Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 6: 6/29-7/5
Sections 13.8 – 13.10, Extrema, Applications, Lagrange Multipliers.

Due this week:

Tuesday 6/30/2020	**Thursday 7/2/2020** WebAssign 13.6 - 13.9	**Sunday 7/5/2020** Weekly Assignment 6, 13.8 – 13.10

Lecture Notes 13.8: Extrema

Part I: Local Extrema

We continue to mirror calculus I, recall from calculus I after learning about derivatives and tangent lines we began to apply what we learned to identify absolute and local or relative maximums and minimums. In the two dimensions that we studied in that course we identified the location of potential extrema by looking for horizontal tangent lines where the derivative of the function would be zero. Well now that we are in three dimensions we can identify potential relative maximums and minimums by seeking out ordered pairs in the domain of $z = f(x,y)$ for which the tangent plane would be horizontal, parallel to the *xy* plane. So now the question becomes how do we know if the tangent plane is horizontal? Think about the equation of a plane, $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, the plane would be horizontal if and only f the normal vector $\langle a,b,c\rangle$ is vertical. Now think about what it means for a vector to be vertical. A vector is vertical any time the *x* and *y* components are zero. That is any vector of the form $\langle 0,0,c\rangle$ would be vertical and thus would be normal to a horizontal tangent plane. Finally, recall how we identified the normal vector for a tangent plane, specifically the *x* and *y* components. For a $z = f(x,y)$ surface the vector normal to the tangent plane would be $\langle f_x(x,y), f_y(x,y), -1\rangle$ and so a necessary condition for a tangent plane to be horizontal would be that **both** $f_x(x,y)=0$ **and** $f_y(x,y)=0$. In this way then we can say that critical values of the function $z = f(x,y)$ are any ordered pairs, (x,y), for which **both** $f_x(x,y)=0$ **and** $f_y(x,y)=0$. This means that to identify the critical values of a function of the form $z = f(x,y)$ we need to find the first partial derivatives and set them each equal to zero and solve that **system** of equations.

Example 1: Critical Values. Find the critical values of $f(x,y)=\sqrt{x^2+y^2+4}$.

Solution: $f_x(x,y)=\dfrac{2x}{2\sqrt{x^2+y^2+4}}=\dfrac{x}{\sqrt{x^2+y^2+4}}$ and $f_y(x,y)=\dfrac{2y}{2\sqrt{x^2+y^2+4}}=\dfrac{y}{\sqrt{x^2+y^2+4}}$ setting both partial derivatives equal to zero leads to the non-linear system of equations:

$$\frac{x}{\sqrt{x^2+y^2+4}}=0$$
$$\frac{y}{\sqrt{x^2+y^2+4}}=0$$

clearly the only solution and thus the only critical value then is the point $(0,0)$.

Example 2: Critical Values. Find the critical values of $f(x,y)=x^3+2x^2y-4y^2$.

Solution: Finding the first partial derivatives and setting them both equal to zero leads to the non-linear system of equations: $f_x(x,y)=3x^2+4xy=0$ and $f_y(x,y)=2x^2-8y=0$. Factoring an *x* from the first equation leads to $f_x(x,y)=x(3x+4y)=0$ and so either $x=0$ or $3x+4y=0$ which leads to $y=\dfrac{-3x}{4}$. Substituting $x=0$ into the partial derivative with respect to y gives $2(0)^2 - 8y = 0$ so $y = 0$ and the point $(0,0)$ is a critical value then substituting $y = \frac{-3x}{4}$ into $f_y(x,y) = 2x^2 - 8y = 0$ gives $2x^2 - 8\left(\frac{-3x}{4}\right) = 2x^2 + 6x = 2x(x+3) = 0$ and so $x = 0$ or $x = -3$ and so then $y = \frac{-3(0)}{4} = 0$ and $y = \frac{-3(-3)}{4} = \frac{9}{4}$ so the points $(0,0)$ and $(-3, \frac{9}{4})$ are critical values also. There is no special significance to the point $(0,0)$ turning up twice as a critical value, duplication like this is not uncommon.

Now that we can find critical values we need to consider how to determine the behavior of the function at these critical values. Recall again from calculus I that when a continuous function, $y = f(x)$, has a horizontal tangent line, $f'(a) = 0$, at a critical value $x = a$ as and is concave down, $f''(a) < 0$, at the critical value then the function must have relative maximum at that critical value. Similarly if $f'(b) = 0$ and $f''(b) > 0$ then the function has a horizontal tangent line and is concave down and thus has a relative minimum at the critical value $x = b$. We have a similar technique for functions in 3-space except we have technically 4 second derivatives (although for us $f_{xy}(x,y) = f_{yx}(x,y)$) and it is possible for a function to be concave up in one vertical cross section and concave down in another at the same location, with horizontal tangent plane.

Suppose $z = f(x,y)$ is a differentiable multivariable function and that (a,b) is a critical value. That is $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let $D = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$. Now if $D > 0$ and $f_{xx}(a,b) > 0$ then the surface has a relative (local) minimum at (a,b), if $D > 0$ and $f_{xx}(a,b) < 0$ then the surface has a relative (local) maximum at (a,b), if $D < 0$ then the surface has a saddle point at (a,b). A saddle point, like the back of a horse, is both a maximum in one cross section and a minimum in another. See page 800 of the text book (chapter 11 section 6) for a drawing of a saddle function formally labeled a hyperbolic paraboloid for those quadric surfaces. If $D = 0$ then this analysis fails. Unlike calculus I where we could appeal to a sign chart to determine the behavior of a function around it's critical values we don't have a fall back unless we can examine the function itself to determine behavior.
You might wonder why we look to the sign of $f_{xx}(a,b)$ when the value of D is greater than zero. In reality it doesn't matter whether we look at $f_{xx}(a,b)$ or $f_{yy}(a,b)$ since they must both have the same sign since $D = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$ can only be positive if the value of the product $f_{xx}(a,b)f_{yy}(a,b)$ is positive and greater than $\left(f_{xy}(a,b)\right)^2$ since as a square $\left(f_{xy}(a,b)\right)^2$ must be positive.

Summary of extrema:

1. Identify critical values of the function by setting $f_x(x,y) = 0$ and $f_y(x,y) = 0$ and solving the system of equations for ordered pairs, (a,b), which are the critical values.
2. Calculate the second partial derivatives, $f_{xx}(x,y)$, $f_{yy}(x,y)$, and $f_{xy}(x,y)$ and evaluate them at the critical values. And let $D = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$
3. $D > 0$ and $f_{xx}(a,b) > 0$ identifies the critical point (a,b) as a local minimum.
   $D > 0$ and $f_{xx}(a,b) < 0$ identifies the critical point (a,b) as a local maximum.
   $D < 0$ identifies the critical point (a,b) as a saddle point.
   $D = 0$ indicates the test fails and we can only determine behavior if we can identify behavior from the form of the function.

Example 3: Extrema. Find the relative maximums, minimums, or saddle points of $f(x,y)=\sqrt{x^2+y^2+4}$.
Solution: From example 1 above the only critical point of this function is $(0,0)$.

$f_x(x,y)=\dfrac{2x}{2\sqrt{x^2+y^2+4}}=\dfrac{x}{\sqrt{x^2+y^2+4}}$ and $f_y(x,y)=\dfrac{2y}{2\sqrt{x^2+y^2+4}}=\dfrac{y}{\sqrt{x^2+y^2+4}}$ and so

$f_{xx}(x,y)=\dfrac{(1)\sqrt{x^2+y^2+4}-\dfrac{x}{\sqrt{x^2+y^2+4}}(x)}{\left(\sqrt{x^2+y^2+4}\right)^2}=\dfrac{\sqrt{x^2+y^2+4}-\dfrac{x^2}{\sqrt{x^2+y^2+4}}}{x^2+y^2+4}$ which can be simplified further but since we are going to evaluate this at $(0,0)$ it is just as easy to do that now. $f_{xx}(0,0)=\dfrac{\sqrt{0+4}-\dfrac{0}{\sqrt{0+4}}}{0+4}=\dfrac{2}{4}=\dfrac{1}{2}$.

Similarly $f_{yy}(x,y)=\dfrac{\sqrt{x^2+y^2+4}-\dfrac{y^2}{\sqrt{x^2+y^2+4}}}{x^2+y^2+4}$ and $f_{yy}(0,0)=\dfrac{\sqrt{0+4}-\dfrac{0}{\sqrt{0+4}}}{0+4}=\dfrac{1}{2}$ and the mixed partial derivative $f_{xy}(x,y)=\dfrac{(0)\sqrt{x^2+y^2+4}-\dfrac{y}{\sqrt{x^2+y^2+4}}(x)}{x^2+y^2+4}=\dfrac{-\dfrac{xy}{\sqrt{x^2+y^2+4}}}{x^2+y^2+4}=\dfrac{-xy}{\left(x^2+y^2+4\right)^{3/2}}$ and

$f_{xy}(0,0)=\dfrac{0}{(0+4)^{3/2}}=0$. So we can calculate $D=f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2=\left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)-0=\dfrac{1}{4}>0$ and with $f_{xx}(0,0)=\dfrac{1}{2}>0$ we know the critical value $(0,0)$ is a local minimum on the surface. The function's value at this point is $z=f(0,0)=\sqrt{0+0+4}=2$ so more precisely the point $(0,0,2)$ is local minimum. This analysis only identifies local extrema but knowing that the square root function is always increasing tells us that this is actually an absolute minimum.

Example 4: Extrema. Find the relative maximums, minimums, or saddle points of $f(x,y)=x^3+2x^2y-4y^2$.
Solution: From example 2 above the critical points of this function are $(0,0)$ and $(-3,\frac{9}{4})$. $f_x(x,y)=3x^2+4xy$ and $f_y(x,y)=2x^2-8y$ so $f_{xx}(x,y)=6x+4y$ and $f_{yy}(x,y)=-8$ and $f_{xy}(x,y)=4x$.
I like to use a table when I have more than one critical value.

Critical Points	$(0,0)$	$(-3,\frac{9}{4})$
$f_{xx}(x,y)=6x+4y$	0	$-18+9=-9$
$f_{yy}(x,y)=-8$	-8	-8
$f_{xy}(x,y)=4x$	0	-12
$D=f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2$	$0-0=0$	$72-144<0$
Conclusion	Test Fails	Saddle Point

Make sure to practice these problems, section 13.8 questions 3 – 24.

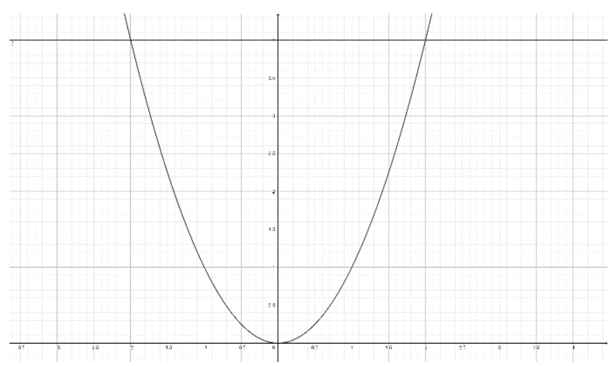
Part II: Absolute Extrema on a Closed Region

Recall again in calculus I when we looked at a continuous function on a closed interval, such as $y = f(x)$ on $[a,b]$. Every continuous function attains an absolute maximum and absolute minimum on a closed interval. We would find critical values by finding all values of *x* for which the derivative was equal to zero, $f'(x) = 0$, or the derivative was undefined, $f'(x)\ DNE$, as these identified possible cusps or sharp turns on the graph. We also would include the endpoints of the interval in the list of critical values. Then we evaluated the original function at each critical value to find the largest and smallest values and reported those as the value and location of the absolute extrema. For example the simple function $f(x) = x^2 + 4x$ on the interval $[-1,3]$ has a derivative of $f'(x) = 2x + 4$ which is equal to zero only at $x = -2$ and is never undefined. Because $x = -2$ is not included in the given interval we throw out that potential critical point and so have only the end points $x = -1$ and $x = 3$ as critical values. Evaluating the function at these values $f(-1) = (-1)^2 + 4(-1) = 1 - 4 = -3$ and $f(3) = (3)^2 + 4(3) = 9 + 12 = 21$ and so we conclude that $f(x) = x^2 + 4x$ has an absolute maximum of 21 at $x = 3$ and an absolute minimum of -3 at $x = -1$.

We need to replicate this process now for a multivariable function $z = f(x, y)$ over a closed region in the *xy* plane. A closed region simply means the curves in the plane bounding the region are included in the region. For a problem such as this we need to use our partial derivatives approach as outline in part I of this lecture. The partial derivatives are used to identify critical points (a,b) on the interior of the region. Then we must also consider the behavior of $z = f(x, y)$ on the cylinder formed by extending the boundary equations vertically until they intersect the surface. We do this by substituting their equations into the function $z = f(x, y)$ for either *x* or *y* thus producing a single variable function for which we use our calculus I approach to absolute extrema. We also include in our list of critical values any points of intersection between the bounding curves. Once we have built a list of critical values we evaluate the original $z = f(x, y)$ function at each critical value and compare results to identify the absolute maximum and minimum values and their locations. Pay close attention to the following example as these can get rather long quickly.

Example 5: Absolute Extrema. Find the absolute maximum and minimum value of the function $f(x, y) = 3x^2 + 2y^2 - 4y$ over the region in the *xy* plane bounded by the graphs of $y = x^2$ and $y = 4$.

Solution: I find it helpful to get a sketch of the region.



__On the interior__: Find the first partial derivatives and set them equal to zero. $f_x(x, y) = 6x = 0$ if $x = 0$ and $f_y(x, y) = 4y - 4 = 0$ if $y = 1$ and so the point $(0,1)$ is a critical point and is inside the region.

On the boundary $y = 4$**:** $f(x, 4) = 3x^2 + 2(4)^2 - 4(4)$ is now a function of *x* only $f(x) = 3x^2 + 16$ and $f'(x) = 6x = 0$ if $x = 0$ so there is a critical point at $(0, 4)$. We know the *y* coordinate because we are only considering $y = 4$.

On the boundary $y = x^2$: $f(x, x^2) = 3x^2 + 2(x^2)^2 - 4(x^2)$ is now a function of *x* only $f(x) = 3x^2 + 2x^4 - 4x^2 = 2x^4 - x^2$ and $f'(x) = 8x^3 - 2x = 2x\left(4x^2 - 1\right) = 0$ if $x = 0$, $x = \frac{1}{2}$, or $x = \frac{-1}{2}$. Now substituting these *x* values into the boundary curve, $y = x^2$, we find critical values of $(0,0)$, $\left(\frac{1}{2}, \frac{1}{4}\right)$, and $\left(\frac{-1}{2}, \frac{1}{4}\right)$.

Vertices (points of intersection of the boundaries): The boundaries $y = x^2$ and $y = 4$ intersect at the points $(2,4)$ and $(-2,4)$.

Now we can make a table of critical points and functional values at each of the critical points.

Critical Points	$(0,1)$	$(0,4)$	$(0,0)$	$\left(\frac{1}{2},\frac{1}{4}\right)$	$\left(\frac{-1}{2},\frac{1}{4}\right)$	$(2,4)$	$(-2,4)$
$f(x,y) = 3x^2 + 2y^2 - 4y$	$0+2-4$ $=-2$	$0+32-16$ $=16$	$0+0-0$ $=0$	$\frac{3}{4}+\frac{2}{16}-1$ $=\frac{-1}{8}$	$\frac{3}{4}+\frac{2}{16}-1$ $=\frac{-1}{8}$	$12+32-16$ $=28$	$12+32-16$ $=28$

From the above table we see the function attains an absolute maximum of 28 at both $(2,4)$ and $(-2,4)$ and an absolute minimum of -2 at $(0,1)$.

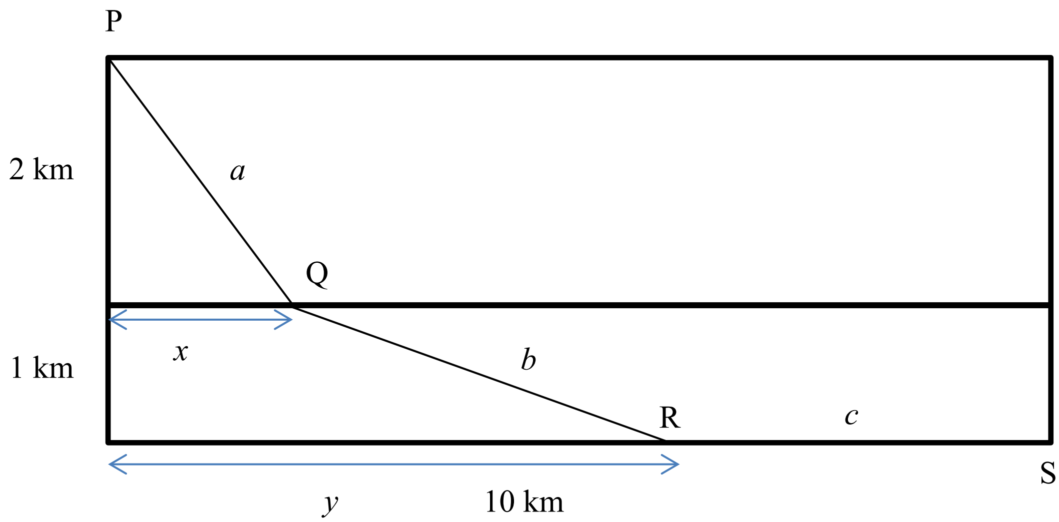
Each time you work through one of these problems you must make sure you have looked for critical points on the interior of the region using partial derivatives, on each boundary equation using calculus I techniques, and don't forget to include any vertices (points of intersection of the boundary equations). The original function then needs to be evaluated at each critical point to determine the absolute maximum and minimum values and where the occur. Any critical points that are not on an included boundary or on the interior need to be rejected as out of bounds.

Make sure to practice these problems, section 13.8 questions 39-46.

Lecture Notes 13.9: Applications

If we can find maximums and minimums it is only natural to apply this situations that are closer to reality. All I can do to help with this section is an example.

Example 1: A power line is to be built from point P to point S and must pass through regions where construction costs differ. The cost per kilometer in dollars is $3k$ from P to Q, $2k$ from point Q to point R and k from point R to point S. See the diagram below. Build a function for the cost of this project in terms of x and y then use optimization techniques from section 13.8 to determine values of x and y that would minimize the total cost.



Solution: From the description the cost of the power line would be $C = 3k(a) + 2k(b) + k(c)$. First we need to express the distances $a, b,$ and c in terms of x and y using the Pythagorean theorem, $a = \sqrt{x^2 + 2^2} = \sqrt{x^2 + 4}$, $b = \sqrt{(y-x)^2 + 1^2} = \sqrt{(y-x)^2 + 1}$, and $c = 10 - y$. Now we can formulate the cost function by multiplying the cost per km by the length of each section. $C(x, y) = 3k\sqrt{x^2 + 4} + 2k\sqrt{(y-x)^2 + 1} + k(10 - y)$ and find the partial derivatives to identify critical values. $C_x(x, y) = 3k\left(\frac{2x}{2\sqrt{x^2 + 4}}\right) + 2k\frac{-2(y-x)}{2\sqrt{(y-x)^2 + 1}} + 0$ then simplifying that we have $C_x(x, y) = k\left(\frac{3x}{\sqrt{x^2 + 4}} + \frac{2x - 2y}{\sqrt{(y-x)^2 + 1}}\right)$ and $C_y(x, y) = 0 + 2k\frac{2(y-x)}{2\sqrt{(y-x)^2 + 1}} + k(-1)$ which simplifies to be $C_y(x, y) = k\left(\frac{2y - 2x}{\sqrt{(y-x)^2 + 1}} - 1\right)$. Setting the partial derivatives equal to 0 give the system

$$k\left(\frac{3x}{\sqrt{x^2 + 4}} + \frac{2x - 2y}{\sqrt{(y-x)^2 + 1}}\right) = 0 \text{ and } k\left(\frac{2y - 2x}{\sqrt{(y-x)^2 + 1}} - 1\right) = 0.$$

Relieve some clutter by dividing off the constant k and rearranging terms.

$\frac{3x}{\sqrt{x^2 + 4}} = \frac{2y - 2x}{\sqrt{(y-x)^2 + 1}}$ and $\frac{2y - 2x}{\sqrt{(y-x)^2 + 1}} = 1$ Now notice the left hand side of the second equation is exactly the same as the right hand side of the first equation so $\frac{3x}{\sqrt{x^2 + 4}} = 1$ then $3x = \sqrt{x^2 + 4}$ and so

$9x^2 = x^2 + 4$. Then $8x^2 = 4$ so $x = \pm\sqrt{\frac{1}{2}}$. We can logically reject the negative case and conclude that $x = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \approx 0.71$ km. Then using that in the equation $\frac{2y-2x}{\sqrt{(y-x)^2+1}} = 1$ we get $\frac{2y-\sqrt{2}}{\sqrt{\left(y-\frac{\sqrt{2}}{2}\right)^2+1}} = 1$ and

simplifying $\frac{2y-\sqrt{2}}{\sqrt{\left(\frac{2y}{2}-\frac{\sqrt{2}}{2}\right)^2+1}} = 1$, $\frac{2y-\sqrt{2}}{\sqrt{\frac{\left(2y-\sqrt{2}\right)^2}{4}+1}} = 1$, $\frac{2y-\sqrt{2}}{\sqrt{\frac{\left(2y-\sqrt{2}\right)^2+4}{4}}} = 1$, $\frac{2y-\sqrt{2}}{\frac{\sqrt{\left(2y-\sqrt{2}\right)^2+4}}{2}} = 1$,

$\frac{2y-\sqrt{2}}{\sqrt{\left(2y-\sqrt{2}\right)^2+4}} = \frac{1}{2}$, $2\left(2y-\sqrt{2}\right) = \sqrt{\left(2y-\sqrt{2}\right)^2+4}$, $\left[2\left(2y-\sqrt{2}\right)\right]^2 = \left[\sqrt{\left(2y-\sqrt{2}\right)^2+4}\right]^2$,

$4\left(2y-\sqrt{2}\right)^2 = \left(2y-\sqrt{2}\right)^2+4$, $3\left(2y-\sqrt{2}\right)^2 = 4$, $\left(2y-\sqrt{2}\right)^2 = \frac{4}{3}$, $2y-\sqrt{2} = \frac{\pm 2}{\sqrt{3}}$, $y = \frac{1}{2}\left(\sqrt{2} \pm \frac{2}{\sqrt{3}}\right)$.

Finally $y \approx 1.28$ km or $y \approx 0.13$ km but $y > x$ allows us to reject $y \approx 0.13$ and so we conclude the minimum cost is attained if $x \approx 0.71$ km and $y \approx 1.28$ km.

And the minimum cost would be $C(0.71, 1.28) = 3k\sqrt{(0.71)^2+4} + 2k\sqrt{(1.28-0.71)^2+1} + k(10-1.28) = \$17.39k$.

Well that was fun!!!

Practice a few of these but don't get too worked up about them.

Section 13.9 problems 7-20.

Lecture Notes 13.10: Lagrange Multipliers

In this final section of chapter 13 we take another look at maximums and minimums. This time we will be looking at extrema over a curve in the *xy* plane rather than a region. Visualize a surface in 3-space then draw a curve in the domain plane and think of the curtain with one end in the plane on the curve with a height that goes up to the surface. French mathematician Joseph-Louis Lagrange was the first to prove that the location of the absolute extrema, maximum and minimum occur when the gradient of the surface is parallel to the gradient of a function which has the plane curve as a level curve. Therefore to locate the critical values of $z = f(x, y)$ subject to a constraint relationship expressed as $g(x, y) = 0$ we need to identify locations where $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$ is parallel to $\nabla g(x, y) = g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j}$. Recall from chapter 11 that two vectors are parallel if one vector is equal to a constant multiple of the other. In honor of Lagrange we use the Greek letter lambda, λ, as the constant multiplier. Hence the title, Lagrange Multipliers. In fact this method generalizes up to higher dimensions as well although the algebra involved gets more complicated as we increase the dimensions.

When asked to maximize and/or minimize a function $f(x, y)$ subject to the constraint $g(x, y) = 0$ we need only solve the system of equations generated by the vector equation $\nabla f(x, y) = \lambda \nabla g(x, y)$. Since $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$ and $\lambda \nabla g(x, y) = \lambda\left(g_x(x, y)\mathbf{i} + g_y(x, y)\mathbf{j}\right) = \lambda g_x(x, y)\mathbf{i} + \lambda g_y(x, y)\mathbf{j}$ the equality of these vectors means that $f_x(x, y) = \lambda g_x(x, y)$ and $f_y(x, y) = \lambda g_y(x, y)$. Now then we have 2 equations but we have 3 unknowns, x, y and λ. From algebra we know that if we have 3 unknowns then we must have 3 equations to determine a unique solution. The third equation we need comes from the constrain, $g(x, y) = 0$. It is worthwhile to note that we need λ to build the equations since the theory only requires that the gradients be parallel but we rarely need or even bother to find a value for λ.

Example 1: Use the method of Lagrange multipliers to maximize the function $f(x, y) = x^2 - y^2$ subject to the constraint $2y - x^2 = 0$. Now what this means is that the only ordered pairs that can used in $f(x, y)$ are those that lie on the graph of $y = \dfrac{x^2}{2}$.

Solution: Let $g(x, y) = 2y - x^2$, the constraint. Then find the gradient of *f* and the gradient of *g*. $\nabla f(x, y) = 2x\mathbf{i} - 2y\mathbf{j}$ and $\lambda \nabla g(x, y) = \lambda(-2x)\mathbf{i} + \lambda(2)\mathbf{j}$ and $\nabla f(x, y) = \lambda \nabla g(x, y)$ if and only if $2x = -2x\lambda$ and $-2y = 2\lambda$ and $2y - x^2 = 0$ thus forming a 3 equation system with 3 unknowns.

$$\begin{aligned} 2x &= -2x\lambda \\ -2y &= 2\lambda \\ 2y - x^2 &= 0 \end{aligned}$$

To solve this system we quickly see from the top equation either $\lambda = -1$ or $x = 0$. We must be careful not to divide both sides of that top equation by the variable *x* without first considering that if $x = 0$ then the value of λ can be anything. The second equation indicates that $\lambda = -y$. Now putting those pieces together, if $x = 0$ then $2y - 0^2 = 0$ and so $y = 0$ then the point $(0,0)$ is a critical point. However if $\lambda = -1$ then the second equation becomes $-2y = -2$ and so $y = 1$ then $2(1) - x^2 = 0$ and we see $x = \sqrt{2}$ or $x = -\sqrt{2}$ so we have critical points of $\left(\sqrt{2}, 1\right)$ and $\left(-\sqrt{2}, 1\right)$. Now to determine the maximum value of $f(x, y) = x^2 - y^2$ we need only evaluate the function at each critical value. $f(\sqrt{2}, 1) = 2 - 1 = 1$, $f(-\sqrt{2}, 1) = 2 - 1 = 1$, and $f(0,0) = 0 - 0 = 0$ so the maximum value of $f(x,y)=x^2-y^2$ subject to the constrain $2y-x^2=0$ is 1 and that occurs at both $\left(\sqrt{2},1\right)$ and $\left(-\sqrt{2},1\right)$

Example 2: Use the method of Lagrange multipliers to maximize the function $f(x,y,z)=xyz$ subject to the constraint $x+y+z=3$.

Solution: Let $g(x,y,z)=x+y+z-3$, the constraint set equal to 0. Then find the gradient of *f* and the gradient of *g*. $\nabla f(x,y,z)=yz\,\mathbf{i}+xz\,\mathbf{j}+xy\,\mathbf{k}$ and $\lambda\nabla g(x,y,z)=\lambda(1)\,\mathbf{i}+\lambda(1)\,\mathbf{j}+\lambda(1)\,\mathbf{k}$ so we can form the system of 4 equations in 4 unknowns.

$$
\begin{aligned}
yz &= \lambda \\
xz &= \lambda \\
xy &= \lambda \\
x+y+z &= 3
\end{aligned}
$$

Since λ must equal itself we can build several equations from the top 3 equations above. $yz=xz$ and $yz=xy$. Again we need to be careful with these, the first tells us that either $z=0$ or $x=y$. Assuming $z=0$ and substituting into the second equation then leads us to $xy=0$ so either $x=0$ or $y=0$. The constrain $x+y+z=3$ then helps up to identify the critical values $(3,0,0)$, $(0,3,0)$, $(0,0,3)$. If we assume that $x=y\neq 0$ then we $xz=xy$ and so $y=z$ and substituting into the constrain gives us $x+x+x=3$ which leads to the conclusion that $x=y=z=1$ and the critical point $(1,1,1)$. Evaluating the function $f(x,y,z)=xyz$ at each critical point gives us $f(3,0,0)=f(0,3,0)=f(0,0,3)=0$ and $f(1,1,1)=1$ so the maximum value of $f(x,y,z)=xyz$ subject to $x+y+z=3$ is 1 which occurs at the point $(1,1,1)$.

This method will also work in cases where there are more than one constraint. To find the extrema of $f(x,y,z)$ subject to $g(x,y,z)=0$ and $h(x,y,z)=0$ we need to find points where all three gradients are parallel so we set up equations based on $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)$. Of course this would lead to a system of 5 equations with 5 unknowns and the algebra involved can quickly become unbearable but these types of systems can be solved nicely through the use of techniques taught in a linear algebra or numerical methods course.