F2 - More Probability Rules (At least one…, Conditional Probability)

"At least one…" problems

The phrase "at least one…" shows up often in probability problems, and while it might seem like a simple phrase, there are some important details we need to consider when calculating "at least one…" probabilities:

- "At least one…" means *one or more*, which includes multiple outcomes that we would need to combine together if we wanted to find the probability directly.
- It is usually easier to find the probability of the complement and subtract that probability from 1. The complement of "at least one…" is "none", so we'll find the probability that none of the selections are what we were looking for (or that all of them are not what we're looking for), and remember to subtract from 1 to get back to the "at least one…" probability we wanted.

Example 1: Marbles are selected randomly from a jar containing 3 red marbles and 5 blue marbles. If 3 marbles are selected without replacement find the probability that at least one of the marbles is red.

0 1 2 3

"At least one red" means that 1 or 2 or all 3 of the selected marbles could be red. This is too many options to be practical to calculate this probability directly at this point, so we'll look at the complement, which just contains the one other possibility that none of the 3 are red, and subtract that probability from 1. But if none of the marbles are red, that means all 3 are blue.

$$P(\text{at least one red}) = 1 - P(\text{none red, all 3 blue}) = 1 - \frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6} = 1 - \frac{60}{336} = 1 - \frac{5}{28} = \frac{23}{28} = 0.821$$

Example 2: To reduce laboratory costs for an expensive blood test for a rare disease, samples from 5 patients are combined and the combined sample is tested. If the combined sample tests negative, then all 5 individual samples would have tested negative. If the combined sample tests positive, that means that at least one of the individual samples would have tested positive, and further testing is required to determine which one(s) are positive. If the probability that an individual sample tests positive is 0.015, find the probability of a combined sample from 5 patients testing positive so that further testing is required.

$$P(\text{at least one positive}) = 1 - P(\text{none positive, all 5 negative})$$

Note that now we need the probability of an individual negative test, so we need another instance of the complement: $P(\text{indiv. negative}) = 1 - P(\text{indiv. positive}) = 1 - 0.015 = 0.985$. Then,

$$P(\text{at least one positive}) = 1 - P(\text{none positive, all 5 negative}) = 1 - (0.985)^5 = 1 - 0.927 = 0.073$$

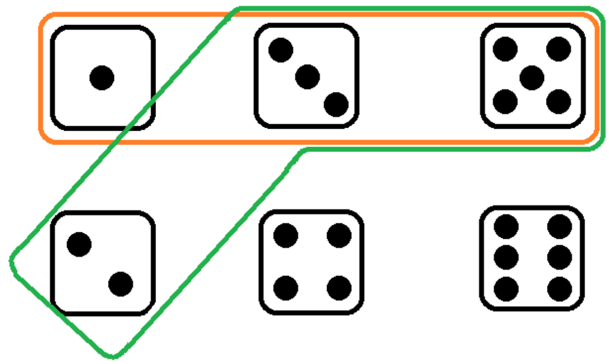
Conditional Probability

We saw conditional probability previously as a part of the dependent event version of the multiplication rule: $P(A \text{ and } B) = P(A)P(B|A)$. The conditional probability $P(B|A)$ represents the probability that event B occurs, "given that" or "if we already know that" event A has occurred. Now we want to be able to find this probability on its own in some situations, so algebraically solving for the conditional probability in the multiplication rule by dividing by $P(A)$, we can come to a formula for conditional probability:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example 3: A fair 6-sided die is rolled one time. Find the probability that an odd number (1,3,5) is rolled given that a prime number (2,3,5) is rolled.

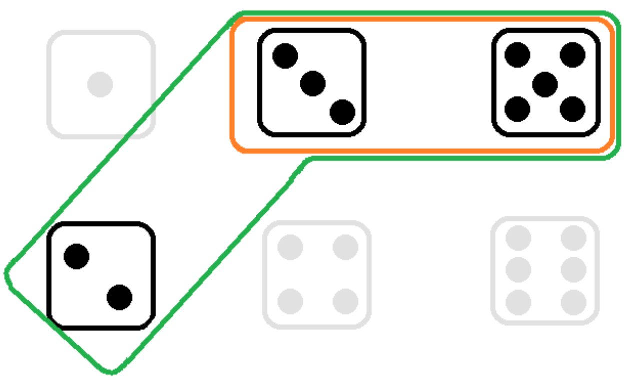
The key phrase to look for to identify this as a conditional probability problem is "given that." The statement is indicating that we already know that the number that was rolled is a prime number (outlined in green to the right), and with that information already known, we want to find the probability that the number is an odd number (outlined in orange). Using our conditional probability notation and formula, we can write this as



$$P(\text{odd}|\text{prime}) = \frac{P(\text{odd and prime})}{P(\text{prime})}.$$

2 of the 6 outcomes are both odd and prime (3 and 5 included in both the orange and green sections), so $P(\text{odd and prime}) = \frac{2}{6}$, and 3 of the 6 outcomes (2,3,5) are in the "given that" prime category, so $P(\text{prime}) = \frac{3}{6}$, and we have:

$$P(\text{odd}|\text{prime}) = \frac{P(\text{odd and prime})}{P(\text{prime})} = \frac{2/6}{3/6} = \frac{2}{6} \cdot \frac{6}{3} = \frac{2}{3}$$



But now, note that we can arrive at this same probability without this formula. Since we were "given that" the number rolled was prime, we could have reduced our sample space from all 6 outcomes to just the 3 prime outcomes, and of those 3, 2 outcomes were odd, for a conditional probability of $P(\text{odd}|\text{prime}) = \frac{2}{3}$. This is a much more practical way of calculating conditional probabilities – make the denominator of our conditional probability the number of outcomes in the "given that" category, and the numerator the number of those outcomes that we're looking for in the other category.

So in practice, we'll calculate conditional probabilities as $P(B|A) = \frac{\#\text{ of outcomes in } A \text{ and } B}{\#\text{ of outcomes in } A}$, where A is the "given that" event.

Example 4: The following contingency table describes the results of a survey regarding cigarette smoking habits among various marital statuses. If one survey participant is randomly selected

a) Find the probability that the person is a smoker given that they are married.

b) Find the probability that the person is divorced given that they are a non-smoker.

	Married	Divorced	Single	Totals
Smoker	54	38	11	103
Non-smoker	146	62	39	247
Totals	200	100	50	350

a) $P(\text{smoker}|\text{married}) = \frac{\#\text{ smoker and married}}{\#\text{ married}} = \frac{54}{200} = \frac{27}{100} = 0.27$

b) $P(\text{divorced}|\text{non-smoker}) = \frac{\#\text{ divorced and non-smoker}}{\#\text{ non-smoker}} = \frac{62}{247} = .251$