Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 4: 6/15-6/21
Sections 13.1 – 13.3, Functions of Several Variables, Limits, Partial Derivatives.

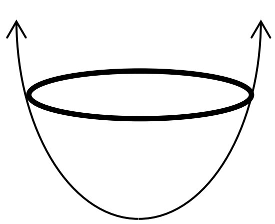
Due this week:

Tuesday 6/16/2020	**Thursday 6/18/2020**	**Sunday 6/21/2020**
Chapter 12 Assessment	WebAssign 12.5, 13.1, 13.2	Weekly Assignment 4, 13.1-13.3

Lecture Notes 13.1: Functions of Several Variables

We are setting aside our study of vectors for the time being and continuing our study of functions. Recall how in previous courses functions were mappings with a domain that was a subset of the real numbers and a range that was also a subset of the reals. Then in the previous chapter we looked at vector valued functions with a domain that was again a subset of the real numbers but the range or output was a subset of the set of vectors in either 2-space or 3-space depending on the function. Now we are going to consider a type of function with a domain that is a subset of the *xy* plane and whose range is a subset of the real numbers. We will also generalize this type of function to those with a domain of any n-dimensional space but the range will continue to be a subset of the reals (one dimensional).

Recall in college algebra or earlier you looked at functions of the form $y = f(x)$. Where the domain was a subset of the real line from which the *x* values were drawn and the range was a subset of the real line made up of the outputs or *y* values of the function. Now we will primarily be looking at functions of the form $z = f(x, y)$ where the domain is a set of ordered pairs (x, y) from $\mathbb{R}^2$ (read R two) which denotes 2-dimensional real space (also known as the *xy* plane) and the outputs or *z* values are real numbers.
Now we have already looked at surfaces in 3-dimensional space like the paraboloid $z = x^2 + y^2$ and we should have a fairly solid idea of that this surface looks like. Horizontal cross sections are circles of radius $\sqrt{z}$ for any given value of *z*. Vertical cross sections parallel to the *xz* plane are parabolas of the form $z = x^2 + a^2$ for all values of $y = a$ and likewise any cross section parallel to the *yz* plane would be a parabola of the form $z = y^2 + b^2$ for all values of $x = b$. This familiar surface can be written in functional notation as $f(x, y) = x^2 + y^2$ and we can use these forms interchangeably like we do with $y = x^2$ and $f(x) = x^2$.

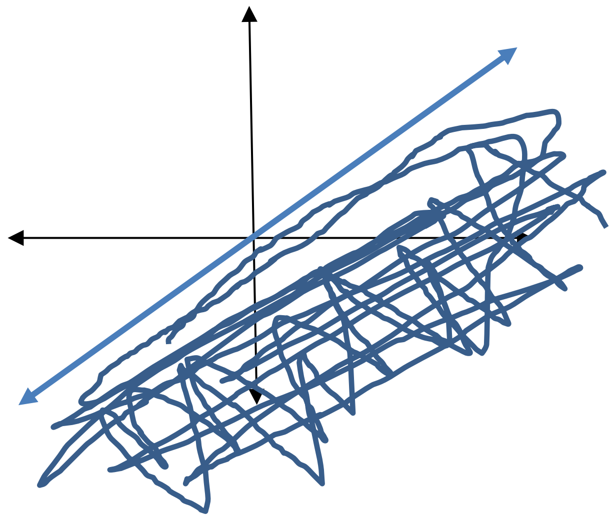


This is an example of a function of two variables. The natural domain would be the set of all ordered pairs (x, y) from $\mathbb{R}^2$ since the sum of two squares is always defined , written $\{(x, y) | -\infty < x < \infty, -\infty < y < \infty\}$ The range however would be $[0, \infty)$ since the sum of squares is always greater than or equal to zero.

The natural domain is the set of ordered pairs for which the function is defined. We will be more interested in the domain than the range but for functions with logical ranges like the above we may ask for both.

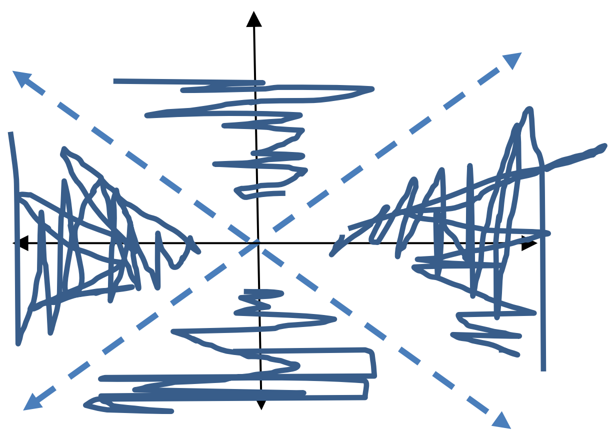
Example 1: Domain and Range. State the domain and range of the function $f(x, y) = \sqrt{x - y}$
Solution: Domain $\{(x, y) \mid y \le x\}$ because we need $x - y \ge 0$ and solving for *y* gives us $y \le x$. Range: $[0, \infty)$.

We can and will also be graphing the domain of these functions. In the above example the domain would be graphed as:



Where we graph the line $y = x$ and shade below the line to show $y \le x$

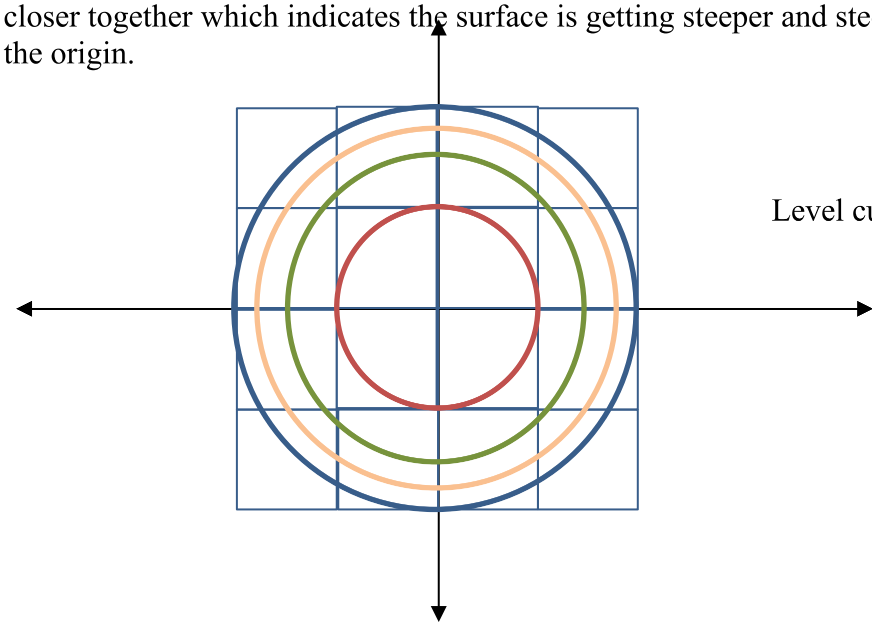
Example 2: Domain and Range. State the domain and range of the function $f(x, y) = \dfrac{1}{x^2 - y^2}$ and sketch a graph of the domain. **Solution:** Domain $\{(x, y) \mid y \ne \pm x\}$ or $\{(x, y) \mid |y| \ne |x|\}$ clearly the domain is restricted to only ordered pairs such that $x^2 - y^2 \ne 0$ and we know that $x^2 = y^2$ if *x* and *y* are either equal or opposite. Range: $(-\infty, 0) \cup (0, \infty)$ since for very small denominators the ratio approaches infinity and for large denominators the ratio would approach zero.



We graph the lines $y = x$ and $y = -x$ as dottedn lines to indicate the domain consists of all values up to but not including these lines.

The domain of a function of two variables is not the only graph we can draw in the *xy* plane. Level curves of a function of two variables can also be graphed in the *xy* plane. A level curve is a horizontal cross section of a 3-dimensional graph that has been projected onto the *xy* plane. That is to say, if we select values of the dependent variable, *z*, and graph the relationship between *x* and *y* for those fixed values of *z* we would have multiple 2-dimensional graphs, each corresponding to a *z* value. This is just like a topographical map where the contour lines indicate altitude or a weather map where the isobars indicate locations where the barometric pressure is the same. None of these ideas are new to us but this is likely the first time we have constructed such a graph for a function. Consider the original example $f(x, y) = x^2 + y^2$ or $z = x^2 + y^2$ for several values of the variable *z* such as $z = 0, 1, 2, 3, 4$ we would have the equations $x^2 + y^2 = 0$, $x^2 + y^2 = 1$, $x^2 + y^2 = 2$, $x^2 + y^2 = 3$, and

$x^2 + y^2 = 4$ each of which graphs a circle of radius $0, 1, \sqrt{2}, \sqrt{3}, 2$ respectively and we can carefully graph these circles. And we see that as the radius increases (for increasing z values) the concentric circles get closer and closer together which indicates the surface is getting steeper and steeper as the ordered pairs move away from the origin.



Level curves of $f(x, y) = x^2 + y^2$

Lecture Notes 13.2: Limits and Continuity

We can take a limit of a function of several variables in the same way we did in calculus I when we were first exposed to them. Recall $\lim_{x\to a} f(x) = f(a)$ provided $f(a)$ existed. Well $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ provided $f(a,b)$ exists. Recall as well that we rarely were asked to find a limit that was so nice. They just aren't that interesting. It was much more fun to try to evaluate a limit like $\lim_{x\to 0} \frac{\sin x}{x}$ or some other indeterminate limit form. By the way, you all do remember that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ right? Also remember how much better indeterminant limits got after we learned L'Hopital's Rule? Remember L'Hopital's Rule, essentially it said $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided either $\lim_{x\to a} f(x) = 0$ **and** $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ **and** $\lim_{x\to a} g(x) = \pm\infty$. Well unfortunately L'Hopital's Rule doesn't apply to multivariable limits because it requires the use of derivatives and derivatives can only be taken with respect to a single independent variable. This mean that if we have an indeterminant limit form we will need to make use of our algebraic manipulation tools such as factoring and canceling.

What does it mean when we say $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$? It means that as an ordered pair (x_0, y_0) gets closer and closer to the point (a,b) regardless of direction then the functional value, $f(x_0, y_0)$, is getting closer and closer to $f(a,b)$. It is important to remember that a limit means (x_0, y_0) approaches but never equals (a,b). To help with this idea lets go back to calculus I. We said that for any error tolerance commonly called epsilon, $\varepsilon > 0$, that would define an open interval on the *y* axis $(L-\varepsilon,\ L+\varepsilon)$ around the outcome *L* we should be able to find an open interval on the *x* axis $(a-\delta,\ a+\delta)$, commonly known as precision, so that for any *x* from the $(a-\delta,\ a+\delta)$ interval except possibly $x = a$ the functional value at that *x* would lie within the $(L-\varepsilon, L+\varepsilon)$ interval. To be precise in calc I $\lim_{x\to a} f(x) = L$ meant that for each epsilon, $\varepsilon > 0$, there is a delta, $\delta > 0$, such that whenever $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$. Now we just extend the idea, the error tolerance, $\varepsilon > 0$ remains just the same and the idea of the $\delta > 0$ precision generalizes to mean any ordered pair (x_0, y_0) within a circle of radius $\delta > 0$, except possibly the point (a,b), when evaluated in the function will have a functional value in the interval $(L-\varepsilon, L+\varepsilon)$.

You may not have seen any of this is calculus I, several instructors no longer take the time to talk about it and that is sad. Others of you may not remember talking about this because it was likely only discussed once. But now you have seen it again and maybe with a few more exposures it will begin to make some sense.

Recall again from calculus I a statement along the lines of this: If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ then $\lim_{x\to a} f(x)$ does not exist. Or in other words if the limit from the left is different from the limit taken from the right then the general limit which is from both directions at the same time can not exist. In this course we will make more extensive use of this concept to prove that limits don't exist. Given a limit $\lim_{(x,y)\to(a,b)} f(x,y)$ that we need to evaluate we should always first find $f(a,b)$ since if that exists then that is the limit. Then if $f(a,b)$ does not exist in some indeterminant way we can consider different "paths" that contain the point (a,b). A path is any equation in 2 variables for which (a,b) is a solution. We can then make a substitution in the original limit based on the path we choose and evaluate the single variable limit using calc I techniques (including L'Hopital's Rule if necessary) to find $\lim_{(x,y)\to(a,b)} f(x,y)$ on that particular path. It is important to note that this result may not be "the limit" since it only exists on one specified path. We can then choose another path containing the point (a,b) and evaluate the limit along that path. If the results are different then we can conclude that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist because the limits are path dependent, just like being different from the left and right from calc I. If the results are the same however we cannot conclude that the limit exists, only that the paths which we chose led to equal limits. We can choose path after path and not matter how many times the results are equal we can never conclude the value of the limit using paths. But we may gain a clue on how to factor and cancel or otherwise show the value of the limit without specifying a path.

Example 1: Multivariable Limits. Find the limit or show the limit does not exist.

A) $\lim\limits_{(x,y)\to(1,2)} \dfrac{xy}{\sqrt{x+y}}$ **Solution**: $f(1,2)=\dfrac{(1)(2)}{\sqrt{1+2}}=\dfrac{2}{\sqrt{3}}$ finished.

B) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x^2+y}$ **Solution**: $f(0,0)=\dfrac{0}{0}$ so the limit is indeterminant. Need 2 paths that include the point (0,0). Path 1: $x=0$ then $\lim\limits_{(0,y)\to(0,0)} \dfrac{0-y}{0^2+y}=\lim\limits_{y\to 0}\dfrac{-y}{y}=-1$ so if the limit exists then the limit is negative one. Path 2: $y=x$ then $\lim\limits_{(x,x)\to(0,0)} \dfrac{x-x}{x^2+x}=\lim\limits_{x\to 0}\dfrac{0}{x^2+x}=0$ so now since a different path resulted in a different limit then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x-y}{x^2+y}$ does not exist. DNE

C) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$ **Solution**: $f(0,0)=\dfrac{0}{0}$ another indeterminant limit. There is no obvious way to factor this so try paths. Path 1: $x=0$ then $\lim\limits_{(0,y)\to(0,0)} \dfrac{0y^2}{0^2+y^2}=\lim\limits_{y\to 0}\dfrac{0}{y^2}=0$ so if the limit exists then the limit must be 0. Path 2: $y=x$ then $\lim\limits_{(x,x)\to(0,0)} \dfrac{x(x)^2}{x^2+(x)^2}=\lim\limits_{x\to 0}\dfrac{x^3}{2x^2}=\lim\limits_{x\to 0}\dfrac{x}{2}=0$ same result from a different path, suspicious but not enough to conclude the limit is 0. Path 3: $y=x^2$ then $\lim\limits_{(x,x^2)\to(0,0)} 0\dfrac{x(x^2)^2}{x^2+(x^2)^2}=\lim\limits_{x\to 0}\dfrac{x^5}{x^2+x^4}=\lim\limits_{x\to 0}\dfrac{x^5}{x^2(1+x^2)}=\lim\limits_{x\to 0}\dfrac{x^3}{1+x^2}=\dfrac{0}{1}=0$ same result. Time to try something different.

Whenever we have a limit of the form $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ where the independent variables are approaching the origin we can try converting the function to polar form. The limit as the ordered pairs approach the origin is the same as the limit as the polar *r* approaches 0. As *r* approaches 0 would include all points inside circles centered at the origin of ever decreasing radii. Look at example 1C after converting to polar form.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}=\lim_{r\to 0}\frac{\left(r\cos(\theta)\right)\left(r\sin(\theta)\right)^2}{r^2}=\lim_{r\to 0}\frac{r^3\cos(\theta)\sin^2(\theta)}{r^2}=$$

$$\lim_{r\to 0} r\cos(\theta)\sin^2(\theta)=0\cos(\theta)\sin^2(\theta)=0$$

Now because we have considered all points within a circle around the origin of ever decreasing radius we have examined all possible paths at once and so we can conclude that the limit is 0.

It is important to note that polar is only effective if the ordered pairs are approaching the origin. If we convert to polar and the limit results in a function of theta then we say the original limit does not exist since if it depends on theta then the result would be different for different theta values and must not exist.

Example 2: Conversion to Polar. Use polar coordinates to find the limit or show the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Solution: $f(0,0)=\frac{0}{0}$ Convert to polar since $(x,y)\to(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r^2} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r^2} = \lim_{r\to 0} \cos\theta\sin\theta = \cos\theta\sin\theta$$ since the limit depends on $\cos\theta\sin\theta$ and is not constant we conclude that $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ Does Not Exist.

Practice these limits from section 13.2

Time for a little foreshadowing, consider these limits, they should remind you of something significant.

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Lecture Notes 13.3: Partial Derivatives.

Recall the definition of derivative from your calculus I class,
$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{dy}{dx}$ presented here in both the Δx and h forms. I want to emphasize the different forms we use to represent the derivative of a function with respect to the independent variable, x. Now look at the derivatives from the end of the notes for section 13.2, $\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$ and $\lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$. These look very much like the definition of derivative and they should. Remember we are now working with functions of multiple variables like $z = f(x, y)$ and so when we evaluate $\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$ we are finding the derivative of $z = f(x, y)$ with respect to the independent variable x and the limit $\lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$ gives the derivative of $z = f(x, y)$ with respect to the independent variable y. Remember we noted in the previous section that a derivative can only be found with respect to one variable at a time. The limits above illustrate this and because of this the prime notation can no longer be used because it would not be clear which of the independent variables was the variable of differentiation. Instead of a prime we use a subscript to identify the variable of differentiation. Also notice in these limits that the "other" variable is untouched by the limit. When calculating the derivative of *f* with respect to *x* we treat the variable *y* as a constant for the differentiation and vice versa, when differentiating with respect to *y* we treat *x* as a constant. We call these partial derivatives since the dependent variable depends on more than one independent variable and we are only measuring the effect on the dependent variable of one of the independent variables at a time, a partial revelation.
There is a lot of "stuff" in the above paragraph that will make much more sense in the context of a few simple examples.

We need just a little notation first. Given a multivariable function like $z = f(x, y)$:

The derivative of *z* with respect to *x* $= \frac{\partial z}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$.

The derivative of *z* with respect to *y* $= \frac{\partial z}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$.

The symbol in the Leibniz form of the derivative is called the partial derivative symbol. We have finally run out of symbols and Greek letters and are now starting to make up new symbols. It takes some practice to make these, they are like a backwards 6 but are really supposed to be more like a script lower case Greek delta.

Example 1: Partial Derivatives. Find all of the first partial derivatives of each of the following.

A) $f(x, y) = 6x^2 + 3y^4 + 4xy + 2x + 7y + 11$

$f_x(x, y) = 12x + 0 + 4y + 2 + 0 + 0$ I add the 0's to show the terms treated as a constant.

Notice that pure *x* terms have the same derivative now as the did before but the product term $4xy$ is treated as constant 4 times constant *y* times variable *x* and so it's derivative is $4y$.

$f_y(x, y) = 0 + 12y^3 + 4x + 0 + 7 + 0$

B) $z = 6x^2 \sin y + y^3 \tan x$

$\frac{\partial z}{\partial x} = 12x \sin y + y^3 \sec^2 x$ since the sine function is only applied to y we treat the entire term as a constant but differentiate $\tan x$.

$\frac{\partial z}{\partial y} = 6x^2 \cos y + 3y^2 \tan x$ treating $\tan x$ as a constant but differentiating the sine term.

C) $f(x, y, z) = \sin^3(xy) + e^{xz^2}$ Here we have a 3-dimensional domain and so will have 3 first partial derivatives. Also note the arguments of sine and the natural exponential contain more than one variable so we will need to be careful but we will NOT need product rule, just chain rule.

$$f_x(x, y, z) = 3\sin^2(xy)\cos(xy)y + e^{xz^2} z^2 = 3y\sin^2(xy)\cos(xy) + z^2 e^{xz^2}$$

$$f_y(x, y, z) = 3\sin^2(xy)\cos(xy)x + 0 = 3x\sin^2(xy)\cos(xy)$$ no y's in the e term.

$$f_z(x, y, z) = 0 + e^{xz^2}(2xz) = 2xze^{xz^2}$$ no z's in the sine.

The differentiation takes a great deal of practice to train yourself to see a variable and not treat it as a variable unless it is **the variable** of differentiation. Also a composite function is still a constant if it does not contain **the variable** of differentiation.

In calculus I the derivative gave us the slope of the tangent line or the rate of change in y with respect to changes in x. That is still the same general case except now the partial derivative of z with respect to x gives us the slope of the tangent line to the surface in a plane parallel to the yz coordinate plane or the rate of change in z with respect to changes in x assuming y to be held constant. If you have ever studied economics this is what economists call marginal analysis. How is the output changing in reaction to changes in one variable holding all others constant.

You may have noticed in the examples that these were first derivatives which of course implies there are second derivatives. Remember the second derivative measured the concavity of a graph and the same is true here. Time for some more notation.

Given a multivariable function like $z = f(x, y)$:

The second derivative of z with respect to x both times is denoted by $\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y)$.

The second derivative of z with respect to y both times is denoted by $\frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y)$ but wait there's more!

$\frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y)$ denotes the case where differentiation with respect to x is applied to the derivative of z with respect to y. And $\frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y)$ denotes differentiation of the x derivative with respect to y. Hopefully you noticed how the order of the variables in Leibniz form and in subscript form are different. This is important in higher level mathematics classes and much less so here but I like to explain why. If we begin with the function,

$z = f(x, y)$ and we apply the differential operator $\frac{\partial}{\partial x}$ which means take the derivative with respect to x to both sides of the equation we get $\frac{\partial}{\partial x}(z) = \frac{\partial}{\partial x} f(x, y)$ which leads to the notation $\frac{\partial z}{\partial x} = f_x(x, y)$. Now if we apply the differential operator for the derivative with respect to y, $\frac{\partial}{\partial y}$, to both sides of this new equation we get $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(f_x(x, y)\right)$ which leads to $\frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y)$ and is designed to show the order in which the derivatives were applied. Now, the reason this is not that critical in this course is because for every function we will see $f_{xy}(x, y) = f_{yx}(x, y)$ or $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$. The "mixed second partial derivatives" will be equal for every function we will see in this course. This is not always true. It is only true for the types of functions we are capable of dealing with at this level.

Example 2: Partial Derivatives. Find all of the first and second partial derivatives of $f(x, y) = \frac{2x^3}{y^2} + e^{6xy^3}$

Solution: First rewrite the function to avoid quotient rule. $f(x, y) = 2x^3 y^{-2} + e^{6xy^4}$

$f_x(x, y) = 6x^2 y^{-2} + 6y^4 e^{6xy^4}$ $\qquad\qquad$ $f_y(x, y) = -4x^3 y^{-3} + 24xy^3 e^{6xy^4}$

Look at $f_x(x, y)$ and do an x derivative:

$$f_{xx}(x, y) = 12xy^{-2} + \left(6y^4\right)6y^4 e^{6xy^4} = 12xy^{-2} + 36y^8 e^{6xy^4}$$

Look at $f_y(x, y)$ and do a y derivative Needs product rule here

$$f_{yy}(x, y) = 12x^3 y^{-4} + 72xy^2 e^{6xy^4} + \left(24xy^3\right)24xy^3 e^{6xy^4} = 12x^3 y^{-4} + 36xye^{6xy^4} + 576x^2 y^6 e^{6xy^4}$$

Look at $f_x(x, y)$ and do a y derivative Needs product rule here

$$f_{xy}(x, y) = -12x^2 y^{-3} + 24y^3 e^{6xy^4} + \left(24xy^3\right)6y^4 e^{6xy^4} = -12x^2 y^{-3} + 24y^3 e^{6xy^4} + 144xy^7 e^{6xy^4}$$

Look at $f_y(x, y)$ and do an x derivative Needs product rule here

$$f_{yx}(x, y) = -12x^2 y^{-3} + 24y^3 e^{6xy^4} + 24xy^3\left(6y^4\right)e^{6xy^4} = -12x^2 y^{-3} + 24y^3 e^{6xy^4} + 144xy^7 e^{6xy^4}$$

And we see above that the mixed partials are equal regardless of order of differentiation.

Review product rule, quotient rule, chain rule and all of the basic derivative rules from calculus I. The next page shows all of the derivatives you should know in their chain rule form. I know that this semester you have open book and notes but if this were an in-class course I would expect you to know all of these and the integrals that go with them. It would help with your time management if you were more comfortable with these.

Columbus State Community College
Math 1151: Calculus and Analytic Geometry I
Derivatives and Derivative Rules

I. The derivative: given $y = f(x)$ the derivative is represented by $y' = f'(x) = \dfrac{dy}{dx}$

The derivative gives the slope of the line tangent to the graph of $y = f(x)$ at any point.

The derivative gives the rate of change in x with respect to y.

If $s(t)$ represents position, then $s'(t) = v(t)$ gives the velocity and $s''(t) = v'(t) = a(t)$ gives the acceleration.

II. Differentiation Rules:

a. $\dfrac{d}{dx}[c] = 0$ Constant Rule

b. $\dfrac{d}{dx}\left[x^n\right] = nx^{n-1}$ Power Rule

c. $\dfrac{d}{dx}[cf(x)] = cf'(x)$ Constant Multiple Rule

d. $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$ Sum and Difference Rules

e. $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$ Product Rule

Alternate form: $\dfrac{d}{dx}[a\,b] = a'b + b'a$

f. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$ Quotient Rule

Alternate form: $\dfrac{d}{dx}\left[\dfrac{a}{b}\right] = \dfrac{a'b - b'a}{b^2}$

g. $\dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$ Chain Rule

Alternate form: $\dfrac{d}{dx}[f(u)] = f'(u)\,u'$

III. Derivatives of common functions:

a. $\frac{d}{dx}\left[u^n\right]= nu^{n-1}\frac{du}{dx} \quad n \neq -1$

Square root shortcut: $\frac{d}{dx}\left[\sqrt{u}\right]=\frac{u'}{2\sqrt{u}}$

b. $\frac{d}{dx}\left[\sin(u)\right]=\cos(u)\frac{du}{dx}$

c. $\frac{d}{dx}\left[\cos(u)\right]=-\sin(u)\frac{du}{dx}$

d. $\frac{d}{dx}\left[\tan(u)\right]=\sec^2(u)\frac{du}{dx}$

e. $\frac{d}{dx}\left[\cot(u)\right]=-\csc^2(u)\frac{du}{dx}$

f. $\frac{d}{dx}\left[\sec(u)\right]=\sec(u)\tan(u)\frac{du}{dx}$

g. $\frac{d}{dx}\left[\csc(u)\right]=-\csc(u)\cot(u)\frac{du}{dx}$

h. $\frac{d}{dx}\left[e^u\right]=e^u\frac{du}{dx}$

i. $\frac{d}{dx}\left[a^u\right]=(\ln a)(a^u)\frac{du}{dx}$

j. $\frac{d}{dx}\left[\ln(u)\right]=\frac{1}{u}\frac{du}{dx}=\frac{u'}{u}$

k. $\frac{d}{dx}\left[\log_a(u)\right]=\frac{1}{(\ln a)u}\frac{du}{dx}=\frac{u'}{u(\ln a)}$

l. $\frac{d}{dx}\left[\sin^{-1}(u)\right]=\frac{u'}{\sqrt{1-u^2}}$

m. $\frac{d}{dx}\left[\cos^{-1}(u)\right]=\frac{-u'}{\sqrt{1-u^2}}$

n. $\frac{d}{dx}\left[\tan^{-1}(u)\right]=\frac{u'}{1+u^2}$

o. $\frac{d}{dx}\left[\cot^{-1}(u)\right]=\frac{-u'}{1+u^2}$

p. $\frac{d}{dx}\left[\sec^{-1}(u)\right]=\frac{u'}{|u|\sqrt{u^2-1}}$

q. $\frac{d}{dx}\left[\csc^{-1}(u)\right]=\frac{-u'}{|u|\sqrt{u^2-1}}$

r. $\frac{d}{dx}\left[\sinh(u)\right]=\cosh(u)\frac{du}{dx}$

s. $\frac{d}{dx}\left[\cosh(u)\right]=\sinh(u)\frac{du}{dx}$

t. $\frac{d}{dx}\left[\tanh(u)\right]=\operatorname{sech}^2(u)\frac{du}{dx}$

u. $\frac{d}{dx}\left[\coth(u)\right]=-\operatorname{csch}^2(u)\frac{du}{dx}$

v. $\frac{d}{dx}\left[\operatorname{sech}(u)\right]=-\operatorname{sech}(u)\tanh(u)\frac{du}{dx}$

w. $\frac{d}{dx}\left[\operatorname{csch}(u)\right]=-\operatorname{csch}(u)\coth(u)\frac{du}{dx}$