Example: Stat 151 Winter 2020: Liking Hockey vs. Liking Soccer

Soccer

Hockey

	Yes	No	Total
Yes	42	21	63
No	30	59	89
Total	72	80	152

c) d) 0.14 rounding

Soccer

b)

Hockey

	Yes	No	Total
Yes	0.28	0.13	0.41
No	0.19	0.40	0.59
Total	0.47	0.53	1

disjoint

$\frac{30}{152} \simeq 0.20$

a)

Suppose we randomly select a student in this class.

Event A: likes hockey

Event B: likes soccer B^c = doesn't like soccer

Calculate:

a) P(Likes soccer) = $P(B) = \frac{72}{152} \simeq 0.47$

b) P(Likes hockey) = $P(A) = \frac{63}{152} \simeq 0.41$

c) P(Likes hockey and soccer) = $P(A \cap B) = \frac{42}{152} \simeq 0.28$

d) P(Likes hockey, but not soccer) = $P(A \cap B^c) = \frac{21}{152} \simeq 0.14$

d) P(Likes hockey or likes soccer)

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$ ← not disjoint

$= \frac{63}{152} + \frac{72}{152} - \frac{42}{152}$

$= \frac{93}{152} \simeq 0.61$

e) P(Likes hockey or doesn't like soccer)

$= P(A \cup B^c)$

$= P(A) + P(B^c) - P(A \cap B^c)$

↖ not disjoint

$= \frac{63}{152} + \frac{80}{152} - \frac{21}{152}$

$= \frac{122}{152} \simeq 0.8$

g) P(Likes hockey | likes soccer)

$= P(A \mid B)$

$= \frac{P(A \cap B)}{P(B)} = \frac{\frac{42}{152}}{\frac{72}{152}} = \frac{42}{72} \simeq 0.58$

g) P(Likes hockey | does not like soccer) $= P(A \mid B^c)$

$\frac{P(A \cap B^c)}{P(B^c)}$

$= \frac{21}{80} \simeq 0.26$

h) P(Likes soccer | likes hockey) $= P(B \mid A) = \frac{42}{63}$

$\simeq 0.67$

$\frac{P(B \cap A)}{P(A)}$

Independence

Two events A and B are said to be **independent** if

$$P(A|B) \underset{\neq}{=} P(A)$$

or equivalently

not true

$$P(B|A) \underset{\neq}{=} P(B)$$

Otherwise, A and B are said to be **dependent**.

- A and B are independent if the occurrence of one has **no effect** on the occurrence of the other.

Example: An urn contains three yellow balls and 4 green balls. Two balls are randomly selected.

Event B: first ball is yellow B^c = 1st ball is green

Event A: second ball is green

With replacement: ← independent

- $P(A|B) = \frac{4}{7} \simeq 0.57 = P(A)$

||

- $P(A|B^c) = \frac{4}{7} = P(A)$

Without replacement: ← dependent

- $P(A|B) = \frac{4}{6} = \frac{2}{3} \simeq 0.67 \neq P(A) = ?$

∦

- $P(A|B^c) = \frac{3}{6} = \frac{1}{2} = 0.5$

Note: Let A and B be events. The following are equivalent:

a) A and B are independent — dependent

b) $P(A|B) = P(A)$ — $P(A|B) \neq P(A)$

c) $P(B|A) = P(B)$ — $P(B|A) \neq P(B)$

d) $P(A|B) = P(A|B^c)$ — $P(A|B) \neq P(A|B^c)$

e) $P(A \cap B) = P(A)P(B)$ — $P(A \cap B) \neq P(A)P(B)$

Example: Liking hockey vs. liking soccer → dependent

P(Likes hockey | likes soccer) $\neq$ P(Likes hockey)

0.58 — 0.47

P(Likes hockey | likes soccer) $\neq$ P(Likes hockey | does not like soccer)

0.58 — 0.26

Example: A dentist's office surveyed all patients under 18 years of age, 40% of whom were under 12, about their fear of visiting the dentist. They found that 25% of all patients under 18 were afraid of visiting the dentist and 15% of all patients under 18 were afraid and under 12.

randomly select patient <18

	Fear	Do not fear	Total
Under 12	0.15	0.25	0.4
Between 12 and 17	0.1	0.5	0.6
Total	0.25	0.75	1

← disjoint

↖ disjoint

$P(\text{fear and} < 12) = 0.15$ ✗

$P(\text{fear})\,P(<12) = (0.25)(0.4) = 0.1$

⇒ fear, <12 dependent

Multiplication Rules:

- **General Multiplication Rule:**

$A \cap B$

$= B \cap A$

$$\begin{aligned} P(A \cap B) &= P(B)P(A|B) \\ &= P(A)\underbrace{P(B|A)} \end{aligned}$$

and

- **Multiplication Rule for Independent Events:**

If A and B are independent events, then

$$P(A \cap B) = P(A)P(B)$$

Example: An urn contains three yellow balls and four green balls.

Two balls are randomly selected **without replacement**. What is the probability that the second ball is yellow? ↪ dependent

Let Y_1 = first is yellow, Y_2 = second is yellow, and G_1 = first is green.

$$P(Y_2) = P(\text{"}\overset{Y_1 Y_2}{Y_1 \text{ and } Y_2}\text{"} \ \underline{\text{or}} \ \text{"}\overset{G_1 Y_2}{G_1 \text{ and } Y_2}\text{"}) \leftarrow \text{disjoint}$$

$$= P(Y_1 \text{ and } Y_2) + P(G_1 \text{ and } Y_2)$$

$$= P(Y_1)P(Y_2|Y_1) + P(G_1)P(Y_2|G_1)$$

$$= \left(\frac{3}{7}\right)\left(\frac{2}{6}\right) + \left(\frac{4}{7}\right)\left(\frac{3}{6}\right) \simeq 0.43$$

Three balls are randomly selected **without replacement**. Find the probability of each of the following: ↪ dependent

a) All three are green.

$$P(1^{st} \text{ green } \underline{\text{and}} \ 2^{nd} \text{ green } \underline{\text{and}} \ 3^{rd} \text{ green})$$

$$= \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} \simeq 0.11$$

conditional probabilities

b) The third ball selected is the first green ball.

yellow

P(1st notgreen and 2nd notgreen and 3rd green)

$= \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{4}{5} \simeq 0.11$

c) There is at least one green ball.

y y y

P(at least one green) = 1 - P(no green)

$= 1 - \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5}$

$\simeq 0.97$

Example: Suppose that the blood types of the citizens of a country occur with the following probabilities:

not Type A

disjoint

P(not Type A)

= 1 - P(Type A)

= 1 - 0.42 = 0.58

Blood Type	O	A	B	AB
Probability	0.47	0.42	0.08	0.03

P(not A)

= P(O) + P(B) + P(AB)

= 0.47 + 0.08 + 0.03

= 0.58

Assume that the blood types of two citizens that are married to each other are independent events. For a married couple, what is the probability that:

a) they are both Type A?

P(1st Type A and 2nd Type A)

= P(1st Type A) P(2nd Type A) = (0.42)(0.42)

= 0.1764

b) at least one is Type A?

P(at least one Type A) = 1 - P(neither Type A) independent

= 1 - P(1st not A) P(2nd not A)

= 1 - (0.58)(0.58) $\simeq$ 0.6636

c) one has Type AB and the other has Type O?

P(one AB and one O) ← disjoint

= P("1st AB and 2nd O" or "1st O and 2nd AB")

= P(1st AB and 2nd O) + P(1st O and 2nd AB)

indep.

= P(1st AB) P(2nd O) + P(1st O) P(2nd AB)

= (0.03)(0.47) + (0.47)(0.03)

$\simeq$ 0.0282

Warning: Do not assume events are independent (without reason to) and do not assume events are disjoint.

→ $A \cap B = \phi$

occurence of one no effect on → occurence of other

Warning: Do not confuse **disjoint** events with **independent** events.

Example: Consider the blood type example.

- For one person, the event that the person is Type A and the event that the person is Type B are disjoint.
- For two people that are married to each other, the event that the first person has Type A and the event that the second person has Type B are independent.

Suppose that A and B are two events such that $P(A), P(B) \neq 0$.

If A and B are **disjoint**, then they are **dependent**.

$$P(A|B) = \frac{\overset{=0}{P(A \cap B)}}{P(B)} = 0 \neq P(A)$$

Equivalently, if A and B are independent, they are not disjoint.

$$P(A \cap B) = \underset{\neq 0}{P(A)}\,\underset{\neq 0}{P(B)} \neq 0$$

Two dependent events may or may not be disjoint.