F1 - Multiplication Rule

The multiplication rule for probability allows us to compute probabilities of events occuring in succession, one after the other, like making multiple selections from a deck of playing cards, multiple trials of an experimental procedure, flipping a coin several times in a row, etc., and finding the probability of various combinations of outcomes of those processes.

When discussing probability, events are *independent* if the occurrence of one does not affect or rely on the occurrence of the other event(s). In other words, the chance of a event happening does not change based on whether the previous event(s) happened or not. For example, separate rolls of a die or flips of a coin are independent events, since what happens on one trial doesn't affect the probability of the next trial. When selecting items from a group, if selections are made *with replacement*, meaning that the items are put back into the group, the selections are independent.

Events are *dependent* if the occurrence of one does affect or rely on the occurrence of the other(s) - the probability of an event happening changes based on whether the previous event(s) happened or not. When selecting items from a group *without replacement*, meaning that previous selections are kept out of the group and can't be chosen again on further selections, the selections are dependent.

The Multiplication Rule(s)

There are two versions of the multiplication rule for finding the probability that multiple events all occur in succession, based on whether the events that are occurring one after the other are independent or dependent, but both versions rely on the same principle, and they are used the same way, in practice. For two events (A, B) that could occur in succession:

$$P(A \text{ and } B) = P(A) \cdot P(B) \quad \text{if } A, B \text{ are independent}$$

$$P(A \text{ and } B) = P(A) \cdot P(B|A) \quad \text{if } A, B \text{ are dependent}$$

$P(B|A)$ in the second version is the probability that event B occurs, given that we already know A has occurred. The vertical bar can be read as "given" or "given that." This probability is called a *conditional probability*, and is necessary for dependent events, since the probability of B might change, depending on whether A happened or not. For independent events, $P(B|A)$ is just the same as $P(B)$, since the probability of B would not change based on A.

These multiplication rules can be extended to as many events occurring in succession as we would like. We'll just keep multiplying the probabilities of each event, however many there are (assuming that each event did happen as we find probabilities of future events, if dependent).

(Note that this $P(A \text{ and } B)$ we're finding with the multiplication rule is different than the one used in the addition rule – this one is for events happening one after the other or multiple trials of a process, the addition rule $P(A \text{ and } B)$ is for the overlap between events A and B happening at the same time on just one trial.)

Example 1: a) A fair coin is flipped two times. Find the probability that both flips result in "heads."

Flips of a coin are *independent* events, as the result of one flip does not affect the result of another. The probability of heads on each individual flip is $P(\text{heads}) = \frac{1}{2}$ or 0.5, and that doesn't change as it is flipped multiple times. We'll let event A represent heads on the first flip, and B heads on the second flip. Then $P(A \text{ and } B) = P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ or 0.25

b) Find the probability that three flips of a coin all result in "heads."

$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$ or 0.125

c) Find the probability that 10 flips of a coin all result in "heads."

$\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$ or 0.000977

Note that the probability of "all heads" decreases with each successive flip. That's the big idea behind the multiplication rule - it's less likely for multiple events all to occur than the probability of any of the individual events. Multiplying probabilities, which are all between fractions/decimals between 0 and 1, results in a lower probability.

Example 2: Marbles are selected randomly from a jar containing 3 red marbles and 5 blue marbles. Find the probability of a blue marble 3 consecutive times if marbles are selected a) with replacement or b) without replacement.

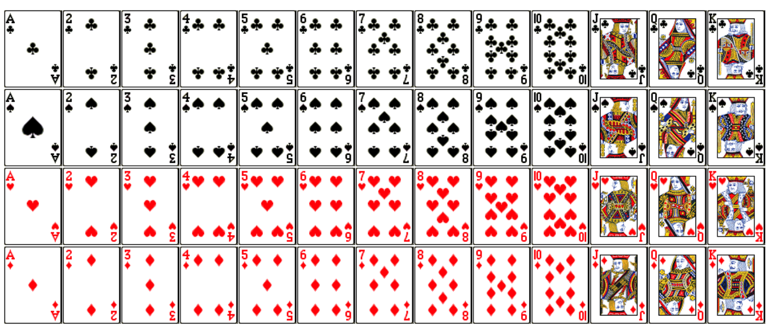
a) With replacement indicates that the marbles are put back into the jar after each selection, so the selections are independent – we're effectively drawing from the same jar all 3 times, so the probability stays the same each time. There are 5 blue marbles, out of $3 + 5 = 8$ total marbles on each selection.

$$P(3\text{ blue}) = P(\text{blue }1^{st}) \cdot P(\text{blue }2^{nd}) \cdot P(\text{blue }3^{rd}) = \frac{5}{8} \cdot \frac{5}{8} \cdot \frac{5}{8} = \left(\frac{5}{8}\right)^3 = \frac{125}{512} = 0.244$$

b) Without replacement indicates that the marbles are not returned to the jar after each selection, so the selections are dependent – the probability will change with each selection. We'll assume that we get what we wanted (blue), so there will be one fewer blue marble and one fewer total marble each time.

$$P(3\text{ blue}) = P(\text{blue }1^{st}) \cdot P(\text{blue }2^{nd}|\text{blue }1^{st}) \cdot P(\text{blue }3^{rd}|\text{blue }1^{st}\&2^{nd}) = \frac{5}{8} \cdot \frac{4}{7} \cdot \frac{3}{6} = \frac{60}{336} = \frac{5}{28} = 0.179$$

Example 3: A standard deck of 52 playing cards has cards of 13 ranks (2-10, J, Q, K, A), and each rank appears 4 times, one of each suit (clubs, spades, hearts, diamonds) – $13 \cdot 4 = 52$ total cards, as pictured below.



Find the probability of drawing a diamond on two consecutive selections from a shuffled deck of playing cards if selections are made a) with replacement or b) without replacement.

a) With replacement: $P(2\text{ diamonds}) = P(\text{diamond}) \cdot P(\text{diamond}) = \frac{13}{52} \cdot \frac{13}{52} = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} = 0.0625$

b) Without replacement: $P(2\text{ diamonds}) = P(\text{diamond }1^{st}) \cdot P(\text{diamond }2^{nd}|\text{diamond }1^{st}) = \frac{13}{52} \cdot \frac{12}{51} = \frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17} = 0.0588$

The 5% Independence Guideline

Notice in comparing the previous two examples that there was a more significant difference between making selections with and without replacement when we were selecting 3 marbles out of a group of 8 (selecting $3/8 = .375 = 37.5\%$ of the group), and the difference was not as significant when selecting 2 cards out of 52 (selecting $2/52 = .038 = 3.8\%$ of the deck). As the percentage of the group that we're selecting decreases, the difference between selecting with and without replacement also decreases. Sometimes, it can become difficult to make multiplication rule probability calculations when selections are made without replacement, but in some of these cases we can make the calculation less complicated without significantly changing the end result by assuming that the selections are made with replacement instead.

If the number of items we're selecting is less than 5% of the overall group, the 5% *Independence Guideline* allows us to assume that selections are independent (with replacement) when they are actually dependent (without replacement) for the purpose of simplifying the calculation.

Example 4: A dentist finds that 30% of her patients have cavities. Find the probability that 4 consecutive patients all do not have cavities.

We can safely assume that the number of selections we're considering, 4, is less than 5% of the population of all the dentist's patients that we're drawing from, since presumably she would have at least 100 patients. So even though the same patient would not be seen multiple times in a row (patients would be selected without replacement), we can assume that the selections are made with replacement for the purpose of this calculation.

Note that we're given the probability of a patient having a cavity, $P(\text{cavity}) = 0.3\ (30\%)$, but we're looking for the 4 patients to not have cavities. So we need the complement, $P(\text{no cavity}) = 1 - 0.3 = 0.7$.

Then the probability of 4 patients in a row not having cavities is

$$P(4\ \text{cavity-free}) = 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7 = (0.7)^4 = 0.2401.$$

Example 5: Acceptance sampling involves selecting a sample of a batch of products and deciding whether to accept or reject the entire batch based on how many from the sample were defective. In a shipment of 5623 calculators, there are 177 defective calculators. A simple random sample of 15 calculators is selected, and the entire shipment is only accepted if all 15 calculators are good. Find the probability that the shipment is accepted.

Since the sample of 15 calculators should be 15 different calculators, these selections are made without replacement, but the calculation becomes cumbersome in that case (see below). Instead, noting that $\frac{15}{5623} = .0027 = 0.27\%$ is less than 5%, we can use the 5% Independence Guideline to assume that the selections are made with replacement without significantly changing the probability.

We're given the number of bad calculators in the shipment, but we're looking for good ones, so we need to subtract to find $5623 - 177 = 5446$ good calculators. Then the probability of selecting a good calculator is $P(\text{good}) = \frac{5446}{5623}$, and to select 15 good calculators in a row, with the assumption that the selections are independent, we'll multiply this probability by itself 15 times, or raise it to the 15^{th} power:

$$P(15\ \text{good}) = \left(\frac{5446}{5623}\right)^{15} = 0.619$$

Just to show how cumbersome the calculation would have been without the 5% Independence Guideline, if we go with the actual scenario of selecting the calculators without replacement, assuming that we're removing one good calculator from the shipment at a time:

$$P(15\ \text{good}) = \left(\frac{5446}{5623}\right)\left(\frac{5445}{5622}\right)\left(\frac{5444}{5621}\right)\left(\frac{5443}{5620}\right)\left(\frac{5442}{5619}\right)\left(\frac{5441}{5618}\right)\left(\frac{5440}{5617}\right)\left(\frac{5439}{5616}\right)\left(\frac{5438}{5615}\right)$$
$$\left(\frac{5437}{5614}\right)\left(\frac{5436}{5613}\right)\left(\frac{5435}{5612}\right)\left(\frac{5434}{5611}\right)\left(\frac{5433}{5610}\right)\left(\frac{5432}{5609}\right) = 0.619$$

We found the same probability (at least to 3 decimal places) with a much simpler calculation by taking advantage of the 5% Independence Guideline.