Binary Numbers

- Digits are 1 and 0
  - 1 = true
  - 0 = false
- MSB – most significant bit
- LSB – least significant bit
- Bit numbering:

MSB	LSB
1 0 1 1 0 0 1 0 1 0 0 1 1 1 0 0	
15	0

Binary Numbers

- Each digit (bit) is either 1 or 0
- Each bit represents a power of 2:

1	1	1	1	1	1	1	1
2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

Every binary number is a sum of powers of 2

Table 1-3 Binary Bit Position Values.

2^n	Decimal Value	2^n	Decimal Value
2^0	1	2^8	256
2^1	2	2^9	512
2^2	4	2^{10}	1024
2^3	8	2^{11}	2048
2^4	16	2^{12}	4096
2^5	32	2^{13}	8192
2^6	64	2^{14}	16384
2^7	128	2^{15}	32768

Translating Binary to Decimal

Weighted positional notation shows how to calculate the decimal value of each binary bit:

$$dec = (D_{n-1} \times 2^{n-1}) + (D_{n-2} \times 2^{n-2}) + \ldots + (D_1 \times 2^1) + (D_0 \times 2^0)$$

D = binary digit

binary 00001001 = decimal 9:

$$(1 \times 2^3) + (1 \times 2^0) = 9$$

Translating Unsigned Decimal to Binary

- Repeatedly divide the decimal integer by 2.
- Each remainder is a binary digit in the translated value:

Division	Quotient	Remainder
37 / 2	18	1
18 / 2	9	0
9 / 2	4	1
4 / 2	2	0
2 / 2	1	0
1 / 2	0	1

37 = 100101

Binary Addition

- Starting with the LSB, add each pair of digits, include the carry if present.

carry:					1				
	0	0	0	0	0	1	0	0	(4)
+	0	0	0	0	0	1	1	1	(7)
	0	0	0	0	1	0	1	1	(11)
bit position:	7	6	5	4	3	2	1	0	

Integer Storage Sizes

Standard sizes:

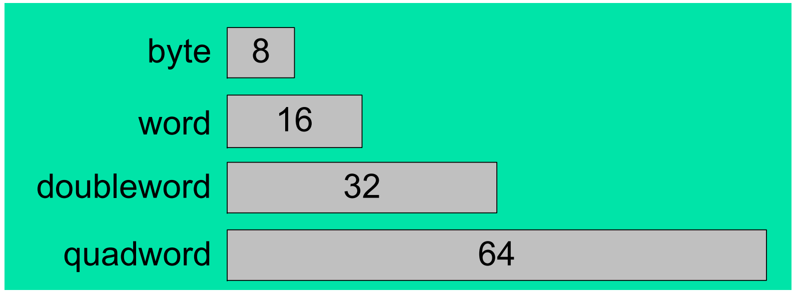


Table 1-4 Ranges of Unsigned Integers.

Storage Type	Range (low–high)	Powers of 2
Unsigned byte	0 to 255	0 to $(2^8 - 1)$
Unsigned word	0 to 65,535	0 to $(2^{16} - 1)$
Unsigned doubleword	0 to 4,294,967,295	0 to $(2^{32} - 1)$
Unsigned quadword	0 to 18,446,744,073,709,551,615	0 to $(2^{64} - 1)$

What is the largest unsigned integer that may be stored in 20 bits?

Large Measurements

- Kilobyte (KB), 2^{10} bytes
- Megabyte (MB), 2^{20} bytes
- Gigabyte (GB), 2^{30} bytes
- Terabyte (TB), 2^{40} bytes
- Petabyte, 2^{50} bytes
- Exabyte, 2^{60} bytes
- Zettabyte, 2^{70} bytes
- Yottabyte, 2^{80} bytes
- Googol, 10^{100}

Hexadecimal Integers

Binary values are represented in hexadecimal.

Table 1-5 Binary, Decimal, and Hexadecimal Equivalents.

Binary	Decimal	Hexadecimal	Binary	Decimal	Hexadecimal
0000	0	0	1000	8	8
0001	1	1	1001	9	9
0010	2	2	1010	10	A
0011	3	3	1011	11	B
0100	4	4	1100	12	C
0101	5	5	1101	13	D
0110	6	6	1110	14	E
0111	7	7	1111	15	F

Translating Binary to Hexadecimal

- Each hexadecimal digit corresponds to 4 binary bits.
- Example: Translate the binary integer 000101101010011110010100 to hexadecimal:

1	6	A	7	9	4
0001	0110	1010	0111	1001	0100

Converting Hexadecimal to Decimal

- Multiply each digit by its corresponding power of 16:

$$\text{dec} = (D_3 \times 16^3) + (D_2 \times 16^2) + (D_1 \times 16^1) + (D_0 \times 16^0)$$

- Hex 1234 equals $(1 \times 16^3) + (2 \times 16^2) + (3 \times 16^1) + (4 \times 16^0)$, or decimal 4,660.

- Hex 3BA4 equals $(3 \times 16^3) + (11 * 16^2) + (10 \times 16^1) + (4 \times 16^0)$, or decimal 15,268.

Powers of 16

- Used when calculating hexadecimal values up to 8 digits long:

16^n	Decimal Value	16^n	Decimal Value
16^0	1	16^4	65,536
16^1	16	16^5	1,048,576
16^2	256	16^6	16,777,216
16^3	4096	16^7	268,435,456

Converting Decimal to Hexadecimal

Division	Quotient	Remainder
422 / 16	26	6
26 / 16	1	A
1 / 16	0	1

decimal 422 = 1A6 hexadecimal

Hexadecimal Addition

- Divide the sum of two digits by the number base (16).
- The quotient becomes the carry value, and the remainder is the sum digit.

		1	1
36	28	28	6A
42	45	58	4B
78	6D	80	B5

21 / 16 = 1, rem 5

Important skill: Programmers frequently add and subtract the addresses of variables and instructions.

Hexadecimal Subtraction

- When a borrow is required from the digit to the left, add 16 (decimal) to the current digit's value:

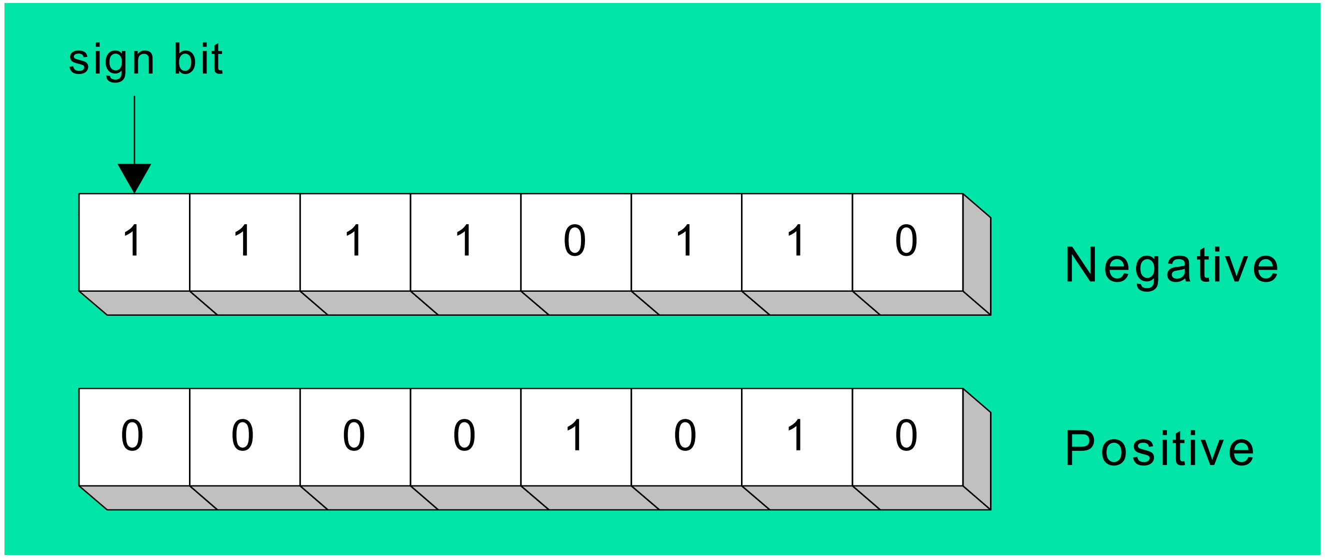
$$\boxed{16 + 5 = 21}$$

$$\begin{array}{rr} & {\color{blue}-1} \\ C6 & 75 \\ A2 & 47 \\ \hline 24 & 2E \end{array}$$

> Practice: The address of var1 is 00400020. The address of the next variable after var1 is 0040006A. How many bytes are used by var1?

Signed Integers

- The highest bit indicates the sign.
- 1 = negative, 0 = positive



If the highest digit of a hexadecimal integer is > 7, the value is negative. Examples: 8A, C5, A2, 9D

Forming the Two's Complement

- Bitwise NOT of the number and add 1

Starting value	00000001
Step 1: reverse the bits	11111110
Step 2: add 1 to the value from Step 1	11111110 +00000001
Sum: two's complement representation	11111111

Note that 00000001 + 11111111 = 00000000

8-bit Two's Complement Integers

sign bit									
0	1	1	1	1	1	1	1	=	127
0	0	0	0	0	0	1	0	=	2
0	0	0	0	0	0	0	1	=	1
0	0	0	0	0	0	0	0	=	0
1	1	1	1	1	1	1	1	=	−1
1	1	1	1	1	1	1	0	=	−2
1	0	0	0	0	0	0	1	=	−127
1	0	0	0	0	0	0	0	=	−128

8-bit two's complement integers

Binary Subtraction

- When subtracting A – B, convert B to its two's complement
- Add A to (–B)

$$
\begin{array}{r}
0\,0\,0\,0\,1\,1\,0\,0 \\
-\;0\,0\,0\,0\,0\,0\,1\,1 \\
\hline
\end{array}
\longrightarrow
\begin{array}{r}
0\,0\,0\,0\,1\,1\,0\,0 \\
1\,1\,1\,1\,1\,1\,0\,1 \\
\hline
0\,0\,0\,0\,1\,0\,0\,1
\end{array}
$$

Advantages for 2's complement:

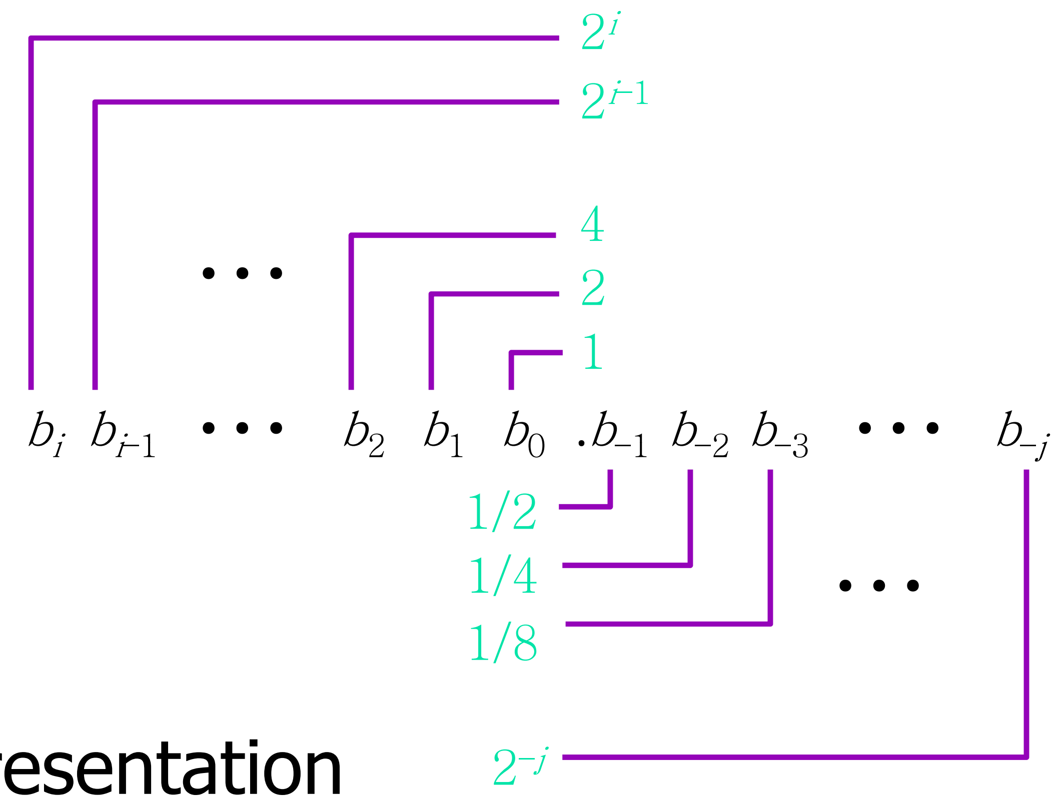
- No two 0's
- Sign bit
- Remove the need for separate circuits for add and sub

Ranges of Signed Integers

- The highest bit is reserved for the sign. This limits the range:

Storage Type	Range (low–high)	Powers of 2
Signed byte	−128 to +127	-2^7 to $(2^7 - 1)$
Signed word	−32,768 to +32,767	-2^{15} to $(2^{15} - 1)$
Signed doubleword	−2,147,483,648 to 2,147,483,647	-2^{31} to $(2^{31} - 1)$
Signed quadword	−9,223,372,036,854,775,808 to +9,223,372,036,854,775,807	-2^{63} to $(2^{63} - 1)$

Fractional Binary Numbers



- Representation
  - Bits to right of “binary point” represent fractional powers of 2
  - Represents rational number: $\displaystyle\sum_{k=-j}^{i} b_k \cdot 2^k$

Examples of Fractional Binary Numbers

- Value　　　　Representation
  - 5-3/4　　　　101.11_2
  - 2-7/8　　　　10.111_2
  - 63/64　　　　0.111111_2

- Observations
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation
  - Can only exactly represent numbers of the form $x \times 2^y$
  - Other numbers have repeating bit representations
- Value Representation

Value	Representation
1/3	$0.0101010101[01]\ldots_2$
1/5	$0.001100110011[0011]\ldots_2$
1/10	$0.0001100110011[0011]\ldots_2$

Converting Real Numbers

- Binary real to decimal real

$$110.011_2 = 4 + 2 + 0.25 + 0.125 = 6.375$$

- Decimal real to binary real

$$\begin{aligned} 0.5625 \times 2 &= 1.125 & \text{first bit} &= 1 \\ 0.125 \times 2 &= 0.25 & \text{second bit} &= 0 \\ 0.25 \times 2 &= 0.5 & \text{third bit} &= 0 \\ 0.5 \times 2 &= 1.0 & \text{fourth bit} &= 1 \end{aligned}$$

$$4.5625 = 100.1001_2$$