UNIT - I

1 Probability and Random Variables

1.1 : Introduction

Q.1 What is random experiment ?

Ans. : An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a random experiment.

Q.2 What is Classical Definition of Probability ?

Ans. : If an experiment has n equally likely simple events and if the number of ways that an event E can occur is m, then the probability of E, P(E), is

$$P(E) = \frac{\text{Number of way that E can occur}}{\text{Number of possible out comes}}$$

$$= \frac{M}{N}$$

Q.3 If we randomly pick two television sets in succession from a shipment of 240 television sets of which 15 are defective, what is the probability that they will be both defective ?

Ans. : • Let A denote the event that the first television picked was defective and B denote the event that the second television picked was defective. Then A B will denote the event that both televisions picked were defective. Using the conditional probability, we can calculate

$$P(A \cap B) = P(A)\,P(B/A)$$

$$= (15/240)\,(14/239)$$

$$= 210 / 57360$$

$$= 7/1912$$

Q.4 A coin is tossed until a head appears. Expect the number of tosses required ?

[JNTU : R15 : March-17, Marks 5]

Ans. : Let X denote the number of tails tossed before the head appears. Then X is a geometric random variable.

Let P(head) = p, where $0 < p \leq 1$, because it is not given whether the coin is balanced. The expectation of this geometric random variable X is

$$E(X) = \frac{1}{(1/2)} = 2$$

Q.5 Write a note on Axioms of Probability.

Ans. : • The subject of probability is based on three commonsense rules, known as **axioms**.

- One way of defining the probability of an event is in terms of its relative frequency. For an experiment, sample space S is repeatedly performed under exactly the same conditions. For each event E of the sample space S, we define n(E) to be the number of times in the first n repetitions of the experiment that the event E occurs. Then P(E), the probability of the event E is denoted by,

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

- The theory of probability starts with the assumption that probabilities can be assigned so as to satisfy the following three basic axioms of probability.
- Suppose we have a sample space S. If S is discrete, all subsets correspond to events and conversely; if S is non-discrete, only special subsets correspond to events.
- To each event A in the class C of events, we associate a real number P(A). The P is called a *probability function*, and P(A) the *probability* of the event, if the following axioms are satisfied.

Axiom 1 : For every event A in class C, $P(A) \geq 0$.

Axiom 2 : For the certain event S in the class C, **P(S) = 1**

Axiom 3 : For any number of mutually exclusive events $A_1, A_2, A_3, \ldots$ in the class C,

$P(A_1 \cup A_2 \cup A_3 \cup ....) = P(A_1)+P(A_2)+P(A_3)$

Particularly for two mutually exclusive events A_1 and A_2,

$$P(A_1 \cup A_2) = P(A_1)+P(A_2)$$

- The first axiom says that the outcome of an experiment is always in the sample space. The second axiom says that the long-run frequency of any event is always between 0 and 100 %. The third axiom says that the probability is either too short or too long.

Q.6 Explain properties of probability.

Ans. : Properties of probability are as follows:

1. $P(\overline{A}) = 1 - P(A)$, which states that the probability of the complement of A is one minus the probability of A.
2. P( ) = 0, which states that the impossible events has probability zero.
3. If A B, then P(A) P(B). that is, if A is a subset of B, the probability of A is at most the probability of B.
4. P(A) 1, which means that the probability of event A is at most 1.
5. For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Q.7 A box of fuses contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random and removed from the box in succession without replacement, what is the probability that all three fuses are defective ?

Ans. : Let A be the event that the first fuse selected is defective. Let B be the event that the second fuse selected is defective. Let C be the event that the third fuse selected is defective. The probability that all three fuses selected are defective is $P(A \cap B \cap C)$. Hence

$$P(A \cap B \cap C) = P(A)P(B/A)P(C/A \cap B)$$

$$= \left[\frac{5}{20}\right]\left[\frac{4}{19}\right]\left[\frac{3}{18}\right]$$

$$P(A \cap B \cap C) = \frac{1}{14}$$

1.2 : Sample Space, Events, Counting Sample Points

Q.8 Define sample space.

Ans. : The set or total of all possible outcomes of an experiment is known as a sample space.

Q.9 If a fair coin is tossed twice, what is the probability of getting at least one head ?

Ans. : The sample space of this experiment is S = {HH, HT, TH, TT}.

Q.10 What is an event ?

Ans. : An event is any collection (subset) of outcomes contained in the sample space. An event is simple if it consists of exactly one outcome and compound if it consists of more than one outcome.

Q.11 What is trial and event? Explain types of events.

Ans. : • In the theory of probability, an experiment is called a trial and its outcome or combination of outcomes is called an event.

- Performing an experiment is known as a trial and the outcomes of the experiment are known as events
- For example : throwing of a coin is a trial and the outcomes of the trial say, 'Getting Tails(T)' are the evets.
- Throwing of a die is a trial and getting 1 or 2 or 3 etc. as the outcomes are the events.
- Exhaustive Events : The total number of possible outcomes in any trial is known as exhaustive events
- In throwing of a die, there are six exhaustive cases, since anyone of the 6 faces, 1, 2, 3, 4, 5, 6 may come uppermost
- Mutually Exclusive Events : Events are said to be mutually exclusive if the happening of any one of the events excludes (or) precludes the happening of all the others i.e. if no two or more of the events can happen simultaneously in the same trial. The joint occurrence is not possible.
- Equally likely events : the outcomes of a trial are said to be equally likely if anyone of them cannot be expected to occur in preference to the others.

Two or more events are said to be equally likely if each one of them has an equal chance of occurring.

- Independent Events : Several events are said to be independent if the happening of an event is not affected by the happening of one or more events

Q.12 What is random experiment ?

Ans. : • In some experiments, we are not able to control the value of certain variables so that the results will vary from one performance of the experiment to the next, even though most of the conditions are the same. These experiments are called as random experiment.

- Random experiment is defined as an experiment whose outcomes are known before the experiment is performed but which outcome is going to happen in a particular trial is not known.

Q.13 Explain sample space with example.

Ans. : • The totality of the possible outcomes of a random experiment is called the sample space of the experiment and it will be denoted by letter 'S'.

- There will be more than one sample space that can describe outcomes of an experiment, but there is usually only one that will provide the most information.
- The sample space is not determined completely by the experiment. It is partially determined by the purpose for which the experiment is carried out.
- **Example 1** : If the experiment consists of flipping two coins, then the sample space consists of the following points:

$$S = \{ (T, T), (T, H), (H, T), (H, H)\}$$

The outcome will be (T, T) if both coins are tails, (T,H) if the first coin is tails and the second heads, (H, T) if the first is heads and the second tails, and (H,H) if both coins are heads.

Example 2 : A die is rolled once. We let X denote the outcome of this experiment. Then the sample space for this experiment is the 6-element set

$$S = \{1, 2, 3, 4, 5, 6\},$$

- It is convenient to classify sample spaces according to the number of elements they contains. If a sample space has a finite number of points, it is called a ***finite sample space***. If it has as many points as there are natural numbers 1, 2, 3, ... , it is called a ***countably infinite sample space***. If it has as many points as there are in some interval on the x axis, such as $0 \le x \le 1$, it is called a ***non-countably infinite sample space***.
- A sample space that is finite or countably finite is often called a *discrete sample space*, while one that is non-countably infinite is called a *non-discrete sample space*.
- The result of a trail in a random experiment is called an **out come**.

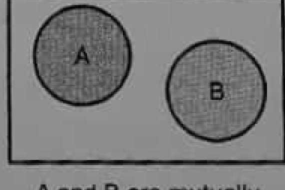
Q.14 What is exhaustive event and mutually exclusive event ? Explain with example.

Ans. : Exhaustive event :

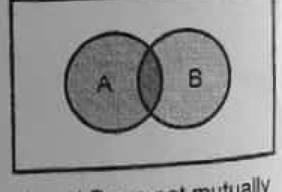
- The events $E_1, E_2, E_3 ..., E_n$ of the sample space S are said to be collectively exhaustive events of the sample space S if they include all possible events i.e. $E_1 \cup E_2 \cup E_3 \cup, ... \cup E_n = S$
- Considering the tossing of a coin then S = { H, T} and the events E_1 = {H} and E_2 = {T} are exhaustive events.

mutually exclusive event :

- The events $E_1, E_2, E_3 ..., E_n$ are said to be manually exclusive events if no two or more of them occur simultaneously in the same experiment.
- Two events are said to be mutually exclusive events when both cannot occur at the same time
- Mutually exclusive events always have a different outcome. Such events are so that when one happens it prevents the second from happening.
- For example, if the coin toss gives you a "Head" it won't give you a "Tail". These are mutually exclusive events.



A and B are mutually exclusive



A and B are not mutually exclusive

Fig. Q.14.1

Q.15 Define union and intersection of events.

Ans. : • The union of the two events A and B, denoted by the symbol AUB, is the event containing all the elements that belong to A or B or both.

• The intersection of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

Q.16 Suppose the data set contains 1.7, 2.2, 3.9, 3.11, and 14.7. Calculate sample mean and median.

Ans. :

$$\text{Sample mean } (\bar{x}) = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{n}$$

$X_1 = 1.7,\quad X_2 = 2.2 \quad X_3 = 3.9 \quad X_4 = 3.11 \quad X_5 = 14.7 \quad n = 5$

$$\text{Sample mean } (\bar{x}) = \frac{1.7 + 2.2 + 3.9 + 3.11 + 14.7}{5} = 5.122$$

Sample median = n/2 = 5/2 2.5 ➡ 3 so third number is 3.11.

Q.17 The following measurements were recorded for the drying time, in hours, of a certain brand of latex paint. 3.4 2.5 4.8 2.9 3.6 2.8 3.3 5.6 3.7 2.8 4.4 4.0 5.2 3.0 4.8

Assume that the measurements are a simple random sample.

(a) What is the sample size for the above sample ?

(b) Calculate the sample mean for this data.

(c) Calculate the sample median.

Ans. : a) sample size = 15

$$\text{b) Sample mean } (\bar{x}) = \frac{3.4+2.5+4.8+2.9+3.6+2.8+3.3+5.6+3.7+2.8+4.4+4.0+5.2+3.0+4.8}{15}$$

Sample mean $(\bar{x})$ = 3.787

c) The sample median = Sample median is the 8^{th} value, after the data is sorted from smallest to largest : 3.6.

Q.18 In one year, three awards will be given for a class of 25 graduate students in a CSE department. If each student can receive at most one award, how many possible selections are there ?

Ans. : Since the award are distinguishable, it is a permutation problem. The total number of sample points is

$$25P^3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = 13800$$

Q.19 Two dice are thrown 120 times. Find the average number of times in which, the number on the first dice exceeds the number on the second dice.

Ans. : The possible combinations that appear when two dice are thrown:

sample space S = { (1, 1), (1, 2), (1,3), ..., (6, 5), (6, 6)} = 36 combinations

Let X be the random variable. From the above combinations, the favorable cases are {(2,1), (3,1), (4,1),(5,1),(6,1), (3,2),(4,2),(5,2), (6,2), (4,3),(5,3),(6,3),(5,4),(6,4),(6,5)}

Total combination = 15

P(Success) = P (number on first dice exceeds that on the second)

$$= \frac{15}{36} = \frac{5}{12}$$

If X is the number of success, then X follow the binomial distribution with parameters (n)

$$E(X) = np = 120 \times \frac{5}{12} = 50$$

Q.20 A coin is flipped until 3 heads in succession occur. List only those elements of the sample space that require 6 or less tosses. Is this a discrete sample space ? Explain.

S = {HHH, THHH, HTHHH, TTHHH, TTTHHH, HTTHHH, THTHHH, HHTHHH, . . . };

The sample space is discrete containing as many elements as there are positive integers.

Q.21 Prove that : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Ans. : Let A and B be any two events. To write $A \cup B$ as the union of three mutually exclusive events : $A \cap B^c$, $A \cap B$ and $A^c \cap B$.

$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B) \quad \text{... (1)}$$

By axioms 3 :

$$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B) \quad \text{... (2)}$$

Now, $A = (A \cap B^c) \cup (A \cap B)$

$B = (A^c \cap B) \cup (A \cap B)$

Therefore

$$P(A) = P(A \cap B^c) + P(A \cap B) \quad \text{... (3)}$$

and $$P(B) = P(A^c \cap B) + P(A \cap B) \quad \text{... (4)}$$

By equations (3) and (4), we get

$$P(A) + P(B) = P(A \cap B^c) + P(A^c \cap B) + 2P(A \cap B) \quad \text{... (5)}$$

Compare the equation (2) and (5),

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

So, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Q.22 For any three arbitrary events A, B, C prove that :
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$. [JNTU : Nov.-09, 10]

Ans. :

$$P(A \cup B \cup C) = P[(A \cup B) \cup C]$$

$$= P(A \cup B) + P(C) - P[(A \cup B) \cap C]$$

$$= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) \cup (B \cap C)]$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) - P((A \cap C) \cap (B \cap C))$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Q.23 If $P(A) = \frac{1}{5}$, $P(B) = \frac{2}{3}$, $P(A \cap B) = \frac{1}{15}$
Find a) $P(A \cup B)$ b) $P(A^c \cap B)$ c) $P(A \cap B^c)$.

Ans. : a) $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{5} + \frac{2}{3} - \frac{1}{15}$$

$$= \frac{3 + 2 \times 5}{15} - \frac{1}{15}$$

$$= \frac{13 - 1}{15} = \frac{12}{15}$$

$$P(A \cup B) = \frac{4}{5}$$

b) $P(A^c \cap B)$

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

$$= \frac{2}{3} - \frac{1}{15}$$

$$= \frac{2 \times 5 - 1}{15} = \frac{10 - 1}{15}$$

$$P(A^c \cap B) = \frac{9}{15} = \frac{3}{5}$$

c) $P(A \cap B^c)$

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$= \frac{1}{5} - \frac{1}{15}$$

$$= \frac{3-1}{15}$$

$$P(A \cap B^c) = \frac{2}{15}$$

1.3 : Probability of an Event, Additive Rules

Q.24 What is an addition rule ?

Ans. : The addition rule states the probability of two events is the sum of the probability that either will happen minus the probability that both will happen.

The addition rule is: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Q.25 Define probability of event for sample.

Ans. : The probability of an event E is defined as the number of outcomes favourable to E divided by the total number of equally likely outcomes in the sample space S of the experiment.

Q.26 A single 6-sided die is rolled. What is the probability of rolling a 2 or a 5 ? How do we find the probabilities of these mutually exclusive events ?

Ans. :

- The number rolled can be a 2.
- The number rolled can be a 5.
- Events : These events are mutually exclusive since they cannot occur at the same time.
- **Addition Rule 1 :** When two events, A and B, are mutually exclusive, the probability that A or B will occur is the sum of the probability of each event.

$P(A \text{ or } B) = P(A) + P(B)$

- Let's use this addition rule to find the probability.

1 : A single 6-sided die is rolled. What is the probability of rolling a 2 or a 5 ?

$$P(2) = \frac{1}{6}$$

$$P(5) = \frac{1}{6}$$

$$P(2 \text{ or } 5) = P(2) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Q.27 A spinner has 4 equal sectors colored yellow, blue, green, and red. What is the probability of landing on red or blue after spinning this spinner ?

Ans. : Probabilities :

$$P(\text{red}) = \frac{1}{4}$$

$$P(\text{blue}) = \frac{1}{4}$$

$$P(\text{red or blue}) = P(\text{red}) + P(\text{blue}) = \frac{1}{4} + \frac{1}{4}$$

$$= \frac{2}{4} = \frac{1}{2}$$

Q.28 If A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{4}$ and $P(A \cup B) = \frac{11}{12}$, find P(A|B) and P(B|A).

Ans. : $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$= \frac{1}{3} + \frac{3}{4} - \frac{11}{12}$$

$$= \frac{4+3\times3}{12} - \frac{11}{12}$$

$$= \frac{13}{12} - \frac{11}{12}$$

$$= \frac{13-11}{12} = \frac{2}{12}$$

$$P(A \cap B) = \frac{1}{6}$$

So, $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{1}{6}}{\frac{3}{4}} = \frac{1}{6} \times \frac{4}{3}$$

$$P(A|B) = \frac{4}{18} = \frac{2}{9}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{6} \times \frac{3}{1}$$

$$P(B|A) = \frac{1}{2}$$

Q.29 Determine i) P(B|A) ii) $P\left(\frac{A}{B^c}\right)$ if A and B are event with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(A \cup B) = \frac{1}{2}$.

Ans. :

i) $P\left(\frac{B}{A}\right)$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

First calculate $P(A \cap B)$ from given data.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{4+3}{12} - \frac{1}{2}$$

$$= \frac{7}{12} - \frac{1}{2} = \frac{7-6}{12}$$

$$P(A \cap B) = \frac{1}{12}$$

Substitute this value into the formula.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{12}}{\frac{1}{3}}$$

$$= \frac{1}{12} \times \frac{3}{1} = \frac{3}{12}$$

$$P\left(\frac{B}{A}\right) = \frac{1}{4}$$

ii) $P\left(\frac{A}{B^c}\right)$

First calculate $P(B^c)$

$$P(B^c) = 1 - P(B)$$

$$= 1 - \frac{1}{4} = \frac{4-1}{4}$$

$$P(B^c) = \frac{3}{4}$$

$$P\left(\frac{A}{B^c}\right) = \frac{P(A \cap B^c)}{P(B^c)}$$

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$= \frac{1}{3} - \frac{1}{12}$$

$$= \frac{4-1}{12} = \frac{3}{12}$$

$$P(A \cap B^c) = \frac{1}{4}$$

Therefore

$$P\left(\frac{A}{B^c}\right) = \frac{P(A \cap B^c)}{P(B^c)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3}$$

$$P\left(\frac{A}{B^c}\right) = \frac{1}{3}$$

Q.30 If P(A) = 1/2, P(B) = 1/3, $P(A \cap B) = \frac{1}{5}$ then find a) $P(A \cup B)$ b) $P(A^c/B)$ c) $P(A/B^c)$ d) $P(A^c/B^c)$ where A^c = The complement of A.

Ans. : Given data :

$$P(A) = \frac{1}{2} \quad P(B) = \frac{1}{3} \quad P\left(\frac{A}{B}\right) = \frac{1}{5}$$

a) $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{5}$$

$$= \frac{15+10-6}{30} = \frac{25-6}{30} = \frac{19}{30}$$

b) $$P(A^c/B) = P(B) - P(A \cap B)$$

$$= \frac{1}{3} - \frac{1}{5}$$

$$= \frac{5-3}{15} = \frac{2}{15}$$

c) $$P(A/B^c) = P(A) - P(A \cap B)$$

$$= \frac{1}{2} - \frac{1}{5} = \frac{5-2}{10} = \frac{3}{10}$$

d) $$P(A^c/B^c) = P((A \cap B)^c)$$

$$= 1 - P(A \cup B)$$

$$= 1 - \frac{19}{30} = \frac{30-19}{30}$$

$$= \frac{11}{30}$$

...complementary event of A then $P(\bar{A}) = 1 - P(A)$ and $P(\bar{A}) \leq 1$.

Ans. : $\bar{A}$ is complementary event of A, i.e. $\bar{A}$ represents the event 'Not A'. Hence A and $\bar{A}$ are mutually exclusive events.

Also, $\quad A \cup \bar{A} = S$

$\therefore \quad P(A \cup \bar{A}) = P(S)$

i.e. $\quad P(A) + P(\bar{A}) = 1 \quad$ using axiom (ii) and (iii)

$\therefore \quad P(\bar{A}) = 1 - P(A)$

$\therefore \quad P(\bar{A}) \leq 1$

Q.32 If $A_1, A_2, \ldots, A_n$ are n events then prove that,

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$$

Ans. : Mathematical induction method is used to prove this. For any event, $E \in S$, $0 \leq P(E) \leq 1$.

Consider the two event (n = 2). Then

$P(A_1 \cup A_2) \leq 1 \quad \because A_1 \cup A_2$ is an event.

By using addition theorem :

$$P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$$

$$P(A_1) + P(A_2) \leq 1 + P(A_1 \cap A_2)$$

We can rewrite the above equation,

$$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$$

$\therefore$ The statement is true for n = 2.

Let us assume that statement is true for n = k.

i.e. $P\left(\bigcap_{i=1}^{k} A_i\right) \geq \sum_{i=1}^{k} P(A_i) - (k-1)$

Then

$$P\left(\bigcap_{i=1}^{k+1} A_i\right) = P\left(\bigcap_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

$$\geq P\left(\bigcap_{i=1}^{k} A_i\right) + P(A_{k+1}) - 1$$

$$\geq \sum_{i=1}^{k} P(A_i) - (k-1) + P(A_{k+1}) - 1$$

$$= \sum_{i=1}^{k+1} P(A_i) - k$$

$$\therefore \quad P\left(\bigcap_{i=1}^{k+1} A_i\right) \geq \sum_{i=1}^{k} P(A_i) - k$$

The statement is also the true for n = k + 1.

1.4 : Conditional Probability, Independence, and the Product Rule

Q.33 What is conditional probability? Explain.

Ans. : • Let A and B be two events such that $P(A) > 0$. We denote $P(B|A)$ the probability of B given that A has occurred. Since A is known to have occurred, it becomes the new sample space replacing the original S. From this, the definition is ,

$$P(B/A) \equiv \frac{P(A \cap B)}{P(A)}$$

OR

$$P(A \cap B) = P(A)\, P(B/A)$$

- The notation $P(B \mid A)$ is read **"the probability of event B given event A"**. It is the probability of an event B given the occurrence of the event A.
- We say that , the probability that both A and B occur is equal to the probability that A occurs times the probability that B occurs given that A has occurred. We call **P(B|A) the conditional probability of B given A**, i.e., the probability that B will occur given that A has occurred.
- Similarly, the conditional probability of an event A, given B by,

$$P(A/B) \equiv \frac{P(A \cap B)}{P(B)}$$

- The probability $P(A|B)$ simply reflects the fact that the probability of an event A may depend on a second event B. If A and B are mutually exclusive $A \cap B = \phi$ and $P(A|B) = 0$.
- Another way to look at the conditional probability formula is :

$$P(\text{Second/First}) = \frac{P(\text{first choice and second choice})}{P(\text{first choice})}$$

- Conditional probability is a defined quantity and cannot be proven.

- The key to solving conditional probability problems is to :
  1. Define the events.
  2. Express the given information and question in probability notation.
  3. Apply the formula.

Q.34 What is joint probability ? Explain.

Ans. : Joint Probability

- A joint probability is a probability that measures the likelihood that two or more events will happen concurrently.
- If there are two independent events A and B, the probability that A and B will occur is found by multiplying the two probabilities. Thus for two events A and B, the special rule of multiplication shown symbolically is :

  P(A and B) = P(A) P(B).
- The general rule of multiplication is used to find the joint probability that two events will occur. Symbolically, the general rule of multiplication is,

  P(A and B) = P(A) P(B|A).
- The probability $P(A \cap B)$ is called the joint probability for two events A and B which intersect in the sample space. Venn diagram will readily shows that

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Equivalently :

$$P(A \cap B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

- The probability of the union of two events never exceeds the sum of the event probabilities.
- A tree diagram is very useful for portraying conditional and joint probabilities. A tree diagram portrays outcomes that are mutually exclusive.

Q.35 Explain independence rule of conditional probability.

Ans. : • Two events A and B are independent if the probability that A occurs is no different whether or not B occurs, and vice versa.

- In other words, knowing that A occurs does not give us any information about whether or not B occurs, and vice versa.
- An intuitive example of independence is an experiment where two fair coins are flipped on opposite sides of a room. Intuitively, the way that one coin lands should have no affect on the way the other coin lands, so the two events are independent.
- Formally two event A and B are independent if and only if : $P(A \cap B) = P(A) \cdot P(B)$

Q.36 If A and B are independent events. Then A^c and B^c are also independent events.

Ans. : Given that A and B are independent events.

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap B) = P(A \cap B)^c) \quad \rightarrow \text{by demorgan law}$$

$$= 1 - P(A \cup B)$$

$$= P - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$= (1 - P(A))\,(1 - P(B))$$

$$= P(A^c)\,P(B^c)$$

Q.37 Determine the probability for the event when a non-defective bolt will be found if out of 600 bolts already examined 12 were defective.

Ans. : Probability of defective bolt $P(D) = \frac{12}{600} = \frac{1}{50}$

Probability (finding a non defective bolt)

$$= P(\overline{D}) = 1 - P(D) = 1 - \frac{1}{50} = \frac{49}{50}$$

Q.38 If P(C) = 0.65, P(D) = 0.40 and $P(C \cap D)$ = 0.24. Are the events C and D independent ?

Ans. : $P(C) \times P(D) = 0.65 \times 0.40$

$$= 0.26 \neq P(C \cap D)$$

hence event C and event D are not independent.

Q.39 The probability that a randomly selected student from a college is a senior is 0.20 and the joint probability that the student is a computer science major and senior is 0.03. Find the conditional probability that a student selected at random is a computer science major given that he/she is a senior.

Ans : Let A = the event that the student selected is a senior $\rightarrow P(A) = 0.20$

B = the event that the student selected is a computer science major → $P(A \cap B) = 0.03$

Required probability = P(B/A)

$$= \frac{P(A \cap B)}{P(A)} = \frac{0.03}{0.20} = 0.15$$

Q.40 Two dice are thrown together. Let A be the event 'getting 6 on the first die' and B be the event 'getting 2 on the second die'. Are the events A and B independent ?

Ans. : A = {(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

B = {(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2)}

$A \cap B$ = {(6, 2)}

$$P(A) = \frac{6}{36} = \frac{1}{6}$$

$$P(B) = \frac{1}{6}$$

$$P(A \cap B) = \frac{1}{36}$$

Events A and B will be independent if

$P(A \cap B) = P(A)\,P(B)$.

LHS : $P(A \cap B) = \frac{1}{36}$

RHS : $P(A)\,P(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

LHS = P A B, RHS = P A P B

Hence, A and B are independent.

Q.41 In a box, there are 100 resistors having resistance and tolerance as shown in the following table. Let a resistor be selected from the box and assume that each resistor has the same like-hood of being chosen. Define three events: A as draw a 47 Ω resistor, B as draw a resistor with 5% tolerance and C as draw a 100 Ω resistor. Find P(A/B), P(A/C), P(B/C).

Resistance (Ω)	Tolerance		Total
	5%	10%	
22	10	14	24
47	28	16	44
100	24	8	32
Total	62	38	100

Ans. : From the table, we have

$$P(A) = P(47\Omega) = \frac{44}{100}$$

$$P(B) = P(5\ \%) = \frac{62}{100}$$

$$P(C) = P(100\ \Omega) = \frac{32}{100}$$

Also, the joint probabilities are

$$P(A \cap B) = P(47\ \Omega\ 5\ \%) = \frac{28}{100}$$

$$P(A \cap C) = P(47\Omega \cap 100\Omega) = P(\phi) = 0$$

being mutually exclusive

$$P(B \cap C) = P(5\ \% \cap 100\ \Omega) = \frac{24}{100}$$

∴ By definition of conditional probability, we have

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{28/100}{62/100} = \frac{28}{62}$$

$$P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{0}{32/100} = 0$$

$$P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{24/100}{32/100} = \frac{24}{32}$$

1.5 : Bayes' Rule

Q.42 Explain Bayes theorem.

Ans. : • Bayes' theorem is a method to revise the probability of an event given additional information. Bayes's theorem calculates a conditional probability called a posterior or revised probability.

- Bayes' theorem is a result in probability theory that relates conditional probabilities. If A and B denote two events, P(A|B) denotes the conditional probability of A occurring, given that B occurs. The two conditional probabilities P(A|B) and P(B|A) are in general different.
- Bayes theorem gives a relation between P(A|B) and P(B|A). An important application of Bayes' theorem is that it gives a rule how to update or revise the strengths of evidence-based beliefs in light of new evidence a posteriori.
- A **prior probability** is an initial probability value originally obtained before any additional information is obtained.

- A **posterior probability** is a probability value that has been revised by using additional information that is later obtained.
- Suppose that $B_1, B_2, B_3 \ldots B_n$ partition the outcomes of an experiment and that A is another event. For any number, k, with $1 \le k \le n$, we have the formula :

$$P(B_k/A) = \frac{P(A/B_k)\cdot P(B_k)}{\sum_{i=1}^{n} P(A/B_i)\cdot P(B_i)}$$

Q.43 Two boxes B_1 and B_2 contain 100 and 200 light bulbs respectively. The first box (B_1) has 15 defective bulbs and the second 5. Suppose a box is selected at random and one bulb is picked out.

a) What is the probability that it is defective ?

b) Suppose we test the bulb and it is found to be defective. What is the probability that it came from box 1 ?

Ans. :

a) **Probability that it is defective :** Box B_1 has 85 good and 15 defective bulbs. Similarly box B2 has 195 good and 5 defective bulbs.
Let D = "Defective bulb is picked out".

Then,

$$P(D/B_1) = \frac{15}{100} = 0.15, \quad P(D/B_2) = \frac{5}{200} = 0.025.$$

Since a box is selected at random, they are equally likely.

$$P(B_1) = P(B_2) = \frac{1}{2}$$

Thus B_1 and B_2 form a partition and using above equation, we obtain

$$P(D) = P(D/B_1)\,P(B_1) + P(D/B_2)\,P(B_2)$$
$$= 0.15\times\frac{1}{2} + 0.025\times\frac{1}{2} = 0.0875$$

Thus, there is about 9% probability that a bulb picked at random is defective.

b) Probability that it came from box 1 :

$$P(B_1/D) = \frac{P(D/B_1)\,P(B_1)}{P(D)} = \frac{0.15\times 1/2}{0.0875} = 0.8571$$

Q.44 A mechanical factory production line is manufacturing bolts using three machines, A, B and C. The total output, machine A is responsible for 25%, machine B for 35% and machine C for the rest. The machines that 5% of the output from machine A is defective, 4% from machine B and 2% from machine C. A bolt is chosen at random from the production line and found to be defective. What is the probability that it came from

I. machine A II. machine B III. machine C ?

Ans. : Let

D = {bolt is defective},

A = {bolt is from machine A},

B = {bolt is from machine B},

C = {bolt is from machine C}.

Given data : $P(A) = 0.25$, $P(B) = 0.35$, $P(C) = 0.4$.

$P(D|A) = 0.05$, $P(D|B) = 0.04$, $P(D|C) = 0.02$

From the Bayes' Theorem :

$$P(A/D) = \frac{P(D/A)\times P(A)}{P(D/A)\times P(A) + P(D/B)\times P(B) + P(D/C)\times P(C)}$$
$$= \frac{0.05\times 0.25}{0.05\times 0.25 + 0.04\times 0.35 + 0.02\times 0.4}$$
$$= \frac{0.0125}{0.0125 + 0.014 + 0.008}$$

P(A/D) = 0.3621

Similarly :

$$P(B/D) = \frac{P(D/B)\times P(B)}{P(D/A)\times P(A) + P(D/B)\times P(B) + P(D/C)\times P(C)}$$
$$= \frac{0.04\times 0.35}{0.05\times 0.25 + 0.04\times 0.35 + 0.02\times 0.4}$$
$$= \frac{0.014}{0.0125 + 0.014 + 0.008} = \frac{0.014}{0.0345}$$

P(B/D) = 0.4057

$$P(C/D) = \frac{P(D/C)\times P(C)}{P(D/A)\times P(A) + P(D/B)\times P(B) + P(D/C)\times P(C)}$$
$$= \frac{0.02\times 0.4}{0.05\times 0.25 + 0.04\times 0.35 + 0.02\times 0.4}$$
$$= \frac{0.008}{0.0125 + 0.014 + 0.008} = \frac{0.008}{0.0345}$$

P(C/D) = 0.2318

Q.45 At a certain university, 4% of men are over 6 feet tall and 1% of women are over 6 feet tall. The total student population is divided in the ratio 3:2 in favour of women. If a student is selected at random from among all those over six feet tall, what is the probability that the student is a woman ?

Ans. : Let us assume following :

$$M = \{\text{Student is Male}\},$$
$$F = \{\text{Student is Female}\},$$
$$T = \{\text{Student is over 6 feet tall}\}.$$

Given data :

$$P(M) = 2/5,$$
$$P(F) = 3/5,$$
$$P(T|M) = 4/100$$
$$P(T|F) = 1/100.$$

We require to find $P(F|T)$?

Using Bayes' Theorem we have :

$$P(F/T) = \frac{P(T/F)\,P(F)}{P(T/F)\,P(F) + P(T/M)\,P(M)}$$

$$= \frac{\frac{1}{100}\times\frac{3}{5}}{\frac{1}{100}\times\frac{3}{5}+\frac{4}{100}\times\frac{2}{5}} = \frac{\frac{3}{500}}{\frac{3}{500}+\frac{8}{500}}$$

$$P(F/T) = \frac{3}{11}$$

Q.46 A pair of dice is rolled. If the sum of 9 has appeared, find the proabability that one of the dice shows 3.

Ans. : Let A = The event that the sum is 9

B = The event the one of dice shows 3.

Exhaustive cases = $6^2 = 36$.

Favorable cases of the event A = (3, 6), (6, 3), (4, 5), (5, 4).

So $P(A) = 4.36$

$$P(A) = \frac{1}{9}$$

Favorable case for the event $A \cap B$ = (3, 6), (6, 3)

Hence $P(A\cap B) = \frac{2}{36} = \frac{1}{18}$

But $P(A\cap B) = P(A)\times P(B/A)$

$$P(B/A) = \frac{P(A\cap B)}{P(A)}$$

$$P(B/A) = \frac{1/18}{1/9} = \frac{1}{18}\times\frac{9}{1}$$

$$P(B/A) = 1/2$$

Q.47 If A_1 and A_2 are quality likely, mutually exclusive and exhaustive events and $P(B/A_1) = 0.2$, $P(B/A_2) = 0.3$, find $P(A_1/B)$.

Ans. : Since A_1 and A_2 are equally likely, we have $P(A_1) = P(A_2)$. Further, they are mutually exclusive.

$$P(A_1) + P(A_2) = 1$$
$$P(A_1) = P(A_2) = 0.5$$
$$= \frac{0.5\times 0.2}{0.5\times 0.2 + 0.5\times 0.3}$$
$$= \frac{0.1}{0.1+0.15} = \frac{0.1}{0.25}$$
$$P(A_1/B) = 0.4$$

Q.48 10% of the bulbs produced in a factory are of red colour and 2% are red and defective. If one bulb is picked up at random, determine the probability of its being defective if it is red.

Ans. :

Let A and B be the events that the bulb is red and detective, respectively.

$$P(A) = 10/100 = 1/10$$
$$P(A\cap B) = 2/100 = 1/50$$
$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1/50}{1/10} = 1/5$$

Thus the probability of the picked up bulb of its being defeftive, if it is red, is 1/5.

1.6 : Random Variables and Probability Distributions : Concept of a Random Variable

Q.49 Define discrete sample space

Ans. : If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a discrete sample space.

Q.50 What is continuous sample space?

Ans. : If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a continuous sample space

Q.51 What is random variable? Explain discrete random variable.

Ans. : • The distribution function F(x) or the density f(x) completely characterizes the behavior of a random variable X. The concept of a random variable will enable us to replace the original probability space with one in which events are set of numbers.

- Whenever you run and experiment, flip a coin, roll a die, pick a card, you assign a number to represent the value to the outcome that you get. This assign is called **a random variable.**
- A random variable is a variable X that assigns a real number [x], for each and every outcome of a random experiment. If S is the sample space containing all the 'n' outcomes $\{e_1, e_2, e_3, ..., e_i ..., e_n\}$ of random experiment, and X is a random variable defined as a function X(e) on S, then for every outcome e_i (where i = 1, 2, 3, ..., n) that is in S the random variable $X(e_i)$ will assign a real value x_i.
- Advantages of random variables is that user can define certain probability functions that make it both convenient and easy to compute the probabilities of various events.

Discrete Random Varaible

- The random variable is called a **discrete random variable** if it is defined over a sample space having a finite or a countable infinite number of sample points. In this case, random variable takes on discrete values and it is possible to enumerate all the values it may assume.
- A discrete random variable can only have a specific (or finite) number of numerical values.
- We can have **infinite discrete random variables** if we think about things that we know have an estimated number. Think about the number of stars in the universe. We know that there are not a specific number that we have a way to count so this is an example of an infinite discrete random variable.
- Another example would be with investments with share market. If you were to invest ₹ 1 lakh at the start of year, you could only estimate the amount you would have at the end of year.

Q.52 Define continuous random variable.

Ans. : A random variable is a continuous random variable if it can take any value in an interval

Q.53 What is the probability distribution for the toss of one fair coin ?

Ans. :

$$P\text{ (Heads)} = \frac{1}{2}$$

$$P\text{ (Tails)} = \frac{1}{2}$$

Let heads denote the coin landing head side up.

Let tails denote the coin landing tail side up.

The possible outcomes are for the coin to land head side up or tail side up.

Using the alternative notation.

$$P(X = \text{Heads}) = \frac{1}{2}$$

$$P(X = \text{Tails}) = \frac{1}{2}$$

X	P (X)
Heads	$\frac{1}{2}$
Tails	$\frac{1}{2}$

Q.54 Probability of a function of the number of Heads from tossing a coin four times. Determine the cumulative distribution function.

Ans. :

$$F(0) = f(0) = \frac{1}{16}$$

$$F(1) = f(0) + f(1)$$

$$= \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$F(2) = f(0) + f(1) + f(2)$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16}$$

$$F(3) = f(0) + f(1) + f(2) + f(3)$$

$$= \frac{1}{16}+\frac{4}{16}+\frac{6}{16}+\frac{4}{16}$$

$$= \frac{1+4+6+4}{16} = \frac{15}{16}$$

$$P(4) = f(0)+f(1)+f(2)+f(3)+f(4)$$

$$= \frac{1}{16}+\frac{4}{16}+\frac{6}{16}+\frac{4}{16}+\frac{1}{16}$$

$$= \frac{1+4+6+4+1}{16} = \frac{16}{16} = 1$$

Q.55 A shipment of 8 similar microcomputers to a. retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Ans. : Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then x can be any of the numbers 0, 1, and 2. Now

Number of ways of choosing any 2 :

$${}^{8}C_2 = \frac{8!}{((8-2)!2!)} = 28$$

(This will be the denominator)

$$P[0] = {}^{3}C_0 \times {}^{5}C_2/28$$

$$= (3!/((3-0)!0!) \times (5! \times ((5-2)!2)))/28$$

$$= 5/14$$

$$P[1] = {}^{3}C_1 \times {}^{5}C_{1/28} = 15/28$$

$$P[2] = {}^{3}C_2 \times {}^{5}C_{0/28} = 3/28$$

Q.56 If a random variable X takes the values 1, 2, 3 and 4 such that 2P (X = 1) = 3P (X = 2) = P (X = 3) = 5P (X = 4), derive the probability distribution function of X.

[JNTU : March-17, Marks 5]

Ans. : Assume P (X = 3) = α By the given equation

$$P(X=1) = \frac{\alpha}{2} \quad P(X=2) = \frac{\alpha}{3} \quad P(X=4) = \frac{\alpha}{5}$$

For a probability distribution (and mass function)

$$\sum P(x) = 1$$

$$P(1)+P(2)+P(3)+P(4) = 1$$

$$\frac{\alpha}{2}+\frac{\alpha}{3}+\alpha+\frac{\alpha}{5} = 1 \quad \Rightarrow \frac{61}{30}\alpha = 1 \quad \Rightarrow \alpha = \frac{30}{61}$$

$$P(X=1) = \frac{15}{61};\ P(X=2) = \frac{10}{61};\ P(X=3) = \frac{30}{61};$$

$$P(X=4) = \frac{6}{61}$$

- The probability distribution given by

X	1	2	3	4
P(X)	15/61	10/61	30/61	6/61

1.7 : Continuous Probability Distributions, Statistical Independence

Q.57 What is Continuous probability distribution ?

Ans. : Continuous probability distribution is a type of distribution that deals with continuous types of data or random variables. The continuous random variables deal with different kinds of distributions.

Q.58 What is a discrete probability distribution? What two conditions determine a probability distribution?

Ans. : • Discrete distribution is a probability distribution where sample space is countable. The random variable in such a case can only take point values. For example: number of people or number of coins etc.

- Discrete Probability distribution consists of the values a random variable can assume and the corresponding probabilities of the values.
- The two conditions are

1. Sum of all probabilities should be equal to 1.
2. Each probability must be greater than and equal to zero.

Q.59 Explain difference between discrete and continuous random variable.

Ans. :

Sr. No.	Discrete	Continuous
1.	It uses countable set	It uses set of interval on R.

2.	F is set of all subset of Ω.	F is made from sub-intervals of Ω with set operations.
3.	For a set A ∈ F, $P(A) = \sum_{\omega \in A} p(\omega)$	For a set A ∈ F, $p(A) = \int_{A}^{-A} f_X(x)\,dx$
4.	Distribution function (Cdf) : $F_X(x) = \sum_{\omega \le x} p\omega$	Distrinbution function (Cdf) : $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dt$

Q.60 Explain statistical independence

Ans. : Two variates A and B are statistically independent if the probability P (A|B) of A given B satisfies P(A|B) = P(A)

in which case the probability of A and B is just

$$P(AB) = P(A \cap B) = P(A)\,P(B)$$

- Statistical independence means one event conveys no information about the other; statistical dependence means there is some information.
- Statistically independent is not the same as mutually exclusive: if A and B are mutually exclusive, then they can't be independent, unless one of them is probability 0 to start with :

$P(r)(()A \cap B) = 0 = \Pr(()A)\Pr(()B)$

Q.61 Three cards are drawn in succession from a deck without replacement. Find the probability distribution for the number of spades.

Ans. : X = number of spades in the three draws.

Let S and N stand for a spade and not a spade, respectively.

P(X = 0) = P(NNN)

= (39/52) (38/51) (37/50)

= 703/1700

P(X = 1) = P(SNN) + P(NSN) + P(NNS)

= 3(13/52)(39/51)(38/50)

= 741/1700

P(X = 3) = P(SSS)

= (13/52)(12/51)(11/50)

= 11/850

$$P(X = 2) = 1 - \frac{703}{1700} - \frac{741}{1700} - \frac{11}{850} = \frac{117}{850}$$

Q.62 From a box containing 4 black balls and 2 green balls, 3 balls are drawn in succession, each ball being replaced in the box before the next draw is made. Find the probability distribution for the number of green balls.

Ans. : • Let X denotes the number of green balls in the three draws.

- Let G and B stand for the colors of green and black, respectively.

Sample Event	x	P(X) = x
BBB	0	(2/3)3 = 8/27
GBB	1	(1/3) * (2/3)2 = 4/27
BGB	1	(1/3) * (2/3)2 = 4/27
BBG	1	(1/3) * (2/3)2 = 4/27
BGG	2	(1/3)2 * (2/3) = 2/27
GBG	2	(1/3)2 * (2/3) = 2/27
GGB	2	(1/3)2 * (2/3) = 2/27
GGG	3	(1/3)3 = 1/27

Q.63 A traffic engineer is interested in the number of vehicles reaching a particular crossroads during periods of relatively low traffic flow. The engineer finds that the number of vehicles X reaching the crossroads per minute is governed by the probability distribution:

x	0	1	2	3	4
P(X = x)	0.37	0.39	0.19	0.04	0.01

Calculate the expected value, the variance and the standard deviation of the random variable X.

Ans. : The expected value, the variance and the standard deviation of the random vaiable X :

x	x^2	P(X = x)	F(x)
0	0	0.37	0.37
1	1	0.39	0.76
2	4	0.04	0.95
3	9	0.09	0.99
4	16	0.01	1.00

$$E(X) = \sum_{x=0}^{4} xP(X = x)$$

$$= 0\times0.37+1\times0.39+2\times0.19+3\times0.04+4\times0.01$$

$$= 0.93$$

$$V(X) = E(X^2)-[E(X)]^2$$

$$= \sum_{x=0}^{4} x^2P(X = x)-\left[\sum_{x=0}^{4} xP(X=x)\right]^2$$

$$= 0\times 0.37+ 1\times 0.39+ 4\times 0.19+ 9\times 0.04+ 16$$

$$\times 0.01 -(0.93)^2$$

$$= 0.8051$$

The standard deviation is given by $\sigma = \sqrt{V(X)} =$ 0.8973

Q.64 Draw a probability histogram of the following probability distribution which represents the number of DVDs a person rents from a video store during a single visit :

x	P(X = x)
0	0.06
1	0.58
2	0.22
3	0.10
4	0.03
5	0.01

Ans. :

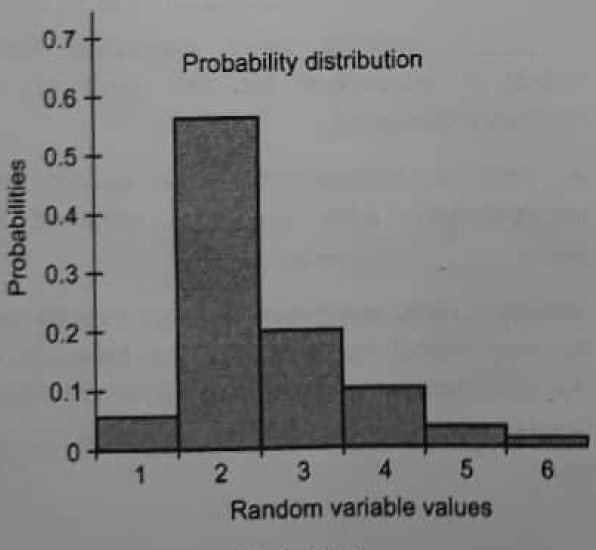


Fig.Q.64.1

Q.65 Let X be a random variable with the following probability distribution

X	-3	6	9
P(X = x)	1/6	1/2	1/3

then evaluate $E(2X + 1)^2$

Ans. : We know that :

$$E(2X+1)^2 = E(4X^2+4X+1) = E4(X^2)+E4(X)+1$$

$$E(X) = (-3)\frac{1}{6}+(6)\frac{1}{2}+(9)\frac{1}{3} = \frac{11}{12}$$

$$E(X^2) = (-3)^2\frac{1}{6}+(6^2)\frac{1}{2}+(9^2)\frac{1}{3} = \frac{93}{2}$$

$$E(2X+1)^2 = (4)\times(93/2)+(4)(11/2) + 1 = 209$$

Q.66 A class contains of 6 girls and 10 boys. If a committee of 3 is chosen at random from the class, find the probability that (i) 3 boys are selected, (ii) exactly 2 girls are selected.

Ans. : Total number of student (n) = 16

Then n(s) = number of ways of choosing 3 from 16

$$= {}^{16}C_3$$

i) 3 boys are selected : We can write ${}^{10}C_3$ here.

$$n(E) = {}^{10}C_3 \text{ Therefore}$$

$$P(E) = \frac{{}^{10}C_3}{{}^{16}C_3} = \frac{10\times9\times8}{16\times15\times14} = \frac{3}{14}$$

ii) exactly 2 girls are selected

$$\text{Then} \quad n(E) = {}^6C_2\times{}^{10}C_1$$

$$P(E) = \frac{{}^6C_2\times{}^{10}C_1}{{}^{16}C_3} = \frac{15}{56}$$

Q.67 Three groups of students contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys; one student is selected at random from each group. Find the probability of selecting 1 girl and 2 boys.

Ans. : If G denotes selection of girls and B denotes selection of a boy then the various cases of selection can be GBB respectively from the three groups, or BBG respectively from the three groups.

So, P(GBB) = P(G from group 1) × P(B from group II) × P(B from group III)

$$P(GBB) = \frac{^3C_1}{^4C_1} \times \frac{^2C_1}{^4C_1} \times \frac{^3C_1}{^4C_1} = \frac{3}{2} \times \frac{2}{4} \times \frac{3}{4} = \frac{9}{32}$$

Similarly,

$$P(BGB) = \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{3}{32}$$

$$P(BBG) = \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{32}$$

Required probability = P(GBB) + P(BGB) + P(BBG)

$$= \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}$$

Q.68 A batch of 100 manufactured components is checked by an inspector who examines 10 components selected at random. If none of the 10 component's is defective, the inspector accepts the whole batch. Otherwise, the batch is subjected to further inspection. What is the probability that a batch containing 10 defective components will be accepted ?

Ans. : • Let the N denote the number of ways of indiscriminately selecting 10 components from a batch of 100 components. Then N is given by

$$N = C(100, 10)$$

$$= \frac{100!}{(100-10)! \times 10!} = \frac{100!}{90! \times 10!}$$

- Let E denote the event "the batch containing 10 defective components is accepted by the inspector".
- The number of ways that E can occur is the number of ways of selecting 10 components from the 90 non-defective components and no components from the 10 defective component's.
- This number, N(E) is given by

$$N(E) = C(90, 10) \times C(10, 0) = C(90, 10)$$

$$= \frac{90!}{(90-10)! \times 10!} = \frac{90!}{80! \times 10!}$$

The probability of event E is given by :

$$P(E) = \frac{N(E)}{N} = \frac{90!}{80! \times 90!} \times \frac{90! \times 10!}{100!} = \frac{90! \times 90!}{100! \times 80!}$$

$$P(E) = 0.3305$$

Q.69 Two coins are tossed. Let A denote the event "at most one head on the two tosses" and let B denote the event "one head and one tail in both tosses". Are A and B independent events ?

Ans. : The sample space of the experiment S = { HH, HT, TH, TT}

Now events are defined as follows :

A = {HT, TH, TT} B = { HT, TH}

and $A \cap B$ = { HT, TH}

Thus :

$$P(A) = \frac{3}{4} \qquad P(B) = \frac{2}{4} = \frac{1}{2}$$

$$P(A \cap B) = \frac{2}{4} = \frac{1}{2}$$

$$P(A)\,P(B) = \frac{8}{3}$$

Since $P(A \cap B)$ P(A) P(B), we conclude that events A and B are not independent.

Q.70 A die is thrown 3 times. If getting a 6 is considered as success find the probability of atleast 2 success.

Ans. :

$$p = \frac{1}{6},\ q = \frac{5}{6},\ n = 3$$

P (at least 2 success) = $P(X \geq 2) = P(X=2) + P(X=3)$

$$= 3C_2\left(\frac{1}{6}\right)^2 \frac{5}{6} + 3C_3\left(\frac{1}{6}\right)^3 = \frac{2}{27}$$

Fill in the Blanks for Mid Term Exam

Q.1 A _____ variable takes numerical value which is determined by the result of the random experiment.

Q.2 A pictorial representation of events and manipulations with events is obtained by using _____ diagrams.

Q.3 Statistical data, generated in large masses, can be very useful for studying the behavior of the distribution if presented in a combined tabular and graphic display called a _____ plot

UNIT - II

2 Mathematical Expectation

2.1 : Mean of Random Variable, Variance and Covariance of Random Variables

Q.1 What is Mean ?

Ans. : • The arithmetic mean of a variable, often called the average, is computed by adding up all the values and dividing by the total number of values.

- It is merely the sum of the observations divided by the number of observations.
- The population mean is represented by the Greek letter μ (mu).

$$\mu = \frac{\sum x_i}{N}$$

- The sample mean is represented by $\bar{x}$(x-bar).

$$\bar{x} = \frac{\sum x_i}{n}$$

Q.2 Define variance.

Ans. : The variance uses the difference between each value and its arithmetic mean. The differences are squared to deal with positive and negative differences. The sample variance (s^2) is an unbiased estimator of the population variance (σ^2), with n-1 degrees of freedom.

Q.3 Compare variance and covariance.

Ans. : Variance refers to the spread of a data set around its mean value, while a covariance refers to the measure of the directional relationship between two random variables.

Q.4 For the following distribution, find the mean of the distribution.

X	− 2	− 1	0	1	2	3
P(X)	1/10	1/15	1/5	2/15	3/10	1/5

Ans. :/Here, X is a discrete random variable

Arithmetic mean or mean of X is

$$\bar{X} = E(X) = \sum X\, P(X)$$

$$\bar{X} = \left[-2\times\frac{1}{10}\right]+\left[-1\times\frac{1}{15}\right]+[0]+\left[1\times\frac{2}{15}\right]+\left[2\times\frac{3}{10}\right]+\left[3\times\frac{1}{5}\right] = \frac{4}{10}+\frac{1}{15}+\frac{3}{5} = \frac{32}{30}$$

Mean $\quad \bar{X} = \frac{16}{15}$

Q.5 Let X be a continuous random variable with mean μX. Then $E(aX + b) = a\,E(X) + b = a\mu X + b$, for any real numbers a, b.

Ans. : For a continuous random variable X, the mean of a function of X, say g(X), is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

So, for $g(X) = aX + b$, we find that

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} ax\, f_X(x)\,dx + \int_{-\infty}^{\infty} b\, f_X(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x\, f_X(x)\,dx + b\int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= a\mu_X + b$$

Q.6 Explain covariance of two random variable.

Ans. : • Covariance is a measure of association between two random variables. It is positive if the deviations of the two variables from their respective means tend to have the same sign and negative if the deviations tend to have opposite signs.

- The **covariance** between two random variables X and Y, denoted by Cov [X, Y] is defined as follows :

$$\text{Cov}[X, Y] = E[(X - E[X])\,(Y - E[Y])]$$

or

$$\text{Cov}[X, Y] = E[(X - \mu_x)\,(Y - \mu_y)]$$

- Covariance indicates how two variables are related. A positive covariance means the variables are positively related, while a negative covariance means the variables are inversely related. The formula for calculating covariance of sample data is shown below.

$$\text{Cov}[X, Y] = \frac{\sum_{i=1}^{n} (X_i - \overline{X})\,(Y_i - \overline{Y})}{n - 1}$$

Where
- x = The independent variable
- y = The dependent variable
- n = Number of data points in the sample
- $\overline{X}$ = The mean of the independent variable x
- $\overline{Y}$ = The mean of dependent variable y

$$\text{Cov}[X, Y] = E(XY - \mu_x Y - \mu_y X + \mu_x \mu_y)$$

$$= E(XY) - \mu_x E[Y] - \mu_y E[X] + \mu_x \mu_y$$

$$= E(XY) - \mu_x \mu_y - \mu_y \mu_x + \mu_x \mu_y$$

$$= E[XY] - E[X]\,E[Y]$$

Q.7 To understand how covariance is used, consider the following table. The rate of economic growth (x_i) and the rate of return of mutual fund.

Economic growth (x_i) %	MF return (y_i) %
2.1	8
2.5	12
4.0	14
3.6	10

Ans. : Calculate mean $\overline{X}$ and $\overline{Y}$

$$\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\overline{X} = \frac{2.1+2.5+4+3.6}{4} = \frac{12.2}{4} = 3.1$$

$$\overline{Y} = \frac{\sum_{i=1}^{n} y_i}{n} = \frac{8+12+14+10}{4} = \frac{44}{4} = 11$$

$$\text{Cov}(X, Y) = \frac{\sum_{i=1}^{n} (x_i - \overline{x})(y_1 - \overline{y})}{n-1}$$

$$= \frac{(2.1-3.1)(8-11)+(2.5-3.1)(12-11)+(4-3.1)(14-11)+(3.6-3.1)(10-11)}{3}$$

$$= \frac{(-1)(-3)+(-0.6)(1)+(0.9)(3)+(0.5)(-1)}{3}$$

$$= \frac{3-0.6+2.7-0.5}{3} = \frac{4.6}{3}$$

$$\text{Cov}(X, Y) = 1.533$$

- Let X be a random variable, then

$$\text{Cov}(X, Y) = \text{Var}[X]$$

Proof : From the definition of variance

$$\text{Cov}(X, Y) = E[(X - E[X])(X - E[X])]$$

$$= E[(X-E[X]^2]$$

$$= \text{Var}[X]$$

- Let X_1 and X_2 be two random variables. Then the variance of their sum is :

$$\text{Var}[X_1 + X_2] = \text{Var}[X_1] + \text{Var}[X_2] + 2\,\text{Cov}[X_1, X_2]$$

Proof :

$$\text{Var}[X_1 + X_2] = E[(X_1 + X_2 - E[X_1 + X_2])^2]$$

$$= E[((X_1 - E[X_1]) + (X_2 - E[X_2]))^2]$$

$$= E[(X_1 - E[X_1])^2 + (X_2 - E[X_2])^2 + 2(X_1 - E[X_1])(X_2 - E[X_2])]$$

$$= E[(X_1 - E[X_1])^2] + E[(X_2 - E[X_2])^2] + 2E[(X_1 - E[X_1])(X_2 - E[X_2])]$$

$$= Var[X_1] + Var[X_2] + 2\,Cov[X_1, X_2]$$

- The covariance operator is symmetric :

$$Cov(X, Y) = Cov(Y, X)$$

Proof : Using the definition of covariance :

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[(Y - E[Y])(X - E[X])] = Cov[Y, X]$$

Q.8 Explain the difference between Standard Deviation and Variance.

Ans. :

Standard Deviation	Variance
Standard deviation is a measure of dispersion of the values of a data set from their mean.	It is the statistical measure of how far the numbers are spread in a data set from their average.
It is a common term in statistical theory to calculate central tendency.	Variance is primarily used for statistical probability distribution to measure volatility from the mean.
It measures the absolute variability of the dispersion.	It helps determine the size of the data spread.
It is calculated by taking the square root of the variance.	It is calculated by taking the average of the squared deviation of each value in the data set from the mean.
The standard deviation is symbolized by the Greek letter sigma "σ" as in lower case sigma	The notation for the variance of a variable is "σ^2" sigma squared.
$\sigma = \sqrt{\sum \frac{(x-M)^2}{n}}$ where M = mean, x = a value in a data set, and n = number of values	$\sigma^2 = \sum \frac{(x-M)^2}{n}$ where M = mean, x = each value in the data set, n = number of values in the data set
Used in finance sector as a measure of market and security volatility.	Used in asset allocation

Q.9 Example: Following data related to the number of telegraphic transfers per day by a bank branch for 300 working days :

Number of telegraphic transfer per day	0	1	2	3	4	5	6	7
Number of days	10	35	45	95	64	32	10	9

Calculate Arithmetic mean, and median.

Ans. :

X_i	$X_1 = 0$	$X_2 = 1$	$X_3 = 2$	$X_4 = 3$	$X_5 = 4$	$X_6 = 5$	$X_7 = 6$	$X_8 = 7$
f_i	$f_1 = 10$	$f_2 = 35$	$f_3 = 45$	$f_4 = 95$	$f_5 = 64$	$f_6 = 32$	$f_7 = 10$	$f_8 = 9$
$X_i f_i$	0	35	90	285	256	160	60	63

$$\text{Arithmetic mean } (\bar{X}) = \frac{\Sigma X_i f_i}{N}$$

$$\text{Arithmetic mean } (\bar{X}) = \frac{X_1 f_1 + X_2 f_2 + X_3 f_3 + X_4 f_4 + X_5 f_5 + X_6 f_6 + X_7 f_7 + X_8 f_8}{N}$$

$$N = \sum f = 10 + 35 + 45 + 95 + 64 + 32 + 10 + 9 = 300$$

$$\text{Arithmetic mean } (\bar{X}) = \frac{0 + 35 + 90 + 285 + 256 + 160 + 60 + 63}{300} = \frac{949}{300} = 3.163$$

Median :

Number of telegraphic transfer per day	Number of days	Cumulative frequency
0	10	10
1	35	45
2	45	90
3	95	185
4	64	249
5	32	281
6	10	291
7	9	300
	$\Sigma f = 300$	

$$N = 300$$

$$N/2 = 300/2 = 150^{\text{th}} \text{ number}$$

We select nearest number i.e. 3

$$\text{Median} = (2 + 3) / 2 = 2.5$$

Q.10 The portfolio manager of industrial investment corporation proposed the following portfolio of securities. Calculate the average return and the standard deviation for this portfolio.

Security	Proportion of funds to be invested (Z_1)	Return in % (X_1)
51	0.15	8
52	0.10	15
53	0.35	25
54	0.30	20
55	0.10	10

Ans. :

$$\text{Average } (\bar{X}) = \sum Z_i X_i = 0.15 \times 8 + 0.10 \times 15 + 0.35 \times 25 + 0.30 \times 20 + 0.10 \times 10 = 18.45$$

$$\text{Variance} = \sum Z_i (X_i - \bar{X})^2$$

$= 0.15(8-18.45)^2 + 0.10(15-18.45)^2 + 0.35(25-18.45)^2 + 0.30(20-18.45)^2 + 0.10(10-18.45)^2$

$= 16.38 + 1.19 + 15.01 + 0.72 + 7.14 = 40.44$

2.2 : Means and Variances of Linear Combinations of Random Variables

Q.11 Let X_1 and X_2 be independent random variables. Suppose the mean and variance of X_1 are 2 and 4, respectively. Suppose, the mean and variance of X_2 are 3 and 5 respectively.

a) What is the mean and variance of $X_1 + X_2$?

b) What is the mean and variance of $X_1 - X_2$?

c) What is the mean and variance of $3X_1 + 4X_2$?

Ans. : a) Mean and variance of $X_1 + X_2$

The mean of the sum : $E(X_1 + X_2) = E(X_1) + E(X_2) = 2 + 3 = 5$

The variance of the sum : $Var(X_1 + X_2) = (1)^2 Var(X_1) + (1)^2 Var(X_2) = 4 + 5 = 9$

b) Mean and variance of $X_1 - X_2$

The mean of the difference : $E(X_1 - X_2) = E(X_1) - E(X_2) = 2 - 3 = -1$

The variance of the difference : $Var(X_1 - X_2) = Var(X_1 + (-1)X_2)$

$= (1)^2 Var(X_1) + (-1)^2 Var(X_2)$

$= 4 + 5 = 9$

That is, the variance of the difference in the two random variables is the same as the variance of the sum of the two random variables.

c) Mean and variance of $3X_1 + 4X_2$

Mean of the linear combination is : $E(3X_1 + 4X_2) = 3E(X_1) + 4E(X_2) = 3(2) + 4(3) = 18$

Variance of the linear combination : $Var(3X_1 + 4X_2)$

$= (3)^2 Var(X_1) + (4)^2 Var(X_2)$

$= 9(4) + 16(5) = 116$

Q.12 What is linear function of random variable ?

Ans. : If X is a random variable with mean μ_X and variance σ_X^2 and a and b are numerical constants, the random variable y defined by y = a + bx is called a linear function of the random variable x.

Q.13 What are the rules for means and variances for linear combinations ?

Ans. : Rules for means :

Rule 1 : If X is a random variable and a and b are fixed numbers, then

$$\mu_{a+bX} = a + b\mu_X$$

Rule 2 : If X and Y are random variables, then

$$\mu_{X+Y} = \mu_X + \mu_Y$$

Rules for variances :

Rule 1 : If X is a random variable and a and b are fixed numbers, then

$$\sigma^2_{a+bX} = b^2\sigma^2_X$$

Rule 2 : If X and Y are independent random variables, then

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$

$$\sigma^2_{X-Y} = \sigma^2_X - \sigma^2_Y$$

This is the addition rule for variances of independent random variables.

Q.14 What is linear combination of X ?

Ans. : If $x_1, x_2, ..., x_n$ are random variables and $a_1, a_2, ..., a_n$ are numerical constants, the random variables y defined as

$$y = a_1x_1 + a_2x_2 + ... + a_nx_n$$

is a linear combination of the $X_i'S$.

Q.15 A distributor of fruit baskets is going to put 4 apples, 6 oranges and 2 bunches of grapes in his small gift basket. The weights, in ounces, of these items are the random variables x_1, x_2 and x_3 respectively with means and standard deviations as given in the following table

	Apples	Oranges	Grapes
Mean (μ)	8	10	7
Standard deviation (σ)	0.9	1.1	2

Find the mean, variance and standard deviation of the random variable y = weight of fruit in a small gift basket.

Ans. :

$$a_1 = 4,\ a_2 = 6,\ a_3 = 2,$$

$$\mu_1 = 8,\ \mu_2 = 10,\ \mu_3 = 7,$$

$$\sigma_1 = 0.9,\ \sigma_2 = 1.1,\ \sigma_3 = 2$$

$$\mu_y = \mu_{a_1x_1+a_2x_2+a_3x_3}$$

$$= a_1\mu_1 + a_2\mu_2 + a_3\mu_3$$

$$= 4(8) + 6(10) + 2(7) = 106$$

$$\sigma^2_y = \sigma^2_{a_1x_1+a_2x_2+a_3x_3}$$

$$= a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + a_3^2\sigma_3^2$$

$$= 4^2(0.9)^2 + 6^2(1.1) + 2^2(2)^2 = 72.52$$

$$\sigma_y = \sqrt{75.52} = 8.5159$$

Q.16 Is $f(x) = \frac{1}{2}X^2e^{-x}$ when $X \geq 0$ can be regarded as a probability function for a continuous random variable ? If, so find mean and variance of the random variable.

Ans. : i) Mean : If X is a continuous random variable then mean = E[x].

$$= \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x\left(\frac{1}{2}x^2e^{-x}\right)dx$$

$$= \frac{1}{2}\int_0^{\infty} x^3e^{-x}dx$$

$$= \frac{1}{2}\left[-x^3e^{-x} + \int 3x^2e^{-x}dx\right]_0^{\infty}$$

$$= \frac{1}{2}\left[-x^3e^{-x} + 3\left[-x^2e^{-x} + \int 2xe^{-x}dx\right]\right]_0^{\infty}$$

$$= \frac{1}{2}\left[-x^3e^{-x} + 3\left[-x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}dx\right]\right]\right]_0^{\infty}$$

$$= \frac{1}{2}\left[-x^3e^{-x} + 3\left[-x^2e^{-x} - 2\,xe^{-x} - 2e^{-x}\right]\right]_0^{\infty}$$

$$= \frac{1}{2}\left[-x^3e^{-x} - 3x^2e^{-x} - 6\,xe^{-x} - 6e^{-x}\right]_0^{\infty}$$

$$= \frac{1}{2}[(-0-0-0-0)-(0-0-0-6)] = \frac{6}{2}$$

Mean = 3

ii) Variance

$$\sigma_x^2 = \text{Var}(x) = E(x)^2 - [E(x)]^2$$

$$E(x^2) = \int_0^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2\left(\frac{1}{2}x^2\right)e^{-x}dx$$

$$= \frac{1}{2}\int_0^{\infty} x^4e^{-x}dx$$

$$= \frac{1}{2}\left[-x^4e^{-x} + \int 4x^3e^{-x}dx\right]$$

$$= \frac{1}{2}\left[-x^4e^{-x} + 4(-x^3)e^{-x} + \int 3x^2e^{-x}dx\right]$$

$$= \frac{1}{2}\left[-x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x}\right]_0^{\infty}$$

$$= \frac{1}{2}(0-4)-12(0)-24(0)-24(0)-[0-4(0)-12(0)-24(0)-24(1)]$$

$$= \frac{1}{2}[0-(-24)] = \frac{24}{2}$$

$$E(x^2) = 12$$

$$\text{Variance} = E(x^2)-[E(x)]^2 = 12-[3]^2 = 12-9$$

$$\text{Variance} = 3$$

2.3 : Chebyshev's Theorem

Q.17 Explain Chebyshev's Theorem.

Ans. : • The Chebyshev inequality is a statement that places a bound on the probability that an experimenta value of a random value of a random variable X with finite mean $E[X] = \mu_X$ and variance σ_X^2 will differ from th mean by more than a fixed number a.

- The statement says that the bound is directly proportional to the variance and inversely proportional to a^2.

$$P[|X-E[X]| \geq a] \leq \frac{\sigma_X^2}{a^2} \; a > 0$$

This is a loose bound that can be obtained as follows :

$$\sigma_X^2 = \int_{-\infty}^{\infty}(x-\mu x)^2 f_x(x)\,dx$$

$$= \int_{-\infty}^{-(\mu_X-a)}(x-\mu_X)^2 f_x(x)dx + \int_{-(\mu_X-a)}^{\mu_X-a}(x-\mu_X)^2 f_x(x)dx + \int_{\mu_X-a}^{\infty}(x-\mu_X)^2 f_x(x)dx$$

$$\therefore \quad \sigma_x^2 \geq \int_{-\infty}^{-(\mu_X-a)}(x-\mu_X)^2 f_x(x)\,dx + \int_{\mu_X-a}^{\infty}(x-\mu_X)^2 f_x(x)dx$$

$$= \int_{|x-\mu_X|\geq a}(x-\mu_X)^2 f_x(x)dx \geq \int_{|x-\mu_X|\geq a} a^2 f_x(x)\,dx$$

$$= a^2\int_{|x-\mu_X|\geq a} f_x(x)dx = a^2 P[|X-\mu_X|\geq a]$$

Q.18 A random variable X has a mean of 4 and a variance of 2. Use the Chebshev inequality of obtain an upper bound for $P[|X-4| \geq 3]$.

Ans. : From the Chebyshev inequality

$$P[|X-4| \geq 3] \leq \sigma^2_{X/(3)^2} = 2/9$$

Q.19 A random variable X has mean 12 and variance 9 and an unknown probability distribution. Using Chebyshev's theorem, estimate

a) $P(6 < X < 18)$

b) $P(3 < X < 21)$

Ans. : a) $P(6 < X < 18)$:

$$P(6 < X < 18) = P(6-\mu < X-\mu < 18-\mu)$$

$= P(6 - 12 < X - 12 < 18 - 12)$

$= P(-6 < X - 12 < 6)$

$= P(|X - 12| < 6)$

We have,

$P(|X - 12| < 6) = P[|X - 12| < 2(3)] > 1 - 1/(2)^2$

$P(6 < X < 18) = P(|X - 12| < 6) > 3/4$

b) $P(3 < X < 21)$

$P(3 < X < 21) = P(3 - \mu < X - \mu < 21 - \mu)$

$= P(3 - 12 < X - 12 < 21 - 12)$

$= P(-9 < X - 12 < 9)$

$= P(|X - 12| < 9)$

We have,

$P(|X - 12| < 9) = P[|X - 12| < 3(3)] > 1 - 1/(3)^2$

$P(3 < X < 21) = P(|X - 12| < 9) > 8/9$

2.4 : Discrete Probability Distributions

Q.20 What is probability distribution ?

Ans. : The behavior of a random variable is characterized by its probability distribution, that is, by the way probabilities are distributed over the values it assumes. A probability mass function are two ways to characterize this distribution for a discrete random variable.

Q.21 What is standard deviation ?

Ans. : The standard deviation measure variability and consistency of the sample or population. In most real-world applications, consistency is a great advantage. In statistical data analysis, less variation is often better. The sample standard deviation will be denoted by "s" and the population standard deviation will be denoted by the Greek letter σ . The standard deviation is the square root of the variance.

Q.22 Define median.

Ans. : The median is the middle number of a set of numbers arranged in numerical order. If the number of values in a set is even, then the median is the sum of the two middle values, divided by 2.

Q.23 What is poisson distribution ?

Ans. : • In a binomial distribution, when the number of trials n is large and the probability of success p is small, the distribution approaches the Poisson distribution.

- In the Poisson distribution, the probability of x successes is given by the equation

$$P(\text{x successes}) = \frac{\mu^x}{X!} e^{-\mu}$$

where μ is the mean.

Q.24 What is Binomial distribution ? Explain its properties. Where binomial distribution is applied ?

Ans. : • The Binomial distribution gives the general form of the probability distribution for the random variable r, whether it represents the number of heads in n coin tosses or the number of hypothesis errors in a sample of n examples.

- The detailed form of the Binomial distribution depends on the specific sample size n and the specific probability p or $error_D(h)$.
- Binomial distribution applies as follows :
  1. There is a base, or underlying, experiment whose outcome can be described by a random variable (Y). The random variable can take on two possible values.
  2. The probability that Y = 1 on any single trial of the underlying experiment is given by some constant p, independent of the outcome of any other experiment.
  3. A series of n independent trials of the underlying experiment is performed, producing the sequence of independent, identically distributed random variables $Y_1, Y_2, \ldots, Y_n$.
- Binomial means 'two numbers'.
- The outcomes of health research are often measured by whether they have occurred or not. For example, recovered from disease, admitted to hospital, died etc.
- The binomial distribution occurs in games of chance, quality inspection, opinion polls, medicine and so on.

- It may be modelled by assuming that the number of events 'n' has a binomial distribution with a fixed probability of event p. Binomial distribution is distribution for a series of Bernoulli trials.
- Properties of binomial distribution :
  1) Experiment consist of n identical trials.
  2) Each trail has only two outcomes.
  3) The probability of one outcome is p and the other is $q = 1 - p$.
  4) The trials are independent.
  5) We are interested in x, the number of success observed during the n trials.
- Binomial distribution written as B (n, p) where n is the total number of events and p = probability of an event.
- A Binomial distribution gives the probability of observing r heads in a sample of n independent coin tosses, when the probability of heads on a single coin toss is p. It is defined by the probability function

$$P(r) = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}$$

- The mean μ (mu) of the binomial distribution is

$$\mu = np$$

- The variance is,

$$\sigma^2 = npq$$

- The mean and variance of binomial distribution with parameters (n, p) are given as,

$$\text{Mean} = \mu = E(X) = \sum_{i=1}^{n} E(X_i)$$

$$= np$$

Q.25 Comment on the following : "the mean of a binomial distribution is 3 and variance is 4".

Ans. : Here mean = 3, variance = 4

$$\frac{\text{Variance}}{\text{Mean}} = \frac{4}{3} = 1.333 = q > 1$$

Which is not possible. Hence, there cannot be a binomial distribution with mean 3 and variance 4.

Q.26 Consider the example of the Binomial distribution

X	0	1	2	3	4	5
P (X = x)	0.004	0.041	0.165	0.329	0.329	0.132

Calculate the mean value of distribution.

Ans. :

$$\mu = xP(X=0) + xP(X=1) + xP(X=2) + xP(X=3) + xP(X=4) + xP(X=5)$$

$$= 0\times(0.004) + 1\times(0.0041) + 2\times(0.165) + 3\times(0.329) + 4\times(0.329) + 5\times(0.132)$$

$$= 0 + 0.0041 + 0.33 + 0.987 + 1.316 + 0.66 = 3.2971$$

Q.27 Find the variance and standard deviation for the following set of test marks T = {75, 80, 82, 87, 96}

Ans. :

$$\text{Mean} = \frac{75+80+82+87+96}{5}$$

$$= \frac{420}{5}$$

$$\text{Mean} = 84$$

$$\text{Variance} = \frac{[(x_1-\mu)^2 + (x_2-\mu)^2 + \ldots + (x_n-\mu)^2]}{n}$$

$$\sigma^2 = \frac{[(75-84)^2 + (80-84)^2 + (82-84)^2 + (87-84)^2 + (96-84)^2}{5}$$

$$= \frac{(-9)^2 + (-4)^2 + (-2)^2 + (3)^2 + (12)^2}{5}$$

$$= \frac{81+16+4+9+144}{5} = \frac{254}{5}$$

$$\sigma^2 = 50.8$$

Standard Deviation (σ)

$$\sigma = \sqrt{\sigma^2} = \sqrt{50.8} = 7.1274$$

Q.28 Find the probability of getting an even number 3 or 4 or 5 times in throwing 10 dice using binomial distribution.

Ans. :

P = Probability of getting even number in throw of a die.

$$P = \frac{3}{6} = \frac{1}{2}$$

$$q = 1 - p = 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

n = 10 (Given data)

x = Probability of getting even number

$$P(X = x) = {}^{10}C_x p^x q^{n-x}$$

Substituting value of p, q, n, we get

$$= {}^{10}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-n}$$

$$= {}^{10}C_x \left(\frac{1}{2}\right)^{10} \text{ (Where } x = 0, 1, 2, 3, \ldots, 10)$$

$$P(X=3) = {}^{10}C_3\left(\frac{1}{2}\right)^{10} = \frac{10!}{3!\,(10-3)!}\times\frac{1}{1024}$$

$$= \frac{3628800}{6\times 5040}\times\frac{1}{1024}$$

$$= \frac{120}{1024}$$

$$P(X=3) = 0.11718$$

$$P(X=4) = {}^{10}C_4\left(\frac{1}{2}\right)^{10} = \frac{10!}{4!\,(10-4)!}\times\frac{1}{1024}$$

$$= \frac{3628800}{24\times 720}\times\frac{1}{1024} = \frac{210}{1024}$$

$$P(X=4) = 0.2050$$

$$P(X=5) = {}^{10}C_5\left(\frac{1}{2}\right)^{10} = \frac{10!}{5!\,(10-5)!}\times\frac{1}{1024}$$

$$= \frac{3628800}{120\times 120}\times\frac{1}{1024} = \frac{252}{1024}$$

$$= 0.246$$

Q.29 A random variable X has the following probability function :

x	0	1	2	3	4	5	6	7
P (x)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Determine :

i) K

ii) Evaluate $P(X<6)$, $P(X\ge 6)$, $P(0<X<5)$ and $P(0\le X\le 4)$.

iii) If $P(X\le K)>\frac{1}{2}$ find the minimum value of K.

iv) Determine the distribution function of X.

v) Mean vi) Variance.

[JNTU : May-10, Nov-...

Ans. : i) K

$$\sum_{x=0}^{7} P(x) = 1$$

$$K+2K+2K+3K+K^2+2K^2+7K^2+K=1$$

$$10K^2+9K-1 = 0$$

$$(K+1)(10K-1) = 0$$

$$K+1 = 0 \quad \text{and} \quad 10K-1=0$$

$$K = -1 \quad \text{and} \quad K=\frac{1}{10}$$

We discard K = – 1 value. Therefore $K = \frac{1}{10} = 0.1$.

ii) $P(X < 6) = P(X=0) + P(X=1) + P(X=2) + \ldots + P(X = 5)$

$= 0 + K + 2K + 2K + 3K + K^2$

Put $K = 0.1 = 0 + 0.1 + 2(0.1) + 2(0.1) + 3(0.1) + (0.1)^2$

$= 0.1 + 0.2 + 0.2 + 0.3 + 0.01$

$P(X < 6) = 0.81$

$P(X \geq 6) = 1 - P(X < 6)$

$= 1 - 0.81$

$P(X \geq 6) = 0.19$

$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= K + 2K + 2K + 3K$

$= 8K$

$= 8 \times 0.1 \qquad (K = 0.1)$

$P(0 \leq X < 5) = 0.8$

$P(0 \leq X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 0 + K + 2K + 2K + 3K$

$= 8K = 8 \times 0.1$

$P(0 \leq X \leq 4) = 0.8$

iii) If $P(X \leq K) > \frac{1}{2}$, minimum value of K.

$P(X \leq 1) = P(X = 0) + P(X = 1) = 0 + K$

$= K$

$= 0.1$

$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0 + K + 2K$

$= 3K = 3 \times 0.1$

$= 0.3$

$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= 0 + K + 2K + 2K$

$= 5K = 5 \times 0.1$

$= 0.5$

$P(X \leq 4) = 0.8$ (We already calculated)

But the condition is $P(X \le K) > \frac{1}{2}$.

So K = 4 is suitable for this minimum value of K = 4.

iv) Distribution function of X.

X	F (X) = P (X≤x)
0	0
1	0.1
2	0.3
3	0.5
4	0.8
5	0.81
6	0.83
7	$9K + 10K^2 = 1$

v) Mean (μ)

$$\mu = \sum_{i=0}^{7} p_i x_i = 0(0) + 1(K) + 2(2K) + 3(2K) + 4(3K) + 5(K^2) + 6(2K^2) + 7(7K^2 + K)$$

$$= 0 + K + 4K + 6K + 12K + 5K^2 + 12K^2 + 49K^2 + 7K$$

$$= 30K + 66K^2$$

Substitute $K = 1/10 = 30 \times \frac{1}{10} + 66 \times \left(\frac{1}{10}\right)^2$

$$= \frac{30}{10} + \frac{66}{100}$$

$$= 3 + 0.66$$

$$\mu = 3.66$$

vi) Variance (σ^2)

$$\sigma^2 = \sum_{i=0}^{7} p_i\, x_i^2 - \mu^2 = K + 8K + 18K + 48K + 25K^2 + 72K^2 + 343K^2 + 49K - (3.66)^2$$

$$= 440K^2 + 124K - 13.3956 = 440\,(0.1)^2 + 124\,(0.1) - 13.3956$$

$$= 4.4 + 12.4 - 13.3956$$

$$\sigma^2 = 3.4044$$

Q.30 The mean of binomial distribution is 3 and variance is 9/4. Find
i) The value of n
ii) $p(x \ge 7)$
iii) $p(1 \le x \le 6)$

[JNTU : Jan.-14]

Ans. : Given data :

$$\mu = 3 \quad \sigma^2 = \frac{9}{4} = npq$$

i) Value of n

$$npq = \frac{9}{4}$$

$$3q = \frac{9}{4}$$

$$q = \frac{9}{4} \times \frac{1}{3} = \frac{3}{4}$$

$$p = 1 - q$$

$$= 1 - \frac{3}{4}$$

$$p = \frac{1}{4}$$

$$np = 3$$

$$n \times \frac{1}{4} = 3$$

$$n = 12$$

ii) $P(x \geq 7) = P(x=7) + P(x=8) + P(x=9) + P(x=10) + P(x=11) + P(x=12)$

Using binomial distribution

$$= {}^nC_x p^x q^{n-x}$$

$$P(x \geq 7) = {}^{12}C_7 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^{12-7} + {}^{12}C_8 \left(\frac{1}{4}\right)^8 \left(\frac{3}{4}\right)^{12-8}$$

$$+ {}^{12}C_9 \left(\frac{1}{4}\right)^9 \left(\frac{3}{4}\right)^{12-9} + {}^{12}C_{10} \left(\frac{1}{4}\right)^{10} \left(\frac{3}{4}\right)^{12-10}$$

$$+ {}^{12}C_{11} \left(\frac{1}{4}\right)^{11} \left(\frac{3}{4}\right)^{12-11} + {}^{12}C_{12} \left(\frac{1}{4}\right)^{12} \left(\frac{3}{4}\right)^{12-12}$$

$$= \frac{1}{(4)^{12}} [792\,(3)^5 + 495\,(3)^4 + 220\,(3)^3 + 66\,(3) + 12\,(3) + 1]$$

$$= \frac{1}{(4)^{12}} [192456 + 40095 + 5940 + 594 + 36 + 1]$$

$$= \frac{239122}{16777216}$$

$$P(x \geq 7) = 0.0142$$

iii) $P(1 \leq x < 6) = P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5)$

Using binomial distribution

$$= {}^{12}C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{12-1} + {}^{12}C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{12-2} + {}^{12}C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{12-3}$$

$$+ {}^{12}C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^{12-4} + {}^{12}C_5\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^{12-5}$$

$$= 0.1267 + 0.2322 + 0.2581 + 0.1935 + 0.1032$$

$$= 0.9137$$

Q.31 A coin is tossed 10 times. Find the probability of getting between 4 and 7 heads inclusive using a) using binomial distribution and b) the normal approximation to the binomial distribution.

Ans. : • Let X be the random variable that denotes the number of heads in 10 tosses of the coin. Then we are required to find $P[4 \le X \ge 7]$.

a) using binomial distribution :

$$p[4 \le X \ge 7] = \sum_{x=4}^{7} P_X(x) = \sum_{x=4}^{7} \binom{10}{x}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{10-x}$$

$$= \left(\frac{1}{2}\right)^{10} \sum_{x=4}^{7} \binom{10}{4}$$

$$= \frac{\frac{10!}{4!6!} + \frac{10!}{5!5!} + \frac{10!}{6!4!} + \frac{10!}{7!3!}}{1024}$$

$$= \frac{792}{1024} = 0.7734$$

b) using normal approximation to the binomial distribution :

$$np = 5$$

$$np(1-p) = 2.5$$

$$p[4 \le X \ge 7] = P\left[\frac{4-5}{\sqrt{2.5}} \le Z \le \frac{7-5}{\sqrt{2.5}}\right]$$

$$= P[-0.63 \le Z \le 1.26]$$

$$= \phi(1.26) - \phi(-0.63)$$

$$= \phi(1.26) + \phi(0.63) - 1$$

$$= 0.8962 + 0.7357 - 1$$

$$= 0.6319$$

Q.32 The mean and variance of a Binomial distribution are 4 and 4/3 respectively. Find $P(X \ge 1)$.

Ans. : Given data :

$$np = 4,$$

$$npq = \frac{4}{3}$$

$$q = \frac{4}{3} \times \frac{1}{np}$$

$$= \frac{4}{3} \times \frac{1}{4} = \frac{1}{3}$$

$$p = 1 - q$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$$\therefore \quad np = 4 \Rightarrow n = \frac{4}{p} = \frac{4 \times 3}{2} = 6$$

$$\therefore \quad P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - q^6$$

$$= 1 - \left(\frac{1}{3}\right)^6 = 1 - \frac{1}{729}$$

$$= 0.9986$$

Q.33 The mean of binomial distribution is 3 and the variance is 9/4. Find
i) the value of n
ii) $P(X > 7)$
iii) $P(1 \leq X \leq 6)$.

Ans. : Given data :

$$\text{mean} = np = 3$$

$$\text{variance} = npq = \frac{9}{4}$$

$$q = \frac{9}{4} \times \frac{1}{np}$$

$$= \frac{9}{4} \times \frac{1}{3}$$

$$q = \frac{3}{4}$$

$$p = 1 - q$$

$$= 1 - \frac{3}{4}$$

$$p = \frac{1}{4}$$

i) The value of n

$$np = 3$$

$$n = 3 \times \frac{1}{p}$$

$$= 3 \times \frac{4}{1}$$

$$n = 12$$

ii) $$P(X > 7) = P(X \geq 8)$$

$$= \sum_{x=8}^{12} 12_{C_x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{12-x}$$

$$= 0.0032$$

iii) $$P(1 \leq X < 6) = P(1 \leq X \leq 5)$$

$$= P(X \leq 5) - P(X = 0)$$

$$= \left[\sum_{x=0}^{5} 15_{C_x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{12-x}\right] - P(X=0)$$

$$= 0.9456 - 0.0317$$

$$= 0.9139$$

Q.34 Assume that X is a continuous random variable with the following PDF :

$$f_X(x) = \begin{cases} A(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

a) What is the value of A ?
b) Find $P[X > 1]$.

Ans. : a) Since, $f_X(x)$ is a PDF, we have that

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} A(2x - x^2)\,dx + \int_{2}^{\infty} 0\,dx$$

$$= \int_{0}^{2} A(2x - x^2)\,dx$$

$$= 1$$

Thus, we obtain,

$$A\left[x^2 - \frac{x^3}{3}\right] = 1$$

$$A\left[4 - \frac{8}{3}\right] = \frac{4A}{3} = 1$$

$$A = \frac{3}{4}$$

b) $$P[X > 1] = \int_{1}^{\infty} f_X(x)\,dx$$

$$= \frac{3}{4}\int_{1}^{2} (2x - x^2)\,dx$$

$$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \frac{3}{4}\left[\frac{4}{3} - \frac{2}{3}\right]$$

$$= \frac{3}{4}\left[\frac{4-2}{3}\right]$$

$$= \frac{1}{2}$$

Q.35 Write short note on Poisson distribution.

Ans. : • Poisson distribution, named after its invertor simeon poisson who was a French mathematician. He found that if we have a rare event (i.e. p is small) and we know the expected or mean (or μ) number of occurances, the probabilities of 0, 1, 2 ... events are given by :

$$\boxed{P(R) = \frac{e^{-\mu}\mu^R}{R!}}$$

Poisson distribution : Is a distribution the number of rare events that occur in a unit of time, distance, space and so on.

Examples :

1. Number of insurance claims in a unit of time.
2. Number of accidents in a ten-mile highway.
3. Number of airplane crash in triangle area.

- When there is a large number of trials, but a small probability of success, binomial calculate becomes impractical. Example : Number of deaths from horse kicks in the army in different years. The mean number of successes from n trials is $\mu = np$.
- If we substitute μ/n for p, and let n tend to infinity, the binomial distribution becomes the Poisson distribution :

$$P(x) = \frac{e^{-\mu}\mu^x}{x!}$$

- Poisson distribution is applied where random events in space or time are expected to occur. Deviation from poisson distribution may indicate some degree of non-randomness in the events under study.
- Example : 64 deaths in 20 years from thousands of soldiers.
- If a mean or average probability of an event happening per unit time/per page/per mile cycled etc., is given and you are asked to calculate a probability of n events happening in a given time/number of pages/number of miles cycled, then the **Poisson distribution** is used.
- If on the other hand, an exact probability of an event happenng is given, or implied, in the question, and you are asked to calculate the probability of this event happening k times out n, then the **Binomial distribution** must be used.

Q.36 What is geometric distribution ?

Ans. : • The geometric distribution represents the number of failures before you get a success in a series of Bernoulli trials. This discrete probability distribution is represented by the probability density function: $f(x) = (1-p)^{x-1}\,p$.

- Instead of counting the number of successes, we can also count the number of trials until a success is obtained. That is, we shall let the random variable X represent the number of trials needed to obtain the first success.
- In this situation, the number of trials will not be fixed. But if the trials are still independent, only two outcomes are available for each trial, and the probability of a success is still constant, then the random variable will have a geometric distribution.
- In a geometric distribution, if p is the probability of a success, and x is the number of trials to obtain the first success, then the following formulas apply.

$$P(X) = p(1-p)^{x-1}$$

Q.37 What is Mean and Variance for geometric distribution ?

Ans. : If X is a geometric random variable with parameter p, then

$$\text{Mean } \mu = E(X) = 1/p$$

$$\text{Variance } \sigma^2 = V(X) = \frac{1-p}{p^2}$$

Q.38 A test of weld strength involves loading welded joints until a fracture occurs. For a certain type of weld, 80% of the fractures occur in the weld itself, while the other 20% occur in the

beam. A number of welds are tested and the tests are independent. Let X be the number of test at which the first beam fracture is observed.

1. Find $P(X \geq 3)$ (i.e. Find the probability that the first beam fracture happens on the third trial or later.)

2. Find E(X) and V (X) and the standard deviation of X.

Ans. : 1. $P(X \geq 3)$

Either a weld fracture or a beam fracture will occur on each Bernoulli trial. We'll call a success a beam fracture. X is a geometric random variable with $p = 0.20$.

$$P(X \geq 3) = 1 - P(X < 3) \quad \text{(complement)}$$

$$= 1 - [P(X = 1) + P(X = 2)]$$

$$= 1 - [0.20 + (0.80)(0.20)]$$

$$= 0.64$$

2) E(X) and V(X) and the standard deviation of X

$$E(X) = \frac{1}{p} = \frac{1}{0.20} = 5$$

$$\sigma^2 = \frac{1-p}{p^2} = \frac{0.8}{(0.2)^2} = 20$$

$$\sigma = \sqrt{(X)} = \sqrt{20} = 4.47$$

Q.39 List assumptions for the geometric distribution

Ans. : The three assumptions are :

1. There are two possible outcomes for each trial (success or failure).
2. The trials are independent.
3. The probability of success is the same for each trial.

Q.40 If your probability of success is 0.2, what is the probability you meet an independent voter on your third try ?

Ans. : Inserting 0.2 as p and with X = 3, the probability density function becomes :

$$f(X) = (1-p)^{x-1} * p$$

$$P(X = 3) = (1-0.2)^{3-1}(0.2)$$

$$P(X = 3) = (0.8)^2 * 0.2 = 0.128$$

Q.41 Explain the distribution function of a geometric random variable X.

Ans. : The distribution function of a geometric random variable X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-(1-p)^{x-1} & \text{if } x \geq 0 \end{cases}$$

- For $x < 0$, $F_X(X) = 0$, because x cannot be smaller than 0. For $x \geq 0$, we have

$$F_X(x) = P(X \leq x) = \sum_{y=0}^{x}(1-p)^7 p$$

$$= p\sum_{y=0}^{x}(1-p)^7 = p\frac{1-(1-p)^{x+1}}{1-(1-p)}$$

$$= 1-(1-p)^{x+1}$$

Q.42 Prove that expected value of a geometric random variable X is $E[X] = \frac{1-p}{p}$.

Ans. : $E[X] = \sum_{x=0}^{\infty}(1-p)^x px$

$$= (1-p)p\sum_{x=0}^{\infty}(1-p)^{x-1}x$$

$$= -(1-p)p\sum_{x=0}^{\infty}\frac{d}{dp}(1-p)^x$$

$$= -(1-p)p\frac{d}{dp}\sum_{x=0}^{\infty}(1-p)^x$$

$$= -(1-p)p\frac{d}{dp}\frac{1}{1-(1-p)}$$

$$= -(1-p)p\frac{d}{dp}\frac{1}{p} = -(1-p)p(-p^{-2})$$

$$= (p-p^2)p^{-2} = \frac{1}{p}-1$$

$$= \frac{1-p}{p}$$

Q.43 Explain the variance of a geometric random variable.

Ans. : Variance formula is as follows

$$Var[X] = E[X^2]-E[X]^2$$

$$= \frac{2-3_p+p^2}{p^2} - \left(\frac{1-p}{p}\right)^2$$

$$= \frac{2-3_p+p^2-(1-2_p+p^2)}{p^2}$$

$$= \frac{2-3p+p^2-1+2p-p^2}{p^2} = \frac{1-p}{p^2}$$

Q.44 What is the difference between binomial and geometric distribution ?

Ans. : Geometric setting counts the number of trials until an event of interest happens. Binomials settings have a fixed set of n trials.

Q.45 If X is a Poisson variate with mean λ show that $E[X^2] = \lambda E[X+1]$.

Ans. :

$$E[X^2] = \lambda^2 + \lambda$$

$$E(X+1) = E(X) + 1$$

$$E[X^2] = \lambda E[X+1]$$

Q.46 A random variable X takes the values 0, 1, 2, 3 and its mean is 13. If P(X = 3) = 2P(X = 1) an P(X = 2) = 0.3, find P(X = 0).

Ans. : Given that

$$\text{Mean} = \sum X_k P(X=k) = 1.3$$

$$X_0P(X=0)+X_1P(X=1)+X_2P(X=2)+X_3P(X=3) = 1.3$$

$\Rightarrow 0.P(X = 0) + 1.P(X = 1) + 2.P(X = 2) + 3.P(X = 3)$

$= 1.3$

$\Rightarrow P(X = 1) + 2(0.3) + 3.2P(X = 1) = 1.3$

$\Rightarrow 7P(X = 1) = 0.7 \Rightarrow P(X = 1) = 0.1$

Now, $P(X = 3)\ 2P(X = 1) = 2(0.1) = 0.2$

Also $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$

$\Rightarrow P(X = 0) + 0.1 + 0.3 + 0.2 = 1 \Rightarrow P(X = 0) = 1 - 0.6$

$= 0.4$

Q.47 If the mean of a Poisson distribution is 1/2 then find the ratio of P(X = 3) to P(X = 2).

Ans. :

Given that $\lambda = \frac{1}{2}$, Now $P(X = n) = \frac{\lambda^n}{n!} e^{\lambda}$

$$\therefore \quad P(X=3) = \frac{\left(\frac{1}{2}\right)^3}{3!} e^{1/2} \text{ and } P(X=2) = \frac{\left(\frac{1}{2}\right)^2}{2!} e^{1/2}$$

$$\frac{P(X=3)}{P(X=2)} = \frac{\dfrac{\left(\frac{1}{2}\right)^3 e^{1/2}}{3!}}{\dfrac{\left(\frac{1}{2}\right)^2 e^{1/2}}{2!}} = \frac{\left(\frac{1}{2}\right)^3 e^{1/2} 2!}{\left(\frac{1}{2}\right)^2 e^{1/2} 3!} = \frac{\frac{1}{2}}{3} = \frac{1}{6}$$

Q.48 It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the number of packets containing at least, exactly and at most 2 defective items in a consignment of 1000 packets using Poisson approximation of binomal distribution.

Ans. : The probability of an item produced on the machine is defective (p) = 0.05

Number of items in a packet (n) = 20

$\therefore$ The parameter of the poisson distribution is

$\lambda = np = 20\ (0.05) = 1$

$\therefore$ the p. m. f. of the distribution is

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1}1^x}{e(x!)},\ x = 0, 1, 2, 3, \ldots$$

Also, there are N = 1000 packets in a consignment.

The probability that a packet will have at least 2 defective is

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - P(X = 0 \text{ or } 1)$$

$$= 1 - [P(0) + P(1)]$$

$$= 1 - \left[\frac{1}{e} * \frac{1}{0!} + \frac{1}{e} * \frac{1}{1!}\right]$$

$$= 1 - \frac{1}{E}(1+1) = 1 - \frac{2}{e}$$

$$P(X \geq 2) = \frac{e-2}{e} = \frac{2.7183-2}{2.7183} = 0.2642$$

$\therefore$ Number of packets containing at least 2 defectives

$$= N * P(X \geq 2)$$

$= 1000 * (0.2642)$

$= 264.2 = 264$

The P(A packet has exactly 2 defective items)

i.e. $P(X = 2) = \frac{1}{e} * \frac{1}{2!} = \frac{1}{2e} = 0.1839$

Number of packets containing exactly 2 defectives

$= 1000 * (0.1839)$

$= 183.9 = 184$

The P(A packet has at most 2 defectives) i.e.

$P(X \leq 2) = P(X = 0 \text{ or } 1 \text{ or } 2)$

$= P(0) + P(1) + P(2)$

$= \frac{1}{e(0!)} + \frac{1}{e(1!)} + \frac{1}{e(2!)}$

$= \frac{1}{e}\left(\frac{1+1+1}{2}\right)$

$= \frac{5}{2e}$

$P(X \leq 2) = 0.9197$

Q.49 Suppose that a trainee soldier shoots a target according to a geometric distribution. If the probability that the target is shot on any one shot is 0.7. What is the probability that it takes him an even.

Ans. :

P = Probability that the soldier shoots a target

P = 0.07

Let X - The number of unsuccessful shots before the first success

Thus X follows Geometric distribution $P(X = x) = q^x p$, $x = 0, 1, 2, 3, \ldots$ and $(x + 1)$ shot is the first successful shot

$= P(x + 1 = \text{even})$

$= P(x = \text{odd})$

$= P(x = 1 \text{ or } 3 \text{ or } 5)$

$= \sum_{r=0}^{\infty} P(X = 2r + 1)$

$= \sum_{r=0}^{\infty} q^{2r+1} p$

$= qp + q^3 p + q^5 p + \ldots$

$= pq(1 + q^2 + q^4 + \ldots)$

$= pq \frac{1}{1-q^2}$

$= \frac{pq}{(1-q)(1+q)}$

$= \frac{q}{1+q}$

$= \frac{0.3}{1+0.3}$

$= 0.2308$

Q.50 For a binomial distribution mean is 6 and standard deviation $\sqrt{2}$. Find the first two terms of distribution.

Ans. : For binomial distribution, we have

Mean $= np = 6$

Standard deviation $= \sqrt{2}$

Variance $= npq$

Standard deviation $= \sqrt{npq}$

$\sqrt{2} = \sqrt{npq}$

$npq = 2$

$q = \frac{2}{np} = \frac{2}{6} = \frac{1}{3}$

$\therefore \quad q = \frac{1}{3}$

$\therefore \quad p = 1 - \frac{1}{3} = \frac{2}{3}$

Also $\quad np = 6$

$n = \frac{6}{p} = \frac{6}{2/3} = \frac{18}{2} = 9$

$\therefore$ The Binomial distribution is

$P(X = x) = n_{C_x} p^x q^{n-x} \quad x = 0,1,2,3,\ldots$

$P(X = x) = 9_{C_x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{q-x} \quad x = 0,1,2,\ldots$

$\therefore$ First term of the distribution is

$$P(X = 0) = {}^{9}C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{9-0}$$

$$= 1.1\left(\frac{1}{3}\right)^9 = \left(\frac{1}{3}\right)^9$$

The second term of the distribution is

$$P(X = 1) = {}^{9}C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^{9-1}$$

$$= 9\cdot\frac{2}{3}\left(\frac{1}{3}\right)^8$$

$$= 3\left(\frac{1}{3}\right)^8$$

3 Continuous Probability Distributions

3.1 : Continuous Uniform Distribution, Normal Distribution

Q.1 What is continuous uniform distribution ?

Ans. : The continuous uniform distribution is one of the simplest probability distributions in statistics. It is a continuous distribution, this means that it takes values within a specified range, e.g. between 0 and 1. The probability density function for a uniform distribution taking values in the range a to b.

The general formula for the probability density function of the uniform distribution is

$$f(x) = \frac{1}{B-A} \qquad \text{for } A \le x \le B$$

where A is the location parameter and (B – A) is the scale parameter.

The case where A = 0 and B = 1 is called the **standard uniform distribution**. The equation for the standard uniform distribution is

$$f(x) = 1 \text{ for } 0 \le x \le 1$$

Q.2 Explain mean and variance of uniform distribution.

Ans. : We have,

$$\text{Mean} = \mu_1' = E(X) = \int_{-\infty}^{\infty} x f(x)dx$$

$$= \int_a^b x \frac{1}{(b-a)} dx = \frac{1}{(b-a)}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)}$$

$$\text{Mean} = \frac{b+a}{2}$$

Again, we have

$$\mu_1' = E(X^2) = \int_{-\infty}^{\infty} x f(x)dx$$

$$= \int_a^b x^2 \frac{1}{(b-a)} dx = \frac{1}{(b-a)}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

$$E(X^2) = \frac{a^2+ab+b^2}{3}$$

$$\therefore \quad Var(X) = E(X^2)-[E(X)]^2$$

$$= \frac{a^2+ab+b^2}{3} - \left[\frac{a+b}{2}\right]^2$$

$$= \frac{(a^2+ab+b^2)}{3} - \frac{(a^2+2ab+b^2)}{4}$$

$$= \frac{4(a^2+ab+b^2)-3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$\therefore \quad Var(X) = \frac{(a-b)^2}{12} = \frac{(b-a)^2}{12}$$

Q.3 If a random variable 'X' has a unifor distribution in (– 3, 3), find

i) $P(X < 2)$

ii) $P(|X| > 1)$

iii) $P(|X - 2| < 2)$

iv) k such that $P(x > k) = 1/4$

v) $P(X = 2)$

Ans. : We have for X ~ U (– a, a) its p.d.f is

$$f(x) = \begin{cases} \frac{1}{2a} & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Here a = 3 $\quad\because x \in (-3,3)$

$$f(x) = \begin{cases} \frac{1}{6} & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Now, $\because$ for a continuous random variable, X we have,

$$P(a < X < b) = \int_a^b f(x)dx$$

i) $$P(X < 2) = P(-3 < X < 2) = \int_{-3}^{2} f(x)dx$$

$$= \int_{-3}^{2} \frac{1}{6}dx = \frac{1}{6}[x]_{-3}^{2} = \frac{1}{6}[2-(-3)] = \frac{5}{6}$$

ii) $$P(|X| > 1) = P(X < -1 \text{ or } X > 1)$$

$$= P(X < -1) + P(X > 1)$$

$$= P(-3 < X < -1) + P(1 < X < 3)$$

$$= \int_{-3}^{-1} f(x)dx + \int_{1}^{3} f(x)dx = \int_{-3}^{-1}\frac{1}{6}dx + \int_{1}^{3}\frac{1}{6}dx$$

$$= \frac{1}{6}[x]_{-3}^{-1} + \frac{1}{6}[x]_{1}^{3} = \frac{1}{6}(-1+3) + \frac{1}{6}(3-1)$$

$$= \frac{2}{6} + \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$$

iii) $$P(|X-2| < 2) = P(-2 < X-2 < 2)$$

$$= P(0 < X < 4) = \int_0^4 f(x)dx$$

$$= \int_0^3 f(x)dx + \int_3^4 f(x)dx = \int_0^3 \frac{1}{6}dx + 0$$

$$= \frac{1}{6}[x]_0^3 = \frac{3}{6} = \frac{1}{2}$$

iv) $$P(X > k) = \frac{1}{4}$$

$$P(k < X < 3) = \frac{1}{4}$$

$$\int_k^3 f(x)\,dx = \frac{1}{4}$$

$$\int_k^3 \frac{1}{6}dx = \frac{1}{4}$$

$$\frac{1}{6}[x]_k^3 = \frac{1}{4}$$

$$\frac{1}{6}(3-k) = \frac{1}{4}$$

$$3-k = \frac{6}{4} = \frac{3}{2}$$

$$k = 3 - \frac{3}{2} = \frac{3}{2}$$

v) $$P(X = 2) = 0$$

$\because$ for continuous random variable $P(X = a) = 0$

Q.4 If X is uniformly distributed over (0, 10) calculate the probability that a) X > 6
b) 3 < X < 8

Ans. : Here $X \sim U(0, 10)$

$\therefore$ Its p.d.f. is

$$f(x) = \begin{cases} \frac{1}{10-0} = \frac{1}{10} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\left[\because f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}\right]$$

a) $\therefore$ $$P(X > 6) = P(6 < X < \infty) = \int_6^\infty f(x)\,dx$$

$$= \int_6^{10} f(x)dx + \int_{10}^{\infty} f(x)dx = \int_6^{10}\frac{1}{10}dx + 0$$

$$= \frac{1}{10}[x]_6^{10} = \frac{1}{10}[4]$$

$\therefore$ $$P(X > 6) = \frac{4}{10} = \frac{2}{5}$$

b) $$P(3 < X < 8) = \int_3^8 f(x)\,dx = \int_3^8 \frac{1}{10}dx = \frac{1}{10}[x]_3^8$$

$$= \frac{1}{10}[8-3] = \frac{1}{10}[5]$$

$$P(3 < X < 8) = \frac{1}{2}$$

Q.5 An electrical firm manufactures lights bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find
a) The probability that a bulb burns more than 834 hours
b) The probability that the bulb burns between 778 and 834 hours

Ans. : Let X be a life of bulb, before it burn out and thus X is normally distributed with mean (μ) = 800 hours and standard deviation (σ) = 40 hours, i.e. variance $(\sigma^2) = (40)^2$

Thus, $X \sim N(800, 40^2)$

$\therefore$ The standard normal variable is

$$z = \frac{x-\mu}{\sigma} = \frac{x-800}{40}$$

a) We have to find

P(Bulb burns more than 834 hours) i.e. $P(X > 834)$

Now, when $x = 834$

$$z = \frac{834-800}{40} = \frac{34}{40} = 0.85$$

$$\therefore \quad P(X > 834) = P(z > 0.85) = P(0.85 < z < \infty) = P(0 < z < \infty) - P(0 < z < 0.85)$$

$$= 0.5 - 0.3023$$ ← using values from table of area under normal curve.

b) P(Bulb burns between 778 and 834 hours) i.e. $P(778 < x < 834)$

Now, when $x = 778,\ z = \frac{778-800}{40} = -0.55$

$$P(778 < x < 834) = P(-0.55 < z < 0.85) = P(-0.55 < z < 0) + P(0 < z < 0.85)$$

$$= P(0 < z < 0.55) + P(0 < z < 0.85) = 0.2088 + 0.3023$$

← by symmetry of standard normal curve

$$= 0.5111$$

Q.6 What is standard normal curve ?

Ans. : The standard normal curve is the normal curve with mean $\mu = 0$ and standard deviation $\sigma = 1$.

Q.7 Define normal distribution

Ans. : • A normal distribution has a bell-shaped density curve described by its mean μ and standard deviation σ. The density curve is symmetrical, centered about its mean, with its spread determined by its standard deviation. The height of a normal density curve at a given point x is given by

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-u}{\sigma}\right)^2}$$

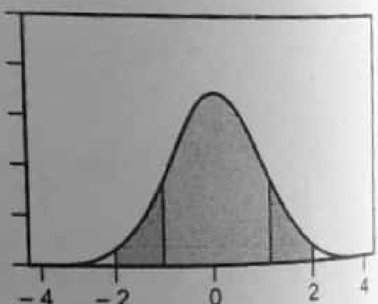


Fig. Q.7.1

- Fig. Q.7.1 shows standard normal curve
- The **Standard Normal** curve, shown here, has mean 0 and standard deviation 1. If a dataset follows a normal distribution, then about 68 % of the observations will fall within σ of the mean μ, which in this case is with the interval (–1,1).
- About 95 % of the observations will fall within 2 standard deviations of the mean, which is the interval (–2,2) for the standard normal, and about 99.7% of the observations will fall within 3 standard deviations of the mean, which corresponds to the interval (–3,3) in this case.
- Although it may appear as if a normal distribution does not include any values beyond a certain interval, the density is actually positive for all values, $(-\infty, \infty)$

- Data from any normal distribution may be transformed into data following the standard normal distribution by subtracting the mean and dividing by the standard deviation.
- The normal distribution is a continuous probability distribution. This has several implications for probability.
  a) The total area under the normal curve is equal to 1.
  b) The probability that a normal random variable X equals any particular value is 0.
  c) The probability that X is greater than a equals the area under the normal curve bounded by a and plus infinity (as indicated by the non-shaded area in the figure Q.7.2 below).
  d) The probability that X is less than a equals the area under the normal curve bounded by a and minus infinity (as indicated by the shaded area in the figure Q.7.2 below).

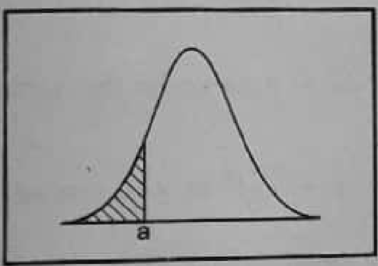


Fig. Q.7.2

Q.8 Write properties of normal curve

Ans. : Properties of a Normal Curve are as follows :

1. All normal curves have the same general bell shape.
2. The curve is symmetric with respect to a vertical line that passes through the peak of the curve.
3. The curve is centered at the mean μ which coincides with the median and the mode and is located at the point beneath the peak of the curve.
4. The area under the curve is always 1.
5. The curve is completely determined by the mean μ and the standard deviation σ. For the same mean, μ, a smaller value of σ gives a taller and narrower curve, whereas a larger value of σ gives a flatter curve.
6. The area under the curve to the right of the mean is 0.5 and the area under the curve to the left of the mean is 0.5.
7. The empirical rule (68 %, 95 %, 99.7 %) for mound shaped data applies to variables with normal distributions. For example, approximately 95 % of the measurements will fall withing 2 standard deviations of the mean, i.e. within the interval $(\mu-2\sigma, \mu+2\sigma)$.
8. If a random variable X associated to an experiment has a normal probability distribution, the probability that the value of X derived from a single trial of the experiment is between two given values x_1 and $x_2 (P(x_1 \le X \le x_2))$ is the area under the associated normal curve between x_1 and x_2.

Q.9 The marks obtained in computer science by 1000 students is normally distributed with mean 78 % and standard deviation 11 %. Determine
a) How many students got marks above 90 %
b) What was the highest mark obtained by the lowest 10 % of the student
c) Within what limits did the middle of 90 % of the students lie.

Ans. : Given data mean (μ) = 78 % = 0.78

Standard deviation (σ) = 11 % = 0.11

$$z = \frac{x-\mu}{\sigma} = \frac{x-0.78}{0.11}$$

a) For x = 0.9

$$Z = \frac{0.9-0.78}{0.11} = 1.09$$

$$P(X > 0.9) = P(Z > 1.09) = 0.5 - 0.3621 = 0.1379$$

$$[\because P(X > \mu) = P(X < \mu) = 0.5]$$

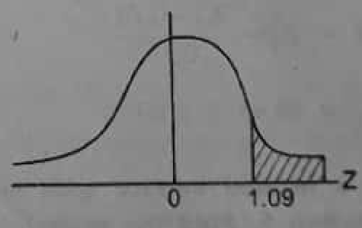


Fig. Q.9.1

The number of student with marks above 90 %

$$= 1000 \times P(X > 0.9) = 1000 \times 0.1379$$

$$= 137.9 \cong 138$$

b) The lowest 10 % student constitute 0.1 area $\left(<\frac{1}{2}\right)$ of extreme left tail so Z_1 must be negative. From table 0.4 = 0.5 − 0.1

0.5 − Area 0.1 from 0 to Z_1

$$Z_1 = -1.28$$

Thus $$-1.28 = \frac{X-0.78}{0.11}$$

$$X = 0.6392$$

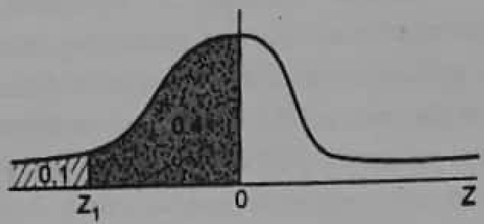


Fig. Q.9.2

The highest mark obtained by the lowest 10 % of students is 63.92 % ≅ 64 %

c) Middle 90 % correspond to 0.9 area, leaving 0.05 area on both sides. Then the corresponding 2 are ± 1.64

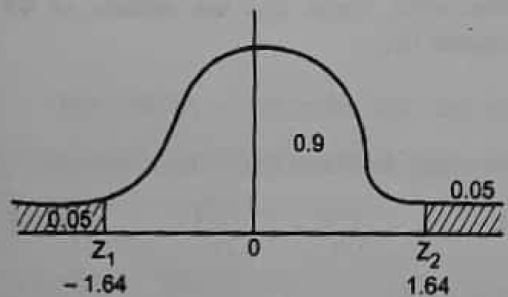


Fig. Q.9.3

$$1.64 = Z_2 = \frac{X_2-0.78}{0.11}$$

$$X_2 = 96.04\ \% \cong 96\ \%$$

$$-1.64 = Z_1 = \frac{X_1-0.78}{0.11}$$

$$X_1 = 59.96\ \% \cong 60\ \%$$

Q.10 If X is a normal variate with mean 30 and standard deviation 5. Find the probability that, a) $26 \le x \le 40$ b) $x \ge 45$.

Ans. : Given data :

Mean $\mu = 30$

Standard deviation $\sigma = 5$.

i) $x_1 = 26$ and $x_2 = 40$

$$Z = \frac{x-\mu}{\sigma}$$

$$Z_1 = \frac{x_1-\mu}{\sigma} = \frac{26-30}{5} = \frac{4}{5} = -0.8$$

$$Z_2 = \frac{x_2-\mu}{\sigma} = \frac{40-30}{5} = \frac{10}{5} = 2$$

$$P(26 \le x \le 40) = P(-0.8 \le z \le 2)$$

ii) $x \ge 45$

$$Z = \frac{x-\mu}{\sigma} = \frac{45-30}{5} = \frac{15}{5} = 3$$

Q.11 Explain mean and variance of norm distribution.

Ans. : • Mean for normal distribution is given as

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{\frac{-(x-\mu)^2}{2\sigma}} dx$$

Since $\int_{-\infty}^{\infty} f(x)\, dx$ = Area under the normal curve = 1

$$\therefore \quad z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \Rightarrow dx = \sigma dz$$

We can write mean as,

$$\text{Mean} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z)\, e^{\frac{-z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\mu \int_0^{\infty} e^{-t} \frac{dt}{\sqrt{2t}}\right] \quad \left[\because t = \frac{z^2}{2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2}\mu \int_0^{\infty} e^{-t} t^{-1/2} dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2}\mu \int_0^{\infty} e^{-t} t^{1/2-1} dt\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\sqrt{2}\mu \left|\frac{1}{2}\right.\right] = \frac{1}{\sqrt{2\pi}}[\sqrt{2}\mu\sqrt{\pi}]$$

$$= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi}\mu$$

$$\text{Mean} = \mu$$

$$\text{Variance} = E((X-\overline{X})^2) = E((X-\mu)^2)$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$

$$\therefore \text{ Variance} = \frac{\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-z^2/2}\,dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\left[2\int_0^{\infty} z^2 e^{-z^2/2}\,dz\right]$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\left[2\int_0^{\infty} (2t)\, e^{-t} \frac{dt}{(2t)^{1/2}}\right] \quad \left[\therefore t = \frac{z^2}{2}\right]$$

$$= \frac{4\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t\, e^{-t} \frac{1}{2\sqrt{t}}\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t^{3/2-1} e^{-t}\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \overline{|3/2} = \frac{2\sigma^2}{\sqrt{\pi}} \times \frac{1}{2}\sqrt{\pi}$$

$$\text{Variance} = \sigma^2$$

$$\text{Standard deviation} = \sqrt{\text{Variance}} = \sqrt{\sigma^2}$$

$$\text{Standard deviation} = \sigma$$

Mode of normal distribution

- Mode is a value of x for which f is maximum. It is a solution of $f'(x) = 0$ and $f''(x) < 0$.
- As per definition of normal distribution :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Differentiating w.r.t. x, we get,

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \times \left[1 - \left(\frac{x-\mu}{\sigma}\right)\right]$$

$$\left(\frac{f'(x)}{f(x)}\right) = \frac{-1}{\sigma^2}(x-\mu) \Rightarrow f'(x) = -\frac{1}{\sigma^2}(x-\mu)\, f(x)$$

Now, $f'(x) = 0 \Rightarrow x \Rightarrow \mu$

$$f''(x) = -\frac{1}{\sigma^2}\left[(x-\mu)\, f'(x) + f(x)\right]$$

$$= -\frac{1}{\sigma^2}\left[(x-\mu) - \frac{(x-\mu)}{\sigma^2} f(x) + f(x)\right]$$

$$= \frac{-f(x)}{\sigma^2}\left[-\frac{(x-\mu)^2}{\sigma^2}\right]$$

Substituting $x = \mu$, we get,

$$f''(x) = \frac{-1}{\sigma^2} \times \frac{1}{\sigma\sqrt{2\pi}} < 0$$

So $x = \mu$ is the mode of the normal distribution.

Median of normal distribution

- If m denotes the medium of the normal distribution, then

$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$$

i.e. $$\int_{-\infty}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

We can rewrite,

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{m} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{2}$$

Substitute $\frac{x-\mu}{\sqrt{2}\,\sigma} = -z$ $\quad\therefore dx = -4\sqrt{2}\,\sigma\,dz$

$$\therefore \quad \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_0^{\infty} e^{[-\mu^2(-\sqrt{2}\,\sigma\,du)]}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-\mu^2} du + \frac{1}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = \frac{1}{2}$$

It shows that $\mu = M$.

Q.12 Let x be a normal variate with mean 30 and standard deviation 5. Then find
a) $P(26 \le x \le 40)$ b) $P(x \ge 45)$

Ans. : Given data :

$$\mu = 30, \quad \sigma = 5 \quad x_1 = 26, \quad x_2 = 40$$

a) $P(26 \le x \le 40)$

$$z_1 = \frac{x_1 - \mu}{\sigma}$$

$$z_1 = \frac{26-30}{5} = -0.8 = 0.8$$

$$z_2 = \frac{x_2-\mu}{\sigma} = \frac{40-30}{5} = 2$$

$$P(26 \le x \le 40) = P(-0.8 < z < 2) = P(z < 2) - P(z < -0.8) = 0.4772 + 0.2881 = 0.7653$$

b) $P(x \ge 45)$

Here $x = 45$ $\quad z = \frac{x-\mu}{\sigma} = \frac{45-30}{5} = 3 \; (z_1)$

$$P(x \ge 45) = P(z_1 \ge 3) = \frac{1}{2} - A(z_1) = 0.5 - A(3) = 0.5 - 0.49865 = 0.00135$$

Q.13 Suppose the length of life of an appliance has an expontial distribution with mean 10 years. What the probability that the average life time of a random sample of the appliances is at least 10.5 ?

Ans. : Given data : $\lambda = \frac{1}{10}$, $f(x) = \lambda e^{-\lambda x}, \; x > 0 \Rightarrow f(x) = \frac{1}{10} e^{-\frac{x}{10}}$

$$P(x > 10.5) = \int_{10.5}^{\infty} f(x)\,dx = \int_{10.5}^{\infty} \frac{1}{10} e^{\frac{-x}{10}}\,dx = e^{-1.05} = 0.3499$$

Q.14 In a binomial distribution the mean is 6 and variance is 1.5. Then find :
i) p [X = 2] and ii) P[X ≤ 2].

Ans. : Given data : Let n and p be the parameters. Then,

$$\text{Mean} = np = 6$$

$$\text{Variance} = npq = 1.5$$

$$\frac{\text{Variance}}{\text{Mean}} = \frac{1.5}{6} = \frac{1}{4}$$

Therefore, $\quad q = 1/4$ and $p = 3/4$

$$\text{Mean} = np$$

$$6 = n \times 3/4$$

$$n = 24/3 = 8$$

$$p(x) = 8C_x \, (3/4)^x \, (1/4)^{8-x}$$

i) $$P(X = 2) = 8C_2 \, (3/4)^2 \, (1/4)^6 = \frac{8!}{2!\,(8-2)!} \times \frac{9}{16} \times \frac{1}{4096} = \frac{40320}{1440} \times \frac{9}{16} \times \frac{1}{4096}$$

$$= \frac{252}{16 \times 4096} = \frac{252}{65536} = 0.003845$$

ii) $$P[X \le 2] = p(0) + p(1) + p(2) = {}^8C_0\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^8 + {}^8C_1\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^7 + {}^8C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^6$$

$$= \frac{8!}{0!\,(8-0)!}(1)\left(\frac{1}{65536}\right) + \frac{8!}{1!\,(8-1)!} \times \frac{3}{4} \times \frac{1}{16384} + \frac{8!}{2!(8-2)!} \times \frac{9}{16} \times \frac{1}{4096}$$

$$= \frac{1}{65536} + \frac{10080}{1680} \times \frac{1}{16384} + \frac{63}{16384}$$

$$= 0.00001525 + 0.000366 + 0.003845$$

$$= 0.004226$$

Q.15 Find the probability of getting 3 and 6 heads inclusive in 10 tosses of a fair coin by using
a) Binomal distribution
b) Normal approximation to the binomal distribution.

Ans. : a) Binomal distribution :

Let X have the random variable giving the number of heads that will turn up in 10 tosses, then

$$P(X = 3) = \binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$$

$$= \frac{10!}{3!(10-3)!} \times \frac{1}{8} \times \frac{1}{128} = \frac{3628800}{6 \times 5040 \times 8 \times 128}$$

$$P(X = 3) = \frac{15}{128}$$

$$P(X = 4) = \binom{10}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6$$

$$= \frac{10!}{4!(10-4)!} \times \frac{1}{16} \times \frac{1}{64} = \frac{3628800}{24 \times 720} \times \frac{1}{16} \times \frac{1}{64}$$

$$= \frac{22680}{24 \times 72 \times 64} = \frac{11340}{12 \times 72 \times 64}$$

$$P(X = 4) = \frac{945}{72 \times 64} = \frac{105}{8 \times 64} = \frac{105}{512}$$

$$P(X = 5) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5$$

$$= \frac{10!}{5!(10-5)!} \times \frac{1}{32} \times \frac{1}{32} = \frac{3628800}{14400 \times 32 \times 32}$$

$$P(X = 5) = \frac{63}{256}$$

$$P(X = 6) = \binom{10}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4 = \frac{10!}{6!(10-4)!} \times \frac{1}{64} \times \frac{1}{16}$$

$$= \frac{3628800}{518400 \times 64 \times 16} = \frac{7}{1024}$$

The required probability is,

$$P(3 \le X \le 6) = \frac{15}{128} + \frac{105}{512} + \frac{63}{256} + \frac{7}{1024}$$

$$= 0.1171 + 0.2050 + 0.246094 + 006836$$

$$= 0.575195$$

b) Normal distribution :

We consider data is continuous.

Then mean $\mu = np = 10 \times \frac{1}{2} = 5$

$$\sigma = \sqrt{npq} = \sqrt{10 \times \frac{1}{2} \times \frac{1}{2}} = \sqrt{5/2} = 1.58$$

Q.16 Let X be uniformly distributed over (α, β) Find a) E(X) b) Var(X)

Ans. : a) E(X)

$$= \int_{-\infty}^{\infty} x f(x) dx = \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx$$

$$= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} = \frac{\beta + \alpha}{2}$$

- The expected value of a random variable uniformly distributed over some interval is equal to the midpoint of that interval.

b) To find Var(X), we first calculate $E(X^2)$

$$E(X^2) = \int_{\alpha}^{\beta} \frac{1}{\beta - \alpha} x^2 dx = \frac{\beta^3 - \alpha^3}{3(\beta - \alpha)} = \frac{\beta^2 + \alpha\beta + \alpha^2}{3}$$

Hence,

$$\mathrm{Var}(X) = \frac{\beta^2 + \alpha\beta + \alpha^2}{3} - \frac{(\alpha + \beta)^2}{4} = \frac{(\beta - \alpha)^2}{12}$$

Therefore, the variance of a random variable that is uniformly distributed over some interval is the square of the length of that interval divided by 12.

Q.17 Let X be the number of times that a fair coin flipped 40 times, lands heads. Find the probability that X = 20. Use the normal approximation and then compare it to the exact solution.

Ans. : We can write

$$P(X = 20) = P(19.5 \le X < 20.5)$$

$$= P\left\{\frac{19.5 - 20}{\sqrt{10}} < \frac{X - 20}{\sqrt{10}} < \frac{20.5 - 20}{\sqrt{10}}\right\}$$

$$= P\left\{-0.16 < \frac{X - 20}{\sqrt{10}} < 0.16\right\}$$

$$= \phi(0.16) - \phi(-0.16) = 0.1272$$

The exact result is

$$P(X = 20) = \binom{40}{20}\left(\frac{1}{2}\right)^{40} = 0.1254$$

3.2 : Gamma Distribution and Exponential Distribution

Q.18 What is gamma distribution and Explain.

Ans. : A random variable is said to have a gamma distribution with parameters $(\alpha, \lambda), \lambda > 0$ and $\alpha > 0$ if its density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(a)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Where $\Gamma(\alpha)$ called the gamma function, is defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-y} y^{\alpha-1} dy$$

The integration by parts of $\Gamma(\alpha)$ yields that

$$\Gamma(\alpha) = -e^{-y} y^{\alpha-1}\Big|_0^{\infty} + \int_0^{\infty} e^{-y}(\alpha-1)y^{\alpha-2}dy$$

$$= (\alpha-1)\int_0^{\infty} e^{-y} y^{\alpha-2}dy \qquad \ldots(1)$$

$$= (\alpha-1)\Gamma(\alpha-1)$$

For integral value of α, sat $\alpha = \eta$, by applying equation (1), we get

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)$$

$$= (n-1)(n-2)\ldots 3.2\,\Gamma(1)$$

Since $\Gamma(1) = \int_0^{\infty} e^{-x}dx = 1,$

it follows that for integral values of n.

$$T(n) = (n-1)!$$

Q.19 What is exponential distributions ?

Ans. : X is said to have an exponential distribution with parameter $\lambda(\lambda > 0)$ if the pdf of X is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- Some sources write the exponential pdf in the form $(1/\beta)e^{-x/\beta}$,so that $\beta = 1/\lambda$
- The expected value of an exponentially distributed random variable X is .

$$E(X) = \int_0^{\infty} x\lambda e^{-\lambda x}dx$$

- Obtaining this expected value necessitates doing an integration by parts. The variance of X can be computed using the fact that $V(X) = E(X^2) - [E(X)]^2$
- The determination of $E(X^2)$ requires integrating by parts twice in succession.
- The results of these integrations are as follows:

$$\mu = \frac{1}{\lambda} \quad \sigma^2 = \frac{1}{\lambda^2}$$

- Both the mean and standard deviation of the exponential distribution equal $1/\lambda$.

Q.20 What does it mean to be exponentially distributed ?

Ans. : The exponential distribution is a continuous probability distribution used to model the time we need to wait before a given event occurs. It is the continuous counterpart of the geometric distribution, which is instead discrete. Sometimes it is also called negative exponential distribution

Q.21 List the applications of the exponential distribution.

Ans. : 1. Time between telephone calls

2. Time between machine breakdowns
3. Time between successive job arrivals at a computing centre

Q.22 Let random variable X have geometric distribution with parameter p. Show that, for any positive m, n, we have P(X > m + n|X > m) - P(X > n). This is the memoryless property of the geometric distribution. Why do you think this property is called memoryless ?

Ans. : • We first calculate the probability $P(X > m)$ for a geometric random variable X with parameter p;

- we know that $X > m$ when the first m trials were failures; therefore, we get: $P(X > m) = (1-p)^m$
- Using this, we can now compute the conditional probability:

$$P(X > m + n \mid X > m) = \frac{P((X > m+n) \cap (X > m))}{P(X > m)}$$

$$= \frac{P(X > m+n)}{P(X > m)}$$

[Since $P(X > m+n) \cap (X > m) = P(X > m+n)$]

$$= \frac{(1-P)^{m+n}}{(1-P)^m} = (1-P)^n = P(X > n)$$

- This property is called memoryless because even knowing that we have waited m trials and have not yet seen a success, the probability of seeing a success in the next n trials is exactly the same as if we had not seen any trials at all

3.3 : Fundamental Sampling Distributions: Random Sampling

Q.23 Define bias.

Ans. : Any sampling procedure that produces inferences that consistently overestimate or consistently underestimate some characteristic of the population is said to be biased.

Q.24 What is a sampling distribution ?

Ans. : • The sampling distribution of the mean refers to the pattern of sample means that will occur as samples are drawn from the population at large.

- Example : I want to perform a study to determine the number of kilo-metres the average person in India drives a car in one day. It is not possible to measure the number of kilo-metres driven by every person in the population, so I randomly choose a sample of 10 people and record how far they have driven

Q.25 What is population ? Explain with example.

Ans. : **Population**

- A population is any entire collection of people, animals, plants or things from which we may collect data. It is the entire group we are interested in, which we wish to describe or draw conclusions about.
- Population is a collection of objects. It may be finite or infinite according to the number of objects in the population.
- A population can be defined as including all people or items with the characteristic one wishes to understand. Because there is very rarely enough time or money to gather information from everyone or everything in a population, the goal becomes finding a representative sample (or subset) of that population.
- In order to make any generalizations about a population, a sample, that is meant to be representative of the population, is often studied. For each population there are many possible samples. A sample statistic gives information about a corresponding population parameter. For example, the sample mean for a set of data would give information about the overall population mean.
- It is important that the investigator carefully and completely defines the population before collecting the sample, including a description of the members to be included.
- **Example** : The population for a study of infant health might be all children born in the UK in the 1980's. The sample might be all babies born on 7th May in any of the years.
- When such measures like the mean, median, mode, variance and standard deviation of a population distribution are computed, they are referred to as parameters. A parameter can be simply defined as a summary characteristic of a population distribution.

Q.26 Write short note on statistics.

Ans. : • Statistics is the science of understanding data and of making decisions in the face of variability and uncertainty. The field of statistics deals with the collection, presentation, analysis and use of data to -

1. Make decisions
2. Solve problems
3. Design products and processes

- Statistical techniques are useful for describing and understanding variability. By variability, we mean successive observations of a system or phenomenon do not produce exactly the same result.
- Statistics gives us a framework for describing this variability and for learning about potential sources of variability.
- Statistics is concerned with the systematic collection of numerical data and its interpretation.
- Three basic methods for collecting data :

1. A **retrospective** study using historical data
2. An **observational** study
3. A **designed experiment**

- One of the main objectives of statistics is drawn inference about a population from the analysis for the sample drawn from that population. The four major branches of statistical inference are

1. Estimation theory
2. Tests of hypothesis
3. Non parametric tests
4. Sequential analysis

Fig. Q.26.1

Q.27 Define central limit theorem.

Ans. : The central limit theorem is a theorem stating that the sum of a large number of independent, identically distributed random variables approximately follows a Normal distribution

Q.28 Find sample Mean and Sample Standard Deviation for the following data set: 5, 10, 15, 20

Ans. : Sample mean $(\bar{x})$:

$$\bar{x} = \frac{\sum x}{n} = \frac{5+10+15+20}{4} = \frac{50}{4} = 12.5$$

Sample standard deviation :

Data	$x-\bar{x}$	$(x-\bar{x})^2$
5	$5-(12.5) = 7.5$	$(-7.5)^2 = 56.25$
10	$10-(12.5) = -2.5$	$(-2.5)^2 = 6.25$
15	$15-(12.5) = 2.5$	$(2.5)^2 = 6.25$
20	$20-(12.5) = 7.5$	$(7.5)^2 = 56.25$
		$\sum(x-\bar{x})^2 = 125.01$

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} = \sqrt{\frac{125.01}{4-1}} = 6.455$$

Q.29 Explain central limit theorem.

Ans. : - The sampling distribution of the sample mean, x is approximated by a normal distribution when the sample is a simple random sample and the sample size, n, is large.

- In this case, the mean of the sampling distribution is the population mean (μ) and the standard deviation of the sampling distribution is the population standard deviation (σ), divided by the square root of the sample size. The later is referred to as the standard error of the mean.
- A sample size of 100 or more elements is generally considered sufficient to permit using the CLT. If the population from which the sample is drawn is symmetrically distributed, $n > 30$ may be sufficient to use the CLT.
- The central limit theorem states that the mean of the sampling distribution of the mean will be the unknown population mean. The standard deviation of the sampling distribution of the mean is called the standard error.
- In fact, it is just another standard deviation, we just call it the standard error so we know we're talking about the standard deviation of the sample means instead of the standard deviation of the raw data. The standard deviation of data is the average distance values are from the mean.

Q.30 Explain sampling distribution of mean.

Ans. : - A theoretical probability distribution of sample means that would be obtained by drawing from the population all possible samples of the same size.

- The standard deviation of the sampling distribution is called the standard error.
- The **sampling error** is the difference between the point estimate (value of the estimator) and the value of the parameter. This is the error caused by sampling only a subset of elements of a population, rather than all elements in a population. A researcher hopes to minimize the sampling error, but all samplés have some such error associated with them.
- The sample is a sampling distribution of the sample means. When all of the possible sample means are computed, then the following properties are true :

1. The mean of the sample means will be the mean of the population
2. The variance of the sample means will be the variance of the population divided by the sample size.
3. The standard deviation of the sample means (known as the standard error of the mean) will be smaller than the population mean and will be equal to the standard deviation of the population divided by the square root of the sample size.
4. If the population has a normal distribution, then the sample means will have a normal distribution.
5. If the population is not normally distributed, but the sample size is sufficiently large, then the sample means will have an approximately normal distribution. Some books define sufficiently large as at least 30 and others as at least 31.

- The formula for a Z-score when working with the sample means is :

$$Z = \frac{\overline{X}-\mu}{\sigma / \sqrt{n}}$$

Finite Population Correction Factor

- If the sample size is more than 5 % of the population size and the sampling is done without replacement, then a correction needs to be made to the standard error of the means.
- In the following, N is the population size and n is the sample size. The adjustment is to multiply the standard error by the square root of the quotient of the difference between the population and sample sizes and one less than the population size.

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} \cdot \sqrt{\frac{N-n}{N-1}}$$

Random sample from a normally distributed population

	Normally distributed population	Sampling distribution of $\overline{x}$ when sample is random
Number of elements	N	n
Mean	μ	μ
Standard deviation	σ	$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$

Classification of Samples

- Samples are classified as two types : Large sample and Small sample.
  1. Large sample : The sample is said to be large if the size of sample ($n \geq 30$).
  2. Small sample : The sample is said to be large if the size of sample ($n < 30$).

Q.31 What is standard error ? List procedure for calculating standard error.

Ans. : • The standard deviation of the sampling distribution is of a statistic. Standard error is a statistical term that measures the accuracy with which a sample represents a population. In statistics, a sample mean deviates from the actual mean of a population; this deviation is the standard error.

- The term "standard error" is used to refer to the standard deviation of various sample statistics such as the mean or median. For example, the "standard error of the mean" refers to the standard deviation of the distribution of sample means taken from a population.

Standard Error Calculation Procedure :

Step 1 : Calculate the mean (Total of all samples divided by the number of samples).

Step 2 : Calculate each measurement's deviation from the mean (i.e. Mean minus the individual measurement).

Step 3 : Square each deviation from mean. Squared negatives become positive.

Step 4 : Sum the squared deviations.

Step 5 : Divide that sum from step 4 by one less than the sample size (n – 1)

Step 6 : Take the square root of the number in step 5. That gives you the "standard deviation (S.D.)."

Step 7 : Divide the standard deviation by the square root of the sample size (n). That gives you the "standard error".

Step 8 : Subtract the standard error from the mean and record that number. Then add the standard error to the mean and record that number. You have plotted mean ± 1 standard error , the distance from 1 standard error below the mean to 1 standard error above the mean.

Let us consider the following table :

Name	Height to nearest	(Step 2) Deviations (m – i)	(Step 3) Squared deviations $(m-i)^2$
Rupali	150	9.6	92.16
Rakshita	170	– 10.4	108.16
Sangeeta	165	– 5.4	29.16
Rutuja	155	4.6	21.16
Rushi	158	1.6	2.56
n = 5	Total = 798 (Step 1) Mean m = 159.6		(Step 4) Sum of squared deviations $\sum (m-i)^2 = 253.2$

Step 5 : Divide by number of measurements – 1 :

$$\frac{\sum (m-i)^2}{n-1} = \frac{253.2}{5-1} = 63.3$$

Step 6 : Standard deviation $= \frac{\text{Square root of} \sum (m-i)^2}{n-1} = \frac{\sqrt{63.3}}{4} = 1.9890$

Step 7 : Standard error $= \frac{\text{Standard deviation}}{\sqrt{n}} = \frac{1.9890}{\sqrt{4}} = 0.9945$

Step 8 :

$$\begin{aligned} m \pm 1SE &= 159.6 \pm 0.9945 \\ &= 159.6 + 0.9945 \text{ or } 159.6 - 0.9945 \\ &= 160.5945 \text{ or } 158.6055 \end{aligned}$$

Q.32 A bowler claims that she has a 215 average. In her latest performance, she scores 188, 214 and 204. Assume that her bowling scores are normally distributed. Calculate the sample mean, variance and standard deviation

Ans. : The sample mean, variance, and standard deviation

$$\text{Sample mean} = \frac{188+214+204}{3} = \frac{606}{3} = 202$$

$$\text{Sample variance} = \frac{(188-202)^2+(214-202)^2+(204-202)^2}{3-1} = \frac{196+144+4}{2} = \frac{344}{2} = 172$$

$$\text{Standard deviation} = \sqrt{172} = 13.11$$

Q.33 The following are the times between six calls for an ambulance in a city and the patient's arrival at the hospital : 27, 15, 20, 32, 18 and 26 minutes. Use these figures to judge the reasonableness of the ambulance services claim that it takes on the average 20 minutes between the call for an ambulance and patient's arrival at the hospital. [JNTU : May-12]

Ans. : Given data : n = 6, Average minutes to reach the hospital $(\mu) = 20$

$x_1 = 27,\ x_2 = 15,\ x_3 = 20,\ x_4 = 32,\ x_5 = 18,\ x_6 = 26$

Then, Arithmetic mean $\bar{x} = \frac{\sum x_i}{n}$

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}{6} = \frac{27 + 15 + 20 + 32 + 18 + 26}{6} = \frac{138}{6} = 23$$

Estimate of variance (S^2) $= \dfrac{\sum (x_i - \bar{x})^2}{n-1}$

$$S^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + (x_4 - \bar{x})^2 + (x_5 - \bar{x})^2 + (x_6 - \bar{x})^2}{6-1}$$

$$= \frac{(27-23)^2 + (15-23)^2 + (20-23)^2 + (32-23)^2 + (18-23)^2 + (26-23)^2}{5}$$

$$= \frac{16+64+9+81+25+9}{5} = \frac{204}{5} = 40.8$$

$$S^2 = 40.8$$

$$S = 6.387$$

$$t = \frac{x - \mu}{S/\sqrt{n}} = \frac{23-20}{6.387/\sqrt{6}} = \frac{3}{6.387/2.449}$$

$$t = 1.15$$

Now, $\quad t_{n-1,\,\alpha} \Rightarrow t_{6-1,\alpha} \Rightarrow t_{5,\alpha} = 2.015$

(for $\alpha = 0.05$)

For $\quad \alpha = 0.05 \quad t_5 = 2.015$

$\alpha = 0.1 \quad t_5 = 1.476$

So $\quad t = 1.15 < 1.476$

So claim is rejected.

Q.34 A random sample of size 144 is taken from an infinite population having the mean 75 and variance 225. What is probability that $\bar{x}$ will be between 72 and 77 ?

Ans. : Given data : Size of sample n = 144, Variance $\sigma^2 = 225$

$$\sigma = \sqrt{225} = 15$$

$$\text{Mean } \mu = 75$$

$$\bar{x}_1 = 72, \quad \bar{x}_2 = 77$$

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$$Z_1 = \frac{\bar{x}_1 - m}{\sigma/\sqrt{n}} = \frac{72-75}{15/\sqrt{144}} = \frac{-3}{15/12} = -2.4$$

$$Z_2 = \frac{\bar{x}_2 - \mu}{\sigma/\sqrt{n}} = \frac{77-75}{15/\sqrt{144}} = \frac{2}{15/12} = 1.6$$

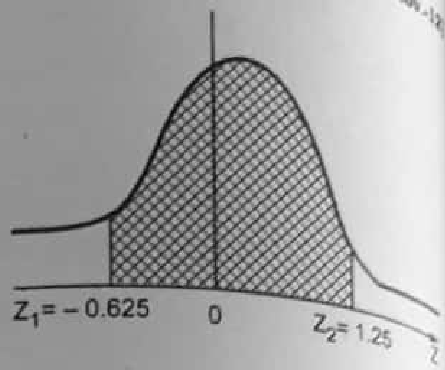
$$P(72 < \bar{x} < 77) = P(\bar{x}_1 < \bar{x} < \bar{x}_2) = P(-2.4 < Z < 0) + P(0 < Z < 1.06)$$

$$= P(0 < Z < 2.4) + P(0 < Z < 1.6) = 0.4918 + 0.4452 = 0.9370$$

Q.35 A random sample of size 100 is taken from an infinite population having the mean μ = 76 and the variance σ^2 = 256. What is the probability that $\bar{x}$ will be between 75 and 78 ?

[JNTU : Nov.-12

Ans. : Given data : Mean $\mu = 76$



$$\text{Variance } \sigma^2 = 256$$

$$\sigma = 16$$

$$n = 100, \quad \bar{x}_1 = 75, \quad \bar{x}_2 = 78$$

$$Z_1 = \frac{\bar{x}_1 - \mu}{\sigma/\sqrt{n}} = \frac{75-76}{16/\sqrt{100}} = \frac{-1}{16/10} = -0.625$$

$$Z_2 = \frac{\bar{x}_2 - \mu}{\sigma/\sqrt{n}} = \frac{78-76}{16/\sqrt{100}} = \frac{2}{16/10} = 1.25$$

$$P(75 < \bar{x} < 78) = P(-0.625 < Z < 1.25) = P(-0.625 < Z < 0) + P(0 < Z < 1.25)$$

$$= P(0 < Z < 0.625) + P(0 < Z < 1.25) = 0.2324 + 0.3944 = 0.6268$$

Q.36 A population consists of four numbers 2, 3, 4, 5. Consider all possible distinct samples of size two with replacement find :

a) The population mean

b) The population standard deviation (s.d)

c) The sampling distribution of means

d) The mean of the S.D of means

e) s.d. of S.D of means. Verify (c) and (e) directly from (a) and (b) by use of suitable formulae.

[JNTU : Jan.-1

Ans. : a) Population mean (μ)

$$\mu = \frac{2+3+4+5}{4} = \frac{14}{4} = 3.5$$

b) The population standard deviation

$$\sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n} = \frac{(2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2}{4} = \frac{2.25 + 0.15 + 0.25 + 2.25}{4}$$

$$\sigma^2 = 1.25$$

$$\sigma = 1.118$$

c) The sampling distribution of means (Sampling with replacement)

$$N^n = (4)^2 = 16 \text{ (sample size = 2)}$$

$$N = \text{Population size}$$

$$n = \text{Sample size listing}$$

$$\text{Sampling distribution is : } \begin{Bmatrix} (2,2),(2,3),(2,4),(2,5) \\ (3,2),(3,3),(3,4),(3,5) \\ (4,2),(4,3),(4,4),(4,5) \\ (5,2),(5,3),(5,4),(5,5) \end{Bmatrix}$$

Sample value	Total of sample values	Distribution means
2, 2	4	2
2, 3	5	2.5
2, 4	6	3
2, 5	7	3.5
3, 2	5	2.5
3, 3	6	3
3, 4	7	3.5
3, 5	8	4
4, 2	6	3
4, 3	7	3.5
4, 4	8	4
4, 5	9	4.5
5, 2	7	3.5
5, 3	8	4
5, 4	9	4.5
5, 5	10	5

$$\mu_{\bar{x}} = \frac{\text{Sum of all sample means}}{16}$$

$$= \frac{2+2.5+3+3.5+2.5+3+3.5+4+3+3.5+4+4.5+3.5+4+4.5+5}{16}$$

$$= \frac{56}{16} = 3.5$$

Considering $\mu_{\bar{x}} = \mu$

d) The mean of the S.D. of means

$$\sigma_{\bar{x}}^2 = \frac{1}{16}\begin{bmatrix}(2-3.5)^2+(2.5-3.5)^2+(3-3.5)^2+(3.5-3.5)^2 \\ +(2.5-3.5)^2+(3-3.5)^2+(3.5-3.5)^2+(4-3.5)^2 \\ +(3-3.5)^2+(3.5-3.5)^2+(4-3.5)^2+(4.5-3.5)^2 \\ +(3.5-3.5)^2+(4-3.5)^2+(4.5-3.5)^2+(5-3.5)^2\end{bmatrix}$$

$$= \frac{\begin{bmatrix}2.25+1+0.25+0+1+0.25+0+0.25+0.25 \\ +0.25+1+0+0.25+1+2.25\end{bmatrix}}{16}$$

$$= \frac{10}{16} = 0.625 = \sqrt{0.625} = 0.79$$

e) s.d. of SD mean

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = \frac{(1.118)^2}{2} = 0.6249$$

Q.37 In a study of an automobile insurance a random sample of 80 body repair costs had a mean of 472.36 and a standard deviation of ₹ 62.35. If $\bar{X}$ is used as a point estimate to the true average repair costs, with what confidence we can assert that the maximum error does not exceed ₹ 10 ?

Ans. : Standard deviation (σ) = 62.35

Sample size (n) = 80

The maximum error (E) = 10

$$\text{The maximum error (E)} = Z_{\alpha/2} \frac{\text{Standard deviation } (\sigma)}{\sqrt{\text{Sample size (n)}}}$$

$$10 = Z_{\alpha/2} \frac{62.35}{\sqrt{80}}$$

$$Z_{\alpha/2} = (10 \times \sqrt{80})/62.35$$

$$Z_{\alpha/2} = 1.43$$

So it is not significant and H_0 is accepted, i.e. the claim is true.

Q.38 Explain test of significance for small samples.

Ans. : • When a small sample (size < 30) is considered, the tests are inapplicable because the assumptions we made for large sample tests, do not hold good for small samples.

- In case of small samples it is not possible to assume
  - i) that the random sampling distribution of a statistics normal
  - ii) the sample values are sufficiently close to population values to calculate the S.E. of estimate.
- Thus an entirely new approach is required to deal with problems of small samples. But one should note that the methods and theory of small samples are applicable to large samples but its converse is not true
- When sample sizes are small, as is often the case in practice, the Central Limit Theorem does not apply. One must then impose stricter assumptions on the population to give statistical validity to the test procedure. One common assumption is that the population from which the sample is taken has a normal probability distribution to begin with.
- Degree of freedom (df) : By degree of freedom we mean the number of classes to which the value can be assigned arbitrarily or at will without voicing the restrictions or limitations placed.
- For example, we are asked to choose any 4 numbers whose total is 50. Clearly we are at freedom to choose any 3 numbers say 10, 23, 7 but the fourth number, 10 is fixed since the total is 50 [50 − (10 + 23 + 7) = 10]. Thus we are given a restriction, hence the freedom of selection of number is 4 − 1 = 3.
- The degree of freedom (df) is denoted by ν (nu) or df and it is given by $\nu = n - k$, where n = number of classes and k = number of independent constrains

Q.39 What is one sided test ?

Ans. : A one-sided test is a statistical hypothesis test in which the values for which we can reject the null hypothesis, H0 are located entirely in one tail of the probability distribution

Q.40 What is mode ?

Ans. : • The mode is the most frequent value in a set. A set can have more than one mode; if it has two, it is said to be bimodal.

- The mode is the most frequently occurring value and is commonly used with qualitative data as the values are categorical.
- Categorical data cannot be added, subtracted, multiplied or divided, so the mean and median cannot be computed. The mode is less commonly used with quantitative data as a measure of centre.

Q.41 Define variability.

Ans. : Variability refers to how spread out a group of data is. The common measures of variability are the range, IQR, variance, and standard deviation. Data sets with similar values are said to have little variability while data sets that have values that are spread out have high variability

Q.42 Define expected value.

Ans. : The expected value (also called the mean) of a random variable is the sum of the product of each possible value and its corresponding probability

Q.43 Explain t-test for single mean.

Ans. : • When the sample values come from a normal distribution, the exact distribution of "t" was worked out by W. S. Gossett. He called it a **t-distribution.**

• Unfortunately, there is not one t-distribution. There are different t-distributions for each different value of n. If n = 7 there is a certain t-distribution but if n = 13 the t-distribution is a little different. We say that the variable t has a t-distribution with n–1 degrees of freedom.

• Suppose a simple random sample of size n is drawn from a population. If the population from which the sample is taken follows a normal distribution, the distribution of the random variable

$$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}$$

follows **Student's t-Distribution** with n – 1 degrees of freedom.

• The sample mean is $\bar{x}$ and the sample standard deviation is s.

• The **degrees of freedom** are the number of free choices left after a sample statistic such as is calculated. When you use a t-distribution to estimate a population mean, the degrees of freedom are equal to one less than the sample size.

$$\text{d.f.} = n - 1$$

Assumptions :

1. Population is normal although this assumption can be relaxed if sample size is "large".
2. Random sample was drawn from the population of interest.

• Based on the comparison of calculated 't' value with the theoretical 't' value from the table, we conclude :

Shape of student's t-distribution

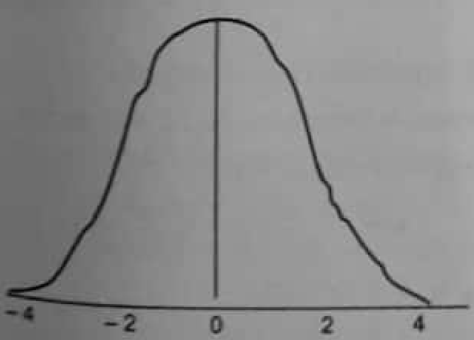


Fig. Q.43.1

Q.44 What are the properties of students t-distribution

Ans. : Properties of Student's t-Distribution

1. The t-distribution is different for different degrees of freedom.
2. The t-distribution is centered at 0 and symmetric about 0.
3. The total area under the curve is 1. The area to the left of 0 is 1/2 and the area to the right of 0 is 1/2.
4. As the magnitude of t increases the graph approaches but never equals 0.
5. The area in the tails of the t-distribution is larger than the area in the tails of the normal distribution.
6. The shape of the t-distribution is dependent on the sample size n.
6. As sample size n increases, the distribution becomes approximately normal.
7. The standard deviation is greater than 1.
8. The mean, median, and mode of the t-distribution are equal to zero.
9. The area in the tails of the t-distribution is a little greater than the area in the tails of the standard normal distribution, because we are using s as an estimate of σ, thereby introducing further variability.
10. As the sample size n increases the density of the curve of t get closer to the standard normal density curve. This result occurs because as the sample size n increases, the values of s get closer to σ, by the Law of Large numbers.

T-critical values

• Critical values for various degrees of freedom for the t-distribution are (compared to the normal)

n	Degrees of freedom	$t_{0.025}$
6	5	2.571
16	15	2.131
31	30	2.042
101	100	1.984
1001	1000	1.962
Normal	"Infinite"	1.960

Q.45 Discuss briefly t-test for correlation coefficients.

Ans. : • The correlation coefficient, ρ (rho), is a popular statistic for describing the strength of the relationship between two variables.

- The correlation coefficient is the slope of the regression line between two variables when both variables have been standardized by subtracting their means and dividing by their standard deviations. The correlation ranges between plus and minus one.
- When ρ is used as a descriptive statistic, no special distributional assumptions need to be made about the variables (Y and X) from which it is calculated.
- When hypothesis tests are made, you assume that the observations are independent and that the variables are distributed according to the bivariate-normal density function.
- However, as with the t-test, tests based on the correlation coefficient are robust to moderate departures from this normality assumption.
- The population correlation ρ is estimated by the sample correlation coefficient r. Note we use the symbol R on the screens and printouts to represent the population correlation.
- t-test for correlation coefficients formula

$$t = r\sqrt{\frac{n-2}{1-r^2}}$$

With degrees of freedom equal to n – 2.

- The steps to be followed for the t-test for correlation coefficient is listed below :

1. State the null hypothesis and alternative hypothesis.

   $H_0: \rho = 0$

   $H_0: \rho \neq 0$

   Here ρ is the population correlation coefficient.
2. State the significance level.
3. Find the test statistic of correlation coefficient with the above-defined formula.
4. To make a decision, use the critical value approach or the p-value approach
5. Finally, state the conclusion.

- The above test is conducted with the supposition that the association is linear between the variables and originate from a normal distribution that is bivariate.
- The t-test is always used for population correlation coefficient of zero. So, in order to test the population correlation coefficient other than zero, z-test for correlation coefficient is used to test the significance of the correlation coefficient.

Q.46 Ten individuals are chosen at random from the population and their heights are found to be inches 63, 63, 64, 65, 66, 69, 69, 70, 70, 71. Discuss the suggestion that the mean height in the universe is 65 inches given that for 9 degree of freedom the value of student's 't' at 0.05 level of significance is 2.262

Ans. : x_i = 63, 63, 64, 65, 66, 69, 69, 70, 70, 71

n = 10

$(x_i - x)^2$
$(63-67)^2 = 16$
$(63-67)^2 = 16$
$(64-67)^2 = 9$
$(66-67)^2 = 1$
$(69-67)^2 = 4$
$(69-67)^2 = 4$
$(70-67)^2 = 9$
$(70-67)^2 = 9$
$(71-67)^2 = 16$
$(63-67)^2 = 9$

∴ $$\bar{x} = \frac{\sum x_i}{n} = \frac{670}{10} = 67$$

and $$S = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{88}{9}} = 3.13 \text{ inches}$$

The null hypothesis $H_0: \mu = 65$ inches

The alternative hypothesis $H_\alpha: \mu \neq 65$ inches

The df = n – 1 = 10 – 1 = 9

$$t = \frac{|\bar{x} - \mu|}{S}\sqrt{n} = \frac{67-65}{3.13}\sqrt{10} = 2.02$$

But $t_{0.05}$ at df = 9 = 2.02

∴ $t = 2.02 < 2.262$

The difference is not significant at a $t_{0.05}$ level thus H_o is accepted and we conclude that the mean height is 65 inches.

Q.47 What is F-Distribution ?

Ans. : • In probability theory and statistics, the **F-distribution** is a continuous probability distribution. It is also known as **Snedecor's F-distribution** or the **Fisher-Snedecor distribution.**

• A random variate of the F-distribution arises as the ratio of two chi-squared variates :

$$\frac{U_1 / d_1}{U_2 / d_2}$$

Where : U_1 and U_2 have chi-square distributions with d_1 and d_2 degrees of freedom respectively and U_1 and U_2 are independent.

• The F-distribution arises frequently as the null distribution of a test statistic, especially in likelihood-ratio tests, perhaps most notably in the analysis of variance;

• The probability density function of an $F(d_1, d_2)$ distributed random variable is given by

$$g(x) = \frac{1}{B(d_1/2, d_2/2)} \left(\frac{d_1 x}{d_1 x + d_2}\right)^{d_1/2} \left(1 - \frac{d_1 x}{d_1 x + d_2}\right)^{d_2/2} x^{-1}$$

for real $x \geq 0$, where d_1 and d_2 are positive integers, and B is the beta function.

• An F random variable is defined as a ratio of two independent chi-square random variables.

Q.48 List the basic properties of F-distribution.

Ans. : Basic Properties of F-distributions

1. The total area under an F-curve equals 1.
2. An F-curve is only defined for $x \geq 0$.
3. An F-curve has value 0 at $x = 0$, is positive for $x > 0$, extends indefinitely to the right, and approaches 0 as $x \to +\infty$
4. An F-curve is right-skewed.

Testing the equality of two variances

1. Test assumption of equal variances that was made in using the t-test
2. Interest in actually comparing the variance of two populations

• Assume we repeatedly select a random sample of size n from two normal populations. Consider the distribution of the ratio of two variances :

$$F = \frac{S_1^2}{S_2^2}$$

• The distribution formed in this manner approximates an F distribution with the following degrees of freedom : $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$.

• The F table gives the critical values of the F-distribution which depend upon the degrees of freedom.

Q.49 Suppose a sample of size 25 is drawn from the normal process which is to be compared to a sample of a new process that has been developed to reduce the variability of impurities.

	Sample 1	Sample 2
n	25	25
S^2	1.04	0.51

Ans. : 1. Formulate the null and alternate hypotheses.

$H_0 : S_1^2 = S_2^2$

$H_1 : S_1^2 > S_2^2$

2. Calculate the F ratio.

$$F(24, 24) = \frac{S_1^2}{S_2^2} = \frac{1.04}{0.51} = 2.039$$

3. Assuming $\alpha = 0.05$ then
Critical value = 1.98 < 2.039

Reject null hypothesis (H_0) and conclude that the variability in the new process (Sample 2) is less than the variability in the original process.

Q.50 For an F-distribution find

i. $F_{0.05}$ with $V_1 = 7$ and $V_2 = 15$

ii. $F_{0.01}$ with $V_1 = 24$ and $V_2 = 19$

iii. $F_{0.95}$ with $V_1 = 19$ and $V_2 = 24$

iv. $F_{0.99}$ with $V_1 = 28$ and $V_2 = 12$

☞[JNTU : Nov.-10]

Ans. : Here we refer critical values of the F-distribution.

i. For $F_{0.05}$ with $V_1 = 7$ and $V_2 = 15$ is 2.71

ii. For $F_{0.01}$ with $V_1 = 24$ and $V_2 = 19$ is 2.92

iii. For $F_{0.95}$ with $V_1 = 19$ and $V_2 = 24$

For $F_{0.05}$ $V_1 = 19$ and $V_2 = 24$ is 2.11.

So $$F_{0.95} = \frac{1}{F_{0.05}(V_1, V_2)} = \frac{1}{2.11} = 0.47393$$

iv. $F_{0.99}$ with $V_1 = 28$ and $V_2 = 12$

$F_{0.01}$ with $V_1 = 28$ and $V_2 = 12$ is 2.90

Then $$F_{0.99}(28, 12) = \frac{1}{F_{0.01}(28, 12)} = \frac{1}{2.90} = 0.344827$$

Q.51 In one sample of 10 observations, the sum of the squares of the deviations of the sample values from sample mean was 120 and in the other sample of 12 observations, it was 314, test whether the difference is significant at 5 % level.

☞[JNTU : Nov.-10]

Ans. : Given data : $n_1 = 10$, $n_2 = 12$, $\alpha = 0.05$,

$$\sum (x-\bar{x})^2 = 120, \quad \sum (y-\bar{y})^2 = 314$$

Null hypothesis (H_0) : $\sigma_1^2 = \sigma_2^2$

Alternative hypothesis (H_1) : $\sigma_1^2 \neq \sigma_2^2$

Calculate S_1^2 and S_2^2 :

$$S_1^2 = \frac{\sum (x-\bar{x})^2}{n_1 - 1} = \frac{120}{10-1} = \frac{120}{9} = 13.333$$

$$S_2^2 = \frac{\sum (y-\bar{y})^2}{n_2 - 1} = \frac{314}{12-1} = \frac{314}{11} = 28.545$$

$S_2^2 > S_1^2$ so null hypothesis (H_0) is true.

Test statistic

$$F = \frac{S_2^2}{S_1^2} = \frac{28.545}{13.333} = 2.14$$

Calculate degree of freedom (d.f.)

$$V_1 = n_1 - 1 = 10 - 1 = 9$$

$$V_2 = n_2 - 1 = 12 - 1 = 11$$

Level of significant at 5 % for ($V_1 = 9$) and ($V_2 = 11$) is 2.90.

Calculated value of (F = 2.14) < 2.90 table value

Null hypothesis (H_0) is accepted.

Conclusion : Difference is not significance at 5 % level.

Fill in the blanks for Mid Term Exam

Q.1 The area under the curve is always ________.

Q.2 When sample sizes are small, as is often the case in practice, the ________ theorem does not apply

Q.3 The ________ value of a random variable is the sum of the product of each possible value and its corresponding probability

Q.4 The degree of freedom (df) is denoted by ________.

Q.5 The ________ is also known as Snedecor's F distribution or the Fisher-Snedecor distribution

UNIT - V

5 Stochastic Processes and Markov Chains

5.1 : Stochastic Processes and Markov Processes

Important Points for Remember

- A stochastic process is the family (ensemble) of random variables defined on specific probability sample space.
- Stocahstic processes can be classified as strictly stationary, wide-sense stationary, markov process, independent process, poisson process or bernall process.
- This chapter mainly concerns with classification of stochastic processes as markov process.
- The stochastic process will be stationary in strict sense if its n^{th} order joint CDF satisfies,

  $F_X(x;t) = F_X(x;t+\tau)$
- A stochastic process $\{X(t); t \in T\}$ is called a markov process if for any time $t_0 < t_1 < t_2 < ... < t_n < t$, the conditional distribution of $X(t)$ depends only on $X(t_n)$.
- The stochastic process is said to be wide sense stationary if its mean and auto-correlation function are independent of time origin.
- All stationary processes are wide sense stationary but converse is not always true.
- The transition probability matrix gives the probabilities of transitions among individual states.

Part A : Short Answered Questions

Q.1 Define stochastic process.

Ans. : The evolvement of random variables over time is represent by a stochastic process. The stochastic process is basically an ensemble or sample space. The sample space consists of sample points whose outcome is random function of time.

Q.2 Define discrete time and continuous time stochastic processes.

Ans. : Discrete time SP : When the state of the system can be observed at discrete instants of time, it is called discrete time SP.

Continuous time SP : When the state of the system can be observed at any time, it is called continuous time SP.

Q.3 Define Markov process.

Ans. : The Markov process is the stochastic process that has following properties.

a) The number of possible outcomes or states are finite.

b) The outcome at any stage depends only upon outcome of the previous stage.

c) The probabilities are constant over time.

Q.4 Define homogeneous markov process.

Ans. : When the conditional distribution function of a process has the time invariance property, it is called time-homogeneous markov process or chain. This means the process has stationary and independent increments.

Q.5 State the properties of probability transition matrix.

Ans. : i) Each probability p_{ij} of a probability transition matrix represents the probability of transition from state 'i' at a previous time to state 'j' at a current time.

ii) The sum of all the probabilities in each row is equal to 1. i.e.,

$$\sum_{j=1}^{s} p_{ij} = 1 \text{ for each } i$$

iii) All the elements of probability transition matrix are non-negative.

Q.6 Define transition probability.

Ans. : The probability of transition from preceding state 'i' to current state 'j' is denoted by p_{ij}. It is called transition probability. It is represented as,

$$p_{ij} = P(X_1 = j/X_0 = i)$$

The transition probability is non negative and lies between 0 and 1.

Part B : Long Answered Questions

Q.7 Explain stochastic processes and its types in detail.

Ans. : Stochastic process : It is the family or ensemble of random variables defined over a given probability sample space. It is represented as $\{X(t);\ t \in T\}$. Here 't' varies over the index set T.

- The values assumed by random variable $X(t)$ are called states. The set of all such possible states is called state space of the process.
- The stochastic process can be discrete time or continuous time. When the states are discrete, it is called stochastic chain.
- **Strictly stationary process :** The stochastic process $\{X(t);\ t \in T\}$ is said to be strictly stationary if its n^{th} order joint CDF satisfies the condition,

$$F_X(x;t) = F_X(x;\ t+\tau)$$

Here 'τ' is the scalar such that $t_i + \tau \in T$.

- **Independent stochastic process :** The stochastic process $\{X(t);\ t \in T\}$ is said to be independent if its n^{th} order joint distribution function satisfies the condition,

$$F_X(x;t) = \prod_{i=1}^{n} F_X(x_i;t_i)$$

- **Markov process :** A stochastic process $\{X(t);\ t \in T\}$ is called markov process if for any $t_0 < t_1 < t_2 < \ldots < t_n < t$, the conditional distribution of $X(t)$ depends only on $X(t_n)$.
- **Wide sense stationary process :** A stochastic process is said to be wide sense stationary if,

a) its mean value is indpendent of time

b) its autocorrelation function is independent of time origin

c) its second moment is finite.

- **Poisson process :** When all the random variables are identically and independently distributed. It is a continuous time discrete state process. It is mainly used in counting the number of events occurring in the interval $(0, t)$.

Q.8 Explain markov process in detail.

Ans. : The stochastic process $\{X(t);\ t \in T\}$ is called a markov process if current state depends only upon previous state. i.e.,

$$P(X_{t+1} = i_{t+1}/X_t = i_t, X_{t-1} = i_{t-1}, \ldots)$$

$$= P(X_{t+1} = i_{t+1}/X_t = i_t).$$

- This means only preceding state determines the current state rest of the past is irrelevant.
- First order markov process satisfies the following property,

$$P(X_{t+1} = i_{t+1}/X_t = i_t, x_{t-1} = i_{t-1}, \ldots)$$

$$= P(X_{t+1} = i_{t+1}/X_t = i_t).$$

- The m^{th} order discrete time markov process satisfies the following properly,

$$P(X_{t+1} = i_{t+1}/X_t = i_t, X_{t-1} = i_{t-1}, \ldots)$$

$$= P(X_{t+1} = i_{t+1}/X_t = i_t \ldots X_{t-m+1} = i_{t-m+1}).$$

This means the current state of the m^{th} order markov process depends upon 'm' number of recent states in the past.

- The markov process is called the markov chain if the state space is discrete, i.e. it is finite or countable states.
- The transition from preceding state to current state is defined by transition probabilities,

$$P(X_{t+1} = i_{t+1}/X_t = i_t)$$

Here 'i_{t+1}' is the current state and 'i_t' is the preceding state.

- The markov process is called time homogeneous if the transition probabilities are independent of time 't'.

Q.9 What is transition probability matrix ? How it is obtained ?

Ans. : Consider the probability $P(X_{t+1} = j/X_t = i)$. This conditional probability represents the probability that the system will be in state 'j' at time $t+1$ given it is in state 'i' at time t.

- A markov chain is called stationary markov chain if $P(X_{t+1} = j/X_t = i)$ is independent of time t. i.e.,

$$P(X_{t+1} = j/X_t = i) = P(X_t = j/X_{t-1} = i)$$
$$= P(X_1 = j/X_0 = i) = p_{ij}$$

Since all the probabilities in above equation does not change, they. are represented by p_{ij}.

- This means 'p_{ij}' is the transition probability form state 'i' at a previous time to state 'j' at a current time.

- If there are total 's' states of the markov chain, then the transition probabilities can be represented by an $s \times s$ matrix.

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & \vdots & & \vdots \\ p_{1s} & p_{2s} & \cdots & p_{ss} \end{bmatrix}$$

- All the probabilities in above matrix are non-negative since probability is $0 \le p \le 1$.

5.2 : Markov Chain

- A discrete time markov process is called markov chain.
- The markov chain is said to be time homogeneous if it shows time invariance.
- Markov chains are used to determine transitions among individual states in a discrete time stochastic process.

Part A : Short Answered Questions

Q.10 Define markov chain.

Ans. : Markov chain is the succession of states of a markov process. Let 'x_0' represents initial state of the system. And let the matrix 'M' represents the matrix of probabilities of transitions. Hence after one iteration, the new state of the system will be Mx_0. After second iteration, the new state will be M^2x_0. Thus we get a chain of states : x_0, Mx_0, M^2x_0, ..., M^nx_0. Such a chain of states is called *Markov chain.*

Q.11 Define first order markov chain.

Ans. : The transition to current state depends only upon preceding state in first order markov chain. i.e.,

$$P(X_{t+1} = i_{t+1}/X_t = i_t, X_{t-1} = i_{t-1}, \ldots)$$
$$= P(X_{t+1} = i_{t+1}/X_t = i_t)$$

For example, if there are four states a, b, c, d and the transition is a → c → d → b This means that in first order markov process the transition to state 'b' will only depend upon state 'd' and not on states 'a' and 'c'.

Part B : Long Answered Questions

Q.12 Explain markov chain with the help of an example.

Ans. : A discrete time stochastic process is a markov chain if the states have following relation :

$$P(X_{t+1} = i_{t+1}/X_t = i_t, X_{t-1} = i_{t-1}, X_1 = i_1, X_0 = i_0)$$
$$= P(X_{t+1} = i_{t+1}/X_t = i_t)$$

for all the states and t = 0, 1, 2, ...

- The left hand side of above equation is the conditional probability. It represents the probability that at time $t+1$, the state is i_{t+1} given that at time '0' the state is i_0; at time '1' the state is i_1; at time 't' the state is i_t.
- The right hand side is also conditional probability. It represents the probability that at time $t+1$, the state is i_{t+1} given that at time 't' the state is i_t.
- Above equation indicates that the probability distribution of state at time '$t+1$' depends only on the state at time 't'. It doesnot depend states before time 't'.

Q.13 Construct the markov chain for the binary communication channel shown in Fig. Q.13.1 also obtain pobability transition matrix.

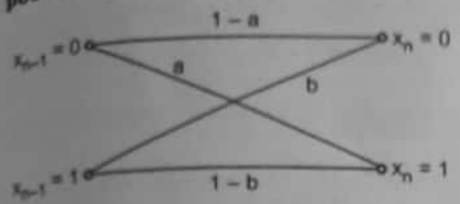


Fig. Q.13.1 Binary communucation channel

Ans. : Here observe that there are two states '0' and '1'. The state '0' to '0' occurs with transition probability $1-a$. Similarly state '0' to '1' occurs with transition probability a.

- And the transition from '1' to '0' occurs with transition probability of 'b'. Similarly the state '1' to '1' transition occurs with probability $1-b$.
- Fig. Q.13.2 shows the markov chain for above description.

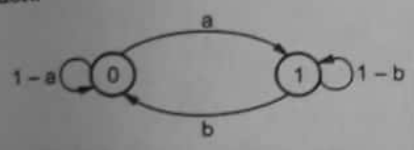


Fig. Q.13.2 Markov chain

- The probability transition matrix can be written as follows :

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \end{array} \quad \text{Here } 0 < a \text{ and } b < 1$$

Q.14 At time '0' you have ₹ 20. You bet ₹ 10 each time in a gamble. The probability of your winning is 0.3 and loosing is 0.7. The game is over when you have ₹ 0 or ₹ 40. Model this gambling problem using markov chain. Also obtain probability transition matrx.

Ans. : Step 1 : Here initially you have ₹ 20. Let this state be $X_0 = 20$. If you win, you will get ₹ 10 and you will have total ₹ 30. Let his state be $X_1 = 30$. The probability of winning is 0.3. Hence probability of transition from state X_0 to X_1 is 0.3. This is shown in Fig. Q.14.1.

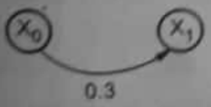


Fig. Q.14.1

Step 2 : From step $X_0 = 20$, you may loose with probability of 0.7. Then you will go to state $X_2 = 10$ because you will loose ₹ 10. This part is shown in Fig. Q.14.2.

Fig. Q.14.2

Step 3 : From state $X_1 = 30$, if you win, you will go to new state with $X_3 = 40$. And if you loose you will come back to state $X_0 = 20$. This is shown in Fig. Q.14.3.

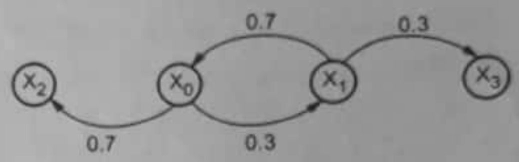


Fig. Q.14.3

Step 4 : When you are in state $X_2 = 10$, if you win, you will go to state $X_0 = 20$ and if you loose, you will go to state $X_4 = 0$. This is shown in Fig. Q.14.4.

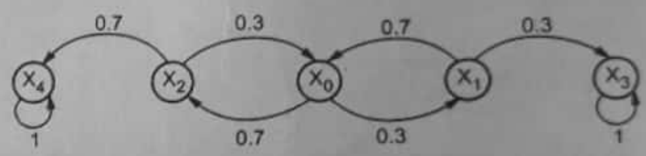


Fig. Q.14.4 Complete markov chain

Step 5 : When you are in state 4, you have ₹ 0 and the game is over. Hence you remain in state X_4 only with probability of 1. Similarly when you are in state X_3, you have ₹ 40 and the game is over. Hence you remain in state X_3 only with probability of 1.

Probability transition matrix can be written as follows :

$$P = \begin{array}{c c} & \begin{array}{ccccc} X_0 & X_1 & X_2 & X_3 & X_4 \end{array} \\ \begin{array}{c} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array} & \begin{bmatrix} 0 & 0.3 & 0.7 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0.3 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

This matrix gives the transition from i^{th} state to j^{th} state.

Q.15 The persion goes from high risk to high risk with probability of 0.6 when the policy renews. And he goes from low risk to high risk with probability 0.15 when the policy renews. Construct the markov chain and obtain probability transition matrix.

Ans. : Let high risk be represented by state 'H' and low risk be represented by state 'L'. Then following are the probabilities among low risk and high risk state. i.e.,

$$P(H \to H) = 0.6, \text{ hence } P(H \to L) = 1 - 0.6 = 0.4$$

$$P(L \to H) = 0.15, \text{ hence } P(L \to L) = 1 - 0.15 = 0.85$$

Fig. Q.15.1 shows the markov chain for above probabilities.

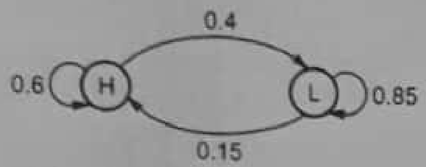


Fig. Q.15.1 Markov chain

- The state transition matrix can be written as follows :

$$P = \begin{array}{c} \\ H \\ L \end{array}\!\!\begin{array}{c} \begin{matrix} H & L \end{matrix} \\ \begin{bmatrix} 0.6 & 0.4 \\ 0.15 & 0.85 \end{bmatrix} \end{array}$$

Q.16 In a basket ball, if the player makes the first shot, he is twice likely to make the second shot. And if he misses the first shot he is three time likely to miss the second shot. Obtain markov chain and state transition matrix.

Ans. : There are two states 'make' and 'miss'. Let these be denoted by 'Mk' and 'Ms'. Following data is given,

$$P(Mk \to Mk) = 2x$$

$$P(Mk \to Ms) = x$$

Since $P(Mk \to Mk) + P(Mk \to Ms) = 1$

$$\therefore \quad 2x + x = 1 \Rightarrow x = \frac{1}{3}$$

$$\therefore P(Mk \to Mk) = 2x = 2 \times \frac{1}{3} = \frac{2}{3}$$

and $P(Mk \to Ms) = x = \frac{1}{3}$

Similarly, $P(Ms \to Mk) = x$

$$P(Ms \to Ms) = 3x$$

Since $P(Ms \to Mk) + P(Ms \to Ms) = 1$

$$\therefore \quad x + 3x = 1 \Rightarrow x = \frac{1}{4}$$

$$\therefore P(Ms \to Mk) = \frac{1}{4}$$

$$P(Ms \to Ms) = 3x = 3 \times \frac{1}{4} = \frac{3}{4}$$

Fig. Q.16.1 shows the markov chain.

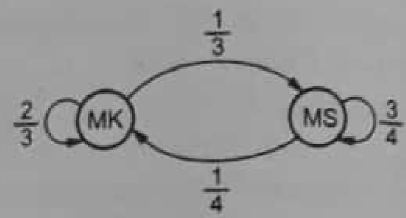


Fig. Q.16.1

The state transition matrix can be written as,

$$P = \begin{array}{c} \\ Mk \\ Ms \end{array}\!\!\begin{array}{c} \begin{matrix} Mk & Ms \end{matrix} \\ \begin{bmatrix} 2x & x \\ x & 3x \end{bmatrix} \end{array} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Q.17 Obtain the markov chain and state transition matrix for error free binary communication channel.

Ans. : Fig. Q.17.1 shows the error free binary communication channel.

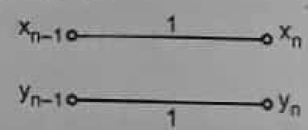


Fig. Q.17.1 Error free channel

Here

$$P(x_{n-1} \to x_n) = 1 \text{ and } P(x_{n-1} \to y_n) = 0$$

$$P(y_{n-1} \to y_n) = 1 \text{ and } P(y_{n-1} \to x_n) = 0$$

The state transition matrix is written as,

$$P = \begin{array}{c} \\ x \\ y \end{array}\!\!\begin{array}{c} \begin{matrix} x & y \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Based on above, the markov chain is written below :

Fig. Q.17.2 Markov chain for error free channel

5.3 : Higher Order Markov Processes

- Higher order markov processes require memory.
- The transition of multiple states can be studied with the help of higher order markov processes.

Part A : Short Answered Questions

Q.18 Define higher order markov chains.

Ans. : Higher order markov chains requires more memory. This means the probability of transition from $(i-1)^{th}$ state to i^{th} state depends upon some of the previous states. i.e.,

$$P(x_i/x_{i-1}, x_{i-2}, \ldots, x_1) = P(x_i/x_{i-1}, x_{i-2}, \ldots, x_{i-n})$$

Thus memory of previous 'n' states determine the transition from $(i-1)^{th}$ to i^{th} state. This is called n^{th} order markov model.

Q.19 Differentiate between first order and second order markov chains.

Ans. :

Sr. No.	First order markov chains	Second order markov chains
1.	The transition to current state depends upon preceding state only.	The transition to current state depends upon previous two states.
2.	Probability of getting in to current state is $P(X_t = i_t/X_{t-1})$.	Probability of getting into current state is $P(X_t = i_t/X_{t-1}, X_{t-2})$.

Part B : Long Answered Questions

Q.20 Explain n^{th} order markov chains in detail.

Ans. : As defined, the n^{th} order markov chain considers the histoty of previous 'n' states from the current state. For example an n^{th} order markov chain over some alphabet 'A' is equivalent to first order markov chain over the alphabet A^n of n - tuples.

- The n^{th} order markov chains are used in applications such as DNA sequences, sales demand predictions, web page predictions and newsboy's problem.
- Fig. Q.20.1 shows the fifth order markov chain for alphabets. Here probability of entering in state *gctac* from begining is $p(gctac)$. From this state, the process can enter in number of other states.

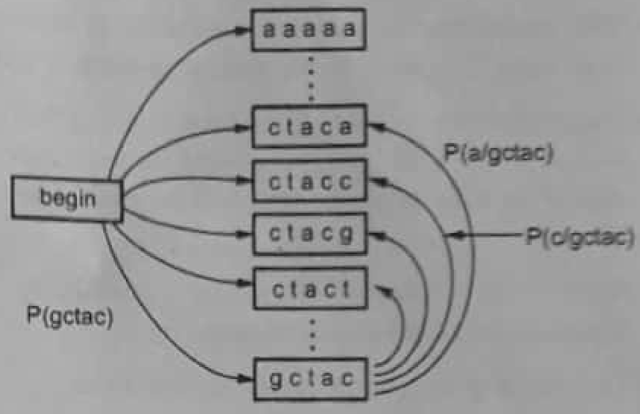


Fig. Q.20.1 Fith order markov chain

- For example, the probability of entering the state *ctaca* from state *gctac* is given as,

$$P(ctaca) = P(gctac) \cdot P(a/gctac)$$

- This is the 5^{th} order markov chain because it evolves around a^5 i.e. five characters combination in each state.
- **Drawback of n^{th} order markov chain :** i) The number of parameters we need to estimate grows exponentially with order for example for modelling DNA by n^{th} order markov chain, we need 4^{n+1} parameters. ii) As the order increases, the parameters are less reliable.

5.4 : n - Step Transition Probabilities

- The transition from current state to n^{th} state can be obtained with the help of n - step transition probabilities.
- It is denoted by $p_{ij}(n) = P(X_{m+n} = j/X_m = i)$.

Part A : Short Answered Questions

Q.21 Define n - step transition probability.

OR Define multi-step transition probability.

Ans. : Let the markov chain be in state 'i' at time t. Then the probability that it will be in state 'j' after 'n' periods is called n - step transition probability.

Part B : Long Answered Questions

Q.22 Explain multistep transition probability.

Ans. : The multi step transition probability is represented by following equation,

$$P(X_{t+n} = j/X_t = i) = P(X_n = j/X_0 = i) = p_{ij}$$

- This represents the n-step probability of transition from state 'i' to state 'j' for stationary markov chain.
- The probability of state 'j' at time '$t + n$' given that it is in state 'i' at time 't' is same as probability of state 'j' at time 'n' given that it is in state 'i' at time '0'.
- Since the n-step transition probability is independent of time, it is denoted by p_{ij}.
- The one step transition matrix is written as

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{22} & \cdots & p_{2s} \\ \vdots & & & \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{bmatrix}$$

- As an example consider the two step transition probability from state 'i' to state 'j' through an intermediate state 'k'. this is shown in Fig. 1.

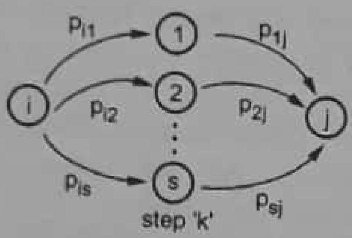


Fig. Q.22.1 Two step chain

- Here 'k' can be any value between 1 to s. Note that the path i - 1 - j mill have probability of 'i' to 'j' transition will be $p_{i1} \times p_{1j}$.
- For the path i - 2 - j the transition from 'i' to 'j' mill have the probability of $p_{i2} \times p_{2j}$
- It is possible to go through any path shown in figure Q.22.1 from 'i' to 'j'. Hence the two step transition probability $p_{ij}(2)$ is equal to the sum of all the probabilities of different paths i.e.,

$$p_{ij}(2) = \sum_{k=1}^{s} p_{ik} p_{kj}$$

- The right hand side of above equation is nothing but ij^{th} element of P^2.

Q.23 Assume that the probability of dry day following rainy day is 0.33 and probability of rainy day following dry day is 0.5. (a) construct a markov chain for this structure. (b) If 1st June is dry day, what is the probability that 3rd and 5th June will be dry day.

Ans. :

a) Markov chain and probability transition matrix

Hence P(Dry following rainy) = P(Rainy to dry) = 0.33

$\therefore$ P(Rainy following rainy) = P(Rainy to rainy)

= 1 − 0.33 = 0.67

And P(Rainy following dry) = P(Dry to rainy) = 0.5

$\therefore$ P(Dry following dry) = P(Dry to dry)

= 1 − 0.5 = 0.5

Fig. Q.23.1 shows the probability transition matrix for given data.

$$P = \begin{array}{c} \\ \text{Dry} \\ \text{Rainy} \\ \uparrow \\ \text{initial state} \end{array} \begin{array}{c} \begin{array}{cc} \text{Dry} & \text{Rainy} \leftarrow \text{Next state} \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.33 & 0.67 \end{bmatrix} \\ \\ \\ \end{array}$$

Let state 'D' represents dry day and state 'R' represents rainy day. Fig. Q.23.1 shows the markov chain for given data.

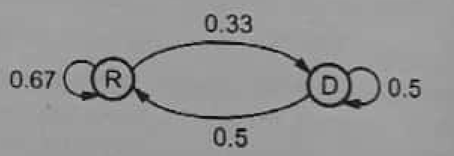


Fig. Q.23.1 Markov chain

b) 1st June dry day P(3rd June day) and P(5th June dry)

Fig. Q.23.2 shows the 1st to 5th June transition of states.

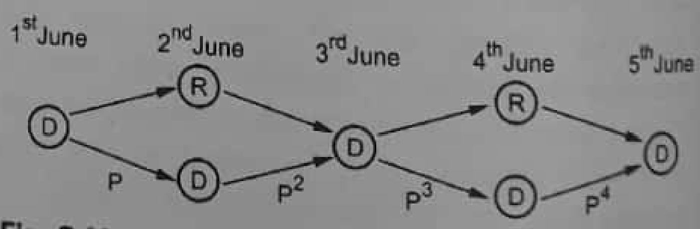


Fig. Q.23.2 Transition of states from 1st June dry to 3rd June dry and 5th June dry

- In the above figure multistep transitions are shown. Let the beginning be with dry day on 1st June. And dry day on 3rd June can be reached through rainy day or dry day on 2nd June. This is a two step transition.

$$\therefore \quad P^2 = P \cdot P = \begin{bmatrix} 0.5 & 0.5 \\ 0.33 & 0.67 \end{bmatrix}\begin{bmatrix} 0.5 & 0.5 \\ 0.33 & 0.67 \end{bmatrix}$$

$$= \begin{bmatrix} 0.415 & 0.585 \\ 0.386 & 0.614 \end{bmatrix}$$

Thus P(1st June dry to 3rd June dry) = 0.415

- Similarly observe that from 3rd June dry to 5th June dry, the transition can take place through rainy day or dry day on 4th June as shown in Fig. Q.23.2. This is 4-step transition from 1st June dry. i.e.,

$$P^4 = P \cdot P \cdot P \cdot P \text{ or } P^2 \cdot P^2$$

$$= \begin{bmatrix} 0.415 & 0.585 \\ 0.386 & 0.614 \end{bmatrix}^2 = \begin{bmatrix} 0.398 & 0.602 \\ 0.397 & 0.603 \end{bmatrix}$$

Thus P(1st June dry to 5th June dry) = 0.398

Q.24 The man either drives his car or takes the train to reach his office each day. He never uses train for two successive days. If he drives to work, then next day he is just likely to drive again as he is to take train. Then

a) Justify this is markov chain

b) Find transition matrix

c) Find the probability that chances his going by train to driving in three days.

Ans. : a) Outcome of each day depends only on previous day. Hence this is markov chain.

b) Let use of train be represented by state '1' and use of car be represented by state '2'.

Since he will not travel by train for two successive days, $p_{11} = 0$. Hence $p_{11} + p_{12} = 1$

$\therefore 0 + p_{12} = 1 \Rightarrow p_{12} = 1$.

When he drives today, then chances of going by train and driving are equal. Hence $p_{21} = \frac{1}{2}$, $p_{22} = \frac{1}{2}$

Hence transition matrix can be written as,

$$P = \begin{array}{c} \\ \text{train} \\ \text{car} \end{array}\begin{array}{c} \begin{array}{cc} \text{train} & \text{car} \end{array} \\ \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

c) Let us find P^3 for 3 days

$$P^3 = P \cdot P \cdot P$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

Form above P^3, the probability that chances his going by train to driving car is $P^3{}_{12} = \frac{3}{4}$.

5.5 : Steady State Condition in Markov Chains

- Under steady state, the probabilities of the states do not change even after transmission.
- Steady state conditions in markov chains are need to study long term behaviour of the states.

Part A : Short Answered Question

Q.25 Define a steady state condition in markov chains.

Ans. : The steady state condition is reached in markov chain, when the probabilities of the state do not change even after transition. i.e.,

$$xP = x$$

This means the state vector remains same even after multiplied by probability transition matrix.

Part B : Long Answered Questions

Q.26 How steady state vectors are obtained ?

Ans. : Consider that the initial vector be 'v'. Then for steady state condition we can write,

$$vP = v$$

$$\therefore \quad vP - v = 0$$

$$\therefore \quad v(P-1) = 0$$

Let $I = 1$ be an identity matrix. Then above equation can be written as,

$$v(P-I) = 0$$

Q.27 For the probability transition matrix $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.15 & 0.85 \end{bmatrix}$, obtain the steady state probabilities.

Ans. :

Here $v(P-I) = 0$ ...(1)

And $v = [x \quad y],\ P = \begin{bmatrix} 0.6 & 0.4 \\ 0.15 & 0.85 \end{bmatrix}$ and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Here v is the unknown steady state vector. Putting values in equation (1).

$$[x \quad y]\left\{\begin{bmatrix} 0.6 & 0.4 \\ 0.15 & 0.85 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = 0$$

$$[x \quad y]\begin{bmatrix} -0.4 & 0.4 \\ 0.15 & -0.15 \end{bmatrix} = 0$$

i.e. $-0.4x + 0.15y = 0$ and

$0.4x - 0.15y = 0$

Above both equations are same. Hence we have one equation,

$0.4\,x - 0.15\,y = 20$ ...(2)

We also know that sum of probabilities 'x' and 'y' equal to 1. i.e.,

$x + y = 1$ ...(3)

Solving equation (2) and equation (3) we get,

$$x = \frac{3}{11} \text{ and } y = \frac{8}{11}$$

Thus $v = [x \quad y] = \begin{bmatrix} \frac{3}{11} & \frac{8}{11} \end{bmatrix}$

Varification of $vP = v$

$$\begin{bmatrix} \frac{3}{11} & \frac{8}{11} \end{bmatrix}\begin{bmatrix} 0.6 & 0.4 \\ 0.15 & 0.85 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{11}\times 0.6 + \frac{8}{11}\times 0.15 & \frac{3}{11}\times 0.4 + \frac{8}{11}\times 0.85 \end{bmatrix} = \begin{bmatrix} \frac{3}{11} & \frac{8}{11} \end{bmatrix}$$

Thus the steady state probabilities are obtained.

Q.28 Bob, Alice and Carol are playing frisbee. Bob always throws to Alice and Alice always throws to Carol. Carol throws to Bob $\frac{2}{3}$ of time and to Alice of $\frac{1}{3}$ of time. In the long run what percentage of the time do each of the players have the frisbee ?

Ans. : Let us form the probability transition matrix of throw among Alice, Bob and Carol.

$$P = \begin{array}{c} \\ \text{Alice} \\ \text{Bob} \\ \text{Carol} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Alice} & \text{Bob} & \text{Carol} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \end{array}$$

Here $v = [x \quad y \quad z]$. Hence consider

$$v(P-I) = 0$$

$$[x \quad y \quad z]\left\{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = 0$$

Solving above matrix we obtain,

$$\left.\begin{aligned} -x+y+\frac{1}{3}z &= 0 \\ -y+\frac{2}{3}z &= 0 \end{aligned}\right\} \text{Adding these two equations give third equation, i.e. } x - z = 0$$

$$x - z = 0$$

Hence we have two equation,

$x - z = 0$ ...(1)

$-y + \frac{2}{3}z = 0$ ...(2)

And the sum of all the probabilities x, y and z will be equal to 1. i.e.,

$x + y + z = 1$ ...(3)

Solving equation (1), (2) and (3) above,

$$x = \frac{3}{8},\ y = \frac{1}{4} \text{ and } z = \frac{3}{8}$$

The percentage of time the players have frisbee is,

Alice $= \frac{3}{8}$, Bob $= \frac{1}{4}$ and Carol $= \frac{3}{8}$