UNIT 1

SETS

Set

A collection of well defined objects.
Eg: A bouquet
Sets are denoted by capital alphabets A,B,...

Elements

The objects of a set.
Eg: Flowers in the bouquet
Elements are denoted by small alphabets a,b,.....

Notations

If a is an element of a set S then we write $a \in S$.
The symbol $\in$ means 'belongs to'.
If a is not an element of S then we write $a \notin S$
The symbol $\notin$ means 'does not belong to'.

Examples of sets

- The set S of Stars in Universe. Here $Sun \in S$ and $Earth \notin S$.
- The set W of all days in a week. Here $Monday \in W$ and $April \notin W$.

Representation of sets

There are two methods of representing a set.

1. Tabulation method
   In this method individual elements are listed, separated by comma and enclosed in braces.
   Eg: The set of all days in a week can be represented as
   $W = \{Sunday, Monday, Tueday, Wednesday, Thursday, Friday, Saturday\}$
2. Set builder method
   In this method instead of listing the elements a rule that the elements obey is given in the braces. Eg: The set $R = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is represented as
   $R = \{x : x \in \mathbb{N}, x < 10\}$ or $R = \{x | x \in \mathbb{N}, x < 10\}$

Types of sets

- *Singleton set*
  A set that contains only one element.
  Eg: $A = \{5\}$
- *Null set or Empty set*
  A set with no element.
  It is denoted by $\{\}$ or ϕ.
  Eg: The set of even prime greater than 5.

Note: The set $\{0\}$ is not a null set but is a singleton set.

- *Universal set*
  A set containing all the elements of a problem under consideration.
  It is denoted by U. $U = \mathbb{N}$ is the universal set for the set $R = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Disjoint sets
Two sets A and B are said to be disjoint if they have no elements in common.
Eg:$A = \{1, 2, 3\}$ and $B = \{4, 6, 9\}$ are disjoint sets

Subsets
A set B is called a subset of another set A if every element of B is in A.
Eg: $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is a subset of $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

Note:

- A Set is a subset of itself.
- Null set is subset of all sets.

Notation
If A is a subset of B then we write $A \subset B$.
The symbol $\subset$ means 'subset of'.
If A is not a subset of B then we write $A \not\subset B$
The symbol $\not\subset$ means 'not a subset of'.

Number of Subsets of a Set
Consider $A = \{x, y\}$. Then the sub sets of A are $\{x\}$, $\{y\}$, $\{x, y\}$, $\{\}$
Number of subsets = 4
Similarly, consider the set $B = \{1, 2.3\}$.
The subsets are $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \phi$

Number of subset = 8

If a set contains n elements, there are 2^n subsets.
Consider the set $Q = \{2, 4, 6, 8\}$.
The number of elements in Q = 4
The number of subsets of Q = $2^4 = 2 \times 2 \times 2 \times 2 = 16$

Power set
The set of all subsets of a given set A. It is denoted by $\wp(A)$
If $A = \{x, y\}$, then power set, $\wp(A) = \{\{x\}, \{y\}, \{x, y\}, \phi\}$

Equal sets
Two sets A And B are said to be equal if every element of A is in B and every element of B is in A.
It is denoted as $A = B$.
The sets A and B are equal if $A \subset B$ and $B \subset A$.
Eg: $A = \{2, 4, 6\}$ and B =Even numbers less than 7

Equivalent sets
Two sets are said to be equivalent if they have same number of elements.
Equal sets are equivalent but not vice-versa.
Eg: $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 8\}$ are equivalent sets as they both have 4 elements but they are not equal.

Set Operations
Given two sets A and B we define operations

1. Union of sets
   The union of sets A and B is the set of all elements which are either in A or in B or in both. The union of sets A and B are denoted by $A \cup B$.
   $A \cup B = \{x | x \in A \text{ or } x \in B\}$
   Eg:
   $*$ Consider $A = \{1, 2, 5, 6, 7\}$ and $B = \{2, 4, 8, 9, 10\}$,
   then $A \cup B = \{1, 2, 4, 5, 6, 7, 8, 9, 10\}$
   $*$ Consider $A = \{2, 4, 7, 11, 15\}$ and $B = \{2, 7, 11\}$,
   the $A \cup B = \{2, 4, 7, 11, 15\}$

   Note: If $B \subset A$ then $A \cup B = A$.

2. Intersection of sets
   The intersection of sets A and B is the set of all elements which are in both A and B. The intersection of sets A and B is denoted by $A \cap B$.
   $A \cap B = \{x | x \in A \text{ and } x \in B\}$

Eg:
* Consider $A = \{1, 2, 5, 6, 7\}$ and $B = \{2, 4, 8, 9, 10\}$,
then $A \cap B = \{2\}$
* Consider $A = \{2, 3, 5, 6\}$ and $B = \{1, 4, 7, 8\}$,
then $A \cap B = \phi$

Note: If A and B are disjoint then $A \cap B = \phi$.

3. Difference of sets
The difference of sets A and B is defined to be the set which contains all those elements in A which are not in B.The difference of sets A and B is denoted by $A - B$.
$A - B = \{x : x \in A, x \notin B\}$
$B - A = \{x : x \in B, x \notin A\}$
Eg:Consider $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5, 7, 9, 10\}$,
then $A - B = \{1, 2, 4\}$ and $B - A = \{7, 9, 10\}$

Note: $A - B \neq B - A$.

Complement of a set
The complement of a set A is defined to be the set of all elements which are in U and not in A.The complement of A is denoted by A^c, $\overline{A}$ or $A^{'}$.
$A^{'} = \{x : x \in U, x \notin A\}$
Eg: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
* If $A = \{1, 2, 4, 6, 8, 10\}$, then $A^{'} = \{3, 5, 7, 9\}$
* If $B = \{1, 4, 6, 8\}$ then $B^{'} = \{2, 3, 5, 7, 9, 10\}$

Excercise

1. Write down the following sets in tabulation method:
   (i) The set A of all odd natural numbers less than 10.
   (ii) The set B of all square numbers less than 100.
   (iii) The set of all prime numbers between 10 and 20.
2. Write down the following sets in set builder method:
   (i) $C = \{3, 6, 9, ...\}$
   (ii) $D = \{1, 3, 5, ...\}$
   (iii) $P = \{5, 10, 15, ...\}$
3. Write down all the subsets of $A = \{a, b, c, d, e\}$.

Answers

1.(i) $A = \{1, 3, 5, 7, 9\}$

(ii) $B = \{1, 4, 9, 16, 25, 36, 49, 64, 81\}$

(iii) $P = \{11, 13, , 17, 19\}$

2.(i) $C = \{x : x \in \mathbb{N}, x \text{ is a multiple of } 3\}$

(ii) $D = \{y : y \in \mathbb{N}, y \text{ is odd}\}$

(iii) $P = \{x : x \in \mathbb{N}, x \text{ is a multiple of } 5\}$

3. The subsets of $A = \{a, b, c, d, e\}$ are
$\{a\}$, $\{b\}$,$\{c\}$,$\{d\}$,$\{e\}$,
$\{a, b\}$,$\{a, c\}$,$\{a, d\}$,$\{a, e\}$,$\{b, c\}$,$\{b, d\}$,$\{b, e\}$,$\{c, d\}$,$\{c, e\}$,$\{d, e\}$,
$\{a, b, c\}$,$\{a, b, d\}$,$\{a, b, e\}$,$\{a, c, d\}$,$\{a, c, e\}$,$\{a, d, e\}$,$\{b, c, d\}$,$\{b, c, e\}$,$\{b, d, e\}$,$\{c, d, e\}$,
$\{a, b, c, d\}$,$\{a, b, c, e\}$,$\{a, c, d, e\}$,$\{b, c, d, e\}$,$\{a, b, d, e\}$,$\{a, b, c, d, e\}$,$\{\}$.

Excercise

1. If $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{2, 3, 4, 6\}$, the find
   (i) $A \cup B$
   (ii) $A \cap B$
   (iii) $A - B$
   (iv) $A \cup (B \cap C)$
   (v) $A - (B \cap C)$

2. Given $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 7\}$, $C = \{2, 5, 6\}$, find
   (i) $A \cup B$
   (ii) $B \cap A$
   (iii) $C - B$
   (iv) $C' \cap A$

3. Given that $A = \{0, 1, 3, 5\}$, $B = \{1, 2, 4, 7\}$, $C = \{1, 2, 3, 5, 8\}$, prove that
   (i) $(A \cap B) \cap C = A \cap (B \cap C)$

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$

(iii) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(iv) $(A \cap C) \cup B = (A \cup B) \cap (C \cup B)$

Answers

1.(i) $A \cup B = \{1, 2, 3, 5\}$

(ii) $A \cap B = \{1, 3\}$

(iii) $A - B = \{2\}$

(iv) $B \cap C = \{3\}$, $A = \{1, 2, 3\}$
$A \cup (B \cap C) = \{1, 2, 3\}$

(v) $B \cap C = \{3\}$, $A = \{1, 2, 3\}$
$A - (B \cap C) = \{1, 2\}$

2.(i) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

(ii) $B \cap A = \{1, 3, 5\}$

(iii) $C - B = \{2, 6\}$

(iv) $C' = \{1, 3, 4\}$, $A = \{1, 2, 3, 4, 5\}$
$C' \cap A = \{1, 3, 4\}$

3.(i) <u>LHS</u>
$A \cap B = \{1\}$,
$(A \cap B) \cap C = \{1\}$
<u>RHS</u>
$B \cap C = \{1\}$,
$A \cap (B \cap C) = \{1\}$
$\therefore LHS = RHS$

(ii) <u>LHS</u>
$A \cup B = \{0, 1, 2, 3, 4, 5, 7\}$,
$(A \cup B) \cup C = \{0, 1, 2, 3, 4, 5, 7, 8\}$
<u>RHS</u> $B \cup C = \{1, 2, 3, 4, 5, 7, 8\}$,
$A \cup (B \cup C) = \{0, 1, 2, 3, 4, 5, 7, 8\}$
$\therefore LHS = RHS$

(iii) <u>LHS</u>
$A \cup B = \{0, 1, 2, 3, 4, 5, 7\}$.
$(A \cup B) \cap C = \{1, 2, 3, 5\}$
<u>RHS</u>
$A \cap C = \{1, 3, 5\}$,
$B \cap C = \{1, 2\}$,
$(A \cap C) \cup (B \cap C) = \{1, 2, 3, 5\}$
$\therefore LHS = RHS$

(iv) LHS

$A \cap C = \{1, 3, 5\}$,

$(A \cap C) \cup B = \{1, 2, 3, 4, 5, 7\}$

RHS$A \cup B = \{0, 1, 2, 3, 4, 5, 7\}$,

$C \cup B = \{1, 2, 3, 4, 5, 7, 8\}$,

$(A \cup B) \cap (C \cup B) = \{1, 2, 3, 4, 5, 7\}$

$\therefore LHS = RHS$

Venn Diagram

Venn Diagram is a method of representing the relationship between sets.
In this method,a rectangle is drawn to represent the Universal set,U and circles are drawn to represent the sets.

Eg:

$*$ Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3, 4\}$, then the Venn diagram representing the set is,

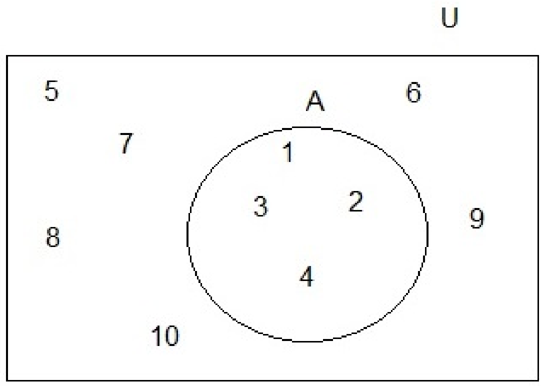

Figure 1: A

$*$ Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 3, 4, 6\}$ and $B = \{2, 3, 8\}$, then the Venn diagram representing the sets is,

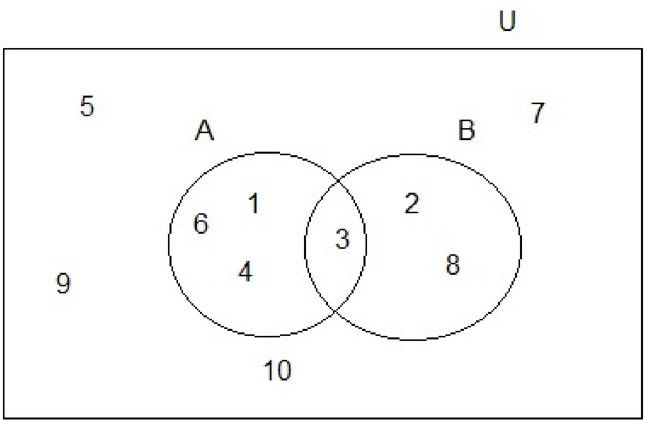

Figure 2: A and B

The Venn diagram for union, intersection, difference and complement of sets are given below.

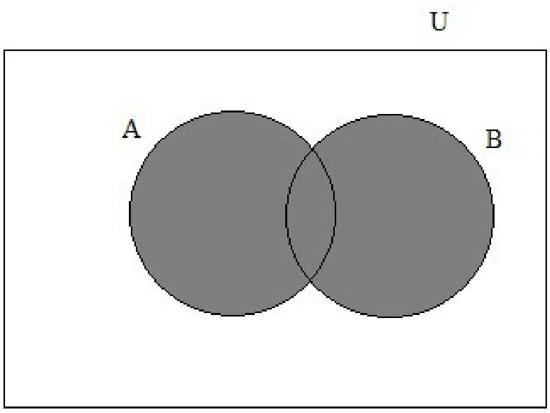

Figure 3: Union $A \cup B$

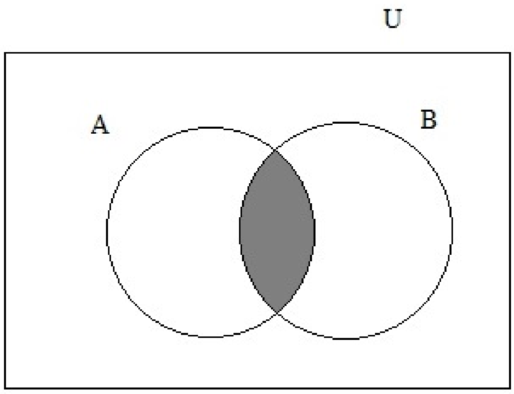

Figure 4: Intersection $A \cap B$

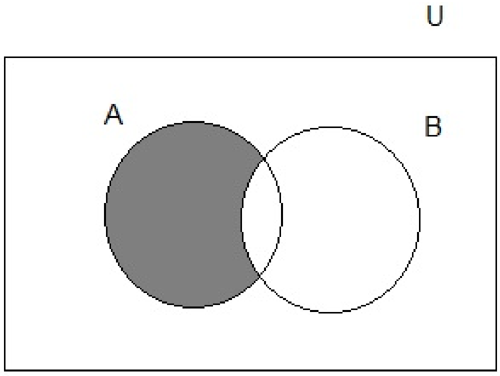

Figure 5: Difference $A - B$

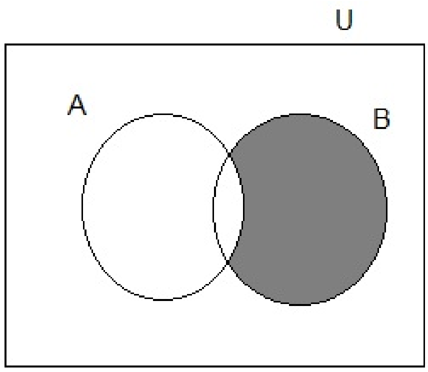

Figure 6: Difference $B - A$

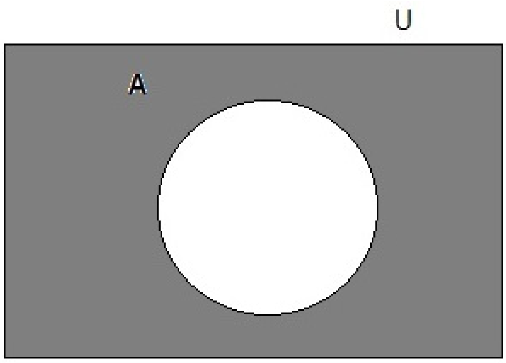

Figure 7: Complement A'

Laws of Sets

1. Commutative Laws

$$\begin{aligned} A \cup B &= B \cup A \\ A \cap B &= B \cap A \\ A - B &\neq B - A \end{aligned}$$

2. Associative Laws

$$\begin{aligned} A \cup (B \cup C) &= (A \cup B) \cup C \\ A \cap (B \cap C) &= (A \cap B) \cap C \end{aligned}$$

3. Distributive Laws

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned}$$

4. De Morgan's Laws

$$\begin{aligned} (A \cup B)^{'} &= A^{'} \cap B^{'} \\ (A \cap B)^{'} &= A^{'} \cup B^{'} \end{aligned}$$

Some important properties

⋄ If $A \subset B$,

$$\begin{aligned} A \cup B &= B \\ A \cap B &= A \\ A - B &= \phi \end{aligned}$$

⋄ If U is the universal set, then

$$\begin{aligned} U^{'} &= \phi \\ \phi^{'} &= U \end{aligned}$$

⋄ For any set A,

$$(A^{'})^{'} = A$$

$\diamond$ For any set A,

$$\begin{aligned} A \cup \phi &= A \\ A \cap \phi &= \phi \\ A \cup U &= U \\ A \cap U &= A \end{aligned}$$

$\diamond$ If $A \subset B$, then $B^{'} \subset A^{'}$

Examples

$\diamond$ Given $A = \{2, 3, 4, 6\} \subset B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4, 5, 6, 7, 8\} = B \\ A \cap B &= \{2, 3, 4, 6\} = A \\ A - B &= \phi \end{aligned}$$

$\diamond$ Given $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{2, 4, 6, 8, 10\}$, then

$$\begin{aligned} A^{'} &= \{1, 3, 5, 7, 9\} \\ (A^{'})^{'} &= \{1, 3, 5, 7, 9\}^{'} = \{2, 4, 6, 8, 10\} = A \end{aligned}$$

$\diamond$ Given $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 3, 4, 7\} \subset B = \{1, 2, 3, 4, 6, 7\}$, then

$$\begin{aligned} A^{'} &= \{1, 5, 6, 8, 9, 10\} \\ B^{'} &= \{5, 8, 9, 10\} \end{aligned}$$

Thus $B^{'} \subset A^{'}$

Number of elements in a set

If $n(A)$ denotes the number of elements in the set A, $n(B)$ denotes the number of elements in the set B, then

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) \end{aligned}$$

If A and B are disjoint sets, $A \cap B = \phi$,
$\therefore n(A \cap B) = 0$
So, $n(A \cup B) = n(A) + n(B)$

Ordered pair

Two elements a and b, listed in a specific order, form an ordered pair, denoted by (a, b).

Eg: $(9,0)$,(e,k),$(0,0)$
* The ordered pair (a,b) is different from the ordered pair (b,a).
* Two ordered pair (a,b) and (c,d) are equal if and only if $a=c$ and $b=d$.

Cartesian Product
The Cartesian product of two sets A and B is defined to be the set of all ordered pair with first element in A and the second element in B. It is denoted as $A\times B$.
$A\times B=\{(a,b):a\in A,b\in B\}$
$B\times A=\{(b,a):a\in A,b\in B\}$.

Note:$A\times B\neq B\times A$

Eg: Let $A=\{a,b,c\}$ and $B=\{1,2\}$, then
$A\times B=\{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$
$B\times A=\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$

Notes:

- $A\times B\neq B\times A$
- If $A=\phi$ and $B=\phi$ then $A\times B=\phi$
- $n(A\times B)=n(B\times A)$

Properties

1. $A\times(B\cup C)=(A\times B)\cup(A\times C)$
2. $A\times(B\cap C)=(A\times B)\cap(A\times C)$

Examples
Let $A=\{1,3\}$, $B=\{1,3,5\}$ and $C=\{3,4\}$.

1.

$$\begin{aligned}
B\cup C &= \{1,3,4,5\}\\
A\times(B\cup C) &= \{(1,1),(1,3),(1,4),(1,5),(3,1),(3,3),(3,4),(3,5)\}\\
A\times B &= \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5)\}\\
A\times C &= \{(1,3),(1,4),(3,3),(3,4)\}\\
(A\times B)\cup(A\times C) &= \{(1,1),(1,3),(1,4),(1,5),(3,1),(3,3),(3,4),(3,5)\}
\end{aligned}$$

2.

$$\begin{aligned}
B \cap C &= \{3\} \\
A \times (B \cap C) &= \{(1,3),(3,3)\} \\
A \times B &= \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5)\} \\
A \times C &= \{(1,3),(1,4),(3,3),(3,4)\} \\
(A \times B) \cap (A \times C) &= \{(1,3),(3,3)\}
\end{aligned}$$

Excercise

1. If $A = \{1,2,3\}$ and $B = \{3,4,5\}$, find $A \times B$ and $B \times A$
2. If $A = \{1,2,3\}$, $B = \{2,3,4\}$, $S = \{1,3,4\}$ and $T = \{2,4,5\}$, verify that $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$

Solutions

1. $A \times B = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5)\}$
   $B \times A = \{(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$

2.

$$\begin{aligned}
\underline{LHS} A \times B &= \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\} \\
S \times T &= \{(1,2)(1,4),(1,5),(3,2),(3,4),(3,5),(4,2),(4,4),(4,5)\} \\
(A \times B) \cap (S \times T) &= \{(1,2),(1,4),(3,2),(3,4)\} \\
\underline{RHS} & \\
A \cap S &= \{1,3\} \\
B \cap T &= \{2,4\} \\
(A \cap S) \times (B \cap T) &= \{(1,2),(1,4),(3,2),(3,4)\}
\end{aligned}$$

$LHS = RHS$, verified.

bfRelations
A relation from a set A to a set B is a subset of the Cartesian product $A \times B$. Hence, a relation R consists of ordered pairs (a, b), where $a \in A$and $b \in B$.

Notation
If $(a, b) \in R$, we say that a is related to b , and we write $a \; R \; b$.
If $(a, b) \notin R$, we say that a is not related to b , and we write $a \not R \; b$.

Eg: Let $A = \{2,3,4\}$ and $B = \{4,6,8,10\}$, then a few examples of relations from A to B are:

$* \ R = \{(x, y) : x \in A, y \in B, y = x + 6\} = \{(2, 8), (4, 10)\}$
$* \ R = \{(x, y) : x \in A, y \in B, y \geq x^2\} = \{(2, 4), (2, 6), (2, 8), (2, 10), (3, 10)\}$

Types of relations

1. *Empty relation* - If no element of set A is related to any element of A,then the relation is known as empty relation. Hence $R = \phi$ which is a subset of $A \times A$.

   Eg:The relation R on the set $A = \{1, 2, 3, 4\}$ defined by $R = \{(a, b) : a + b = 10\}$. But $a + b \neq 10$ for any two elements of set A . Therefore, $(a, b) \in R$ for any $a, b \in A$.

2. *Reflexive relation*- A relation R in a set A is said to be Reflexive if and only if $a \ R \ a$, for all a A. It means, every element of A is related to itself.

   Eg:Let $A = \{3, 4, 5\}$ and $R = \{(a, a) : a \in A\}$.Hence, $R = \{(3, 3), (4, 4), (5, 5)\}$ is a reflexive relation.

3. *Symmetric relation* - A relation R in a set A is said to be Symmetric if and only if $a \ R \ b \Rightarrow b \ R \ a$ or $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

   Eg:

4. *Transitive relation* - A relation in a set A is called Transitive if and only if, $a \ R \ b$ and $b \ R \ c \Rightarrow a \ R \ c$ or $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

   Eg:

5. *Equivalence relation* - A relation R in a set A is called an Equivalence relation if and only if

   - it is Reflexive that is, $(a, a) \in R$ for all $a \in A$.
   - it is Symmetric that is, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.
   - it is Transitive that is, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

   Eg: Let $A = \{1, 2, 3, 4\}$ then
   $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$
   is not an equivalence relation since the pairs $(4, 1)$ and $(4, 2)$ are not in R.

Let $A = \{3, 4, 5\}$ then $R = \{(3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$ is an equivalence relation.

6. *Identity relation* - Let A be any set, the relation $R = \{(a, a) : a \in A\}$ is called the identity element.
   Every identity relation is a reflexive relation but the reverse need not be true. Also every identity relation is an equivalence relation. It is denoted as I_A.

   Eg: $A = \{1, 2, 3\}$ then $I_A = \{(1, 1), (2, 2), (3, 3)\}$

7. *Inverse relation*- An inverse relation is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original relation.
   $R^{-1} = \{(b, a) : (a, b) \in R\}$

   Eg: Let $A = \{a, b, c\}$ and $R = \{(a, b), (a, c), (b, c)\}$, then
   $R^{-1} = \{(b, a), (c, a), (c, b)\}$

Excercise

1. Let L be the set of all lines in X-Y plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1 \parallel L_2\}$. Show that R is an equivalence relation.

2. Let $W = \{1, 2, 3, 4\}$. Consider the following relations in W.

$$\begin{aligned} R_1 &= \{(1, 1), (4, 3), (2, 2), (2, 1), (3, 1)\} \\ R_2 &= \{(2, 2), (2, 3), (3, 2)\} \\ R_3 &= \{(1, 3)\} \end{aligned}$$

   Determine whether each relation is

   (i) symmetric
   (ii) transitive

Solutions

1. Every line is parallel to itself.$\Rightarrow L_1 \; R \; L_1$, $\forall L_1 \in L$.
   $\Rightarrow$ R is reflexive.
   Let $L_1 \parallel L_2$
   $\Rightarrow (L_1, L_2) \in R$,
   then $L_2 \parallel L_1$,

$\Rightarrow (L_2, L_1) \in R$
$\Rightarrow$ R is symmetric.
Let $L_1 \parallel L_2$ and $L_2 \parallel L_3$
$\Rightarrow (L_1, L_2), (L_2, L_3) \in R$
then $L_1 \parallel L_3$,
$\Rightarrow (L_1, L_3) \in R$
$\Rightarrow$ R is transitive.
Since the relation R is reflexive, symmetric and transitive,it is an equivalence relation.

2. (i) A relation is not symmetric if there exist an ordered pair $(a, b) \in R$ such that $(b, a) \notin R$.
R_1 is not symmetric since $(4, 3) \in R_1$ but $(3, 4) \notin R_1$.
R_2 is symmetric as both $(2, 3)$ and $(3, 2) \in R_2$.
R_3 is not symmetric as $(1, 3) \in R$ but $(3, 1) \notin R$.

   (ii) A relation is not transitive if there exist elements a,b and c,such that $(a, b) \in R$ and$(b, c) \in R$ but $(a, c) \notin R$.
R_1 is not transitive since $(4, 3) \in R_1$ and $(3, 1) \in R_1$ but $(4, 1) \notin R_1$.
R_2 is not transitive since $(3, 2) \in R_2$ and $(2, 3) \in R_2$ but $(3, 3) \notin R$.
R_3 is transitive as all set with single element are transitive. **Functions**
A function f (or mapping) is a rule of correspondence between two sets A and B such that each element of the set A (x) is assigned to only one element of the set B $(f(x))$ and we write as $f : A \to B$.

Note:
⋄ Every element of A is assigned to an element of B and no element of A is left without an assignment in B.
⋄ No element of A can be assigned to more than one element of B.

We can use arrow diagram to represent a function.
Eg:
It is a function.
Domain and Range of a function
Let $f : A \to B$, then A is called the domain of f, and B is called the co-domain of f.
The set$f(A) = \{f(x) : x \in A\}$ is called the range of f.
Eg: Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16, 25\}$.Also let f be a function defined by $f(x) = x^2$.
Here domain of f is $\{1, 2, 3, 4\}$ range of f is $\{1, 4, 9, 16\}$ and co-domain of f is $\{1, 4, 9, 16, 25\}$.

Types of Function

(a) *One-one function* - Let $f : A \to B$. If distinct elements in A have distinct images in B, then f is said to be a one-one or an injective function or mapping.
Let $f : A \to B$ and let $x_1, x_2 \in A$. If f is one to one then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
Eg: Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$ and $f : A \to B$ is defined as $f(x) = 2x$. Then $f(1) = 2, f(2) = 4, f(3) = 6$.
f is a function from A to B such that different elements in A have different images in B. Hence f is one-one.

Or

$$\begin{aligned} f(x) &= 2x \\ \text{if} \quad f(x_1) &= f(x_2) \\ \text{then} \quad 2x_1 &= 2x_2 \\ \Rightarrow \quad x_1 &= x_2 \end{aligned}$$

Since $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, f is one-one function.

(b) *Onto function* - Let $f : A \to B$. If every element in B has at least one pre-image in A, then f is said to be an onto or a surjective function.
f is onto if and only if its range is equal to the co-domain.
If $f : X \to Y$,then f is said to be surjective if
$\forall\ y \in Y, \exists\ x \in X, f(x) = y$
Eg: Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$.
$y = 2x + 1 \Rightarrow x = \dfrac{y-1}{2}$.i.e. for any real number y we will get a real number x.
Let $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = 2x$ is not onto.
$y = 2x \Rightarrow x = \dfrac{y}{2}$. If $y = 1$, $x = \dfrac{1}{2} = 0.5$, which is not possible because $0.5 \notin \mathbb{N}$.

(c) *Bijective function* - Let $f : A \to B$. f is said to be bijective or bijection, if a it satisfies both the injective (one-one function) and surjective (onto function) properties.
Eg: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x - 5$.

Suppose

$$\begin{aligned} f(x_1) &= f(x_2) \\ 3x_1 - 5 &= 3x_2 - 5 \\ 3x_1 &= 3x_2 \\ \Rightarrow x_1 &= x_2 \end{aligned}$$

$\therefore f$ is one-one.
$y = 3x - 5 \Rightarrow x = \dfrac{y+5}{3}$. Since this is a real number, f is onto.
Thus

(d) *Many to one function* - A function f from A to B is called many to one if more than one element of A is mapped into the same element in B.
Eg: Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. When $x = 2$,
$y = (2)^2 = 4$
and when $x = -2$,
$y = (-2)^2 = 4$.
i.e., the elements 2 and -2 in A is mapped to the element 4 in B.

(e) *Constant function* - A function f from A to B is called a constant function, if every element of A has the same image in B.
Eg: Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(x) = 5$, then for any natural value of x, $f(x) = 5$.

(f) *Identity function* - The function f is called the identity function if each element of set A has an image on itself i.e. $f(x) = x$, $\forall$ $x \in A$. It is denoted by I.

Excercise

(a) If $X = \{x : 1 < x < 5, x \text{ is a prime number}\}$, find the domain, co-domain and range of the function $f : X \to \mathbb{R}$ given by $f(x) = \dfrac{x-1}{x+1}$.

(b) Is $f = \{(2,1),(1,2),(2,3)\}$ is a function from A to A where $A = \{1,2,3\}$?

(c) What type of functions are represented by the sets:

(i) $f_1 = \{(-2,-4),(-1,0),(0,-1),(1,0)\}$

(ii) $f_2 = \{(-2,-1),(-1,-1),(0,-1),(1,-1)\}$

(iii) $f_3 = \{(-2,-2),(-1,-1),(0,0),(1,1)\}$

(iv) $f_4 = \{(-2,1),(-1,1),(0,-2),(1,4)\}$

Given the domain $A = \{-2,-1,0,1\}$ and co-domain $B = \{-2,-1,0,1,4\}$.

(d) If $A = \{4,6,8,12\}$, $B = \{3,4,5,6,7\}$ and $f : A \to B$ is defined as $f(x) = \frac{1}{2}(x+2)$, what type of function is f?

(e) If $A = \{3,6,9,12\}$, $B = \{1,2,3,4,5,6\}$ and $f : A \to B$ is defined by $f(x) = \dfrac{1}{3}x + 1$. Express f as a set of ordered pairs and an arrow diagram.

Solutions

1. $X = \{2,3\}$ and $f(x) = \dfrac{x-1}{x+1}$

$$\begin{aligned} f(2) &= \frac{2-1}{2+1} = \frac{1}{3} \\ f(3) &= \frac{3-1}{3+1} = \frac{2}{4} = \frac{1}{2} \end{aligned}$$

Domain of $f = X = \{2,3\}$
Co-domain of $f = \mathbb{R}$
Range of $f = \left\{\dfrac{1}{3}, \dfrac{1}{2}\right\}$

2. f is not a function since the element 3 in A is not mapped to any element and 2 is mapped to both 1 and 3.

3. (i) f_1 is a many-one function since -1 and 1 in A are mapped to 0 in B.

(ii) f_2 is a constant function as all the elements in A are mapped to -1 in B.

(iii) f_3 is an one-one function as every element in A has distinct images in B.

(iv) f_4 is a many-one function as -2 and -1 in A are mapped to 1 in B.

4. The given function is$f(x) = \frac{1}{2}(x+2)$,then,

$$\begin{aligned} f(4) &= \frac{1}{2}(4+2) = \frac{1}{2}(6) = 3 \\ f(6) &= \frac{1}{2}(6+2) = \frac{1}{2}(8) = 4 \\ f(8) &= \frac{1}{2}(8+2) = \frac{1}{2}(10) = 5 \\ f(12) &= \frac{1}{2}(12+2) = \frac{1}{2}(14) = 7 \end{aligned}$$

$\therefore f = \{(4,3),(6,4),(8,5),(12,7)\}$.
f is a one-one function as all elements in A is mapped to distinct elements in B.

5. The given function is $f(x) = \frac{1}{3}x + 1$,then

$$\begin{aligned} f(3) &= \frac{1}{3}(3) + 1 = 1 + 1 = 2 \\ f(6) &= \frac{1}{3}(6) + 1 = 2 + 1 = 3 \\ f(9) &= \frac{1}{3}(9) + 1 = 3 + 1 = 4 \\ f(12) &= \frac{1}{3}(12) + 1 = 4 + 1 = 5 \end{aligned}$$

$\therefore f = \{(3,2),(6,3),(9,4),(12,5)\}$

Arrow diagram of f

Composition of functions
Consider $f : A \to B$ and $g : B \to C$.
Let $a \in A$.Then its image $f(a)$ is in B and B is the domain of g. Then the

image f $f(a)$ under the function g is $g[f(a)]$.
The function from $A \to C$ which assigns to each element of A, the element $g[f(a)]$ in C is called the composition of f and g and is denoted as $g \circ f$.
$\therefore (g \circ f)a = g[f(a)]$

Note: Composition of functions is not commutative. But it is associative.

Eg: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x + 5$ and $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = 3x - 1$. Then,

$$\begin{aligned}
(g \circ f)x &= g[f(x)] = g(x^2) = x^2 + 5 \\
(f \circ g)x &= f[g(x)] = f(x+5) = (x+5)^2 \\
g \circ f &\neq f \circ g
\end{aligned}$$

$$\begin{aligned}
(g \circ h)x &= g[h(x)] = g(3x-1) = 3x - 1 + 5 = 3x + 4 \\
f \circ (g \circ h)x &= f[(g \circ h)x] = f(3x+4) = (3x+4)^2 \\
(f \circ g)x &= f[g(x)] = f(x+5) = (x+5)^2 \\
[(f \circ g) \circ h]\, x &= (f \circ g)[h(x)] = (f \circ g)(3x-1) = (3x-1+5)^2 = (3x+4)^2 \\
f \circ (g \circ h) &= (f \circ g) \circ h
\end{aligned}$$