Columbus State Community College
Math 2153: Calculus and Analytic Geometry III **Summer 2020 Remote**

Week 10: 7/27-8/2
Sections 15.5 – 15.8, Parametric Surfaces, Surface Integrals, Divergence Theorem, Stokes Theorem.

Due this week:

Tuesday 7/28/2020	**Thursday 7/30/2020**	**Sunday 7/19/2020**
Chapter 14 Assessment	WebAssign 15.3 – 15.6	Weekly Assignment 10, 15.5 – 15.8

Lecture Notes 15.5: Parametric Surfaces

Recall parametric equations $x = x(t),\ y = y(t),\ z = z(t)$ where for each value of the parameter *t*, a real number, the parametric equations generate a point in 3 dimensional space, $\left(x(t), y(t), z(t)\right)$, the collection of which produces a curve in space. Now let us extend this idea to include a 2-dimensional domain space for an ordered pair of parameters, generally denoted by *u* and *v*. Then $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ is called a parametric surface with $x = x(u,v),\ y = y(u,v),\ z = z(u,v)$ as the parametric equations of the surface.

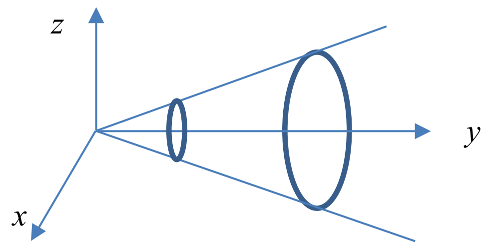
To illustrate how we can use these parametric surfaces to first remember the parametric equations of a helix, $x = 3\cos t,\ \ y = 3\sin t,\ \ z = t$. Clearly $x^2 = 9\cos^2 t,\ y^2 = 9\sin^2 t$ and so $x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9$ indicates that the space curve lies on the cylinder $x^2 + y^2 = 9$. The value of *z* being equal to the value of the parameter, *t,* indicates the curve spirals up the cylinder, a helix. Now this is just the curve, that is this set of parametric equations generates the spring like shape that is the helix but not the cylinder itself. Now consider the parametric surface $\mathbf{r}(u,v) = \left(3\cos u\right)\mathbf{i} + \left(3\sin u\right)\mathbf{j} + v\,\mathbf{k}$ for real numbers *u* and *v* as parameters. As before it should be clear that $x^2 + y^2 = 9\cos^2 u + 9\sin^2 u = 9$ indicates that the *x* and *y* coordinates lie on a circle of radius 3 but unlike before the *z* coordinate does not depend on the same parameter and so is free to take on any real number for each value of *u* which then fills in the sides of the cylinder rather than being constrained to a single *z* value for each *x, y* pair.

Given a parametric surface we can often eliminate the parameters to find the equation of the surface in a non-parametric form. Like we did with parametric equations of curves we look for a way to relate the *x*'s, *y*'s, and *z*'s directly.

Example 1: Given the parametric surface $\mathbf{r}(u,v) = \left(3\cos v\cos u\right)\mathbf{i} + \left(3\cos v\sin u\right)\mathbf{j} + \left(5\sin v\right)\mathbf{k}$ find the rectangular equation of the surface by eliminating the parameters.
Solution: Begin with the parametric equations, $x = 3\cos v\cos u$, $y = 3\cos v\sin u$, and $z = 5\sin v$. The trigonometry suggests that squaring each term may be helpful. $x^2 = 9\cos^2 v\cos^2 u$, $y^2 = 9\cos^2 v\sin^2 u$, and $z^2 = 25\sin^2 v$. Looking at the terms we see that $x^2 + y^2 = 9\cos^2 v\cos^2 u + 9\cos^2 v\sin^2 u$ and factoring we get $x^2 + y^2 = 9\cos^2 v\left(\cos^2 u + \sin^2 u\right) = 9\cos^2 v$ now since the coefficients 9 and 25 are different we can solve for the trig functions $\dfrac{z^2}{25} = \sin^2 v$, and $\dfrac{x^2 + y^2}{9} = \cos^2 v$ and so $\dfrac{z^2}{25} + \dfrac{x^2 + y^2}{9} = \sin^2 v + \cos^2 v = 1$ and finally then we have the rectangular equation $\dfrac{z^2}{25} + \dfrac{x^2 + y^2}{9} = 1$ or in standard form $25x^2 + 25y^2 + 9z^2 = 225$ an ellipsoid.

We can also parameterize a surface from rectangular equtions by building a parametric, vector valued function of the form, $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ which generates points on the surface.

Example 2: Find a vector valued function whose graph is the elliptic cone $y = \sqrt{4x^2 + 9z^2}$.



Solution: The graph is not necessary but its nice. Consider the double cone $y^2 = 4x^2 + 9z^2$ first and recognize the least common multiple between the coefficients on the right is 36. So if we let $x = 3\cos u$ and $z = 2\sin u$ then we see that $4x^2 + 9z^2 = 4\left(9\cos^2 u\right) + 9\left(4\sin^2 u\right) = 36\cos^2 u + 36\sin^2 u = 36$ is a 2-dimensional parameterization of the ellipse but it doesn't relate the ellipse to *y*. So lets begin with $4x^2 + 9z^2 = 4\left(9\cos^2 u\right) + 9\left(4\sin^2 u\right) = 36$ and multiply through by the second parameter, *v*. $4v\left(9\cos^2 u\right) + 9v\left(4\sin^2 u\right) = 36v$. Hopefully everyone now sees $y^2 = 4x^2 + 9z^2$ and $36v = 4v\left(9\cos^2 u\right) + 9v\left(4\sin^2 u\right)$ leads to the parametric surface $\mathbf{r}(u,v) = \left(3\sqrt{v}\cos u\right)\mathbf{i} + \left(6\sqrt{v}\right)\mathbf{j} + \left(2\sqrt{v}\sin u\right)\mathbf{k}$. Where $x = 3\sqrt{v}\cos u$, $y = 6\sqrt{v}$, and $z = 2\sqrt{v}\sin u$. Substituting these parameterizations into $y^2 = 4x^2 + 9z^2$ gives $\left(6\sqrt{v}\right)^2 = 36v = 4\left(3\sqrt{v}\cos u\right)^2 + 9\left(2\sqrt{v}\sin u\right)^2 = 4\left(9v\cos^2 u\right) + 9\left(4v\sin^2 u\right)$. We could have let $y = 6v$ and we would not have had the radicals but then we would need to restrict the domain to $v \ge 0$ to just get the positive half of the cone but by using $y = 6\sqrt{v}$ the square root's natural domain limits the cone to just the positive half without impacting the equation.

For any parameterized surface, $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ we can find a Normal vector at any point on the surface (a,b,c) by calculating $\mathbf{r}_u(a,b) \times \mathbf{r}_v(a,b) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}$.

Lecture Notes 15.6: Surface Integrals

Let S be a surface and let R be the image of the surface in the xy plane. Think of R as domain of S or the shadow that S casts on the xy plane. There is a decent drawing of this on the first page of section 15.6. Further let S be given by $z = g(x,y)$ such that g, g_x, and g_y are all continuous at all points in R. Then for a function, $f(x,y,z)$ defined on S, meaning the ordered triplets (x,y,z) are points on S, the surface integral

$$\iint_S f(x,y,z)\,dS = \iint_R f\left(x,y,g(x,y)\right)\sqrt{1+\left(g_x(x,y)\right)^2+\left(g_y(x,y)\right)^2}\,dA$$

Example 1: Evaluate the surface integral $\iint_S \left(2x-y+z\right)dS$ for $S: z=5y,\ 0\le x\le 2,\ 0\le y\le x$.

Solution: Using the above definition $\iint_S f(x,y,z)\,dS = \iint_R f\left(x,y,g(x,y)\right)\sqrt{1+\left(g_x(x,y)\right)^2+\left(g_y(x,y)\right)^2}\,dA$

we can build $\iint_S \left(2x-y+z\right)dS = \iint_R \left(2x-y+5y\right)\sqrt{1+0^2+5^2}\,dA$ where R is the region in the xy plane

$0\le x\le 2,\ 0\le y\le x$ and so $\iint_S \left(2x-y+z\right)dS = \int_0^2\int_0^x \left(2x+4y\right)\sqrt{26}\,dydx = \sqrt{26}\int_0^2 \left(2xy+2y^2\right)\Big|_0^x dx$

$$\iint_S \left(2x-y+z\right)dS = \sqrt{26}\int_0^2 \left(2x^2+2x^2-0\right)dx = \sqrt{26}\int_0^2 \left(4x^2\right)dx = \sqrt{26}\left(\frac{4x^3}{3}\right)\Big|_0^2 = \frac{32\sqrt{26}}{3}.$$

Example 2: Evaluate the surface integral $\iint_S \sqrt{x^2+y^2+z^2}\,dS$ for $S: z=\sqrt{x^2+y^2},\ x^2+y^2\le 4$.

Solution: $g(x,y)=\sqrt{x^2+y^2}$ so $g_x(x,y)=\dfrac{x}{\sqrt{x^2+y^2}}$ and $g_y(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$ so

$$\iint_S \sqrt{x^2+y^2+z^2}\,dS = \iint_R \sqrt{x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2}\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dA$$

$$= \iint_R \sqrt{2x^2+2y^2}\sqrt{1+\left(\frac{x^2}{x^2+y^2}\right)+\left(\frac{y^2}{x^2+y^2}\right)}\,dA = \iint_R \sqrt{2x^2+2y^2}\sqrt{1+\left(\frac{x^2+y^2}{x^2+y^2}\right)}\,dA = \iint_R \sqrt{2x^2+2y^2}\sqrt{2}\,dA$$

$= \iint_R 2\sqrt{x^2+y^2}\,dA$ Now with R being defined by $x^2+y^2\le 4$ it would be easier to convert this integral to

polar coordinates to evaluate, $= \iint_R 2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^2 2r\,rdrd\theta = \int_0^{2\pi}\left(\frac{2r^3}{3}\right)\Big|_0^2 d\theta = \int_0^{2\pi}\left(\frac{16}{3}\right)d\theta = \frac{32\pi}{3}.$

When the surface S is of the form $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ over a region D in the uv plane then

$$\iint_S f(x,y,z)\,dS = \iint_R f\big(x(u,v), y(u,v), z(u,v)\big)\left\| \mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v) \right\| dA$$

Example 3: Evaluate the surface integral $\displaystyle\iint_S (x+y)\,dS$

$S : \mathbf{r}(u,v) = 3u\cos v\,\mathbf{i} + 3u\sin v\,\mathbf{j} + 6u\,\mathbf{k}, \quad 0 \le u \le 3, \quad 0 \le v \le \pi.$

Solution: $\mathbf{r}_u(u,v) = 3\cos v\,\mathbf{i} + 3\sin v\,\mathbf{j} + 6\,\mathbf{k}$ and $\mathbf{r}_v(u,v) = -3u\sin v\,\mathbf{i} + 3u\cos v\,\mathbf{j} + 0\,\mathbf{k}$

$$\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos v & 3\sin v & 6 \\ -3u\sin v & 3u\cos v & 0 \end{vmatrix} = (0 - 18u\cos v)\,\mathbf{i} - (0 - -18u\sin v)\,\mathbf{j} + \left(9u\cos^2 v - -9u\sin^2 v\right)\mathbf{k}$$

$$\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v) = (-18u\cos v)\,\mathbf{i} + (-18u\sin v)\,\mathbf{j} + (9u)\,\mathbf{k}$$

$$\left\| \mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v) \right\| = \sqrt{(-18u\cos v)^2 + (-18u\sin v)^2 + (9u)^2} = \sqrt{324u^2\cos^2 v + 324u^2\sin^2 v + 81u^2}$$

$$\left\| \mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v) \right\| = \sqrt{81u^2\left(4\cos^2 v + 4\sin^2 v + 1\right)} = 9u\sqrt{5}\,,$$

$f(x,y,z) = x + y$ so $f\big(x(u,v), y(u,v), z(u,v)\big) = (3u\cos v + 3u\sin v)$ and finally

$$\iint_S (x+y)\,dS = \int_0^{\pi}\int_0^{3} (3u\cos v + 3u\sin v)\left(9u\sqrt{5}\right)du\,dv = 27\sqrt{5}\int_0^{\pi}\int_0^{3} u^2(\cos v + \sin v)\,du\,dv$$

$$= 27\sqrt{5}\int_0^{\pi} \left(\frac{u^3}{3}\right)_0^3 (\cos v + \sin v)\,dv = 243\sqrt{5}\int_0^{\pi} (\cos v + \sin v)\,dv = 243\sqrt{5}\left[\sin v - \cos v\right]_0^{\pi}$$

$$= 243\sqrt{5}\left[(0 - -1) - (0 - 1)\right] = 486\sqrt{5}\,.$$

Everything we have looked at thus far has been "how to evaluate" a surface integral. Now I just want to provide a little exposure on what these integrals are used for. Applications are primarily in the realm of physics. If $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a velocity field for a fluid on a surface S that is oriented by a normal vector valued function $\mathbf{N}(x,y,z)$ then $\displaystyle\iint_S \mathbf{F}\bullet\mathbf{N}\;dS = \iint_R \mathbf{F}\bullet\left[-g_x(x,y)\,\mathbf{i} - g_y(x,y)\,\mathbf{j} + \mathbf{k}\right]dA$ is known as the Flux Integral for a surface that is oriented upward. If the orientation is downward the signs of the components of **N** are reversed. The orientation of the normals is arbitrary and up to whomever is doing the calculations but the final results will be the same regardless.

For any surface $z = g(x,y)$ let $G(x,y,z) = z - g(x,y)$ then the unit normal vector valued function

$$\mathbf{N} = \frac{\nabla G(x,y,z)}{\left\|\nabla G(x,y,z)\right\|} = \frac{-g_x(x,y)\,\mathbf{i} - g_y(x,y)\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(g_x(x,y)\right)^2 + \left(g_y(x,y)\right)^2}}$$ would give an upward orientation (coefficient of **k** is +1)

and $$\mathbf{N} = \frac{-\nabla G(x,y,z)}{\left\|\nabla G(x,y,z)\right\|} = \frac{g_x(x,y)\,\mathbf{i} + g_y(x,y)\,\mathbf{j} - \mathbf{k}}{\sqrt{1 + \left(g_x(x,y)\right)^2 + \left(g_y(x,y)\right)^2}}$$ gives a downward orientation. (note the sign of **k**)

Lecture Notes 15.7: Divergence Theorem

If Q is a closed surface oriented outward by unit vector valued function, **N**, and **F** is a vector field whose components have continuous partial derivatives on Q then $\iint\limits_S \mathbf{F}\bullet\mathbf{N}\ dS = \iiint\limits_Q div(\mathbf{F})\,dV$. This is a 3-dimensional version of Green's Theorem and is used in physics in the study of hydrodynamics.

Lecture Notes 15.8: Stoke's Theorem

Another 3-dimensional variant of Green's Theorem is Stoke's Theorem. If S is a oriented surface with unit normal vector **N**, and S is bounded by a piecewise smooth simple closed curve, C, with a positive (counter clockwise) orientation and **F** is a vector field whose components have continuous partial derivatives on an open region containing S and C then $\int\limits_C \mathbf{F}\bullet d\mathbf{r} = \iint\limits_S \left(curl\,\mathbf{F}\right)\bullet\mathbf{N}\,dS$.

Clearly (hopefully) if C is a curve in the plane then this is just Green's Theorem but Stoke's Theorem comes into play when C is a curve in space forming the edge of a surface.

This is another topic from physics most commonly related to hydrodynamics and relates the tendency of a fluid to circulate around a surface compared to the flow over the surface.

There will be no questions from sections 15.7 or 15.8 on the weekly assignment, chapter 15 assessment, or final exam, I'll leave that for your physics instructors.