Chapter 20: Paired Samples and Blocks

In Chapter 19, we compared the means of two populations by using independent samples. In this chapter, we will compare such means using **paired** or **matched** samples.

Data are paired when the observations are collected in pairs or the observations in one group are naturally related to observations in the other group. → No way to test if paired. Need to think about it. (not enough just to have the same sample size)

Paring in an experiment is a type of **blocking**.

Paring in an observational study is a form of **matching**.

Example: → twins

We could have two samples in which the paired observations consist of:

- the ages of a husband and wife
- the starting salaries after graduation for a male and female with the same major and similar GPA
- the stopping times for the same vehicle and driver on wet pavement and dry pavement
- the average of the daily high temperatures in January in a city in two different years
- an individual's cholesterol levels before and after taking a medication
- an individual's memory test scores before and after learning to how play chess

When we have paired samples, we examine the pairwise differences. We treat the **differences as the data** sampled from a population of differences.

data set

Example: Ages of husbands and wives:

Couple	Husband	Wife	Difference (d)
1	59	53	6
2	21	22	-1
3	33	36	-3
4	78	74	4
5	70	64	6
⋮	⋮	⋮	⋮
40	52	44	8

Compute:
- mean $\bar{d}$
- standard deviation s_d

$y_{husband}$ $\quad$ y_{wife} $\quad$ $y_{husband} - y_{wife}$

In this situation, we have **one data set** and hence we can use a one-sample t-test.

A **paired t-test** is mechanically a one-sample t-test for the mean of the pairwise differences.

Notation:

n = the number of pairs (sample size)

d = a paired difference

$\bar{d}$ = the mean of the pairwise differences in the sample $(\bar{y}_1 - \bar{y}_2)$

s_d = the standard deviation of the pairwise differences in the sample

$\mu_d = \mu_1 - \mu_2$, the difference in the population means
(also, the mean of the population of differences)

Assumptions and Conditions:

a) **Paired Data Condition:** The data must be quantitative and paired.

b) **Independence Assumption:** The **differences** must be independent of each other. (not the individual groups)

   i) **Randomization Condition:** With paired data, randomness can arise in many ways and often depends on what we want to know.

      ∗ The pairs may be a random sample. (The n differences should represent a random sample from the population of differences.)

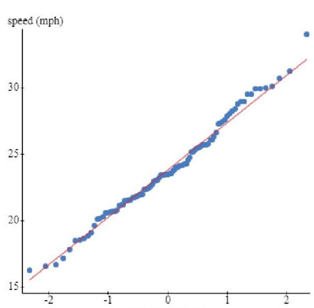
      ∗ In an experiment, the order of two treatments may be randomly assigned or treatments may be randomly assigned to one member of each pair.

   ii) **10% Condition:** sample size should be less than 10% of population size (when drawn without replacement).

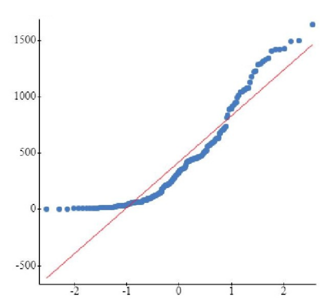
c) **Normal Population Assumption:** We require either

   i) The number of pairs is large ($n \geq 30$), or

   ii) **Nearly Normal Condition:** The population of **differences** is (approximately) normally distributed. To check this, we can look at a histogram or Normal probability plot of the differences of the data. approximately normal = roughly straight diagonal line



approx. normal



not normal

Confidence Interval: Paired t-Interval for μ_d

Recall: a confidence interval has the form

$$\text{point estimate} \pm \text{margin of error}$$

$$= \text{point estimate} \pm \left(\text{critical value} \times \text{standard error of the estimate}\right)$$

$\bar{d}$ $\quad$ t^*_{n-1} $\quad$ $SE(\bar{d}) = \frac{s_d}{\sqrt{n}}$

When the relevant assumptions are made and applicable conditions are met, a $100(1-\alpha)\%$ confidence interval for μ_d is

$$\bar{d} \pm t^*_{n-1} \frac{s_d}{\sqrt{n}}$$

where n is the number of pairs and where t^*_{n-1} is the critical value corresponding to the $100(1-\alpha)\%$ confidence level based on $n-1$ degrees of freedom.

Hypothesis Test: Paired t-Test for μ_d

The general form of a test statistic is:

$$\text{test statistic} = \frac{\text{estimate} - \text{null hypothesis value}}{\text{standard error of the estimate}}$$

A paired t-test (hypothesis test) for the mean of paired differences μ_d has five steps:

1. **Assumptions/Conditions:**
   - The samples are paired.
   - The n differences represent a random sample from the population of differences.
   - The n differences are independent of each other.
   - Either $n \geq 30$ or the population of differences is approximately normally distributed.

often
$d_0 = 0$

2. **Hypotheses:**

$$H_0 : \mu_d = d_0$$

choose one

$$H_A : \begin{cases} \mu_d \neq d_0 & \text{(two-tailed test)} \\ \mu_d < d_0 & \text{(lower-tailed test)} \\ \mu_d > d_0 & \text{(upper-tailed test)} \end{cases}$$

3. **Test Statistic:**

if $d_0 = 0$
$\bar{d}$

$$t_0 = \frac{\bar{d} - d_0}{\left(\frac{s_d}{\sqrt{n}}\right)}$$

When the conditions are met and the null hypothesis is true, this statistic follows a Student's t-model with $n-1$ degrees of freedom.

4. ***P*-value:** Compute one of the following using software or the t-table, with $df = n - 1$:

Test	P-value
Two-tailed Test	$2P(t_{n-1} > \|t_0\|)$
Lower-tailed Test	$P(t_{n-1} < t_0)$
Upper-tailed Test	$P(t_{n-1} > t_0)$

If using a t-table, we will likely only be able to find a range in which the P-value lies.

5. **Conclusion:** Given a significance level α,

- if P - value $\leq \alpha$, we reject H_0 at level α (results are statistically significant at level α).
- if P- value $> \alpha$, we do not reject H_0 at level α (results are not statistically significant at level α).

Example: Medical researchers were concerned that a certain medication had the undesirable side effect of lowering the blood pressure of the user. A random sample of 15 adults who were prescribed the medication was selected. The researchers recorded each patient's blood pressure prior to beginning the medication and then again after six months of taking the medication regularly. The patients' diastolic measurements (in mm Hg) are given in the table below:

	Subject														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Before	70	80	72	76	76	76	72	78	82	64	74	92	74	68	84
After	68	72	62	70	58	66	68	52	64	72	74	60	74	72	74
Difference	2	8	10	6	18	10	4	26	18	-8	0	32	0	-4	10

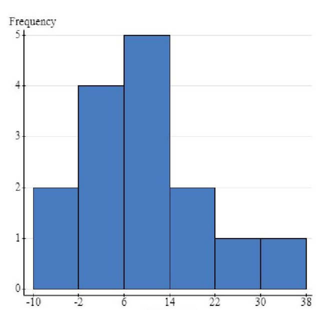
Note: For $d = y_{\text{before}} - y_{\text{after}}$, we have $\bar{d} = 8.8$ and $s_d = 10.98$.

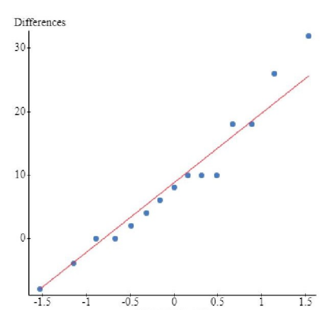
a) Do the data substantiate the claim that the medication reduces blood pressure? Use $\alpha = 0.02$.

$n = 15$

1. **Assumptions/Conditions:**

- Samples are paired
- 15 pairwise differences are a random sample
- assume 15 pairwise differences are independent
- $n = 15 < 30$, so we will assume population distribution is approximately normal





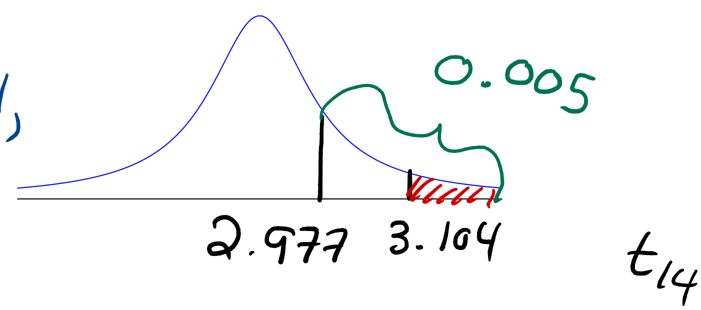
2. **Hypotheses:** $\left.\begin{aligned} H_0&: \mu_d = 0 \\ H_A&: \mu_d > 0 \end{aligned}\right\}$ upper-tailed test

3. **Test Statistic:**

$$t_0 = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}} = \frac{8.8}{\frac{10.98}{\sqrt{15}}} = 3.104$$

4. ***P*-value:** df = 15 − 1 = 14

Using t-table with df = 14,



P-value = $P(t_{14} > 3.104) < 0.005$
≤ 0.02

Paired T hypothesis test:

$\mu_D = \mu_1 - \mu_2$: Mean of the difference between Before and After
$H_0 : \mu_D = 0$
$H_A : \mu_D > 0$
Hypothesis test results:

Difference	Mean	Std. Err.	DF	T-Stat	P-value
Before - After	8.8	2.8338095	14	3.1053605	0.0039

≤ 0.02

P-value = 0.0039

5. **Conclusion:**

Since P-value ≤ $\alpha = 0.02$, we reject Ho at the 0.02 significance level, that is, there is enough statistical evidence to conclude that the medication reduces blood pressure.

b) Construct a 90% confidence interval for the mean reduction in blood pressure μ_d.

df = 14

$t^*_{14} = 1.761$

$$\bar{d} \pm t^*_{n-1} \frac{s_d}{\sqrt{n}}$$

$$= 8.8 \pm (1.761) \frac{10.98}{\sqrt{15}}$$

$$= 8.8 \pm 4.9925$$

$$= (3.81, 13.79)$$

∴ we are 90% confident that $\mu_d \in (3.81, 13.79)$. Since all values in the interval are positive, we are 90% confident that medication reduces blood pressure.

Paired T confidence interval:

$\mu_D = \mu_1 - \mu_2$: Mean of the difference between Before and After

90% confidence interval results:

Difference	Mean	Std. Err.	DF	L. Limit	U. Limit
Before - After	8.8	2.8338095	14	3.8087826	13.791217

$$\underbrace{\hphantom{3.8087826 \quad 13.791217}}_{(3.81,\ 13.79)}$$

Example: Trace metals in drinking water affect the taste of water and, in high concentrations, can pose a health hazard. Researchers collected samples of both bottom water and surface water at six randomly selected locations along a river. Data for the concentration of zinc (in mg/L) is given below:

Location	Bottom Water	Surface Water	Difference
1	0.43	0.415	0.015
2	0.266	0.238	0.028
3	0.567	0.39	0.177
4	0.531	0.41	0.121
5	0.707	0.605	0.102
6	0.716	0.609	0.107

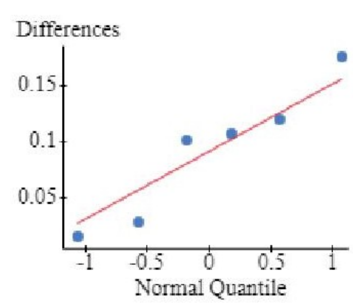
Note: For $d = y_{\text{bottom}} - y_{\text{surface}}$, we have $\bar{d} = 0.0917$ and $s_d = 0.0607$.

a) At the 1% significance level, is there enough statistical evidence to conclude that there is a difference between the mean zinc concentration in bottom and surface water?

$n = 6$

1. **Assumptions/Conditions:**

- Samples are paired
- 6 pairwise differences are a random sample
- assume 6 pairwise differences are independent
- $n = 6 < 30$, so we will assume population distribution is approximately normal



2. **Hypotheses:** $H_0: \mu_d = 0$

$H_A: \mu_d \neq 0$

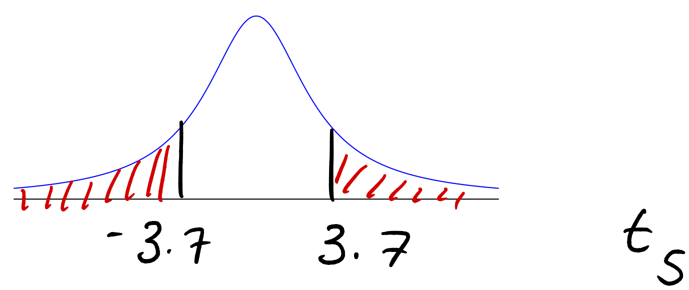
two-tailed test

3. **Test Statistic:**

$$t_o = \frac{\bar{d}}{\frac{s_d}{\sqrt{n}}} = \frac{0.0917}{\frac{0.0607}{\sqrt{6}}} = 3.7$$

4. ***P*-value:** $df = 6 - 1 = 5$

Using t-table with $df = 5$,



$0.1 < \text{P-value} = 2P(t_5 > 3.7) < 0.2$

$\alpha = 0.1$

Paired T hypothesis test:

$\mu_D = \mu_1 - \mu_2$: Mean of the difference between Bottom Water and Top Water

$H_0 : \mu_D = 0$

$H_A : \mu_D \neq 0$

Hypothesis test results:

Difference	Mean	Std. Err.	DF	T-Stat	P-value
Bottom Water - Top Water	0.091666667	0.024775884	5	3.6998343	0.014

> 0.1

P-value = 0.014

5. **Conclusion:**

Since P-value > $\alpha = 0.01$, we do not reject H_0 at the 0.01 significance level, that is, there is not enough statistical evidence to conclude that there is a difference between the mean zinc concentration in bottom and surface water.

two-tailed test using $\alpha = 0.01$

b) Construct a 99% confidence interval for the mean difference between bottom and surface water zinc concentrations μ_d.

df = 5

$$\bar{d} \pm t^*_{n-1} \frac{s_d}{\sqrt{n}}$$

$$t^*_5 = 4.032$$

$$= 0.0917 \pm (4.032) \frac{0.0607}{\sqrt{6}}$$

$$= 0.0917 \pm 0.0999$$

$$= (-0.008, 0.192)$$

$\ni 0 \rightarrow$ $H_0: \mu_d = 0$ not rejected

$\therefore$ we are 99% confident that $\mu_d \in (-0.008, 0.192)$

Paired T confidence interval:

$\mu_D = \mu_1 - \mu_2$: Mean of the difference between Bottom Water and Top Water

99% confidence interval results:

Difference	Mean	Std. Err.	DF	L. Limit	U. Limit
Bottom Water - Top Water	0.091666667	0.024775884	5	-0.008233634	0.19156697

(-0.008, 0.192)