Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 7: 7/6-7/12
Sections 14.1 – 14.4, Iterated Integrals, Double Integrals, Polar Coordinates, Center of Mass.

Due this week:

Tuesday 7/7/2020	**Thursday 7/9/2020**	**Sunday 7/12/2020**
Chapter 13 Assessment	WebAssign 13.10 – 14.2	Weekly Assignment 7, 14.1 – 14.4

Lecture Notes 14.1: Iterated Integrals

Please review your calculus II notes, books, and tests on integration by substitution and integral forms. I will include a table of integrals that we should all know as an extra page.

We are taking our study of multivariable functions to the next level now. While we have gotten rather good at partial derivatives it is important to realize there is no such thing as a partial integral. Integration must always be done with a single variable at a time. However we will be borrowing the strategy we used with partial derivatives of treating a variable as a constant if it is not **the** variable of integration.

The following are what are known as iterated integrals and we evaluate them in much the same way we do most mathematics, from the inside out, one integral at a time.

Example 1: Evaluate $\int_1^5 \int_1^3 (2x+4y)\ dydx$

Solution: We see that y is the variable of integration for the inner integral so we treat x as a constant and integrate and evaluate using the Fundamental Theorem of Calculus never losing the outer integral symbol or its differential. **PLEAE PAY ATTENTION TO NOTATION AND KEEP ALL INTEGRAL SYMBOLS IN THE PROCESS UNTIL THAT INTEGRAL IS EVALUATED.**

$$\int_1^5 \int_1^3 (2x+4y)\ dydx = \int_1^5 \left[2xy + \frac{4y^2}{2} \right]_1^3 dx \quad \text{Treating 2x as a constant.}$$

$$= \int_1^5 \left[2xy + \frac{4y^2}{2} \right]_0^3 dx = \int_1^5 \left[2xy + 2y^2 \right]_1^3 dx = \int_1^5 \left[2x(3) + 2(3)^2 \right] - \left[2x(1) + 2(1)^2 \right] dx =$$

$$= \int_1^5 \left(6x + 18 - (2x+2) \right) dx = \int_1^5 \left(4x + 16 \right) dx = 2x^2 + 16x \Big|_1^5 = \left(2(25) + 16(5) \right) - (2+16) = 112$$

Example 2: Evaluate $\int_0^4 \int_0^1 (x^2 + 4xy)\ dxdy$

Solution: This time the inner variable of integration is x.

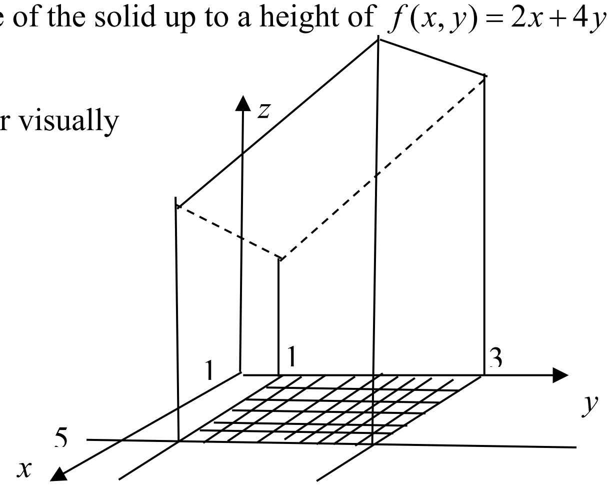
$$\int_0^4 \int_0^1 (x^2 + 4xy)\ dxdy = \int_0^4 \left(\frac{x^3}{3} + 2x^2 y \right)\Bigg|_0^1\ dy = \int_0^4 \left(\frac{1}{3} + 2y - 0 \right)\ dy$$

$$= \int_0^4 \left(\frac{1}{3} + 2y \right)\ dy = \left(\frac{y}{3} + y^2 \right)\Bigg|_0^4 = \left(\frac{4}{3} + 16 \right) - (0) = \frac{52}{3}$$

Tedious and time consuming but these are not too challenging. Everyone must watch their notation and be careful and neat. Not as much for me but for yourselves. **Pay particular attention to the order of integration**.

But what did we find in these integrals? From calculus II we know $\int_a^b f(x)\ dx$ is the signed area between the graph of $f(x)$ and the x axis on the interval from a to b. And from this semester we know that $z = f(x, y)$ is function which takes each point in the xy plane and maps it to a z value in three dimensional space. So it should not be too much of a stretch to see an iterated integral as the signed volume above a rectangle or other closed region in the xy plane up to a height of $z = f(x, y)$.

In this way then $\int_0^5 \int_1^3 (2x + 4y)\ dydx$ represents the volume of the solid up to a height of $f(x, y) = 2x + 4y$ over the rectangular domain $\left\{ (x, y) \,\middle|\, 1 \le y \le 3,\ 0 \le x \le 5 \right\}$ or visually

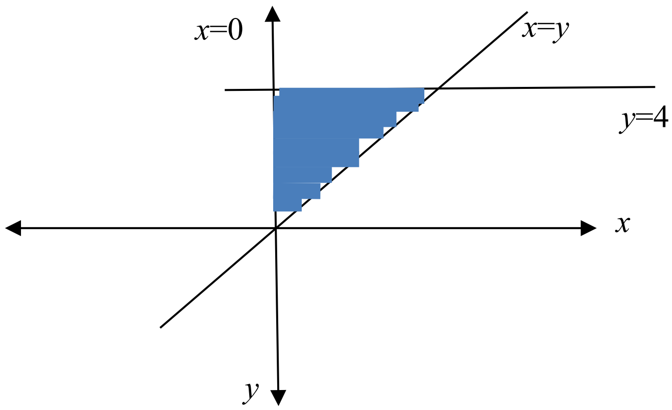


Then what about an iterated integral like $\int_0^4 \int_0^y (4xy)\ dxdy$ Limits of integration need not be constants.

The outer limits must always be constants and the inner limits can only have variables that will be the variable of integration later in the process.

$$\int_0^4 \int_0^y (4xy)\ dxdy = \int_0^4 2x^2y\Big|_0^y\ dx = \int_0^4 \left(2y^2y-0\right)\ dy = \int_0^4 2y^3\ dy = \frac{2y^4}{4}\Big|_0^4 = \frac{2(4)^4}{4} - 0 = 128$$

This iterated integral can be interpreted in exactly the same fashion, only the region in the xy plane is not a rectangle.

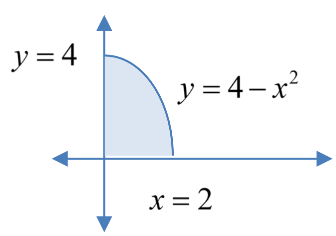


There is not much ambiguity in this region but note that because the x limits go from $x=0$, the lower limit, to $x=y$, the upper limit, that the region is bounded on the left by $x=0$ and on the right by $x=y$. This attention to left, right, lower, and upper will become critical to what we are doing as the regions become more complicated and move to higher dimensions. Visually we are generally more interested in the shape of the region in the domain space of the integral than we are in surface. It is the region in the domain that we need to form the limits of integration.

An iterated integral with an integrand of 1 gives the area of the region in the xy plane described by the limits of integration.

Example 3: Find the area of the region in the first quadrant bounded by the graph of $y=4-x^2$ using an iterated integral.

Solution: Sketch the region and set up the limits of integration. We see the easiest description of the region would be to let $0 \le y \le 4-x^2$ as $0 \le x \le 2$ then the area of the region would be



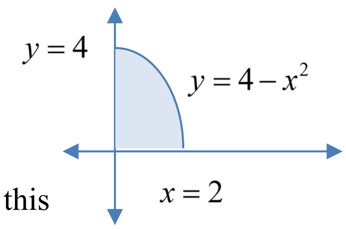
given by $A = \int_0^2 \int_0^{4-x^2} 1\,dydx = \int_0^2 [y]_0^{4-x^2}\ dx = \int_0^2 \left(4-x^2\right)-(0)\,dx$

$$= \left(4x - \frac{x^3}{3}\right)_0^2 = \left(8-\frac{8}{3}\right) - 0 = \frac{16}{3}$$

We can also use the sketch of the region to help us change the order of integration. In example 3 above it was very natural to set up the integral in *dydx* order because the boundary was given as $y = f(x)$. In general we don't have to fight the problem when we use the dependent variable from the given function as the first variable of integration but sometimes we may need switch that around.

Example 4: Find the area of the region in the first quadrant bounded by the graph of $y = 4 - x^2$ using an iterated integral in *dxdy* order.

Solution: Sketch the region and set up the limits of integration.



This time we are told to use *x* as the inner or first variable. To do this we will need to solve the equation $y = 4 - x^2$ for *x*.

Clearly $x = \pm\sqrt{4-y}$ but now, since we are only in the first quadrant we can eliminate the negative possibility and have $x = \sqrt{4-y}$. Because this equation bounds the right side of the region and $x = 0$ bounds the left side we would set up the iterated integral as $A = \int_0^4 \int_0^{\sqrt{4-y}} 1\,dxdy$. Notice the outer limits are 0 and 4 because these are the extreme values of the *y* variable on this region. The order of integration if changed correctly should never alter the value of the integral. $A = \int_0^4 \int_0^{\sqrt{4-y}} 1\,dxdy = \int_0^4 x\Big|_0^{\sqrt{4-y}} dy = \int_0^4 \left(\sqrt{4-y} - 0\right) dy$

$$A = \left[-\frac{2}{3}(4-y)^{3/2} \right]_0^4 = \frac{-2}{3}\left[(0) - (4)^{3/2} \right] = \frac{-2}{3}(-8) = \frac{16}{3}$$ which is exactly the same area we found in example 3.

When changing the order of integration or setting up the integral in a specified order we must make sure that the same pair of functions bound the left and right sides of the region for *x* first order or the top and bottom of the region for a *y* first order. If there is a change in bounding function then we may need to use two or more iterated integrals to cover the region.

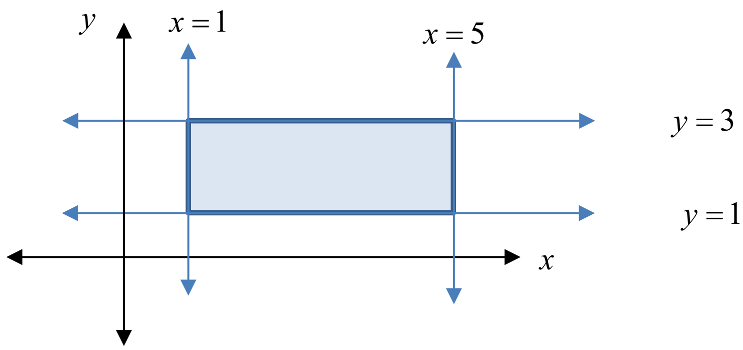
Lecture Notes 14.2: Double Integrals

Double Integrals are more of a conceptual statement than an iterated integral.

$\iint_R (2x+4y)\,dA$, read as the double integral over the region *R*, is a double integral that represents the integral over a region *R* up a height of $f(x,y)=2x+4y$. When *R* is defined as the region in the xy plane given by $\{ (x,y) \mid 1 \le y \le 3, 1 \le x \le 5 \}$ then the double integral can be rewritten as an iterated integral and evaluated as before.

Example 1: Evaluate $\iint_R (2x+4y)\,dA$ where *R* is the region in the xy plane $\{(x,y) \mid 1 \le x \le 5,\ 1 \le y \le 3\}$

Solution: Sketching the region in the plane we can set up the iterated integral from the rectangle we see.



$$\iint_R (2x+4y)\,dA = \int_1^5 \int_1^3 (2x+4y)\ dydx = \int_1^5 \left(2xy+2y^2\right)\Big|_1^3 dx = \int_1^5 \left(2xy+2y^2\right)\Big|_1^3 dx$$

$$= \int_1^5 \left[(6x+18)-(2x+2)\right]dx = \int_1^5 (4x+16)\,dx = \left[2x^2+16x\right]_1^5 = (50+80)-(2+16) = 112$$

Essentially a double integral with a domain description puts the pressure on us to set up the iterated integral to evaluate. In many ways this is like setting up the area and volume integrals we saw in calculus II where we had to determine the integral from the description.

It is important to note here that we can set up iterated integrals in either a $dydx$ or $dxdy$ order depending on the region and the integrand. We can also take an iterated integral and change the order of integration to make the computations easier.

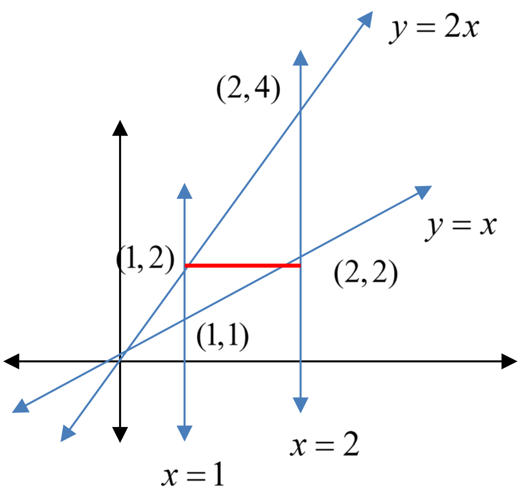
Additional conceptual ideas: $\iint_R f(x,y)\ dA$ means we are integrating a height, $z=f(x,y)$, with respect to area, dA. From geometry we know that the area of the base times the height give volume. So $\iint_R f(x,y)\ dA$

represents volume, which we have said before. But also note that if the height is a constant value of one then the volume and the area are numerically equivalent which we also mentioned in the previous section. So $\iint_R 1\ dA$ is numerically equivalent to the area of region R. Thus we can use a double integral to find the area of a region in the plane.

All of the basic properties of definite integrals that we learned in chapter five are still valid. Check the table of properties in section 14.2 of the text book (page 980 in the seventh edition)

Example 2: . Set up iterated integrals in both orders of integration to find the area of region R which is the trapezoid bounded by the graphs of $y = x$, $y = 2x$, $x = 1$, and. $x = 2$

Solution: I like to sketch the region for visualization. This is not a required step but I strongly recommend it. It is also useful to find the coordinates of the intersections, here we can get these rather easily.



The easier order to set up would have y first because of the equations $y = x$ and $y = 2x$ and the x variable bounds are constant. $A = \int_1^2 \int_x^{2x} 1\,dydx$

When looking at integrating x first we need to see that the left boundary of the region changes from $x = 1$ to the line $y = 2x$ at the point $(1,2)$ and that the right boundary changes from $x = 2$ to the line $y = x$ at the point $(2,2)$. We are fortunate that both exchanges occur at the same y value. If the line connecting the exchange points was not horizontal then we would need to use 3 iterated integrals since there would be 3 sub-regions.

Also to integrate x first we need to change the form of the equations of the lines through the origin so that x is the dependent variable. $y = x$ becomes $x = y$ (trivial) and $y = 2x$ becomes $x = \frac{y}{2}$ (harder but still easy).

Putting all this together for the x first order then we get the following sum of iterated integrals:

$A = \int_1^2 \int_1^y 1\,dxdy + \int_2^4 \int_{\frac{y}{2}}^2 1\,dxdy$. Clearly then the first area integral would be the easier to evaluate so let's do that one first. $A = \int_1^2 \int_x^{2x} 1\,dydx = \int_1^2 (y)\Big|_x^{2x} dx = \int_1^2 (2x - x)\,dx = \int_1^2 x\,dx = \left(\frac{x^2}{2}\right)\Bigg|_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$.

Now let's try the second set -up. Remember the area should be the same regardless.

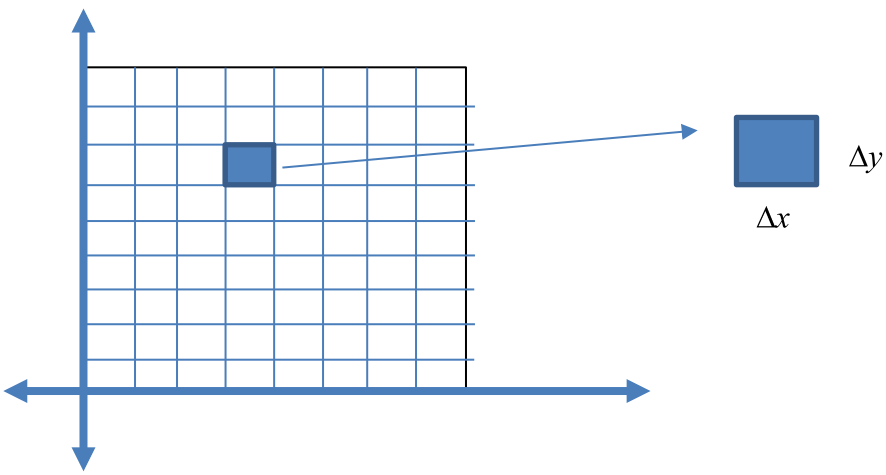
$$A=\int_1^2\int_1^y 1\,dxdy+\int_2^4\int_{\frac{y}{2}}^2 1\,dxdy=\int_1^2 \left(x\right)\Big|_1^y\,dy+\int_2^4 \left(x\right)\Big|_{y/2}^2\,dy=\int_1^2 \left(y-1\right)dy+\int_2^4\left(2-\frac{y}{2}\right)dy$$

$$A=\left(\frac{y^2}{2}-y\right)\Bigg|_1^2+\left(2y-\frac{y^2}{4}\right)\Bigg|_2^4=\left[\left(\frac{4}{2}-2\right)-\left(\frac{1}{2}-1\right)\right]+\left[\left(8-\frac{16}{4}\right)-\left(4-\frac{4}{4}\right)\right]=\frac{1}{2}+1=\frac{3}{2}$$

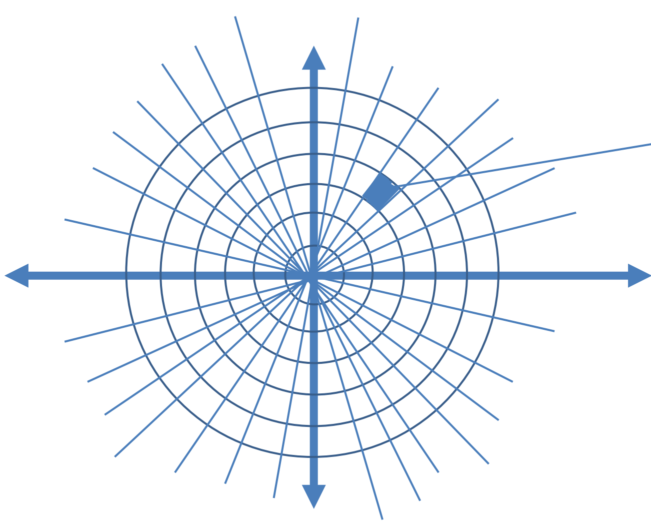
This second order was definitely more difficult to set up and to evaluate but the area was the same.

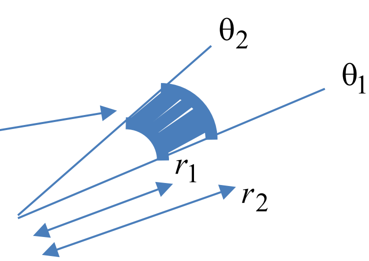
Lecture Notes 14.3: Double Integrals in Polar Coordinates

This week we are extending our work with double integrals to the situation where the *xy* plane is being described through the use of polar coordinates but the *z* dimension remains a subset of the reals. So rather than $z = f(x, y)$ we will have $z = f(r, \theta)$. When we look at a double integral in rectangular coordinates we see $\iint\limits_R f(x,y)\,dA = \int\limits_{x=a}^{x=b} \int\limits_{y=g(x)}^{y=h(x)} f(x,y)\,dydx$ and one of the more subtle aspects of this double integral is the notion that dA corresponds to $dydx$ and that in rectangular coordinates $dydx$ is the limiting form of $\Delta y \Delta x$ which represents the area of an infinitely small rectangle within the domain of the integral. Think back to Riemann sums in calculus I with the $\Delta x \to 0$. Now with a 2-dimensional domain we have the area product $\Delta y \Delta x$ as $\Delta x \to 0$ and $\Delta y \to 0$ and this product evolves into $dydx$ which represents dA.



Now the question becomes what happens when we replace the rectangular coordinates in the domain plane by polar coordinates?





Recall the area of a sector of a circle of radius r subtended by a central angle θ is $A = \frac{1}{2}\theta r^2$. In this diagram the central angle $\theta = \theta_2 - \theta_1$ or the difference between the rotational angles. We are only interested in the space between the arcs of the circle so we would want to find the area of the larger sector and subtract the area of the smaller sector. $A = \frac{1}{2}(\theta_2 - \theta_1)(r_2)^2 - \frac{1}{2}(\theta_2 - \theta_1)(r_1)^2$ now we can rename $\theta_2 - \theta_1 = \Delta\theta$ so $A = \frac{1}{2}\Delta\theta(r_2)^2 - \frac{1}{2}\Delta\theta(r_1)^2$ and then factor out common terms $A = \frac{1}{2}\left((r_2)^2 - (r_1)^2\right)\Delta\theta$. Now we can factor the difference of squares $A = \frac{1}{2}(r_2 + r_1)(r_2 - r_1)\Delta\theta$.

Renaming $\Delta r = \left(r_2 - r_1\right)$ and rewriting we have $A = \frac{1}{2}\left(r_2 + r_1\right)\Delta r \Delta\theta$. Finally as the difference between radii and the difference between rotational angles approaches zero in a Riemann sum approach we note that $r_1 \to r_2$ and so $\left(r_2 + r_1\right) \to 2r$ and thus $A \to \frac{1}{2}(2r)\Delta r \Delta\theta$ and $dA = r\,\Delta r \Delta\theta$.

The bottom line is simply $\iint\limits_R f(x,y)\,dA = \int\limits_{x=a}^{x=b}\int\limits_{y=g(x)}^{y=h(x)} f(x,y)\,dydx = \int\limits_{\theta=a}^{\theta=b}\int\limits_{r=g(\theta)}^{r=h(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,drd\theta$

Every time we convert double integral to polar coordinates we MUST remember that $dA = dydx = r\,drd\theta$ with that extra *r* from the differential.

Example 1: Use a double integral in polar coordinates to find the area of a circle of radius 5.
Solution: We know the area is $\pi r^2 = 25\pi$ but that is not the real goal, can we find this using a double integral?

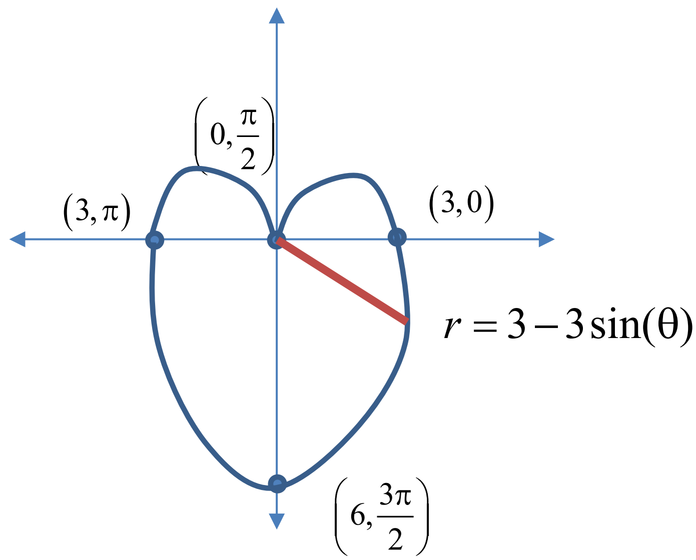
$Area = \iint\limits_R 1\,dA$ and in polar coordinates then this would be $A = \int\limits_{\min\theta}^{\max\theta}\int\limits_{\min r}^{\max r} 1r\,drd\theta$ clearly I hope, the minimal radius inside a circle of radius 5 is 0 and the maximum is 5, $0 \le r \le 5$ as theta wraps around the circle $0 \le \theta \le 2\pi$. So $A = \int\limits_0^{2\pi}\int\limits_0^5 1r\,drd\theta = \int\limits_0^{2\pi}\left.\left(\frac{r^2}{2}\right)\right|_0^5 d\theta = \int\limits_0^{2\pi}\left(\frac{25}{2} - 0\right)d\theta = \int\limits_0^{2\pi}\left(\frac{25}{2}\right)d\theta = \left.\left(\frac{25}{2}\theta\right)\right|_0^{2\pi} = 25\pi$

Note, many polar curves have some sort of symmetry that we can take advantage of to simplify our work. In the above example since each quadrant of a circle is exactly the same we could have set the theta limits to be $0 \le \theta \le \frac{\pi}{2}$ and then multiply the final result by 4. When using symmetry we must be careful to make sure the theta values are in order. Consider the next example.

Example 2: Use a double integral in polar coordinates to find the area inside the cardioid $r = 3 - 3\sin(\theta)$.
Solution: Recall how to graph cardioids, limacons, and dimpled limacons from the end of calculus II. Generally the only points you would need for any of these are the *r* values on the coordinate axes. In this case we can complete the table below

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$r = 3 - 3\sin(\theta)$	$r = 3 - 0 = 3$	$r = 3 - 3 = 0$	$r = 3 - 0 = 3$	$r = 3 - (-3) = 6$	$r = 3 - 0 = 3$

Now plot these points and connect them to complete the graph. Label the exact coordinates of at least 4 points (this will be required). Don't laugh at my artistic non-skill.



The area integral then would be $A = \int_{\min\theta}^{\max\theta} \int_{\min r}^{\max r} 1 r \, dr d\theta = \int_0^{2\pi} \int_0^{3-3\sin(\theta)} 1 r \, dr d\theta$ now we can use symmetry to reduce the work but we need to be careful. The left and right sides of the cardioid are symmetric.

$$A = \int_0^{2\pi} \int_0^{3-3\sin(\theta)} 1 r \, dr d\theta = 2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{3-3\sin(\theta)} 1 r \, dr d\theta \text{ or } 2 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^{3-3\sin(\theta)} 1 r \, dr d\theta \text{ or } 2 \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \int_0^{3-3\sin(\theta)} 1 r \, dr d\theta$$

or any other paring of theta values that cover the rotation from the straight up to or from straight down but the value of the upper limit of integration must be greater than the value of the lower limit.

$$A = \int_0^{2\pi} \int_0^{3-3\sin(\theta)} 1 r \, dr d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} \right) \Bigg|_0^{3-3\sin(\theta)} d\theta = \frac{1}{2} \int_0^{2\pi} \left[\left(3 - 3\sin(\theta)\right)^2 \right] d\theta$$

$$A = \frac{1}{2} \int_0^{2\pi} \left[9 - 18\sin(\theta) + 9\sin^2(\theta) \right] d\theta = \frac{9}{2} \int_0^{2\pi} \left[1 - 2\sin(\theta) + \sin^2(\theta) \right] d\theta = \frac{9}{2} \left[\theta + 2\cos(\theta) + \frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right]_0^{2\pi}$$

$$A = \frac{9}{2} \left[2\cos(\theta) + \frac{3\theta}{2} - \frac{\sin(2\theta)}{4} \right]_0^{2\pi} = \frac{9}{2} \left[\left(2\cos(2\pi) + \frac{3(2\pi)}{2} - \frac{\sin(4\pi)}{4} \right) - \left(2\cos(0) + \frac{0}{2} - \frac{\sin(0)}{4} \right) \right]$$

$$A = \frac{9}{2} \left[\left(2 + 3\pi - 0\right) - \left(2 + 0 - 0\right) \right] = \frac{27\pi}{2}$$

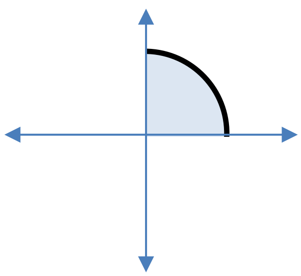
I made use of a common integration formula for polar integrals which everyone is allowed to look up and use.

$\int \sin^2(u) \, du = \frac{u}{2} - \frac{\sin(2u)}{4}$ and $\int \cos^2(u) \, du = \frac{u}{2} + \frac{\sin(2u)}{4}$ no point reinventing this every time it comes up.

Sometimes we will need to translate an integral from rectangular coordinates to polar coordinates in order to evaluate the integral. Consider the following example.

Example 3: Convert to polar coordinates and evaluate iterated integral $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\, dydx$.

Solution: Fortunately we have no need of the 3-dimensional graph but we will benefit from understanding and graphing the region in the *xy* plane described by the limits of integration. The position and order of integration tells us $0 \le y \le \sqrt{4-x^2}$ and $0 \le x \le 2$. Pay attention to limits of this form as they will be very common moving forward, particularly the graph of $y = \sqrt{4-x^2}$. Think about squaring both sides of the equation and moving all the variables to one side, $y^2 = 4 - x^2$ so $x^2 + y^2 = 4$ which we all have to recognize as a circle centered at the origin of radius 2. Now because it started out as $0 \le y \le \sqrt{4-x^2}$ this describes all of the points inside the top half of the circle only, and then since $0 \le x \le 2$ we would only have the first quadrant points inside the circle.



From the graph and thinking about polar coordinates it is easy to formulate the polar limits of integration, $0 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$. Now we just need to directly translate the integrand and since $x^2 + y^2 = r^2$ the integrand $e^{x^2+y^2}$ can be represented by e^{r^2} and finally the differentials $dydx$ become $r\,dr d\theta$ in polar form. Thus the rectangular integral can be converted to polar form

$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\, dydx = \int_0^{\frac{\pi}{2}} \int_0^2 e^{r^2}\, r\,drd\theta$ and an impossible integral in rectangular form is now a simple u-substitution integration in polar form.

$\int_0^{\frac{\pi}{2}} \int_0^2 e^{r^2}\, r\,drd\theta$, let $u = r^2$ then $du = 2r\,dr$ and so $\frac{1}{2}du = r\,dr$ so $\int e^{r^2}\, r\,dr = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u = \frac{1}{2}e^{r^2}$

Then $\int_0^{\frac{\pi}{2}} \int_0^2 e^{r^2}\, r\,drd\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{r^2}\Big|_0^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(e^4 - 1\right) d\theta = \frac{1}{2}\left(e^4 - 1\right)\theta\Big|_0^{\frac{\pi}{2}} = \frac{\left(e^4-1\right)\pi}{4}$.

This result could be the volume of the solid lying above the quarter circle of radius 2 in the first quadrant with a height given by the function $f(x,y) = e^{x^2+y^2}$. But it could also be the mass of a lamina (thin layered solid) of variable density with a density function $\rho(x,y) = e^{x^2+y^2}$, the Greek letter rho (ρ) is the standard representative of a density function.

Lecture Notes 14.4: Center of Mass

We saw in the last example how a double integral can be used to calculate the mass of a region in the *xy* plane with a non-constant density function $\rho(x, y)$. The ability to calculate mass and the first moments and the center of mass as well as second moments (moments of inertia) are critical to a career in engineering. Simply put, a moment can be thought of as mass times distance from an axis of rotation. The center of mass is the point at which an irregularly shaped lamina can be balanced or in the case of a rotating body it would be the point around which the lamina would rotate smoothly and without vibration. Eliminating vibration from rotating drive shafts and cam shafts is one of the greatest reasons why modern engines have such a longer life cycle than those of the past.

The brief (calculus versus physics) version of center of mass is really a just a collection of formulas.

Given a region, *R*, in the *xy* plane with density function $\rho(x, y)$ we define

$$\text{Mass} = M = \iint_R \rho(x, y)\, dA$$

$$\text{First Moment about the } x \text{ axis} = M_x = \iint_R y\,\rho(x, y)\, dA$$

$$\text{First Moment about the } y \text{ axis} = M_y = \iint_R x\,\rho(x, y)\, dA$$

Coordinates of the Center of Mass or Centroid $= (\bar{x}, \bar{y})$ where $\bar{x} = \dfrac{M_y}{M}$ and $\bar{y} = \dfrac{M_x}{M}$.

To help make a little sense of these formulas since it seems like the subscripts are all wrong the moment about the *x* axis is the mass times the distance from the *x* axis which is *y*. Similarly when finding the center of mass we want the average weighted distance from the *y* axis to be the *x* coordinate so we need M_x to be divided by mass since it is the moment about the *y* axis and therefore involves the distance from the therefor involves the distance from *y* axis and distance from that axis is measured by the variable *x*. It can give you a headache.

Moments of Inertia are second moments.

$$\text{The moment of inertia about the } x \text{ axis} = I_x = \iint_R y^2\, \rho(x, y)\, dA$$

$$\text{The moment of inertia about the } y \text{ axis} = I_y = \iint_R x^2\, \rho(x, y)\, dA$$

$$\text{The moment of inertia about the origin} = I_0 = \iint_R \left(x^2 + y^2\right) \rho(x, y)\, dA = I_x + I_y$$

These are fun problems with three double integrals for each question so there is a lot of practice. Five double integrals per question if the moments of inertial are also required.