P-values: How Small is Small?

If the P-value is high

- we don't reject the null hypothesis
- **doesn't** prove the null hypothesis is true (insufficient evidence to disprove it)
- data are consistent with the model from null hypothesis
- null hypothesis doesn't appear to be false

A small P-value indicates that the statistic we observed would be very unlikely if H_0 is true. If the P-value is "small enough", we reject the null hypothesis. To determine if the P-value is small enough to reject the null hypothesis, we used a significance level. Common significance levels are 0.1, 0.05, and 0.01. → arbitrary

problems: • p-value close to cut-off α
• reject with one α, but not another.

Note: Always report the P-value with your conclusion to show the strength of the evidence against the null hypothesis. Give the reader the opportunity to make their own conclusion.

Note: Choose the significance level before looking at the data.

Note: The P-value of a hypothesis test is the smallest value of α for which H_0 can be rejected.

How small the P-value has to be to reject the null hypothesis is **context-dependent**:

- long standing hypothesis → Small p-value
- importance of issued being tested

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General Guidelines for Reporting Strength of Evidence Against H_0 using P-value:

P-value	Strength of evidence against the null hypothesis
$P\text{-value} < 0.001$	Very Strong
$0.001 < P\text{-value} < 0.01$	Strong
$0.01 < P\text{-value} < 0.05$	Moderate
$0.05 < P\text{-value} < 0.1$	Weak
$P\text{-value} > 0.1$	None

Example: Which of the following pairs can be used for a hypothesis test?

a) $H_0 : \mu = 300,\ H_A : \mu > 300$ Yes

b) $H_0 : \hat{p} = 0.33,\ H_A : \hat{p} > 0.33$ No $\hat{p}$ not population parameter

c) $H_0 : p > 0.45,\ H_A : p = 0.45$ No

d) $H_0 : \mu = 50,\ H_A : \mu \neq 50$ Yes

e) $H_0 : p = \underline{0.1},\ H_A : p > \underline{0.2}$ No

f) $H_0 : p = 0.85,\ H_A : p < 0.85$ Yes

g) $H_0 : \bar{y} = 35.4,\ H_A : \bar{y} \neq 35.4$ No $\bar{y}$ not population parameter

h) $H_0 : \mu = 11,\ H_A : \mu = 12$ No

i) $H_0 : p = 2,\ H_A : p > 2$ No $0 \leq p \leq 1$

j) $H_0 : \mu \leq 47,\ H_A : \mu > 47$ Yes

$\Leftrightarrow$ $H_0 : \mu = 47$, $H_A : \mu > 47$

Chapter 21: Comparing Two Proportions

Given two populations, we often want to compare the proportion of one population that has a specific characteristic with the proportion of the other population that has the same characteristic by considering the difference of the two population proportions. $p_1 - p_2$ parameter

	Population proportion	Sample size	Number of Successes	Sample Proportion
Population 1	p_1	n_1	x_1	$\hat{p}_1 = \frac{x_1}{n_1}$
Population 2	p_2	n_2	x_2	$\hat{p}_2 = \frac{x_2}{n_2}$

	Mean	Standard Deviation	Sample Size for CLT
Sampling Distribution of $\hat{p}_1$ with Sample Size n_1	$\mu_{\hat{p}_1} = p_1$	$\sigma_{\hat{p}_1} = \sqrt{\frac{p_1(1-p_1)}{n_1}}$	$n_1 p_1 \geq 10$ $n_1(1-p_1) \geq 10$
Sampling Distribution of $\hat{p}_2$ with Sample Size n_2	$\mu_{\hat{p}_2} = p_2$	$\sigma_{\hat{p}_2} = \sqrt{\frac{p_2(1-p_2)}{n_2}}$	$n_2 p_2 \geq 10$ $n_2(1-p_2) \geq 10$

Key Fact: If X and Y are independent Normal random variables, then $X - Y$ is Normal with mean

$$\mu_{X-Y} = E(X - Y) = E(X) - E(Y) = \mu_X - \mu_Y$$

and variance

$$\sigma^2_{X-Y} = \mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = \sigma^2_X + \sigma^2_Y$$

The Sampling Distribution of the Difference Between Two Sample Proportions

If independent random samples of sizes n_1 and n_2 have been selected from two populations with population proportions p_1 and p_2, respectively, and if the sampled responses are independent, then the sampling distribution of the difference between sample proportions

$$\hat{p}_1 - \hat{p}_2 = \frac{x_1}{n_1} - \frac{x_2}{n_2}$$

has the following properties:

- The mean is

$$\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2$$

- The standard deviation is $\sigma^2_{\hat{p}_1 - \hat{p}_2} = \sigma^2_{\hat{p}_1} + \sigma^2_{\hat{p}_2}$ Variance

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

- If the sample sizes n_1 and n_2 are both large enough, that is, we have

$$n_1 p_1 \geq 10 \qquad n_1(1-p_1) \geq 10$$
$$n_2 p_2 \geq 10 \qquad n_2(1-p_2) \geq 10$$

  then the sampling distribution of $\hat{p}_1 - \hat{p}_2$ is approximately Normally distributed.

Note: The difference between two proportions ranges from -1 to 1.

Assumptions and Conditions When Comparing Two Proportions

To make inferences about $p_1 - p_2$ by building confidence intervals or conducting hypothesis tests, we will require some assumptions and conditions. Since we don't know the true proportions p_1 and p_2 in these situations, we will estimate these values using $\hat{p}_1$ and $\hat{p}_2$, respectively.

Independence Assumptions

- **Independent Responses Assumption:** for each population, the data sampled should come from independently responding individuals.
- **Independent Groups Assumption:** the two samples, one from each population, must be independent of each other.
- **Randomization Condition:** for each sample, the data should be drawn independently, using random selection from the population (or come from a randomized experiment). random Selection of participants.
- **10% Condition:** if the data are sampled without replacement, the sample size should not exceed 10% of the population.

Sample Size Condition

The sample size for each population must be large enough.

Success/Failure Condition: there should be least ten successes and at least ten failures in each sample, that is, we require

$$n_1\hat{p}_1 = x_1 \geq 10 \qquad n_1(1-\hat{p}_1) = n_1 - x_1 \geq 10$$
$$n_2\hat{p}_2 = x_2 \geq 10 \qquad n_2(1-\hat{p}_2) = n_2 - x_2 \geq 10$$

Large Sample Confidence Intervals for the Difference Between Two Population Proportions

Recall: a large sample confidence interval has the form

$$\text{point estimate} \pm \text{margin of error}$$

$$= \text{point estimate} \pm \left(\text{critical value} \times \text{standard error of the estimate}\right)$$

When the previously mentioned conditions are met, a $100(1-\alpha)\%$ confidence interval for the difference of two population proportions $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2) \pm z^* \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

where z^* is the critical value corresponding to the $100(1-\alpha)\%$ confidence level.

Is $0 \in CI$?

$p_1 - p_2 = 0 \iff p_1 = p_2$

$p_1 - p_2 < 0 \iff p_1 < p_2$

$p_1 - p_2 > 0 \iff p_1 > p_2$

Example: Researchers examined the effects of a certain weed-killing herbicide on dogs. The dogs, some of whom were from homes that regularly used the herbicide, were examined for malignant lymphoma. A sample of 827 dogs who were exposed to the herbicide had 473 dogs with lymphoma and a sample of 130 dogs who were not exposed to the herbicide had 19 dogs with lymphoma. Construct a 90% confidence interval for $p_1 - p_2$.

p_1 = proportion of dogs exposed to herbicide who developed lymphoma.

p_2 = proportion of dogs not exposed to herbicide who developed lymphoma.

$\hat{p}_1 = \frac{473}{827}$, $n_1 = 827$

$\hat{p}_2 = \frac{19}{130}$, $n_2 = 130$

$n_1 \hat{p}_1 = 473 \geq 10$

$n_1(1-\hat{p}_1) = 354 \geq 0$

$n_2 \hat{p}_2 = 19 \geq 10$

$n_2(1-\hat{p}_2) = 111 \geq 10$

$$(\hat{p}_1 - \hat{p}_2) \pm z^* \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$= \left(\frac{473}{827} - \frac{19}{130}\right) \pm 1.645 \sqrt{\frac{\left(\frac{473}{827}\right)\left(\frac{354}{827}\right)}{827} + \frac{\left(\frac{19}{130}\right)\left(\frac{111}{130}\right)}{130}}$$

$= 0.42579 \pm 0.058299$

$= (0.3675, 0.4841)$ * by 36.8% to 48.4%

$\therefore$ we are 90% confident that $p_1 - p_2 \in (0.3675, 0.4841)$

Since $0 \notin (0.3675, 0.4841)$, we are 90% confident that $p_1 > p_2$. With 90% confidence we conclude the prop. of exposed dogs who develope lymphoma exceeds the prop. of unexposed dogs. *

Hypothesis Tests for the Difference Between Two Population Proportions: Two-Proportion z-Test

The null hypothesis for a hypothesis test to compare two population proportions is

$$H_0 : p_1 - p_2 = 0$$

If we assume H_0 is true, then $p_1 = p_2$ and the sampling distribution for $\hat{p}_1 - \hat{p}_2$ has mean

$$\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2 = 0$$

and standard deviation

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

$$= \sqrt{\frac{p(1-p)}{n_1} + \frac{p(1-p)}{n_2}}$$

$p = p_1 = p_2$

where p is the common value of p_1 and p_2. Unlike with a hypothesis test for one population proportion, we don't know the value of p, so we will estimate it by **pooling** the data from the two samples.

The pooled sample proportion is

$$\hat{p}_{\text{pooled}} = \frac{x_1 + x_2}{n_1 + n_2}$$

estimate of standard deviation

For simplicity, we will denote this as $\hat{p}$. The standard error of $\hat{p}_1 - \hat{p}_2$ is then

$$SE(\hat{p}_1 - \hat{p}_2) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n_1} + \frac{\hat{p}(1-\hat{p})}{n_2}}$$

$$= \sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

A hypothesis test for the difference of two population proportions $p_1 - p_2$ has five steps:

1. **Assumptions/Conditions:** → binary categorical variable.
   - Data collected using randomization and the sampled values are independent.
   - The two samples are independent of each other.
   - Both sample sizes n_1 and n_2 are large enough:

$$n_1\hat{p}_1 = x_1 \geq 10 \qquad n_1(1-\hat{p}_1) = n_1 - x_1 \geq 10$$
$$n_2\hat{p}_2 = x_2 \geq 10 \qquad n_2(1-\hat{p}_2) = n_2 - x_2 \geq 10$$

2. **Hypotheses:**

$$H_0 : p_1 - p_2 = 0$$

choose one
$$H_A : \left\{\begin{array}{ll} p_1 - p_2 \neq 0 & \text{(two-tailed test)} \\ p_1 - p_2 < 0 & \text{(lower-tailed test)} \\ p_1 - p_2 > 0 & \text{(upper-tailed test)} \end{array}\right.$$

3. **Test Statistic:**

$\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2 = 0$

$$z_0 = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where $\hat{p}$ is the pooled estimate

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

$$= \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$$

4. ***P*-value:**

Test	P-value
Two-tailed Test	$2P(z < -\|z_0\|)$
Lower-tailed Test	$P(z < z_0)$
Upper-tailed Test	$P(z > z_0)$

5. **Conclusion:** Report and interpret the P-value in context. Given a significance level α,

   - if P - value $\leq \alpha$, we reject H_0 at level α
   - if P- value $> \alpha$, we do not reject H_0 at level α

$n_1 = 120, \; x_1 = 51 \qquad n_2 = 150, \; x_2 = 88$

Example: A comparative study examined the cure rates of two new medications, Drug X and Drug Z. In a random sample of 120 patients who were treated with Drug X, 51 were cured, and in a random sample of 150 patients who were treated with Drug Z, 88 were cured. Are these results evidence that there is a higher cure rate with Drug Z than Drug X? Use $\alpha = 0.05$.

p_1 = proportion of patients cured by Drug X

p_2 = proportion of patients cured by Drug Z

1. **Assumptions/Conditions:**
   - data collected randomly
   - samples independent of each other
   - $n_1 \hat{p}_1 = 51 \geq 10, \quad n_1(1-\hat{p}_1) = 69 \geq 10$

     $n_2 \hat{p}_2 = 88 \geq 10, \quad n_2(1-\hat{p}_2) = 62 \geq 10$

2. **Hypotheses:**

$$H_0: p_1 - p_2 = 0$$

$$H_A: p_1 - p_2 < 0$$

lower-tailed test

3. **Test Statistic:**

$$z_0 = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \quad , \quad \text{where } \hat{p} = \frac{51+88}{120+150} = \frac{139}{270}$$

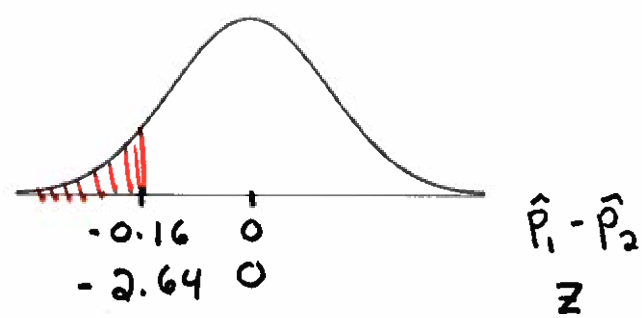
$$= \frac{\frac{51}{120} - \frac{88}{150}}{\sqrt{\frac{139}{270}\left(\frac{131}{270}\right)\left(\frac{1}{120}+\frac{1}{150}\right)}}$$

$$= \frac{-0.16167}{0.06121} = -2.64$$

4. ***P*-value:**

$P(z < -2.64)$

$= 0.0041 \leq 0.05$



5. **Conclusion:** Since P-value $\leq \alpha = 0.05$, we reject H_0 at the 0.05 significance level, that is, there is enough statistical evidence to conclude that there is a higher cure rate with Drug Z.