Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 3: 6/8-6/14
Sections 12.2 – 12.5, Calculus of Vector Valued Functions, Velocity and Acceleration, Tangent lines and Normal vectors, Arc length and Curvature.

Due this week:

Tuesday 6/9/2020	**Thursday 6/11/2020**	**Sunday 6/14/2020**
Chapter 11 Assessment	WebAssign 12.1-12.4	Weekly Assignment 3, 12.1-12.5

Lecture Notes 12.2: Calculus of Vector Valued Functions

In the preceding section we looked at evaluating vector valued functions and limits of these functions. Now we will look at applying our derivative limit. No you will not have to find any derivatives by limit but it is good to know that you could. We define the derivative with respect to the domain variable, t, of a vector valued function $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}$ in the plane and $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$ in space as $\mathbf{r}'(t)=f'(t)\mathbf{i}+g'(t)\mathbf{j}$ or $\mathbf{r}'(t)=f'(t)\mathbf{i}+g'(t)\mathbf{j}+h'(t)\mathbf{k}$. Since the derivative of a vector valued function is defined in terms of the derivatives of the real functions that serve as coefficients to the basis vectors **i**, **j**, and **k** we only have to remember our differentiation rules from calculus I. We can also apply second and higher order derivatives. In the next few sections we will add interpretations of these derivatives but for now we are just going to focus on the mechanics of finding derivatives.

Example 1: Derivatives of Vector Valued Functions. Find the derivative of $\mathbf{r}(t)=\sqrt{t+2}\ \mathbf{i}+\dfrac{4}{t-5}\ \mathbf{j}+(t^2-4)\mathbf{k}$. **Solution**: Using the chain rule from calculus I we get $\mathbf{r}'(t)=\dfrac{1}{2\sqrt{t+2}}\ \mathbf{i}-\dfrac{4}{(t-5)^2}\ \mathbf{j}+2t\mathbf{k}$. Note, the derivative of a vector valued function is always another vector valued function.

Logically, if we can find a derivative we should also be able to find an antiderivative or integral. The antiderivative of $\mathbf{r}(t)=f(t)\mathbf{i}+g(t)\mathbf{j}$ can be found by finding the antiderivative of the individual real valued coefficient functions, $\int \mathbf{r}(t)\,dt=\int f(t)\,dt\,\mathbf{i}+\int g(t)\,dt\,\mathbf{j}$ and of course we can generalize this up to higher dimensional vector valued functions. Recall that any time we calculated an antiderivative we had to remember the constant of integration, c and this is still the case although since we finding the antiderivative of a vector valued function our c term takes on the form of a constant vector in the same dimensional space as the original vector valued function.

Example 2: Antiderivatives of Vector Valued Functions. Find the antiderivative of $\mathbf{r}(t)=\sqrt{t+2}\ \mathbf{i}+\dfrac{4}{t-5}\ \mathbf{j}+(t^2-4)\mathbf{k}$. **Solution**: Using integration techniques learned in calculus I we find $\int \mathbf{r}(t)\,dt=\int \sqrt{t+2}\,dt\ \mathbf{i}+\int \dfrac{4}{t-5}dt\ \mathbf{j}+\int (t^2-4)\,dt\,\mathbf{k}$. Now we evaluate the individual integrals to get $\int \mathbf{r}(t)\,dt=\dfrac{2(t+2)^{3/2}}{3}\ \mathbf{i}+4\ln|t-5|\ \mathbf{j}+\left(\tfrac{1}{3}t^3-4t\right)\mathbf{k}+\mathbf{C}$ where $\mathbf{C}$ is a constant vector in 3-space, $\mathbf{C}=\langle c_1,c_2,c_3\rangle$.

We can also evaluate a definite integral of a vector valued function in in exactly the same way, only in such a case the individual integrals would have limits of integration.

Example 3: Solving Differential Equations Involving Vector Valued Functions. Given the derivative of a vector valued function $\mathbf{r}'(t) = (t+2)\,\mathbf{i} + \left(4t^3 - 6t\right)\mathbf{j} + (t^2 + 3t)\,\mathbf{k}$ and information about the value of the vector function $\mathbf{r}(2) = \langle 4, 2, -5\rangle$ find the vector valued function.

Solution: Just like solving a differential equation like we did in calculus I and II we first find the antiderivative. $\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \int (t+2)\,dt\ \mathbf{i} + \int \left(4t^3 - 6t\right)dt\,\mathbf{j} + \int (t^2 + 3t)\,dt\,\mathbf{k}$ now calculate the antiderivatives so $\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 2t + c_1\right)\mathbf{i} + \left(t^4 - 3t^2 + c_2\right)\mathbf{j} + \left(\frac{1}{3}t^3 + \frac{3}{2}t^2 + c_3\right)\mathbf{k}$. This would be what we called a general solution because each vector component contains a constant of integration. I would then evaluate my antiderivative at the given value of *t*. $\mathbf{r}(2) = \left(\frac{1}{2}(2)^2 + 2(2) + c_1\right)\mathbf{i} + \left((2)^4 - 3(2)^2 + c_2\right)\mathbf{j} + \left(\frac{1}{3}(2)^3 + \frac{3}{2}(2)^2 + c_3\right)\mathbf{k}$ and simplify $\mathbf{r}(2) = (6 + c_1)\mathbf{i} + (4 + c_2)\mathbf{j} + \left(\dfrac{26}{3} + c_3\right)\mathbf{k} = \langle 4, 2, -5\rangle$ then equate the components to the given vector. $6 + c_1 = 4$, $4 + c_2 = 2$, and $\dfrac{26}{3} + c_3 = -5$ then solve for the constants of integration $c_1 = -2$, $c_2 = -2$, and $c_3 = -\dfrac{41}{3}$.

Finally we can rewrite the vector valued function with the constants in place to form the particular solution. $\mathbf{r}(t) = \left(\frac{1}{2}t^2 + 2t - 2\right)\mathbf{i} + \left(t^4 - 3t^2 - 2\right)\mathbf{j} + \left(\frac{1}{3}t^3 + \frac{3}{2}t^2 - \frac{41}{3}\right)\mathbf{k}$.

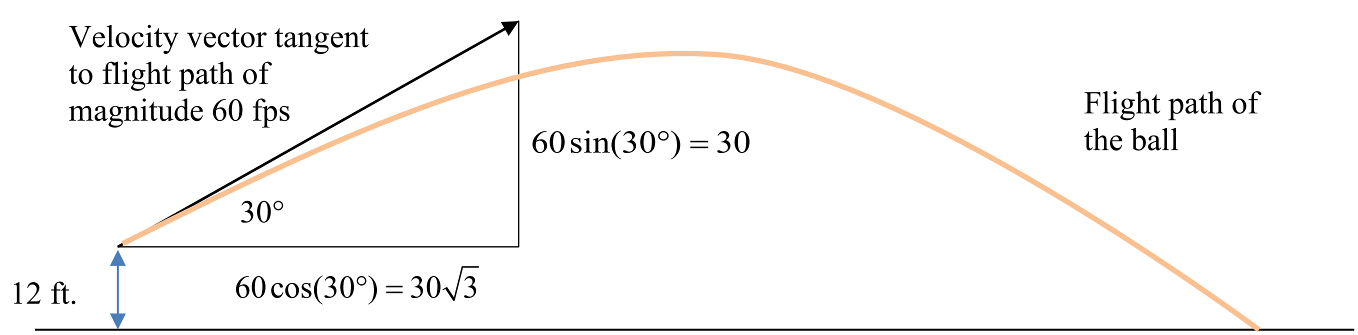
Lecture Notes 12.3: Velocity and Acceleration, Projectile Motion

Recall in calculus I we talked about rectilinear motion. Most people just ignored the rectilinear part but it is important to note that we were talking about motion in a straight line. One independent variable that could be thought of as time and one dependent variable for location along a line. So when we find the first derivative we would be finding how location changes with respect to time, change in position over time or **velocity**. And if we calculate the second derivative we are finding the change in velocity over time or **acceleration.** Now with vector valued functions the concepts of velocity and acceleration are even easier to understand. As a particle travels along a path generated by a vector valued function, the **position** of the particle at any time t is given by the function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ for a curve in the plane or $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ for a space curve. Then the derivative $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ is a vector function in the same dimension which gives the particle's velocity and the second derivative $\mathbf{r}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j}$ is another vector in the same dimension which gives the particle's acceleration. These concepts apply to both the 2-dimensional plane and 3-dimensional space.

Definitions and Formulas:
Position vector - $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ With tail at the origin this vector identifies points on a plane or space curve with the head points.
Velocity vector - $\mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$ is a vector in the direction tangential to the curve.
Acceleration vector - $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}$ is a vector in the direction of the acceleration, this vector will always point in the direction the curve is turning.
For a particle traveling on the curve defined by $\mathbf{r}(t)$ the **Speed** of the particle is the magnitude of the velocity vector, $\|\mathbf{v}(t)\|$. Recall velocity has direction and speed does not.

Speed - $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2 + \left(h'(t)\right)^2}$

Now let's use this information to build something useful. We have studied projectile motion for many semesters. Recall the function we were given for the height of a projectile at any time t has always been $h(t) = v_0 t - \frac{1}{2}gt^2 + h_0$ where v_0 is the initial vertical velocity, g is the acceleration due to gravity, and h_0 is any initial height. For example, if a ball is thrown upward with an initial velocity of 60 feet per second from a height of 12 feet above ground level what is the equation for the ball's height at any time, t? Using the formula above $h(t) = 60t - \frac{1}{2}(32)t^2 + 12 = 60t - 16t^2 + 12$ gives the height of the ball after t seconds. But that is the old way and all we were calculating is the height. More realistically consider throwing that same ball with the initial velocity of 60 feet per second but rather than throwing it straight up this time lets throw it at an angle of $30°$ with the horizontal from the same initial height of 12 feet above ground. Now our motion takes place in 2 dimensions, horizontal and vertical. This means that the velocity is divided into a vertical component and a horizontal component based on the angle of projection.



We can build the equation of motion knowing only that the acceleration due to gravity in English units is 32 feet per second per second. Since this vector is pointing downward only we begin with the acceleration vector,

$\mathbf{a}(t) = 0\,\mathbf{i} - 32\,\mathbf{j}$. Using integration we get $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int 0\,\mathbf{i}\,dt - \int 32\,\mathbf{j}\,dt = c_1\,\mathbf{i} + (-32t + c_2)\,\mathbf{j}$. And $\mathbf{v}(0) = c_1\,\mathbf{i} + (0 + c_2)\,\mathbf{j}$. But we know that the initial velocity has vector equation $\mathbf{v}(0) = 30\sqrt{3}\,\mathbf{i} + 30\,\mathbf{j}$ from the diagram above. So now we can equate these to find the constants of integration. $c_1 = 30\sqrt{3}$ and $c_2 = 30$ so the velocity equation becomes $\mathbf{v}(t) = 30\sqrt{3}\,\mathbf{i} + (-32t + 30)\,\mathbf{j}$. Finally we integrate again to get to the position vector and use the fact that the initial position was given by $\mathbf{r}(0) = 0\,\mathbf{i} + 12\,\mathbf{j}$ to find the constants of integration.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int 30\sqrt{3}\,dt\,\mathbf{i} + \int (-32t + 30)\,dt\,\mathbf{j} = \left(30\sqrt{3}(t) + c_3\right)\mathbf{i} + \left(-16t^2 + 30t + c_4\right)\mathbf{j}$ then

$\mathbf{r}(0) = \left(0 + c_3\right)\mathbf{i} + \left(0 + c_4\right)\mathbf{j} = (0)\,\mathbf{i} + (12)\,\mathbf{j}$ and so $c_3 = 0$, $c_4 = 12$, and so $\mathbf{r}(t) = \left(30\sqrt{3}(t)\right)\mathbf{i} + \left(-16t^2 + 30t + 12\right)\mathbf{j}$.

We don't expect everyone to rederive this vector equation for every projectile motion question so let's take a look at the equation in a more standard form.

Projectile Motion: Position is given by $\mathbf{r}(t) = v_0 \cos(\theta)\, t\ \mathbf{i} + \left(-\frac{1}{2} g t^2 + v_0 \sin(\theta)\, t + h_0\right)\mathbf{j}$ where v_0 is the initial velocity, θ is the angle with the horizontal from which the projectile is fired, $g = 32\ ft/s^2$ if English units are used or $g = 9.8\ m/s^2$ if metric units are used, finally h_0 represents any initial height.

Notes: To find out when a projectile hits the ground set the **j** component to zero and solve for *t*. To find out how far a projectile travels in *t* seconds evaluate the **i** component at the given *t* value. You can work with these two components independently to solve for whichever variable you need based on what your given information is.

Lecture Notes 12.4: Tangent Vectors and Normal Vectors.

We have already seen that for any curve with position function $\mathbf{r}(t)$ that the derivative $\mathbf{r}'(t)$ is tangent to the curve either in space or in the plane. While $\mathbf{r}'(t)$ is often a suitable tangent vector there are times when we don't want the magnitude of the tangent vector just the direction. For those instances we define the **<u>Unit Tangent Vector</u>** $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$. As we saw earlier, a vector scaled by the reciprocal of it's own magnitude is always a unit vector.

Now with $\mathbf{T}(t)$ as a unit vector tangent to any curve at any point on the curve we are next interested in what is known as the **<u>Principal Normal Vector</u>**, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$. $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are a pair of orthogonal unit vectors much like **i** and **j** which define the *xy* plane. Unit tangent vectors and principal normal vectors form a basis for curves in the plane relative to the curve. That is to say, they are just like **i** and **j** but instead of being based on an ultimate origin these are based at points on the curve. If we have a 3-dimensional space curve we would need a third unit vector to complete the reference frame. This third vector is known as the **<u>Binormal Vector</u>**, $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. Thus the binormal vector is also a unit vector orthogonal to both the tangent vector and the principal normal vector. When trying to visualize these vectors think about an airplane doing acrobatics in the air. The path of the plane is a space curve, $\mathbf{T}(t)$ is always going to be pointing in the direction the airplane is traveling, $\mathbf{N}(t)$ will be pointing out one of the wings away from the body of the airplane, $\mathbf{N}(t)$ provides a sense of left and right although it changes as the airplane (curve) changes how it is turning and $\mathbf{N}(t)$ will always be in the direction of curve. The binormal finishes this with a sense of up and down relative to the airplane and regardless of the absolute origin. Since $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ and cross product follows the right hand rule (if you lay your right hand down so the fingers are in the direction of $\mathbf{T}(t)$ and curl toward $\mathbf{N}(t)$ then $\mathbf{B}(t)$ will be in the direction of your thumb), $\mathbf{B}(t)$ will also change depending on the way the curve is turning.

Finally, for any curve we can separate the acceleration into components just like we did with velocity in the projectile motion application in the previous section. The acceleration can be thought of as consisting of a tangential component, acceleration that is in the direction of motion, and a normal component, acceleration that is devoted to turning the curve. The normal component of acceleration is denoted by $a_{\mathbf{N}}$ and the tangential component is denoted as $a_{\mathbf{T}}$. These are NOT vectors, they are scalars because the notion of dividing the acceleration into tangential and normal components means we want the acceleration vector to be $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$.

Computationally $a_{\mathbf{T}} = \dfrac{\mathbf{v}(t)\bullet\mathbf{a}(t)}{\|\mathbf{v}(t)\|} = \dfrac{\mathbf{v}\bullet\mathbf{a}}{\|\mathbf{v}\|} = \dfrac{\mathbf{r}'\bullet\mathbf{r}''}{\|\mathbf{r}'\|}$ and $a_{\mathbf{N}} = \dfrac{\|\mathbf{v}(t)\times\mathbf{a}(t)\|}{\|\mathbf{v}(t)\|} = \dfrac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \dfrac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|}$.

I will be posting some examples of these calculations this week as they can be rather intense at times.

Lecture Notes 12.5: Arc Length and Curvature.

We wrap up our initial discussions of vector valued functions with the ideas of arc length and curvature. Much like the previous section we are going to be looking at some straightforward formulas. In a traditional face to face class I would require everyone to know these formulas but because you are all already disadvantaged by having to learn upper level calculus at a distance you will always have access to these formulas. After all, in the "real" world if you ever need the formula for the length of a curve you would certainly look it up.

For all of these formulas we are assuming we have a curve, either in the plane, $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, or in space, $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$.

Arc Length: The length of a curve is denoted by the letter s which itself is strange because there is no s in arc length but s is the letter of the alphabet with the smoothest curves and since we are measuring the length of a curve we use s. The arc length over an interval from $t = t_1$ to $t = t_2$ is given by $s = \int_{t=t_1}^{t=t_2} \|\mathbf{r}'(t)\|\, dt$. Hopefully this formula makes sense in the context of a definite integral being the sum of infinitely many infinitely small increments, in this case the increments are $\|\mathbf{r}'(t)\| = \|\mathbf{v}(t)\|$ which was defined in section 12.3 to be speed. So we are adding up infinitely many infinitely small products of Rate times Time which is of course, distance!

Now if we want to form a function for arc length for a smooth curve over an interval from $t = a$ to any value of the parameter t we need only evaluate $s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du$. Here s is known as the arc length parameter because the value of $s(t)$ is the length of the curve over the interval from a to t. Using the second fundamental theorem of calculus from calculus I we know that $\frac{d}{dt}\left(s(t)\right) = \frac{d}{dt}\left(\int_a^t \|\mathbf{r}'(u)\|\, du\right) = \|\mathbf{r}'(t)\|$ and so $s'(t) = \|\mathbf{r}'(t)\|$.

Further, if $s = s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du = Ct$ we can solve for t in terms of s. $t = \frac{s}{C}$ and then rewrite the vector valued function for the curve, $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, in terms of the arclength parameter, s. $\mathbf{r}(s) = f\left(\frac{s}{C}\right)\mathbf{i} + g\left(\frac{s}{C}\right)\mathbf{j} + h\left(\frac{s}{C}\right)\mathbf{k}$ where $\mathbf{r}(s)$ gives the coordinates of the point on the curve which is a distance along the curve of s units away from a fixed beginning.

Example 1: Given the helix $\mathbf{r}(t) = (5\cos t)\mathbf{i} + (5\sin t)\mathbf{j} + 4t\mathbf{k}$ (A) find the arc length of the helix on the interval from $t = 0$ to $t = 2\pi$. (B) find the arc length parameter, s. (C) express the helix in terms of it's arc length parameter. (D) find the point on the curve for an arc length of 30 units.

Solution (A) $s = \int_0^{2\pi} \|\mathbf{r}'(t)\|\, dt = \int_0^{2\pi} \sqrt{(-5\sin t)^2 + (5\cos t)^2 + 4^2}\; dt = \int_0^{2\pi} \sqrt{41}\; dt = \sqrt{41}\, t\Big|_0^{2\pi} = \left(2\sqrt{41}\right)\pi \approx 40.23$

Solution (B) $s = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t \sqrt{(-5\sin u)^2 + (5\cos u)^2 + 4^2}\; du = \int_0^t \sqrt{41}\; du = \sqrt{41}\, u\Big|_0^t = \sqrt{41}\, t,\ \ s = \sqrt{41}\, t.$

Solution (C) $t = \frac{s}{\sqrt{41}},\ \ \mathbf{r}(s) = 5\cos\left(\frac{s}{\sqrt{41}}\right)\mathbf{i} + 5\sin\left(\frac{s}{\sqrt{41}}\right)\mathbf{j} + \left(\frac{4s}{\sqrt{41}}\right)\mathbf{k}$

Solution (D) $\mathbf{r}(30) = 5\cos\left(\frac{30}{\sqrt{41}}\right)\mathbf{i} + 5\sin\left(\frac{30}{\sqrt{41}}\right)\mathbf{j} + \left(\frac{120}{\sqrt{41}}\right)\mathbf{k}$ or approximately $(-0.14, -5.00, 18.74)$

Our last goal is to find a way to measure how sharply a curve is curving. We want a curve to curving sharply if it has a large curvature value and to be nearly straight if it has a curvature value that is close to 0. Nearly everyone understands a circle, it is the ultimate in a consistent curve. A circle of radius 1 in. is curving much more sharply than a circle of radius 10 inches and so forth. But this is exactly backwards from what we wanted so to reconcile this we define the curvature, K, to be the rate of curve associated with a circle of radius $\frac{1}{K}$. In that way when a space curve has a curvature of 7 units we know it is curving as sharply as a circle of radius $\frac{1}{7}$ units and if the space curve has a curvature value of 0.1 units we picture a circle of radius $\frac{1}{0.1}=10$ units. Now that the concept is in place we need to examine the calculations associated with curvature. There are several formulas, some dealing with the rate of change in the arc length parameter, but the clearest and easiest to use in general is the following.

<u>**Curvature**</u> - $K=\dfrac{\left\|\mathbf{v}\times\mathbf{a}\right\|}{\left\|\mathbf{v}\right\|^3}=\dfrac{\left\|\mathbf{r}'\times\mathbf{r}''\right\|}{\left\|\mathbf{r}'\right\|^3}$

Example 2: Find the curvature of $\mathbf{r}(t)=4t\,\mathbf{i}+\left(3\cos t\right)\mathbf{j}+\left(3\sin t\right)\mathbf{k}$.

Solution: $\mathbf{r}'(t)=4\,\mathbf{i}+\left(-3\sin t\right)\mathbf{j}+\left(3\cos t\right)\mathbf{k}=\mathbf{v}(t)$ and $\mathbf{r}''(t)=0\,\mathbf{i}+\left(-3\cos t\right)\mathbf{j}+\left(-3\sin t\right)\mathbf{k}=\mathbf{a}(t)$ then

$$K=\frac{\left\|\left\langle 4,-3\sin t,3\cos t\right\rangle\times\left\langle 0,-3\cos t,-3\sin t\right\rangle\right\|}{\left\|\left\langle 4,-3\sin t,3\cos t\right\rangle\right\|^3}=\frac{\left\|\left\langle 9\sin^2 t--9\cos^2 t,-\left(-12\sin t-0\right),-12\cos t-0\right\rangle\right\|}{\left(\sqrt{16+9\sin^2 t+9\cos^2 t}\right)^3}$$

$$K=\frac{\left\|\left\langle 9\sin^2 t--9\cos^2 t,-\left(-12\sin t-0\right),-12\cos t-0\right\rangle\right\|}{\left(\sqrt{16+9\sin^2 t+9\cos^2 t}\right)^3}=\frac{\left\|\left\langle 9,12\sin t,-12\cos t\right\rangle\right\|}{\left(\sqrt{16+9}\right)^3}$$

$K=\dfrac{\sqrt{81+144\sin^2 t+144\cos^2 t}}{(5)^3}=\dfrac{\sqrt{225}}{125}=\dfrac{15}{125}=\dfrac{3}{25}$ and so everywhere on this curve the curvature is the same as that of a circle of radius $\frac{25}{3}$.

You may note that this space curve is a helix (spring shape) that spirals out around the x axis. The curve is bounded by the circular cylinder of radius 3 in perpendicular to the yz plane. Why then is the curvature not $\frac{1}{3}$? Because as the curve circles the cylinder it also moves away from the yz plane reducing the rate of curve.