Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 5: 6/22-6/28
Sections 13.4 – 13.7, Differentials, Chain Rule, Directional Derivative, Tangent Planes and Normal Lines.

Due this week:

Tuesday 6/23/2020	**Thursday 6/25/2020**	**Sunday 6/28/2020**
	WebAssign 13.3 - 13.5	Weekly Assignment 5, 13.4 – 13.7

Lecture Notes 13.4: Differentials

Recall a brief discussion from calculus I about differentials. In case you don't remember, we can express the differential of y, dy, in terms of the derivative, $f'(x)$, and the differential of x, dx.

Since $\dfrac{dy}{dx} = f'(x)$ then $dy = f'(x)\,dx$. This relationship was important to us as the basic principal behind antiderivatives and integrals. Another useful variation of this is a differential approximation also known as a linear approximation $f(x+\Delta x) \approx f(x) + f'(x)\Delta x$. This type of approximation assumes that the tangent line is not too different from the function for small differences in x.

That was the calc I version. Now with multiple variable we have what is known as the total differential. For a function of the form $z = f(x, y)$ then the total differential of z would be
$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$ or for an even higher dimension,
if $w = g(x, y, z)$ then the total differential of w would be given by the equation

$$dw = g_x(x,y,z)\,dx + g_y(x,y,z)\,dy + g_z(x,y,z)\,dz = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz .$$

There is some good mathematics in this section with regard to differentiability and continuity but the relationships above outlining the total differential is really all you will need going forward.

Example 1: Total Differential. Given $w = x^2yz^3 + \sin(yz)$ find the total differential.
Solution: $dw = g_x(x,y,z)\,dx + g_y(x,y,z)\,dy + g_z(x,y,z)\,dz$

$$dw = 2xyz^3dx + \left(x^2y^3 + z\cos(yz)\right)dy + \left(3x^2yz^2 + y\cos(yz)\right)dz$$

Lecture Notes 13.5: Chain Rule

This is not the calculus I chain rule although the calculus I chain rule is consistent with this method. Suppose $z = f(x, y)$ and further that $x = g(r,t)$ and $y = h(r,t)$. Here *z* ultimately depends on *r* and *t* but the effects of *r* and *t* on *z* are generated through the functions *g* and *h*. Then we can set up the chain rule for the derivative of *z* with respect to *r* and with respect to *t* can be expressed as:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \qquad \text{and} \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

If we were to replace the *x* and *y* variables with the functions above $z = f(g(r,t), h(r,t))$ then we could calculate the above partial derivatives using our knowledge of chain rule and partial derivatives. Generally this is a more complicated process than the above even though it is more familiar to us.
Consider a more complicated form,
$M = f(w, x, y, z)$ where $w = g(r,s,t)$, $x = h(r,s,t)$, $y = k(r,s,t)$, $z = l(r,s,t)$ and $r = R(u,v)$, $s = S(u,v)$, and $t = T(u,v)$ then we can express the partial derivative

$$\frac{\partial M}{\partial u} = \frac{\partial M}{\partial w}\frac{\partial w}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial M}{\partial w}\frac{\partial w}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial M}{\partial w}\frac{\partial w}{\partial t}\frac{\partial t}{\partial u} + \frac{\partial M}{\partial x}\frac{\partial x}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial M}{\partial x}\frac{\partial x}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial M}{\partial x}\frac{\partial x}{\partial t}\frac{\partial t}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial s}\frac{\partial s}{\partial u}$$
$$+ \frac{\partial M}{\partial y}\frac{\partial y}{\partial t}\frac{\partial t}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial t}\frac{\partial t}{\partial u}$$

<u>Getting a little crazy right?</u> You always know you have the correct set up if you look to treat the partial derivatives as fractions and look at the cancelations. It is important to know that we can not actually cancel these pieces but we want the set up to be such that if we did then each term would have the correct derivative.

$$\frac{\partial M}{\partial u} = \frac{\partial M}{\cancel{\partial w}}\frac{\cancel{\partial w}}{\cancel{\partial r}}\frac{\cancel{\partial r}}{\partial u} + \frac{\partial M}{\cancel{\partial w}}\frac{\cancel{\partial w}}{\cancel{\partial s}}\frac{\cancel{\partial s}}{\partial u} + \frac{\partial M}{\cancel{\partial w}}\frac{\cancel{\partial w}}{\cancel{\partial t}}\frac{\cancel{\partial t}}{\partial u} + \frac{\partial M}{\cancel{\partial x}}\frac{\cancel{\partial x}}{\cancel{\partial r}}\frac{\cancel{\partial r}}{\partial u} + \frac{\partial M}{\cancel{\partial x}}\frac{\cancel{\partial x}}{\cancel{\partial s}}\frac{\cancel{\partial s}}{\partial u} + \frac{\partial M}{\cancel{\partial x}}\frac{\cancel{\partial x}}{\cancel{\partial t}}\frac{\cancel{\partial t}}{\partial u} + \frac{\partial M}{\cancel{\partial y}}\frac{\cancel{\partial y}}{\cancel{\partial r}}\frac{\cancel{\partial r}}{\partial u} + \frac{\partial M}{\cancel{\partial y}}\frac{\cancel{\partial y}}{\cancel{\partial s}}\frac{\cancel{\partial s}}{\partial u}$$
$$+ \frac{\partial M}{\cancel{\partial y}}\frac{\cancel{\partial y}}{\cancel{\partial t}}\frac{\cancel{\partial t}}{\partial u} + \frac{\partial M}{\cancel{\partial z}}\frac{\cancel{\partial z}}{\cancel{\partial r}}\frac{\cancel{\partial r}}{\partial u} + \frac{\partial M}{\cancel{\partial z}}\frac{\cancel{\partial z}}{\cancel{\partial s}}\frac{\cancel{\partial s}}{\partial u} + \frac{\partial M}{\cancel{\partial z}}\frac{\cancel{\partial z}}{\cancel{\partial t}}\frac{\cancel{\partial t}}{\partial u}$$

Each term would reduce to be $\frac{\partial M}{\partial u}$ if we could reduce. Most text books and probably webassign as well expect us to substitute the functions for *w, x, y, z* or whatever the variables are so that the final partial derivative is only in terms of the independent variables, *u* and *v* in this example. I do not require you to do this and so you can leave your partial derivative in a mixed format as in the example below on any written assignment.

Example 1: Chain Rule. Given $w = xyz$ with $x = s + t$, $y = s - t$, $z = st^2$. Find $\frac{\partial w}{\partial t}$ and evaluate the partial derivative at $s = 3$, $t = 2$.

Solution: $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$ $\qquad$ $\frac{\partial w}{\partial t} = (yz)(1) + (xz)(-1) + (xy)(2st)$

At $s = 3$, $t = 2$ then $x = 3 + 2 = 5$, $y = 3 - 2 = 1$, $z = 3(2^2) = 12$ and so

$$\frac{\partial w}{\partial t} = (1)(12)(1) + (5)(12)(-1) + (5)(1)(2(3)(2)) = 12 - 60 + 60 = 12$$

Lecture Notes 13.6: Directional Derivative

Consider a 3-dimensional surface, $z = f(x, y)$ and imagine yourself standing on this surface at some point (x_0, y_0). Facing in a direction parallel to the y axis the slope of the line tangent to the surface would be $f_y(x_0, y_0)$. The partial derivative of the function with respect to y gives the slope of the tangent line in any plane parallel to the yz plane. Similarly if you turn to face in the direction parallel to the x axis you would find the slope of the line tangent to the surface in that direction to be $f_x(x_0, y_0)$. Wonderful, we can find the slope of the line tangent to our surface in space in 2 directions. But what if we need to know the slope of the line tangent to the surface in a different direction? How can we specify a direction? We will answer all of these questions and more after defining a new mathematical "thing", known as the gradient function or gradient vector.

The **Gradient Function** or **Gradient Vector** is denoted $\nabla f(x, y)$ for a 3 dimensional function $z = f(x, y)$. The symbol, ∇, which looks like an upside down delta is called del. (Mathematicians are not the most creative thinkers when it comes to names.) This is another new symbol, like the partial derivative symbol, which is not part of any alphabet other than the mathematical one.

Given a multivariable function then like $z = f(x, y)$ we can calculate $\nabla f(x, y)$ or ∇z to be the vector valued function $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$ or if you prefer $\nabla z = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle$. In higher dimensions such as a function with a 3-dimensional domain space, $W = g(x, y, z)$ we generalize the gradient vector to be $\nabla g(x, y, z) = g_x(x, y, z)\mathbf{i} + g_y(x, y, z)\mathbf{j} + g_z(x, y, z)\mathbf{k}$ or $\nabla W = \left\langle \frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z} \right\rangle$ and so forth for higher dimensions. Often my students ask how a function like $W = g(x, y, z)$ can exist since with a 3-dimensional domain this would be a -4-dimensional function. So often we think of time as the fourth dimension but that doesn't need to be the case. There are two examples I like to use to illustrate this idea of a 4-dimensional function. The first is temperature. Imagine a room, we can coordinatize the room using the *x, y,* and *z* axis then at any point in the 3-dimensional room we would have a temperature and that temperature may be able to be described using a function of *x, y,* and *z.*

The Gradient Vector is very important for us and as such we need to know just what it is and what it gives.

Given $z = f(x, y)$ the gradient vector is $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$

1. $\nabla f(x, y)$ is a vector valued function in the **domain space** of the function.
2. $\nabla f(x_0, y_0)$ is a vector in the direction of greatest rate of increase in $z = f(x, y)$ at (x_0, y_0).
3. A vector in the direction of greatest rate of decrease would be $-\nabla f(x_0, y_0)$
4. The greatest rate of increase in $z = f(x, y)$ is the magnitude of $\nabla f(x, y)$, $\| \nabla f(x, y) \|$.
5. The greatest rate of decrease in $z = f(x, y)$ is $-\| \nabla f(x, y) \|$.
6. $\nabla f(x, y)$ is always normal (perpendicular) to the level curves of $z = f(x, y)$.
7. The rate of change in $z = f(x, y)$ in the direction of vector ***v*** is $D_{\mathbf{v}} f(x, y) = \frac{\nabla f(x, y) \bullet \mathbf{v}}{\| \mathbf{v} \|}$

Property 7 above is known as the Directional Derivative of a function in the direction of a vector, **v**, and is worthy of being restated on its own.

The directional derivative gives the slope of the line tangent to a surface in any direction if the surface exists in Euclidean space (our conventional 3-dimensions) or more generally the rate of change in any function with respect to a given direction in the domain of the function.

Directional Derivative: $D_{\mathbf{v}} f = \dfrac{\nabla f \bullet \mathbf{v}}{\|\mathbf{v}\|}$ gives the derivative of *f* in the direction of domain vector **v**. Text books traditionally express this slightly differently as $D_{\mathbf{u}} f = \nabla f \bullet \mathbf{u}$ where the vector **u** is a unit vector in the desired direction. Basically the difference between the formulas is computational, either convert the direction vector to a unit vector first or do it all at once. You may do this calculation any way you wish.

Examples:

1. Given the function $f(x,y) = x^2 + 3y^2$ find the gradient vector at the point $(4,2)$.
   Solution: $\nabla f(x,y) = 2x\,\mathbf{i} + 6y\,\mathbf{j}$ and $\nabla f(4,2) = 2(4)\,\mathbf{i} + 6(2)\,\mathbf{j} = \langle 8,12 \rangle$.

2. Given the function $f(x,y,z) = xz^2 + 4xy^2z$ find the gradient vector at the point $(-1,3,2)$.
   Solution: $\nabla f(x,y,z) = \left(z^2 + 4y^2z\right)\mathbf{i} + 8xyz\,\mathbf{j} + \left(2xz + 4xy^2\right)\mathbf{k}$ and
   $\nabla f(-1,3,2) = \left(4 + 4(9)(2)\right)\mathbf{i} + 8(-1)(3)(2)\,\mathbf{j} + \left(2(-1)(2) + 4(-1)(9)\right)\mathbf{k} = \langle 76,-48,-40 \rangle$.

3. Find a vector in the direction of greatest rate of increase in $g(x,y) = \dfrac{3x}{x^2+y^2}$ at the point $(-1,3)$. Then find the greatest rate of increase at that point.
   Solution: $\nabla f(x,y) = \left(\dfrac{3(x^2+y^2) - 2x(3x)}{\left(x^2+y^2\right)^2}\right)\mathbf{i} + \left(\dfrac{0 - 2y(3x)}{\left(x^2+y^2\right)^2}\right)\mathbf{j} = \left(\dfrac{3y^2 - 3x^2}{\left(x^2+y^2\right)^2}\right)\mathbf{i} - \left(\dfrac{6xy}{\left(x^2+y^2\right)^2}\right)\mathbf{j}$
   $\nabla f(-1,3) = \left(\dfrac{3(9) - 3(1)}{(1+9)^2}\right)\mathbf{i} - \left(\dfrac{6(-1)(3)}{(1+9)^2}\right)\mathbf{j} = \left(\dfrac{24}{100}\right)\mathbf{i} + \left(\dfrac{18}{100}\right)\mathbf{j} = \left(\dfrac{6}{25}\right)\mathbf{i} + \left(\dfrac{9}{50}\right)\mathbf{j} = \left\langle \dfrac{6}{25}, \dfrac{9}{50} \right\rangle$
   The greatest rate of increase is $\left\| \left\langle \dfrac{6}{25}, \dfrac{9}{50} \right\rangle \right\| = \sqrt{\left(\dfrac{6}{25}\right)^2 + \left(\dfrac{9}{50}\right)^2} = \sqrt{\dfrac{9}{100}} = \dfrac{3}{10}$ at the point $\left(-1, 3, \dfrac{-3}{10}\right)$

4. Find a vector perpendicular to the level curve of $f(x,y) = x^2y^3$ at the point $(4,-2)$.
   Solution: $\nabla f(x,y) = 2xy^3\,\mathbf{i} + 3x^2y^2\,\mathbf{j}$ and $\nabla f(4,-2) = -64\,\mathbf{i} + 192\,\mathbf{j} = \langle -64,192 \rangle$ is perpendicular to the level curve of $f(x,y) = x^2y^3$ at the point $(4,-2)$.

5. Find the derivative of $f(x,y) = x^2 + 3y^2\sin(x) - \sin(\pi y)$ at the point $(0,2)$ in the direction of $\mathbf{v} = \langle 3,7 \rangle$
   Solution: $\nabla f(x,y) = \left(2x + 3y^2\cos(x)\right)\mathbf{i} + \left(6y\sin(x) - \pi\cos(\pi y)\right)\mathbf{j}$,
   $\nabla f(0,2) = \left(0 + 12\cos(0)\right)\mathbf{i} + \left(12\sin(0) - \pi\cos(2\pi)\right)\mathbf{j} = \langle 12,-\pi \rangle$
   $D_{\mathbf{v}} f(0,2) = \dfrac{\langle 12,-\pi \rangle \bullet \langle 3,7 \rangle}{\|\langle 3,7 \rangle\|} = \dfrac{36 - 7\pi}{\sqrt{58}} \approx 1.84$

Lecture Notes 13.7: Tangent Planes and Normal Lines

Recall from chapter 11 how we found the equation of a plane and the parametric equations of a line in space.

For the equation of a plane we needed a point on the plane which we denoted (x_0, y_0, z_0) and a vector oriented **perpendicular** to the plane denoted by $\langle a, b, c \rangle$. Then the equation of the plane was generated to be $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ or in the more desirable simplified form $ax+by+cz=ax_0+by_0+cz_0=d$.

For the parametric equations of a line in space we needed a point on the line denoted (x_0, y_0, z_0) and a vector oriented **parallel** to the line denoted $\langle a, b, c \rangle$. Then the parametric equations of the line were $x = x_0+at,\ y = y_0+bt,\ z = z_0+ct$ but now that we have studied vector valued functions we can also give the equation of the line in function form as $\mathbf{r}(t)=(x_0+at)\mathbf{i}+(y_0+bt)\mathbf{j}+(z_0+ct)\mathbf{k}$.

Now we want to combine our earlier knowledge base with multivariable functions and gradient vectors to develop a strategy for finding the equation of tangent planes and normal lines, planes that are tangent to surfaces, and lines that are oriented perpendicular to the surface. The key to making this work is the fact that gradient vectors are always oriented perpendicularly to level curves of a function. So given a surface of the form $z = f(x, y)$ level curves of the surface would be 2-dimensional curves in a horizontal plane and that just won't do for what we want but if we create a function in 4-dimensions with a 3-dimensional domain by letting $g(x, y, z) = f(x, y) - z$ then the level "curves" of $g(x, y, z)$ are actually surfaces in 3-space and the gradient vector function, $\nabla g(x, y, z) = g_x(x, y, z)\mathbf{i} + g_y(x, y, z)\mathbf{j} + g_z(x, y, z)\mathbf{k}$, would always be oriented normal (perpendicular) to the surface which is simply $z = f(x, y)$. Further, since $g(x, y, z) = f(x, y) - z$ we can say that $\nabla g(x, y, z) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} - \mathbf{k}$. Thus for any point on the surface $z_0 = f(x_0, y_0)$ we have the point (x_0, y_0, z_0) and the vector $\nabla g(x_0, y_0, z_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k} = \langle a, b, -1 \rangle$ where $a = f_x(x_0, y_0)$ and $b = f_y(x_0, y_0)$ would be normal (perpendicular) to the surface at the point. With those we can then form the equation of the plane tangent to the surface $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ or the simplified form, $ax+by+cz=ax_0+by_0+cz_0=d$. Now since the vector perpendicular to the tangent plane is also normal to the surface then that same vector is correctly oriented for the parametric equations of the normal line at that same point on the surface, $x = x_0+at,\ y = y_0+bt,\ z = z_0+ct$. I prefer the parametric form of the line but we could also form the vector valued function of the line, $\mathbf{r}(t)=(x_0+at)\mathbf{i}+(y_0+bt)\mathbf{j}+(z_0+ct)\mathbf{k}$.

If the equation of the surface is not in functional form, such as an ellipsoid like $x^2+2y^2+4z^2=16$, we can still find the equation of the tangent plane and normal line at a point by forming a function with a 3-dimensional domain. In cases such as these we let set the given equation equal to zero, $x^2+2y^2+4z^2-16=0$ and then let $g(x, y, z)$ equal the left hand side of the equation, $g(x, y, z) = x^2+2y^2+4z^2-16$, and this function which exists in 4-dimensions has a level surface the original 3-dimensional surface. Then the gradient vector $\nabla g(x, y, z) = g_x(x, y, z)\mathbf{i} + g_y(x, y, z)\mathbf{j} + g_z(x, y, z)\mathbf{k}$ would be normal to the original surface and so would be the normal vector for the tangent plane and direction vector for the normal line.

It is critically important to be aware of how your surface is presented. In functional form, $z = f(x, y)$, the normal vector takes the form $\nabla g(x, y, z) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} - \mathbf{k}$ but in equation form we need to create our 4-dimensional function by setting the given equation equal to zero before declaring our function, $g(x, y, z)$, and finding the gradient vector, $\nabla g(x, y, z) = g_x(x, y, z)\mathbf{i} + g_y(x, y, z)\mathbf{j} + g_z(x, y, z)\mathbf{k}$, to use to form the plane equation and parametric equations of the normal line. In reality we don't have to set the given equation equal to zero, any constant would be do just as well. Think about the ellipsoid example, $g(x, y, z) = x^2 + 2y^2 + 4z^2 - 16$ and $h(x, y, z) = x^2 + 2y^2 + 4z^2$ would have the exact same gradient vector since the derivative of a constant is 0. The ellipsoid in question would correspond to the level surface $g(x, y, z) = 0$ as well as the level surface $h(x, y, z) = 16$.

Examples:

1. Find the equation of the tangent plane and parametric equations of the normal line to the surface $f(x, y) = x^2 + 3y^2$ at $(4, 2)$.
   Solution: The surface is in functional form so a 4-dimensional function with $f(x, y) = x^2 + 3y^2$ as a level surface would be $g(x, y, z) = x^2 + 3y^2 - z$ and the gradient vector $\nabla g(x, y, z) = 2x\,\mathbf{i} + 6y\,\mathbf{j} - \mathbf{k}$ would be normal to the surface. At the point $(4, 2)$ we find $z = f(4, 2) = 16 + 12 = 28$ and so $\nabla g(4, 2, 28) = 8\,\mathbf{i} + 12\,\mathbf{j} - \mathbf{k} = \langle 8, 12, -1 \rangle$.
   So the equation of the tangent plane is $8(x - 4) + 12(y - 2) - (z - 28) = 0$
   which simplifies to $8x + 12y - z = 28$.
   And the normal line would be $x = 4 + 8t,\ \ y = 2 + 12t,\ \ z = 28 - t$.

2. Find the equation of the tangent plane and normal line to the surface $xy^2 + yz^2 = 66$ at the point $(2, 3, 4)$ on the surface.
   Solution: The surface is in equation form so let $g(x, y, z) = xy^2 + yz^2 - 66$ or just $g(x, y, z) = xy^2 + yz^2$ then the gradient function is $\nabla g(x, y, z) = y^2\,\mathbf{i} + \left(2xy + z^2\right)\mathbf{j} + 2yz\,\mathbf{k}$ and at the given point $\nabla g(2, 3, 4) = 9\,\mathbf{i} + 28\,\mathbf{j} + 24\,\mathbf{k} = \langle 9, 28, 24 \rangle$.
   Tangent Plane: $9(x - 2) + 28(y - 3) + 24(z - 4) = 0$ or $9x + 28y + 24z = 198$
   Normal Line: $x = 2 + 9t,\ \ y = 3 + 28t,\ \ z = 4 + 24t$

We can also use the gradient vector to find the angle of inclination of the tangent plane. The angle of inclination is the measure of the angle between the tangent plane and the *xy* plane. To find this angle we need only find the angle between the vector normal to the tangent plane which is of course the gradient vector and the vector normal to the *xy* plane which would be $\langle 0, 0, 1 \rangle$. Recall from chapter 11 again that the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \cos^{-1}\left(\dfrac{|\mathbf{u} \bullet \mathbf{v}|}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$. We generally restrict this to be the acute angle, $0 \le \theta \le 90°$, which is the reason for the absolute values in the numerator. Now if the surface in question is $g(x, y, z) = 0$ then $\mathbf{u} = \nabla g(x, y, z) = g_x(x, y, z)\,\mathbf{i} + g_y(x, y, z)\,\mathbf{j} + g_z(x, y, z)\,\mathbf{k}$ and $\mathbf{v} = \langle 0, 0, 1 \rangle$ so $\theta = \cos^{-1}\left(\dfrac{g_z(x, y, z)}{\|\nabla g(x, y, z)\|}\right)$.

Examples:

1. Find the angle of inclination of the tangent plane to the surface $z = x^2 + 3y^2$ at $(4, 2, 28)$.
   Solution: The surface is in equation form so let $g(x, y, z) = x^2 + 3y^2 - z$ so then $\nabla g(x, y, z) = 2x\,\mathbf{i} + 6y\,\mathbf{j} - \mathbf{k}$ and $\nabla g(4, 2, 28) = 8\,\mathbf{i} + 12\,\mathbf{j} - \mathbf{k} = \langle 8, 12, -1 \rangle$. Now $\theta = \cos^{-1}\left(\dfrac{|-1|}{\sqrt{64 + 144 + 1}}\right)$ and then $\theta = \cos^{-1}\left(\dfrac{1}{\sqrt{209}}\right) \approx 86.0°$