A STUDY ON

MAGNETIC RECONNECTION

By

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pursued under the guidance of

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I came across certain papers, articles and other content which were helpful in getting me acquainted with the concepts explored throughout the report. They have been duly cited with references.

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CONTENTS

CHAPTER

1

INTRODUCTION TO MAGNETIC RECONNECTION

What is magnetic reconnection?
Two field lines being frozen in and carried along with the fluid, until they come close to each other at some point where, due to weak nonideal effects in Ohm's law, they are cut and reconnected in a different way.[1]
-D.Biskamp, Magnetic Reconnection in Plasmas

In simple terms, it refers to the topological rearrangement of the magnetic field in a plasma. The process releases a large amount of energy stored in the field configuration in a violent eruption and comes to a lower energy state. The energy released is either dissipated as heat or transferred to the kinetic energy of the particles. It drives plasma flow.

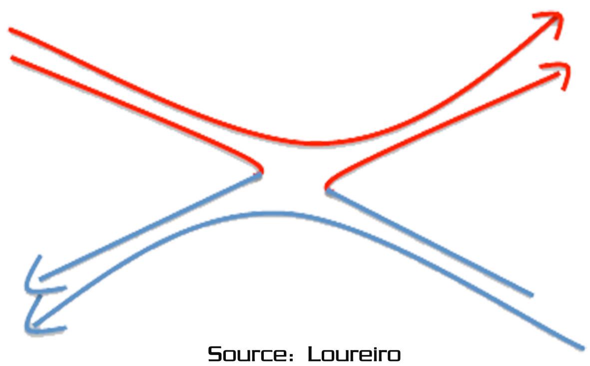

Figure 1.1: Elementary picture of field line reconnection

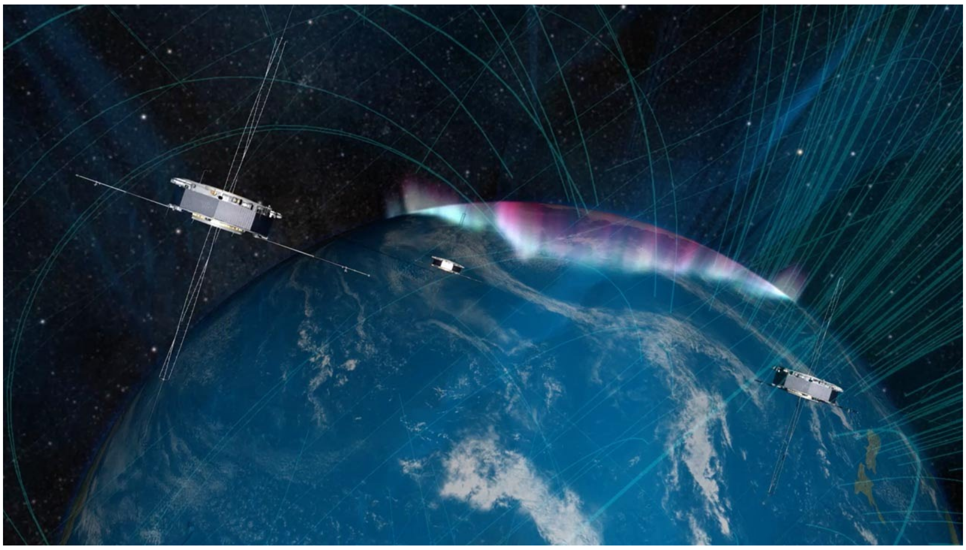

Figure 1.2: Reconnection drives the Northern Lights [Source:NASA]

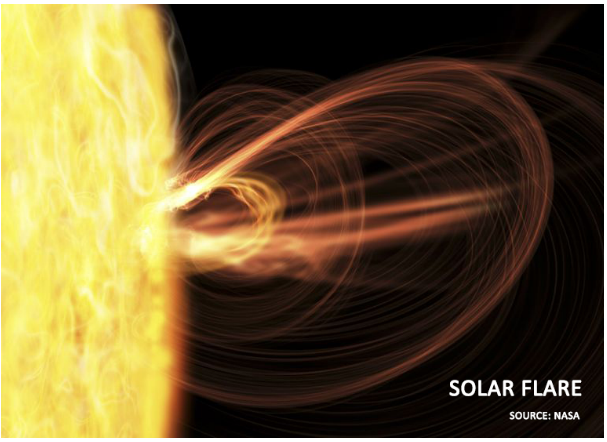

Figure 1.3: Reconnecting magnetic field lines in a solar flare

Reconnection - as we (don't) know it in our analytical labs:

- Reconnection rates as observed from solar flares are much faster than predicted by theory.
- While magnetic field lines break on microscopic scales, energy is stored and plasma responds on global scales. We try to understand how the two scales are coupled.

Can I see reconnection?
Such energetic eruptions are seen as solar flares in the photosphere of the Sun. Reconnection also occurs in the Earth's magnetosphere, as the magnetic field lines of the Sun reconnect with those of the Earth, enabling the solar wind to enter the Earth's atmosphere and manifest itself as the beautiful auroras. We have observed reconnection in the laboratory sawtooth crashes in tokamaks.

In order to appreciate this fully, let us go back a step and understand what these terms imply.

CHAPTER

2

MAGNETOHYDRODYNAMICS

In astrophysical settings, large magnetic fields are associated with the plasma (partially or fully-ionised gas) and the interaction between the two creates rather interesting phenomena that physicists are trying to decipher. Solar flares, accretion discs, planetory auroras, galactic dynamos are namely a few, where such spectacular interplay of matter and energy calls upon our attention. The implications of these processes in our daily lives is found in the magnetic substorms, sawtooth crashes in tokamaks, etc. and have driven research in this field of study beyond purely astrophysical interest. Reconnection is one such physical problem in order to discuss which we first need to mathematically describe the situation we are looking into.

Magnetohydrodynamics refers to studying the interaction between magnetic fields and plasma. Plasma here, is treated as a fluid. The fluid limit is valid when the collision rate of each species with itself is large compared to the macroscopic rates of change. This implies that the mean free paths are short compared to the macroscopic scales. Thus, before plasma conditions change significantly, each species collisionally relaxes to a local Maxwellian with its own mean density ρ, temperature T and mean velocity V. These parameters characterize the plasma state and are described by the MHD equations.[2] In MHD we combine Maxwell's equations of electrodynamics with the fluid equations, including the Lorentz forces due to the electro-magnetic fields present.

The magnetic Reynold's number is given by:

$$R_M = \frac{VL}{\eta}$$

where, V = Bulk velocity of the plasma, L = Length scale of Magnetic field, η = Spitzer resistivity. In astrophysical plasmas, this dimensionless quantity is usually large and hence resistive effects are often negligible.

2.1 MHD Equations

Induction Equation:

The induction equation gives the evolution of the magnetic field given the velocity field. Gaussian units have been used. The symbols denote the following:
$\mathbf{E}$ = Electric field, $\mathbf{V}$ = Bulk velocity of plasma, $\mathbf{B}$ = Magnetic field, $\mathbf{j}$ = Current density, c = speed of light in vacuum, σ = Conductivity of plasma.
Faraday's Induction Equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

Ohm's Law (in fixed frame of reference):

$$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} + \rho_q \mathbf{V} = \frac{\mathbf{j}}{\sigma}$$

Poisson's Equation:

$$\nabla.\mathbf{E} = 4\pi\rho_q$$

Ampere's Law:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\ \mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

Substituting, we get the Induction Equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \eta\nabla^2\mathbf{B}$$

where, $\eta = \dfrac{c^2}{4\pi\sigma}$

Here the Faraday time, $\tau_F = \frac{\eta}{c^2}$ is found to be very small as compared to the advective time scale, $\frac{L}{v}$, hence the length rate of change of the Electric field can be neglected. Also, the Faraday time is small as compared to the typical time over which the Electric field varies, hence the time rate of change of the Electric field can be neglected too.[2]

Divergence $\mathbf{B}$ Equation:

$$\nabla.\mathbf{B} = 0$$

Momentum Equation:

The evolution of the bulk velocity of the plasma is obtained. Ideal fluid treatment has been done. The symbols denote the following:
p = plasma pressure, ρ = density, g = gravitational potential, γ = adiabatic constant.
Continuity equation (Law of conservation of mass):

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{V}) = 0$$

For incompressible plasma, $\nabla.(\rho\mathbf{V}) = 0$

Equation of motion:

$$\rho\frac{d\mathbf{V}}{dt} = \rho(\tfrac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\ \mathbf{V}) = \frac{\mathbf{j} \times \mathbf{B}}{c} - \nabla p + \rho\mathbf{g}$$

The plasma obeys the ideal gas equation, $pM = \rho RT$. Taking adiabatic flow,
$\frac{p}{\rho^\gamma} = constant.$
Entropy equation:

$$\frac{d}{dt}(\tfrac{p}{\rho^\gamma}) = 0$$

2.2 Flux Freezing

The change in flux through a fluid moving from a surface **S** (bounded by a curve C) to a surface **S'** in a time dt, with velocity **V**, is given by:

$$d\phi = \int_{S'} \mathbf{B}(t+dt).d\mathbf{S} - \int_S \mathbf{B}(t).d\mathbf{S}$$

Using $\int \nabla.\mathbf{B}(t+dt)dV = 0$ for the flux tube swept by the moving surface **S** in time $dt \to 0$:

$$\int_{S'} \mathbf{B}(t+dt).d\mathbf{S} - \int_S \mathbf{B}(t+dt).d\mathbf{S} + \oint_C \mathbf{B}(t+dt).(d\mathbf{l} \times \mathbf{V}dt) = 0$$

Substituting into the equation for flux:

$$d\phi = \int_S (\mathbf{B}(t+dt) - \mathbf{B}(t)).d\mathbf{S} - \oint_C \mathbf{B}(t+dt).(d\mathbf{l} \times \mathbf{V}dt)$$

Using $\mathbf{B}.(d\mathbf{l} \times \mathbf{V}) = (\mathbf{V} \times \mathbf{B}).d\mathbf{l}$ and Green's theorem for curl of a vector:

$$d\phi = \int_S \frac{\partial \mathbf{B}}{\partial t} dt.d\mathbf{S} - \int_S \nabla \times (\mathbf{V} \times \mathbf{B}) dt.d\mathbf{S}$$

For a highly conducting plasma, as is the case in astrophysical contexts, the resistive term is dropped and the induction equation becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})$$

Hence, we get $\frac{d\phi}{dt} = 0$. This is called magnetic flux freezing.[2]

Reconnection implies breaking the frozen flux constraint, i.e., going beyond Ohm's law.

$$\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} = \frac{\mathbf{j}}{\sigma}$$

Flux freezing: η is small in highly conducting plasma, RHS goes to zero.
Magnetic reconnection: The RHS becomes important not because resistivity is large, but because sharp gradients of the magnetic field give rise to a large current(current sheet).

CHAPTER 3

SWEET-PARKER MODEL

Highly conducting astrophysical plasmas show fast rate of reconnection. Taking into account resistive dissipation alone, the large scale magnetic field of the Sun would decay so slowly that it could outlast the age of the universe. One must ask, then how does the magnetic field annihilate during reconnection? The order of the resistive diffusion time τ_R for field lines can be estimated using the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} \sim \eta \nabla^2 \mathbf{B} \Rightarrow \tau_R = \frac{L^2}{\eta}$$

But a typical solar flare takes only 15 mins to 1 hr to happen! That rules out the simple resistive diffusion mechanism. So we need something faster.

How did we begin understanding reconnection?
NASA has renamed the Solar Probe Plus spacecraft − humanity's first mission to a star, which will launch in 2018 − as the Parker Solar Probe in honor of astrophysicist Eugene Parker. At a conference in 1956, Peter Sweet pointed out that by pushing two plasmas with oppositely directed magnetic fields together, resistive diffusion is able to occur on a length scale much shorter than a typical equilibrium length scale. Eugene Parker was in attendance at this conference and developed scaling relations for this model during his return travel.

Sweet-Parker Current Sheet
It's a long(length $\sim 2L$) and thin(width $\sim 2\delta$) current sheet with inflow plasma velocity v_{in} along x axis from both sides and outflow plasma velocity v_{out} along both negative and positive y directions. Magnetic field outside the sheet is given by $\mathbf{B} = B_0\mathbf{y}$ and is zero at the centre, i.e., B=0 at x=0, hence there is a singularity of the current density. The sharp gradient of the magnetic field across the width of the sheet gives rise to a large current.

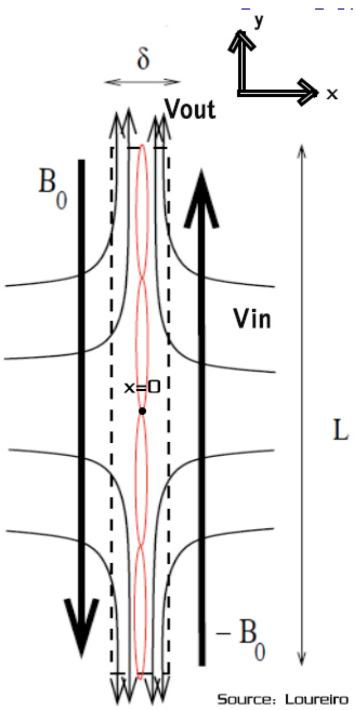

Figure 3.1: Sweet-Parker Current Sheet.

Conserving the mass flow of the incompressible plasma using continuity equation gives:(since $v_z = 0$)

$$\nabla.\mathbf{V} = 0 \Rightarrow v_{in}L = v_{out}\delta$$

Since the system is in steady state, the time rate of change of magnetic field is really small ~ 0, the induction equation becomes:

$$\nabla \times (\mathbf{V} \times \mathbf{B}) + \eta\nabla^2\mathbf{B} = 0 \rightarrow \text{y component} \rightarrow -\frac{v_{in}B_0}{\delta} + \eta\frac{B_0}{\delta^2} = 0 \Rightarrow v_{in} = \frac{\eta}{\delta}$$

Pressure balance in the sheet: $v_{out} = v_A$, $v_A = \dfrac{B_0}{\sqrt{4\pi\rho}}$ = Alfven Velocity

Alfven time: $\tau_A = \dfrac{L}{v_A}$; Resistive Diffusion time: $\tau_R = \dfrac{L^2}{\eta}$

S=Lundquist Number= $\dfrac{Lv_A}{\eta}$

$$\frac{v_{in}}{v_{out}} = \frac{\eta}{\delta v_A} = \frac{\delta}{L} \Rightarrow \delta = \sqrt{\frac{L\eta}{v_A}} = L\sqrt{\frac{\eta}{Lv_A}} = LS^{-\frac{1}{2}} \Rightarrow v_{in} = v_{out}S^{-\frac{1}{2}}$$

Aspect Ratio $= \dfrac{\delta}{L} = S^{-\frac{1}{2}}$

Reconnection time: $\tau_{SP} = \dfrac{L}{v_{in}} = S^{\frac{1}{2}}\tau_A = \sqrt{\tau_A\tau_R}$

Sweet Parker Rate: ${\tau_R}^{-1} < {\tau_{SP}}^{-1} << {\tau_A}^{-1}$

Hence, the Sweet-Parker rate is faster than resistive dissipation. However, it's not fast enough to account for reconnection.
Typical solar corona parameters: S$\sim 10^8$, $v_A = 100$ km/s, L $= 10^4$ km; this theory predicts that flares should last ∼15 days; in fact, flares last 15min to 1h.

CHAPTER

4

TEARING MODE INSTABILITY

4.1 Magnetic field and current profile

For theoretical estimation of the reconnection rate in natural systems, analysis is started off with a current sheet configuration, as shown in Fig 3.1. The Sweet Parker sheet parameters are assumed and it is asked if such a sheet can be stable.[3]
In order to determine the structure of the current sheet, a stationary solution of the induction equation is to be found:

$$\partial_t \mathbf{B} + \mathbf{u}.\nabla\mathbf{B} = \mathbf{B}.\nabla\mathbf{u} + \eta\nabla^2\mathbf{B}$$

where **u** is the velocity and **B** is the magnetic field, measured in velocity units. The desired resistive equilibrium is stationary, not static. If incompressibility is assumed and reconnection is considered in two-dimensional sheet (x, y) along the y-axis, then:

$$\nabla.v = 0 \Rightarrow \partial_x v_x + \partial_y v_y = 0$$

Consistent with this condition, flows can be taken of the form:

$$v_x = -\Gamma_0 x, v_y = \Gamma_0 y; \Gamma_0 = 2\frac{v_A}{L_{CS}}$$

where v_A is the Alfven velocity and L_{CS} is the length of the SP current sheet. At equilibrium, $\partial_t B = 0$. The change in parallel component of the magnetic field $B_{0_z} = 0$. Then the induction equation becomes:

$$(v_x\partial_x + v_y\partial_y)\mathbf{B}_0 = (\mathbf{B}_0.\nabla)\mathbf{v} + \eta\nabla^2\mathbf{B}_0$$

Since reconnection is along y-axis, say $B_{0_x} = 0, \mathbf{B} = B_0(x)\mathbf{y}$:

$$-\Gamma_0 x B'_{0_y} = \Gamma_0 B_{0_y} + \eta\nabla^2 B_{0_y}$$

Width of the current sheet, $\delta_{CS} = \sqrt{\frac{\eta}{\Gamma_0}}$:

$$\delta_{CS}^2 \partial_x^2 B_{0_y} = \partial_x(x B_{0_y})$$

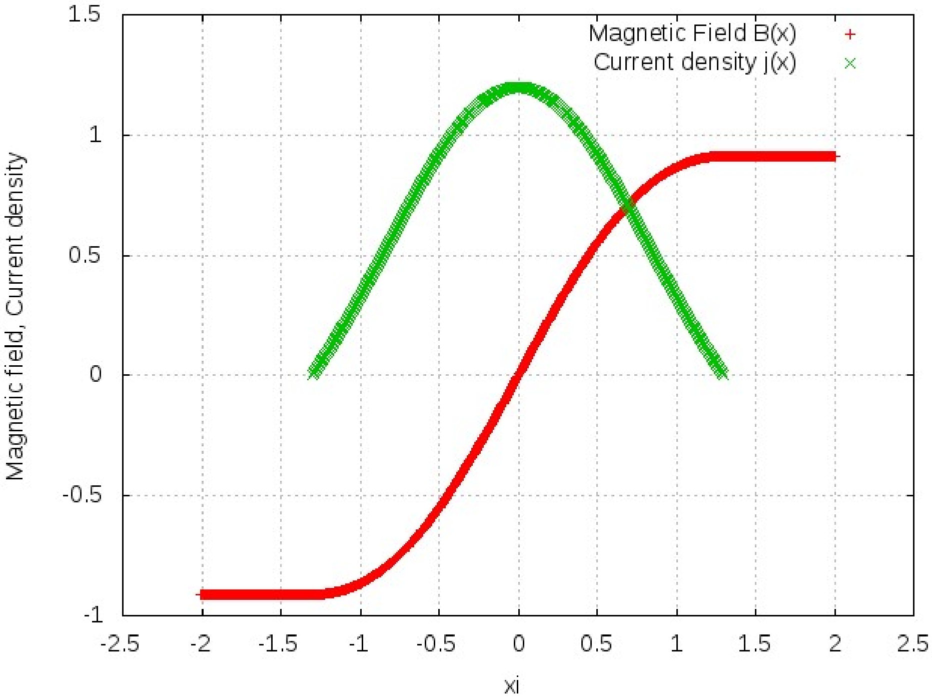

Figure 4.1: Magnetic field and Current density profile in the current sheet.

Substituting $x = \xi\delta_{CS}$ and solving the second order ODE, we get:

$$B_{0_y} = v_A f(\xi); f(\xi) = \alpha e^{\frac{-\xi^2}{2}} \int_0^\xi e^{\frac{z^2}{2}} dz$$

The integration constant α is chosen by matching this solution with the magnetic field outside the sheet: $B_{0_y} = \pm v_A$. The point $x = x_0$ at which the solutions inside and outside the current sheet can be matched is where B_{0_y} has its maximum, i.e., $\partial_x B_{0_y}(\pm x_0) = 0$, and the current vanishes. This gives, $x_0 = \xi_0\delta_{CS}$, where $\xi_0 \approx 1.3$. We require, $f(\pm\xi_0) = \pm 1 \Rightarrow \alpha = \xi_0$.The normalised equilibrium magnetic field and current are plotted in Figure 4.1. Next, a fast linear instability in the current sheet will be investigated.

4.2 Linear Tearing Mode

- In the absence of resistivity, plasma instabilities in such field configurations are stable.
- For non-zero resistivity, plasma instabilities grow and are termed as tearing modes.

Two dimensional perturbations to the Sweet-Parker Equilibrium in the plane of the current sheet, i.e., along x and y axes are of the form:

$$\zeta = \zeta_0(x)\exp(iky + \gamma t)$$

Here, ζ denotes the form of all perturbations in magnetic field and velocity.

Now, the magnetic field in the sheet is given as: $\mathbf{B} = \mathbf{B}_\perp + B_0(x)\mathbf{y}$

where, $\mathbf{B}_\perp = B_{0_x}(x)\exp(iky + \gamma t)\mathbf{x} + B_{0_y}(x)\exp(iky + \gamma t)\mathbf{y}$

The equilibrium plasma velocity is zero. The velocity in the sheet is given by : $\mathbf{v} = \mathbf{v}_\perp$

where, $\mathbf{v}_\perp = v_{0_x}(x)\exp(iky + \gamma t)\mathbf{x} + v_{0_y}(x)\exp(iky + \gamma t)\mathbf{y}$

Substituting the perturbed magnetic field and velocity into the MHD equations and linearizing them, we get:

- Induction Equation: $B_x\gamma = ikB_0v_x + \eta({\partial_x}^2 - k^2)B_x$
- curl of Euler Equation: $\gamma\rho_0(\partial_x^2 - k^2)v_x = ikB_{0_y}(\partial_x^2 - k^2 - \frac{B''_{0_y}(x)}{B_{0_y}(x)})B_x$

Two functions are defined to simplify the equations:

Stream function: ϕ; Flux function: ψ

Hydromagnetic timescale: $\tau_H = \frac{1}{kB_{0_y}}$; $\mathbf{v} = \mathbf{z} \times \nabla\phi$; $\mathbf{B} = \mathbf{z} \times \nabla\psi$;

Substituting to make the quantities dimensonless,

$\frac{B_{0_y}(x)}{B_0} = F(x), x \to \frac{x}{a}, k \to ka, F' \to \frac{dF}{dx}, \phi \to \frac{ikv_x}{\gamma}, \psi \to \frac{B_x}{B_0}, \gamma \to \gamma\tau_A$

where B_0 is the typical magnetic field and a is the typical length scale in the plasma, we get the Reduced MHD equations (see Appendix A for derivation):

- $\gamma(\psi - F\phi) = S^{-1}({\partial_x}^2 - k^2)\psi$

  RHS represents the plasma resistivity.

- $\gamma^2({\partial_x}^2 - k^2)\phi = -k^2F({\partial_x}^2 - k^2 - \frac{F''}{F})\psi$

  LHS represents the plasma inertia.

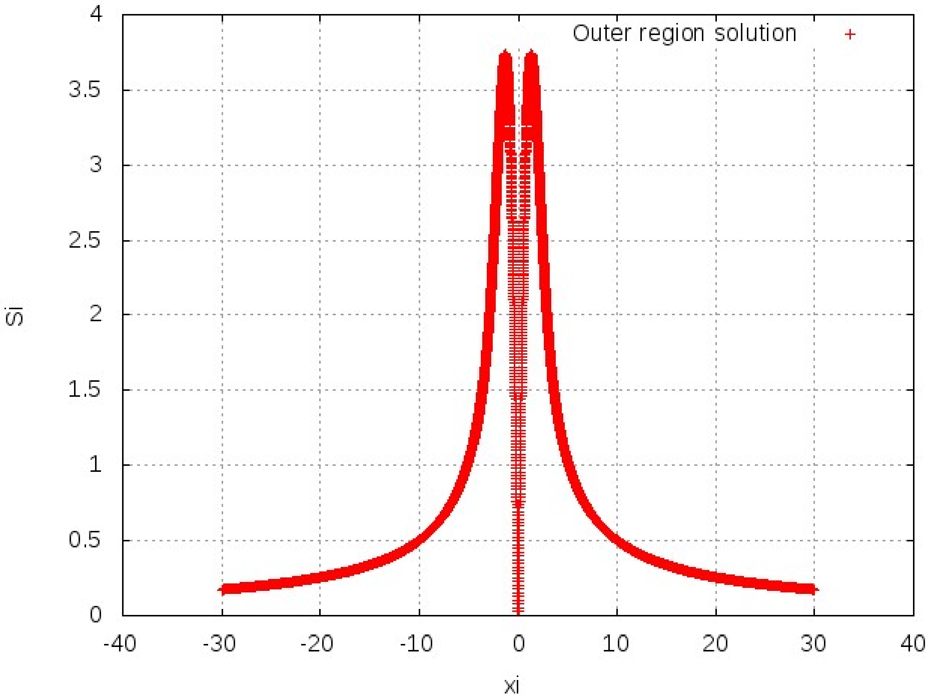

Figure 4.2: Solution: The magnetic flux function in the outer region.

Outer region

Flux freezing condition holds for the plasma. This is the 'ideal' MHD region.
Time scale of growth of instability, γ, is intermediate between the advective rate and the resistive diffusion rate: $\tau_A << \gamma^{-1} << \tau_R \rightarrow$ After normalising: $\gamma << 1 << S\gamma; S = \frac{\tau_R}{\tau_A}$
(Effects of plasma inertia and resistivity are negigible). The following equations are obtained:

$$\psi = F\phi$$
$$(\partial_x^2 - k^2 - \frac{F''}{F})\psi = 0$$

It can be solved for ψ perturbatively, using $k^2 << 1$. Neglecting k^2 to lowest order, one of the solutions is F(x). The second solution is found, to get the general solution. The constant coefficients can be found by matching the solution to be continuous at x = 0. Outside the current sheet, F$''$ = 0 and $\psi'' = k^2\psi \Rightarrow \psi = C_3^{\pm} e^{\mp kx}$. Matching the former solution to this at the edge of the current sheet, rest of the coefficients can be found. The solution comes out to be:(substituting B_{0_y} as the equilibrium magnetic field profile as discussed in the previous section; symbols have their usual meanings.)

$$\psi^{\pm}(\xi) = \pm\frac{\alpha\psi(0)}{k} f(\xi) - \alpha\psi(0) f(\xi) \int_{\pm\xi_0}^{\xi} \frac{dz}{f^2(z)}$$

This solution has been plotted in Figure 4.2 for $k\delta_{CS} = 0.34$. This value is of the wave number, k, corresponding to the maximum growth rate of the instability(more on this, later).[3]

For standard tearing mode calculations, it can be seen that the differential equations have a non-trivial solution such that: $\psi(x) = \psi(-x)$ and $\phi(x) = -\phi(-x)$
At the edge of the resistive layer, the solution can be given as:

$$\psi(x) = \psi_0(1 + \tfrac{\Delta'}{2}|x|) \rightarrow \text{even parity.}$$
$$\phi(x) = \frac{\psi(x)}{x} \rightarrow \text{odd parity.}$$

At the interface, the derivative of the solution is discontinous. This gives the 'tearing mode instability':

$$\Delta' = \frac{[\partial_x \psi]_{0^-}^{0^+}}{\psi(0)}$$

which can be derived from the equilibrium field profile.

Inner region

Resistive effects become important and ideal MHD breaks down at the interface, x = 0 where $B_{0_y} = 0$. For small x, the equilibrium magnetic field, $F \sim x \Rightarrow F'' \sim 0$
In a thin current sheet, $x << L, \partial_x >> 1$. The following equations are obtained:

$$\gamma(\psi - x\phi) = S^{-1}\partial_x^2\psi$$
$$\gamma^2\partial_x^2\phi = -x\partial_x^2\psi$$

Upon solving for ϕ and ψ by taking Fourier transform of these differential equations, the following solution is obtained at the edge of the resistive layer:

$$\phi(x) = a_{-1}\tfrac{\pi}{2}S^{1/3}sgn(x) + \frac{a_0}{x} + ..$$
$$\psi = x\phi \Rightarrow \psi(x) = a_{-1}\tfrac{\pi}{2}S^{1/3}sgn(x)x + a_0$$

$\psi(x)$ changes sign at $x = 0$ and it can be seen that $\psi'(x)$ is discontinuous at the interface. This disconitinuity is given by:

$$\Delta(\gamma, S) = \pi\frac{a_{-1}}{a_0}S^{\frac{1}{3}}$$

the coefficients a_{-1} and a_0 are determined from the asymptotic behaviour of the Fourier transformed layer solution at the edge of the resistive layer. The growth-rate, γ, of the tearing instability is determined by the matching criterion:

$$\Delta(\gamma, S) = \Delta'$$

Apparently, the Fourier transformed equations are exactly solvable and the most general solution is expressed as:[4]

$$\hat{\phi}(k_x) = A\left[1 - 2\,\frac{\Gamma(3/4)}{\Gamma(1/4)}\,Q^{1/4}\,k_x + O(k_x^2)\right].$$

where $Q = \gamma\,\tau_H^{2/3}\,\tau_R^{1/3}$; $Q > 0$ for unstable tearing mode.
Upon matching the solutions, the coefficients can be found and it is obtained:[4]

$$\Delta = 2\pi\,\frac{\Gamma(3/4)}{\Gamma(1/4)}\,S^{1/3}\,Q^{5/4}.$$

Alternatively, the order of γ may be estimated using:

$$\Delta' = \int_{x^-}^{x^+} \frac{\psi''(x)}{\psi(0)} dx = -\int_{x^-}^{x^+} \frac{\phi''(x)}{x\psi(0)} \gamma^2 dx$$

Substituting the equations and solving, the growth rate of the instability is obtained as:

$$\gamma \sim \frac{\Delta'^{\frac{4}{5}}}{\tau_H^{\frac{2}{5}} \tau_R^{\frac{3}{5}}}$$

The tearing mode is unstable when $\Delta' > 0$ and it grows on the hybrid time scale $\tau_H^{\frac{2}{5}} \tau_R^{\frac{3}{5}}$.
Thickness of the current sheet, $\delta \sim (\frac{\gamma}{S})^{\frac{1}{4}}; S = \frac{\tau_R}{\tau_H}$
For large S, the unstable current sheet is extremely narrow and it breaks up into plasmoids.

4.3 Non-linear stage

APPENDIX

A

REDUCED MHD EQUATIONS

The magnetic field in the sheet is given as: $\mathbf{B} = \mathbf{B}_\perp + B_0\mathbf{z}$

Divergence equation: $\nabla.\mathbf{B} = 0 \Rightarrow \partial_x B_x + \partial_y B_y = 0$

Then, $B_x = -\partial_y\psi$ and $B_y = \partial_x\psi$

or, $\mathbf{B}_\perp = \mathbf{z} \times \nabla\psi$; ψ is Flux function.

Induction Equation: $\dfrac{\partial\mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \dfrac{4\pi\eta}{c}\nabla \times \mathbf{j}$

Uncurling the induction equation: $\dfrac{\partial\mathbf{A}}{\partial t} = (\mathbf{V} \times \mathbf{B}) - \dfrac{4\pi\eta}{c}\mathbf{j} - \nabla\xi$

From Ampere's Law, we have: $j_z = \dfrac{c}{4\pi}\nabla^2\psi$ and from the definition of the flux function, we get: $\psi = -A_z$

$\nabla_\perp\xi = \mathbf{v}_\perp \times B_0\mathbf{z}$

Say $\phi = \dfrac{\xi}{B_z} \Rightarrow \mathbf{v}_\perp = \mathbf{z} \times \nabla\phi$; ϕ is Stream function.

Then, $\partial_z\xi = B_0\partial_z\phi$

The Induction Equation has reduced to:

$$\frac{\partial\psi}{\partial t} = \{\phi, \psi\} + \nabla^2\psi + B_0\frac{\partial\phi}{\partial z}$$

Using $\mathbf{v} = \mathbf{z} \times \nabla\phi \Rightarrow v_x = -\partial_y\phi, v_y = \partial_x\phi$

Then, $\nabla.\mathbf{v} = 0 \Rightarrow$ Continuity equation is satisfied for incompressible plasma.

Momentum Equation: $\rho\dfrac{d\mathbf{V}}{dt} = \rho(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}.\nabla\ \mathbf{V}) = \dfrac{\mathbf{j} \times \mathbf{B}}{c} - \nabla p + \rho\mathbf{g}$

Taking a curl: ($\nabla \times \mathbf{v} = \mathbf{w} =$ Vorticity)

$w_z = \nabla^2 \phi$

Taking only z-component of the equation, the momentum equation has reduced to:

$$\rho_0 \frac{\partial \nabla^2 \phi}{\partial t} + \rho_0 \{\phi, \nabla^2 \phi\} = B_0 \frac{\partial \nabla^2 \psi}{\partial z} + \{\psi, \nabla^2 \psi\}$$

where $\{P, Q\} = \nabla P \times \nabla Q$; P, Q are some scalars.

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