An **event** is a subset of the sample space. Simple : one outcome event

↳ capital letters

Example: Flipping a coin three times: # heads

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$ 0, 1, 2, 3

Consider the following events:

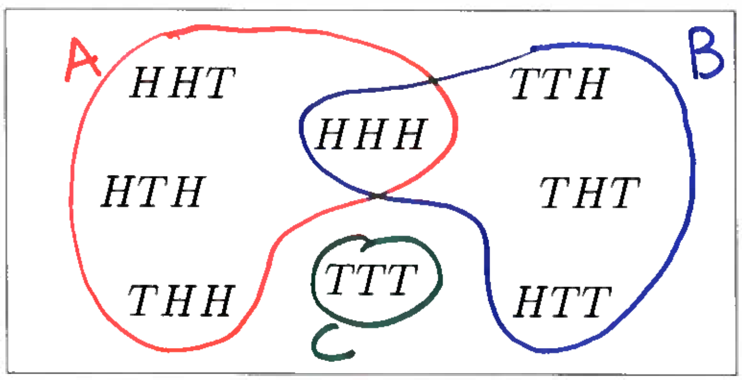
$A =$ at least two heads $= \{HHH, HHT, HTH, THH\}$

≥ 2, so 2, 3

$B =$ an odd number of heads $= \{HHH, TTH, THT, HTT\}$

1, 3

$C =$ no heads $= \{TTT\}$ ← Simple



$A^c = \{TTH, THT, HTT, TTT\}$ at most one head : 0, 1

$B^c = \{HHT, HTH, THH, TTT\}$ even # of heads : 0, 2

$C^c = \{HHH, HHT, HTH, THH, TTH, THT, HTT\}$ at least one head

or

$A \cup B = \{HHT, HTH, THH, HHH, TTH, THT, HTT\}$

$A \cup C = \{HHH, HHT, HTH, THH, TTT\}$

$B \cup C = \{TTH, THT, HTT, HHH, TTT\}$

and

$A \cap B = \{HHH\}$

$A \cap C = \emptyset$ ← empty set { }

$B \cap C = \emptyset$

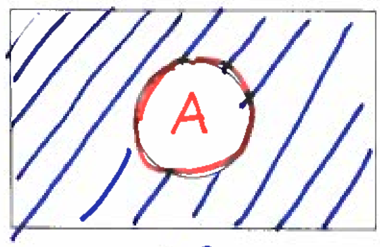
Forming New Events

Let S be a sample space of a random phenomenon and let A and B be two events consisting of outcomes of S.

event

a) **Complement of A:** set consisting of all outcomes in S which are **not** in A. Denoted A^c. or $\bar{A}$

- A^c occurs means that A does not occur.



A^c

$A \cap A^c = \emptyset$

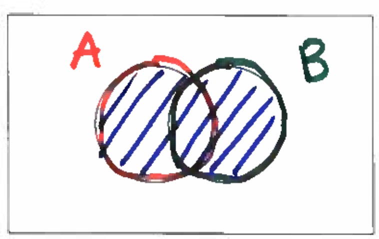
$A \cup A^c = S$

event

b) **Union of A and B:** set consisting of all outcomes in S which are in A or in B (or both). Denoted $A \cup B$.

- $A \cup B$ occurs means that either A occurs or B occurs (or both occur).

or

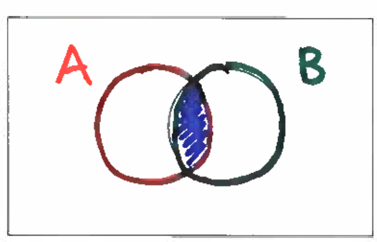


$A \cup B$

c) **Intersection of A and B:** set consisting of all outcomes in S which are in A and in B. Denoted $A \cap B$. → event

- $A \cap B$ occurs means that both A and B occur.

and



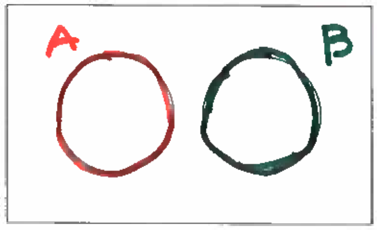
$A \cap B$ $= B \cap A$

Two events A and B are said to be **disjoint** or **mutually exclusive** if they have no outcomes in common, that is, $A \cap B = \emptyset$.

- When one event occurs, the other does not.

→ Simple events disjoint

· A, A^c disjoint



Probability

The **probability** of an event A of a chance experiment, denoted $P(A)$, is the proportion of times that the event would occur if the experiment were performed an arbitrarily large number of times. This value is guaranteed to exist by the **Law of Large Numbers** (LLN) .

- probability measures the likelihood of an event's occurrence.
- $P(A) =$ sum of the probabilities of the outcomes in A (Simple events)
- $0 \leq P(A) \leq 1$
- $P(A) = 1$ if event is certain to occur.
- $P(A) = 0$ if event is impossible.

} S finite or countably infinite

If S consists of **equally likely** outcomes, then

watch for this!!

$$P(A) = \frac{\#\text{ of outcomes in } A}{\underbrace{\#\text{ of outcomes in } S}_{\text{total \# of outcomes}}}$$

Example: Flipping a fair (balanced) coin once: $S = \{H, T\}$.

$P(\{H\}) = \frac{1}{2}$ $P(\{T\}) = \frac{1}{2}$ equally likely

Example: Rolling a fair die once: $S = \{1, 2, 3, 4, 5, 6\}$.

Event A: rolling a 6 $= \{6\}$

Event B: rolling an even number $= \{2, 4, 6\}$

$P(A) = \frac{1}{6}$

$P(B) = \frac{3}{6} = \frac{1}{2}$

Example: Flipping an **unbalanced** coin two times: not equally likely

$S = \{HH, HT, TH, TT\}$

$P(S) = 1$

Suppose the outcomes have the following probabilities:

Outcome	HH	HT	TH	TT
Probability	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

Event A: at most one head $= \{HT, TH, TT\}$ $A^c = \{HH\}$

≤ 1, so 0, 1

$P(A) = 1 - P(A^c) = 1 - \frac{4}{9} = \frac{5}{9}$

* individual outcomes

$P(A) = P(\{HT\}) + P(\{TH\}) + P(\{TT\})$

$= \frac{2}{9} + \frac{2}{9} + \frac{1}{9} = \frac{5}{9} \approx 0.56$

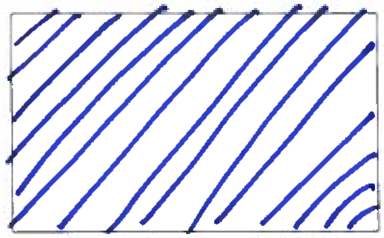
Probability Rules:

→ area in Venn diagram.

Let S be a sample space of a random phenomenon and let A and B be two events consisting of outcomes of S.

- **Total Probability Rule:** $P(S) = 1$ → Sum of probabilities of <u>all</u> outcomes

S

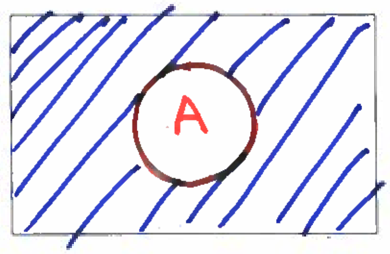


area = 1

- **Complement Rule:** $P(A) = 1 - P(A^c)$ → watch for $P(\text{at least one}) = 1 - P(\text{none})$

$P(A) + P(A^c) = 1$

$P(A^c) = 1 - P(A)$



A^c

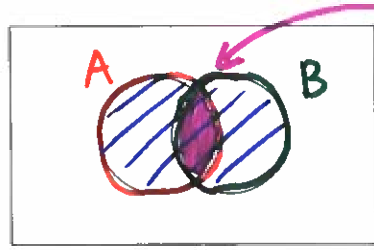
$A \cup A^c = S$

$P(A \cup A^c) = 1$

$A \cap A^c = \emptyset$

$P(A \cap A^c) = 0$

- **General Addition Rule:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



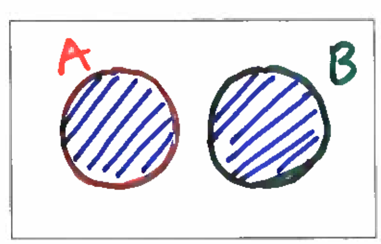
$A \cap B$ counted twice

$A \cup B$

- **Addition Rule for <u>Disjoint</u> Events:** $P(A \cup B) = P(A) + P(B)$

$A \cap B = \emptyset$

$P(A \cap B) = 0$



$A \cup B$

→ equally likely outcomes

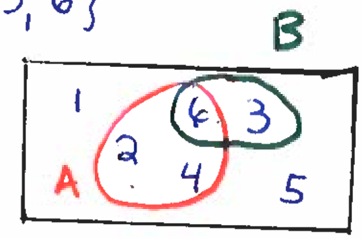
Example: Rolling a fair die once: $S = \{1, 2, 3, 4, 5, 6\}$. $P(S) = 1$

Event A: rolling an even number $= \{2, 4, 6\}$

Event B: rolling a number that is divisible by 3 $= \{3, 6\}$

$A^c = \{1, 3, 5\}$ $B^c = \{1, 2, 4, 5\}$

$A \cap B = \{6\}$ $A \cup B = \{2, 3, 4, 6\}$



Compute the following probabilities:

$P(A) = \frac{3}{6} = \frac{1}{2}$

$P(B) = \frac{2}{6} = \frac{1}{3}$

$P(A \cap B) = \frac{1}{6}$

$P(A \cup B) = \frac{4}{6} = \frac{2}{3}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$

$P(A \cap B^c) = \frac{2}{6} = \frac{1}{3}$

$A \cap B^c = \{2, 4\}$

$P(A^c) = \frac{3}{6} = \frac{1}{2}$

$P(A^c) = 1 - P(A)$
$= 1 - \frac{1}{2}$
$= \frac{1}{2}$

$P(B^c) = \frac{4}{6} = \frac{2}{3}$

$P(B^c) = 1 - P(B)$
$= 1 - \frac{1}{3}$
$= \frac{2}{3}$

Event C = rolling a number that is divisible by 4 $= \{4\}$

Are A and C disjoint? No!

$A \cap C = \{4\} \neq \emptyset$

Are B and C disjoint? Yes!

$B \cap C = \emptyset$

Sums: 2, 3, ..., 12

Example: Rolling two fair dice: one red, one blue. $|S| = 36$

Event A: sum of the two dice is 5

$= \{ (1,4), (4,1), (2,3), (3,2) \}$

← disjoint

Event B: sum of the two dice is 10

$= \{ (5,5), (4,6), (6,4) \}$

Event C: sum of the two dice is at most 10 ≤ 10, so $2, 3, \dots, 10$

$C^c =$ sum ≥ 11 $= \{ (5,6), (6,5), (6,6) \}$

(so 11, 12)

What is the probability that the sum of the two dice is 5 or 10?

→ union

$$P(A \cup B) = P(A) + P(B)$$ (disjoint formula)

$$= \frac{4}{36} + \frac{3}{36}$$

$$= \frac{7}{36} \approx 0.19$$

approx 19% of outcomes have sum 5 or 10

What is the probability that the sum of the two dice is at most 10?

$$P(C) = 1 - P(C^c)$$

$$= 1 - \frac{3}{36}$$

$$= \frac{33}{36}$$

$$\simeq 0.92$$

approx 92% of outcomes have sum ≤ 10

Chapter 12: Probability Rules

Conditional Probability

Let A and B be events such that $P(B) \neq 0$.

The probability that A occurs given that B has already occurred is called the **conditional probability of A given B** and is denoted $P(A|B)$.

Example: Tossing a fair coin and then a fair die. $|S| = 12$

Event A: getting a head on the coin

Event B: getting a 3 on the die

$P(A|B) = \frac{1}{2} = P(A)$

$P(B|A) = \frac{1}{6} = P(B)$

Example: Flipping a fair coin twice.

$\{HH, HT, TH, TT\}$

$P(H \text{ on second flip} \,|\, \underbrace{T \text{ on first flip}}_{TH \quad TT}) = \frac{1}{2} = P(H \text{ second})$

Example: Draw two cards (randomly) from a deck of 52 cards:

Event A: the first card is a 7

Event B: the second card is an ace

With replacement: $P(B|A) = \frac{4}{52} = \frac{1}{13}$

Without replacement: $P(B|A) = \frac{4}{51}$

Formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example: Rolling two fair dice. $|S| = 36$

Event A: a 2 appears on at least one dice $A^c =$ 2 doesn't appear

$= \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)\}$

$P(A) = 1 - P(A^c)$

easier to compute.

Event B: sum of the two dice is 5

$= \{(1,4),(4,1),(2,3),(3,2)\}$

What is the probability of rolling a 2 on at least one of the dice given that the sum is 5?

$A \cap B = \{(2,3),(3,2)\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{4}{36}} = \frac{2}{4} = \frac{1}{2}$$