8Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 8: 7/13-7/19
Sections 14.5 – 14.8, Surface Area, Triple Integrals, Cylindrical and Spherical Integrals, Jacobians.

Due this week:

Tuesday 7/14/2020	**Thursday 7/16/2020** WebAssign 14.3 – 14.6	**Sunday 7/19/2020** Weekly Assignment 8, 14.5 – 14.8

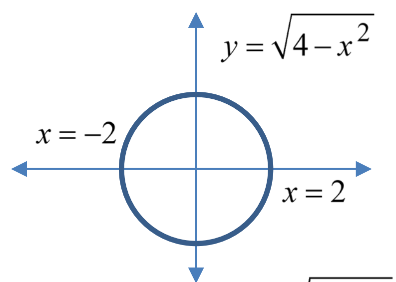
Lecture Notes 14.5: Surface Area

Take just a moment to catch your breath from double integrals and polar coordinates and center of mass. In this section we look at surface area of a surface of the form $z = f(x, y)$ over a region R in the xy plane. This is formula driven.

$$\text{Surface Area} = \iint_R 1\, dS = \iint_R \sqrt{1+\left(f_x(x,y)\right)^2+\left(f_y(x,y)\right)^2}\, dA$$

Example 1: Find the area of the surface given by $f(x,y) = 6+4x+3y$ over the region $R = \left\{(x,y): x^2+y^2 \le 4\right\}$

Solution: The region lies inside the circle centered at the origin of radius 2.
$f_x(x,y) = 4$ and $f_y(x,y) = 3$ so we can build the iterated integral



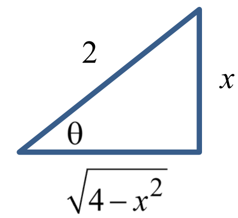
$$\text{Surface Area} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+4^2+3^2}\, dydx$$ (be careful assuming symmetry because while the circle has vertical and horizontal symmetry the function may not share that symmetry. It is generally safer to leave the iterated integral in full form without using symmetry.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+4^2+3^2}\, dydx = \sqrt{26}\int_{-2}^{2} \left[y\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \sqrt{26}\int_{-2}^{2}\left[\sqrt{4-x^2}-\left(-\sqrt{4-x^2}\right)\right] dx$$

$= \sqrt{26}\int_{-2}^{2} 2\sqrt{4-x^2}\, dx$ This is not a terribly pleasant integral, we can evaluate it using a trigonometric substitution with $x = 2\sin(\theta)$, $dx = 2\cos(\theta)\,d\theta$ Putting the limits of integration to the side for now then

$$= \sqrt{26}\int 2\sqrt{4-x^2}\, dx = 2\sqrt{26}\int \sqrt{4-4\sin^2(\theta)}\left(2\cos(\theta)\right)d\theta = 2\sqrt{26}\int \left(2\cos(\theta)\right)\left(2\cos(\theta)\right)d\theta$$

$= 8\sqrt{26}\int \cos^2(\theta)\,d\theta = 8\sqrt{26}\left[\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right] = \sqrt{26}\left[4\theta + 2\left(2\sin(\theta)\cos(\theta)\right)\right]$ and so now we form a right triangle to help us get back to the original variables, from the trig sub: $\sin(\theta) = \frac{x}{2}$.



$$\sqrt{26}\left[4\theta + 2\left(2\sin(\theta)\cos(\theta)\right)\right] = \sqrt{26}\left[4\arcsin\left(\frac{x}{2}\right)+4\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right]$$

And so finally the surface area is $\sqrt{26}\left[4\arcsin\left(\frac{x}{2}\right)+4\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right]_{-2}^{2}$

$$= \sqrt{26}\left[\left(4\arcsin(1)+4(1)(0)\right)-\left(4\arcsin(-1)+4(-1)(0)\right)\right] = \sqrt{26}\left[4\left(\frac{\pi}{2}\right)-4\left(\frac{-\pi}{2}\right)\right] = \sqrt{26}\,(4\pi) = 4\pi\sqrt{26} \approx 64.076$$

Well that was fun but I wonder if it could have been easier. Let's go back to the original iterated integral and try converting to polar coordinates.

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+4^2+3^2}\;dydx = \int_{0}^{2\pi}\int_{0}^{2} \sqrt{26}\;r\,drd\theta = \frac{\sqrt{26}}{2}\int_{0}^{2\pi}\left[r^2\right]_0^2 d\theta = \frac{\sqrt{26}}{2}\int_{0}^{2\pi} 4\,d\theta = \frac{\sqrt{26}}{2}\left[4\theta\right]_0^{2\pi} = 4\pi\sqrt{26}$$

And that was much easier. Keep your eyes open for circular regions that lend themselves so nicely to polar form or integrands involving x^2+y^2 because they also translate and simplify nicely in polar form.

Do not forget the proper conversion of $dydx = r\,drd\theta$ however.

Lecture Notes 14.6: Triple Integrals

To begin, a triple integral is just one more integral than a double. Hold on to that thought. Because we need to evaluate all 3 integrals these can get tedious rather quickly but just as before we work our way out from the inside.

The general form of a triple integral in rectangular coordinates is:

$$\iiint_Q f(x,y,z)\,dV$$

Recall a double integral was evaluated over a region in the *xy* plane generally called *R* and we integrated with respect to area, dA. Here we are integrating over a solid generally called *Q* and because it is a solid we are integrating with respect to volume, *dV*. And as in the case of double integrals when the integrand is just 1 we are finding volume, but when the integrand is a function in 3-space then we are finding what can be called hyper-volume. If the integrand is a density function on the solid then the result of the integration would be a mass. This interpretation is a little easier to swallow than "hyper-volume".

The most common application of a triple integral is volume. Consider the volume we found on the last assignment where the region was bounded above by $z = 9 - x^2 - y^2$ and below by the *xy* plane inside the cylinder $x^2 + y^2 = 1$. As a double integral we use the *z* top – *z* bottom as the integrand. As a triple integral we move these quantities to the upper and lower limits on the inner most integral and then form *x* and *y* bounds based on the circle $x^2 + y^2 = 1$.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{9-x^2-y^2} 1\ dzdydx$$

The key idea is to NOT try to do too much in 3-space. Find the upper bound and the lower bound, usually of *z,* as functions and then smash the 3 dimensional solid into its image in the *xy* plane and set up *x* and *y* limits from there.

Example 1: Evaluate $\displaystyle\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln(z)\,dydzdx$

Solution: Work from the inside out, the first variable of integration is *y* so $\ln(z)$ is treated as a constant.

$$\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln(z)\,dydzdx = \int_1^4\int_1^{e^2} \left[y\ln(z)\right]_0^{1/xz} dzdx = \int_1^4\int_1^{e^2} \left[\frac{1}{xz}\ln(z)-0\right]dzdx$$ for the next integration the variable is *z* and so in order to integrate $\ln(z)$ we must use a *u* substitution.

$\displaystyle\int \frac{1}{xz}\ln(z)\,dz$ let $u=\ln(z)$ then $du=\frac{1}{z}dz$. Remember for this integration the *x* is treated as a constant and so in *u* form $\displaystyle\int \frac{1}{x}u\ du = \frac{1}{x}\frac{u^2}{2}$ and going back to the original variables we have

$$\int_1^4\int_1^{e^2}\left[\frac{1}{xz}\ln(z)-0\right]dzdx=\int_1^4\left[\frac{(\ln(z))^2}{2x}\right]_1^{e^2}dx=\int_1^4\left[\frac{\left(\ln\left(e^2\right)\right)^2}{2x}-\frac{\left(\ln(1)\right)^2}{2x}\right]dx=\int_1^4\left[\frac{(2)^2}{2x}-\frac{(0)^2}{2x}\right]dx$$

$=\int_1^4 \frac{2}{x}\,dx=\left[2\ln|x|\right]_1^4=2(\ln 4-\ln 1)=2\ln 4$ or $4\ln 2$ or $\ln 16$. Any of these would be correct but do not make this into a decimal, we are stressing exact solutions and anything the calculator gives would only be an approximation.

Example 2: Use a triple integral to find the volume of the solid in the first octant bounded above by the plane $z=x+y+2$ and below by the *xy* plane over the region in the *xy* plane bounded by the graphs of $y=x^2$ and $y=9$

Solution: Identify the upper and lower boundaries as functions of *z*. $z=x+y+2$ and $z=0$. The most common order of integration places *z* on the innermost integral and these form the limits of integration. Remember the only variables that can appear as part of the limits of integration are ones which will be integrated later in the process. Thus we can use *x* and *y* here but not *z*. Now consider the region in the plane, $y=x^2$ and $y=9$. We will not have to do anything with these equations if we integrate with respect to *y* next and we finish with the limits of *x* which come from equating $y=x^2$ and $y=9$. Remember we are only in the first octant where *x, y,* and *z* are all greater than or equal to zero.

$V=\int_0^3\int_{x^2}^9\int_0^{x+y+2} 1\,dzdydx$ Now we can evaluate the triple $V=\int_0^3\int_{x^2}^9\left[\,z\,\right]_0^{x+y+2}dydx=\int_0^3\int_{x^2}^9 x+y+2\,dydx$

$$V=\int_0^3\left[xy+\frac{y^2}{2}+2y\right]_{x^2}^9 dx=\int_0^3\left[\left(9x+\frac{81}{2}+18\right)-\left(x^3+\frac{x^4}{2}+2x^2\right)\right]dx=\int_0^3\left[\left(9x-x^3-\frac{x^4}{2}-2x^2+\frac{117}{2}\right)\right]dx$$

$$V=\left[\frac{9x^2}{2}-\frac{x^4}{4}-\frac{x^5}{10}-\frac{2x^3}{3}+\frac{117x}{2}\right]_0^3=\frac{81}{2}-\frac{81}{4}-\frac{243}{10}-\frac{54}{3}+\frac{351}{2}=\frac{3069}{20}$$

We can calculate the mass and center of mass for 3-dimensional solids in much the same we that we did for laminas (thin layers) in the previous section. Of course in 3-dimensions the center of mass will have 3 coordinates given by $(\overline{x}, \overline{y}, \overline{z})$ and we will have 3 first moments. In 2-dimensions we measured moments about a coordinate axis, in 3-dimensions we measure moments about a coordinate plane. The following set of formulas are all we need, along with our visualization and integration skills, to find mass, moments and centers of mass.

Given a solid region Q whose density is given by the density function $\delta(x, y, z)$. Many books including ours use the Greek letter ρ (rho) for the density function but I am hoping to avoid any confusion with the upcoming section.

Mass $= M = \iiint\limits_Q \delta(x, y, z)\, dV$

The first moment about the yz plane is $M_{yz} = \iiint\limits_Q x\,\delta(x, y, z)\, dV$

The first moment about the xz plane is $M_{xz} = \iiint\limits_Q y\,\delta(x, y, z)\, dV$

The first moment about the xy plane is $M_{xy} = \iiint\limits_Q z\,\delta(x, y, z)\, dV$

Center of Mass: $\overline{x} = \dfrac{M_{yz}}{M}, \quad \overline{y} = \dfrac{M_{xz}}{M}, \quad \overline{z} = \dfrac{M_{xy}}{M}$

Moments of Inertia are measured about the coordinate axes as follows:

The moment of inertia about the x axis is $I_x = \iiint\limits_Q \left(y^2 + z^2\right)\delta(x, y, z)\, dV$

The moment of inertia about the y axis is $I_y = \iiint\limits_Q \left(x^2 + z^2\right)\delta(x, y, z)\, dV$

The moment of inertia about the z axis is $I_z = \iiint\limits_Q \left(x^2 + y^2\right)\delta(x, y, z)\, dV$

Lecture Notes 14.7: Triple Integrals In Cylindrical and Spherical Coordinates

Recall the conversion formulas we had when we visited these coordinate systems earlier this term.

Cylindrical Coordinates	Spherical Coordinates
$x = r\cos(\theta)$	$x = \rho\sin(\phi)\cos(\theta)$
$y = r\sin(\theta)$	$y = \rho\sin(\phi)\sin(\theta)$
$z = z$	$z = \rho\cos(\phi)$
$x^2 + y^2 = r^2$	$x^2 + y^2 + z^2 = \rho^2$
$\tan(\theta) = \frac{y}{x}$	$\tan(\phi) = \frac{r}{z}$
	$r = \rho\sin(\phi)$

Cylindrical Coordinates: Triple integrals in cylindrical coordinates are not much of a stress. As cylindrical coordinates are polar coordinates in the *xy* plane with a rectangular (straight) measure for *z*. When we look at a triple integral in cylindrical form we see the same "package" we saw in polar in terms of the conversion of the differentials $dV = dzdydx = r\ dzdrd\theta$. A cylindrical triple is generally in *z, r,* θ order.

$$\int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r,\theta,z)\ r\ dzdrd\theta$$

Spherical Coordinates: Triple integrals in spherical coordinates are wonderful. First say this out loud, rho squared sine phi d rho d phi d theta. I feel like I pick up a few IQ points every time I say that. That is the spherical "package" that is always part of the *dV* differential. It isn't hard to see where it comes from if you relate it to the polar $dA = rdrd\theta$. A rectangular measurement of area in polar form becomes a radius times the product of the differentials. In cylindrical integrals we already have $rdzdrd\theta$ and when we move to express *z* in a polar form it is reasonable we need another radius, ρ, so we think of radius relative to the origin times radius relative to the *z* axis times the product of the differentials or $\rho\, r\, d\rho\, d\phi\, d\theta$ except *r* is not a variable in spherical coordinates. However we recall that $r = \rho\sin(\phi)$ so the *dV* becomes

$r\, dzdrd\theta = \rho\ r\ d\rho d\phi d\theta = \rho\rho\sin(\phi)\, d\rho d\phi d\theta = \rho^2\sin(\phi) d\rho d\phi d\theta$.

$$\int_{\theta_1}^{\theta_2} \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho,\theta,\phi)\ \rho^2\sin(\phi)\ d\rho d\phi d\theta$$

We will also need to brush up on our visualization skills in both cylindrical and spherical coordinates.

Example 1: Express the mass of a solid with density function $f(x,y,z) = x^2 + y^2$ located in the first octant bounded above by the paraboloid $z = 9 - x^2 - y^2$ in rectangular, cylindrical and spherical coordinates and evaluate the most convenient form.

Solution: Recall the in the first octant *x,y,* and *z* are greater than or equal to zero. So our region is that section of the downward opening paraboloid located in the first octant. Also recall that to find the mass of a solid we need a triple integral with an integrand function, $f(x,y,z) = x^2 + y^2$, in this case.

Finally we know $0 \le z \le 9 - x^2 - y^2$ has an image in the *xy* plane of the circle $x^2 + y^2 = 9$ and the first octant corresponds to the first quadrant in the plane.

Thus we have $M = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \left(x^2 + y^2\right) dzdydx$ in Rectangular Coordinates.

Then in Cylindrical form $M = \int_0^{\pi/2} \int_0^3 \int_0^{9-r^2} \left(r^2\right) r\, dzdrd\theta$ since $z = 9 - x^2 - y^2$ translates to $z = 9 - r^2$ in cylindrical form and the quarter circle of radius 3 in the first quadrant is described by $0 \le r \le 3,\ 0 \le \theta \le \dfrac{\pi}{2}$.

To construct the Spherical triple integral for the mass we need to apply a direct translation to the integrand, $r^2 = \left(\rho \sin(\phi)\right)^2 = \rho^2 \sin^2(\phi)$. The solid needs to be described as a distance from the origin or $0 \le \rho \le$ (the paraboloid) so we need to find the spherical form of $z = 9 - r^2$. Substituting the known conversions for *z* and *r* we arrive at the equation $\rho \cos(\phi) = 9 - \rho^2 \sin^2(\phi)$. Unfortunately we must solve this equation for ρ. Moving all terms to one side we have a quadratic equation with non-constant coefficients and we can use the quadratic formula to solve for ρ in terms of ϕ. $\rho^2 \sin^2(\phi) + \rho \cos(\phi) - 9 = 0$ so

$$\rho = \frac{-\cos(\phi) \pm \sqrt{\cos^2(\phi) - 4\sin^2(\phi)(-9)}}{2\sin(\phi)} = \frac{-\cos(\phi) \pm \sqrt{\cos^2(\phi) + 36\sin^2(\phi)}}{2\sin(\phi)} = \frac{-\cos(\phi) \pm \sqrt{1 + 35\sin^2(\phi)}}{2\sin(\phi)}$$

Now we only want to consider non-negative values for the radius ρ and clearly $\sqrt{1 + 35\sin^2(\phi)} \ge 1$ so we can safely reject the negative case of the quadratic formula so $\rho = \dfrac{-\cos(\phi) + \sqrt{1 + 35\sin^2(\phi)}}{2\sin(\phi)}$. Therefore in spherical coordinates the mass of this solid is given by $M = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\frac{-\cos(\phi)+\sqrt{1+35\sin^2(\phi)}}{2\sin(\phi)}} \left(\rho^2 \sin^2(\phi)\right) \rho^2 \sin(\phi)\, d\rho d\phi d\theta$

or $M = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\frac{-\cos(\phi)+\sqrt{1+35\sin^2(\phi)}}{2\sin(\phi)}} \left(\rho^4 \sin^3(\phi)\right) d\rho d\phi d\theta$. I know the very ugly upper limit on ρ has probably grabbed everyone's attention but don't fail to notice the integration limits for ϕ. Because we are only in the first octant and $\phi = 0$ is the positive z axis and ϕ measures the rotation down from vertical we have the first octant being represented by $0 \le \phi \le \frac{\pi}{2}$.

Alright, time to evaluate the least offensive of these triple integrals. Rectangular version, $M = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \left(x^2+y^2\right) dzdydx$ or cylindrical version, $M = \int_0^{\pi/2} \int_0^3 \int_0^{9-r^2} \left(r^3\right) dzdrd\theta$ or spherical version $M = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\frac{-\cos(\phi)+\sqrt{1+35\sin^2(\phi)}}{2\sin(\phi)}} \left(\rho^4 \sin^3(\phi)\right) d\rho d\phi d\theta$. Usually over half the class will prefer rectangular just because the variable are more comfortable but for sheer ease of integration you can't beat cylindrical in this case.

$$M = \int_0^{\pi/2} \int_0^3 \int_0^{9-r^2} \left(r^3\right) dzdrd\theta = \int_0^{\pi/2} \int_0^3 \left(r^3 z\right)\Big|_0^{9-r^2} drd\theta = \int_0^{\pi/2} \int_0^3 r^3\left(9-r^2\right) drd\theta = \int_0^{\pi/2} \int_0^3 \left(9r^3 - r^5\right) drd\theta$$

$$= \int_0^{\pi/2} \left(\frac{9r^4}{4} - \frac{r^6}{6}\right)\Bigg|_0^3 d\theta = \int_0^{\pi/2} \frac{243}{4}\, d\theta = \left(\frac{243}{4}\theta\right)_0^{\pi/2} = \frac{243\pi}{8}$$ for the mass of the solid.

Example 2: Use a spherical triple integral to find the volume of the solid inside the sphere $x^2+y^2+z^2 = 25$ and above (inside) the cone $z = \sqrt{x^2+y^2}$.

Solution: Express the sphere in spherical coordinates, $\rho = 5$. Inside the sphere indicates $0 \le \rho \le 5$. Express the cone in spherical coordinates, $z = \sqrt{r^2} = r$ so $\rho\cos(\phi) = \rho\sin(\phi)$ or $\cos(\phi) = \sin(\phi)$ so $\phi = \frac{\pi}{4}$, we will be above the cone so $0 \le \phi \le \frac{\pi}{4}$. We are not limited by octants so $0 \le \theta \le 2\pi$. Now just don't forget the spherical differential package and we are set.

$$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^5 1\,\rho^2 \sin(\phi)\, d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/4} \left(\frac{\rho^3}{3}\sin(\phi)\right)_0^5 d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/4} \left(\frac{125}{3}\sin(\phi)\right) d\phi d\theta = \int_0^{2\pi} \left(-\frac{125}{3}\cos(\phi)\right)_0^{\pi/4} d\theta$$

$$= \int_0^{2\pi} \left(-\frac{125}{3}\left(\cos\left(\frac{\pi}{4}\right) - \cos(0)\right)\right) d\theta = \int_0^{2\pi} \left(-\frac{125}{3}\left(\frac{\sqrt{2}}{2} - 1\right)\right) d\theta = \frac{125}{3}\left(1 - \frac{\sqrt{2}}{2}\right) \int_0^{2\pi} (1)\, d\theta$$

$$= \frac{125}{3}\left(1 - \frac{\sqrt{2}}{2}\right)(2\pi) = \frac{125\left(2-\sqrt{2}\right)\pi}{3}$$

Lecture Notes 14.8: Jacobians

The topic of Jacobians is an optional topic and I think we would be better served spending more time on multiple integrals without worrying about section 14.8.