MATHEMATICS FOR COMPUTER SCIENCE ENGINEERS

Generation of Random Variates

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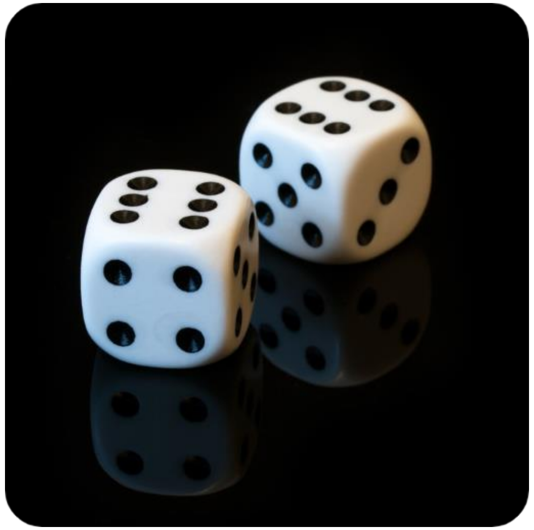
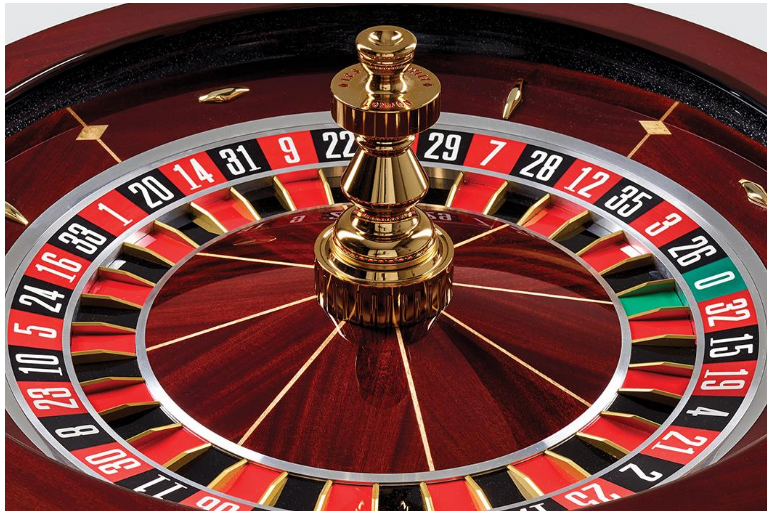
Topics to be covered

- ❖ Random Numbers
- ❖ Random Variate Generator
- ❖ Random Variates
- ❖ Techniques for generating Random Variates

Random Numbers

- It is a random sequence of numbers obtained from a stochastic process.
- A random number is a number chosen as if by chance from some specified distribution such that selection of a large set of these numbers reproduces the underlying distribution.
- A random number is chosen using methods which give equal probability to all numbers occurring in the specified distribution.
- In real world, random numbers may be generated using a dice or a roulette wheel.





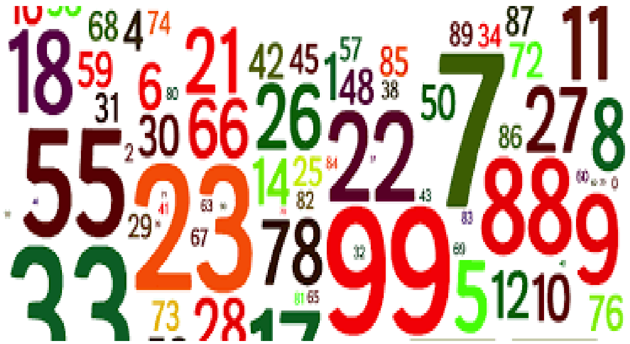
Source: tcsjohnhuxley.com, amazon.in

Random Numbers

- Random numbers are most commonly produced with the help of a random number generators.
- Random numbers have **important applications**, especially in **cryptography** where they act as ingredients in encryption keys.
- One of the most **important prerequisites of a random number** is to be **independent**, as this helps in establishing no correlations between successive numbers.
- It must be ensured that the frequency of the occurrence of these random numbers should be approximately be the same. As a result, theoretically, it is not easy to generate a long random number.

Random Numbers

- Random numbers are also very important for a simulations.
- All the randomness required by a simulation model is obtained by a random number generator.
- The output of a random number generator is assumed to be a sequence of independent and identically (uniformly) distributed random numbers between 0 and 1.
- These random numbers are transformed into required probability distributions.
- Example: The most common set from which random numbers are derived is the set of single-digit decimal numbers {**0, 1, 2, 3, 4, 5, 6, 7, 8, 9**}.



Source: medium.com

Generating Random Numbers

- **Problem:**
  Generate sample of a random variable X with a given density f. (The sample is called a random variate)
- **Answer:**
  Develop an algorithm such that if one used it repeatedly (and independently) to generate a sequence of samples X1, X2, . . . , Xn then as n becomes large, the proportion of samples that fall in any interval [a, b] is close to P(X ∈ [a, b]), i.e

$$\boxed{\{X_i \in [a, b]\}/n \approx P(X \in [a, b])}$$

Generating Random Numbers

- **Solution:** 2-step process
  - Generate a random variate uniformly distributed in [0, 1], also called a random number.
  - Use an appropriate transformation to convert the random number to a random variate of the correct distribution.
- **Why is this approach good ?**

  Answer: It focuses on generating samples from ONE distribution only.

Random Number Generators

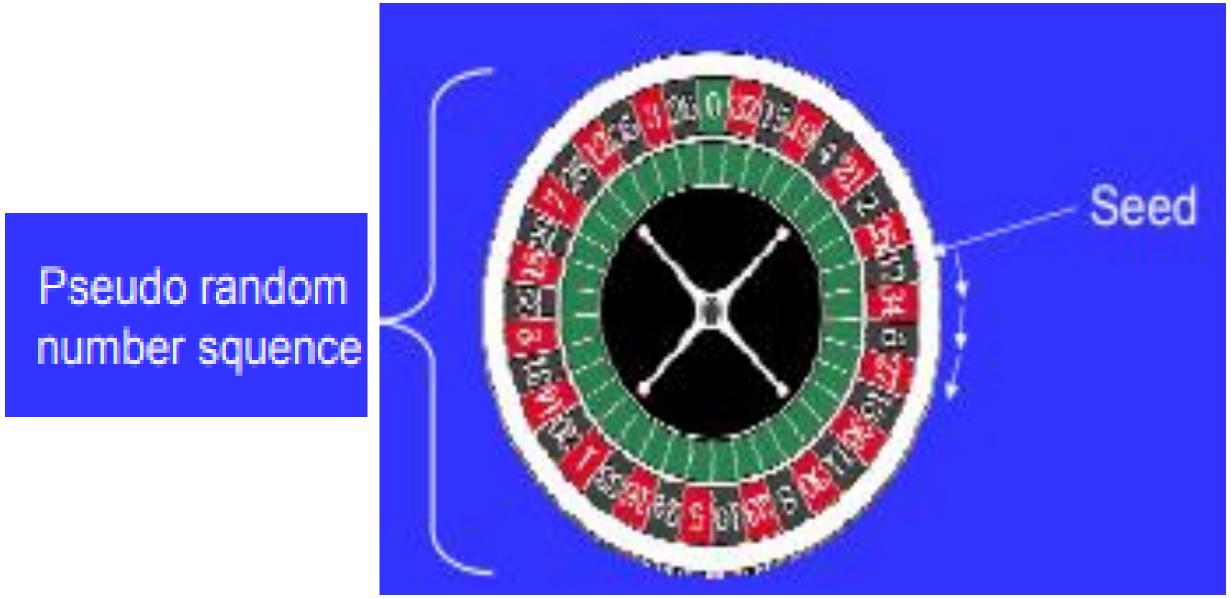
- A random number generator is **a hardware device or software algorithm** that generates a **number** that is taken from a **limited or unlimited distribution** and outputs it.
- The two main types of random number generators are pseudo random number generators and true random number generators.
- The numbers or sequence of numbers generated must lack any pattern (i.e. must appear random).
- **True random number generator:**
  - It measures some physical phenomenon that is expected to be random and then compensates for possible biases in the measurement process.
  - Example sources include measuring atmospheric noise, thermal noise, and other external electromagnetic and quantum phenomena.

Random Number Generators

- **Pseudo random number generator:**
  - It uses computational algorithms that can produce long sequences of apparently random results.
  - But these results are in fact completely determined by a **shorter initial value**, known as a **seed value or key**.
  - As a result, the entire seemingly random sequence can be reproduced if the seed value is known.
  - Properties that pseudo-random number generators should possess:
    1. It should be fast and not memory intensive
    2. It must be able to reproduce a given stream of random numbers.
    3. provision for producing several different independent streams of random numbers
- The random numbers generated must meet some statistical tests for randomness intended to ensure that they do not have any easily discernible patterns.

Random Number Seed

- A random seed (or seed state, or just seed) is **a number (or vector) used to initialize a pseudorandom number generator**.
- Computer-based generators use random number seeds for setting the starting point of the random number sequence.
- For a seed to be used in a pseudorandom number generator, it does not need to be random.
- These seeds are often initialized using a computer's real time clock in order to have some external noise.



Source: http://www.cs.bilkent.edu.tr/~cagatay/cs503/_M&S_04_Random_Variate_Generation.pdf

Random Variate

- A **random variate** is a variable generated from uniformly distributed pseudorandom numbers.
- It is a particular outcome of a random variable.
- The random variates which are other outcomes of the same random variable might have different values.
- Random variates are used when simulating processes are driven by random influences (stochastic processes).
- They are frequently used as the input to simulation models.
- Procedures to generate random variates corresponding to a given distribution are known as procedures for **random variate generation or pseudo-random number sampling.**
- Depending on how they are generated, a **random variate** can be **uniformly or non-uniformly distributed**.
- Examples: Inter-arrival time and service time.

Random Variate Generation

- Random variate generation is a fundamental aspect of simulation modeling and analysis.
- The objective of random variate generation is to produce observations that have the stochastic properties of a given random variable.
- Various methods and algorithms have been developed to generate random variates that are accurate (representative of the target distribution) and computationally efficient.
- The **distribution** from which random variates are generated is assumed to be completely specified.
- We wish to generate samples from this distribution as input to a simulation model.
- Random variate generation relies on generating uniformly distributed random number in the closed interval [0,1].
- Random variate generators use as starting point, random numbers distributed in U[0,1].

Random Variate Generation : Objectives

- The objective of random variate generation is to produce sample observations that have the stochastic properties of a given random variable, X, having distribution function

  $$F(x) = \Pr(X \le x)\ ,\quad \text{where } -\infty < x < \infty$$

- The development of the theory/concepts surrounding random variate generation via computer algorithms is based on the following two key assumptions :
  - **Assumption 1 :** There exists a perfect uniform (0,1), U(0,1), random number generator that can produce a sequence of independent random variables uniformly distributed on (0,1).
  - **Assumption 2 :** Computers can store and manipulate real numbers.
- Although Assumptions 1 and 2 are used for developing Random Variate Generation theory, the assumptions are violated when implementing Random Variate Generation algorithms on digital computers.

Factors to be considered for random variate generation

1. **Exactness:**

- Exactness or accuracy refers to how well the generator produces random variates with the characteristics of the desired distribution.
- This refers to the theoretical exactness of the random variate generator itself, as well as the error that is induced by the U(0,1) random number generator and the error induced by digital computer calculations.

2. **Speed:**

- Speed refers to the computational set-up and execution time required to generate random variates. Contributions to time are:
   a. Setup time
   b. Variable generation time

Factors to be considered for random variate generation

3. Space:

- Space refers to computer memory that is required for the generator.
- Although space is not typically a major consideration for modern computers, computer memory was an important consideration in the early days of Random Variate Generation development.

4. Simplicity:

- Simplicity refers to the both the simplicity of the algorithm as well as the simplicity of implementation.
- This includes the number of lines of code, support routines required, number of mathematical operations, as well as portability across platforms and interaction with other simulation methods such as variance reduction techniques.

The importance of each of these factors will vary depending on the particular situation or simulation application.

Random Variate Generation Techniques

- We assume that a pseudo random number generator RN(0,1) producing a sequence of independent values between 0 and 1 is available.
- General methods:
  - Inverse transform method
  - Acceptance-rejection method
  - Composite method
  - Translations and other simple transforms

Acceptance and Rejection Method: Poisson Distribution

Procedure of generating a Poisson random variate N is as follows:

1. Set $n=0$, $P=1$
2. Generate a random number R_{n+1}, and replace P by $P \times R_{n+1}$
3. If $P < \exp(-\alpha)$, then accept $N=n$.

   Otherwise, reject the current n, increase n by one, and return to step 2.

Acceptance and Rejection Method: Poisson Distribution example

- Example: Generate three Poisson variates with mean $\alpha=0.2$
  - $\exp(-0.2) = 0.8187$
- Variate 1
  - Step 1: Set $n=0, P=1$
  - Step 2: $R1 = 0.4357, P = 1 \text{ x } 0.4357$
  - Step 3: Since $P = 0.4357 < \exp(-0.2)$, accept $N = 0$
- Variate 2
  - Step 1: Set $n=0, P=1$
  - Step 2: $R1 = 0.4146, P = 1 \text{ x } 0.4146$
  - Step 3: Since $P = 0.4146 < \exp(-0.2)$, accept $N = 0$
- Variate 3
  - Step 1: Set $n=0, P=1$
  - Step 2: $R1 = 0.8353, P = 1 \text{ x } 0.8353$
  - Step 3: Since $P = 0.8353 > \exp(-0.2)$, reject $n = 0$ and return to Step 2 with $n = 1$
  - Step 2: $R2 = 0.9952, P = 0.8353 \text{ x } 0.9952 = 0.8313$
  - Step 3: Since $P = 0.8313 > \exp(-0.2)$, reject $n = 1$ and return to Step 2 with $n = 2$
  - Step 2: $R3 = 0.8004, P = 0.8313 \text{ x } 0.8004 = 0.6654$
  - Step 3: Since $P = 0.6654 < \exp(-0.2)$, accept $N = 2$

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

Acceptance and Rejection Method: Poisson Distribution example

- It took five random numbers to generate three Poisson variates
- In long run, the generation of Poisson variates requires some overhead!

N	R_{n+1}	P	Accept/Reject		Result
0	0.4357	0.4357	$P < \exp(-\alpha)$	Accept	N=0
0	0.4146	0.4146	$P < \exp(-\alpha)$	Accept	N=0
0	0.8353	0.8353	$P \geq \exp(-\alpha)$	Reject	
1	0.9952	0.8313	$P \geq \exp(-\alpha)$	Reject	
2	0.8004	0.6654	$P < \exp(-\alpha)$	Accept	N=2

Source:https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

References

http://cs.bilkent.edu.tr/~cagatay/cs503/_M&S_04_Random_Variate_Generation.pdf

https://www.mi.fu-berlin.de/inf/groups/ag-tech/teaching/2012_SS/L_19540_Modeling_and_Performance_Analysis_with_Simulation/07.pdf

[1] Kuhl, Michael E. "History of random variate generation." *2017 Winter Simulation Conference (WSC)*. IEEE, 2017.

Text Book:

Statistics for Engineers and Scientists, William Navidi.

THANK YOU

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