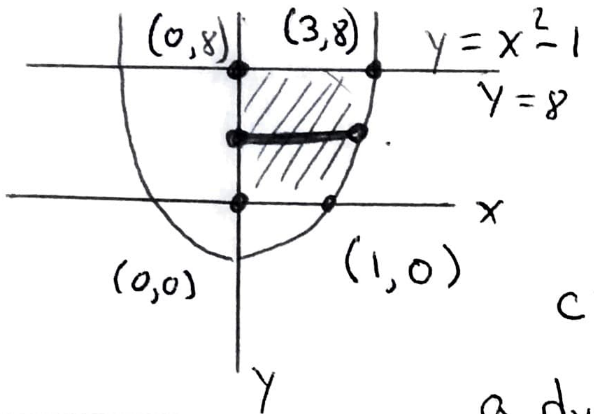
Example: Center of Mass

Find the mass, the first moments and the coordinates of the center of mass of the lamina in the **first quadrant** bounded by $y = x^2 - 1$ and $y = 8$ if the density of the lamina is given by $\rho(x,y) = x^2 y$.

Solution: Sketch



Find points of Intersection.

$$x^2 - 1 = 8 \qquad x^2 - 1 = 0$$
$$x^2 - 9 = 0 \qquad x^2 = 1$$
$$x = \pm 3 \qquad x = \pm 1$$

Choose order of integration.

a dy dx order would require 2 double integrals because lower bounds change at (1,0). Therefore dx dy would be better. So Need to solve for x. $y = x^2 - 1 \rightarrow x = \sqrt{y+1}$ in 1^{st} Q

$$0 \le x \le \sqrt{y+1}$$
$$0 \le y \le 8$$

Formulas:

$$M = \iint_R \rho(x,y)\, dA$$

$$M_y = \iint_R x\rho(x,y)\, dA$$

$$M_x = \iint_R y\rho(x,y)\, dA$$

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

Calculations:

$$M = \int_0^8 \int_0^{\sqrt{y-1}} x^2 y \, dx\, dy$$

$$M = \int_0^8 \frac{1}{3} x^3 y \Big|_0^{\sqrt{y+1}} dy = \frac{1}{3}\int_0^8 (\sqrt{y+1})^3 y \, dy \qquad \begin{array}{ll} u = y+1 & y = u-1 \\ du = dy & \end{array}$$

Integrate

$$\int u^{3/2}(u-1)\, du$$

oops!

Center of Mass Example page 2.

$$\int u^{5/2} - u^{3/2}\, du = \frac{2}{7}u^{7/2} - \frac{2}{5}u^{5/2}$$

so $$M = \frac{1}{3}\left[\frac{2(y+1)^{7/2}}{7} - \frac{2(y+1)^{5/2}}{5}\right]_0^8 = \frac{1}{3}\left[\frac{2}{7}(9)^{7/2} - \frac{2}{5}(9^{5/2}) - \frac{2}{7}(1) + \frac{2}{5}\right]$$

$$= \frac{1}{3}\left[\frac{2(2187)}{7} - \frac{2(243)}{5} - \frac{2}{7} + \frac{2}{5}\right] = \frac{1}{3}\left(\frac{18472}{35}\right) = \frac{18472}{105}$$

$$M_y = \int_0^8\int_0^{\sqrt{y-1}} x(x^2y)\,dx\,dy = \int_0^8\int_0^{\sqrt{y-1}} x^3y\,dx\,dy = \int_0^8 \frac{1}{4}x^4y\Big|_0^{\sqrt{y+1}}dy$$

$$= \frac{1}{4}\int_0^8 (y+1)^2 y\,dy$$ using same u-sub. $$\int u^2(u-1)\,du$$

$$= \int u^3 - u^2\,du = \frac{u^4}{4} - \frac{u^3}{3}$$

$$= \frac{1}{4}\left[\frac{(y+1)^4}{4} - \frac{(y+1)^3}{3}\right]_0^8 = \frac{1}{4}\left[\frac{9^4}{4} - \frac{9^3}{3} - \frac{1}{4} + \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[\frac{6561}{4} - \frac{729}{3} - \frac{1}{4} + \frac{1}{3}\right] = \frac{1}{4}\left(\frac{16768}{12}\right) = \frac{1048}{3}$$

$$M_x = \int_0^8\int_0^{\sqrt{y+1}} y(x^2y)\,dx\,dy = \int_0^8 \frac{1}{3}x^3y^2\Big|_0^{\sqrt{y+1}}dy$$

$$= \frac{1}{3}\int_0^8 (y+1)^{3/2}y^2\,dy$$ same u-sub $$\int u^{3/2}(u-1)^2\,du$$

Center of Mass Example page 3.

$$\int u^{3/2}(u^2-2u+1)\,du = \int u^{7/2}-2u^{5/2}+u^{3/2}\,du$$

$$= \frac{2}{9}u^{9/2} - 2\left(\frac{2}{7}\right)\left(u^{7/2}\right) + \frac{2}{5}u^{5/2}$$

$$M_x = \frac{1}{3}\left[\frac{2}{9}(y+1)^{9/2} - \frac{4}{7}(y+1)^{7/2} + \frac{2}{5}(y+1)^{5/2}\right]_0^8$$

$$= \frac{1}{3}\left[\frac{2}{9}(3^9) - \frac{4}{7}(3^7) + \frac{2}{5}(3^5) - \frac{2}{9} + \frac{4}{7} - \frac{2}{5}\right]$$

$$= \frac{1}{3}\left[4374 - \frac{8748}{7} + \frac{486}{5} - \frac{2}{9} + \frac{4}{7} - \frac{2}{5}\right]$$

$$= \frac{1}{3}\left(\frac{1014752}{315}\right) = \frac{1014752}{945}$$

$$M = \frac{18472}{105} \qquad M_y = \frac{1048}{3} \qquad M_x = \frac{1014752}{945}$$

$$\bar{x} = \frac{1048}{3} \Big/ \frac{18472}{105} = \frac{1048}{3}\cdot\frac{105}{18472} = \frac{4585}{2309} \approx 1.986$$

$$\bar{y} = \frac{1014752}{945}\cdot\frac{105}{18472} = \frac{1014752}{9(18472)} = \frac{126844}{20781} \approx 6.104$$

$$\text{Center of Mass} \approx \underbrace{(1.986,\ 6.104)}_{\text{Reasonable.}}$$

Fun!