

Different surfaces have simpler equations in different coordinate systems is the moral of this story and we will be revisiting these systems again in later chapters. This is just the beginning of our work with such equations and coordinate systems.

Be sure to practice translations between systems and try to visualize graphs of these equation if possible.

Lecture Notes 12.1: Vector Values Functions

The title of this course is Multivariable Calculus but it could just as easily been name the evolution of function. Up until now we have really only dealt with one type of function, those with domains that subsets of the real numbers and ranges that are also subsets of the real numbers such $y = f(x)$. Last semester we looked at polar functions $r = f(\theta)$ but again the domain and range were subsets of the real numbers. Now we are going to introduce a new type of function known as vector valued functions generally expressed as $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ in the plane and $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ in space. Note the boldface type used to denote vectors in typed text. What makes these functions different is their range as both types have a domain which is a subset of the real numbers, $t \in \mathbb{R}$, however the range is a set of vectors in either the plane or space depending on the dimension of the function. Each of these vectors are thought of as emanating from the origin. That is they have their tail at the origin. The collection of the points located at the head (arrow part) of each vector forms a curve in either the plane or in space. In the case of 3-dimensional vector valued functions we call the resulting curve a space curve.

Domain: The domain of a vector valued function is the intersection of the domains of the individual coefficient functions, $f(t)$, $g(t)$, and if we are in space then $h(t)$ as well.

Example 1: Find the domain of $\mathbf{r}(t) = \sqrt{t+2}\ \mathbf{i} + \frac{4}{t-5}\ \mathbf{j} + (t^2 - 4)\mathbf{k}$. **Solution**: The **i** coefficient has a domain of $[-2, \infty)$, the **j** coefficient has a domain of $(-\infty, 5) \cup (5, \infty)$ and the **k** coefficient has a domain of all reals, $\mathbb{R}$. The intersection of these then would be the set $[-2, 5) \cup (5, \infty)$ since any *t* value selected from this set would result in a defined vector from the range.

We can think of the graph of a vector valued function by thinking of the ordered triple generated by the vectors for fixed values of *t*. That is if we are given a vector valued function we can plot points based on values of *t* until we have an idea of what the curve would look like.

Example 2: Graph the plane curve generated by the vector valued function $\mathbf{r}(t) = t^2\,\mathbf{i} + t\,\mathbf{j}$. **Solution**: Generating some vectors $\mathbf{r}(-2) = \langle 4, -2 \rangle$, $\mathbf{r}(-1) = \langle 1, -1 \rangle$, $\mathbf{r}(0) = \langle 0, 0 \rangle$, $\mathbf{r}(1) = \langle 1, 1 \rangle$, $\mathbf{r}(2) = \langle 4, 2 \rangle$, and $\mathbf{r}(3) = \langle 9, 3 \rangle$. Plotting the end points of the vectors then in the plane generates a parabola opening to the right.

Example 3: Describe the space curve generated by the vector valued function $\mathbf{r}(t) = 3\sin(t)\,\mathbf{i} + 3\cos(t)\,\mathbf{j} + t\,\mathbf{k}$. **Solution**: Recall from parametric equations that $x = 3\sin(t)$, $y = 3\cos(t)$ generates a circle of radius 3 centered at the origin. So the **i** and **j** components are forming a circle centered at the origin but the **k** component is causing the curve to move up or down depending on the sign of *t* so we are going to see a spiral with all points on the vertical right cylinder of radius 3. The curve will wrap around and around this cylinder like a barber's pole. This particular curve is common and known as a helix.

Finally we can define limits for vector valued functions. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ then $\lim_{t \to a}\left(\mathbf{r}(t)\right) = \lim_{t \to a}\left(f(t)\right)\,\mathbf{i} + \lim_{t \to a}\left(g(t)\right)\mathbf{j}$. Limits can be applied to higher dimensional vector valued functions as well simply by applying the limit to each component.

Logically now, if we can apply limits to vector valued functions then we know that waiting on the horizon (next week) we will likely be finding derivatives and integrals of vector valued functions and defining what this calculus gives us.