UNIT- I

First order of ODE.

Derivative formulas:-

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$

(2) $\frac{d}{dx}(c) = 0$ [where c is a constant]

(3) $\frac{d}{dx}(x) = 1$

(4) $\frac{d}{dx}(a^x) = a^x \log a$.

(5) $\frac{d}{dx}(|x|) = \frac{|x|}{x}$

(6) $\frac{d}{dx}(\cos x) = -\sin x$

(7) $\frac{d}{dx}(\sin x) = \cos x$

(8) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(9) $\frac{d}{dx}(\tan x) = \sec^2 x$.

(10) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

(11) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(12) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

(13) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

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(14) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

(15) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(16) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

(17) $\frac{d}{dx}(\sec^{-1}x) = \frac{+1}{|x|\sqrt{x^2-1}}$

(18) $\frac{d}{dx}(x^x) = x^x(1+\log x)$

Integration formulas.

(1). $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$.

(2). $\int x^n = \frac{x^{n+1}}{n+1} + c$.

(3) $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c$.

(4) $\int \frac{1}{x}\,dx = \log|x| + c$.

(5) $\int \frac{1}{x^2}\,dx = \frac{-1}{x} + c$

(6) $\int e^x\,dx = e^x + c$.

(7) $\int a^x\,dx = \frac{a^x}{\log a} + c$.

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(8) $\int \sin x \, dx = -\cos x + C.$

(9) $\int \cos x \, dx = \sin x + C.$

(10) $\int \sec^2 x \, dx = \tan x + C$

(11) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C.$

(12) $\int \sec x . \tan x \, dx = \sec x + C.$

(13) $\int \operatorname{cosec} x \cdot \cot x \cdot dx = -\operatorname{cosec} x + C.$

(14) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$ (or) $-\cos^{-1} x + C.$

(15) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$ (or) $-\cot^{-1} x + C.$

(16) $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + C$ (or) $-\operatorname{cosec}^{-1} x + C$

(17) $\int k \cdot f(x) \, dx = k \int f(x) dx + C.$

(18) $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + C.$

(19) $\int k \, dx = kx + C$

(20) $\int \tan x \, dx = \log |\sec x| + C.$

(21) $\int \cot x \, dx = \log |\sin x| + C$

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(22) $\int \sec x\,dx = \log|\sec x + \tan x| + c.$

(23) $\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + c.$

(24) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

(25) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\frac{x}{a} + c = \log\left|x+\sqrt{x^2-a^2}\right| + c.$

(26) $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a} + c = \log\left|x+\sqrt{x^2+a^2}\right| + c.$

(27) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$

(28) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(29) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

(30) $\int e^x\{f(x)+f'(x)\}\,dx = e^x f(x) + c$

(31) * $\int e^x[f(x)-f''(x)]\,dx = e^x[f(x)-f'(x)] + c.$

(32) * $\int e^x f(x)\,dx = e^x\{f(x) - f'(x) + f''(x) - f'''(x) \ldots\}$
where $f(x)$ is a polynomial in x.

(33) *** $\int u\,dv = uv - \int v\,du$ (Integration by parts formula)

(34) $\int \log x\,dx = x\log x - x + c.$

(35) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$

(or)

$$\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c.$$

(36) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c.$

(or)

$$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c.$$

(37) $\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$

(38) $\int e^{ax}\,dx = \frac{e^{ax}}{a} + c.$

Differential equation

⇒ An equation involving derivaties of one or more dependent variables with respective one (or) more independent variables is called a differential equation.

Types of differential equations.

⇒ There are two types of differential equations

1. ordinary differential equations (ODE)
2. partial differential equations (PDE)

ordinary differential equation

⇒ A differential equation is said to be ordinary if the derivates in the equation are ordinary differential equation.

Example $\left(\frac{dy}{dx}\right)^3 - \left(\frac{dy}{dx}\right)^2 + 7y = \cos x$

single independent variable

$\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right) = 2x + y$

two dependent variables

partial differential equation

⇒ A differential equation is said to be partial if the derivates in the equation have reference to two or more independented variable

example:- $\frac{\partial^2 y}{\partial x^2} = \frac{1}{C^2} \cdot \frac{\partial^2 y}{\partial t^2}$

two independent variable

$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$

two dependent variables

order and degree of a differential equations

⇒ The order of the highest order derivate involved in a differential equation is called the order of the differential equation

⇒ A differential equation is said to be of order "n", if the "nth" order derivate is the highest order derivate in that equation

example ① $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2$

* The first order derivate is "$\frac{dy}{dx}$" is the highest derivate in the above equation.

* The order of the above differential equation is "1"

$$\left[\frac{dy}{dx}\right]^{1}$$ order, → degree

(2) $x \cdot \frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$.

* The second order derivate is "$\frac{d^2y}{dx^2}$" is the highest derivate in the above equation.

* The order of the above differential equation is "2"

$$\left(\frac{d^2y}{dx^2}\right)^{1}$$ order, → degree

Problems

1) Form the differential equation by elimating arbitary constant (removing of "c" value) $y = ce^{\sin^{-1}x}$

Given equation $y = ce^{\sin^{-1}x}$ —— (1)

diff w.r.t to "x"

$$\frac{dy}{dx} = c \cdot e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}} \quad \text{—— (2)}$$

$$\frac{dy}{dx} = y \cdot \frac{1}{\sqrt{1-x^2}} \quad [\because \text{from } (1)]$$

$$\sqrt{1-x^2}\,\frac{dy}{dx} - y = 0$$

∴ It is a required solution

→ variable separable :-

The general solution is $\boxed{\int f(x)\,dx - \int g(y)\,dy = c}$

where "c" is constant.

problem

① $\frac{dy}{dx} = e^{2x-3y} + x^2 e^{-3y}$ $\boxed{\therefore e^{a-b} = e^{a} \cdot e^{-b}}$

$\frac{dy}{dx} = e^{2x} \cdot e^{-3y} + x^2 e^{-3y}$

$\frac{dy}{dx} = e^{-3y}(e^{2x} + x^2)$

$\frac{dy}{e^{-3y}} = (e^{2x} + x^2)\,dx$.

$e^{3y}\,dy = (e^{2x} + x^2)\,dx$.

Integrate on both sides,

* $\boxed{\int f(x)\,dx - \int g(y)\,dy = c}$

$\int (e^{2x} + x^2)\,dx - \int e^{3y}\,dy = c$.

$\frac{e^{2x}}{2} + \frac{x^3}{3} - \frac{e^{3y}}{3} = c$

② $\frac{dy}{dx} = \frac{-x}{y}$

$y \cdot dy = -x \cdot dx$

in. o. b. s

$$\int^{x} x \cdot dx + \int y \cdot dy = c$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

$$x^2 + y^2 = 2c$$

$$\int y \cdot dy = -\int x \cdot dx + c$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + c$$

③ $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$

$e^y \cos x\, dx + \cos x\, dx + e^y \sin x\, dy = 0$

$e^y (\sin x + \cos x$

$$\frac{\cos x}{\sin x} dx + \frac{e^y}{e^y + 1} dy = 0$$

$$\cot x\, dx + \left(\frac{e^y}{e^y+1}\right) dy = 0$$

I.O.B.S.

$$\int \cot x\, dx + \int \frac{e^y}{e^y+1} dy = c.$$

$$\int \frac{f'(x)}{f(x)} = \log|f(x)|$$

$\log|\sin x| + \log|e^y + 1| = \log|c|$

$(\sin x)(e^y + 1) = c.$

Exact differential equation (5m)

$\Rightarrow$ let $M(x,y) + N(x,y) = 0$ be a first order & first degree differential equation, where M, N are real valued functions for some x, y. then the equation $Mdx + Ndy = 0$ is said to be an exact differential equation. If there exist a function "f" such that

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N$$

eg:- The differential equation $2xy\,dx + x^2\,dy = 0$ is an exact equation since there exist a function $f(x,y) = x^2 y$

$M = 2xy$ & $N = x^2$ [from question $\underset{M}{2xy}\,dx + \underset{N}{x^2}\,dy = 0$]

$f(x,y) = x^2 y$.

$\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2$.

Condition for exactness:-

If $M(x,y)$ & $N(x,y)$ are two real valued function which have continuous partial derivaty then a necessary and sufficient condition for the differential equation (To be exact) $\boxed{M\,dx + N\,dy = 0}$ to be exact: $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Working rule

To solve an exact differential equation.

(1). let the given differential eqn. be of the form $M\,dx + N\,dy = 0$ after check the condition for the exactness "$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$"

(2). The general solution $\boxed{\underbrace{\int M\,dx}_{(y\ \text{const})} + \underbrace{\int N\,dy}_{\text{terms of } N \text{ not containing "}x\text{"}} = c}$ Regardless

Problem: ① Solve differential equation $(2x-y+1)dx + (2y-x+1)dy = 0$

$$(2x-y+1)dx + (2y-x-1)\,dy = 0 \quad \text{--- ①}$$

$M \qquad\qquad N$

$$M\,dx + N\,dy = 0 \quad \text{--- ②}$$

Step:-1

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$\frac{\partial}{\partial y}(2x-y+1) = -1$$

$$\frac{\partial}{\partial x}(2y-x-1) = -1$$

where :- $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Step:-2

$$\boxed{\int M\,dx + \int N\,dy = c} \rightarrow \text{General solution.}$$

$$\int_{(y\text{ const})} (2x-y+1)\,dx + \int (2y-1)\,dy = c.$$

(terms of N not containing x)

$$\frac{\cancel{2}x^2}{\cancel{2}} - yx + x + \frac{\cancel{2}y^2}{\cancel{2}} - y = c$$

$$\boxed{x^2+y^2+x-y-yx = c}$$

② Solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

$$(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$$

$M \qquad\qquad N$

$M = e^y \cos x + \cos x$.

$N = e^y \sin x$

Step:-1

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(e^y \cos x + \cos x\right)$$

$$\frac{\partial M}{\partial y} = e^y \cos x.$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(e^y \sin x\right)$$

$$= e^y \cos x$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$

Step:-2 General solution

$$\int M \cdot dx + \int N \cdot dy = c$$

$$\int \left(e^y \cos x + \cos x\right) dx + \int 0 \cdot dy = c$$

$$\boxed{e^y \sin x + \sin x = c.}$$

(3) Solve $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$

(4) $(x^2 - y^2)\,dx = 2xy\,dy$.

(5) $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

(6) $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$

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(5) Solution

$(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

$\downarrow M \qquad \downarrow N$

Step:-1

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y\cos x + \sin y + y) = \cos x + \cos y + 1$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\sin x + x\cos y + x) = \cos x + \cos y + 1$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

Step:-2 General solution.

$$\int M\,dx + \int N\,dy = C$$

$$\int (y\cos x + \sin y + y)\,dx + \int 0\,dy = C$$

$y\sin x + \sin y \cdot x + y \cdot x = C$

$$\boxed{y\sin x + x\sin y + xy = C}$$

(7) $x^3\sec^2 y\,\frac{dy}{dx} + 3x^2\tan y = \cos x$

$x^3\sec^2 y\,\frac{dy}{dx} + 3x^2\tan y - \cos x = 0$

$x^3\sec^2 y\,dy + (3x^2\tan y - \cos x)dx = 0$

$(3x^2\tan y - \cos x)\,dx + x^3\sec^2 y\,dy = 0$

$\downarrow M \qquad \downarrow N$

$M = 3x^2\tan y - \cos x, \quad N = x^3\sec^2 y$

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Step:-1

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(3x^2\tan y - \cos x\right)$$

$$= 3x^2\sec^2 y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x^3\sec^2 y\right)$$

$$= 3x^2\sec^2 y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step:-2 General solution

$$\int M \cdot dx + \int N \cdot dy = C$$

$$\int\left(3x^2\tan y - \cos x\right)dx + \int 0 \cdot dy = C$$

$$\frac{\cancel{3}x^3}{\cancel{3}} \cdot \tan y - \sin x = C$$

$$\boxed{x^3\tan y - \sin x = C}$$

(6) $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y + 3x^2y^2 - 5y^4)\,dy = 0$

$M = 5x^4 + 3x^2y^2 - 2xy^3$

$N = 2x^3y - 3x^2y^2 - 5y^4$

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Step:-1

$$\frac{d}{dy}(5x^4+3x^2y^2-2xy^3) = 6x^2y-6xy^2$$

$$\frac{dM}{dy} = 6x^2y-6xy^2$$

$$\frac{dN}{dx} = \frac{d}{dx}(2x^3y-3x^2y^2-5y^4) = 6x^2y-6xy^2$$

$$\frac{dM}{dy} = \frac{dN}{dx}.$$

Step:- General Solution

$$\int M\,dx + \int N\,dy = c.$$

$$\int(5x^4+3x^2y^2-2xy^3)\,dx + \int(-5y^4)\,dy = c.$$

$$\frac{\cancel{5}x^5}{\cancel{5}} + \frac{\cancel{3}x^3}{\cancel{3}}y^2 - \frac{\cancel{2}x^2}{\cancel{2}}y^3 - \frac{\cancel{5}y^5}{\cancel{5}} = c$$

$$x^5+x^3y^2-x^2y^3-y^5 = c.$$

③ $(y^2-2xy)\,dx = (x^2-2xy)\,dy$

$$\underbrace{(y^2-2xy)}_{M}\,dx + \underbrace{(-x^2+2xy)}_{N}\,dy = 0$$

Step:-1

$$\frac{dM}{dy} = \frac{dN}{dx}$$

$$\frac{d}{dy}(y^2-2xy) = 2y-2x \quad \Big| \quad \frac{d}{dx}(-x^2+2xy) = 2y-2x$$

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Step:-2 General solution

$$\int M \cdot dx + \int N \cdot dy = c$$

$$\int (y^2 - 2xy)\, dx + \int 0 \cdot dy = c$$

$$xy^2 - \cancel{2} \cdot \frac{x^2}{\cancel{2}} y = c$$

$$xy^2 - x^2 y = c$$

4) $(x^2 - y^2)\, dx = 2xy\, dy$

$$(x^2 - y^2)\, dx + (-2xy)\, dy = 0$$

$$M\, dx + N\, dy = 0$$

Step:-1

$$\frac{dM}{dy} = \frac{d}{dy}(x^2 - y^2)$$

$$\frac{dM}{dy} = -2y$$

$$\frac{dN}{dx} = \frac{d}{dx}(-2xy) = -2y$$

$$\frac{dM}{dy} = \frac{dN}{dx}$$

Step:- 2 General Solution

$$\int M \cdot dx + \int N \cdot dy = C$$

$$\int (x^2 - y^2) dx + \int 0 \cdot dy = C$$

$$\frac{x^3}{3} - xy^2 = C$$

$$\frac{x^3}{3} - \frac{3xy^2}{3} = C$$

Integrating factors (equations reducable to exact equations).

⇒ ~~let~~ there are some differential equations that are not exact, but they can be made exact by multiplying the given equation by a suitable factor known as integrating factor.

Integrating factors

⇒ let $M(x,y)\,dx + N(x,y)\,dy = 0$ be not an exact differential equation.

⇒ If $M\,dx + N\,dy = 0$ can be made exact by multipling it with a suitable function $u(x,y) \neq 0$ then

Example- let $y\,dx - x\,dy = 0$ —— (1)

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$M\,dx + N\,dy = 0$$

$u(x,y) \neq 0$.

Ex:- Let $y\,dx - x\,dy = 0$ —— (1)

$M\,dx + N\,dy = 0$

$M = y, \quad N = -x$

$$\frac{dM}{dy} = \frac{dM}{dy} = \frac{d}{dy}(y) = 1$$

$$\frac{dN}{dx} = \frac{d}{dx}(-x) = -1$$

$\therefore \frac{dM}{dy} \neq \frac{dN}{dx}$ is not Exact differencial equation

M. w. $\frac{1}{x^2}$ eqn (1)

$$\frac{y}{x^2}\,dx - \frac{x}{x^2}\,dy = 0$$

$$\frac{y}{x^2}\,dx - \frac{1}{x}\,dy = 0 \quad \text{—— (2)}$$

$M_1\,dx + N_1\,dy = 0$

$M_1 = \frac{y}{x^2}, \quad N_1 = \frac{-1}{x}$

$$\frac{dM_1}{dy} = \frac{d}{dy}\left(\frac{y}{x^2}\right) = \frac{1}{x^2}(1) = \frac{1}{x^2}$$

$$\frac{dN_1}{dx} = \frac{d}{dx}\left(\frac{-1}{x}\right) = \left(-\left(\frac{-1}{x^2}\right)\right) = \frac{1}{x^2}$$

$\frac{dM_1}{dy} = \frac{dN_1}{dx}$ is Exact differential equation

$\therefore$ Suitable ~~function~~ formula $= \frac{1}{x^2}$

Methods to find integrating factors

Method:-1

To find an integrating factor $Mdx+Ndy=0$

1. $d(xy) = xdy+ydx$
2. $d(y/x) = \dfrac{xdy-ydx}{x^2}$
3. $d(x/y) = \dfrac{ydx-xdy}{y^2}$
4. $d\left(\dfrac{x^2+y^2}{2}\right) = xdx+ydy$
5. $d\left[\log\left(\dfrac{y}{x}\right)\right] = \dfrac{xdy-ydx}{xy}$
6. $d\left[\log(x/y)\right] = \dfrac{ydx-xdy}{xy}$
7. $d\left[\tan^{-1}(x/y)\right] = \dfrac{ydx-xdy}{x^2+y^2}$
8. $d\left[\tan^{-1}(y/x)\right] = \dfrac{xdy-ydx}{x^2+y^2}$
9. $d\left[\log(xy)\right] = \dfrac{ydx+xdy}{xy}$
10. $d\left[\log(x^2+y^2)\right] = \dfrac{2\left[xdx+ydy\right]}{x^2+y^2}$
11. $d\left[\dfrac{e^x}{y}\right] = \dfrac{ye^xdx-e^xdy}{y^2}$

problems:-

① Solve $x\,dx + y\,dy = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ ①

Suitable formula

$d\left[\dfrac{x^2+y^2}{2}\right] = d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right]$ (∵ ④ & ⑧ formulas)

Integrating on both sides

$\int d\left[\dfrac{x^2+y^2}{2}\right] - \int d\left[\tan^{-1}\left(\dfrac{y}{x}\right)\right] = c$

$\dfrac{x^2+y^2}{2} - \tan^{-1}\left(\dfrac{y}{x}\right) = c$

② solve $(1+xy)x\,dy + (1-yx)y\,dx = 0$

$x\,dy + x^2y\,dy + y\,dx - y^2x\,dx = 0$

$x\,dy + y\,dx + x^2y\,dy - y^2x\,dx = 0$

$x\,dy + y\,dx + xy[x\,dy - y\,dx] = 0$

M. w. $\dfrac{1}{x^2y^2}$ eqn ②

$\dfrac{x\,dy + y\,dx}{(xy)^2} + \dfrac{xy[x\,dy - y\,dx]}{(xy)^2} = 0$

$\dfrac{d(xy)}{(xy)^2} + \dfrac{x\,dy - y\,dx}{xy} = 0$ (∵ from eqn ①)

$$\frac{d(xy)}{(xy)^2} + d\left[\log\left(\frac{y}{x}\right)\right] = 0 \quad \left[\text{from equation } 5\right]$$

I. on. b.s.

$$\int \frac{d(xy)}{(xy)^2} + \int d\left[\log\left[\frac{y}{x}\right]\right] = c \longleftarrow \text{I}$$

$\cancel{\text{dt}}$ $xy = t$ in eqn I

$$\int \frac{dt}{t^2} + \log\left[\frac{y}{x}\right] = c$$

$$\int \frac{1}{t^2}\, dt + \log\left[\frac{y}{x}\right] = c$$

$$\cancel{\int t^{-2}\, dt} \quad \frac{-1}{t} + \log\left[\frac{y}{x}\right] = c$$

$$\therefore \frac{-1}{xy} + \log\left[\frac{y}{x}\right] = c \quad \left[\text{from I}\right]$$

3 Solve $y\left[2x^2y + e^x\right]dx = \left(e^x + y^3\right)dy$

$$y\left[2x^2y + e^x\right]dx - \left(e^x + y^3\right)dy = 0$$

$$\underline{2x^2y^2\,dx} + \underline{ye^x\,dx - e^x\,dy} - \underline{y^3\,dy} = 0$$

$$2x^2y^2\,dx - y^3\,dy + ye^x\,dx - e^x\,dy = 0$$

M. by $\frac{1}{y^2}$

$$\frac{2x^2\cancel{y^2}}{\cancel{y^2}}\,dx - \frac{y^3}{y^2}\,dy + \frac{ye^x\,dx - e^x\,dy}{y^2} = 0$$

$$2x^2\,dx - y\,dy + d\left(\frac{e^x}{y}\right) = 0 \quad (\because 11)$$

I.O.B.S.

$$\int 2x^2\,dx - \int y\,dy + \int\!\!\int d\left(\frac{e^x}{y}\right) = c$$

$$\frac{2x^3}{3} - \frac{y^2}{2} + \frac{e^x}{y} = c.$$

(4) multiply with $\left(\frac{1}{x^2}\right)$

$$(y - x^2)\,dx + (x^2\cot y - x)\,dy = 0$$

$$y\,dx - x^2\,dx + x^2\cot y\,dy - x\,dy = 0$$

$$y\,dx - x\,dy - x^2(dx - \cot y\,dy) = 0$$

$$-\left(\frac{x\,dy - y\,dx}{x^2}\right) - 1(dx - \cot y\,dy) = 0$$

$$-\left\{d\left(\frac{y}{x}\right)\right\} - dx + \cot y\,dy = 0.$$

I.O.B.S

$$-\int d\left(\frac{y}{x}\right) - \int dx + \int \cot y\,dy = \int c$$

$$-\frac{y}{x} - x + (\operatorname{cosec}^2 y) = c.$$

$$-\frac{y}{x} - x - \operatorname{cosec}^2 y = c.$$

$$-\frac{y}{x} = c + x + \operatorname{cosec}^2 y$$

$$y = -x(x + \operatorname{cosec}^2 y) + cx$$

Method-2   homogeneous ⇒ degree same then it is Homogeneous

To find an integrating factor $Mdx+Ndy=0$

if $M(x,y)\,dx + N(x,y)\,dy = 0$ is a homogeneous differential equation and $Mx+Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx+Ndy=0$

** problem (5M)

⇒ 1 solve $x^2y\,dx - (x^3+y^3)\,dy = 0$ (1)

$M\,dx + N\,dy = 0$!

$M(x,y)\,dx + N(x,y)\,dy = 0$.

$Mx+Ny \neq 0$ then $\frac{1}{Mx+Ny}$.

(S) $x^2y = M$; $N = -x^3 - y^3$

$$\frac{dM}{dy} = \frac{d}{dy}(x^2y) = x^2$$

$$\frac{dN}{dx} = \frac{d}{dx}(-x^3-y^3) = -3x^2$$

$\frac{dM}{dy} \neq \frac{dN}{dx}$ is not Exact differential equation.

But eqn (1) is Homogeneous equation $Mx+Ny \neq 0$

the I.F $\frac{1}{Mx+Ny}$.

$$Mx+Ny = (x^2y)x + (-x^3-y^3)y$$
$$= \cancel{x^3y} - \cancel{x^3y} - y^4$$
$$= -y^4 \neq 0$$

Integration formula $= \dfrac{1}{-y^4}$ m. w eqn (1)

$$\frac{x^2 \cancel{y}}{-y^4}\,dx + \frac{(x^3+y^3)}{-y^4}\,dy = 0$$

$$\frac{-x^2}{y^3}\,dx + \left(\frac{x^3+y^3}{y^4}\right)dy = 0 \quad \text{— (2)}$$

$$M_1\,dx + N_1\,dy = 0$$

Eqn (2) $M_1 = \dfrac{-x^2}{y^3}$, $N_1 = \dfrac{x^3+y^3}{y^4}$

$$\frac{dM_1}{dy} = \frac{d}{dy}\left(\frac{-x^2}{y^3}\right) = -x^2\frac{d}{dy}\left(\frac{1}{y^3}\right)$$

$$= -x^2\frac{d}{dy}(y^{-3})$$

$$= -x^2(-3)y^{-4}$$

$$\frac{dM_1}{dy} = 3x^2y^{-4}$$

$$= \frac{3x^2}{y^4}$$

$$\frac{dN_1}{dx} = \frac{d}{dx}\left(\frac{x^3+y^3}{y^4}\right) = \frac{1}{y^4}\frac{d}{dx}(x^3+y^3)$$

$$= \frac{1}{y^4}\,3x^2 = \frac{3x^2}{y^4}$$

$$\frac{dM_1}{dy} = \frac{dN_1}{dx}$$

$\therefore$ It is the exact differential equation.

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SO, I.O.b.s.

$$\int M_1\, dx + \int N_1\, dy = c$$

$$\int -\frac{x^2}{y^3} dx + \int \frac{1}{y}\, dy = c$$

$$-\frac{x^3}{3y^3} + \log|y| = c$$

② Solve $y^2 dx + (x^2 - xy - y^2) dy = 0$

$$M\, dx + N\, dy = 0.$$

$$M = y^2, \quad N = x^2 - xy - y^2$$

$$\frac{dM}{dy} = \frac{d}{dy}(y^2) = 2y$$

$$\frac{dN}{dx} = \frac{d}{dx}(x^2 - xy - y^2) = 2x - y$$

$\boxed{\frac{dM}{dy} \neq \frac{dN}{dx}}$ is Not e.d.een.

$$Mx + Ny = \cancel{y^2 x} + x^2 y - \cancel{xy^2} - y^3$$

$$= x^2 y - y^3 = y[x^2 - y^2] \neq 0.$$

I.f $\frac{1}{Mx+Ny} = \frac{1}{y[x^2-y^2]}$ M.w. eqn ①

$$\frac{y}{(x^2-y^2)} dx + \left(\frac{x^2 - xy - y^2}{y[x^2-y^2]}\right) dy = 0 \quad ②$$

$$M_1\,dx + N_1\,dy = 0$$

$$M_1 = \frac{y}{x^2-y^2}, \qquad N_1 = \frac{x^2-xy-y^2}{y(x^2-y^2)}$$

$$\frac{dM_1}{dy} = \frac{d}{dy}\left[\frac{y}{x^2-y^2}\right] \qquad \boxed{\frac{u}{v} = \frac{u'v-uv'}{v^2}}$$

$$\frac{dM_1}{dy} = \frac{(x^2-y^2)(1)-y\cdot(-2y)}{(x^2-y^2)^2}$$

$$\frac{dM_1}{dy} = \frac{x^2-y^2+2y^2}{(x^2-y^2)^2} = \frac{x^2+y^2}{(x^2-y^2)^2}$$

$$\frac{dN_1}{dx} = \frac{1}{y}\cdot\frac{d}{dx}\left(\frac{x^2-xy-y^2}{x^2-y^2}\right)$$

$$= \frac{1}{y}\left[\frac{(x^2-y^2)(2x-y)-(x^2-xy-y^2)(2x)}{(x^2-y^2)^2}\right]$$

$$= \frac{1}{y}\left[\frac{\cancel{2x^3}-x^2y-\cancel{2xy^2}+y^3-\cancel{2x^3}+2x^2y+\cancel{2xy^2}}{(x^2-y^2)^2}\right]$$

$$= \frac{1}{y}\left[\frac{y^3+x^2y}{(x^2-y^2)^2}\right]$$

$$= \frac{1}{\cancel{y}}\left\{\frac{\cancel{y}(y^2+x^2)}{(x^2-y^2)^2}\right\} = \frac{x^2+y^2}{(x^2-y^2)^2}$$

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$\frac{dM_1}{dy} = \frac{dN_1}{dx}$ is exact differential equation.

$\int M_1 dx + \int N_1 dy = c$

I.o.b.s →

$$\int \frac{y}{x^2-y^2}\,dx + \int \frac{1}{y}\,dy = c$$

$$\cancel{y}\cdot\frac{1}{2\cancel{y}}\log\left|\frac{x-y}{x+y}\right| + \log|y| = c$$

$$\frac{1}{2}\log\left|\frac{x-y}{x+y}\right| + \log|y| = c$$

is a General solution.

METHOD-3

To find an integrating factor of $Mdx+Ndy=0$

of if the equation $Mdx+Ndy=0$ is

of the form $\boxed{y f(xy)\,dx + x g(xy)\,dy = 0}$

and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an

integrating factor of $Mdx+Ndy=0$

Q) Solve $y(x^2y^2+2)dx + x(2-2x^2y^2)dy$ ①

$Mdx+Ndy=0$

$M = x^2y^3+2y$, $N = 2x-2x^3y^2$

$$\frac{dM}{dy} = \frac{d}{dy}(x^2y^3+2y) = 3x^2y^2+2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x - 2x^3y^2) = 2 - 6x^2y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So, $Mx - Ny \neq 0$ (proof)

$$(x^2y^3 + 2y) - (2x - 2x^3y^2)y = Mx - Ny$$

$$Mx - Ny = 3x^3y^3 \neq 0$$

then I.F $\frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$ M.W Eqn (1)

$$\frac{\cancel{y}(x^2y^2+2)dx}{3x^3y^{\cancel{3}}} + \frac{\cancel{x}(2-2x^2y^2)dy}{3x^{\cancel{3}}y^{3}} = 0.$$

$$\left(\frac{x^2y^2+2}{3x^3y^2}\right)dx + \left(\frac{2-2x^2y^2}{3x^2y^3}\right)dy = 0 \quad (2)$$

$M_1 = \frac{x^2y^2+2}{3x^3y^2}$, $N_1 = \frac{2-2x^2y^2}{3x^2y^3}$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2y^2+2}{3x^3y^2}\right) = \frac{2x^2y(3x^3y^2) - (x^2y^2+2)(6x^3y)}{(3x^3y^2)^2}$$

$$= \frac{\cancel{6x^5y^3} - \cancel{6x^5y^3} - 12x^3y}{\cancel{9x^6y^4}}$$

$$\pm \frac{-4}{y}$$

$$= \frac{\cancel{6x^5y^3} - \cancel{6x^5y^3} - 12x^3y}{(3x^3y^2)^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{-12x^3y}{(3x^3y^2)^2} = \frac{-12x^3y}{9x^6y^4} = \frac{-4}{9x^3y^3}$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2-2x^2y^2}{3x^2y^3}\right) \frac{u}{v}$$

$$= \frac{(-4xy^2)(3x^2y^3) - (2-2x^2y^2)(6xy^3)}{(3x^2y^3)^2}$$

$$= \frac{\cancel{-12x^3y^5} - 12xy^3 + \cancel{12x^3y^5}}{(3x^2y^3)^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{-12xy^3}{9x^4y^6} = \frac{-4}{3x^3y^3}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

(i)

$$\int M_1\,dx + \int N_1\,dy = C$$

$$\int\left[\frac{x^2y^2+2}{3x^3y^2}\right]dx + \int \cancel{\left[\frac{2-2x^2y^2}{3x^2y^3}\right]}\,dy = C$$

(x terms is not mention

$$\int \frac{x^2y^2+2}{3x^3y^2}dx + \int 0 \cdot dy = C$$

$$\int \frac{x^2y^2}{3x^3y^2}dx + \int \frac{2}{3x^3y^2}dx + \int 0 \cdot dy = C$$

$$\int \frac{1}{3x}dx + 2\int \frac{1}{3x^3y^2}dx = C$$

$$\cancel{-\frac{1}{3x^2} + \frac{2}{3y^2}\left(-\frac{1}{2x^2}\right) = C}$$

$$\cancel{\frac{-1}{3x^2} - \frac{2}{6x^2y^2} = C}$$

wrong

$$\cancel{-\frac{1}{3}\left(\frac{1}{x^2} + \frac{1}{x^2y^2}\right) = C}$$

$$\frac{1}{3}\log|x| - \frac{1}{3x^2y^2} = C$$

② solve $(xy(\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$

$$y(xy\sin xy + \cos xy)dx + x(xy\sin xy - \cos xy)dy = 0 \quad (1)$$

$$M = xy^2\sin xy + y\cos xy$$
$$N = x^2y\sin xy + x\cos xy$$

$$\frac{dM}{dy} = \frac{\partial}{\partial y}\left(\underset{u}{xy^2}\,\underset{v}{\sin xy} + \underset{u}{y}\,\underset{v}{\cos xy}\right)$$

$$= x\left[y^2(\cos xy)\cdot x + (\sin xy)(2y)\right] + y(-\sin xy)\,x + \cos xy(1)$$

$$\frac{\partial M}{\partial y} = \cos xy(x^2y^2+1) + xy\sin xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\Big(\underset{u}{x^2 y}\,\underset{v}{\sin xy} - \underset{u}{x}\,\underset{v}{\cos xy}\Big)$$

constant

$$= y\left[x^2(\cos xy)y + \sin xy(2x)\right] - \left[x(-\sin xy)\,y + \cos xy(1)\right]$$

$$\frac{\partial N}{\partial x} = x^2y^2\cos xy + 2xy\sin xy + xy\sin xy - \cos xy$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ is not exact differential equation

But eqn (1) is $y f(xy)\,dx + x\,g(xy)\,dy = 0$ & $Mx - Ny \neq 0$

then I.F $\frac{1}{Mx - Ny}$

$$Mx - Ny = (xy^2\sin xy + y\cos xy)x - (x^2y\sin xy + x\cos xy)y$$

$$Mx - Ny = \cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy$$

$$Mx - Ny = 2xy\cos xy \neq 0$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{2xy\cos xy} \quad \text{M. w eqn (1)}$$

$$\frac{\frac{1}{x}(xy\sin xy + \cos xy)\,dx}{2xy\cos xy} + \frac{x(xy\sin xy - \cos xy)\,dy}{2xy\cos xy} = 0$$

$$\left(\frac{xy\sin xy + \cos xy}{2x\cos xy}\right)dx + \left(\frac{xy\sin xy - \cos xy}{2y\cos xy}\right)dy = 0$$

$$M_1 = \frac{xy\sin xy + \cos xy}{2x\cos xy}$$

$$N_1 = \frac{xy\sin xy - \cos xy}{2y\cos xy}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}\left[\frac{xy\sin xy + \cos xy}{2x\cos xy}\right] \quad \left(\frac{u}{v} = \frac{u'v - v'u}{v^2}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{xy\sin xy}{2x\cos xy} + \frac{\cos xy}{2x\cos xy}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial y}\left(y\tan xy + \frac{1}{x}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial y}\left(\underset{u}{y}\,\underset{v}{\tan xy} + \frac{1}{x}\right)$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2}\left[y(\sec^2 xy)(x) + \tan xy - 0\right]$$

$$\frac{\partial M_1}{\partial y} = \frac{xy}{2}\sec^2 xy + \frac{\tan xy}{2}$$

$$\frac{\partial N_1}{dx} = \frac{\partial}{\partial x}\left[\frac{x\cancel{y}\sin xy}{2\cancel{y}\cos xy} - \frac{\cos xy}{2y\cos xy}\right]$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left[\underset{u}{x}\,\underset{v}{\tan xy} - \frac{1}{y}\right]$$

$$= \frac{1}{2}\left[x(\sec^2 xy)(y) + \tan xy - 0\right]$$

$$= \frac{xy}{2}\sec^2 xy + \frac{\tan xy}{2}$$

$$\frac{\partial M_1}{\partial x} = \frac{\partial N_1}{\partial y}$$

$$\int M_1\, dx + \int N_1\, dy = c$$

$$\int \frac{xy\sin xy}{2x\cos xy}dx + \int \frac{\cos xy}{2x\cos xy}dx + \int \cancel{\frac{xy\sin xy}{2y\cos xy}}\, dy\ \ 0\int \cancel{\frac{\cos xy}{2y\cos xy}}\, dy = c$$

$$\int \frac{y}{2}\tan xy\, dx + \int \frac{1}{2x}dx + \int -\frac{1}{2y}\, dy = c.$$

$$\frac{1}{2}\int y\tan xy\, dx + \int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{y}dy = c$$

$$\frac{y}{2}\log|\sec xy| + \log|x| - \frac{1}{2}\log|y| = \log c.$$

(3) $y(1+xy)\,dx + x(1-xy)\,dy = 0.$

(4) $y(x^4y^4 + x^2y^2 + xy)\,dx + x(x^4y^4 - x^2y^2 + xy)\,dy = 0$

③ $y(1+xy)dx + x(1-xy)dy = 0 \quad \text{---} ①$

ⓐ $M = y+xy^2 \quad , \quad N = x-x^2y$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y+xy^2) = 1+2xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x-x^2y) = 1-2xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$Mx - Ny = (y+xy^2)x - (x-x^2y)y$$

$$= \cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2$$

$$= 2x^2y^2 \neq 0$$

$$I.F = \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2} \text{ in eqn } ①$$

$$\left(\frac{y+xy^2}{2x^2y^2}\right)dx + \left(\frac{x-x^2y}{2x^2y^2}\right)dy = 0$$

$$M_1 = \frac{y+xy^2}{2x^2y^2} \quad N_1 = \frac{x-x^2y}{2x^2y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y+xy^2}{2x^2y^2}\right) = \frac{(1+2xy)(2x^2y^2) - (y+xy^2)(4x^2y)}{(2x^2y^2)^2}$$

$$= \frac{2x^2y^2 + \cancel{4x^3y^3} - 4x^2y^2 - \cancel{4x^3y^3}}{(2x^2y^2)^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{-2x^2y^2}{(2x^2y^2)^2} = \frac{-1}{2x^2y^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x - x^2y}{2x^2y^2}\right) \frac{u}{v}$$

$$= \frac{(1-2xy)(2x^2y^2) - (x - x^2y)(4xy^2)}{(2x^2y^2)^2}$$

$$= \frac{2x^2y^2 - \cancel{4x^3y^3} - 4x^2y^2 + \cancel{4x^3y^3}}{(2x^2y^2)^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{-2x^2y^2}{(2x^2y^2)^2} = \frac{-1}{2x^2y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Integrate.

$$\int M_1\,dx + \int N_1\,dy = c.$$

$$\int \frac{y+xy^2}{2x^2y^2}\,dx + \int \frac{-1}{2y}\,dy = c$$

$$\int \frac{1}{2x^2y}\,dx + \int \frac{1}{2x}\,dx - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

$$\frac{1}{2y}\int \frac{1}{x^2}\,dx + \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

$$\frac{-1}{2yx} + \frac{1}{2}\log|x| - \frac{1}{2}\log|y| = c$$

$$\frac{-1}{2xy} + \frac{1}{2}\log\left|\frac{x}{y}\right| = c$$

(4) $y(x^4y^4+x^2y^2+xy)\,dx + x(x^4y^4 - x^2y^2+xy)\,dy = 0$

(8)

$M = x^4y^5+x^2y^3+xy^2$, $N = x^5y^4 - x^3y^2 + x^2y$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^4y^5+x^2y^3+xy^2) = 5x^4y^4+3x^2y^2+2xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^5y^4 - x^3y^2+x^2y) = 5x^4y^4 - 3x^2y^2+2xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$Mx - Ny = (x^4y^5+x^2y^3+xy^2)x - (x^5y^4-x^3y^2+x^2y)y = 0$$

$$\cancel{x^5y^5}+x^3y^3+\cancel{x^2y^2} - \cancel{x^5y^5}+x^3y^3\cancel{-x^2y^2} = 0$$

$Mx - Ny = 2x^3y^3 \neq 0$

$Mx - Ny \neq 0$

$$\text{I.F} \ \frac{1}{Mx-Ny} = \frac{1}{2x^3y^3} \text{ in eqn (1)}$$

$$\frac{x^4y^5+x^2y^3+xy^2}{2x^3y^3}dx + \frac{x^5y^4 - x^3y^2+x^2y}{2x^3y^3}dy = 0$$

$$M_1 = \frac{x^4y^5+x^2y^3+xy^2}{2x^3y^3} , \quad N_1 = \frac{x^5y^4-x^3y^2+x^2y}{2x^3y^3}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^4y^5+x^2y^3+xy^2}{2x^3y^3}\right) \frac{u}{v}$$

$$= \frac{(5x^4y^4+3x^2y^2+2xy)(2x^3y^3) - (x^4y^5+x^2y^3+xy^2)(6x^3y^2)}{(2x^3y^3)^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{10x^7y^7 + \cancel{6x^5y^5} + 4x^4y^4 - 6x^7y^7 - \cancel{6x^5y^5} - 6x^4y^4}{(2x^3y^3)^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{4x^7y^7 - 2x^4y^4}{4x^6y^6}$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^5y^4 - x^3y^2 + x^2y}{2x^3y^3}\right)$$

$$\frac{\partial N_1}{\partial x} = \frac{(5x^4y^4 - 3x^2y^2 + 2xy)(2x^3y^3) - (x^5y^4 - x^3y^2 + x^2y)(6x^2y^3)}{(2x^3y^3)^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{4x^7y^7 - 2x^4y^4}{4x^6y^6}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$\int M_1 dx + \int N_1 dy = C$$

$$\int \frac{x^4y^5 + x^2y^3 + xy^2}{2x^3y^3} dx + \int \frac{-1}{2y} dy = C$$

$$\frac{1}{2}\int \frac{xy^2}{2} dx + \frac{1}{2}\int \frac{1}{x} dx + \frac{1}{2}\int \frac{1}{x^2y} dx + \left(\frac{-1}{2}\right)\int \frac{1}{y} dy = C.$$

$$\frac{1}{2}\left(\frac{x^2y^2}{2}\right) + \frac{1}{2}\log|x| + \frac{1}{2y}\left(\frac{-1}{x}\right) - \frac{1}{2}\log|y| = C.$$

$$\frac{x^2y^2}{4} + \frac{1}{2}\log\left|\frac{x}{y}\right| - \frac{1}{2xy} = C$$

$$\frac{x^2y^2}{2} + \log\left[\frac{x}{y}\right] - \frac{1}{xy} = C$$

Method - 4

To find an integrating factor $M dx + N dy = 0$

If there exist a continuous single variable function $f(x)$. such that $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then $e^{\int f(x)\,dx}$ is an integrating factor of $Mx + N dy = 0$.

problem

(1) solve $2xy\,dy - (x^2+y^2+1)dx = 0$

$$(-x^2-y^2-1)dx + 2xy\,dy = 0 \quad \text{—(1)}$$

$$M dx + N dy = 0$$

$$M = -x^2-y^2-1$$

$$N = 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x^2-y^2-1) = -2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

$$\frac{\cancel{2}y - \cancel{2}y}{\cancel{2}xy} = f(x)$$

$$\frac{-2y}{xy} = f(x)$$

$$-\frac{2}{x} = f(x)$$

$$I.f = e^{\int f(x)\,dx} = e^{-2\int 1/x\,dx} = e^{-2\log_e x}$$

$$= e^{\log_e x^{-2}}$$

$$= x^{-2} = \frac{1}{x^2}$$

$$I.f = \frac{1}{x^2}.$$

$\frac{1}{x^2}$ M.ω eqn (1)

$$\left(\frac{-x^2 - y^2 - 1}{x^2}\right)dx + \frac{2xy}{x^2}\,dy = 0$$

$$\left(\underbrace{-1 - \frac{y^2}{x^2} - \frac{1}{x^2}}_{M_1}\right)dx + \underbrace{\frac{2y}{x}}_{N_1}dy = 0 \quad (2)$$

$M_1\,dx + N_1\,dy$

$$\int\left(-1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right)dx + \int 0\,dy = C$$

$$-x - y^2\left(\frac{-1}{x}\right) - \left(\frac{-1}{x}\right) = C \qquad \frac{y^2}{x} + \frac{1}{x} - x = C$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}\left(-1-\frac{y^2}{x^2}-\frac{1}{x^2}\right) = -\frac{1}{x^2}2y-0$$

$$= -\frac{2y}{x^2}$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2y}{x}\right) = 2y\left(\frac{-1}{x^2}\right) = \frac{-2y}{x^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$\int M_1\,dx + \int N_1\,dy = c$$

$$\int\left(-1-\frac{y^2}{x^2}-\frac{1}{x^2}\right)dx + \int 0\,dy = c$$

$$-x+\frac{y^2}{x}+\frac{1}{x} = c$$

(2) solve $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$

$M = 3xy - 2ay^2$, $N = x^2 - 2axy$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3xy-2ay^2) = 3x - 4ay$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2-2axy) = 2x - 2ay$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3x - 4ay - 2x + 2ay}{x^2 - 2axy}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x - 2ay}{x^2 - 2axy} = f(x) - \frac{1}{x}$$

$$= \frac{\cancel{x - 2ay}}{x(\cancel{x - 2ay})} = \frac{1}{x}$$

$$e^{\int f(x)\,dx} = e^{\int \frac{1}{x} dx} = e^{\log_e x} = x$$

I.F is "x"

multiply w.t to Eqn (1)

$$x(3xy - 2ay^2)dx + x(x^2 - 2axy)\,dy = 0$$

$$(3x^2y - 2axy^2)dx + (x^3 - 2ax^2y)\,dy = 0 \quad \text{(2)}$$

$$M_1\,dx + N_1\,dy = 0$$

$$M_1 = 3x^2y - 2axy^2, \quad N_1 = x^3 - 2ax^2y$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}(3x^2y - 2axy^2) = 3x^2 - 4axy$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}(x^3 - 2ax^2y) = 3x^2 - 4axy$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

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$$\int M_1 dx + \int N_1 dy = c$$

$$\int [3x^2y - 2axy^2] dx + \int 0 \cdot dy = c$$

$$\cancel{3}\frac{x^3}{\cancel{3}}y - \cancel{2}a\frac{x^2}{\cancel{2}}y^2 = c$$

$$x^3y - ax^2y^2 = c$$

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Method-5

To find $Mdx+Ndy=0$; if there exist a continuous and differentiable single variable function $g(y)$

Such that

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=g(y) \quad \text{there } \int g(y)\,dy$$

integrating factor $Mdx+Ndy=0$

Solve :- ① $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$ ①

$M = y^4+2y \qquad N = xy^3+2y^4-4x$

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(y^4+2y) = 4y^3+2$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left[(xy^3+2y^4-4x)\right] = y^3-4$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=g(y)$$

$$\frac{y^3-4-4y^3-2}{y^4+2y}=$$

$$\frac{-3y^3-6}{y^4+2y}=\frac{-3(y^3+2)}{y(y^3+2)}=\frac{-3}{y}=g(y)$$

$$I.F = e^{\int g(y)dy} \Rightarrow e^{-3\int \frac{1}{y} dy} = e^{-3\log y}$$

$$= e^{\log y^{-3}} = y^{-3}$$

$$= \frac{1}{y^3}$$

I.F $\frac{1}{y^3}$ m. w. eqn (1)

$$\frac{y^4+2y}{y^3} dx + \frac{xy^3 + 2y^4 - 4x}{y^3} dy = 0$$

$$\left(\frac{y^3+2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0$$

$$M_1 = \frac{y^3+2}{y^2}, \quad N_1 = x + 2y - \frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y^3+2}{y^2}\right) \frac{u}{v} = \frac{3y^2(y^2) - 2y(y^3+2)}{(y^2)^2}$$

$$= \frac{3y^4 - 2y^4 - 4y}{y^4}$$

$$= y' - \frac{4}{y^3} = 1 - \frac{4}{y^3}$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left(x + 2y - \frac{4x}{y^3}\right) = 1 - \frac{4}{y^3}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$\int M_1\, dx + \int N_1\, dy = C$$

$$\int \left(\frac{y^3+2}{y^2}\right) dx + \int 2y\, dy = C$$

$$\frac{y^3+2}{y^2}\int 1 \cdot dx + \cancel{2}\frac{y^2}{\cancel{2}} = C$$

$$\left(\frac{y^3+2}{y^2}\right) x + y^2 = C$$

② solve $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3)\,dy = 0$

$M = xy^2 - x^2$

$N = 3x^2y^2 + x^2y - 2x^3$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(xy^2 - x^2) = 2xy$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(3x^2y^2 + x^2y - 2x^3) = 6xy^2 + 2xy - 6x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$$

$$\frac{6xy^2 + \cancel{2xy} - 6x^2 - \cancel{2xy}}{xy^2 - x^2} \quad \frac{6(xy^2 - x^2)}{xy^2 - x^2} = 6$$

$= g(y)$

$I.f = e^{\int g(y)\,dy} = e^{\int 6\,dy} = e^{6y}$

e^{6y} I.f multiply with eqn ①

$e^{6y}(xy^2 - x^2)\,dx + e^{6y}(3x^2y^2 + x^2y - 2x^3)\,dy = 0$

$M_1\,dx + N_1\,dy = 0$

$M_1 = e^{6y}(xy^2 - x^2)$

$N_1 = e^{6y}(3x^2y^2 + x^2y - 2x^3)$

$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}\left(e^{6y}(xy^2 - x^2)\right)$

$= e^{6y}.xy^2 - e^{6y}x^2$

$= (2xy)\,e^{6y} + 6xy^2e^{6y} - 6x^2e^{6y}$

$\frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left(e^{6y}(3x^2y^2 + x^2y - 2x^3)\right)$

$= e^{6y}(6xy^2 + 2xy - 6x^2)$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\int M_1\,.dx + \int N_1.\,dy = c.$

$\int e^{6y}(xy^2 - x^2)\,dx + \int 0.\,dy = c.$

$e^{6y}\left(\frac{x^2}{2}y^2 - \frac{x^3}{3}\right) = c$

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Linear differential equations

Working rule

① Given equation form $\frac{dy}{dx} + P(x)y = Q(x)$

② $I.F = e^{\int P(x)dx}$

③ General solution $y \times I.F = \int Q(x) \times I.F\, dx + c$

(or)

① $\frac{dx}{dy} + P(y)x = Q(y)$

② $I.F = e^{\int P(y)dy}$

③ $G.S = x \cdot I.F = \int Q(y) \times I.F\, dy + c$.

problem

① solve $x \cdot \frac{dy}{dx} + y = \log x$.

$$\frac{dy}{dx} + y \cdot \left(\frac{1}{x}\right) = \frac{\log x}{x}$$

$P(x) = \frac{1}{x}$, $Q(x) = \frac{\log x}{x}$.

$I.F = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log_e x} = x$

$G.S = y \times x = \int \frac{\log x}{\cancel{x}} \times \cancel{x}\, dx + c$.

$y x = x \log x - x + c$.

2) Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$ — ①

$$\frac{dy}{dx} + \left(\frac{-y}{x+1}\right) = \frac{e^{3x}(\cancel{x+1})^2}{\cancel{(x+1)}}$$

$$\frac{dy}{dx} + \left(\frac{-y}{x+1}\right) = e^{3x}(x+1)$$ — ②

This form of $\frac{dy}{dx} + P(x)y = Q(x)$ — ③

Compare ② & ③ Eqns

$P = \frac{-1}{x+1}$, $Q = e^{3x}(x+1)$

$I.F = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x+1}dx}$

$= e^{-\int \frac{1}{x+1}dx}$

$= e^{-\log(x+1)}$

$= e^{\log_e(x+1)^{-1}}$

$= \frac{1}{x+1} = I.F$

G.S. $y \times I.F = \int Q(x) \times I.F\,dx + C$

$y \times \frac{1}{x+1} = \int e^{3x}(\cancel{x+1}) \cdot \frac{1}{\cancel{x+1}}\,dx + C$

$$\frac{y}{x+1} = \int e^{3x}\,dx + c$$

$$\frac{y}{x+1} = \frac{e^{3x}}{3} + C \Rightarrow \text{General solution.}$$

3. $$(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0 \quad \text{— (1)}$$

$$\frac{dx}{dy}(1+y^2) + x - e^{\tan^{-1}y} = 0$$

$$\frac{dx}{dy} + \frac{x - e^{\tan^{-1}y}}{(1+y^2)} = 0$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{(1+y^2)} \quad \text{— (2)}$$

cf $$\frac{dx}{dy} + P(y)\,x = Q(y) \quad \text{— (3)}$$

compare (2) & (3)

$$P = \frac{1}{1+y^2} \qquad Q = \frac{e^{\tan^{-1}y}}{1+y^2}$$

I.F $$e^{\int P(y)\,dy} = e^{\int \frac{1}{1+y^2}\,dy} = e^{\tan^{-1}y} = \text{I.f}$$

$$x \times \text{I.F} = \int Q(y) \times \text{I.F}\,dy + C$$

$$x e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y}\,dy + C \quad \text{— (4)}$$

from eqn (4) $\Rightarrow$

let $\tan^{-1} y = t$

diff w.r. to "t"

$$\frac{1}{1+y^2} dy = dt$$

sub in eqn (4)

$$x \times e^t = \int e^t \times e^t \, dt + c$$

$$xe^t = \int e^{2t} dt + c$$

$$xe^t = \frac{e^{2t}}{2} + c$$

$$xe^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + c$$

4. $\frac{dy}{dx} + y\sec x = \tan x$

$$\frac{dy}{dx} + p(x) y = Q(x)$$

$p(x) = \sec x$

$Q(x) = \tan x$

$$\int p(x)\, dx = \int \sec x \, dx = \log_e (\sec x + \tan x)$$

$$\text{I.F} = e^{\log_e(\sec x + \tan x)} = \sec x + \tan x$$

$G.S = y \cdot I \times F = \int Q(x) \times I.F\, dx$

$$y(\sec x + \tan x) = \int \tan x \times (\sec x + \tan x)\, dx$$

$$\cancel{y(\sec x + \tan x) = y \log|\sec x| + c}$$

$$y(\sec x + \tan x) = \int \sec x \tan x\, dx + \int \tan^2 x\, dx$$

$$\Downarrow$$

$$\int (\sec^2 x - 1)\, dx$$

$$y(\sec x + \tan x) = \sec x + \tan x - x + C.$$

5. $(x + 2y^3)\dfrac{dy}{dx} = y$

$$\cancel{\frac{dy}{dx} - \frac{y}{x+2y^3}}$$

$$\frac{x+2y^3}{y} = \frac{dx}{dy}$$

$$\cancel{\frac{dx}{dy}(x + 2y^3) = \frac{y}{dy}}$$

$$\frac{x}{y} + 2y^2 = \frac{dx}{dy}$$

$$\frac{dx}{dy} + \left(\frac{-1}{y}\right)(x) = 2y^2$$

$$\frac{dx}{dy} + p(y)x = Q(y).$$

$$p = \frac{-1}{y}, \quad Q = 2y^2$$

$$I.F = e^{\int P(y)dy} = e^{-\int \frac{1}{y} dy}$$

$$= e^{-\log_e y}$$

$$= e^{\log_e y^{-1}} = y^{-1}$$

$I.F = \frac{1}{y}$

G.S $\quad x \times I.F = \int g(y) \cdot I.F \, dy + c$

$$x \cdot \frac{1}{y} = \int 2 \cdot y^{\cancel{2}} \cdot \frac{1}{\cancel{y}} dy + c$$

$$\frac{x}{y} = 2 \int y \, dy + c$$

$$\frac{x}{y} = \cancel{2} \cdot \frac{y^2}{\cancel{2}} + c$$

$$x = y^3 + cy$$

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Bernoulli's equation.

⇒ An equation of the form $\frac{dy}{dx} + Py = Qy^n$ (1)

is called Bernoulli's equation.

⇒ If P & Q are constants (or) functions of 'x' alone and 'n' is real constant.

Case:- 1

⇒ If "n = 1" by Bernoulli's equation

(inverse are taken whichever is $\boxed{y}$)

$$\frac{dy}{dx} + Py = Qy^{1}$$

$$\frac{dy}{dx} + Py - Qy = 0$$

$$\frac{dy}{dx} + y(P-Q) = 0$$

$$\frac{dy}{y} + (P-Q)\,dx = 0 \text{ (Variable separable)}$$

$$\boxed{\int \frac{dy}{y} + \int (P-Q)\, dx = C}$$

case :- (ii) :-

⇒ if $n \neq 1$ by bernoulli's equation $\frac{dy}{dx} + Py = Qy^n$

dividing with y^n

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{Py}{y^n} = \frac{Qy^n}{y^n}$$

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \quad \text{--- (1)}$$

from eqn (1)

let $y^{1-n} = u$

$$(1-n)y^{1-n-1}\frac{dy}{dx} = \frac{du}{dx}$$

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{du}{dx} \quad \text{(variable separable)}$$

$$y^{-n}\frac{dy}{dx} = \left(\frac{1}{1-n}\right)\frac{du}{dx}$$

substituting eqn (1)

$$\frac{1}{1-n}\frac{du}{dx} + Pu = Q$$

$$\frac{du}{dx} + (1-n)Pu = Q(1-n)$$

There is a linear eqn of first order in 'u' and this can be solved as discribed as earlyer and the end we substitute and get the required solution.

problems:-

(1) $x\frac{dy}{dx}+y=x^3y^6$

(12)

D. w. x

$$\frac{dy}{dx}+\frac{y}{x}=x^2y^6 \quad \text{—— (1)}$$

D. w. y^6

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{y}{y^6x}=\frac{x^2\cancel{y^6}}{\cancel{y^6}}$$

$$y^{-6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2 \quad \text{—— (2)}$$

Eq^n (2) let $y^{-5}=u$

$$-5y^{-5-1}\frac{dy}{dx}=\frac{du}{dx}$$

$$\boxed{y^{-6}\frac{dy}{dx}=\frac{-1}{5}\frac{du}{dx}} \to (3)$$

Substitute Eqn (2) Eqn (3) value.

$$\frac{-1}{5}\frac{du}{dx}+\frac{u}{x}=x^2.$$

$$\frac{du}{dx}-\frac{5u}{x}=-5x^2 \quad \text{—— (4)}$$

$$\frac{du}{dx}+P(x)u=Q(x) \quad \text{—— (5)}$$

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4 & 5

$$P(x) = -\frac{5}{x} , Q(x) = -5x^2$$

$$I.F \; e^{\int P(x) dx}$$

$$= e^{\int -5/x \, dx}$$

$$= e^{-5 \int 1/x \, dx}$$

$$= e^{-5 \log_e x}$$

$$= \cancel{e^{\log_e}} x^{-5}$$

$$I.F = \frac{1}{x^5}$$

General solution.

$$u \times I.F = \int Q(x) \times I.F \, dx + C$$

$$y^{-5} \cdot \frac{1}{x^5} = \boxed{\int -5x^{\cancel{2}} \cdot \frac{1}{\cancel{x^5}} dx + C}$$

$$\frac{1}{x^5 y^5} = -5 \int \frac{1}{x^3} dx + C.$$

$$\frac{1}{x^5 y^5} = \cancel{-}5 \left(\frac{1}{\cancel{-}2x^2} \right) + C$$

$$\frac{1}{x^5 y^5} = \frac{5}{2x^2} + C.$$

differential equations reducable to linear equation by subtitution.

(1) $$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y \quad \text{—— (1)}$$

D.w.t. $\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x\sin 2y}{\cos^2 y} = \frac{x^3\cos^2 y}{\cos^2 y}$$

$$\sec^2 y\frac{dy}{dx} + \frac{x\cdot 2\sin y\cancel{\cos y}}{\cancel{\cos^2 y}} = x^3$$

$$\sec^2 y\cdot\frac{dy}{dx} + 2x\cdot\tan y = x^3 \quad \text{—— (2)}$$

Eqn (2) let $\tan y = u$

$$\sec^2 y \frac{dy}{dx} = \frac{du}{dx}$$

Eqn (2) $\frac{du}{dx} + 2x \cdot u = x^3$ —— (3)

$$\frac{du}{dx} + P(x)u = Q(x)$$ —— (4)

(3) & (4)

$P(x) = 2x$, $Q(x) = x^3$

$I.F = e^{\int P(x)dx}$

$= e^{\int 2x\,dx} = e^{2\int x\,dx} = e^{2x^2/2}$

$= e^{x^2} = I.F$

G.S $u \times I.F = \int Q(x)\, I.F\, dx + C$

$\tan y \times e^{x^2} = \int x^3 e^{x^2} dx + C$

$= \int x^2 \cdot e^{x^2} \cdot x\,dx + c$ —— (5)

Eqn (5) let $x^2 = t$

$2x\,dx = dt$

$x \cdot dx = \frac{1}{2} dt$

Eqn (5)

$$\int t \cdot e^t \tfrac{1}{2} dt + c$$

$$\boxed{\int uv = u\int v - \int\left(u' \int v\right) dx}$$

$$= \left[t\int e^t dt - \int 1 \cdot e^t dt + c\right]\tfrac{1}{2}$$

$$\left\{te^t - e^t + c\right\}\tfrac{1}{2}$$

$$= \left[e^t(t-1) + c\right]\tfrac{1}{2}$$

$$\tan y \cdot e^{x^2} = \frac{e^{x^2}(x^2-1)+c}{2}$$

(2) solve $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x \sec y$ —— (1)

D. w. $\sec y$

$$\frac{1}{\sec y} \cdot \frac{dy}{dx} - \frac{\sin y}{\cancel{\cos y}(1+x)(\cancel{\sec y})} = (1+x)e^x \frac{\cancel{\sec y}}{\cancel{\sec y}}$$

$$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x \quad \text{—— (2)}$$

let $\sin y = u$

$$\cos y \frac{dy}{dx} = \frac{du}{dx}.$$

Eqn (2) $\dfrac{du}{dx} - \dfrac{u}{1+x} = (1+x)e^x$ —— (3)

$$\frac{du}{dx} + P(x)u = Q(x) \quad \text{—— (4)}$$

③ & ④

$P(x) = -1/1+x$, $Q(x) = (1+x)e^x$

$I.F = e^{\int P(x)dx}$

$= e^{-\int \frac{1}{1+x} dx}$

$= e^{-\log(1+x)}$

$= e^{\log_e (1+x)^{-1}}$.

$= (1+x)^{-1}$

$I.F = \frac{1}{1+x}$.

G.S.

$U \times I.F = \int Q(x) \times I.F \, dx + C$.

$\frac{\sin y}{1+x} = \int e^x (1+x) \cdot \frac{1}{1+x} dx + C$

$\frac{\sin y}{1+x} = e^x + C$.

(3) $\cos x\, dy = y(\sin x - y)\, dx$

$$\frac{dy}{dx} + \tan x = 1 - y^2 \sec x$$

$$2.\frac{dy}{dx} - y\sec x = y^3 \tan x$$

d. by y^3

$$\frac{2}{y^3}\cdot\frac{dy}{dx} - \frac{y}{y^3}\sec x = \frac{\cancel{y^3}}{\cancel{y^3}}\tan x$$

$$\frac{2}{y^3}\frac{dy}{dx} - \frac{1}{y^2}\sec x = \tan x$$

$$\frac{2}{y^3}\frac{dy}{dx} + (\sec x)\frac{-1}{y^2} = \tan x \quad \text{(1)}$$

$u = -\sec x$

$$\frac{du}{dx} = -\sec x \tan x \; dx$$

$$u = \frac{-1}{y^2}$$

$$\frac{du}{dx} = -(-2)y^{-3}\frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{2}{y^3}\frac{dy}{dx} \quad \text{(2)}$$

Sub (2) in (1)

$$\frac{du}{dx} + \sec x \cdot u = \tan x$$

$$\frac{dy}{dx} + P(x) \cdot u = Q(x)$$

$P_{(x)} = \sec x$, $Q_{(x)} = \tan x$

$$I.F = e^{\int P(x)dx} = e^{\int \sec x \, dx} = e^{\log_e |\sec x + \tan x|}$$

$$= \sec x + \tan x$$

G.S. $u \times I.F = \int Q(x) \times I.F \, dx + c$

$$u \times (\sec x + \tan x) = \int (\tan x)(\sec x + \tan x) dx$$

$$u(\sec x + \tan x) = \int \tan x \sec x \, dx + \int \tan^2 x \, dx + c$$

$(1 - \sec^2 x)$

(4) $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$ —(1)

$u = \tan y$

$$\frac{du}{dx} = \sec^2 y \frac{dy}{dx}$$

sub this in eqn (1)

$$\frac{du}{dx} + 2xu = x^3$$

$$\frac{du}{dx} + p(x)u = Q(x)$$

compare

$P_{(x)} = 2x$, $Q(x) = x^3$

$$I.F = e^{\int p(x)dx} = e^{2\int x\,dx} = e^{x^2}.$$

Gns

$$u \times I.F = \int Q(x) \times I.F\, dx + C.$$

$$u \times e^{x^2} = \int x^3 \times e^{x^2} dx + c.$$

$$ue^{x^2} = \int x^2 \cdot e^{x^2} \cdot x\, dx + c$$

let $x^2 = t$

$2x\,dx = dt$

$x \cdot dx = \frac{dt}{2}$

$$ue^{x^2} = \int t \cdot e^{t} \cdot \frac{dt}{2} + c.$$

$$ue^{x^2} = \frac{1}{2}\int \underset{u}{t} \cdot \underset{v}{e^t}\, dt + c$$

$$\int uv = u\int v - \int (u' \textstyle\int v)\, dx.$$

$$ue^{x^2} = \frac{1}{2}\left[t(e^t) - \int 1 \cdot e^t\, dt\right] + c$$

$$ue^{x^2} = \frac{1}{2}\left[te^t - e^t\right] + c$$

$$ue^{x^2} = \frac{1}{2}\left[x^2e^{x^2} - e^{x^2}\right] + c.$$

$$ue^{x^2} = \frac{1}{2}e^{x^2}(x^2 - 1) + c$$

** (IOM problem)

⇒ Newtons law of cooling

⇒ Statement :- The rate of change of the [illegible] of a body is proportional to the difference of the temperature of the body and that surrounding medium (usually air)

Proof let 'θ' be that temperature of the body at time 't' and 'θ_0' be the temperature of its surrounding medium

By the newtons law of cooling be have

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\text{i.e } \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

where 'k' is a partially constant.

[variable separable]

$$\frac{d\theta}{\theta - \theta_0} = -k\,dt$$

Integrating on both side

$$\int \frac{d\theta}{\theta - \theta_0} = -k\int dt$$

$$\log(\theta - \theta_0) = -kt + \log c$$

$$\log(\theta - \theta_0) - \log c = -kt$$

$$\log\left[\frac{\theta - \theta_0}{c}\right] = -kt$$

$$\theta - \theta_0 = c\, e^{-kt} \quad (1)$$

initially

$\theta = \theta_1$, $t = 0$ in Eqn (1)

$$\theta_1 - \theta_0 = c\, e^{0}$$

$$\boxed{c = \theta_1 - \theta_0} \rightarrow \text{in Eqn (1)}$$

$$\boxed{\theta - \theta_0 = (\theta_1 - \theta_0)\, e^{-kt}}$$

General solution.

$$\boxed{\theta = \theta_0 + (\theta_1 - \theta_0)\, e^{-kt}}$$

which gives the temperature of the body at any time "t".

1. problem

(1) A body is orginally 80°C and cools down to 60°C in 20 min. If the temp of

the air $40^\circ c$, find the temp of the body after 40 mins.

let "θ" be the temperature of a body at time 't'

let "θ_0" be the temperature of a air

By newton's law of cooling $\frac{d\theta}{dt} = -k(\theta-\theta_0)$ — (1)

Given $\theta_0 = 40^\circ c$

— (1) subtitute

$$\frac{d\theta}{dt} = -k(\theta-40)$$

Variable separable

$$\frac{d\theta}{\theta-40} = -k\,dt.$$

Integrating on both sides

$$\int \frac{d\theta}{\theta-40} = -k\int 1.dt$$

$$\log(\theta-40) = -kt + \log c$$

$$\log(\theta-40) - \log(c) = -kt.$$

$$\log\left(\frac{\theta-40}{c}\right) = -kt$$

$$\theta-40 = c.e^{-kt} \quad \text{— (2)}$$

$\theta = 80^\circ, t = 0$

Eqn (2)

$80 - 40 = C \cdot e^{-k(0)}$

$\boxed{C = 40^\circ}$

Sub in Eqn (2)

$\boxed{\theta - 40 = 40 \cdot e^{-kt}}$ (3)

where $60 = \theta$, $t = 20\,min$.

Eqn (3)

$60 - 40 = 40 \cdot e^{-k(20)}$

$\frac{\cancel{20}^{1}}{\cancel{40}_{2}} = e^{-20k}$

$k = \frac{1}{20} \log 2$ (4)

$t = 40\,min$, $\theta = ?$ Eqn (3)

$\theta - 40 = 40 e^{-(\frac{1}{\cancel{20}} \log 2)(\cancel{40}^{2})}$

$\theta - 40 = 40 e^{-\log 4}$

$\theta - 40 = 40 e^{\log 4^{-1}}$

$$\theta - 40 = \overset{10}{\cancel{40}} \times \frac{1}{\cancel{4}}$$

$$\boxed{\theta = 50^\circ C}$$

2. If the air maintain at $15^\circ c$, and the temperature of the body drops from $70^\circ C$ to $40^\circ C$ in 10 mins. what will be its temp after 30 mins

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \quad \text{--- (1)} \qquad \theta_0 = 15^\circ c.$$

$$\frac{d\theta}{dt} = -k(\theta - 15)$$

$$\frac{d\theta}{\theta - 15} = -k\,dt$$

I.o.b.s.

$$\log(\theta - 15) = -kt + \log c$$

$$\log\left(\frac{\theta - 15}{c}\right) = -kt$$

$$\theta - 15 = c \cdot e^{-kt} \quad \text{--- (2)}$$

$\theta = 70^\circ$, $t = 0$ in Eq (2)

$$70 - 15 = c \cdot e^{-k(0)}$$

$\boxed{55 = c}$ sub in (2)

$$\theta - 15 = 55e^{-kt} \quad \text{—— (3)}$$

$$\theta = 40, \ t = 10$$

$$40 - 15 = 55e^{-k(10)}$$

$$\cancel{25}\,5 = \cancel{55}\,11\, e^{-10k}$$

$$\frac{5}{11} = e^{-10k}$$

$$k = \frac{1}{10}\log\left(\frac{11}{5}\right) \quad \text{—— (4)}$$

Subtitute the values in eqn (3)

$$\theta - 15 = 55e^{-\left(\frac{1}{\cancel{10}}\log\left(\frac{11}{5}\right)\right)\times 3\cancel{0}}$$

$$\theta - 15 = 55e^{-3\log_e(11/5)}$$

$$\theta - 15 = 55e^{\log_e\left(\frac{5}{11}\right)^3}$$

$$= \cancel{55}\,5 \times \frac{5\times5\times5}{\cancel{11}\times11\times11}$$

$$\theta - 15 = \frac{625}{121}$$

$$\theta - 15 = 5.165$$

$$\boxed{\theta = 20.16}$$

3) A body can in air with temp 25°c cools from 140°c to 80°c in 20min find time when the body cools down to 35°c.

(20)

$\theta_0 = 25°$

$\theta = 140°c \quad t = 0$

$\theta = 80°c \quad t = 20$

$\theta = 35°c \quad t = ?$

$\frac{d\theta}{dt} = -k(\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -k(\theta - 25)$

$\frac{d\theta}{\theta - 25} = -k\,dt$

I.O.B.S

$\log(\theta - 25) = -kt + \log c$ (1)

$\theta = 140°c, \; t = 0$

Eqn (1)

$\log(140 - 25) = -k(0) + \log c$

$\log 115 = \log c$

Eqn (1) $\log(\theta - 25) = -kt + \log 115$

$\log(115) - \log(\theta - 25) = kt$ ... (2)

$\theta = 80^\circ c$, $t = 20m$ in eqn (1)

$\log(80-25) = -k \cdot 20 + \log 115$

$\log 55 = -20k + \log 115$

$\log 115 - \log 55 = 20k$ ... (3)

divide eqn (2) & (3)

$$\frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55} = \frac{\cancel{k}t}{20\cancel{k}}$$

$$\frac{\log\left(\frac{115}{\theta - 25}\right)}{\log\left(\frac{115}{55}\right)} = \frac{t}{20}$$

$$\log\left(\frac{55}{\theta - 25}\right) = \frac{t}{20}$$

$\theta = 35^\circ$ subtitute in above

$$\frac{55}{35-25} = \frac{t}{20}$$

$$\frac{55}{10} = \frac{t}{20}$$ $\boxed{t = 110\ \text{min}}$

Solvable for p.

The differential equation of first order and degree higher then first class $\frac{dy}{dx}$ is occuring in higher degree, it is convinent to denote $\frac{dy}{dx}$ by P.

so, $P = \frac{dy}{dx}$ then such differential equation take from $f(x, y, P) = 0$.

Case!-(i) Equation solvable for 'p'

→ A diff eqn of 1^{st} order and n^{th} degree is of the form $p^n + P_1 \cdot p^{n-1} + P_2 p^{n-2} + \ldots\ldots P_n = 0$

where $P_1, P_2 \ldots P_n$ of function of x & y.

1. $p^2 - 7p + 12$

$$p^2 - 3p - 4p + 12 = 0$$

$$p(p-3) - 4(p-3) = 0$$

$$p = 3, 4$$

$$\frac{dy}{dx} = 3$$

variable are separable

$$dy = 3 \cdot dx$$

I.B.S.

$$\int dy = 3\int dx + c$$

$y = 3x + c$

$y - 3x - c = 0$ — (1)

let $p = 4$.

$\frac{dy}{dx} = 4$.

variables are separable.

$dy = 4 . dx$

I.O.B.S.

$\int dy = 4\int dx + c$

$y = 4x + c$

$y - 4x - c = 0$ — (2)

from (1) & (2) eqn s.t.

$(y - 3x - c)(y - 4x - c) = 0$

2. Solve the $p^2 - 2p \cosh x + 1 = 0$

$p^2 - 2p\left[\frac{e^x + e^{-x}}{2}\right] + 1 = 0$

$p^2 - \cancel{2}p\left[\frac{e^x + e^{-x}}{\cancel{2}}\right] + e^x e^{-x} = 0$

$p^2 - p.e^x - pe^{-x} + e^x e^{-x} = 0.$

$p(p - e^x) - e^{-x}(p - e^x) = 0$

$p = e^x, e^{-x}$

let $p = e^x$

$$\frac{dy}{dx} = e^{x}$$

$$dy = e^{x}.dx$$

I.O.B.S.

$$y = e^{x} + c$$

$$y - e^{x} - c = 0 \quad (1)$$

Let $p = e^{-x}$

$$\frac{dy}{dx} = e^{-x}$$

$$dy = e^{-x}.dx$$

I.O.B.S.

$$y = \frac{e^{-x}}{-1} + c$$

$$y + e^{-x} + c = 0 \quad (2)$$

from (1) & (2)

$$(y - e^{x} - c)(y + e^{-x} + c) = 0$$

(3) Solve $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$

$$yp^2 + (x-y)p - x = 0$$

$$yp^2 + px - py - x = 0$$

$$yp^2 - py + px - x = 0$$

$$yp(p-1) + x(p-1) = 0$$

$(p-1)(yp+x)=0$

let $p=1$

$$\frac{dy}{dx}=1.$$

$dy=dx$

I.O.B.S.

$$\int dy=\int dx$$

$$y=x+c$$

$$y-x-c=0 \quad (1)$$

let $yp+x=0$

$$p=-\frac{x}{y}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

$$y\cdot dy=\int -x\cdot dx+c.$$

I.O.B.S

$$\int y\cdot dy=-\int x\cdot dx+c.$$

$$\frac{y^2}{2}=\frac{-x^2}{2}+c.$$

$$\frac{y^2}{2}+\frac{x^2}{2}-c=0$$

combine equation (1) & (2)

$$(y-x-c)\left(\frac{y^2}{2}+\frac{x^2}{2}-c\right)=0.$$

(4) Solve $x^2p^2+xyp-6y^2=0$.

$$x^2p^2+3xyp-2xyp-6y^2=0$$

$$xp(xp+3y)-2y(xp+3y)=0.$$

$$(xp-2y)(xp+3y)=0$$

let $xp+3y=0$

$$xp=-3y$$

$$p=\frac{-3y}{x}$$

$$\frac{dy}{dx}=\frac{-3y}{x}$$

$$\frac{dy}{y}=\frac{-3}{x}dx$$

I.O.B.S.

$$\int\frac{1}{y}dy=-\int 3/x\,dx$$

$$\log y=-3\log x+c.$$

$$\log y=\log x^{-3}+c.$$

$$\log y=\frac{1}{\log x^3}+c$$

$$\left(\log y-\frac{1}{\log x^3}\right)-c=0.$$

(1)

let $xp - 2y = 0$

$$p = \frac{2y}{x}$$

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\frac{dy}{\cancel{2}y} = \frac{2\,dx}{x}$$

I.O.B.S.

$$\int \frac{1}{y}\,dy = 2\int \frac{1}{x}\,dx + \log c$$

$$\log y = 2\log x + \log c$$

$$\log y = \log x^2 + \log c$$

$$\log y - \log x^2 = \log c \quad \text{— (2)}$$

~~from~~ (1) + (2)

$$\left(\log y - \frac{1}{\log x^2} - c\right)\left(\log y - \log x^2 - \log c\right) = 0$$

(5). To solve $p^3 = ax^4$

Given equation $p^3 = ax^4$

$$p = (ax^4)^{1/3}$$

$$p = a^{1/3}x^{4/3}$$

$$p = \frac{dy}{dx} = a^{1/3}x^{4/3}$$

$$dy = a^{1/3}x^{4/3}\,dx$$

I.O.B.S

$$\int dy + c = \int a^{1/3} x^{4/3}\, dx$$

$$y + c = a^{1/3}\frac{x^{7/3}}{7/3}$$

$$y + c = \frac{3a^{1/3}x^{7/3}}{7}$$

(6) To solve $p(p+y) = x(x+y)$

$$p^2 + py = x^2 + xy$$

$$p^2 + py - x^2 - xy = 0$$

$a = 1,\ b = y,\ c = -x^2 - xy$ $\quad \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(or)

$$(p^2 - x^2) + y(p - x) = 0$$

$$(p+x)(p-x) + y(p-x) = 0$$

$$(p-x)(p+x+y) = 0$$

$p = x,\ p = -x - y$

let $p = x$

$$\frac{dy}{dx} = x$$

$$dy = x\, dx$$

I.O.B.S.

$$\int dy = \int x\, dx$$

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$$y = \frac{x^2}{2} + c$$

$$y - \frac{x^2}{2} - c = 0 \quad (1)$$

$$p = -x - y.$$

$$p + y = -x.$$

$\frac{dy}{dx} + y = -x$ which is linear equation.

I.F $\frac{dy}{dx} + p(x)y = g(x)$.

$$p(x) = 1, \; g(x) = -x.$$

$$I.F = e^{\int p(x)dx} = e^{\int 1 \cdot dx} = e^x.$$

General solution.

$$y \times I.F = \int g(x) \cdot \times I.F \, dx + c.$$

$$y e^x = \int -x \times e^x \, dx + c.$$

$$y \cdot e^x = -\int \underset{u}{x} \cdot \underset{v}{e^x} \, dx + c$$

$$\boxed{\int uv = u\int v - \int \left(u' \int v \, dx\right) dx + c}$$

$$y \cdot e^x = -\left[xe^x - \int 1 \left(\int e^x dx \right) + c \right]$$

$$y \cdot e^x = -\left[xe^x - \int e^x dx + c \right]$$

$$y \cdot e^x = -\left[xe^x - e^x + c \right]$$

$ye^x = -e^x(x-1) + c$ divide with e^x on both sides

$$y = -(x-1) + \frac{c}{e^{x}}$$

$$y = -(x-1) + ce^{-x}$$

$$y + (x-1) - ce^{-x} = 0 \quad \text{— (2)}$$

general solution is

$$(2y - x^2 - c)\left(y + (x-1) - ce^{-x}\right) = 0$$

Clairaut type (or) equation.

→ An equation of the form $y = px + f(p)$ is known as clairaut equation.

→ General solution of clairaut equation.

→ To show that the solution of clairauts equation $y = p(x) + f(p)$ is $y = cx + f(c)$ obtained by p/c (or) p by c were "c" is an abitary constant.

Problems

Solve $y = px + p^n$ — ①

This form of clariraut equation $y = px + f(p)$

$\cancel{y = px + p^n}$ — ①

$y = px + f(p)$

so, replacing p by c.

Eqn ① $y = cx + c^n$.

①) $y = px + \log x$

This form of clariraut equation $y = px + f(p)$

① $\cancel{y = px + \log x}$ — ①

so, replacing p by c.

Eqn ① $y = cx + \log x$.

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(3) $p = \log(px - y)$

$$e^p = px - y$$

$$-y = e^p - px$$

$$y = px - e^p \quad (1)$$

This form of $y = px + f(p)$ so, clairaut equation

p by c

so, replace p by c

$$y = cx - e^c$$

(4) $p = \tan(px - y)$

$$\boxed{y = p(x) + f(p)}$$

$$\tan^{-1} p = px - y$$

$$-y = \tan^{-1} p - px$$

$$y = px - \tan^{-1} p \quad (1)$$

This form of $y = px + f(p)$, so. clairaut equation.

so, replace p by c in eqn (1)

$$y = cx - \tan^{-1} c$$

(5) $\cos y \cos px + \sin y \sin px = p$.

$$\cos(y - px) = p.$$

$$\cos^{-1} p = y - px$$

$$y = px + \cos^{-1} p$$

This form of $y = p(x) + f(p)$ so, clairaut equation

so, replace p by c.

$$y = cx + \cos^{-1} c.$$

(6) $\sin px \cos y = \cos px \sin y + p$

$$\sin px \cos y - \cos px \sin y = p$$

$$\sin(px - y) = p.$$

$$px - y = \sin^{-1}(p)$$

$$y = px - \sin^{-1} p$$

$$y = cx - \sin^{-1} c. \qquad p = c$$

Singular solution

If be eliminate between the given eqn and relation $x + f(p) = 0$, we get soln which is free from an arbitally constant and it is called singular solution of the equation.

Working rule

The solution of clairaut's equation obtained replaced p by singular sol^n (of clairaut) perform the following,

a) find the general sol^n of the clairauts equation by replacing p by c

i.e $y = cx + f(c)$ — ①

from eqn ① diff w.r.t c.

we get. $x + f'(c) = 0$ — ②

eliminate 'c' from ① & ② eq^n which will be the singular sol^n.

① Solve $y = px + p^2$ and obtain singular solution.

$$y = px + p^2 \quad \text{(1)}$$

p by c.

$$y = cx + c^2 \quad \text{(2)}$$

Diff w.r.to c.

$$0 = 1 \cdot x + 2c$$

$$x + 2c = 0$$

$$\boxed{c = -x/2}$$

Eqn (2)

$$y = \left(-\frac{x}{2}\right)x + \left(-\frac{x}{2}\right)^2$$

$$y = -\frac{x^2}{2} + \frac{x^2}{4}$$

$$y = \frac{-2x^2 + x^2}{4}$$

$$y = \frac{-x^2}{4}$$

$$4y = -x^2$$

$$4y + x^2 = 0.$$

(3) $y = px - \sqrt{1+p^2}$ —— (1)

p by c

$y = cx - \sqrt{1+c^2}$ —— (2)

Diff w.r.to c

$0 = x - \frac{1}{\cancel{2}\sqrt{1+c^2}}(\cancel{2}c)$

$0 = x - \frac{c}{\sqrt{1+c^2}}$

$\sqrt{1+c^2} = c/x$

S.O.B.S.

$1+c^2 = c^2/x^2$.

$x^2 + c^2x^2 = c^2$

$x^2 + c^2x^2 - c^2 = 0$

$x^2 + c^2(x^2-1) = 0$

$x^2 = -c^2(x^2-1)$

$\frac{x^2}{x^2-1} = -c^2$.

$c^2 = -\left(\frac{x^2}{x^2-1}\right)$

$c = -\sqrt{\frac{x^2}{x^2-1}}$ → sub in eqn (2)

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$$y = -\left(\sqrt{\frac{x^2}{x^2-1}}\right)x - \sqrt{1+\left(-\sqrt{\frac{x^2}{x^2-1}}\right)}$$

(3) $\sin(px-y) = p$ — (1)

p by c.

$\sin(cx-y) = c$ — (2)

Diff w.r.t to c.

$\cos(cx-y) = 1$.

$px - y = \sin^{-1}p$

$px - \sin^{-1}p = y$

$y = px - \sin^{-1}p$. (3)

p by c

$y = cx - \sin^{-1}c$. — (2)

Diff w.r.t c

$0 = x - \frac{1}{\sqrt{1-c^2}}$.

$\frac{1}{\sqrt{1-c^2}} = x$

$\sqrt{1-c^2} = \frac{1}{x}$

$1-c^2 = \frac{1}{x^2}$

(3) in (2)

$1 - \frac{1}{x^2} = c^2$.

$\frac{x^2-1}{x^2} = c^2$.

$\frac{1}{x}\sqrt{x^2-1} = c$ — (3)

Sub in (3) in (2)

$y = \frac{1}{x}\cdot x\sqrt{x^2-1} - \sin^{-1}\left(\frac{1}{x}\sqrt{x^2-1}\right)$

$y = \sqrt{x^2-1} - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$

(4) $y = px + a/p$ — (1)

p by c

$y = cx + a/c$ — (2)

diff w.r.to c.

$$0 = x + \frac{a(-1)}{c^2}$$

$$0 = x - \frac{a}{c^2}$$

$$\frac{a}{c^2} = x$$

$$\frac{a}{x} = c^2$$

$$c = \sqrt{\frac{a}{x}} \quad (3)$$

sub (3) in (2)

$$y = \sqrt{\frac{a}{x}}\,x + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$y = \sqrt{\frac{a}{x}}\,x + a\sqrt{\frac{x}{a}}$$

$$y = \frac{\sqrt{a}}{\cancel{\sqrt{x}}} \cdot \cancel{\sqrt{x}} \cdot \sqrt{x} + \sqrt{a} \cdot \cancel{\sqrt{a}} \cdot \frac{\sqrt{x}}{\cancel{\sqrt{a}}}$$

$$y = \sqrt{ax} + \sqrt{ax} = 2\sqrt{ax}$$

$$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$$

Application of clairauts problems

problems

1) solve $x^2(y-px) = yp^2$ —— (1)

put $x^2 = u$, $y^2 = v$.

$2x\,dx = du$ —— (2) ; $2y\,dy = dv$ —— (3)

(3) ÷ (2)

$$\frac{\cancel{2}y\,dy}{\cancel{2}x\,dx} = \frac{dv}{du}.$$

$$\frac{y}{x}p = P \quad \left(\because \frac{dy}{dx} = p,\ \frac{dv}{du} = P\right)$$

$$p = \frac{x}{y}P$$

Eqn (1)

$$x^2\left(y - (x/y)P.x\right) = y\,(x/y)^2 P^2$$

$$x^2\left(\frac{y^2 - x^2 P}{y}\right) = \cancel{y}.\frac{x^2}{y^{\cancel{2}}}P^2.$$

$$x^2(y^2 - x^2 P) = x^2 P^2.$$

$$u(v - uP) = uP^2.$$

P by c

$u(v - uc) = uc^2$

p by c.

$uv - u^2c = uc^2$

$$\boxed{x^2y^2 - x^4c = x^2c^2}$$

Nature growth or decay

let $x(t)$ be the among of a substance and time 't'. and let the substance. we getting chemically law of chemical conversion, state that rate of change of amount $x(t)$ of a chemically change in substance proportional to the amount of substance of the time 't'.

$$\frac{dx}{dt} \propto x, \quad \frac{dx}{dt} = -kt$$

problem:-

① A bacterial culture, growing exponentially increases from 200 to 500 gms in the period 6 am to 9 am. how many gms will be present at noon?

Sol. Given,

$N = 200$ gr, $t = 0$.

$N = 500$ gr, $t = 3$

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = -k \cdot dt$$

I.O.B.S

$$\int \frac{dN}{N} = -k \int 1 \cdot dt$$

$$\log N = -kt + \log c$$

$$\log N - \log c = -kt$$

$$\log \frac{N}{c} = -kt$$

$$\boxed{N = c \cdot e^{-kt}} \quad (1)$$

$N = 200$ gms, $t = 0$

$$200 = c \cdot e^{0}$$

$\boxed{c = 200}$ in eqn (1)

$$\boxed{N = 200\, e^{-kt}} \quad (2)$$

$N = 500$ gm, $t = 3$ in eqn (2)

$$500 = 200\, e^{-k(3)}$$

$$\frac{5}{2} = e^{-3k}$$

$$\boxed{\frac{1}{3} \log\left(\tfrac{5}{2}\right) = k} \rightarrow \text{sub in } (2)$$

Sub in eqn

$t = 6$ hours (from 6 am to noon)

$$N = 200\, e^{-\frac{1}{3} \log(2.5)(6)\, 2} = 1249.89$$

(2) A bacterial culture, growing exponential increases from 100 to 400 grms in 10 hrs. How much mass present after 3 hrs from the intial instant?

Given

$N = 100$ grs ; $t = 0$

$N = 400$ grs ; $t = 10$

$t = 3$ hrs.

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

I.O.B.S.

$$\int \frac{dN}{N} = k \int dt$$

$$\log N = kt + \log c$$

$$\log \frac{N}{c} = kt$$

$$N = c \cdot e^{kt} \quad \text{---(1)}$$

$N = 100$ grms, $t = 0$

$$100 = c \times e^{0}$$

$$\boxed{c = 100} \rightarrow \text{in (1)}$$

$$N = 100 \cdot e^{kt} \quad \text{---(2)}$$

$N = 400, t = 10$ in Eqn (2)

$$400 = 100 \cdot e^{-10k}$$

$$4 = e^{-10k}.$$

$$-\frac{1}{10}\log 4 = k$$

$\rightarrow$ Eqn (2)

$t = 3$ hrs in Eq (2)

$$N = 100 \cdot e^{-(3)\left(-\frac{1}{10}\log 4\right)}$$

$$\cancel{N = 100 \cdot e^{-\left(\frac{1}{10}\log 4\right)(3)}}$$

$$\cancel{N = 100 \cdot e^{-\frac{1}{10}\log 2^2 (3)}}$$

$$\cancel{= \log e^{\frac{2}{10}\log 2(3)}}$$

$$\cancel{= \log e^{\log 2}}$$

$$N = 100 \cdot e^{(\log_e 4)^{3/10}}$$

$$N = 100 \times (4)^{3/10}$$

$$N = 100 \times 1.515\cancel{72}$$

$$N = 151.5\cancel{72}$$

Solvable for y :-

In the given equation Solvable from 'y' then we can express 'y'. Expansion is a function x and y. This an equation Solvable from 'y'.

Can be put in $y = f(x, p)$

* Differential with 'x' and writing p for $\frac{dy}{dx}$

we get $\frac{dy}{dx}$. $\frac{dy}{dx} = p = f\left(x, p, \frac{dp}{dx}\right)$

⇒ which is diff eqn of first order in two variable p.

⇒ let x its solution be $\phi(x, p, c) = 0$ — (3)

⇒ c being an orbitary constant eliminating p being (1) & (2) equation.

⇒ we will get relation between x, y and c which will be the required solution.

⇒ In sum cases the elimination 'p' b/w (1) & (3) Equation is not possible.

⇒ In that case (1) & (3) Eqn together constitue the solution.

⇒ The given values of x & y in terms of p & c.

⇒ That is in the form $x = f_1(p, c)$, $y = f_2(p, c)$.

This two equations together form general solution of Eq^n ① in the paramatric form where 'p' is parameteric

*Note :-

⇒ In some problems equation 2 can be expressed as $f_1(x, p)$ which does not involve $\frac{dp}{dx}$ and proceed with $f_2(x, p)\frac{dp}{dx} = 0$.

Solvable for y:-

⇒ solve $y = (x-a)p - p^2$.

$$\frac{dy}{dx} = (x-a)\frac{dp}{dx} + p(1) - 2p\frac{dp}{dx}$$

$$\cancel{p} = (x-a)\frac{dp}{dx} + \cancel{p} - 2p\frac{dp}{dx}$$

$$\frac{dp}{dx}(x - a - 2p) = 0$$

$$\frac{dp}{dx} = 0$$

$$dp = 0$$

I.O.B.S.

$$\int dp = c \Rightarrow \boxed{p = c}$$

can be

not involve

) =0

$=(x-a)p-p^2$

Eliminating with ① & ② eqns we get

$$\boxed{y=(x-a)c-c^2}$$

② Solve $y=(1+p)x+p^2$

Sol. Given equation

$$y=(1+p)x+p^2 \rightarrow ①$$

Diff. w.r.to 'x'

$$\frac{dy}{dx} = (1+p)1 + x\left(\frac{dp}{dx}\right)+2p\frac{dp}{dx}$$

$$\cancel{p} = 1+\cancel{p} + \frac{dp}{dx}\left(x+2p\right)$$

$$\frac{dp}{dx}\left(x+2p\right) = -1$$

$$\frac{dp}{dx} = \frac{-1}{x+2p}$$

$$\frac{dx}{dp} = -x-2p$$

$$\frac{dx}{dp}+x=-2p$$

$P=1 \quad Q=-2p$

$$I.f = e^{\int P(p)\,dp}$$

$$= e^{\int 1\,dp}$$

$$\boxed{I.f = e^{p}}$$

G.S

$$\boxed{x \cdot I.f = \int Q \cdot I.f \, dP + C}$$

$$x \cdot e^P = \int -2P \, e^P dP + C$$

$$x \cdot e^P = -2\int P \cdot e^P dP + C \qquad \boxed{\because \int e^P \cdot P = (P-1)e^P}$$

$$x \cdot e^P = -2\left(e^P(P-1)\right) + C$$

$$x \cdot e^P + 2(P-1)e^P = C$$

$$e^P\left(x + 2(P-1)\right) = C$$

$$x + 2(P-1) = \frac{C}{e^P}$$

$$x = \frac{-C}{e^P(\cancel{2}(P-1))} \rightarrow (3)$$

from (3) sub in (1)

$$\boxed{y = (1+P)\frac{-C}{e^P(2(P-1))} + P^2}$$

(3) $y = 2Px + P^4x^2$

Sol $y = 2Px + P^4x^2 \rightarrow (1)$

Diff w.r.to 'x' $\quad [uv = uv' + vu']$

$$\frac{dy}{dx} = 2\left[P.(1) + x \cdot \frac{dP}{dx}\right] + P^4(2x) + x^2 4P^3\frac{dP}{dx}$$

$$\frac{dy}{dx} = 2P + 2x\frac{dP}{dx} + P^4 2x + 4x^2P^3\frac{dP}{dx}$$

$$p-2p-p^4 2x = \frac{dp}{dx}\left[2x+4x^2p^3\right]$$

$$-p-p^4 2x = \frac{dp}{dx}\left(2x(1+2xp^3)\right)$$

$$-p(1+p^3 2x) - \frac{dp}{dx}\left(2x(1+2xp^3)\right) = 0$$

$$\left(-p-\frac{dp}{dx}2x\right)(1+2xp^3) = 0$$

neglecting the second factor

$$-p-\frac{dp}{dx}2x = 0$$

$$-p = \frac{dp}{dx}2x$$

$$\frac{-dx}{2x} = \frac{dp}{p}$$

I.O.B

$$-\frac{1}{2}\int\frac{dx}{x} = \int\frac{dp}{p}+c$$

$$-\frac{1}{2}\log x = \log p + \log c$$

$$-\log x = 2\log p + \log c$$

$$-\cancel{\log}\, x = \cancel{\log}\, p^2 . c$$

$$-x = p^2 c$$

$$x = -p^2 c$$

$$-p^2 = \frac{x}{c}$$

$$p = \frac{\sqrt{x}}{c}$$

From ①

$$\boxed{y = 2\left(\frac{-\sqrt{x}}{c}\right)x + \left(\frac{-\sqrt{x}}{c}\right)^4 x^2}$$

solve $y = 2px + f(xp^2)$ –

Given equation

$$y = 2px + f(xp^2) \rightarrow ①$$

Diff. w.r. to x

$$\frac{dy}{dx} = 2\left[p(1) + x\frac{dp}{dx}\right] + f'(xp^2)\left(x\, 2p\frac{dp}{dx} + p^2(1)\right)$$

$$p = 2p + 2x\frac{dp}{dx} + f'(xp^2)\left(x\, 2p\frac{dp}{dx}\right) + f'(xp^2)(p^2)$$

$$p - 2p - f'(xp^2)\,p^2 = \frac{dp}{dx}\left(2x + f'(xp^2)\, 2xp\right)$$

$$-p - f'(xp^2)\,p^2 = \frac{dp}{dx}\, 2x\left(1 + f'(xp^2)\,p\right)$$

$$-p\left(1 + f'(xp^2)\,p\right) - \frac{dp}{dx}\, 2x\left(1 + f'(xp^2)\,p\right) = 0$$

$$\left(-p - \frac{dp}{dx}\, 2x\right)\left(1 + f'(xp^2)\,p\right) = 0$$

Neglecting the second factor

$$-p - \frac{dp}{dx}\, 2x = 0$$

$$-p = \frac{dp}{dx}\, 2x$$

$$-\frac{dx}{2x} = \frac{dp}{p}$$

I.O.B

$$-\frac{1}{2}\int \frac{dx}{x} = \int \frac{dp}{p}$$

$$p\frac{dp}{dx} + p^2(1)$$

$$f'(xp^2)(p^2)$$

$$-\frac{1}{2}\log x = \log p + c$$

$$-\log x = \log p^2 + c$$

$$-\log x = \cancel{2}\log p^2 \cdot c$$

$$-x = p^2 c$$

$$x = -p^2 c$$

$$\boxed{p = \frac{-\sqrt{x}}{c}}$$

from ①

$$\boxed{y = 2\left(\frac{-\sqrt{x}}{c}\right)x + f\left(x\left(\frac{-\sqrt{x}}{c}\right)^2\right)}$$

④ $y = p\tan p + \log\cos p$

Sol Given eqn

$$y = p\tan p + \log\cos p \rightarrow ①$$

Diff w.r.to 'x'

$$\frac{dy}{dx} = p\sec^2 p\frac{dp}{dx} + \tan p\frac{dp}{dx} + \frac{1}{\cos p}(-\sin p)\frac{dp}{dx}$$

$$p = p\sec^2 p\frac{dp}{dx} + \cancel{\tan p\frac{dp}{dx}} - \cancel{\tan p\frac{dp}{dx}}$$

$$\cancel{p} = \cancel{p}\sec^2 p\frac{dp}{dx}$$

$$\sec^2 p\frac{dp}{dx} = 1$$

$$\sec^2 p\, dp = 1\cdot dx$$

I.O.B

$$\int \sec^2 P\,dP = \int dx + c$$

$$\boxed{\tan P = x + c}$$

Equations ~~given~~ ① & ② together form the required solution in parametric 'p'

Solve $y = p\sin p + \cos p$

$y = p\sin p + \cos p \rightarrow$ ①

Diff w.r.to 'x'

$$\frac{dy}{dx} = p\cos p\frac{dp}{dx} + \cancel{\sin p\frac{dp}{dx}} \cancel{- \sin p\frac{dp}{dx}}$$

$$\cancel{p} = \cancel{p}\cos p\frac{dp}{dx}$$

$$\cos p\frac{dp}{dx} = 1$$

$$dp\cos p = 1 \cdot dx$$

I.O.B.S

$$\int \cos p\,dp = \int 1 \cdot dx$$

$$\boxed{\sin p = x + c}$$

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* Home work problems

(1) Solve $x^2 = 1+p^2$

Sol:

Given Equation

$$x^2 = 1+p^2 \rightarrow (1)$$

$$p^2 = x^2-1$$

$$p = \sqrt{x^2-1}$$

$$\frac{dy}{dx} = \sqrt{x^2-1}$$

By variable & seperable method

$$dy = \sqrt{x^2-1}\, dx.$$

Integrating O.B.S

$$\int dy = \int \sqrt{x^2-1}\, dx$$

$$\boxed{y = \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\log(x+\sqrt{x^2-1}) + c}$$

(2) $p^2 = x^3$ solve the equation

Sol:

$$p = (x^3)^{\frac{1}{2}}$$

$$p = x^{3/2}$$

$$\frac{dy}{dx} = x^{3/2}$$

variable & separable method

$$dy = x^{3/2}\, dx$$

Integration O.B.S

$\int dy = \int x^{3/2} dx$

$y = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$

$y = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$

$\frac{2y}{5} = x^{\frac{5}{2}} + c$

$2y = 5 \cdot x^{\frac{5}{2}} + c$

$\boxed{y = \frac{5x^{\frac{5}{2}}}{2} + c}$

(3) $4xp^2 = (3x-a)^2$

$4xp^2 = (3x-a)^2 \rightarrow (1)$

Sol

$p^2 = \frac{(3x-a)^2}{4x}$

$p = \frac{3x-a}{16x^2}$

$\frac{dy}{dx} = \frac{3\cancel{x}}{16x^{\cancel{2}}} - \frac{a}{16x^2}$

$\frac{dy}{dx} = \frac{3}{16x} - \frac{a}{16x^2}$

$dy = \left(\frac{3}{16x} - \frac{a}{16x^2}\right) dx$

I.B.S

$\int dy = \int\left(\frac{3}{16x} - \frac{a}{16x^2}\right) dx$

$\frac{\cancel{xy}\sin xy + \cos xy}{2x\cos xy} =$

$\frac{\cancel{\sin xy}}{2\cancel{x}\cos xy} + \frac{\cancel{\cos xy}}{2x\cancel{\cos xy}}$

$\frac{y}{2\cos}$

$\frac{y}{2}\tan xy + \frac{1}{2x}$.

(4) (P-2

Sol Let

d

$\int$

y

y-

L

Sub in (1)

$$y - c = \frac{3}{16}\log x + \frac{a}{16x}$$

$= \frac{x^{-1}}{-1}$

$= \frac{-1}{x}$.

$$y - c = \frac{1}{16}\left(3\log x + \frac{a}{x}\right)$$

$$y - c = \frac{1}{16}\left(\log x^3 + \frac{a}{x}\right)$$

(4) $(p-2x)(p+y)$

Sol

Let $p - 2x = 0$

$p = 2x$

$\frac{dy}{dx} = 2x$

$dy = 2x\,dx$

I.B.S

$\int dy = \int 2x\,dx$

$y = x^2 + c$

$y - x^2 - c \rightarrow$ (1)

Let $p + y = 0$

$p = -y$

$\frac{dy}{dx} = -y$

$\frac{\partial M}{\partial y} = \{x^2 y \sin xy + \cos xy\} y\,dx +$

$= x[\;xy^2 \sin xy + y\cos xy$

$\sin xy \cdot 2y] +$

$= x[y^2 \cos xy \cdot x + \sin xy \cdot 2y] +$

$(-\sin xy)(x) + \cos xy)$

$= x^2y^2 \cos xy + 2x\,\sin xy$

$- xy\sin xy + \cos xy$,

$Mx - Ny = x^2y^2 \sin xy + xy\cos xy$

$- x^2y^2 \sin xy + xy\cos xy$.

$= 2xy\cos xy + 0$

$\frac{y[xy\sin xy + \cos xy]\,dx +}{2xy\cos xy}$

$\frac{(xy\sin xy - \cos xy)\,x}{2xy\cos xy}$.

$$\frac{dy}{y} = -dx$$

I.O.B.S

$$\int \frac{1}{y} dy = \int -dx$$

$$\log y = -x + c$$

$$\log y + x - c \rightarrow (2)$$

from (1) & (2)

combined equations

$$\boxed{(y - x^2 - c)(\log y + x - c) = 0}$$

Solve $p^2 - 6p + 8 = 0$

$$p^2 - 6p + 8 = 0$$

$$p^2 - 2p - 4p + 8 = 0$$

$$p(p-2) - 4(p-2) = 0$$

$$(p-2)(p-4) = 0$$

$$(p-2) = 0$$

$$p = 2$$

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

I.B.S

$$\int dy = \int 2dx + c$$

$$y = 2x + c$$

$$y - 2x - c = 0 \rightarrow (1)$$

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Let $P-4=0$

$P=4$

$\frac{dy}{dx}=4$

$dy=4dx$

I.O.B.S

$\int dy=\int 4dx$

$y=4x+c$

$y-4x-c=0 \rightarrow (2)$

from (1) & (2)

$\boxed{(y-2x-c)(y-4x-c)=0}$

(6) $P^2+P-6=0$ solve the Equation

Sol: $P^2+P-6=0$

$P^2-2P+3P-6=0$

$P(P-2)+3(P-2)=0$

$(P-2)(P+3)=0$

Let $P-2=0$ $\quad$ $P+3=0$

$P=2$ $\quad$ $P=-3$

$\frac{dy}{dx}=2$ $\quad$ $\frac{dy}{dx}=-3$

$dy=2dx$ $\quad$ $dy=-3dx$

I.O.B.S $\quad$ I.O.B.S

$\int dy=2dx$ $\quad$ $\int dy=\int -3dx$

$y-2x-c=0 \rightarrow (1)$ $\quad$ $y+3x-c=0 \rightarrow (2)$

from (1) & (2)

$\boxed{(y-2x-c)(y+3x-c)=0}$

(7) IMP AA

Solve $y = x + a\tan^{-1}p$

SOl: Given Equation

$$y = x + a\tan^{-1}p \rightarrow (1)$$

Diff w.r.to 'x'

$$\frac{dy}{dx} = 1 + a \cdot \frac{1}{1+p^2}\frac{dp}{dx}$$

$$p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$$

$$p-1 = \frac{a}{1+p^2}\frac{dp}{dx}$$

$$\frac{1}{a}dx = \frac{dp}{(p-1)(1+p^2)}$$

I.O.B.S

$$\frac{1}{a}\int 1 \cdot dx = \int \frac{1}{(p-1)(1+p^2)} dp + c \rightarrow (2)$$

To find partial fractions

$$\frac{\cancel{dp}}{(p-1)(1+p^2)} = \left(\frac{A}{p-1} + \frac{Bp+C}{1+p^2}\right)\cancel{dp}$$

$$1 = A(1+p^2) + (Bp+C)(p-1)$$

$$1 = Ap^2 + A + Bp^2 - Bp + Cp - C$$

$$1 = p^2(A+B) + p(C-B) + A - C$$

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$A+B=0 \rightarrow ③$

$C-B=0 \rightarrow ④$

$A-C=1 \rightarrow ⑤$

Solving ③ & ④

$A+\cancel{B}=0$

$C-\cancel{B}=0$

$A+C=0 \rightarrow ⑥$

⑤ & ⑥

$A-\cancel{C}=1$

$A+\cancel{C}=0$

$2A=1$

$\boxed{A=\frac{1}{2}}$ sub in ③

$B+\frac{1}{2}=0$

$\boxed{B=-\frac{1}{2}}$ sub in ④

$C+\frac{1}{2}=0$

$\boxed{C=-\frac{1}{2}}$

$\boxed{\int \frac{f'(x)}{f(x)} dx = \log f(x)}$

$A=\frac{1}{2} \quad B=-\frac{1}{2} \quad C=-\frac{1}{2}$

$$\frac{1}{a}\int 1\cdot dx = \int \frac{1}{2(P-1)} - \frac{\frac{1}{2}P-\frac{1}{2}}{(P^2+1)} dP + C$$

$$\frac{x}{a} = \frac{1}{2}\int\left(\frac{1}{P-1} - \frac{P}{P^2+1} - \frac{1}{P^2+1}\right)dP + C$$

$$x = \frac{a}{2}\left[\log(P-1) - \frac{1}{2}\log(P^2+1) - \tan^{-1}P + C\right)$$

$$\boxed{x = \frac{a}{2}\ \frac{2\log(p-1)}{(p^2+1)} - \tan^{-1}p + c}$$

(1) Solve $4y = x^2 + p^2$

(2) Solve $x = yp + ap^2$

(3) Solve $y = 3px + 4p^2$

(4) Solve $y = xp^2 + p$

(5) Solve $x^3p^2 + x^2py + a^3 = 0$

(1) $4y = x^2 + p^2$

Sol:

$$y = \frac{x^2 + p^2}{4} \rightarrow (1)$$

Diff w.r.to 'x'

$$\frac{dy}{dx} = \frac{x^2 + p^2}{4} \text{(w) r.t 4}$$

$$\frac{dy}{dx} = \frac{x^2}{4} + \frac{p^2}{4}$$

$$\frac{dy}{dx} = \frac{2x}{4} + \frac{2p}{4} \cdot \frac{dp}{dx}$$

$$\frac{dy}{dx} = \frac{x}{2} + \frac{p}{2} \cdot \frac{dp}{dx}$$

P

$$4y - x^2 = p^2$$

$$16y^2 - x = p$$

$$\frac{dy}{dx} = 16y^2 - x$$

$$\frac{dy}{16y^2} = -x\,dx$$

I.O.B.S

$$\int \frac{dy}{16y^2} = \int -x\,dx$$

$$\frac{1}{16}\left(\frac{-1}{y}\right) = -\frac{x^2}{2} + c$$

$$\frac{-1}{{}_8 16y} = \frac{-x^2}{2} + c$$

$$\frac{-1}{8y} + x^2 = c$$

$$\boxed{\frac{1}{8y} - x^2 = c}$$

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(2) $x = yp + ap^2$

Sol: $x - ap^2 = yp$

$$y = \frac{x - ap^2}{p} \rightarrow (1)$$

$$\frac{dy}{dx} = \frac{(x-ap^2)\frac{dp}{dx} - p[1 - a2p\cdot\frac{dp}{dx}]}{p^2}$$

$$p = \frac{x\frac{dp}{dx}}{p^2} - \frac{\cancel{p}^2 a\frac{dp}{dx}}{p^2} - \frac{\cancel{p}}{\cancel{p^2}} + \frac{2a\cancel{p}^2\frac{dp}{dx}}{p^2}$$

$$p = \frac{x\cdot\frac{dp}{dx}}{p^2} - a\cdot\frac{dp}{dx} - \frac{1}{p} + 2a\frac{dp}{dx}$$

$$p + \frac{1}{\cancel{p}} x p^{\cancel{2}} = x\cdot\frac{dp}{dx} + a\frac{dp}{dx}$$

$$2p = \frac{dp}{dx}(x+a)$$

$$2p\,dx = dp(x+a)$$

$$\frac{dp}{2p} = \frac{dx}{x+a}$$

I.O.B.S

$$\int\frac{dp}{2p} = \int\frac{dx}{x+a}$$

$$\frac{1}{2}\log p = \log(x+a)$$

$$\log p = 2\log(x+a)$$

$$\cancel{\log} p = \cancel{\log}(x+a)^2$$

$$p = (x+a)^2$$

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From ①

$$\boxed{y = \frac{x - a(x+a)^2}{(x+a)^2}}$$

$y = 3px + 4p^2 \rightarrow ①$

Differentiate w.r.to 'x'

$$\frac{dy}{dx} = 3\left[p(1) + x\cdot\frac{dp}{dx}\right] + 4\, 2p\cdot\frac{dp}{dx}$$

$$p = 3p + 3x\frac{dp}{dx} + 8p\cdot\frac{dp}{dx}$$

$$p - 3p = 3x\frac{dp}{dx} + 8p\frac{dp}{dx}$$

$$-2p = \frac{dp}{dx}(3x+8p)$$

$$-2p\,dx = dp(3x+8p)$$

$$\frac{dp}{-2p} = \frac{dx}{3x+8p}$$

I.O.B.S

$$\int\frac{dp}{-2p} = \int\frac{dx}{3x+8p}$$

$$-\frac{1}{2}\log p = \int\frac{1}{3x+8p}$$

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(4) $y = xp^2 + p$

Sol

Diff. w.r.t. to 'x'

$$\frac{dy}{dx} = x\,2p\cdot\frac{dp}{dx} + p^2(1) + \frac{dp}{dx}$$

$$p = \frac{dp}{dx}(2px+1) + p^2$$

$$p - p^2 = \frac{dp}{dx}(2px+1)$$

$$p(1-p) = \frac{dp}{dx}(2px+1)$$

$$1-p = \frac{\frac{dp}{dx}(2px+1)}{p}$$

$$1-p = \frac{dp}{dx}(2x) + \frac{dp}{dx} + \frac{1}{p}$$

$$1-p = \frac{dp}{dx}\left(2x+\frac{1}{p}\right)$$

$$(1-p)dx = \left(2x+\frac{1}{p}\right)dp$$

$$\frac{dp}{1-p} = \frac{dx}{\left(2x+\frac{1}{p}\right)}$$

$$\int\frac{1}{1-p}\,dp = \int\frac{1}{2x+\frac{1}{p}}\,dx$$

$$\log(1-p) =$$

(5) $x^3p^2 + x^2py + a^3 = 0$

$$x^2py = -x^3p^2 - a^3$$

$$y = -\frac{(a^3 + x^3p^2)}{x^2p}$$

$$y = \frac{-a^3 - x^3p^2}{x^2p}$$

$$y = \frac{-a^3}{x^2p} - \frac{x^3p^2}{x^2p}$$

$$y = \frac{-a^3}{x^2p} - xp$$

$$\frac{dy}{dx} = \frac{x^2p(-2a^2) - (-a^3)\left(2x\frac{dp}{dx}\right)}{(x^2p)^2}$$

$$x\left(\frac{dp}{dx}\right) + p(1)$$

$$p = \frac{-2x^2a^2p}{x^4p^2} + \frac{a^3\,2x\frac{dp}{dx}}{x^4p^2} - x\left(\frac{dp}{dx}\right) + p$$

$$p = \frac{-2x^2a^2}{p} + \frac{a^3\,2\frac{dp}{dx}}{x^3p^2} - x\cdot\frac{dp}{dx} + p$$

$\cancel{p \times p \times x^3 p^2}$ -

$$p - p = \frac{-2x^2a^2}{p} + \frac{2a^3\frac{dp}{dx}}{x^3p^2} - x\cdot\frac{dp}{dx} + p$$

$$-2x^2a^2 + 2a^3\frac{dp}{dx} - x\cdot\frac{dp}{dx}$$

$$x = 2a^3$$

$$\boxed{y = \frac{-a^3 + (2a^3)^3p^2}{(2a^3)^2p}}$$

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* **Solvable for 'x':**

→ If the given equation is solvable for 'x' then we can express 'x' expressively as a function of 'y' and 'p'. Thus the equation of this type can be written as $x = f(y, p)$ →①

→ Differentiating w.r.to 'y' and writing $\frac{1}{p}$ for $\frac{dx}{dy}$

→ we get $\frac{dx}{dy} = \frac{1}{p} = F(y, p, \frac{dp}{dy})$ →②

→ This equation when solve will give us a relation of the form $\phi(y, p, c) = 0$ →③

→ where 'c' being arbitary constant

→ Eliminating 'p' between ① & ③ equations we get the required solution eqn of ① in the form $\phi(x, y, c) = 0$

→ If the elimination of p between ① & ③ eqns is not possible, then we solve ① & ③ eqns to express x and y in terms of p and c in the form $x = f_1(p, c)$, $y = f_2(p, c)$

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→ this two eq^ns together will be the solution of equation ① interms of parameter 'p'

→ Some times even the form eq^n ④ desired solution is not possible in that case eq^ns ① & ③ together constitute the solution giving x and y interms of parameter 'p'

NOTE-1

① In some problems eq^n ② we can Express
$f_1(y,p)\, f_2(y,p,\frac{dp}{dy})=0$

② In such cases, we ignore the first factor $f_1(y,p)$ which does not involve $\frac{dp}{dy}$ and proceed with $f_2(y,p,\frac{dp}{dy})=0$

* NOTE :

→ If instead of ignoring the factor $f_1(y,p)$ we eliminate 'p' b/w eq^n ① and $f_1(y,p)=0$ we obtain an equation involving no constant 'c'

→ This is known as Singular solution of equation ①

(1) Solve $x = 3y - \log p$

Sol. Given equation

$$x = 3y - \log p \rightarrow (1)$$

Diff w.r.to 'y'

$$\frac{dx}{dy} = 3 - \frac{1}{p} \cdot \frac{dp}{dy}$$

$$\frac{dx}{dy} = 3 - \frac{1}{p}\frac{dp}{dy}$$

$$\cancel{\frac{1}{p}} = 3 - \cancel{\frac{1}{p}}\frac{dp}{dy}$$

$$\cancel{3 + \frac{dp}{dy} = 0} \qquad \frac{1}{p} + \frac{1}{p}\frac{dp}{dy} = 3$$

$$\cancel{\frac{dp}{dy} =} \qquad \frac{1}{p}\left[1 + \frac{dp}{dy}\right] = 3$$

$$\cancel{\frac{dp}{dy} = 3} \qquad 1 + \frac{dp}{dy} = 3p$$

$$\cancel{dp = 3dy} \qquad \frac{dp}{dy} = 3p - 1 \quad = \quad dy = \frac{dp}{3p-1}$$

I.O.B.S

$$\int dy = \int \frac{dp}{3p-1} + C$$

$$y = \frac{1}{3}\log(3p-1) + C$$

$$y - C = \frac{1}{3}\log(3p-1)$$

$$3y = \log(3p-1)$$

$$e^{3y} = 3p-1$$

(2)

Sol:

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$e^{3y} - c = 3p - 1$

$e^{3y} - c + 1 = 3p$

$p = \dfrac{e^{3y} + 1 - c}{3} \rightarrow (2)$

from (1) & (2)

$$\boxed{x = 3y - \log\left[\frac{e^{3y} + 1 - c}{3}\right]}$$

(2) solve $y = 2px + p^3y^2$

Sol: Given Equation

$y = 2px + p^3y^2$

$2px + p^3y^2 - y = 0$

$2px = y - p^3y^2$

$2x = \dfrac{y}{p} - p^2y^2 \rightarrow (1)$

Diff w.r.to y

$$2\frac{dx}{dy} = \frac{p(1) - y\frac{dp}{dy}}{p^2} - p^2 2y - y^2 2p\frac{dp}{dy}$$

$$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2yp^2 - y^2 2p\frac{dp}{dy}$$

$$\frac{2}{p} - \frac{1}{p} + 2yp^2 = \frac{-dp}{dy}\left(\frac{y}{p^2} + 2py^2\right)$$

$$\left(\frac{1}{p} + 2yp^2\right) + \frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + 2yp^2\right) = 0$$

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UNIT-II

ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Linear differential equation of second and higher order.

The eqn of the form.

$$\frac{d^n y}{dx^n} + P_1 x \frac{d^{n-1} y}{dx^{n-1}} + P_2 x \frac{d^{n-2} y}{dx^{n-2}} + \dots P_n(x) y = \phi(x)$$

$P_1(x), P_2(x) \dots P_n(x)$ & $\phi(x)$ are all continuous real value function of 'x' is called a linear differential eqn of order 'n'.

linear diff eqns with constant coefficient.

An equation of the form

$$\frac{d^n y}{dx^n} + P_1 x \frac{(d^{n-1}) y}{dx^{n-1}} + \dots P_n(x) y = \phi(x) \quad (1)$$

where $P_1, P_2 \dots P_n$ are real constants and $\phi(x)$ is a continuous function of 'x' and is called an ordinary linear eqn of order 'n' with constant coeff.

General solution

$y = C.F + P.I$

where C.F is complementary function and

P.I is particular integral.

Find the G.S complementary function of

$F(D)y = 0$

The Algebric equation $F(m) = 0$ that is

i.e $m^n + P_1 m^{n-1} + P_2 m^{n-2} + \dots \dots P_n = 0$.

where $P_1, P_2 \dots P_n$ are real constants is

called Auxillary equation (A.E) of $F(D)y=0$

Since, the Auxillary equation $F(m) = 0$ is a

polynomial equation of degree 'n' it will

have n^{th} roots $m_1, m_2 \dots \dots m_n$.

Auxillary equation (A.E)	complementary function (C.F)
1) $m_1, m_2 \dots m_n$ is i.e, all roots are real and distinct	1) $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$
2) $m_1, m_2 \dots, m_n$ i.e, two roots are real & equal and rest are real & different	2) $(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
3) $m_1, m_2 \dots m_n$ i.e, three roots are real & equal & rest are real & different.	3) $(c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
4) Two roots of A.E are complex say $\alpha + i\beta$ & $\alpha - i\beta$ and the remaining roots are real & different.	4) $e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \dots c_n e^{m_n x}$
5) A pair of conjugate complex roots $\alpha \pm i\beta$ are repeated twice & the remaining roots are real & different	5) $e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$
6) A pair of conjugate complex roots $\alpha \pm i\beta$ are repeated thrice & the remaining roots are real & different	6) $e^{\alpha x}[(c_1 + c_2 x + c_3 x^2)\cos\beta x + (c_4 + c_5 x + c_6 x^2)\sin\beta x] + c_7 e^{m_7 x} + c_8 e^{m_8 x} + \dots + c_n e^{m_n x}$

Note

If $\alpha+\sqrt{\beta}$ is a real, irrational roots of $F(m)=0$ $F(m)=0$, $\alpha-\sqrt{\beta}$ is also a root of the equation. the part of the complementary function is corresponding to this roots can also be put in the form $\boxed{e^{\alpha x}(c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x)}$

problems

Solve the differental equation

① $(D^2-3D+4)y=0$

$F(D)y=0$

A.E. $F(m)=0$.

$m^2-3m+4=0$. ①

$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$a=1, b=-3, c=4$

$m=\frac{3\pm\sqrt{9-4\times1\times4}}{2\times1}$

$m=\frac{3\pm\sqrt{9-16}}{2}$

$m=\frac{3\pm\sqrt{-7}}{2}$

$m=\frac{3\pm\sqrt{7i^2}}{2}$

$m=\frac{3\pm i\sqrt{7}}{2}$

$\alpha\pm i\beta$ of form

$\alpha=3/2$, $\beta=\sqrt{7}/2$

$$C.F = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right)$$

$$\alpha = \frac{3}{2}, \quad \beta = \frac{\sqrt{7}}{2}$$

$$C.F = e^{3/2 x}\left[C_1 \cos \frac{\sqrt{7}}{2} x + C_2 \sin \frac{\sqrt{7}}{2} x\right]$$

② find the solution $y'' + 2y' = 0$.

The given equation $y'' + 2y' = 0$ — ①

equation ① is symbolic form

$$(D^2 + 2D)\, y = 0$$

This form of $F(D)y = 0$

$$F(m) = 0$$

$$m^2 + 2m = 0$$

$$m(m+2) = 0$$

$$m = 0, \quad m = -2.$$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots$$

$$m_1 = 0, \quad m_2 = -2$$

$$C.F = c_1 e^{0} + c_2 e^{-2x}$$

③ solve $y'' + 6y' + 9y = 0$, given values $y(0) = -4$, $y'(0) = 14$

The given equation $y'' + 6y' + 9y = 0$ — ①

Eqn ① is a symbolic form

$$(D^2 + 6D + 9)y = 0 \quad \text{— ②}$$

A.E $F(m) = 0$

$$m^2 + 6m + 9 = 0$$

$$m^2 + 3m + 3m + 9 = 0$$

$$m(m+3) + 3(m+3) = 0$$

$$(m+3)(m+3) = 0$$

$m+3=0$, $m+3=0$,

$$m = -3, -3.$$

$$C.F = (C_1 + C_2 x)e^{mx}$$

$$C.F = (C_1 + C_2 x)e^{-3x} \quad \text{— ③}$$

$$y = (C_1 + C_2 x)e^{-3x} \quad \text{— ④}$$

Diff w.r.t 'x'

$$y' = (C_1 + C_2 x)e^{-3x}(-3) + e^{-3x}(C_2)$$

$$y' = -3(C_1 + C_2 x)e^{-3x} + C_2 e^{-3x} \quad \text{— ⑤}$$

Given that $y(0) = -4$ in Eqn (4)

$$-4 = (c_1 + c_2 \cdot 0)\, e^{-3 0}$$

$$-4 = c_1$$

$$\boxed{c_1 = -4} \rightarrow (6)$$

Given $y'(0) = 14$ in Eqn (5) & (6)

$$14 = -3\,(-4 + c_2(0))\, e^{-3(0)} + c_2\, e^{-3(0)}$$

$$14 = 12 + 3c_2 e^{-3(0)} + c_2 e^{-3(0)}$$

wrong

$$14 = 12 + 3c_2 + c_2$$

$$14 = 12 + c_2$$

$$2 = 2c_2$$

$$\boxed{c_2 = -1}$$

$$14 = 12 + c_2 e^{-3(0)}$$

$$* \boxed{2 = c_2} \;(7)$$

$c_1 = -4$, $c_2 = 2$ in Eqn (4)

$$y = (-4 + 2x)\, e^{-3x}$$

$$\boxed{y = (2x - 4)\, e^{-3x}}$$

④ solve $(D^3 - 14D + 8)y = 0$

A.E $F(m) = 0$

$m^3 - 14m + 8 = 0$

by synthetic Division.

$$\begin{array}{r|rrrr} m=-4 & 1 & 0 & -14 & 8 \\ & 0 & -4 & 16 & -8 \\ \hline & 1 & -4 & 2 & \boxed{0} \end{array}$$

$(m+4)(m^2 - 4m + 2) = 0$

$m+4=0, \quad m^2 - 4m + 2 = 0$

$m = -4, \quad m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$m = \dfrac{4 \pm \sqrt{16 - 4 \times 1 \times 2}}{2}$

$m = \dfrac{4 \pm \sqrt{8}}{2}$

$m = \dfrac{4 \pm 2\sqrt{2}}{2}$

$m = 2 \pm \sqrt{2}$

$\boxed{m = -4, \; m = 2 \pm \sqrt{2}}$

$\alpha \pm \sqrt{\beta}$. (note)

$C.F = c_1 e^{m_1 x} + e^{\alpha x}\left[c_2 \cosh\sqrt{\beta}x + c_3 \sinh\sqrt{\beta}x\right]$

$C.F = c_1 e^{-4x} + e^{2x}\left[c_2 \cosh\sqrt{2}x + c_3 \sinh\sqrt{2}x\right]$

(5) solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

A.E $F(m) = 0$.

$m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & -3 & 4 & 4 \\ & 0 & -1 & 3 & 0 & -4 \\ \hline -1 & 1 & -3 & 0 & 4 & \boxed{0} \\ & 0 & -1 & 4 & -4 & \\ \hline & 1 & -4 & 4 & \boxed{0} & \end{array}$$

$(m+1)(m+1)(m^2 - 4m + 4) = 0$.

$(m+1)(m+1)(m-2)(m-2) = 0$

$m = -1, -1, 2, 2$.

$C.F = (c_1 + c_2 x)e^{mx} + (c_3 + c_4 x)e^{mx}$

$C.F = (c_1 + c_2 x)e^{-1x} + (c_3 + c_4 x)e^{2x}$

$y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}$

(6) $(D^3 - 1)y = 0$.

$F(D)y = 0$

A.E $F(m) = 0$

$m^3 - 1^3 = 0$ $\quad \boxed{a^3 - b^3 = (a-b)^3 + 3ab[a-b]}$

$(m-1)^3 + 3m(1)[m-1] = 0$

$m-1[(m-1)^2 + 3m] = 0$

$(m-1) = 0, \quad m^2 + 1 - 2m + 3m = 0$

$m = 1$ $\quad m^2 + m + 1 = 0$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)}$$

$$m = \frac{-1 \pm \sqrt{-3}}{2(1)}$$

$$m = \frac{-1 \pm \sqrt{3(-1)}}{2(1)} = \frac{-1 \pm \sqrt{3i^2}}{2}$$

$$\boxed{m = \frac{-1 \pm i\sqrt{3}}{2}}$$

$$= \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\alpha = -1/2$

$\sqrt{\beta} = \frac{\sqrt{3}}{2}$

$$y = c_1 e^{1x} + e^{-x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

General method of finding particular integral (P.I)

$\Rightarrow$ A par. I of $F(D)y = Q(x)$, when $\frac{1}{F(D)}$ is expressed as partial fractions

$$P.I = \frac{1}{F(D)} \times Q = \frac{1}{(D-\alpha_1)(D-\alpha_2)\cdots(D-\alpha_n)} \times Q$$

$\Rightarrow$ resolving into partial fractions

$$= \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \cdots + \frac{A_n}{D-\alpha_n} \right)$$

General Solution of a P.I.

(i) $A_1 e^{\alpha_1 x} \int Q \cdot e^{-\alpha_1 x} dx + A_2 e^{\alpha_2 x} \int Q \cdot e^{-\alpha_2 x} dx + \cdots A_n e^{\alpha_n x} \int Q \cdot e^{-\alpha_n x} dx.$

problems

$F(D)y = Q(x)$

① solve $(D^2 - 5D + 6)y = xe^{4x}$

$F(D)y = 0$

A·E $F(m) = 0$

$m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$

$m_1 = 2, 3 = m_2$

$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$y = C_1 e^{2x} + C_2 e^{3x}$ — (2)

$\frac{1}{F(D)} Q = e^{\alpha x} \int Q \cdot e^{-\alpha x} dx$ — (I)

$$\boxed{\frac{1}{F(D)} Q = \left[\frac{1}{(D-2)(D-3)}\right] x e^{4x}}$$

$Q = xe^{4x}$

L.H.S

$$\left[\frac{1}{(D-2)(D-3)}\right] x \cdot e^{4x} = \left[\frac{A}{D-2} + \frac{B}{D-3}\right] x \cdot e^{4x}$$ — (3)

$$\left[\frac{1}{(D-2)(D-3)}\right] \cancel{x \cdot e^{4x}} = \left(\frac{A(D-3) + B(D-2)}{(D-2)(D-3)}\right)$$

$A(D-3) + B(D-2) = 1$

put $D = 2$

$\boxed{A = -1}$

put $D = 3$,

$A(3-3) + B(3-2) = 1$

$\boxed{B = 1}$

$A = -1, \; B = 1.$

[Eqn (3)]

$$= \left[\frac{-1}{D-2} + \frac{1}{D-3} \right] x e^{4x} \quad (4)$$

from eqn (I)

$$\left[\frac{1}{D-3}\right] x e^{4x} = e^{3x} \int x e^{4x} e^{-3x} \, dx$$

$\alpha = 3$

$$= e^{3x} \int x e^{x} \, dx$$

$$= e^{3x} \left[x e^{x} - e^{x} \right]$$

$$= e^{4x} [x - 1] \quad (5)$$

$$\left[\frac{-1}{D-2}\right] x e^{4x} = e^{\alpha x} \int \phi \cdot e^{-\alpha x} \, dx$$

$\alpha = 2$

$$= e^{2x} \int x e^{4x} \cdot e^{-2x} \, dx$$

$$= e^{2x} \int x \cdot e^{2x} \, dx$$

$$= e^{2x} \left[x \int e^{2x} dx - \int \left(x' \int e^{2x} dx \right) dx \right]$$

$$= e^{2x} \left[x \cdot \frac{e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2} \, dx \right]$$

$$= e^{2x} \left[x \cdot \frac{e^{2x}}{2} - \frac{1}{2} \cdot \frac{e^{2x}}{2} \right]$$

$= e^{2x} \cdot \frac{e^{2x}}{2}\left[x - \frac{1}{2}\right]$

$= \frac{e^{4x}}{2}\left[\frac{2x-1}{2}\right]$

$\left[\frac{-1}{D-2}\right] \varphi = \frac{e^{4x}}{4}(2x-1)$ — (6)

(4), (5), (6)

$\left[\frac{-1}{D-2} + \frac{1}{D-3}\right] x e^{4x}$

$P.I = \frac{e^{4x}}{4}(2x-1) + e^{4x}(x-1)$

$P.I = e^{4x}\left[\frac{2x-1}{4} + x - 1\right]$

$P.I = e^{4x}\left[\frac{2x-1+4x-4}{4}\right]$

$P.I = \frac{e^{4x}}{4}[6x-5]$

$y = C.F + P.I$

$y = c_1 e^{2x} + c_2 e^{3x} + \frac{e^{4x}}{4}(6x-5)$

→ solve. $(D^2+4D+3)\,y = e^{e^x}$

$F(D)\,y = \phi(x)$

$F(D)\,y = 0$

$F(m) = 0$.

A.E. $m^2+4m+3=0$

$m^2+3m+m+3=0$

$m(m+3)+1(m+3)=0$

$m = -1, -3$.

$\boxed{C.F = c_1 e^{-x} + c_2 e^{-3x}}$ — (1)

$$\boxed{P.I = \frac{1}{F(D)}\phi = e^{\alpha x}\int \phi e^{-\alpha x}\,dx}$$

L.H.S

$$\left[\frac{1}{(D+1)(D+3)}\right]e^{e^x} = \left[\frac{A}{D+1} + \frac{B}{D+3}\right]e^{e^x} \quad \text{— (I)}$$

one step here missed

$1 = A(D+3) + B(D+1)$ — (2)

put $D=-1$ in eqn (2)

$1 = A(-1+3) + B(0)$

$\boxed{A = 1/2}$

put $\boxed{D=-3}$ in eqn (2)

$\boxed{B = -1/2}$

eqn (I) replace above value

$$\left[\frac{1}{2(D+1)} - \frac{1}{2(D+3)}\right] e^{e^x} \quad \text{--- (3)}$$

$$\left[\frac{1}{2(D+1)}\right] e^{e^x} = \frac{1}{2}\left(e^{-x}\int e^{e^x} e^x \, dx\right)$$

$$e^x = t$$

$$e^x dx = dt$$

$$= \frac{1}{2}\left[e^{-x}\int e^t \, dt\right]$$

$$= \frac{1}{2}\left[e^{-x}\left(e^t\right)\right]$$

$$\left[\frac{1}{2(D+1)}\right] = \frac{e^{-x}\left(e^{e^x}\right)}{2} \quad \text{--- (4)}$$

$$\left(\frac{1}{2(D+3)}\right) e^{e^x} = \frac{1}{2}\left[e^{-3x}\int e^{e^x} e^{3x} \, dx\right] \quad \text{--- (5)}$$

$$e^x = t$$

$$e^x dx = dt$$

$$dx = \frac{dt}{e^x}$$

values in eqn

$$= \frac{1}{2}\left[e^{-3x}\int e^{t} t^{3} \frac{dt}{e^{x}}\right]$$

$$= \frac{1}{2} e^{-3x}\left(\int e^{t} t^{3} \frac{dt}{t}\right)$$ from below $e^{x} = t$

$$= \frac{1}{2} e^{-3x}\left(\int e^{t} t^{2} dt\right)$$

$$= \frac{1}{2} e^{-3x}\left[t^{2}e^{t} - \int 2t e^{t} dt\right]$$

$$= \frac{1}{2} e^{-3x}\left[t^{2}e^{t} - 2\left[e^{t}(t-1)\right]\right]$$

$$= \frac{1}{2} e^{-3x}\left[e^{t}\left[t^{2} - 2t + 2\right]\right]$$

$$= \frac{1}{2} e^{-3x}\left[e^{e^{x}}\left(e^{2x} - 2e^{x} + 2\right)\right]$$

$$\left[\frac{1}{2(D+3)}\right]e^{e^{x}} = \frac{e^{-3x}\left[e^{e^{x}}\left(e^{2x} - 2e^{x} + 2\right)\right]}{2} \quad \text{——— (6)}$$

(3), (4), (6)

$$P.I = e^{-x}\left[e^{e^{x}}\right] - \frac{e^{-3x}\left[e^{e^{x}}\left(e^{2x} - 2e^{x} + 2\right)\right]}{2}.$$

$$G.S = C.F + P.I$$

$$G.S = c_1 e^{-x} + c_2 e^{-3x} + \frac{e^{-x}\left[e^{e^x}\right]}{2} - \frac{e^{-3x}\left[e^{e^x}(e^{2x} - 2e^x \ldots\right.}{2}$$

Solve v.v.imp

$$(D^2+4)y = \tan 2x$$

$$(D^2+4)y = 0$$

$$F(D)y = 0$$

A.E $\quad F(m) = 0$

$$m^2 + 4 = 0$$

$$m = \pm 2i$$

$$m = \pm 2i$$

$$C.F = e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$$

$$\alpha = 0, \quad \beta = 2$$

$$C.F = e^0\left[c_1 \cos 2x + c_2 \sin 2x\right] \quad \text{--- (1)}$$

$$P.F = \frac{1}{F(D)} Q = e^{\alpha x}\int Q e^{-\alpha x}\, dx.$$

$$\left[\frac{1}{(D+2i)(D-2i)}\right]\tan 2x = \left[\frac{A}{D+2i} + \frac{B}{D-2i}\right]\tan 2x.$$

$$\left[\frac{1}{(D+2i)(D-2i)}\right]\tan 2x = \left[\frac{A(D-2i)+B(D+2i)}{(D+2i)(D-2i)}\right]\tan 2x$$

$$1 = A(D-2i)+B(D+2i)$$

put $D=-2i$

$$1 = A(-2i-2i)+B(0)$$

$$1 = A(-4i)$$

$$\boxed{A = \frac{-1}{4i}}$$

put $D = 2i$

$$1 = A\cdot(0)+B(2i+2i)$$

$$1 = B(4i)$$

$$\boxed{B = \frac{1}{4i}}$$

$$\left[\frac{A}{D+2i}+\frac{B}{D-2i}\right]\tan 2x = \left[\frac{-1}{4i(D+2i)}+\frac{1}{4i(D-2i)}\right]\tan 2x$$

$$\left[\frac{-1}{4i(D+2i)}\right]\tan 2x = \frac{1}{4i}\left[e^{\alpha x}\int \phi\cdot e^{-\alpha x}\,dx\right]$$

$$= \frac{-1}{4i}\left[e^{+2i\cdot x}\int \tan 2x\cdot e^{-2i\cdot x}\,dx\right]$$

$$\boxed{e^{i\theta} = \cos\theta - i\sin\theta}$$

$$= e^{2ix}\left(\int \frac{\sin 2x}{\cos 2x}\left[\cos 2x - i\sin 2x\right]dx\right) \times \frac{1}{4i}$$

$$= e^{2ix}\left[\int \sin 2x - \frac{i(\sin^2 2x)}{\cos x}\,dx\right] \times \frac{1}{4i}$$

$$= e^{2ix}\left[\int \sin 2x - \frac{i\left[1-\cos^2 2x\right]}{\cos 2x}\,dx\right] \times \frac{1}{4i}$$

$$= e^{2ix}\left[\int (\sin 2x - i\sec 2x + i\cos 2x)\,dx\right] \times \frac{1}{4i}$$

$$\boxed{\int \sec 2\theta = \frac{\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{2}}$$

$$= e^{2ix}\left(\frac{-\cos 2x}{2} - \frac{i}{2}\left(\log\tan\left(\pi/4 + x\right)\right) + \frac{i\sin 2x}{2}\right) \times \frac{1}{4i}$$

$$= e^{2ix}\left[\frac{-\cos 2x}{2} + \frac{i}{2}\sin 2x - \frac{i}{2}\left(\log\tan\left(\pi/4 + x\right)\right)\right] \times \frac{1}{4i}$$

$$= \frac{e^{2ix}}{2}\left(\cos 2x - i\sin 2x + i\log\tan\left(\pi/4 + x\right)\right) \times \frac{1}{4i}$$

$$= \frac{-e^{2ix}}{2}\left[e^{-i2x} + i\left(\log\tan\left(\pi/4 + x\right)\right)\right] \times \frac{1}{4i}$$

$$\left[\frac{1}{4i(D-2i)}\right]Q = \frac{e^{2ix}}{8i}\left[e^{-2ix} + i\left(\log\tan\left(\frac{\pi}{4}+x\right)\right)\right] \quad \text{--- (6)}$$

$$\cancel{\left[\frac{1}{4i(D+2i)}\right]Q = \frac{1}{4i}\left[e^{\alpha x}\int Q\cdot e^{-\alpha x}\,dx\right]}$$

put $i = -i$ in eqn (6)

$$\cancel{\frac{1}{4i}\left[e^{-2ix}\right.}$$

$$\left[\frac{1}{4i(D+2i)}\right]Q = \frac{e^{-2ix}}{-8i}\left[e^{2ix} - i\left[\log\tan\left(\frac{\pi}{4}+x\right)\right]\right] \quad \text{--- (7)}$$

$$\cancel{\left[\frac{1}{D+2i} + \frac{13}{D-2i}\right]\tan 2x}$$

(7) & (6)

$$\Rightarrow \left[\frac{-1}{4i(D+2i)} + \frac{1}{4i(D-2i)}\right]Q$$

$$= \frac{e^{-2ix}}{8i}\left[e^{2ix} - i\left[\log\tan\left(\frac{\pi}{4}+x\right)\right]\right] +$$

$$\frac{e^{2ix}}{8i}\left[e^{-2ix} + i\left(\log\tan\left(\frac{\pi}{4}+x\right)\right)\right]$$

$GS = C.F + P.I$

$GS = c_1 \cos 2x + c_2 \sin 2x +$

$\frac{1}{8i}\left[e^{2ix}\left(e^{-i2x} + i\left(\log \tan\left(\frac{\pi}{4}+x\right)\right)\right) + e^{-2ix}\left(e^{2ix} + i\left(\log\tan\left(\frac{\pi}{4}+x\right)\right)\right)\right.$

Particular Integral method:

If $F(D)y = \phi(x)$, where $\phi(x) = e^{ax}$

where 'a' a constant.

formula (Method-1)

1. $\frac{1}{F(D)} e^{ax} = \frac{e^{ax}}{F(a)}$ if $F(a) \neq 0$

2. $\frac{1}{F(D)} e^{ax} = \frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$ if $\phi(a) \neq 0$

problems

(1) Solve $(D^2-4D+13)y = e^{2x}$.

$[D^2-4D+13]y = 0$

$F(D)y = 0$

A.E $F(m) = 0$

$m^2-4m+13=0$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a=1$, $b=-4$, $c=13$

$m = \frac{4 \pm \sqrt{16-4\times1\times13}}{2\times1} = \frac{4 \pm \sqrt{6^2 i^2}}{2}$

$m = \frac{4 \pm 6i}{2}$

$m = 2 \pm 3i$

$\alpha \pm \beta i$, $\alpha = 2$, $\beta = 3$.

$C.F = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$

$= e^{2x}[c_1 \cos 3x + c_2 \sin 3x]$ — (1)

$$P.I = \left[\frac{1}{F(D)}\right] e^{ax} = \frac{e^{ax}}{F(a)} \quad \text{if } F(a) \neq 0.$$

$$P.I = \left[\frac{1}{D^2 - 4D + 13}\right] e^{2x} = \left[\frac{e^{2x}}{2^2 - 4 \cdot 2 + 13}\right] = \frac{e^{2x}}{9}$$

$$G.S = C.F + P.I$$

$$G.S = e^{2x}\left[c_1 \cos 3x + c_2 \sin 3x\right] + \frac{e^{2x}}{9}$$

(2) Solve $(D^2+16)y = e^{-4x}$

$$F(D)\,y = 0$$

A E $F(m) = 0$

$$m^2 + 16 = 0$$

$$m = \pm 4i$$

$\alpha \pm \beta i$ $\alpha = 0,\ \beta = 4$

Complementry function

$$= e^{0x}\left[c_1 \cos 4x + c_2 \sin 4x\right]$$

$$C.F = c_1 \cos 4x + c_2 \sin 4x$$

$$P.I = \frac{1}{F(D)} \cdot e^{ax} = \frac{e^{ax}}{F(a)}$$

$$\frac{1}{(D^2+16)} e^{-4x} = \frac{e^{-4x}}{(-4)^2+16} = \frac{e^{-4x}}{32}$$

$$P.I. = \frac{e^{-4x}}{32}.$$

$G.S = C.F + P.I$

$$= \cancel{\frac{e^{ax}}{F(a)}}$$

$$G.S = c_1 \cos 4x + c_2 \sin 4x + \frac{e^{-4x}}{32}.$$

(3) $(4D^2 - 4D + 1)y = 100$

$$(4D^2 - 4D + 1)\, y = e^{0.x} \cdot 100.$$

$$(4D^2 - 4D + 1)y = 0$$

$$F(D)y = 0$$

$A.E \quad F(m) = 0$

$$4m^2 - 4m + 1 = 0.$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$a = 4, \quad b = -4, \; c = 1.$

$$m = \frac{4 \pm \sqrt{16 - 4 \times 4 \times 1}}{2 \times 4}$$

$$m = \frac{1}{2}, \frac{1}{2}$$

$$C.F = \cancel{c_1 e^{1/2 x}} \; (c_1 + c_2 x)\, e^{+1/2 x}$$

$$P.I = \frac{1}{F(D)} e^{ax} = \frac{e^{ax}}{F(a)} \quad (\because F(a) \neq 0)$$

$$= \frac{1}{(4D^2-4D+1)} e^{0.x}.100 = \frac{e^{0.x} . 100}{4(0)^2-4(0)+1} = \frac{100}{1} = 100$$

$P.I = 100$

$G.S = C.F + P.I$

$$= (C_1+C_2x)e^{x/2} + 100$$

④ solve $[D^3 - 5D^2 + 8D - 4]y = e^{2x}$

$F(D)y = 0$

A.E $F(m) = 0$

$$m^3 - 5m^2 + 8m - 4 = 0.$$

$$\begin{array}{c|cccc} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$(m-1)(m^2-4m+4) = 0$

$m = 1 \qquad m^2-4m+4 = 0$

$$m = \frac{4 \pm \sqrt{16-4(1)(4)}}{2(1)}$$

$m = 2$

$(m-2)^2 = 0$

$m = 1, \quad m = 2, 2$

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$m = 1, 2, 2$

C.F $= C_1 e^{x} + [C_2 + x C_3] e^{2x}$

$$P.I = \frac{1}{F(D)} e^{ax} = \frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$$

$$\frac{1}{(D-1)\underbrace{(D-2)^2}} \cdot e^{2x} = \frac{e^{2x}}{(2-1)} \cdot \frac{x^2}{2!}$$

(Same)

$$= \frac{e^{2x}}{1} \cdot \frac{x^2}{2!}$$

G.S = C.F + P.I

G.S = $C_1 e^{x} + [C_2 + C_3 x] e^{2x} + e^{2x} \frac{x^2}{2!}$

5) $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

$F(D)\, y = 0$

A.E $F(m) = 0$

$(m+2)(m-1)^2 = 0$

$m = -2, 1, 1$

C.F is $y = C_1 e^{-2x} + [C_2 + C_3 x] e^{x}$

$$P.I = \frac{e^{-2x} + 2\sinh x}{(D+2)(D-1)^2} = \frac{e^{-2x} + e^{x} - e^{-x}}{(D+2)(D-1)^2}$$

$$P.I = \frac{e^{-2x}}{(D+2)(D-1)^2} + \frac{e^{x}}{(D+2)(D-1)^2} + \frac{e^{-x}}{(D+2)(D-1)^2}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$y_{p_1} \qquad\qquad y_{p_2} \qquad\qquad y_{p_3}$$

$$y_{p_1} = \frac{e^{-2x}}{(D+2)(D-1)^2} \qquad \frac{e^{ax}}{F(D)} = \frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$$

$$y_{p_1} = \frac{e^{-2x}}{9} \cdot \frac{x^1}{1!} = \frac{e^{-2x} \cdot x}{9}$$

$$\text{(or)} \quad \frac{e^{-2x}}{(D+2)(D^2-1+2D)} = \frac{e^{-2x}}{D^3-3D+2} = \frac{e^{-2x} \cdot x}{3D^2-3} = \frac{e^{-2x} \cdot x}{3(-2)^2-3} = \frac{e^{-2x} \cdot x}{9}$$

$$y_{p_2} = \frac{e^{x}}{(D+2)(D-1)^2} \quad \{\text{Same}\}$$

$$= \frac{e^{ax}}{F(D)} = \frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$$

$$= \frac{e^{x}}{(D+2)(D-1)^2} = \frac{e^{x}}{3} \cdot \frac{x^2}{2!}$$

$$y_{p_3} = \frac{e^{-x}}{(D+2)(D-1)^2}$$

$$= \frac{e^{ax}}{F(D)} = \frac{e^{ax}}{F(a)}$$

$$= \frac{e^{-x}}{(D+2)(D-1)^2} = \frac{e^{-x}}{(1)(-2)^2} = \frac{e^{-x}}{4}$$

$$G.S = C.F + P.I$$

$$G.S = c_1 e^{-2x} + (c_2 + c_3 x) e^{x} + \frac{x e^{-2x}}{9} + \frac{x^2 e^{x}}{6} - \frac{e^{-x}}{4}$$

6) $(D^3-1)y = (e^x+1)^2$

A.E $m^3-1=0$

$(m-1)(m^2+m+1)=0$

$m-1=0, \quad m^2+m+1=0$

$m=1, \quad \frac{-1\pm i\sqrt{3}}{2}$

C.F $= y_c = c_1e^x + e^{-x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$

P.I. $y_p = \frac{1}{D^3-1}(e^x+1)^2$

$= \frac{e^{2x}+2e^x+1}{D^3-1}$

$= \frac{e^{2x}}{D^3-1} + \frac{2e^x}{D^3-1} + \frac{1}{D^3-1}\cdot e^{0\cdot x}$

$= \frac{e^{2x}}{8-1} + \frac{2e^x}{(D-1)(D^2+D+1)} + \frac{1}{0-1}\cdot e^{0\cdot x}$

$= \frac{e^{2x}}{7} + \frac{2x\cdot e^x}{3} - 1$

(7) $y'' - 4y' + 3y = 4e^{3x}$, $y(0) = -1$, $y'(0) = 3$

$(D^2 - 4D + 3)y = 4e^{3x}$

$\therefore F(D)y = 0$

$F(m) = 0$

$m^2 - 4m + 3 = 0$

$m = 3, 1$

$C.F = c_1 e^{3x} + c_2 e^{x}$

$P.I = \frac{4e^{3x}}{D^2 - 4D + 3} = \frac{4e^{3x}}{(D-1)(D-3)}$

$= \frac{4e^{3x}}{(D-1)(D-3)} = \frac{\cancel{4}e^{3x}}{\cancel{2}} \cdot \frac{x}{1!} = 2xe^{3x}$

$y = (G.S) = c_1 e^{3x} + c_2 e^{x} + 2xe^{3x}$ —— (1)

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Given $y(0) = -1$ sub in Eq ①

eq① $c_1 e^{0} + c_2 e^{0} + 2 \cdot 0 \cdot e^{0} = -1$

$c_1 + c_2 = -1$ —— ②

$y'(x) = 3c_1 e^{3x} + c_2 e^{x} + 2e^{3x} + 6x e^{3x}$

$y'(0) = 3c_1 + c_2 + 2 + 0.$

$3 = 3c_1 + c_2 + 2$ ——.

$3c_1 + c_2 = 1$ —— ③

solve ② & ③

$$\begin{aligned} c_1 + c_2 &= -1 \\ 3c_1 + c_2 &= 1 \\ \hline -2c_1 &= -2 \end{aligned}$$

$\boxed{c_1 = 1}$ sub in ②

$1 + c_2 = -1$

$\boxed{c_2 = -2}$

Subtitute the above values in ①

$y = e^{3x} - 2e^{x} + 2xe^{3x}$

$y = -2e^{x} + (1+2x)e^{3x}$

Method:-2

(1) $\frac{1}{F(D)} \sin bx = \frac{1}{\phi(D^2)} \sin bx = \frac{\sin bx}{\phi(-b^2)}$

$\left[\text{if } (\phi(-b^2)) \neq 0\right]$

(2) $\frac{1}{F(D)} \cos bx = \frac{1}{\phi(D^2)} \cos bx = \frac{\cos bx}{\phi(-b^2)}$ (if $\phi(-b^2) \neq 0$)

(3) $\frac{1}{D^2+b^2} \sin bx = \frac{-x\cos bx}{2b}$ (if $(\phi(-b^2)) = 0$)

(4) $\frac{1}{D^2+b^2} \cos bx = \frac{x \sin bx}{2b}$ (if $\phi(-b^2) = 0$)

problems

(1) Solve $(D^2+3D+2)y = \sin 3x$

$(D^2+3D+2)\,y = 0.$

$m^3+3m+2=0.$

$m^2+2m+m+2=0.$

$m(m+2)+1(m+2)=0$

$(m+1)(m+2)=0$

$m=-1,-2$

$C.F = c_1 e^{-x} + c_2 e^{-2x}$

$$P.I = \frac{1}{F(D)} \cdot \sin bx$$

$\phi(-b^2)$

$\phi(-3\times3) = -9$

$$= \frac{1}{(D^2+3D+2)} \sin 3x$$

differentiation $D = \frac{d}{dx}$
$\frac{1}{D} = \int$ integration

$$= \frac{\sin 3x}{-9+3D+2}$$

$$= \frac{\sin 3x}{3D-7} \times \frac{3D+7}{3D+7}$$

$$= \frac{3D\sin 3x + 7\sin 3x}{(3D)^2 - 7^2}$$

$$= \frac{3D\sin 3x + 7\sin 3x}{9D^2 - 49}$$

$$= \frac{3 \cdot \frac{d}{dx}\sin 3x + 7\sin 3x}{9\times(-9) - 49}$$

$$= \frac{(3 \cdot \cos 3x \cdot 3 + 7\sin 3x)}{-130}$$

$$= \frac{-[9\cos 3x + 7\sin 3x]}{130}$$

$G.S = C.F + P.I.$

$$G.S = c_1 e^{-x} + c_2 e^{-2x} + \frac{-[9\cos 3x + 7\sin 3x]}{130}$$

(2) solve $(D^2 - 3D + 2)y = \cos 3x$

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m = +1, +2$$

$$m = 1, 2$$

$$C.F = c_1 e^{x} + c_2 e^{2x}.$$

$b = 3$

$\phi(-b^2) = -9$

$$P.I = \frac{1}{F(D)} \cos bx$$

$$= \frac{1}{D^2 - 3D + 2} \cos 3x.$$

$$P.I = \frac{\cos 3x}{-9 - 3D + 2}$$

$$= \frac{\cos 3x}{-7 - 3D} \times \frac{-7 + 3D}{-7 + 3D}$$

$$= \frac{-7\cos 3x + 3D\cos 3x}{(-7)^2 - (3D)^2}$$

$$P.I = \frac{[3D\cos 3x - 7\cos 3x]}{-[9D^2 - 49]}$$

$$P.I = \frac{-[3\frac{d}{dx}\cos 3x - 7\cos 3x]}{9(-9) - 49}$$

$$P.I = \frac{-[3(-\sin 3x)3 - 7\cos 3x]}{-81-49}$$

$$P.I = \frac{-[-9\sin 3x - 7\cos 3x]}{-130}$$

$$P.I = \frac{+[9\sin 3x + 7\cos 3x]}{+130}$$

$$P.I = \frac{[9\sin 3x + 7\cos 3x]}{130}$$

$G.S = C.F + P.I.$

$$G.S = c_1 e^{x} + c_2 e^{2x} + \left[\frac{9\sin 3x + 7\cos 3x}{130}\right]$$

⇒ Solve $(D^2-4)y = 2\cos^2 x$

$(D^2-4)y = 1+\cos 2x$ $\quad$ [∵ $2\cos^2 x = 1+\cos 2x$]

A.E → $F(m) = 0$

$m^2 - 4 = 0$

$m = \pm 2$

$C.F = c_1 e^{2x} + c_2 e^{-2x}$

$$P.I = \frac{1+\cos 2x}{D^2-4}$$

$$= \frac{1}{D^2-4} + \frac{\cos 2x}{D^2-4}$$

$$= \frac{e^{0\cdot x}\cdot 1}{D^2-4} + \frac{\cos 2x}{D^2-4}$$

$$= \frac{e^{0\cdot x}\cdot 1}{0-4} + \frac{\cos 2x}{-4-4}$$

(∵ e^{ax} formula, $\cos bx$ formula, $b = 2$)

$$= -\frac{1}{4} - \frac{\cos 2x}{8}$$

$G.S = C.F + P.I$

$$G.S = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} - \frac{\cos 2x}{8}$$

Solve $(D^2-4D+3)y = \sin 3x \cos 2x$

multiply & divide by 2

$(D^2-4D+3)y = \frac{1}{2}(2\sin 3x\cos 2x)$

$\sin(a+b) + \sin(a-b)$

A·E $F(m)=0$

$m^2-4D+3=0$

$m = 1, 3$

$C\cdot F = c_1 e^{x} + c_2 e^{3x}$

$$P.I = \frac{1}{2}\left[\frac{\sin 5x + \sin 3x}{D^2-4D+3}\right]$$

$b=5$ $b=1$
$\phi(-b^2) = -25,$
$\phi(-b^2) = -1$

$$P.I = \frac{1}{2}\left[\frac{\sin 5x}{-25-4D+3} + \frac{\sin x}{-1-4D+3}\right]$$

$$= \frac{1}{2}\left[\frac{\sin 5x}{-22-4D} + \frac{\sin x}{-4D+2}\right]$$

$$= \frac{-1}{4}\left[\frac{\sin 5x}{2D+11} + \frac{\sin x}{2D-1}\right]$$

$$= \frac{-1}{4}\left[\frac{\sin 5x}{2D+11}\times\frac{2D-11}{2D-11} + \frac{\sin x}{2D-1}\times\frac{2D+1}{2D+1}\right]$$

$$= \frac{-1}{4}\left[\frac{2D\sin 5x - 11\sin 5x}{4D^2-121} + \frac{2D\sin x + \sin x}{4D^2-1}\right]$$

$$= -\frac{1}{4}\left[\frac{2\frac{d}{dx}\sin 5x - 11\sin 5x}{4(-25) - 121} + \frac{2\frac{d}{dx}\sin x + \sin x}{4(-1) - 1}\right]$$

$$= -\frac{1}{4}\left[\frac{2\cos 5x \cdot 5 - 11\sin 5x}{-221} + \frac{2\cos x + \sin x}{-5}\right]$$

$$= \frac{1}{4}\left[\frac{10\cos 5x - 11\sin 5x}{221} + \frac{2\cos x + \sin x}{5}\right]$$

$$G.S = C.F + P.I$$

$$= C_1 e^{x} + C_2 e^{3x} + \frac{1}{4}\left[\frac{10\cos 5x - 11\sin 5x}{221} + \frac{2\cos x + \sin x}{5}\right]$$

$(y'' + 4y' + 4)y = 4\cos x + 3\sin x, \quad y(0) = 0,$

$y'(0) = 0.$

Given equation is symbolic form

$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$

A.E. $\quad m^2 + 4m + 4 = 0$

$m^2 + 2m + 2m + 4 = 0$

$m = -2, -2$

C.F. $= (C_1 + C_2 x)\, e^{-2x}$

$$P.I. = \frac{4\cos x + 3\sin x}{D^2 + 4D + 4}$$

$$= \frac{4\cos x}{D^2+4D+4} + \frac{3\sin x}{D^2+4D+4}$$

$b = 1 \qquad\qquad b = 1$

$\phi(-b^2) = -1 \qquad \phi(-b^2) = -1$

$$= \frac{4\cos x}{-1+4D+4} + \frac{3\sin x}{-1+4D+4}$$

$$= \frac{4\cos x}{4D+3} + \frac{3\sin x}{4D+3}$$

By rati

$$= \frac{4\cos x}{4D+3} \times \frac{4D-3}{4D-3} + \frac{3\sin x}{4D+3} \times \frac{4D-3}{4D-3}$$

$$= \frac{16D\cos x - 12\cos x}{16D^2 - 9} + \frac{12D\sin x - 9\sin x}{16D^2 - 9}$$

$$= \frac{16D\cos x - 12\cos x}{16(-1) - 9} + \frac{12D\sin x - 9\sin x}{16(-1) - 9}$$

$$= \left[\frac{16\frac{d}{dx}\cos x - 12\cos x}{-25}\right] + \left[\frac{12\frac{d}{dx}\sin x - 9\sin x}{-25}\right]$$

$$= \frac{-[16\sin x + 12\cos x]}{-25} + \frac{12\cos x - 9\sin x}{-25}$$

$$= \frac{-16\sin x - \cancel{12\cos x} + \cancel{12\cos x} - 9\sin x}{-25}$$

$$= \frac{\cancel{-25}\sin x}{\cancel{-25}}$$

$P.I = \sin x$

$G.S = C.F + P.I$

$$y = (C_1 + C_2 x)e^{-2x} + \sin x \quad \text{—— (I)}$$

$$y(0) = (C_1 + C_2(0))e^{-2(0)} + \sin(0) = 0$$

$$y(0) = C_1 = 0$$

$$\boxed{C_1 = 0}$$

$$y(x) = (c_1 + c_2 x)\, e^{-2x} + \sin x$$

$$y'(x) = (c_1 + c_2 x)(-2)\, e^{-2x} + e^{-2x}(c_2) + \cos x$$

$$y'(0) = (c_1 + c_2 \cdot 0)(-2)\, e^{-2(0)} + e^{-2(0)}(c_2) + \cos(0) = 0$$

$$y'(0) = -2c_1 + c_2 + 1 = 0$$

$$-2c_1 + c_2 + 1 = 0$$

$$0 + c_2 + 1 = 0$$

$$\boxed{c_2 = -1}$$

Subtitube c_2 value in

Equation (I)

$$y = -x e^{-2x} + \sin x.$$

(7) $\Rightarrow (D^2+5D-6)\, y = \sin 4x \sin x$

A.E is $m^2+5m-6=0$

$(m+6)(m+1)=0$

$m=-6,1$

The roots are real & distinct

C.F is $y_c = c_1 e^{-6x} + c_2 e^{x}$.

$\cos(a-b) + \cos(a+b)$

P.I $y_p = \dfrac{\sin 4x \sin x}{D^2+5D-6} = \dfrac{1}{2} \cdot \dfrac{2\sin 4x \sin x}{D^2+5D-6}$

$= \dfrac{1}{2}\left[\dfrac{\cos 3x - \cos 5x}{D^2+5D-6}\right]$

$= \dfrac{1}{2}\left[\dfrac{\cos 3x}{D^2+5D-6} - \dfrac{\cos 5x}{D^2+5D-6}\right]$

$= \left(\begin{array}{ll} D^2=-9, & -b^2=-9 \\ D^2=-25, & -b^2=-25 \end{array}\right)$

$= \dfrac{1}{2}\left[\dfrac{\cos 3x}{-9+5D-6} - \dfrac{\cos 5x}{-25+5D-6}\right]$

$= \dfrac{1}{2}\left(\dfrac{\cos 3x}{5D-15} - \dfrac{\cos 5x}{5D-31}\right)$

$$= \frac{1}{2}\left[\frac{\cos 3x}{5D-15} \times \frac{5D+15}{5D+15} - \frac{\cos 5x}{5D-31} \times \frac{5D+31}{5D+31}\right]$$

$$= \frac{1}{2}\left[\frac{\cos 3x(5D+15)}{25D^2-225} - \frac{\cos 5x\,(5D+31)}{25D^2-961}\right]$$

$$= \frac{1}{2}\left[\frac{\cos 3x \cdot 5D + \cos 3x \cdot 15}{25D^2-225} - \frac{\cos 5x \cdot 5D + \cos 5x \cdot 31}{25D^2-961}\right]$$

$$= \frac{1}{2}\left[\frac{5 \cdot \frac{d}{dx}\cos 3x + 15\cos 3x}{25D^2-225} - \frac{5 \cdot \frac{d}{dx}\cos 5x + 31\cos 5x}{25D^2-961}\right]$$

$$= \frac{1}{2}\left[\frac{-15\sin 3x + 15\cos 3x}{-225-225} - \frac{(-25\sin 5x + 31\cos 5x)}{-625-961}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{-\cancel{15}\sin 3x + \cancel{15}\cos 3x}{-\cancel{450}\ 60}\right] - \frac{1}{2}\left[\frac{31\cos 5x - 25\sin 5x}{1586}\right]$$

$$= \frac{\sin 3x - \cos 3x}{120} + \frac{25\sin 5x - 31\cos 5x}{3172}$$

$G.S = C.F + P.I$

$$y = C_1 e^{-6x} + C_2 e^{x} + \frac{\sin 3x - \cos 3x}{120} + \frac{25\sin 5x - 31\cos 5x}{3172}$$

⇒ Method - 3

⇒ particular integral of $F(D)y = \phi(x)$ when $\phi(x) = x^k$, where 'k' is a positive integer

⇒ $\boxed{P.I = \frac{1}{F(D)} x^k}$

Problems ①

Formulas :-

① $\frac{1}{1-D} = (1-D)^{-1} = 1+D+D^2+D^3+\dots$

② $\frac{1}{1+D} = (1+D)^{-1} = 1-D+D^2-D^3+\dots$

③ $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1+2D+3D^2+4D^3+\dots$

④ $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1-2D+3D^2-4D^3+\dots$

⑤ $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1+3D+6D^2+10D^3+\dots$

⑥ $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1-3D+6D^2-10D^3$

problem (7)

Find the P.I of $(D^2+1)y = x$.

$A.E \quad f(m) = 0$

$m^2+1 = 0$

$m = \pm i$

$\alpha \pm \beta \quad \alpha = 0, \ \beta = \pm 1$

$C.f = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

$C.F = c_1 \cos x + c_2 \sin x$

$P.I = \frac{1}{F(D)} \cdot x^k$

$P.I = \frac{1}{D^2+1} \cdot x = \frac{1}{1+D^2} \cdot x$

$P.I = (1+D^2)^{-1} \cdot x$ here $D = D^2$.

$\boxed{[1+D]^{-1} = 1 - D + D^2 + D^3 \ldots}$

$P.I = [1 - D^2 + D^4 - D^6 + D^8 + \ldots]x$

($\because$ neglecting higher power of D)

$P.I = x - 0 \ldots$

$y = C.F + P.I$

$y = c_1\cos x + c_2 \sin x + x$

② $(D^2+1)y = x^2$

A.E $F(m) = 0$.

$m^2+1=0$

$m = \pm 1 \cdot i \qquad \beta = 1$

$C.F = c_1 \cos x + c_2 \sin x$

$P.I = \frac{1}{F(D)} \cdot x^k$

$P.I = \frac{1}{D^2+1} \cdot x^2 = \frac{1}{1+D^2} \cdot x^2 = (1+D^2)^{-1} x^2$

$(1+D^2)^{-1} = [1 - D^2 + (D^2)^2 - (D^2)^3 - \dots] x^2$

$= x^2 - 2$.

$= x^2 - 2$ (neglected higher powers)

$y = C.F + P.I$

$y = c_1 \cos x + c_2 \sin x + x^2 - 2$.

③ $(D^2+1)y = x^3$

A.E $F(m) = 0$

$m^2+1=0$

$m = \pm 1 \cdot i$

$\beta = 1, \quad \alpha = 0$

$C.F = c_1 \cos x + c_2 \sin x$.

$$P.I = \frac{1}{D^2+1} \cdot x^3 = \frac{1}{1+D^2} \cdot x^3 = (1+D^2)^{-1} x^3.$$

$$= \left[1 - D^2 + (D^2)^2 - (D^2)^3 \ldots \ldots\right] x^3$$

$$P.I = x^3 - 6x + \ldots \ldots$$

$$y = C.F + P.I$$

$$y = c_1 \cos x + c_2 \sin x + x^3 - 6x$$

(4) $(D^2+4)y = x^3$

A.E $\quad F(m) = 0$

$m^2+4 = 0$

$m = \pm 2i \qquad \beta = 2$

C.F $= c_1 \cos 2x + c_2 \sin 2x$

P.I $= \dfrac{1}{F(D)} \cdot x^3$

$= \dfrac{1}{(D^2+4)} \cdot x^3$

$= \dfrac{1}{4\left(\frac{D^2}{4}+1\right)} \cdot x^3$

$= \dfrac{1}{4}\left(\dfrac{D^2}{4}+1\right)^{-1} \cdot x^3$

$(1+D)^{-1} = 1 - D + D^2 - D^3 \ldots$

where

$D = \dfrac{D^2}{4}$

$= \dfrac{1}{4}\left[1 - \dfrac{D^2}{4} + \dfrac{D^4}{16} - \left(\dfrac{D^2}{4}\right)^3 + \ldots\right] x^3$

$= \dfrac{x^3}{4} - \dfrac{1}{4\times 4} 6x + \ldots$

$= \dfrac{1}{4}\left[x^3 - \dfrac{3x}{2}\right]$

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$y = C.F + P.I$

$y = c_1 \cos 2x + c_2 \sin 2x + \frac{x^3}{4} - \frac{3x}{2}$

5) $(D^2+D+1)y = x^3$

$A \cdot E \quad f(m) = 0$

$m^2+m+1 = 0$

$m^2+m+1 = 0$

$a=1, \; b=1, \; c=1$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$m = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$\beta = \frac{\sqrt{3}}{2}$

$\alpha = -\frac{1}{2}$

$$C.F = e^{-\frac{1}{2}x}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right]$$

$$C.F = e^{-x/2}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right]$$

$$P.I = \frac{1}{F(D)} x^k$$

$$P.I = \frac{1}{1+(D+D^2)} \cdot x^3$$

$$P.I = \left[1 - (D+D^2) + (D+D^2)^2 \cdots\cdots\right] x^3$$

$$P.I = x^3 - 3x^2 - \cancel{6x} + \cancel{6x} + 2 \cdot 6 + 0 \ldots$$

(higher powers negel

$$P.I. = x^3 - 3x^2 + 12$$

$$y = C.F + P.I$$

$$y = e^{-x/2}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right] + x^3 - 3x^2 + 12$$

(6) Solve $(D^3 + 2D^2 + D)\,y = x^3$.

(16)

$$A.E \quad f(m) = 0$$

$$m^3 + 2m^2 + m = 0$$

$$m(m^2 + 2m + 1) = 0$$

$$m(m+1)^2 = 0$$

$$m = 0, \quad (m+1)^2 = 0$$

$$m = -1, -1$$

$$C.F = C_1 e^{0 \cdot x} + (C_2 + C_3 x)e^{-x}.$$

$$P.I = \frac{1}{F(D)} x^k$$

$$P.I = \frac{1}{[D^3+2D^2+D]} \cdot x^3$$

$$P.I = \frac{1}{[D^2+2D+1]D} x^3 = \left(\frac{1}{(D+1)^2 D}\right) x^3$$

$$= \frac{1}{(D+1)^2} \cdot \frac{1}{D} \cdot x^3$$

$$= \frac{1}{(D+1)^2} \int x^3 \, dx$$

$$= (1+D)^{-2} \frac{x^4}{4}$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + 5D^4 \ldots$$

$$= \frac{1}{4}\left[1 - 2D + 3D^2 - 4D^3 + 5D^4 \ldots\right] x^4$$

$$= \frac{1}{4}\left[x^4 - 2(4x^3) + 36x^2 - 4(24)x + 120 \ldots\right]$$

$$P.I = \frac{1}{4}\left[x^4 - 8x^3 + 36x^2 - 96x + 120 \ldots\right]$$

$$y = C.F + P.I$$

$$y = c_1 e^{0x} + (c_2 + c_3 x) e^{-x} + \frac{1}{4}\left[x^4 - 8x^3 + 36x^2 - 96x + 120\right]$$

④ Method

P. I of $F(D)\,\phi(x) = e^{ax}V$ where "a" is a constant and "V" is a function of "x".

$$P.I = \frac{1}{F(D)}\cdot e^{ax}V = e^{ax}.V.\frac{1}{F(D+a)}$$

$$= e^{ax}.\frac{1}{F(D+a)}.V$$

Solve:-

① $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}(1+x)$

$(D^2-7D+6)y = e^{2x}(1+x)$

A.E $F(m)=0$.

$m^2-7D+6=0$

$m = 1, 6$

$C.F = c_1e^{x} + c_2e^{6x}$

$$P.I = \frac{1}{F(D)}\cdot e^{ax}.V = e^{ax}.\frac{1}{F(D+a)}.V$$

$$= \frac{1}{(D^2-7D+6)}.e^{2x}.(1+x)$$

$$P.I = \frac{e^{2x}(1+x)}{D^2-7D+6} = e^{2x}\frac{1}{(D+2)^2-7(D+2)+6}\cdot(1+x)$$

$$= e^{2x}\frac{1}{D^2+4+4D-7D-14+6}\cdot(1+x)$$

$$= e^{2x}\frac{1}{D^2-3D-4}(1+x)$$

$$= e^{2x}(1+x)\frac{1}{-4\left[1-\left(\frac{D^2+3D}{4}\right)\right]}$$

$$= \frac{e^{2x}}{-4}\left[1-\left(\frac{D^2+3D}{4}\right)\right]^{-1}(1+x)$$

"D" place in $\frac{D^2+3D}{4}$.

$$(1-D)^{-1} = 1+D+D^2+D^3+\cdots\cdots$$

$$= \frac{-e^{2x}}{4}\left[1+\frac{D^2+3D}{4}+\left(\frac{D^2+3D}{4}\right)^2+\left(\frac{D^2+3D}{4}\right)^3+\cdots\right](1+x)$$

$$= \frac{-e^{2x}}{4}\left[1+\frac{D^2+3D}{4}+\left(\frac{D^2+3D}{4}\right)^2+\left(\frac{D^2+3D}{4}\right)^3+\cdots\right](1+x)$$

$$= \frac{-e^{2x}}{4}\left[1+x+\frac{3}{4}(1)\cdots\right]$$

$$= -\frac{e^{2x}}{4}\left[1+x+\frac{3}{4}\right]$$

(4) $y = C.F + P.I$

$$y = c_1 e^{x} + c_2 e^{6x} - \frac{e^{2x}}{4}\left[1+x+\frac{3}{4}\right]$$

(2) $(D^4-1)y = e^{x}\cos x$

A.E $f(m)=0$.

$$D^4-1=0$$

$$m^4-1=0$$

$$(m^2)^2-(1)^2=0$$

$$(m^2+1)(m^2-1)=0$$

$$m=\pm i, \quad m=\pm 1$$

$$C.F = c_1 e^{x} + c_2 e^{-x} + c_1 \cos x + c_2 \sin x$$

$$P.I = \frac{1}{F(D)} \cdot e^{ax} \cdot V = e^{ax} \cdot \frac{1}{F(D+a)} \cdot V$$

$$P.I = \frac{1}{D^4-1} \cdot e^{x} \cdot \cos x$$

$$P.I = e^{x} \cdot \frac{1}{(D+1)^4-1} \cdot \cos x$$

$$P.I = e^{x} \cdot \frac{1}{(D^2+1+2D)^2-1} \cos x$$

$$P.I = e^{x} \frac{1}{(-1+1+2D)^2 - 1} (\cos x)$$

$$P.I = e^{x} \frac{1}{4D^2 - 1} \cos x$$

$$P.I. = e^{x} \frac{1}{4(-1) - 1} \cos x$$

$$P.I = e^{x} \frac{1}{-5} \cos x$$

$$P.I = \frac{-e^{x}}{5} \cos x$$

$$y = C.F + P.I$$

$$y = c_1 e^{x} + c_2 e^{-x} + c_1 \cos x + c_2 \sin x - \frac{e^{x}}{5} \cos x$$

(8) $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

$A.E \; f(m) = 0$

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$C.F = c_1 e^{x} + c_2 e^{2x}$$

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$$P.I = \frac{1}{F(D)} . e^{ax} . V = e^{ax} \frac{1}{F(D+a)} . V$$

$$P.I = \frac{1}{(D-1)(D-2)} x e^{3x} + \sin 2x$$

$$P.I = \frac{x e^{3x}}{(D^2+3D+2)} + \frac{\sin 2x}{D^2+3D+2}$$

$$= e^{3x} \frac{1}{(D^2+3D+2)} . x + \frac{\sin 2x}{D^2+3D+2}$$

$$= e^{3x} \frac{1}{(D+3)^2+3(D+3)+2} . x + \frac{\sin 2x}{-4-3D+2}$$

$$= e^{3x} \frac{1}{D^2+9+6D+3D+9+2} . x + \frac{\sin 2x}{-2-3D}$$

$$= e^{3x} \frac{1}{D^2+3D+2} . x - \left[\frac{\sin 2x}{3D+2} \times \frac{3D-2}{3D-2}\right]$$

$$= e^{3x} \frac{1}{2\left[1+\frac{D^2+3D}{2}\right]} . x - \left[\frac{3D \sin 2x - 2\sin 2x}{9D^2-4}\right]$$

$$= \frac{e^{3x}}{2}\left[1+\frac{D^2+3D}{2}\right]^{-1} . x - \left[\frac{3\frac{d}{dx}\sin 2x - 2\sin 2x}{-36-4}\right]$$

here $D = \frac{D^2+3D}{2}$

$$= \frac{e^{3x}}{2}\left[1 - \frac{D^2+3D}{2} + \cdots\right]x - \frac{3\cos x \cdot 2 + 2\sin 2x}{-40}$$

$$PI = \frac{e^{3x}}{2}\left[x - \frac{3}{2}\cdot 1\right] + \frac{6\cos x + 2\sin 2x}{4}$$

$$y = C.F + P.I$$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{3x}}{2}\left[x - \frac{3}{2}\right] + \frac{6\cos x + 2\sin 2x}{4}$$

Method - V

If $f(D) \cdot y = \phi(x)$; where $x^m \cdot v$ where m is a +ve integer and v is a function of x

Working Rules:

1. finding P.I of $\frac{1}{F(D)} x^m \sin ax$ (or) $x^m \cos ax$

1. $P.I = \frac{1}{F(D)} x^m \sin ax =$ I.P of $\frac{1}{F(D)} x^m e^{iax}$

(Imaginary part)

(or)

$=$ I.P of $\frac{1}{F(D)} x^m (\cos ax + i \sin ax)$

(Real part)

2. $P.I = \frac{1}{F(D)} x^m \cos ax =$ R.P of $\frac{1}{F(D)} x^m e^{iax}$

(or)

R.P of $\frac{1}{F(D)} x^m (\cos ax + i \sin ax)$

problem

1) solve $(D^2+4)y = x\sin x$

$m^2+4=0$

$m = \pm 2i$

$\alpha = 0, \beta = 2$

$C.F = c_1\cos 2x + c_2\sin 2x$

$P.I = \frac{1}{F(D)} x^m \sin ax = I.P \text{ of } \frac{1}{F(D)} x^m e^{iax}$

$P.I = \frac{1}{D^2+4} x^1 e^{i(1).x}$

I.P = Imaginary part.

$P.I.P = \frac{1}{D^2+4} . e^{ix} . x$

IV. method

$P.I.P = \frac{1}{D^2+4} e^{ix} . x = \frac{e^{ix}}{(D+i)^2+4} . x$

$= \frac{e^{ix}}{D^2+i^2+2iD+4} . x$

$= \frac{e^{ix}}{D^2+2iD+3} x \quad (i^2=-1)$

$= \frac{e^{ix}}{3\left(1+\frac{D^2+2Di}{3}\right)} . x$

$I.P \text{ of } \frac{e^{ix}}{3}\left(1+\frac{D^2+2Di}{3}\right)^{-1} . x.$

$(1+D)^{-1} = 1 - D + D^2 \cdots$

$$\text{I.P. of } \frac{e^{ix}}{3}\left(1 - \left(\frac{D^2+2iD}{3}\right) + \cdots\right)x$$

$$\frac{e^{ix}}{3}\left[x - 0 + \frac{2}{3}i(1) + \cdots\right]$$

$$\frac{e^{ix}}{3}\left[x + \frac{2i}{3} \cdots\right]$$

$$\text{I.P. } \frac{e^{ix}}{3}\left[x - \frac{2i}{3}\right]$$

$$\text{I.P. of } \frac{1}{3}\left[\cos x + i\sin x\right]\left(x - \frac{2i}{3}\right)$$

$$\text{I.P of } \frac{1}{3}\left[x\cos x - \frac{2}{3}i\cos x + ix\sin x + \frac{2}{3}\sin x\right]$$

$$\text{I.P of } \frac{1}{3}\left[x\cos x + \frac{2}{3}\sin x - i\left[\frac{2}{3}\cos x - x\sin x\right]\right]$$

$$\text{I.P.} = \frac{2}{9}\cos x - \frac{1}{3}x\sin x.$$

$y = C.F + P.I$

$$y = c_1\cos 2x + c_2\cos 2x + \frac{2}{9}\cos x - \frac{1}{3}x\sin x$$

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Solve $\frac{d^2y}{dx^2} - y = x\sin x + (1+x^2)e^x$

$(D^2-1)y = x\sin x + (1+x^2)e^x$

$m^2 - 1 = 0$

$m = \pm 1$

$C.F = c_1e^x + c_2e^{-x}$

$P.I = \frac{1}{F(D)} x\sin x + (1+x^2)e^x$

IP of $\frac{1}{(D^2-1)} \cdot x e^{ix} + e^x \cdot \frac{(1+x^2)}{D^2-1}$

IP of $\frac{e^{ix}\cdot x}{(D+i)^2-1} + \frac{e^x(1+x^2)}{(D+1)^2-1}$

I.P of $\frac{e^{ix}\cdot x}{D^2+2iD-1-1} + \frac{e^x(1+x^2)}{D^2+\cancel{1}+2D-\cancel{1}}$

I.P of $\frac{e^{ix}\cdot x}{D^2+2iD-2} + \frac{e^x(1+x^2)}{D^2+2D}$

$\frac{e^{ix}\cdot x}{-2\left(1-\left[\frac{D^2+2iD}{2}\right]\right)} + \frac{e^x(1+x^2)}{2D\left[1+\frac{D}{2}\right]}$

$\frac{e^{ix}\cdot x}{-2\left[1-\left[\frac{D^2+2iD}{2}\right]\right]} + \frac{e^x(1+x^2)}{D[2+D]}$

(3)

$$I.P \text{ of } \frac{e^{ix}\cdot x}{-2}\left[1-\left(\frac{D^2+2iD}{2}\right)\right]^{-1} + \frac{e^x}{(D+2)}\int (1+x)$$

$$I.P \text{ of } \frac{-e^{ix}}{2}\left(1-\left(\frac{D^2+2iD}{2}\right)\right]^{-1} x + \frac{e^x}{(D+2)}\cdot\left(x+\frac{x^2}{2}\right)$$

$$I.P \text{ of } \frac{-e^{ix}}{2}\left[1+\frac{D^2+2iD}{2}+\dots\right]x + \frac{e^x}{(D+2)}\left(x+\frac{x^2}{2}\right)$$

$$= \frac{-e^{ix}}{2}\left[x+0+i(1)+\dots\right] + \frac{e^x}{D+2}\left(x+\frac{x^2}{2}\right)$$

$$= \left(\frac{-1}{2}\right)e^{ix}\left[x+i(1)\right] + \frac{e^x}{D+2}\left(x+\frac{x^2}{2}\right)$$

$$= -\frac{1}{2}(\cos x+i\sin x)(x+i) + \frac{e^x}{2(1+D/2)}\left(x+\frac{x^2}{2}\right)$$

$$= -\frac{1}{2}(\cos x+i\sin x)(x+i) + \frac{e^x}{2}\left[1+D/2\right]^{-1}\left(x+\frac{x^2}{2}\right)$$

$$= -\frac{1}{2}\left[x\cos x+ix\sin x+i\cos x-\sin x\right] +$$

$$\frac{e^x}{2}\left[D-\frac{D}{2}+\left(\frac{D}{2}\right)^2+\dots\right]\left[x+\frac{x^2}{2}\right]$$

I.P

$$= -\frac{1}{2}\left[x\cos x-\sin x\right] + i(x\sin x+x\cos x) + \frac{e^x}{2}\left[x-\frac{1}{2}+\frac{x^2}{2}+\frac{x^2}{2}+\frac{1}{12}+\frac{1}{4}+0\right]$$

$$= -\frac{1}{2}\left[x\cos x-\sin x\right]+i(x\sin+x\cos x) + \frac{e^x}{2}\left[\frac{3x}{2}+\frac{1}{2}+\frac{x^2}{2}+\dots\right]$$

$y = C.F + P.I$

(5) $\left(\frac{d^2y}{dx^2}+9\right)y = x\cos 2x$

$(D^2+9)y = x\cos 2x$

$m^2+9=0$

$m=\pm 3i$

$\alpha=0,\ \beta=3$

$C.F = c_1\cos 3x + c_2\sin 3x$

$P.I.\ \frac{x\cos 2x}{D^2+9}.$

$R.P\ \text{of}\ \frac{e^{i2x}}{D^2+9}.x$

$R.P\ \text{of}\ \frac{e^{2ix}}{(D+2i)^2+9}.x$ (IV method)

$R.P\ \text{of}\ \frac{e^{2ix}}{D^2-4+4iD+9}.x.$

$R.P\ \text{of}\ \frac{e^{2ix}}{D^2+4iD+5}.x$

$R.P\ \text{of}\ \frac{e^{2ix}}{5\left[1+\frac{D^2+4iD}{5}\right]}.x$ $D=\frac{D^2+4iD}{5}$

$R.P\ \text{of}\ \frac{e^{2ix}}{5}\left[1+\frac{D^2+4iD}{5}\right]^{-1}.x$

R.P of $\frac{e^{2ix}}{5}\left[1-\left(\frac{D^2+4iD}{5}\right)\cdots\cdots\right]x$

R.P of $\frac{e^{2ix}}{5}\left[x-0-\frac{4}{5}i\right]$

R.P of $\frac{e^{2ix}}{5}\left[x-\frac{4}{5}i\right]$

R.P of $\frac{1}{5}[\cos 2x+i\sin 2x]\left[x-\frac{4i}{5}\right]$

R.P of $\frac{1}{5}\left(x\cos 2x-i\frac{4}{5}\cos 2x+xi\sin 2x+\frac{4}{5}\sin 2x\right)$

R.P of $\frac{1}{5}\left[x\cos 2x+\frac{4}{5}\sin 2x\right]$

$y = C.F + P.I$

$y = C_1\cos 3x + C_2\cos 3x + \frac{1}{5}\left[x\cos 2x+\frac{4}{5}\sin 2x\right]$

Variation of parameters :- (LAQ 10M)

working rule

⇒ To Solve $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ by the method of variation of parameters following steps

Steps:-

(1) Reduce the given equation to the standard form, if necessary

① Find the solution of $\frac{d^2y}{dx^2} + P\cdot\frac{dy}{dx} + Qy = 0$

and let the solution be complement function (C.F)

$$\boxed{C.F = y_c = c_1 u(x) + c_2 v(x)}$$

② Take particular integral (P.I)

$$\boxed{P.I = AU + BV}$$ where "A", "B" are

functions of small "x"s.

③ Find $$\boxed{\omega(u,v) = u\frac{dv}{dx} - v\cdot\frac{du}{dx}}$$

④ Find A & B using $$\boxed{A = -\int \frac{vR\,dx}{\omega(u,v)}\cdot dx}$$

$$\boxed{B = \int \frac{uR\,dx}{\omega(u,v)}}$$

⑤ write general solution of the given equation

as $$\boxed{y = y_c + y_p}$$

problems:-

① Apply the method of variation of parameters to solve $\left(\frac{d^2y}{dx^2} + y = \operatorname{cosec} x\right)$ — ①

Solution: given eqn of the form $(D^2+1)y = \operatorname{cosec} x$

Comparing with standard form of the

equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ——— (2)

(1) & (2) equations

$P = 0, Q = 1, R = \operatorname{cosec} x$

$f(m) = 0$

$m^2 + 1 = 0$

$m = \pm i$

$C.F = y_c = c_1 \cos x + c_2 \sin x$ ——— (3)

from (3) $u(x) = \cos x$

$v(x) = \sin x$

$w(u, v) = u\frac{dv}{dx} - v\frac{du}{dx}$

$= \cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)$

$= \cos x(\cos x) - \sin x(-\sin x)$

$= \cos^2 x + \sin^2 x = 1$

$w(u, v) = 1$

(1) → $A = -\int \frac{\sin x \cdot \operatorname{cosec} x}{1} dx$

$$B = \int \frac{\cos x - \operatorname{cosec} x}{1}\, dx$$

$$B = \int \cot x \, dx$$

$$B = \log|\sin x| + c.$$

$$y_p = P.I = (-x)\cos x + (\log|\sin x|)\sin x$$

$$y_p = -x\cos x + \sin x \log|\sin x|$$

$$y = y_c + y_p$$

$$y = c_1\cos x + c_2 \sin x - x\cos x + \sin x \log|\sin x|$$

(2) $(y''+1)y = \sec x$. (1)

$$(D^2+1)y = \sec x.$$

$$\frac{d^2y}{dx^2} + P\cdot\frac{dy}{dx} + Q\cdot y = R \quad (2)$$

(1) & (2)

$P = 0,\ Q = 1,\ R = \sec x$

A.E $m^2 + 1 = 0$

$m = \pm i$

$$C.F = y_c = c_1\cos x + c_2 \sin x \quad (3)$$

eqn ③

$y_c = c_1 u(x) + c_2 v(x)$

$u = \cos x, \quad v = \sin x$

$w(u,v) = u\frac{dv}{dx} - v\cdot\frac{du}{dx}$

$= \cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)$

$= \cos^2 x + \sin^2 x$

$w(u,v) = 1$

$A = -\int \frac{v \cdot R \, dx}{w(u,v)}$

$A = -\int \frac{\sin x \cdot \sec x}{\cancel{w(u,v)}\, 1} dx$

$A = -\log|\sec x| \, dx$,

$B = \int \frac{u \cdot R \cdot dx}{w(u,v)}$

$B = \int \frac{\cos x \cdot \sec x}{1} dx$

$B = x \, dx$

$y_p = Au + Bv$

$y_p = \left(-\log|\sec x|\right)\cos x + x \cdot (\sin x)$

$y_p = -\cos x \log|\sec x| + x\sin x$

$y = y_c + y_p$

$y = c_1\cos x + c_2\sin x - \cos x\log|\sec x| + x\sin x.$

Wronskian of functions.

Suppose ~~the two~~ functions. formulas

The wronskian of two functions

$y_1(x)$ & $y_2(x)$

$w.\ (y_1, y_2)$

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If $w(y_1, y_2) = 0$, y_1 & y_2 are linearly dependent otherwise, y_1 & y_2 are linearly independent.

problems

1. $w(e^x, e^{2x}) = |w(y_1, y_2)|$

let $y_1 = e^x$, $y_2 = e^{2x}$

$$w(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$$= \left[2e^{3x} - e^{3x}\right] = e^{3x} \neq 0$$

So, linearly independent.

$w(y_1, y_2) = w(\log x, \log x^n)$

$y_1 = \log x, \quad y_2 = \log x^n$

$$w(y_1, y_2) = \begin{vmatrix} \log x & \log x^n \\ \frac{1}{x} & \frac{1}{x^n}(nx^{n-1}) \end{vmatrix}$$

$$= \log x \left[\frac{1}{\cancel{x^n}} \cdot n\cancel{x^n} \cdot \frac{1}{x}\right] - \frac{1}{x} \log x^n$$

$$= \frac{n}{x} \log x - \frac{1}{x} \log x^n$$

$$= \cancel{\frac{n}{x} \log x} - \cancel{\frac{n}{x} \log x} = 0$$

So, linearly dependent.

Cauchy ~~Euler~~ Equation (or homogeneous linear equation)

$\Rightarrow$ An equation ~~of the form~~ $\dfrac{x^n d^n y}{dx^n} + P_1 \dfrac{x^{n-1} d^{n-1} y}{dx^{n-1}} + \cdots$

$$\cdots + P_n y = \phi(x)$$

where $P_1, P_2, \ldots, P_n$ are real constants and $\phi(x)$ is a function of "x" is called a homogeneous ~~linear~~ equation (or) euler equation of order "n".

Cauchy's linear equation

denote $\dfrac{d}{dx} = D$ & $\dfrac{d}{dz} = \theta$

$xD = \theta$

$x^2D^2 = \theta(\theta - 1)$

$x^3D^3 = \theta(\theta-1)(\theta-2)$

→ problems

① Solve

$$\left[x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 1\right]y = \log x$$

$$(x^2D^2 - xD + 1)\,y = \log x \quad \text{--- ①}$$

let $x = e^z \Rightarrow z = \log x$

$xD = \theta$

$x^2D^2 = \theta(\theta - 1)$

Eqn ①

$$[\theta(\theta-1) - \theta + 1]\,y = z$$

$$[\theta^2 - 2\theta + 1]\,y = z \quad \text{--- (I)}$$

$f(D)y = 0$

A.E $f(m) = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1.$

$$C.F = (C_1 + C_2 z)e^z$$

$$P.I = \frac{1}{\theta^2 - 2\theta + 1} \cdot z$$

$$P.I = \frac{z}{(\theta-1)^2} = \frac{z}{(1-\theta)^2}$$

$$P.I = + z(1-\theta)^{-2}$$

$$P.I = +[1+2\theta+3\theta^2+\cdots]z$$

$$P.I = +[z+2+0\cdots]$$

$$G.S = C.F + P.I$$

$$G.S = (c_1+c_2 z)e^z + [z+2]$$

$$= (c_1+c_2\log x)x + \log x + 2$$

3) solve $\left[x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2\right]y = 10(x+1/x)$

$$[x^3D^3 + 2x^2D^2 + 2]y = 10[x+1/x]$$

$$[\theta(\theta-1)(\theta-2) + 2(\theta)(\theta-1) + 2]y = 10[x+1/x] \quad \text{— (I)}$$

$x = e^z \Rightarrow z = \log x$

sub in (I)

$$[(\theta^2-3\theta+2)\theta + 2\theta^2-2\theta+2]y = 10(e^z+e^{-z})$$

$$[\theta^3 - 3\theta^2 \cancel{+2\theta} + 2\theta^2 \cancel{-2\theta} + 2]y = 10e^z + 10e^{-z}$$

$$(\theta^3-\theta^2+2)y = 10e^z + 10e^{-z}$$

AE $F(m) = 0.$

$m^3 - m^2 + 2 = 0$

$$\begin{array}{r|cccc} -1 & 1 & -1 & 0 & 2 \\ & 0 & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$(m+1)\ (m^2 - 2m + 2) = 0$

$(m+1)\ (m^2 - 2m + 2) = 0$

$(m+1) \left[\dfrac{2 \pm \sqrt{4 - 4(1) \times (1)}}{2 \times 1} \right] = 0$

$(m+1) \left(\dfrac{2 \pm \sqrt{-4}}{2} \right) = 0$

$(m+1) \left(\dfrac{2 \pm 2i}{2} \right) = 0$

$(m+1)\ (1 \pm i) = 0$

$m = -1;\ \alpha = 1,\ \beta = 1$

$C.F = c_1 e^{-z} + e^{z} [c_2 \cos z + c_3 \sin z]$

$P.I = \dfrac{10e^{z}}{\theta^3 - \theta^2 + 2} = \dfrac{10e^{z}}{1 - 1 + 2} = \dfrac{10e^{z}}{2}$

$P.I = \frac{10e^{-z}}{\theta^3 - \theta^2 + 2} = \frac{10e^{-z} \cdot z}{\cancel{-1-1+2}\; 3\theta^2 - 2\theta} = \frac{10e^{-z} \cdot z}{5}$

$P.I = \frac{10e^{z}}{\theta^3 - \theta^2 + 2} + \frac{10e^{-z}}{\theta^3 - \theta^2 + 2}$

$P.I = 5e^{z} + 2ze^{-z}$

$G.S = C.F + P.I$

$= \cancel{c_1 e^{-z} + e^{z}[c_2 \cos z}$

$= c_1 e^{-z} + e^{z}\left[c_2 \cos z + c_3 \sin z\right] + 5e^{z} + 2ze^{-z}$

$= c_1 e^{-\log x} + e^{\log x}\left[c_2 \cos \log x + c_3 \sin \log x\right]$

$+ 5x + 2\left(\frac{1}{x}\right)\log x$

(3) $\left(x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2\right)y = x \log x.$

$\left[x^2D^2 - xD + 2\right]y = x \log x$ $\quad x = e^z, \; z = \log x$

$\left[\theta(\theta-1) - \theta + 2\right]y = x \log x$

$x = e^z \Rightarrow z = \log x$ $\quad \boxed{e^{\log_e x} = x}$

$\left[\theta^2 - \theta - \theta + 2\right]y = \left[\cancel{x \log x}\right] = e^z \cdot z$

$(\theta^2 - 2\theta + 2)y = \left[\cancel{x \log x}\right] e^z \cdot z$

A.E $m^2-2m+2=0$

$$m=\frac{2\pm\sqrt{4-4(1)(2)}}{2(1)}$$

$$m=\frac{2\pm\sqrt{4-8}}{2}$$

$$m=\frac{2\pm i2}{2}=1\pm i$$

$\alpha=1,\ \beta=1$

$$C.F=e^{z}\left[c_1\cos z+c_2\sin z\right]$$

$$P.I=\frac{e^{z}\cdot z}{\theta^2-2\theta+2}$$

$$=\frac{e^{z}\cdot z}{(\theta+1)^2-2(\theta+1)+2}$$

$$=\frac{e^{z}\cdot z}{\theta^2+1}$$

$$=e^{z}\cdot\left[1+\theta^2\right]^{-1}\cdot z$$

$$=e^{z}\cdot\left[1-\theta^2+(\theta^2)^2-\cdots\right]z$$

$$x=e^{\log_e x}$$

$$=e^{z}\left[z-0+\right]$$

$$=e^{z}z$$

$G.S = C.F + P.I$

$y = y_c + y_p$

$y = x\left[c_1 \cos \log x + c_2 \sin \log x\right] + x \cdot \log x$

Legendre's equations.

An equation of the form $\left[(a+bx)^n \frac{d^n y}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots P_n\right] y = \phi(x)$

where $P_1, P_2, P_3 \dots, P_n$ are real constants and $\phi(x)$ is a function of "x" is called legendre's equation.

$\Rightarrow$ This can be solved by the substitution

$a+bx = e^z$ (or) $z = \log(a+bx)$ and

$\theta = \frac{d}{dz}$ then $(a+bx)D = b\theta$.

$(a+bx)^2 D^2 = b^2\theta(\theta-1)$

$(a+bx)^3 D^3 = b^3\theta(\theta-1)(\theta-2)\dots$

problems

① Solve $\left[(x+1)^2\frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4\right]y = x^2+x+1$ — ①

$u = e^z$

let $x+1 = u \Rightarrow x = u-1$

$\frac{dx}{du} = 1.$

$e^z = x+1$

$z = \log(a+b$

$\left[(x+1)^2D^2 - 3(x+1)D + 4\right]y = x^2+x+1$

$\left[u^2D^2 - 3uD + 4\right]y = (u-1)^2 + \cancel{u} \cancel{-1} \cancel{+1}$

$$\left[u^2\frac{d^2y}{dx^2} - 3u\frac{dy}{du} + 4\right]y = u^2 + 1 - u \quad \text{②}$$

$$\left[\theta(\theta-1) - 3\theta + 4\right]y = u^2 - u + 1$$

$$u^2\frac{d^2y}{du^2} \Rightarrow (a+bx)D$$

$$(x+1)^2\left(\frac{d^2y}{du^2}\right) = (a+bx)^2D^2 = \theta(\theta-1)$$

$$\boxed{(a+bx) = e^z}$$

$$\left[\theta^2 - 4\theta + 4\right]y = e^{2z} - e^z + 1$$

$$f(m) = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$C.F = (c_1 + c_2 z)e^{+2z}.$$

$$P.I = \frac{e^{2z} - e^z + 1}{\theta^2 - 4\theta + 4}$$

$$= \frac{e^{2z}}{\theta^2 - 4\theta + 4} - \frac{e^z}{\theta^2 - 4\theta + 4} + \frac{e^{0.z}\,1}{\theta^2 - 4\theta + 4}$$

$$= \frac{z^2 e^{2z}}{2} - \frac{e^z}{1-4+4} + \frac{1}{0-0+4}$$

$$= \frac{z^2 e^{2z}}{2} - e^z + \frac{1}{4} \quad \text{P.A.}$$

$$G.S = \left[c_1 + c_2\log(x+1)\right]e^{2\log(x+1)}$$

$$+ \frac{\log(x+1)^2 \cdot e^{2\log(x+1)}}{} - (x+1) + \frac{1}{4}$$

(2) $\left[(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2\right] y = x$ — (1)

let $2x - 1 = u = e^z$ $\qquad (2x-1) = u$

$x = \frac{u+1}{2}$ $\qquad 2 = \frac{du}{dx}$

$\frac{dx}{du} =$ $\qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\boxed{\frac{dy}{dx} = 2 \cdot \frac{dy}{du}}$

$\left[2^3 u^3 \frac{d^3y}{du^3} + 2u \frac{dy}{du} - 2\right] y = \frac{u+1}{2}$

$\left[8u^3 D^3 + 2uD - 2\right] y = \frac{u+1}{2}$ $\qquad \left(\begin{array}{l} u = e^z \\ z = \log u \end{array}\right.$

$\left[8\left[\theta(\theta-1)(\theta-2)\right] + 2(\theta) - 2\right] y = \frac{e^z + 1}{2}$

$\left[8(\theta^3 - 3\theta^2 + 2\theta) + 2\theta - 2\right] y = \frac{e^z + 1}{2}$

$\left[8\theta^3 - 24\theta^2 + 16\theta + 2\theta - 2\right] y = \frac{e^z + 1}{2}$

$\left[8\theta^3 - 24\theta^2 + 18\theta - 2\right] y = \frac{e^z + 1}{2}$

A.E $\quad f(m) = 0$

$8m^3 - 24m^2 + 18m - 2 = 0$

$$\begin{array}{c|cccc} 1 & 8 & -24 & 18 & -2 \\ & 0 & 8 & -16 & 2 \\ \hline & 8 & -16 & 2 & \boxed{0} \end{array}$$

$$(m-1)(4m^2 - 8m + 1) = 0$$

$$(m-1)\left[\frac{8 \pm \sqrt{64 - 4\times 4\times 1}}{2\times 4}\right] = 0$$

$$\begin{array}{r|l} 2 & 48 \\ 2 & 24 \\ 2 & 12 \\ 2 & 6 \\ & 3 \end{array}$$

$$= 4\sqrt{3}.$$

$$(m-1)\left[\frac{8 \pm 4\sqrt{3}}{8}\right] = 0.$$

$$(m-1)\left[\frac{2 \pm \sqrt{3}}{2}\right] = 0.$$

$$m = 1, \quad m = \frac{2 \pm \sqrt{3}}{2}$$

$$\alpha = 1, \quad \beta = \frac{\sqrt{3}}{2}.$$

$$C.F = c_1 e^{z} + e^{z}\left(c_2 \cos \frac{\sqrt{3}}{2} z + c_3 \sin \frac{\sqrt{3}}{2} z\right)$$

$$C.F = c_1 e^{z} + e^{z}\left[c_2 \cosh \frac{\sqrt{3}}{2} z + c_3 \sinh \frac{\sqrt{3}}{2} z\right]$$

$$P.I = \frac{e^{z} + e^{0\cdot x}\cdot 1}{2(8\theta^3 - 24\theta^2 + 18\theta - 2)} + \frac{e^{0\cdot x}\cdot 1}{}$$

$$= \frac{e^{z}\cdot z}{2[24\theta^2 - 48\theta + 18]} + \frac{e^{0\cdot z}\cdot 1}{\cancel{2[24\theta^2 - 48\theta + 18]}\ \ 2[8\theta^3 - 24\theta^2 + 18\theta - 2]}$$

$$= \frac{z e^{z}}{2[24\theta^2 - 48\theta + 18]} + \frac{e^{0z} \cdot 1}{2[8\theta^3 - 24\theta^2 + 18 - 2]}$$

$$= \frac{z \cdot e^{z}}{2(-6)} + \frac{e^{0z} \cdot 1}{2(-2)}$$

$$= \frac{z \cdot e^{z}}{-12} + \frac{1}{-4}$$

$$= \frac{\log[2x-1] \cdot e^{\log(2x-1)}}{-12} + \frac{1}{-4}$$

$$G.S = c_1 e^{z} + e^{z}\left[c_2 \cosh \frac{\sqrt{3}}{2} z + c_3 \sinh \frac{\sqrt{3}}{2} z\right] + \frac{\log(2x-1)(2x-1)}{-12} + \left(-\frac{1}{4}\right)$$

$$G.S = c_1 (2x-1) + (2x-1)\left[c_2 \cosh \frac{\sqrt{3}}{2} \log(2x-1) + c_3 \sinh \frac{\sqrt{3}}{2} \log(2x-1)\right]$$

$$- \frac{\log(2x-1)(2x-1)}{12} - \frac{1}{4}.$$

UNIT - III

MULTI VARIABLE CALCULUS (Integration)

double Integrals

→ consider a region "R" in the xy plane bonded by one (or) more curves.

→ let $f(x,y)$ be a function defined at all points of "R".

→ let the region "R" be divided into small subregions $\delta R_1, \delta R_2 \ldots \delta R_n$ each of area which are pair wise non-overlapping &

→ let (x_1, y_1) be an orbitary point with the sub region (δR_i)

→ consider the sum $f(x_1, y_1)\delta R_1 + f(x_2, y_2)\delta R_2 + f(x_3, y_3)\delta R_3 + \ldots f(x_n, y_n)\delta R_n$

→ If this sum tends to a finite units as n tends to Infinity such that maximum (δR_i) tends ($\rightarrow$) to 0 irrespective the choice of (x_i, y_i), the limit is called the double integral of $f(x,y)$ over the region R and it is denoted by the symbol.

$$\iint_R f(x,y)\,dR \quad (\text{or}) \quad \iint_R f(x,y)\,dx\,dy$$

Problems

(1) Evaluate $\int_{y=0}^{2}\int_{x=0}^{3} xy \, dx \, dy$

$$\int_{y=0}^{2}\left(\int_{x=0}^{3} xy \, dx\right) dy$$

$$\int_{y=0}^{2}\left(\left[y.\frac{x^2}{2}\right]_0^3\right) dy$$

$$\int_{y=0}^{2}\left[y.\frac{3^2}{2}-\frac{0^2}{2}\right] dy$$

$$\int_{y=0}^{2}\frac{9}{2}y \, dy$$

$$\frac{9}{2}\int_{y=0}^{2} y \, dy$$

$$\frac{9}{2}\left[\frac{y^2}{2}\right]_0^2$$

$$\frac{9}{\cancel{2}}\left[\frac{\cancel{4}-0}{\cancel{2}}\right] = 9$$

$$\int_0^2\int_0^x y\,dy\,dx$$

$$\int_0^2\left[\int_0^x y\,dy\right]dx$$

$$\int_0^2\left[\frac{y^2}{2}\right]_0^x dx$$

$$\int_0^2 \frac{x^2}{2}\,dx$$

$$\frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left[\frac{8}{3}\right] = 4/3$$

***
(3) evalute $\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dx\,dy$

$$\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dy\,dx.$$

$$\int_0^1\left[x^2 \cdot y+\frac{y^3}{3}\right]_x^{\sqrt{x}} dx$$

$$\int_0^1\left(x^2\cdot\sqrt{x}+\frac{(\sqrt{x})^3}{3}-x^2\cdot x+\frac{x^3}{3}\right)dx.$$

$$\int_0^1 x^{5/2}\,dx+\frac{1}{3}\int_0^1 x^{3/2}\,dx-\frac{4}{3}\int_0^1 -x^3\,dx.$$

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$$\left[\frac{x^{5/2+1}}{\frac{5}{2}+1}\right]_0^1 + \frac{1}{3}\left[\frac{x^{3/2+1}}{\frac{3}{2}+1}\right]_0^1 - \frac{4}{3}\left(\frac{x^4}{4}\right)_0^1$$

$$2 \cdot \frac{1^{7/2}}{7} + \frac{1}{3}\,2\,\frac{(1)^{5/2}}{5} - 4\left(\frac{1}{3}\right)$$

$$\frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$0.28 + 0.13 - 0.33 = 0.71 = 0.71$$

(4)

$$\Rightarrow \int_0^3 \int_1^2 (xy + x^2y + xy^2)\,dy\,dx$$

$$\int_0^3 \left[\int_1^2 (xy + x^2y + xy^2)\,dy\right] dx$$

$$\int_0^3 \left[x\frac{y^2}{2} + x^2y + x\frac{y^3}{3}\right]_1^2 dx$$

$$\int_0^3 \left[2x + 2x^2 + \frac{8}{3}x\right]dx - \left[\frac{x}{2} + \frac{x^2}{2} + \frac{x}{3}\right]dx$$

$$\int_0^3 \left(\frac{14x}{3} + 2x^2\right)dx - \left(\frac{5x}{6} + \frac{x^2}{2}\right)dx$$

$$\int_0^3 \left[\frac{23}{6}x + \frac{3}{2}x^2\right] dx = \left[\frac{23}{6}\cdot\frac{x^2}{2} + \frac{3}{2}\cdot\frac{x^3}{2}\right]_0^3$$

$$= \left[\frac{23}{12}x^2 + \frac{3}{4}x^3\right]_0^3$$

$$= \left[\frac{23}{12_4}(9) + \frac{3}{4}(27) - 0 - 0\right]$$

$$= \frac{69}{4} + \frac{74}{4} = \frac{123}{4}.$$

$$\begin{array}{r} 69 \\ 74 \\ \hline 123 \end{array}$$

$$\int_0^3\int_1^2 (xy + x^2y + xy^2)\, dy\, dx.$$

5) $$\int_0^2\int_0^x \left[e^{x+y}\right] dy\, dx.$$

$$\int_0^2 \left[\int_0^x (e^{x+y})\,dy\right] dx.$$

$$\int_0^2 \left[\int_0^x e^x e^y \cdot dy\right] dx.$$

$$\int_0^2 \left[e^x \left[e^y\right]_0^x\right] dx.$$

$$\int_0^2 e^x(e^x - e^0)\, dx$$

$e^0 = 1$

$$\int_0^2 \left[e^{2x} - e^x\right] dx.$$

$$\left[\frac{e^{2x}}{2} - e^x\right]_0^2$$

$$\frac{e^4}{2} - e^2 - \frac{e^0}{2} + e^0$$

$$\frac{e^4}{2} - e^2 + \frac{1}{2} + 1$$

$$\frac{e^4}{2} - e^2 + \frac{3}{2}$$

$$\frac{e^4}{2} - e^2 + \frac{1}{2}$$

$$\frac{1}{2}\left[e^4 - 2e^2 + 1\right]$$

$$\frac{1}{2}\left(e^2 - 1\right)^2.$$

(6) $$\int_0^{\pi/2}\int_{-1}^{1} x^2y^2\,dy\,dx$$

$$\int_0^{\pi/2}\left(\int_{-1}^{1} x^2y^2\,dy\right)dx$$

$$\int_0^{\pi/2}\left[x^2\frac{y^3}{3}\right]_{-1}^{1} dx$$

$$\int_0^{\pi/2} x^2\cdot\left(\frac{1}{3}+\frac{1}{3}\right)dx$$

$$\frac{2}{3}\int_0^{\pi/2} x^2\,dx$$

$$\frac{2}{3}\left(\frac{x^3}{3}\right)_0^{\pi/2} = \frac{2}{9}\left(\frac{\pi/8}{4}\right) = \frac{\pi}{96}$$

$$\int_0^4\int_0^{x^2} e^{y/x}\,dy\,dx$$

$$\int_0^4\left[\int_0^{x^2} e^{y/x}\,dy\right]dx$$

$$\int_0^4\left[\frac{e^{y/x}}{1/x}\right]_0^{x^2}dx$$

$$\int_0^4\left(xe^{x^2/x} - xe^{0/x}\right)dx$$

One (1)

$$\int_0^4\left(xe^{x} - x\right]dx$$

$$\left[xe^{x} - \frac{x^2}{2}e^{x} + \frac{x^2}{2}\right]_0^4$$

$$4e^4 - \frac{16}{2}e^4 - \frac{16}{2} - x(e^0) - 0$$

$$4e^4 - 8e^4 - 8 - \cancel{x} - 0$$

$$4e^4$$

$$\left[e^{x}(x-1) - \frac{x^2}{2}\right]_0^4$$

$$e^4(4-1) - \frac{16}{2} - e^0(0-1) + 0$$

$$3e^4 - 8 + 1$$

$$3e^4 - 7$$

(8) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$$\int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2} \right] dx$$

$\sqrt{1+x^2} = p$

$1+x^2 = p^2$

$$\int_0^1 \left[\int_0^p \frac{dy}{p^2+y^2} \right] dx$$

$$\int_0^1 \left[\frac{1}{p} \tan^{-1}(y/p) \right]_0^p dx$$

$$\int_0^1 \left[\frac{1}{p} \left[\tan^{-1}(p/p) - \tan^{-1} 0 \right] \right] dx$$

$$\int_0^1 \frac{1}{p} \left[\pi/4 - 0 \right] dx$$

$$\pi/4 \int_0^1 \frac{1}{p}\, dx$$

$$\pi/4 \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx \qquad (\because p = \sqrt{1+x^2})$$

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$$\frac{\pi}{4}\left[\log(x+\sqrt{x^2+1})\right]_0^1$$

$$\cancel{\frac{\pi}{4}\left[\log 1 + \sqrt{1+1} - \log 0 - \sqrt{0+1}\right]}$$

$$\cancel{\frac{\pi}{4}\left[0+\sqrt{2}-1-1\right]} \quad \frac{\pi}{4}\log\left[1+\sqrt{1+1}\right]$$

$$\cancel{\frac{\pi}{4}\left[\sqrt{2}-2\right]} \quad \frac{\pi}{4}\log\left[1+\sqrt{2}\right]$$

(9) $\iint x^2\,dx\,dy$ over the region bounded by

$xy=4, \quad y=0, \quad x=1, \quad x=4$

$y = 4/x$

$$\int_1^4\int_0^{4/x} x^2\,dy\,dx$$

I. w.r to y

$$\int_1^4 (x^2 \cdot y)_0^{4/x}\,dx$$

$$\int_1^4 \left(x^2\cdot\frac{4}{x} - 0\right)dx$$

$$\int_1^4 4x\,dx$$

$$4\left[\frac{x^2}{2}\right]_1^4 \Rightarrow 4\left[\frac{16}{2}-\frac{1}{2}\right] = 30$$

(10) $$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$$

$$\int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2} dx\, dy$$

$$\int_0^\infty e^{-y^2} \left[\int_0^\infty e^{-x^2} dx\right] dy$$

$$\boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$$

$$\int_0^\infty e^{-y^2} \frac{\sqrt{\pi}}{2} \cdot dy$$

$$\frac{\sqrt{\pi}}{2} \int_0^\infty e^{-y^2} dy \qquad \boxed{\int_0^\infty e^{-y^2} dx = \frac{\sqrt{\pi}}{2}}$$

$$\frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

** Evaluate: (i) $\iint_R y\, dx\, dy$

(ii) $\iint_R y^2\, dx\, dy$ where "R" is the region bounded by parabolas $y^2 = 4x$, $x^2 = 4y$

$y^2 = 4x$ — (1) | $x^2 = 4y$ — (2)

$y = \frac{x^2}{4}$

$\frac{x^4}{16} = 4x$.

$x^4 - 64x = 0$

$x(x^3 - 4^3) = 0$

$x = 0,\ x = 4$

x limits 0 to 4.

y limits $\frac{x^2}{4}$ to $2\sqrt{x}$.

I. w.r. to y.

$$\int_0^4 \int_{x^2/4}^{2\sqrt{x}} y \, dx \, dy$$

$$\int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$$

$$\frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left[4 \cdot \frac{x^2}{2} - \frac{x^5}{16 \times 5}\right]_0^4$$

$$\frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4$$

$$\frac{1}{2}\left[32 - \frac{64}{5}\right]$$

$$\frac{1}{2}\left[\ \right] = \frac{48}{5}$$

(ii) $\iint_R y^2 \cdot dx\,dy$

x limits 0 to 4

y limits $\frac{x^2}{4}$ to $2\sqrt{x}$

$$\int_0^4 \left[\int_{x^2/4}^{2\sqrt{x}} y^2 \cdot dy\right] dx$$

$$\int_0^4 \left[\frac{y^3}{3}\right]_{x^2/4}^{2\sqrt{x}} dx$$

$$\frac{1}{3}\int_0^4 \left(8x^{3/2} - \frac{x^6}{64}\right) dx$$

$$\frac{1}{3}\left[\frac{8x^{5/2}}{5/2} - \frac{x^7}{64 \times 7}\right]_0^4$$

$$\frac{1}{3}\left[\frac{8(4)^{5/2}}{5/2} - \frac{4^7}{7 \times 64} - 0\right]$$

$$\frac{1}{3}\left[\frac{8 \times 2 \times 2^5}{5} - \frac{1}{64 \times 3}\left[\frac{4^7}{7}\right]\right]$$

$$\frac{8 \times 64}{15} - \left(\frac{4^4}{21}\right) \quad 21.94$$

5M $\iint xy\,dx\,dy$ taken over the positive quadrant — ①

of the ellipse equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ — ②

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(\frac{a^2-x^2}{a^2}\right)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

Eqn ②

let $y=0$.

$$\frac{x^2}{a^2}=1$$

$$x^2=a^2$$

$$x=\pm a.$$

($\because$ positive quadrant)

x limits 0 to a

y limits 0 to $\frac{b}{a}\sqrt{a^2-x^2}$

$$\int_0^a\int_0^{b/a\sqrt{a^2-x^2}} xy\,dx\,dy$$

I. w.r.to y.

$$\int_0^a \left[\int_0^{b/a\sqrt{a^2-x^2}} xy\,dy\right] dx$$

$$\int_0^a x\left[\frac{y^2}{2}\right]_0^{b/a\sqrt{a^2-x^2}} dx$$

$$\int_0^a x\left(\frac{b^2/a^2\,(a^2-x^2)}{2} - 0\right) dx.$$

$$\frac{b^2}{2a^2}\int_0^a (a^2x - x^3)\,dx$$

I. w. to. x

$$\frac{b^2}{2a^2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$\frac{b^2}{2a^2}\left[a^2\cdot\frac{a^2}{2} - \frac{a^4}{4} - 0\right]$$

$$\frac{b^2}{2a^2}\left[\frac{2a^4 - a^4}{4}\right] \Rightarrow \frac{b^2}{8a^2}\left[a^4\right] = \frac{a^2b^2}{8}$$

evalute double integration $\int_0^{\pi}\int_0^{a\sin\theta} r \cdot dr \cdot d\theta$.

$$\int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$\frac{1}{2}\int_0^{\pi} (a^2\sin^2\theta - 0.)\, d\theta$$

$$\frac{a^2}{2}\int_0^{\pi} \sin^2\theta\, d\theta.$$

$$\frac{a^2}{2}\int_0^{\pi} \sin^2\theta\, d\theta \implies \frac{a^2}{2}\int_0^{\pi}\left[\frac{1-\cos 2\theta}{2}\right] d\theta.$$

$$\frac{a^2}{4}\int_0^{\pi}(1-\cos 2\theta)\, d\theta = \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[\pi - 0 - \frac{\sin 2\pi}{2} + \frac{\sin 0}{2}\right]$$

$$= \frac{a^2}{4}\left[\pi - 0 - 0 + 0\right]$$

$$= \frac{\pi a^2}{4}$$

$\Rightarrow$ Evaluate double Integration $\int_0^{\infty}\int_0^{\pi/2} e^{-r^2} r\,d\theta\,dr$

$$\int_0^{\infty}\left[\int_0^{\pi/2} 1 \cdot d\theta\right] e^{-r^2} r\,dr$$

I. w.r to θ

$$\int_0^{\infty} (\theta)_0^{\pi/2} e^{-r^2} r \cdot dr$$

$$\int_0^{\infty} \left[\frac{\pi}{2} - 0\right] e^{-r^2} \cdot r\,dr$$

$$-\frac{\pi}{2\times 2}\int_0^{\infty} e^{-r^2}(-2r)\,dr$$

$$\boxed{\int e^{f(x)} f'(x)\,dx = e^{f(x)}}$$

$$-\frac{\pi}{4}\left[e^{-r^2}\right]_0^{\infty}$$

$$-\frac{\pi}{4}\left[e^{-\infty^2} - e^{0}\right] = -\frac{\pi}{4}[0-1] = \frac{\pi}{4}$$

$\Rightarrow$ Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$

$$\int_0^{\pi/2}\left[\int_0^{\infty} \frac{r\,dr}{(r^2+a^2)^2}\right] d\theta$$

I. w.r. to r.

$$\frac{1}{2}\int_0^{\pi/2}\left[\int_0^{\infty}\frac{2r\cdot dr}{(r^2+a^2)^2}\right]d\theta \Rightarrow \boxed{\int \frac{f'(x)}{(f(x))^2}\,dx = \frac{-1}{f(x)}}$$

$$\frac{1}{2}\int_0^{\pi/2}\left[\frac{-1}{r^2+a^2}\right]_0^{\infty} d\theta$$

$$\frac{1}{2}\int_0^{\pi/2}\left[\frac{-1}{\infty^2+a^2} + \frac{1}{0^2+a^2}\right]$$

$$\frac{1}{2}\int_0^{\pi/2}\left(0+\frac{1}{a^2}\right)d\theta$$

$$\frac{1}{2a^2}\int_0^{\pi/2} 1\cdot d\theta$$

$$\frac{1}{2a^2}\left[\theta\right]_0^{\pi/2} \Rightarrow \frac{1}{2a}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{4a^2}.$$

⇒ double Integral by change of order of Integration.

Working rule

⇒ The change of order is nothing but change of limits of the integration.

⇒ To change the order of integration for "x" from a to b, "y" from $f_1(x)$ to $f_2(x)$

i.e $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dx\, dy$

⇒ To draw region of integration by drawing to curves $y = f_1(x)$, $y = f_2(x)$ and $x = a$, $x = b$

⇒ If this curves and lines intersect then draw a line parallel to x-axis to get various sub regions.

⇒ In each of these sub regions, draw eliminatory strips parallel to x-axis

⇒ Obtain the limits for x in term of y and then limits of y as constants then the equation $y = a$ to b, x equal to $f_1(y)$, $f_2(y)$

$$\int_{y=a}^{b} \int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$$

⇒ If region of integration consist of a vertical strip and along x-axis then the change of order is a horizontal strip on y-axis problems:-

1) BY changing of order of integration. evaluate $\int_{0}^{4a}\int_{x^2/4a}^{2\sqrt{ax}} 1 \cdot dy \cdot dx$.

Given limits $x = 0$ to $4a$

& $y = x^2/4a$ to $2\sqrt{ax}$

curve:- $\boxed{x=0}$, $\boxed{x=4a}$

$y = x^2/4a$, $y = 2\sqrt{ax}$

$\boxed{x^2 = 4ay}$, $\boxed{y^2 = 4ax}$

1):- Here the limits of "x" are fixed and limits of "y" are varied.

2) To Change the order of integration fix "y". and we change the elements of "x" in terms of "y".

i.e, $y = 0$ to $4a$

$x = y^2/4a$ to $2\sqrt{ay}$

$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dy \, dx$$

I. w.r to x

$$\int_{y=0}^{4a} \left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$\int_{y=0}^{4a} \left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy$$

$$\int_{y=0}^{4a} 2\sqrt{ay}\, dy - \int_{y=0}^{4a} \frac{y^2}{4a}\, dy$$

$$2\sqrt{a}\int_{y=0}^{4a} y^{1/2}\, dy - \frac{1}{4a}\int_{y=0}^{4a} y^2\, dy$$

$$2\sqrt{a}\left[\frac{y^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{4a}$$

$$\frac{2\sqrt{a}}{1} \times \frac{2}{3}\left[y^{3/2}\right]_0^{4a} - \frac{1}{12a}\left[y^3\right]_0^{4a}$$

$$\frac{4\sqrt{a}}{3}\left[(4a)^{3/2} - 0^{3/2}\right] - \frac{1}{12a}\left[(4a)^3 - 0\right]$$

$$\frac{4\sqrt{a}}{3}\left[(4a)^{3/2}\right] - \frac{1}{\cancel{12}_{3}\cancel{a}}\left[\cancel{64}^{16}\,\cancel{a^3}^{a^2}\right]$$

$$\frac{4\sqrt{a}}{3}\left[\left(2^{2}\right)^{3/2}\left(a^{3/2}\right)\right] - \frac{16}{3}a^{2}$$

$$\frac{4a^{1/2}}{3}\left[8\,a^{3/2}\right] - \frac{16}{3}a^{2}$$

$$\frac{4\times 8\,a^{\frac{1}{2}+\frac{3}{2}}}{3} - \frac{16}{3}a^{2}$$

$$\frac{32}{3}a^{2} - \frac{16}{3}a^{2} = \frac{16}{3}a^{2}$$

By changing of order of integration.

evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$

Given limits $x = 0$ to 1

$y = 0$ to $\sqrt{1-x^2}$.

curve

$x=0$ & $x=1$.

$y=0$ & $y=\sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2 = 1$

Here the limits of "x" are fixed and limits of "y" are varied.

2) To change the order of integration be fixed "y" and we change the elements of "x" in terms of "y"

$x = 0$ to $\sqrt{1-y^2}$
$y = 0$ to 1

$$\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2 \, dy \cdot dx$$

I. w.r.to x

$$\int_{y=0}^{1} y^2 \cdot [x]_0^{\sqrt{1-y^2}} \, dy.$$

$$\int_{y=0}^{1} y^2 \left[\sqrt{1-y^2}-0\right] dy$$

$$\int_{y=0}^{1} y^2\sqrt{1-y^2} \, dy.$$

let $y = \sin\theta$
$dy = \cos\theta \, d\theta$
$\theta = 0$ to $\pi/2$

$$\int_{0}^{\pi/2} \sin^2\theta \sqrt{1-\sin^2\theta} \, \cos\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^2\theta \cdot \cos\theta \cdot \cos\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta \, d\theta$$

$$\int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right) d\theta .$$

$$\frac{1}{4}\int_0^{\pi/2} (1^2 - \cos^2 2\theta)\, d\theta$$

$$\frac{1}{4}\int_0^{\pi/2} \left[1 - \left(\frac{1+\cos 4\theta}{2}\right)\right] d\theta$$

$$\frac{1}{4}\left[\int_0^{\pi/2} 1 \cdot d\theta - \frac{1}{2}\int_0^{\pi/2} 1 \cdot d\theta - \frac{1}{2}\int_0^{\pi/2} \cos 4\theta \, d\theta\right]$$

$$\frac{1}{4}\left[\theta\right]_0^{\pi/2} - \frac{1}{4}\left(\frac{1}{2}\left[\theta\right]_0^{\pi/2}\right) - \frac{1}{4}\left(\frac{1}{2}\left[\frac{\sin 4\theta}{4}\right]_0^{\pi/2}\right)$$

$$\frac{1}{4}\left[\frac{\pi}{2}\right] - \frac{1}{4}\left(\frac{1}{2}\left[\frac{\pi}{2}\right]\right) - \frac{1}{32}\left[\sin 4\left(\frac{\pi}{2}\right) - \sin 4(0)\right]$$

$$\frac{\pi}{8} - \frac{\pi}{16} = \frac{\pi}{16}$$

3) By change of order of integration evaluate $\displaystyle\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dx\,dy$

Given limits

$x = 0$ to 1

$y = x^2$ to $2-x$

Gi curve

$x=0$, $x=1$

$y=x^2$, $y=2-x$

1) Here, the limits of "x" are fixed and limits of "y" are varied.

2) To change the order of integration, we fix "y" and we change the element of "x" in terms of "y".

$$\int_0^1 \int_{\sqrt{y}}^{2-y} xy\,dx\,dy$$

(a) $y=x^2$ $x=y^2$

$$\int_{y=0}^{1} y\cdot\left[\frac{x^2}{2}\right]_{y^2}^{2-y} dy$$

$$\int_{y=0}^{1} \frac{y}{2}\left[(2-y)^2-(y^2)^2\right]dy$$

$$\int_0^1 \frac{y}{2}\left[(4+y^2-4y)-y^4\right]dy$$

$$\frac{1}{2}\int_0^1 \left[4y+y^3-4y^2-y^5\right]dy$$

$$\frac{1}{2}\left[4\cdot\frac{y^2}{2}+\frac{y^4}{4}-\frac{4y^3}{3}-\frac{y^6}{6}\right]_0^1$$

$$\frac{1}{2}\left[\cancel{4}^{2}\frac{1}{\cancel{2}}+\frac{1}{4}-\frac{4}{3}-\frac{1}{6}\right]$$

$$\frac{1}{2}\left(2+\frac{1}{4}-\frac{4}{3}-\frac{1}{6}\right)$$

$$\frac{1}{2}\left(2-\frac{\cancel{3}}{\cancel{4}}-\frac{1}{6}\right)$$

$$\frac{1}{2}\left(2-\frac{3}{4}-\frac{1}{6}\right)$$

$$\frac{1}{2}\left(2-\frac{1}{4}-\frac{1}{6}\right) \Rightarrow \frac{1}{2}\left[\frac{21}{12}\right]=\frac{7}{8}$$

$24-9-2$

$\frac{2(6,4}{3,2}$

24

11

$\frac{1}{2}\left[\frac{17}{12}\right]$

4. VECTOR DIFFERENTIATION

(1)

⇒ Vector differentiation & vector operators

Let $A = a_1 i + a_2 j + a_3 k$

$B = b_1 i + b_2 j + b_3 k$.

(1) $A \times B = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(2) $A \cdot B = a_1 b_1 + a_2 b_2 + a_3 b_3$

$(i \cdot i = 1,\ j \cdot j = 1,\ k \cdot k = 1)$

Unit vector

let $A = a_1 i + a_2 j + a_3 k$ be any vector, the unit vector of "A" is denoted by "$\bar{e}$" that is

$$\boxed{\bar{e} = \frac{A}{|A|}} = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

Vector differential operator (∇)

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

⇒ The vector differential operator

Gradient :- let $\phi(x, y, z)$ be any function, the gradient of ϕ is given by* $\boxed{grad\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}}$

Normal vector of a Surface

let $\phi(x, y, z)$ be a any Surface, the normal vector of "ϕ" is given

2

* $\boxed{\nabla\phi = grad\phi}$

unit normal vector is given by $\frac{\nabla\phi}{|\nabla\phi|}$

5m Angle between any two Surfaces

let $\phi_1(x, y, z)$ and $\phi_2(x, y, z)$ are any two Surfaces and "θ" is angle b/w the Surfaces $\boxed{\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}}$

*5m

Directional derivative

let $\phi(x, y, z)$ be any surface the directional of "ϕ" in the directional vector

* $\boxed{A = a_1 i + a_2 j + a_3 k \text{ is given by } \bar{e} \cdot \nabla\phi}$

where "$\bar{e}$" is unit vector of "A"

Divergent of a Vector

(3)

let $\bar{F} = f_1 i + f_2 j + f_3 k$ be a any vector then the divergent of F is given as

$$* \quad \boxed{\operatorname{div} \bar{F} = \nabla \cdot \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

Curl of a vector

let $\bar{F} = f_1 i + f_2 j + f_3 k$ be a any vector the curve of $\bar{F}$ is given by

$$* \quad \boxed{\operatorname{curl} \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}}$$

Irrotational Vector

If $\operatorname{curl} \bar{F} = 0$, then F is called Irrotational vector

Solenoidal vector If divergent $F = 0$ then $\operatorname{div} \bar{F} = 0$

F is called Soleniodal vector

Note :- let $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ are any two points then the vector joining

$r \; \boxed{(A, B) \text{ is } (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k}$

C

$Ok: 2$ let $\bar{r} = xi + yj + zk$ then $\boxed{\bar{r} = |\bar{r}|}$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}}$$

4

problems (I) → Vector differential problem (∇)

1) find. $\nabla.(x^2 + y^2 z)$

let $f = x^2 + y^2 z$

$$\bar{F} = \nabla.\bar{F} = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} \quad (1)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 z) = 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 z) = 2yz$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2 + y^2 z) = y^2.$$

Subtitute the values in Eqn (1)

$$\nabla f = (2x)\bar{i} + (2yz)\bar{j} + (y^2)\bar{k}$$

$$\boxed{\nabla f = \bar{i}(2x) + \bar{j}(2yz) + \bar{k}(y^2)}$$

2) $\nabla.(x^2 + y^2 + z^2)$

(2) $\nabla(x^2 - yz + z^2)$

5

let $f = x^2 - yz + z^2$

$$\nabla F = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2x \quad , \quad \frac{\partial f}{\partial y} = -z \quad , \quad \frac{\partial f}{\partial z} = 2z - y$$

$$\nabla F = 2x\bar{i} - z\bar{j} + (2z-y)\bar{k}$$

$$\nabla F = 2x\bar{i} - z\bar{j} + (2z-y)\bar{k}$$

(3) Prove that $\nabla(r^n) = n r^{n-2} \cdot \bar{r}$

$$\nabla(r^n) = \bar{i}\frac{\partial}{\partial x}(r^n) + \bar{j}\frac{\partial}{\partial y}(r^n) + \bar{k}\frac{\partial}{\partial z}(r^n)$$

$$= \bar{i}\left[n r^{n-1}\frac{\partial r}{\partial x}\right] + \bar{j}\left[n r^{n-1}\frac{\partial r}{\partial y}\right] + \bar{k}\left[n r^{n-1}\frac{\partial r}{\partial z}\right]$$

w.k.T $r = |\bar{r}|$

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2.$$

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~~$2r\,dr = 2x\,dr$~~

6

F

$r^2 = x^2 + y^2 + z^2$

$\cancel{2} r\,dr = \cancel{2} x \cdot dx \qquad 2r\,dr = 2y \cdot dy$

$\boxed{\frac{r}{x} = \frac{dx}{dr}} \qquad \boxed{\frac{r}{y} = \frac{dy}{dr}}$

$2r\,dr = 2z \cdot dz$

$\boxed{\frac{r}{z} = \frac{dz}{dr}}$

$= \bar{i}\left[n r^{n-1}\left(\frac{x}{r}\right)\right] + \bar{j}\left[n r^{n-1}\left(y/r\right)\right] + \bar{k}\left[n r^{n-1}\left(\frac{z}{r}\right)\right]$

$= \bar{i}\left[n r^{n-2}(x)\right] + \bar{j}\left[n r^{n-2}(y)\right] + \bar{k}\left[n r^{n-2}(z)\right]$

$= n r^{n-2}\left[x\bar{i} + y\bar{j} + z\bar{k}\right]$

$= n r^{n-2}\,\bar{r}$

L.H.S = R.H.S

⇒ prove that $\nabla(f(r)) = \frac{f'(r)}{r}\,\bar{r}$

L.H.S

$\nabla(f(r)) = \bar{i}\,\frac{\partial[f(r)]}{\partial x} + \bar{j}\,\frac{\partial[f(r)]}{\partial y} + \bar{k}\,\frac{\partial[f(r)]}{\partial z}$

$= \bar{i}\left[f'(r)\frac{\partial r}{\partial x}\right] + \bar{j}\left[f'(r)\frac{\partial r}{\partial y}\right] + \bar{k}\left[f'(r)\cdot\frac{\partial r}{\partial z}\right]$

$= \bar{i}\left[f'(r)\cdot\frac{x}{r}\right] + \bar{j}\left[f'(r)\frac{y}{r}\right] + \bar{k}\left[f'(r)\cdot\frac{z}{r}\right]$

$= \frac{f'(r)}{r}[x\bar{i} + y\bar{j} + z\bar{k}]$

$= \frac{f'(r)}{r}\bar{r}$

= R.H.S

⇒ find the unit Normal vector to the Surface $z = x^2+y^2$ at point $(-1, -2, 5)$

$U.N.V = \frac{\nabla f}{|\nabla f|}$

$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

let $z = x^2+y^2$

$f = x^2+y^2-z$

$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = -1$

$\nabla f = \bar{i}(2x) + \bar{j}(2y) + \bar{k}(-1)$

$\nabla f = 2x\bar{i} + 2y\bar{j} - \bar{k} = -2\bar{i} - 4\bar{j} - \bar{k}$

unit vector $= \frac{\nabla f}{|\nabla f|} = \cancel{\frac{2x\bar{i}+2y\bar{j}-\bar{k}}{\sqrt{4+4+1}}} =$

$= \cancel{\frac{2x\bar{i}+2y\bar{j}-\bar{k}}{3}}$

$$= \frac{-2\bar{i}-4\bar{j}-\bar{k}}{\sqrt{4+16+1}} = \frac{-(2\bar{i}+4\bar{j}+\bar{k})}{\sqrt{21}}$$

Evaluate the angle b/w the normal to the surface $xy = z^2$ at the points $(4,1,2)$ & $(3,3,-3)$

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$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

let $f_1 = xy - z^2$

$$\nabla f_1 = \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_1}{\partial y} + \bar{z}\frac{\partial f_1}{\partial z}$$

$$\frac{\partial f_1}{\partial x} = y \cdot \quad \frac{\partial f_1}{\partial y} = x, \quad \frac{\partial f_1}{\partial z} = -2z$$

$$\nabla F = y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$\nabla F_{1\,(4,1,2)} = \bar{i} + 4\bar{j} - 4\bar{k} = \bar{F}_1$$

$$\nabla F_{2\,(3,3,-3)} = 3\bar{i} + 3\bar{j} + 6\bar{k} = \bar{F}_2$$

$$\cos\theta = \frac{(\bar{i}+4\bar{j}-4\bar{k})(3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{1+16+16}\,\sqrt{9+9+36}}$$

$$= \frac{3(\bar{i}\cdot\bar{i}) + 12(\bar{j}\cdot\bar{j}) - 24(\bar{k}\cdot\bar{k})}{\sqrt{33}\,\sqrt{54}}$$

$$\cos\theta = \frac{3+12-24}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

$$\theta = \cos^{-1}\left[\frac{-9}{\sqrt{33}\sqrt{54}}\right]$$

$$\theta = \cos^{-1}\left[\frac{-9}{\sqrt{11}\sqrt{3}\;3\cdot\sqrt{6}}\right]$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{11}\sqrt{3}\;3\cdot\sqrt{2}\sqrt{3}}\right)$$

$$\theta = \cos^{-1}\left[\frac{-9}{9\sqrt{22}}\right] = \cos^{-1}\left(\frac{-1}{\sqrt{22}}\right)$$

$x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ $(2,-1,2)$

(10)

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1| \cdot |\nabla f_2|}$$

let $f_1 = x^2+y^2+z^2=9$, $f_2 = x^2+y^2-z-3$

$$\nabla f_1 = \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_1}{\partial y} + \bar{k}\frac{\partial f_1}{\partial z} \quad \text{— (1)}$$

$$\frac{\partial f_1}{\partial x} = 2x, \quad \frac{\partial f_1}{\partial y} = 2y, \quad \frac{\partial f_1}{\partial z} = 2z$$

Eqn (1) $\nabla f_1 = \bar{i}\,2x + \bar{j}\,2y + \bar{k}\,2z$ — (2) $\begin{pmatrix} 2, & -1, & 2 \\ x & y & z \end{pmatrix}$

$$\nabla f_1 = 4\bar{i} - 2\bar{j} + 4\bar{k}$$

$$|\nabla f_1| = \sqrt{16+4+16} = \sqrt{36} = 6.$$

$$\nabla f_2 = \bar{i}\frac{\partial f_2}{\partial x} + \bar{j}\frac{\partial f_2}{\partial y} + \bar{k}\frac{\partial f_2}{\partial z}.$$

$$\frac{\partial f_2}{\partial x} = 2x, \quad \frac{\partial f_2}{\partial y} = 2y, \quad \frac{\partial f_2}{\partial z} = -1. \quad (3)$$

Eq (3) $\nabla f_2 = \bar{i}\,2x + \bar{j}\,2y + \bar{k}(-1)$ — (4) $\begin{pmatrix} 2, & -1, & 2 \\ x, & y, & z \end{pmatrix}$

$$\nabla f_2 = 4\bar{i} - 2\bar{j} - \bar{k}$$

$$|\nabla f_2| = \sqrt{16+4+1} = \sqrt{21}$$

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1||\nabla f_2|} = \frac{(4\bar{i}-2\bar{j}+4\bar{k})(4\bar{i}-2\bar{j}-\bar{k})}{6\sqrt{21}}$$

$$= \frac{16(1) + 4(1) - 4(1)}{6\sqrt{21}} = \frac{\overset{8}{\cancel{16}}}{\underset{3}{\cancel{6}}\sqrt{21}}$$

$\cos\theta = \frac{8}{3\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

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→ In the direction of vector $f = xy + yz + 3x$ ①

$\bar{i} + 2\bar{j} + 2\bar{k}$: $(1, 2, 0)$

Directional derivative $= \bar{e} \cdot \nabla f$.

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$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\cdot\frac{\partial f}{\partial z} \quad ①$$

$$\frac{\partial f}{\partial x} = y + z, \quad \frac{\partial f}{\partial y} = x + z, \quad \frac{\partial f}{\partial z} = x + y.$$

Eqn ①

$$\nabla f = (y+z)\bar{i} + (x+z)\bar{j} + (x+y)\bar{k}$$

$(1,2,0)$ $\quad \begin{pmatrix} x, & y, & z \\ 1, & 2, & 0 \end{pmatrix}$

$$\nabla f = 2\bar{i} + \bar{j} + 3\bar{k}$$

$$\bar{e} = \frac{A}{|A|} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

$$\nabla f \cdot \bar{e} = \left(\frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}\right)\left[2\bar{i} + \bar{j} + 3\bar{k}\right]$$

$$\nabla f \cdot \bar{e} = \frac{2 + 2 + 6}{3} = \frac{10}{3}$$

→ Find the directional derivate of $f(x,y,z) = xy^2 + yz^2$

at $(2, -1, 1)$ $\quad \bar{i} + 2\vec{j} + 2\bar{k}$

$$D.D = \nabla f \cdot \bar{e}.$$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}.$$

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$\frac{\partial f}{\partial x} = y^2$, $\frac{\partial f}{\partial y} = 2xy + z^2$, $\frac{\partial f}{\partial z} = 2yz$

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$\nabla F = \bar{i}\, y^2 + \bar{j}\,(2xy + z^2) + \bar{k}\,(3yz^2)$

(x, y, z) points are

$(2, -1, 1)$

$\nabla F = \bar{i} + (-3)\bar{j} + (-2)\bar{k} = \bar{i} - 3\bar{j} - 3\bar{k}$

$\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{9}} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}$

$D.D = \nabla f \cdot \bar{e}$

$= \left(\bar{i} - 3\bar{j} - 3\bar{k}\right)\left[\frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}\right]$

$= \frac{1 - 6(1) - 6(1)}{3} = \frac{-11}{3}$

⇒ Find the directional derivative of

$f - \phi = x^2yz + 4xz^2$ $(1, 2, -1)$ $2\bar{i} - \bar{j} - 2\bar{k}$

$D.D = \nabla f \cdot \bar{e}$

$\nabla f = \bar{i}\,\frac{\partial f}{\partial x} + \bar{j}\,\frac{\partial f}{\partial y} + \bar{k} \cdot \frac{\partial f}{\partial z}$

$\frac{\partial f}{\partial x} = 2xyz + 4z^2$, $\frac{\partial f}{\partial y} = x^2 z$, $\frac{\partial f}{\partial z} = x^2y + 4x \cdot 2z$

$2(1)(2)(-1) + 4(-1)^2$ $(1)^2(-1)$ $(1)^2(2) + 4(1)(-1)$

$\nabla f = -0 - \bar{j} + 6\bar{k} = -\bar{j} + 6\bar{k}$

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$$\nabla f = -\bar{j} + 6\bar{k}$$

$$\bar{e} = \frac{\bar{A}}{|\bar{A}|} = \frac{2\bar{i}-\bar{j}-2\bar{k}}{\sqrt{4+4+1}} = \frac{2\bar{i}-\bar{j}-2\bar{k}}{3}$$

$$\nabla f \cdot \bar{e} = \left(-\bar{j}+6\bar{k}\right)\left[\frac{2\bar{i}-\bar{j}-2\bar{k}}{3}\right]$$

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$$= \frac{\cancel{4}\,0+1+12}{3} = \frac{13}{3}$$

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Find the greatest value of the directional derivative of the function $f = x^2 y z^3$ at the point $(2,1,-1)$

Given $F = x^2 y z^3$ and given point $(2,1,-1)$

Greatest value of the directional derivative $|\nabla F|$

$$\nabla F = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2xyz^3, \quad \frac{\partial f}{\partial y} = x^2 z^3, \quad \frac{\partial f}{\partial z} = 3x^2 y z^2$$

$$\nabla f = \bar{i}\underset{2.2.1.(-1)}{(2xyz^3)} + \bar{j}\underset{4.(-1)}{(x^2 z^3)} + \bar{k}\underset{3.4.1.(-1)^2}{(3x^2yz^2)}$$

$$= \bar{i}(-4) + \bar{j}(-4) + \bar{k}(12)$$

$$= -4\bar{i} - 4\bar{j} + 12\bar{k}$$

$$\nabla F = 4[-\bar{i} - \bar{j} + 3\bar{k}]$$

$$|\nabla F| = 4[-\bar{i} + \bar{j} + 3\bar{k}] = 4\sqrt{1+1+9} = 4\sqrt{11}$$

$\Rightarrow$ In what direction the point $\pi(-1,1,2)$ is Directional derivative DD of $\phi = xy^2z^2$ a maximum.

what the magnitude of this maximum.

Given $\phi = x^2 y^2 z^3$ $(-1, 1, 2)$

$|\nabla\phi| = ?$

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$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\left[y^2 z^3\right] + \bar{j}\left[2xyz^3\right] + \bar{k}\left[3xy^2z^2\right]$$

$(1)^2(2)^3$ $\quad 2(-1)(1)(2)^3$ $\quad 3(-1)(1)^2$

$$= \bar{i}\left[8\right] + \bar{j}\left[-16\right] + \bar{k}\left[-12\right]$$

$$= 8\bar{i} - 16\bar{j} - 12\bar{k}$$

$$\nabla\phi = 4\left[2\bar{i} - 4\bar{j} - 3\bar{k}\right]$$

$$|\nabla\phi| = 4\sqrt{4+16+9} = 4\sqrt{29}$$

$\Rightarrow$ Find the Directional derivative $xyz^2 +$ at point $(1, 1, 1)$ in the direction of the normal to the surface $3xy^2 + y = z$ at (

let given

$f = xyz^2 + xz$ at $(1,1,1)$ (16)

$g = 3xy^2 + y - z$ at $(0,1,1)$

$D.D = \nabla f \cdot \bar{e}$

$\nabla g = \bar{i}\frac{\partial g}{\partial x}(3xy^2+y-z) + \bar{j}\frac{\partial}{\partial y}(3xy^2+y-z)$

$+ \bar{k}\frac{\partial}{\partial z}[3xy^2+y-z]$

$\nabla g = 3\bar{i} + \bar{j} - \bar{k}$

Normal to the surface $\bar{e} = \frac{\nabla g}{|\nabla g|}$

$\bar{e} = \frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{9+1+1}}$

$\bar{e} = \frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{11}}$

$\boxed{D.D} = \nabla f \cdot \bar{e}$

$\nabla f = \left(\bar{i}\cdot\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\cdot\frac{\partial f}{\partial z}\right)\cdot\bar{e}$

$\nabla f = \left(\bar{i}[yz^2+z] + \bar{j}[xz^2+x] + \bar{k}(2xyz+x)\right)\bar{e}$

$\nabla f = 2\bar{i} + 2\bar{j} + 3\bar{k}\left(\frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{11}}\right) = \frac{6+2-3}{\sqrt{11}}$

$= \frac{4}{\sqrt{11}}$

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⇒ Find the directional derivative of $\frac{1}{r}$ in direction of $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ at $(1, 1, 2)$

$$\bar{r} = \nabla\left(\frac{1}{r}\right)\cdot\frac{\bar{r}}{|\bar{r}|}$$

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Sol Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2} \qquad \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

hence the directional derivative of $\frac{1}{r}$ in the direction of $\bar{r} = \nabla\left(\frac{1}{r}\right)\frac{\bar{r}}{|\bar{r}|}$

$$\nabla\left(\frac{1}{r}\right) = \nabla f = \bar{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \bar{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \bar{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \quad \text{—— (1)}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} \qquad \cancel{x^{n-1}} = -x$$

$$= -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}x)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -(x^2+y^2+z^2)^{-3/2}\cdot x = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

Simillarly

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \quad \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Subtitute Eqn (1) the above values

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$$\nabla\left(\frac{1}{r}\right) = \bar{i}\left[\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right] + \bar{j}\left[\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right] + \bar{k}\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$\nabla(1/r) = \frac{-1}{(x^2+y^2+z^2)^{3/2}}\left[x\bar{i}+y\bar{j}+z\bar{k}\right]$$

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$$\nabla(1/r) = \frac{-1}{(x^2+y^2+z^2)^{3/2}}\left[\bar{r}\right]$$

$$\bar{r} = \nabla\cdot\left(\frac{1}{r}\right)\cdot\frac{\bar{r}}{|\bar{r}|}$$

$$= \nabla\left(\frac{-1}{(x^2+y^2+z^2)^{3/2}}(\bar{r})\right)\frac{(\bar{r})}{|\bar{r}|}$$

$$= \frac{-1}{(x^2+y^2+z^2)^{3/2}}\left[x\bar{i}+y\bar{j}+z\bar{k}\right]\frac{\left[x\bar{i}+y\bar{j}+z\bar{k}\right]}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2+1/2}} = \frac{-1}{x^2+y^2+z^2}$$

$$= \frac{-1}{1+1+4} = \frac{-1}{6}$$

+/# (1) TO (find) $\phi = xyz$ find $\nabla\phi$

* (2) $f(x,y,z) = zx^2 - xyz$ at the point $(1,3,1)$ in the direction of the vector $3\bar{i} - 2\bar{j} + \bar{k}$

$\nabla F\cdot\bar{e}$

(3) Find the directional derivative of the scalar function $f(x,y,z) = xyz$ at point $(1,4,9)$ in the direction of the lying from $(1,2,3)$ to $(1,-1,-3)$ $\quad \nabla F \cdot \bar{e}$

(4) Find a unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$ $\quad \dfrac{\nabla F}{|\nabla F|}$

(5) find the angle b/w $xy = z^2$ at the point $(4,1,2)$ & $(3,3,-3)$

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(1) $\phi = xyz \qquad \nabla\phi$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\,[yz] + \bar{j}\,[xz] + \bar{k}\,(xy)$$

(2) $f(x,y,z) = zx^2 - xyz$ at point $(1,3$

$9\bar{i} - 2\bar{j} + \bar{k} \doteq \bar{e} \qquad \nabla \cdot F \cdot \bar{e}$

$$\nabla F = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$$\nabla F = \bar{i}\left[2xz - yz\right] + \bar{j}\left[-xz\right] + \bar{k}\left(x^2 - xy\right)$$

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$\nabla f \cdot \bar{e} = \underset{(1)(1)\ \ (3)(1)}{[2x3 - y3]}\,\bar{i} - [x3]\,\bar{j} + \underset{1-(1)(3)}{[x^2 - xy]}\,\bar{k}$

$\nabla f_{(1,3,1)} = [-1]\bar{i} - \bar{j} - 2\bar{k} = -\bar{i} - \bar{j} - 2\bar{k}$

$$\nabla f \cdot \bar{e} = (-\bar{i} - \bar{j} - 2\bar{k}) \frac{[3\bar{i} - 2\bar{j} + \bar{k}]}{\sqrt{9+4+1}}$$

$$= \frac{-3+2-2}{\sqrt{14}} = \frac{-3}{\sqrt{14}}$$

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(3) $f(x,y,z) = xyz$ at point $(1,4,9)$

$(1,2,3), (1,-1,-3)$

$\bar{A} = (x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}$

$\bar{A} = -3\bar{j} - 6\bar{k} =$

$$\nabla F = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

$\nabla f = \bar{i}\underset{(4)(9)}{[yz]} + \bar{j}\underset{(1)(9)}{(xz)} + \bar{k}\underset{(1)(4)}{[xy]}$

$\nabla f_{(1,4,9)} = 36\bar{i} + 9\bar{j} + 4\bar{k}$

$$\nabla f \cdot \bar{e} = 36\bar{i} + 9\bar{j} + 4\bar{k} \frac{(-3\bar{j} - 6\bar{k})}{\sqrt{9+36}}$$

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5m div F where $\bar{F}$ ~~or~~ $r^n \bar{r}$. find n, if it is Solenoidal

(or)

P.T $r^n \bar{r}$ is Solenoidal if $n = -3$

P.T $div(r^n \bar{r}) = (n+3) r^n$ where

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

Given that

$\bar{F} = r^n \bar{r}$

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w.k.T $r = x\bar{i} + y\bar{j} + z\bar{k}$

about $r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

Diff w.r.to x

$\cancel{2} r \frac{\partial r}{\partial x} = \cancel{2} x + 0 + 0 = \frac{x}{r}$

Similarly

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{F} = r^n \bar{r}$

$\bar{F} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

$\bar{F} = r^n x\bar{i} + r^n y\bar{j} + r^n z\bar{k}$ — (1)

$div.F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ — (2)

$\cancel{\frac{\partial f_1}{\partial x} = r^n}, \quad \cancel{\frac{\partial f_2}{\partial y} = r^n}, \quad \cancel{\frac{\partial f_3}{\partial y} = r^n}$

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$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}(r^n x) = r^n(1) + n r^{n-1} \frac{\partial r}{\partial x} \cdot x$$

$$\frac{\partial f_1}{\partial x} = r^n + n r^{n-1} \frac{x}{r} \cdot x$$

$$= r^n + n r^{n-1} \frac{x^2}{r}.$$

Similarly, we can know that

$$\frac{\partial f_2}{\partial y} = r^n + n r^{n-1} \frac{y^2}{r}$$

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$$\frac{\partial f_3}{\partial z} = r^n + n r^{n-1} \frac{z^2}{r}$$

$$\text{div}.\bar{F} = \frac{r^n + n r^{n-1}}{r}\left[x^2 + y^2 + z^2\right]$$

Wrong

$$\text{div}.\bar{F} = \frac{r^n + n r^{n-1}}{r}(r^2)$$

$$\text{div}.\bar{F} = \left[r^n + n r^{n-1}\right] r.$$

$$\text{div}.\bar{F} = 3r^n + \frac{n r^{n-1}}{r}\left[x^2 + y^2 + z^2\right]$$

$$\text{div}.\bar{F} = 3r^n + \frac{n r^n}{\cancel{r^2}}\left[\cancel{r^2}\right]$$

$$\text{div}.\bar{F} = r^n\left[3 + n\right]$$

Since $\bar{F}$ is solenoidal

$\text{div}.\bar{F} = 0$

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$r^n(3+n)=0$

$n+3=0 \rightarrow n=-3$

(23)

$\boxed{\text{div}.\bar{F} \text{ (or) } \nabla.\bar{F}}$

Same formula

→ Evaluate $\nabla\left[\frac{\bar{r}}{r^3}\right]$ where $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$

& $r=|\bar{r}|$

(or)

Show that $\frac{\bar{r}}{r^3}$ is solenoidal.

(or)

If $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$ & $r=|\bar{r}|$, S.T $\nabla\left[\frac{\bar{r}}{r^3}\right]=0$

$\nabla.\left(\frac{\bar{r}}{r^3}\right) = \text{div}.\left[\frac{\bar{r}}{r^3}\right]$

$\nabla.\left[\frac{\bar{r}}{r^3}\right] = \frac{\partial}{\partial x}\left[\frac{\bar{r}}{r^3}\right]+\frac{\partial}{\partial y}\left[\frac{\bar{r}}{r^3}\right]+\frac{\partial}{\partial z}\left[\frac{r}{r^3}\right]$ — (1)

$\frac{\partial}{\partial x}\left[\frac{\bar{r}}{r^3}\right] = \frac{\partial}{\partial x}\left(\frac{x\bar{i}+y\bar{j}+z\bar{k}}{r^3}\right)$ $\quad \bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$

$r=\sqrt{x^2+y^2+z^2}$

$= \frac{\partial}{\partial x}\left(\underset{u}{r^{-3}}\,\underset{v}{(x\bar{i}+y\bar{j}+z\bar{k})}\right)$ $\quad r^2=x^2+y^2+z^2$

$= r^{-3}+(-3)r^{-3-1}\frac{\partial r}{\partial x}.x$

$= r^{-3}-3r^{-4}\frac{\partial r}{\partial x}.x$ ⟵ (2)

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where $\frac{\partial r}{\partial x} = \frac{x}{r}$ sub in eqn (2)

$$\frac{\partial}{\partial x}\left(r^{-3}\cdot\bar{r}\right) = r^{-3} - 3r^{-4}\,\frac{x}{r}\cdot x = \frac{\partial f_1}{\partial x}$$

$$= r^{-3} - 3r^{-4}\,\frac{x^2}{r}$$

$$= r^{-3} - 3r^{-5}\cdot x^2$$

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Simillary

$$\frac{\partial}{\partial y}\left(r^{-3}\cdot\bar{r}\right) = r^{-3} - 3r^{-5}\cdot y^2 = \frac{\partial f_2}{\partial y}$$

$$\frac{\partial}{\partial z}\left(r^{-3}\cdot\bar{r}\right) = r^{-3} - 3r^{-5}\,z^2 = \frac{\partial f_3}{\partial z}$$

$\nabla\cdot\bar{F} = \text{div}\cdot\bar{F}$

$$\text{div}\cdot\bar{F} = 3r^{-3} - 3r^{-5}\left(x^2+y^2+z^2\right)$$

$$= 3r^{-3} - 3r^{-5}\left[r^2\right]$$

$$= 3r^{-3} - 3r^{-3}$$

$$= 0$$

So, it is Solenoidal

$$\nabla\left(\frac{\bar{r}}{r^3}\right) = 0$$

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⇒ prove that the position vector of any point in space, then [$r^n \bar{r}$ is] irrotational.

(or)

Show that curl $(r^n \bar{r}) = 0$

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Given that $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|\bar{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

curl F

let $\bar{f} = r^n \bar{r}$

$\bar{f} = r^n (x\bar{i} + y\bar{j} + z\bar{k})$

$$\text{curl } (r^n \bar{r}) = 0$$

$$\text{curl } F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] - \bar{j}\left[\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right]$$

$$\text{curl } F = \bar{i}\left[z\, n r^{n-1}\frac{\partial r}{\partial y} - y \cdot n r^{n-1}\frac{\partial r}{\partial z}\right) - \bar{j}\left[z \cdot n r^{n-1}\frac{\partial r}{\partial x} - x \cdot n r^{n-1}\frac{\partial r}{\partial z}\right]$$

$$+ \bar{k}\left[y \cdot n r^{n-1}\frac{\partial r}{\partial x} - x \cdot n r^{n-1}\frac{\partial r}{\partial y}\right]$$

$$\text{curl } \bar{F} = \bar{i}\left[z \cdot n \cdot r^{n-1}(y/r) - y \cdot n r^{n-1}(z/r)\right] - \bar{j}\left[z \cdot n r^{n-1}(x/r) - x \cdot n r^{n-1}(z/r)\right]$$

$$+ \bar{k}\left[y \cdot n r^{n-1}(x/r) - x \cdot n r^{n-1}(y/r)\right]$$

$\boxed{\text{So, it is irrotational.}}$

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$$\text{curl}\,\bar{F} = \hat{i}\left\{ n r^{n-2}\left[\cancel{yz - yz}\right]\right\} - \hat{j}\, n r^{n-2}\left[\cancel{zx - zx}\right] + \hat{k}\, n r^{n-2}\left[\cancel{xy - xy}\right]$$

$$\text{curl}\,\bar{F} = 0$$

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So, it is irrotational.

$\Rightarrow$ $div.F = \nabla.F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ —— (1)

$\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ at point $(1,-1,1)$

$f_1 = xy^2$, $f_2 = 2x^2yz$, $f_3 = -3yz^2$ (27)

$\frac{\partial f_1}{\partial x} = y^2$, $\frac{\partial f_2}{\partial y} = 2x^2z$, $\frac{\partial f_3}{\partial z} = -6yz$

Eqn (1)

$div.F = y^2 + 2x^2z - 6yz$

$div.F = 1+2+6 = 9$

$\Rightarrow$ Find divergent $\bar{F}$ when $\bar{F} = grad(x^3+y^3+z^3-3xyz)$

$f = x^3+y^3+z^3-3xyz)$

$grad.\bar{F} = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) = 3x^2-3yz$

$\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) = 3y^2-3xz$

$\vdots$

$\frac{\partial f}{\partial z} = 3z^2-3xy$

$\bar{F} = (3x^2-3yz)\bar{i} + (3y^2-3xz)\bar{j} + (3z^2-3xy)\bar{k}$

$div.F = 6x + 6y + 6z$ | $\frac{\partial f_1}{\partial x} = 6x$, $\frac{\partial f_2}{\partial y} = 6y$, $\frac{\partial f_3}{\partial z} = 6z$

$= 6x+6y+6z$

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$\Rightarrow$ If $\bar{F} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is

Solenoidal, find "p"

$$\text{diver}.F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0$$

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$f_1 = x+3y$, $f_2 = y-2z$, $f_3 = x+pz$

$\frac{\partial f_1}{\partial x} = 1$, $\frac{\partial f_2}{\partial y} = 1$, $\frac{\partial f_3}{\partial z} = p$

$\boxed{\text{div}.\bar{F} = 0}$

$$1+1+p = 0$$

$$p = -2 .$$

$\Rightarrow$ Curl.F Problems

(1) If $\bar{F} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find

Curl.F at the point $(1, -1, 1)$

$$\text{Curl}F = \nabla\times F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

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$= \bar{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right]$

$+ \bar{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] +$

$\bar{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$

$= \bar{i}\left[-3z^2 - 2x^2y\right] - \bar{j}\left[0\right] + \bar{k}\left[4xyz - 2xy\right]$

$\text{curl}F = \bar{i}\left[-3z^2 - 2x^2y\right] + \bar{k}\left[4xyz - 2xy\right]$ ①

Give points $(1,-1,1)$ sub in ①

$\text{curl}F = \bar{i}\left[-3+2\right] + \bar{k}\left[-4+2\right] = -\bar{i} - 2\bar{k}$

② find curlF where $F = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

$\text{grad } F = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$\frac{\partial f}{\partial x} = 3x^2 - 3yz, \quad \frac{\partial f}{\partial y} = 3y^2 - 3xz,$

$\frac{\partial f}{\partial z} = 3z^2 - 3xy.$

$\bar{F} = (3x^2 - 3yz)\bar{i} + (3y^2 - 3xz)\bar{j} + (3z^2 - 3xy)\bar{k}$

$$\text{Curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

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$$\text{curl}\,F = \bar{i}\left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right] +$$

$$\bar{j}\left[\frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz)\right] +$$ 30

$$\bar{k}\left[\frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz)\right]$$

$$\text{curl}\,F = \bar{i}\left[-3x + 3x\right] + \bar{j}\left(-3x + 3x\right)$$

$$+ \bar{k}\left[-3z + 3z\right]$$

$$\text{curl}\,F = 0$$

③ If $\bar{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$

Show that $\bar{F} \cdot \text{curl}\,\bar{F} = 0$.

$f_1 = x+y+1, \quad f_2 = 1, \quad f_3 = -(x+y)$

$$\text{curl}\,F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$\text{curl}\,F = \bar{i}\left[\frac{\partial}{\partial y}(-x-y) - \frac{\partial}{\partial z}(1)\right] - \bar{j}\left[\frac{\partial}{\partial x}(-x-y) - \frac{\partial}{\partial z}(x+y+1)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+1)\right]$$

$$\text{curl } F = \bar{i}[-1] - \bar{j}[-1] + \bar{k}[-1]$$

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$$= -\bar{i} + \bar{j} - \bar{k}$$

$$\bar{F}\cdot\text{curl}\bar{F} = [(x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}][-\bar{i}+\bar{j}-\bar{k}]$$

$$\bar{F}\cdot\text{curl}\bar{F} = -(x+y+1) + 1 + x + y.$$

$$\bar{F}\cdot\text{curl}\bar{F} = \cancel{-x}\cancel{-y}\cancel{-1}\cancel{+1}\cancel{+x}\cancel{+y} = 0$$

$$\boxed{\bar{F}\cdot\text{curl}\bar{F} = 0}$$

∴ Hence proved.

Q) Prove that $\bar{F} = yz\bar{i} + zx\bar{j} + yx\bar{k}$ is Irrotational.

$$\text{curl } F = 0$$

$$f_1 = yz, \quad f_2 = zx, \quad f_3 = yx.$$

$$\text{curl }\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & yx \end{vmatrix}$$

curl $\bar{F} = 0$
So, it is Irrotational

$$\text{curl }\bar{F} = \bar{i}\left[\frac{\partial}{\partial y}(yx) - \frac{\partial}{\partial z}(zx)\right] - \bar{j}\left[\frac{\partial}{\partial x}(yx) - \frac{\partial}{\partial z}(yz)\right]$$
$$+ \bar{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$\text{curl }\bar{F} = \bar{i}[x-x] + \bar{j}[y-y] + \bar{j}[z-z] = 0$$

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$\rightarrow$ If $\bar{F} = x\bar{i} + y^2\bar{j} + z^3\bar{k}$, find curl $\bar{F}$

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$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y^2 & z^3 \end{vmatrix}$$

$$\text{curl } \bar{F} = \bar{i}\left[\frac{\partial}{\partial y}(z^3) - \frac{\partial}{\partial z}(y^2)\right] - \bar{j}\left[\frac{\partial}{\partial x}(z^3) - \frac{\partial}{\partial z}(x)\right] + \bar{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x)\right]$$

curl $\bar{F} = 0$

so, it is irrotational.

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V. Vector Integration

line integral

→ defination (text)

line integral $= \int_C \bar{F} \cdot \delta r$

In general any integral which is be evaluated along a curve is called line ~~cur~~ integral

① $\int_C \bar{F}\cdot d\bar{r}$ when $\bar{F} = x^2\bar{i} + y^2\bar{j}$ and C is the curve $y = x^2$ in the xy plane from $(0,0)$ to $(1,1)$

$\bar{r} = x\bar{i} + y\bar{j}$

$d\bar{r} = dx\bar{i} + dy\bar{j}$

$F\cdot d\bar{r} = (x^2\bar{j} + y^2\bar{j})(dx\bar{i} + dy\bar{j})$

$[\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = 1 / \bar{i}\cdot\bar{j} = \bar{j}\cdot\bar{i} = 0]$

$\bar{F}\cdot d\bar{r} = x^2dx + y^2dy$

Here given $y = x^2$

$dy = 2x\,dx$

x limits 0 to 1

$$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} x^2dx + (x^2)^2(2x)dx$$

$$= \int_{x=0}^{1} x^2dx + 2x^5\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^6}{6}\right]_0^1 = \frac{1-0}{3} + 2\left[\frac{1-0}{6}\right]$$

$$= \frac{1\times 2}{3\times 2} + \frac{2}{6}$$

$$\int_C \bar{F}\cdot d\bar{r} = \frac{4}{6} = \frac{2}{3}$$

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$\bar{F} = 3xy\bar{i} - y^2\bar{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$

the curve $y = 2x^2$ in the xy plane from

$(0,0)$ to $(1,2)$

$\bar{F} = 3xy\bar{i} - y^2\bar{j}$

$\bar{r} = x\bar{i} + y\bar{j}$

$d\bar{r} = dx\bar{i} + dy\bar{j}$

$\bar{F}\cdot d\bar{r} = (3xy\bar{i} - y^2\bar{j})(dx\bar{i} + dy\bar{j})$

$\boxed{\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = 1 \,/\, \bar{i}\cdot\bar{j} = 0, \bar{j}}$

$\bar{F}\cdot d\bar{r} = 3xy\,dx - y^2 dy$

Here given

$y = 2x^2$

$dy = 4x\,dx$

x limits 0 to 1

$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} [3x(2x^2)dx - (2x^2)^2 4x\,dx]$

$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} 6x^3 dx - 16x^5 dx$

$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1$

$= \frac{6}{4}[1-0] - \frac{16}{6}[1-0] = \frac{6}{4\times3} - \frac{16}{6\times2}$

$\int_C \bar{F}\cdot d\bar{r} = \frac{18-32}{12} = \frac{-14}{12} = \frac{-7}{6}$

① work done by force defination

If $\bar{V}$ represents the force vector acting on a particle moving along an arc AB, then the work done during a small displacement $\delta\bar{r}$ is $\bar{F}.\delta\bar{r}$. Hence the total work done by $\bar{F}$ during displacement from A to B is given by the line integral $\int_A^B \bar{F}.d\bar{r}$.

If the force $\bar{F}$ is conservative (i.e) $\bar{F} = \nabla\phi$ then the work dom is independent of the path & vice versa. In this case

Curl $\bar{F}$ = Curl (grad ϕ) = $\bar{0}$ & ϕ is Scalar potential

② Find the work dom in moving a particle in the force field points (0,0,0) to (2,3,1)

ⓐ $\bar{F} = x^3\bar{i} + \bar{j} + z\bar{k}$

ⓑ $\bar{F} = 5x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$

points (0,0,0) to (2,3,1)

ⓐ Given $\bar{F} = x^3\bar{i} + \bar{j} + z\bar{k}$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$\bar{F}.d\bar{r} = x^3 dx + dy + z dz$$

$$\int_C \bar{F}.d\bar{r} = \int_C x^3 dx + dy + z dz$$

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$$\int_C \bar{F} \cdot d\bar{r} = \int_0^2 x^3\,dx + \int_0^3 dy + \int_0^1 z\,dz$$

$$= \left[\frac{x^4}{4}\right]_0^2 + \left[y\right]_0^3 + \left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{\cancel{16}^{\,4} - 0}{\cancel{4}} + 3 - 0 + \frac{1-0}{2}$$

$$= 4 + 3 + \frac{1}{2}$$

$$\int_C \bar{F} \cdot d\bar{r} = 7 + \frac{1}{2} = \frac{15}{2}$$

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$\bar{F} = 5x^2\bar{i} + (2xz-y)\bar{j} + z\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F}\cdot d\bar{r} = 5x^2dx + (2xz-y)\,dy + z\,dz$

str. lin eqn from $(0,0,0)$ to $(2,3,1)$

$$\frac{x-0}{2-0} = \frac{y-0}{3-0} = \frac{z-0}{1-0} = t$$

$x = 2t,\quad y = 3t,\quad z = t$

$dx = 2\,dt,\quad dy = 3\,dt,\quad dz = dt$

The above values sub. in $\bar{F}\cdot d\bar{r}$

$$\bar{F}\cdot d\bar{r} = 5(2t)^2 2\,dt + \big(\overset{4t^2}{2(2t)(t)} \overset{-3t}{-3t}\big)3dt + t\,dt$$

$\bar{F}\cdot d\bar{r} = 40t^2dt + 12t^2dt - 9t\,dt + t\,dt$

$\bar{F}\cdot d\bar{r} = 40t^2dt + 12t^2dt - 8t\,dt$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C 52t^2dt + (-8t)\,dt$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} 52t^2\,dt + \int_{t=0}^{1} -8t\,dt$$

$$\int_C \bar{F}\cdot d\bar{r} = 52\left[\frac{t^3}{3}\right]_0^1 - 8\left[\frac{t^2}{2}\right]_0^1$$

$$= 52\left[\frac{1-0}{3}\right] - 8\left[\frac{1-0}{2}\right]$$

$$= \frac{52}{3} - \frac{8}{2}$$

$$= \frac{52\times 2}{3\times 2} - \frac{8\times 3}{2\times 3}$$

$$= \frac{104}{6} - \frac{24}{6} = \frac{80}{6} = \frac{40}{3}$$

$$\cancel{\frac{88}{6}} \quad \cancel{\frac{44}{3}}$$

(3) If $\phi = x^2 y z^3$, evaluate $\int_C \phi \, d\bar{r}$ along curve $x=t$, $y=2t$, $z=3t$ from $t=0$ to $t=1$

$\phi = x^2 y z^2$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

given that $x=t$, $y=2t$, $z=3t$ from $t=0$ to 1

$dx = dt$, $dy = 2dt$, $dz = 3dt$

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$$\int_c \phi \, d\bar{r} = \int x^2 y z^3 (dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k})$$

$$= \int_0^1 t^2 (2t)(3t)^3 (dt\,\bar{i} + 2dt\,\bar{j} + 3dt\,\bar{k})$$

$$= \int_0^1 54t^6 \, dt \, [\bar{i} + 2\bar{j} + 3\bar{k}]$$

$$\int_c \phi \, d\bar{r} = 54\left[\frac{t^7}{7}\right]_0^1 [\bar{i} + 2\bar{j} + 3\bar{k}]$$

$$= 54\left[\frac{1-0}{7}\right] [\bar{i} + 2\bar{j} + 3\bar{k}]$$

$$= 54\left[\frac{1}{7}\right] = \frac{54}{7} [\bar{i} + 2\bar{j} + 3\bar{k}]$$

Green's Theorem

① Green's theorem statement

Statement:- If $M(x,y)$, $N(x,y)$, $\frac{\partial N}{\partial x}$ + $\frac{\partial M}{\partial y}$ are continuous function over region R bounded by closed curve C in xy plane, then

$$\oint_C [M\,dx + N\,dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

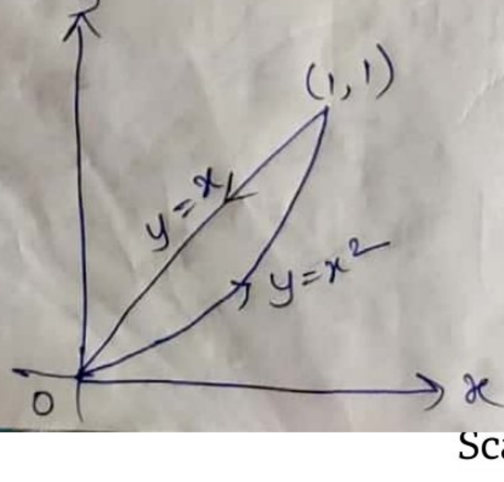
- Line integral is to be taken along entire boundary C of R

② Verify Green's theorem for

$$\int_C \left[[xy+y^2]\,dx + x^2\,dy\right], \text{ where } C \text{ is}$$

bounded by $y=x$ and $y=x^2$

⇒ The intersection of $y=x$ & $y=x^2$ is at (0,0) to (1,1) positive direction in traversing C is as shown in dig



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Along $y = x^2$

$dy = 2x\,dx$

$$\int_c [x^3 + x^4]\,dx + x^2\,2x\,dx.$$

$$= \left[\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 + 2\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{5} + 2\left[\frac{1}{4}\right]$$

$$= \frac{9}{20} + \frac{2}{4}$$

$$= \frac{9+10}{20} = \frac{19}{20}$$

Along $y = x$

$dy = dx$

$$\int_c [x^2 + x^2]\,dx + x^2\,dx$$

$$\int_1^0 2x^2\,dx + x^2\,dx.$$

$$2\left[\frac{x^3}{3}\right]_1^0 + \left[\frac{x^3}{3}\right]_1^0$$

$$= 2\left[\frac{-1}{3}\right] + \left[\frac{-1}{3}\right]$$

$$\frac{-2 + (-1)}{3} = \frac{-3}{3} = -1$$

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Adding both ; $\frac{19}{20} + (-1) = \frac{19-1}{20} = \frac{-1}{20}$

$\frac{-1}{20}$ is the line integral (L.H.S)

Now Surface integral

$$\int_0^1 \int_{x^2}^{x} \left[\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y}(xy + y^2) \right] dy\, dx$$

$$\int_0^1 \int_{x^2}^{x} \left[2x - x - 2y \right] dy\, dx$$

$$\int_0^1 \int_{x^2}^{x} \left[x - 2y \right] dy\, dx$$

$$\int_0^1 \left[xy - \cancel{2}\frac{y^2}{\cancel{2}} \right]_{x^2}^{x} dx = \int_0^1 \left(\cancel{x^2} - \cancel{x^2} - x^3 + x^4 \right) dx$$

$$= \left[\frac{-x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= -\frac{1}{4} + \frac{1}{5} = \frac{-1}{20}$$

(R.H.S)

∴ Hence prove

L.H.S = R.H.S

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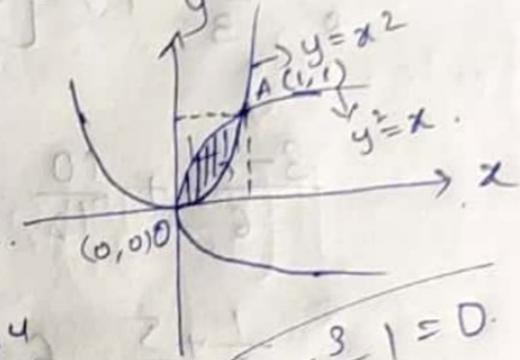
(2) Verify Green's theorem for

$$\int_C (2xy - x^2)\,dx + (x^2+y^2)\,dy,$$

where 'c' is the closed curve of the region bounded by $y=x^2$ & $y^2=x$.



$y=\sqrt{x}$ & $y=x^2$

$y^2=x$.

$\sqrt{x}=y=x^2$ $\quad \sqrt{x}=x^2$.

$x=x^4$

$x^4-x=0$

$x(x^3-1)=0$, $x=0,1$

are $(0,0)$ & $(1,1)$ $\quad x(x^3-1)=0$

$x^3-1=0$

$x^3=1$

$x=0,1$, $x^2=\pm 1$

let $M=2xy-x^2$, $N=x^2+y^2$.

$$\frac{\partial M}{\partial y}=2x \qquad \frac{\partial N}{\partial x}=2x.$$

LHS: $\oint_C M\cdot dx + N\,dy$.

Along $y=x^2$ $\quad dy=2x\,dx$

$$\int_0^1 (2x\,x^2 - x^2)\,dx + (x^2+x^4)\,2x\,dx$$

$$\int_0^1 (2x^3 - x^2)\,dx + 2(x^3+x^5)\,dx$$

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$$\left[2\frac{x^4}{4}-\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^4}{4}+\frac{x^6}{6}\right]_0^1$$

$$2\left[\frac{1}{4}\right]-\left[\frac{1}{3}\right] + 2\left[\frac{1}{4}+\frac{1}{6}\right]$$

$$\frac{1}{2}-\frac{1}{3}+\cancel{2}\left[\frac{6+4}{\cancel{24}_{12}}\right]$$

$$\frac{3-2}{6}+\frac{10}{12}$$

$$\frac{1}{6}+\frac{5}{6}=\frac{6}{6}=1.$$

Along $y^2=x$

$2y\,dy=dx$

$$\int_1^0 (2xy-x^2)dx+(x^2+y^2)dy$$

$$\int_1^0 \left[2\cdot y^2 y-y^4\right]2y\,dy+\left[y^4+y^2\right]dy$$

$$2\left[2\frac{y^5}{5}-\frac{y^6}{6}\right]_1^0+\left[\frac{y^5}{5}+\frac{y^3}{3}\right]_1^0$$

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$$2\left[2\left(\frac{-1}{5}\right)-\left(\frac{-1}{6}\right)\right]+\left[\frac{-1}{5}-\frac{1}{3}\right]$$

$$2\left[\frac{-2}{5}+\frac{1}{6}\right]+\left[\frac{-3-5}{15}\right]$$

$$\cancel{2}\left[\frac{-7}{\cancel{30}_{15}}\right]+\left[\frac{-8}{15}\right]=\frac{-15}{15}=-1$$

Adding both: $1-1=0$

"0" is the line integral (R.H.S)

Now surface integral

$$\int_0^1\int_{x^2}^{\sqrt{x}}\left[\frac{\partial}{\partial x}(x^2+y^2)-\frac{\partial}{\partial y}\left((2xy-x^2)\right)\right]dy\,dx$$

$$\int_0^1\int_{x^2}^{\sqrt{x}}\left[2x-2x\right]dy\,dx=0\ (\text{R.H.S})$$

$\therefore$ Hence proved.

L.H.S = R.H.S.

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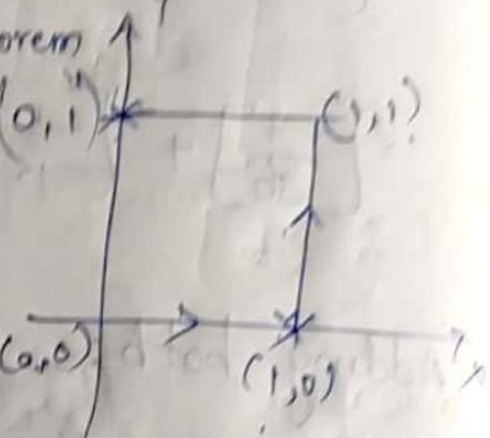
④ evaluate $\int_C [(x^2+xy)dx + (x^2+y^2)dy]$

where C is the boundary of the region bounded by the lines $x=0, x=1, y=0, y=1$

<u>Sol</u> By the Green's theorem



$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

given $\int_C [(x^2+xy)dx + (x^2+y^2)dy]$

$M = x^2 + \cancel{2}xy$, $N = x^2 + y^2$

$\frac{\partial M}{\partial y} = x \cdot 1 = x$, $\frac{\partial N}{\partial x} = 2x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x.$

x limits from 0 to 1

& y limits from 0 to 1

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$$\left.\begin{aligned}&\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy \\ &= \int_{x=0}^{1}\int_{y=0}^{1} x\,dx\,dy\end{aligned}\right\}$$ By the green's theorm.

$$= \int_{x=0}^{1} x\,[y]_0^1\,dx$$

$$= \int_{x=0}^{1} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}[1-0] = \frac{1}{2}.$$

$\therefore$ Surface integral $= \frac{1}{2}$

$$\therefore \int_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \frac{1}{2}.$$

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(7) Define Surface integral & formula?

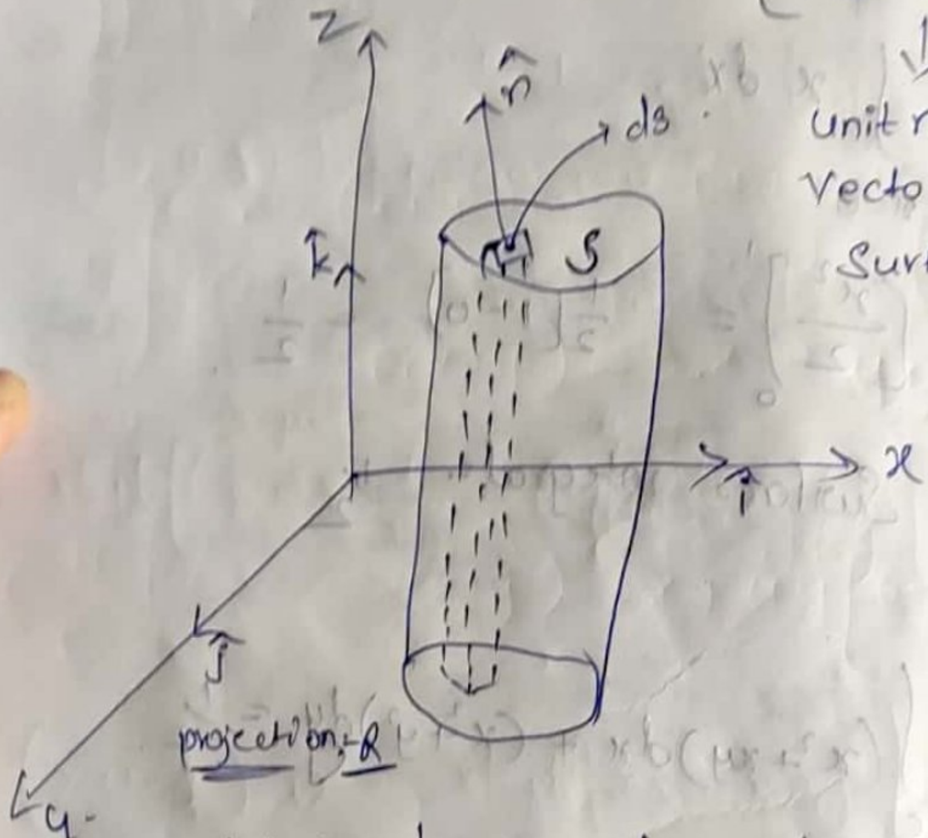
Let "S" be surface of finite area & F(...) is defined over S. then ~~...~~

$$\boxed{\iint_S F\cdot d\vec{S} = \iint_S F\cdot \hat{n}\, ds}$$

$\left[d\vec{S} = \hat{n}\cdot ds \right]$

$\downarrow$

unit normal vector to the surface



Let R be orthogonal projection on xy plane, then

$$* \quad \boxed{ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}} \quad (k \to \text{unit vector along z-axis})$$

If it is in yz plane, then

$$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

If it is in zx plane, then

$$ds = \frac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$$

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Significance

If vector field F represents flow of fluid, the surface integral.

② If $\bar{F} = (x+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$

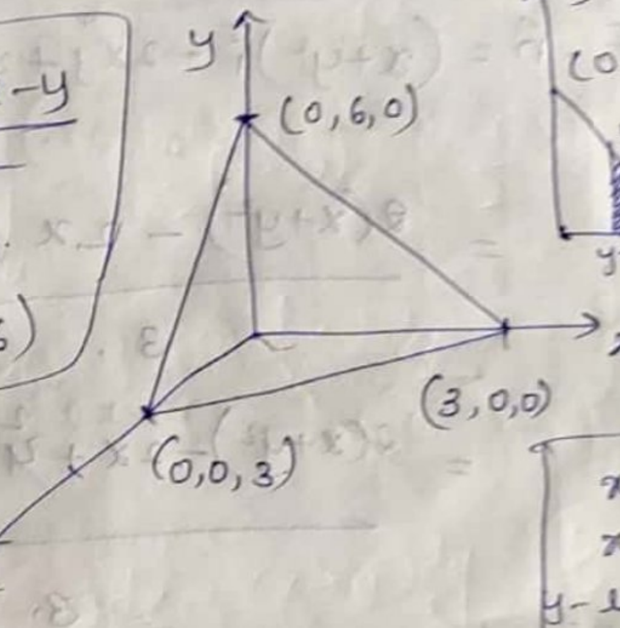
Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where S is of the surface of the plane $2x+y+2z=6$ in the first octant

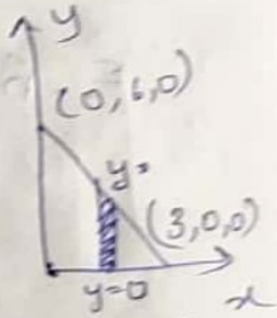
$z = \frac{6-2x-y}{2}$

if $z=0$.

$y = -(2x-6)$

$y = 6-2x$





x-limits

$x=0$ to $x=3$

y-limits

$y=0$ to $y=2x-6$

$\phi = 2x+y+2z-6=0$

$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{2i+j+2k}{\sqrt{4+1+4}}$

$= \frac{2\bar{i}+\bar{j}+2\bar{k}}{\sqrt{9}} = \frac{2\bar{i}+\bar{j}+2\bar{k}}{3}$

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$$\hat{n}\cdot\hat{k} = \left(2\bar{i}+\bar{j}+2\bar{k}/3\right)\hat{k}$$

$$\hat{n}\cdot\hat{k} = \frac{2}{3}$$

$$ds = dx\,dy\left(\frac{3}{2}\right)$$

$$\iint_S F\cdot\hat{n}\,ds.$$

$$F\cdot\hat{n} = (x+y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}\left[\frac{2\bar{i}+\bar{j}+2\bar{k}}{3}\right]$$

$$= \frac{2(x+y^2) - 2x + 4yz}{3}$$

$$= \frac{2(x+y^2) - 2x + \cancel{4}^{2}y\left[\frac{6-2x-y}{\cancel{2}}\right]}{3}$$

$$= \frac{\cancel{2x}+2y^2 - \cancel{2x} + 12y - 4xy - \cancel{2y^2}}{|3|}$$

<u>Where</u> $\iint_S F\cdot\hat{n}\,ds$

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$$\int_0^{6-2x}\int_0^3 \frac{\cancel{12}^{6}x^{2}y-\cancel{4}^{2}xy}{\cancel{3}}\left(\frac{\cancel{3}}{\cancel{x}}\right)dx\,dy$$

$$\int_0^3\int_0^{6-2x} 6y-2xy\;dy\,dx$$

$$\cancel{2}\int_0^3\left[3\left[\frac{y^2}{\cancel{2}}\right]_0^{2x-6}-x\left[\frac{y^2}{\cancel{2}}\right]_0^{2x-6}\right]dx$$

$$\cancel{2}\int_0^3\left[3\left[2x-6\right]^2-x\left[2x-6\right]^2\right]dx.$$

$$\int_0^3\left[12\,(x-3)^2-4x\left[x-3\right]^2\right]dx$$

$$\int_0^3\left[12\left[x^2+9-6x\right]-4x\left[x^2+9-6x\right]\right]dx$$

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$$4\int_0^3 \left(3\left[x^2+9-6x\right] - x\left[x^2+9-6x\right]\right) dx$$

$$4\int_0^3 \left(3x^2+27-18x-x^3-9x+6x^2\right) dx$$

$$4\int_0^3 \left(9x^2-x^3-27x+27\right) dx$$

$$4\int_0^3 \left(-x^3+9x^2-27x+27\right) dx$$

$$4\left[-\frac{x^4}{4}+9\cdot\frac{x^3}{3}-27\left[\frac{x^2}{2}\right]+27x\right]_0^3$$

$$4\left(\frac{-81}{4}+9\left(\frac{27}{3}\right)-27\left[\frac{9}{2}\right]+81\right]$$

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$$4\left[-\frac{81}{4}+81-\frac{243}{2}+81\right]$$

$$\cancel{4}\left[\frac{-81+324-486+324}{\cancel{4}}\right]=81$$

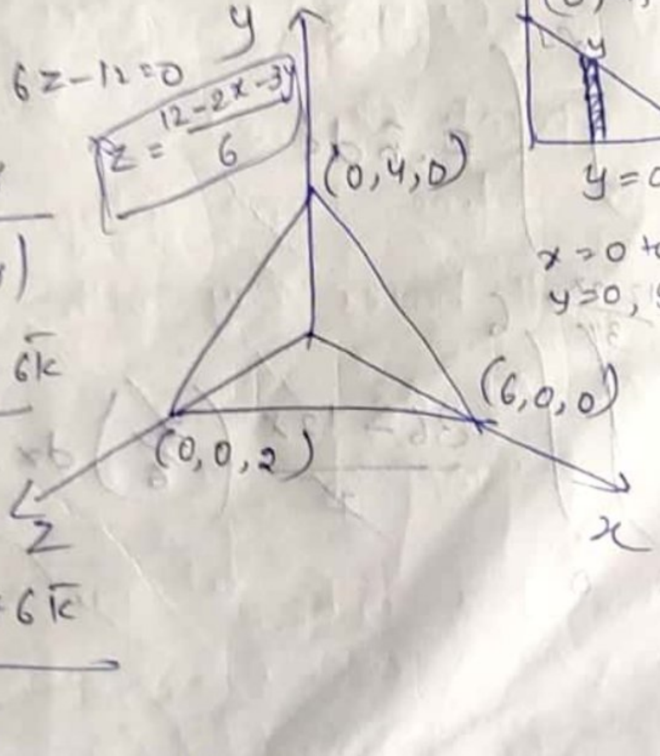
(1) Evaluate $\int \bar{F}\cdot\bar{n}\,ds$ where $\bar{F}=18z\bar{i}-12\bar{j}+xy\bar{k}$ and s is the part of the surface of the plane $2x+2y+6z=12$ located in the first octant.

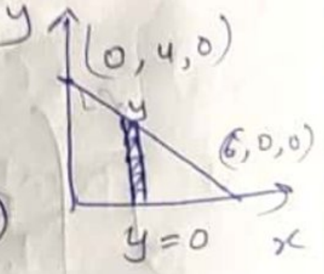
$\phi = 2x+3y+6z-12=0$

$$\boxed{z=\frac{12-2x-3y}{6}}$$

$$\hat{n}=\frac{\text{grad }\phi}{|\text{grad }\phi|}$$

$$\hat{n}=\frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{49}}$$





$x=0$ to $x=6$

$y=0,\ y=\frac{12-2x}{3}$

$$\hat{n}=\frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$$

$$\hat{n}\cdot\hat{k}=\left(\frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}\right)\cdot\bar{k}=\frac{6}{7}$$

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$$ds = dx \cdot dy \left(\tfrac{7}{6}\right)$$

$$\iint_S \bar{F} \cdot \hat{n}$$

$$\bar{F} \cdot \hat{n} = \left(18z\,\bar{i} - 12\,\bar{j} + 3y\,\bar{k}\right)\left(\frac{2\bar{i} + 3\bar{j} + 6\bar{k}}{7}\right)$$

$$= \frac{36z - 36 + 18y}{7}$$

$$= 36\left(\frac{12 - 2x - 3y}{6}\right) - 36 + 18y/7$$

$$= 72 - 12x - 18y - 36 + 18y/7$$

$$\bar{F} \cdot \hat{n} = 36 - 12x/7$$

$$\iint F \cdot \hat{n}\, ds$$

$$\int_0^{\frac{12-2x}{3}} \int_0^{6} \frac{36 - 12x}{\cancel{7}} \left(\frac{\cancel{7}}{6}\right) dx\, dy$$

$$\int_0^6 \int_0^{\frac{12-2x}{3}} \frac{36-12x}{6}\, dy\, dx$$

$$\int_0^6 \left[6y - 2xy \right]_0^{\frac{12-2x}{3}} dx$$

$$\int_0^6 \left[6\left(\frac{12-2x}{3}\right) - 2x\left(\frac{12-2x}{3}\right)\right] dx$$

$$\int_0^6 \left[24 - 4x - \frac{24x + 4x^2}{3} \right] dx$$

$$\int_0^6 \left[24 - 4x - 8x + \frac{4}{3}x^2 \right] dx$$

$$\left[24x - \frac{12x^2}{2} + \frac{4}{3}\cdot\frac{x^3}{3} \right]_0^6$$

$$\left[24\cdot 6 - \frac{12}{2}\cdot 36 + \frac{4}{9}\cdot 216 \right] = \left[144 - 216 + 96 \right]$$

$$= 24$$

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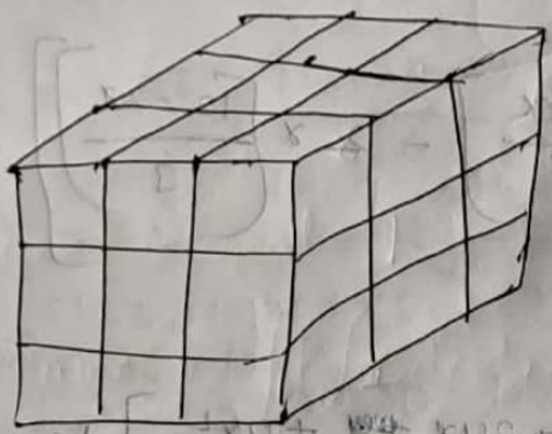
Volume Integral

→ It refers to 3-D domain

→ Consider closed surface in space enclosing volume V. then $\iiint_V F \cdot dv = \iiint_V F \cdot dx\,dy\,dz$

Significance Mass = density · V

$\Delta V = \Delta x \Delta y \Delta z$



⇒ let $f(x, y, z)$ be density at 3D solid at point (x, y, z)

⇒ To define triple integral, divide into small boxes, say with dimensions $\Delta x, \Delta y, \Delta z$

∴ volume of each box $\Delta V = \Delta x \cdot \Delta y \Delta z$

⇒ For box ijk, at point $(x_{ijk}, y_{ijk}, z_{ijk})$

density of box $= f(x_{ijk}, y_{ijk}, z_{ijk})$

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$\Rightarrow$ Mass is its density times it volume

$$\boxed{\text{mass} \doteq f(x_{ijk}, y_{ijk}, z_{ijk}) \cdot \Delta V}$$

$\rightarrow$ Summing these approximatly masses, we get total mass of solid.

Ex
$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} F \, dx\, dy\, dz$$

(1) Evaluat integral $\int_0^1\int_0^x\int_0^{1+x+y} f(x,y,z)\,dz\,dy\,dx$

where $f(x,y,z) = 1$

$$\iiint_V F\,dV = \int_0^1\int_0^x\int_0^{1+x+y} 1 \,.\, dz\,dy\,dz$$

$$= \int_0^1\int_0^x \left[z\right]_0^{1+}$$

$$= \int_0^1\int_0^x [1+x+y]\,dy\,dz$$

$$= \int_0^1 \left[y + xy + \frac{y^2}{2}\right]_0^x dz$$

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$$= \int_0^1 \left[x + x^2 + \frac{x^2}{2}\right] dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^3}{6}\right]_0^1$$

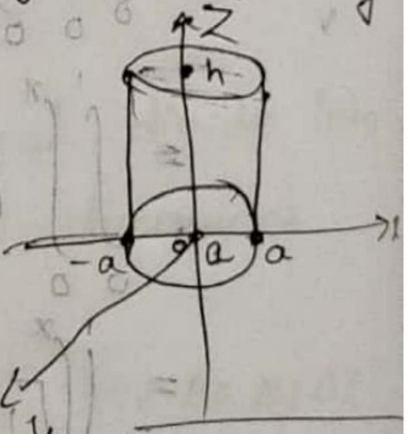
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

$$= \frac{3+2+1}{6} = \frac{6}{6} = 1.$$

Volume integral = 1

(2) Volume = $V = \iiint_V \cancel{1}\, dx\, dy\, dz$

find the volume of cylinder by using triple integral.



Sol Equation of cylinder in

$x^2 + y^2 = a^2$, $0 \le h \le a$

$$\text{Volume} = \int_{z=0}^{h} \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 1 \cdot dy\, dx\, dz$$

$x^2 + y^2 = a^2$

$y^2 = a^2 - x^2$

$y = \pm\sqrt{a^2 - x^2}$

$$\boxed{\begin{array}{c} -a \le x \le a \\ 0 \le z \le h \\ -\sqrt{a^2-x^2} \le y \le +\sqrt{a^2-x^2} \end{array}}$$

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$$V = \int_{z=0}^{h} 2 \int_{x=0}^{a} 2 \cdot [y]_{0}^{\sqrt{a^2-x^2}} \, dx \, dz$$

$$V = 4 \int_{z=0}^{h} \int_{x=0}^{a} \sqrt{a^2-x^2} \, dx \, dz$$

$$V = 4 \int_{z=0}^{h} \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_0^a dz$$

$$\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C}$$

$$V = 4 \int_{z=0}^{h} \left[0 + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) - (0) \right] dz$$

$$V = 4 \int_{z=0}^{h} \frac{a^2}{2} \sin^{-1}(1) \, dz$$

$$V = 4\frac{a^2}{2} \int_{z=0}^{h} \frac{\pi}{2} \, dz$$

$$\boxed{\because \sin^{-1}(1) = \frac{\pi}{2}}$$

$$V = 4\frac{a^2}{2} \cdot \frac{\pi}{2} \int_{z=0}^{h} 1 \cdot dz$$

Volume of cylinder $= \pi a^2 h$

we know mainly $= \pi r^2 h$

$$V = a^2 \cdot \pi \, [z]_0^h = a^2 \pi h = \pi a^2 h.$$

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5 - UNIT

1. Evaluate $\iiint (2x+y)\,dv$, where v is the closed region bounded by the cylinder $z = 4x^2$ and planes $x=0, y=0, y=2, z=0$

x limits are 0 to 2

y limits are 0 to 2

z limits are 0 to $4-x^2$

$z = 4-x^2$

$4-x^2=0$

$x^2=4$

$x=\pm 2$

x limits 0 to 2

$$\int_0^2 \int_0^2 \int_0^{4-x^2} (2x+y)\,dy\,dx\,dz$$

$$\int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{4-x^2} (2x+y)\,dz\,dx\,dy$$

$$\int_{x=0}^{2} \int_{y=0}^{2} \left[2x z + y z\right]_0^{4-x^2} dx\,dy$$

$$\int_{x=0}^{2} \int_{y=0}^{2} \left[2x(4-x^2) + y(4-x^2)\right] dx\,dy$$

$$\int_{x=0}^{2} \int_{y=0}^{2} \left[8x - 2x^3 + 4y - x^2 y\right] dy\,dx$$

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$$\int_{x=0}^{2}\left[8xy - 2x^3(y) + 4\left(\frac{y^2}{2}\right) - x^2\left(\frac{y^2}{2}\right)\right]_0^2 dx$$

$$\int_{x=0}^{2}\left(16x - 4x^3 + 8 - 2x^2\right) dx$$

$$\left[16\frac{x^2}{2} - \cancel{4}\frac{x^4}{\cancel{4}} + 8x - 2\frac{x^3}{3}\right]_0^2$$

$$16\frac{(4)}{2} - \cancel{(2)^4} + \cancel{8(2)} - 2\frac{(8)}{3}$$

$$32 - \frac{16}{3} = \frac{80}{3}$$

(2) If $\phi = 45x^2y$ evaluate $\iiint \phi \, dv$ where v is the closed region v bounded by the planes $4x + 2y + z = 8$, $y = 0$, $z = 0$

$4x + 2y + z = 8 \quad y = 0, z = 0$ — (1)

$4x + 2(0) + z(0) = 8$

$\boxed{x = 2}$ [x limits 0 to 2]

$4x+2y+z=8$

put $z=0$

$4x+2y=8$

$2y=8-4x$

$y=4-2x$

[y limits 0 to $4-2x$]

put $z=8-2y-4x$

[z limits 0 to $8-2y-4x$]

$$\int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-2y-4x} 45x^2y \, dz\,dy\,dx$$

$$\int_{x=0}^{2}\int_{y=0}^{4-2x} 45x^2y[z]_0^{8-2y-4x} dy\,dx$$

$$\int_{x=0}^{2}\int_{y=0}^{4-2x} 45x^2y\left[8-2y-4x\right] dy\,dx$$

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$$= \int_0^2 \int_0^{4-2x} \left(360x^2y - 180x^3y - 90x^2y^2\right) dx\,dy$$

$$= \int_0^2 \left[360x^2\left(\frac{y^2}{2}\right) - 180x^3\left(\frac{y^2}{2}\right) - 90x^2\left(\frac{y^3}{3}\right)\right]_0^{4-2x} dx$$

$$= \int_0^2 \frac{\cancel{360}\,180\,x^2(4-2x)^2}{\cancel{2}} \; 90x^3(4-2x)_0^2 - 30x^2[4-2x]^3 \, dx$$

$$= \int_0^2 180x^2\left[16+4x^2-16x\right] - 90x^3\left[4-2x\right]^2 - 30x^2(4-2x)^3 dx$$

$$= (16+4x^2-16x) - 30x^2(64-96x+48x^2-8x^3)$$

$$= \int_0^2 \Big[2880x^2 + 720x^4 - \cancel{2880x^3} - 1440x^3 - 360x^5$$
$$\cancel{+1440x^4} - 1920x^2 +$$
$$\cancel{2880x^3} - \cancel{1440x^4} + 240x^5\Big] dx$$

$$= \int_0^2 \left(960x^2 - 120x^5 + 720x^4 - 1440x^3\right) dx$$

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$$\int_0^2 \left(-120x^5 + 720x^4 - 1440x^3 + 960x^2\right) dx$$

$$\left[-120\left[\frac{x^6}{6}\right] + 720\left[\frac{x^5}{5}\right] - 1440\left[\frac{x^4}{4}\right] + 960\left[\frac{x^3}{3}\right]\right]_0^2$$

$$\overset{60}{\cancel{120}}\left[\frac{64}{\cancel{6}}\right] + \overset{144}{\cancel{720}}\left[\frac{32}{\cancel{5}}\right] - \overset{360}{\cancel{1440}}\left[\frac{16}{\cancel{4}}\right] + \overset{320}{\cancel{960}}\left[\frac{80}{\cancel{3}}\right]$$

$$= -1280 + 4608 - 5760 + 2560$$

$$= 128.$$

(3) Evaluate $\int \bar{F}\, dv$ when $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ and v is the region bounded by

$x=0,\ x=2,\ y=0,\ y=6,\ z=4,\ z=x^2$

$$\int_0^2\int_0^6\int_4^{x^2} (x\bar{i} + y\bar{j} + z\bar{k})\, dx\, dy\, dz$$

$$\bar{i}\int_0^2\int_0^6\int_4^{x^2} x\, dx\, dy\, dz + \bar{j}\int_0^2\int_0^6\int_4^{x^2} y\, dx\, dy\, dz + \bar{k}\int_0^2\int_0^6\int_4^{x^2} z\, dx\, dy\, dz$$

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$$= \bar{i}\int_0^2\int_0^6 x(z)_4^{x^2}\,dx\,dy + \bar{j}\int_0^2\int_0^6 y(z)_4^{x^2}\,dx\,dy + \bar{k}\int_0^2\int_0^6\left[\frac{z^2}{2}\right]_4^{x^2} dx\,dy$$

$$= \bar{i}\int_0^2\int_0^6 x(x^2-4)\,dy\,dx + \bar{j}\int_0^2\int_0^6 y(x^2-4)\,dy\,dx + \bar{k}\int_0^2\int_0^6 \frac{x^4-16}{2}\,dy\,dx$$

$$= \bar{i}\int_0^2 x^3-4x\,[y]_0^6\,dx + \bar{j}\int_0^2 (x^2-4)\left(\frac{y^2}{2}\right)_0^6 dx + \bar{k}\int_0^2\left[\frac{x^4-16}{2}\right][y]_0^6\,dx$$

$$= \bar{i}\int_0^2 (x^3-4x)[6]\,dx + \bar{j}\int_0^2 [x^2-4]\,\frac{\overset{18}{\cancel{36}}}{\cancel{2}}\,dx + \bar{k}\int_0^2 \frac{x^4-16}{\cancel{2}}\cdot\overset{3}{\cancel{6}}\,dx$$

$$= \bar{i}\int_0^2 (6x^3-24x)\,dx + \bar{j}\int_0^2 (18x^2-72)\,dx + \bar{k}\int_0^2 (3x^4-48)\,dx$$

$$= \bar{i}\left[6\left(\frac{16}{4}\right)-24\left(\frac{4}{2}\right)\right] + \bar{j}\left[18\left[\frac{8}{3}\right]-72(2)\right] + \bar{k}\left[3\left(\frac{32}{5}\right)-48(2)\right]$$

$$= i(-24) + j(-96) + \bar{k}\left(-\frac{384}{5}\right)$$

$$= -24\bar{i} - 96\bar{j} - \frac{384}{5}\bar{k}$$

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STOKE'S Theorem

(1) (Transformation b/w surface & line integral)

→ It is generalisation of Green's theorem from circulation in along the Surface.

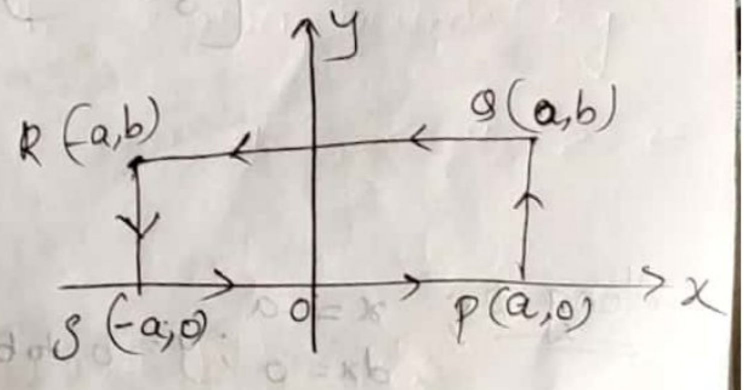
→ It states "line Integral of F over closed curve 'c' in Surface integral of curl F,

$$\boxed{\oint_c F\, dr = \iint_S \text{curl } F \cdot dS} \rightarrow \boxed{dS = \hat{n}\, ds}$$

$$\oint F \cdot dr = \iint_S \text{curl} F \cdot \hat{n}\, ds$$

(2) Verify Stokes theorem for $\vec{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ the taken around rectangle bounded by lines $x = \pm a$ & $y = 0$ to b.

$$\int_c \vec{F} \cdot dr$$



$$\int_c \bar{F} \cdot d\bar{r} = \int \left[(x^2+y^2)\bar{i} + 2xy\bar{j}\right]\left[dx\bar{i} + dy\bar{j}\right]$$

$$= (x^2+y^2)\, dx + 2xy\, dy$$

Along PQ ; $x = a$; $y = 0$ to b

$dx = 0$

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$$\int_{PQ} (x^2+y^2)\,dx - 2xy\,dy$$

$$\int_0^b -2xy\,dy = \left[-\cancel{2}x\frac{y^2}{\cancel{2}}\right]_0^b = -xb^2 = -ab^2$$

<u>Along QR;</u> $y=b$; $x = a$ to $-a$
$dy=0$

$$\int_{QR} (x^2+y^2)\,dx = \int_a^{-a} (x^2+y^2)\,dx$$

$$= \left[\frac{x^3}{3} + y^2x\right]_a^{-a} = \frac{-a^3}{3} - ay^2 - \frac{a^3}{3} - \dots$$

$$= -\frac{2a^3}{3} - 2ay^2$$

$$= -\frac{2a^3}{3} - 2ab^2$$

<u>Along RS;</u> $x=-a$
$dx=0$; $y = 0$ to b.

$$\int_{RS} -2xy\,dy = -ab^2.$$

<u>Along SP;</u> $y=0$
$dy=0$; $x = -a$ to $+a$

$$\int_{SP} (x^2+y^2)\,dx = \int_{-a}^{+a} (x^2+y^2)\,dx = \frac{+2a^3}{3} + 2ay^2 = \frac{+2a^3}{3}$$

(where $y=0$)

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Adding $= -ab^2 - \cancel{\frac{2a^3}{3}} - 2ab^2 - ab^2 + \cancel{\frac{2a^3}{3}}$

$= -4ab^2$ L.H.S.

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}.$$

$$= \hat{i}\left[0 - \frac{\partial}{\partial z}(-2xy)\right].$$

$$-\hat{j}\left[0 - \frac{\partial}{\partial z}(x^2+y^2)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2+y^2)\right]$$

$$= \hat{k}\left[\cancel{\frac{x^2}{2}} - 2y - 2y\right]$$

$$= -4y\hat{k}$$

$$\text{curl } F \cdot \hat{n} = -4y\hat{k} \cdot \hat{k} = -4y$$

$$\iint_S \text{curl } F \cdot \hat{n} \, ds = \int_{-a}^{a}\int_{0}^{b} -4y \, dy \, dx$$

$$= \int_{-a}^{a} -\cancel{4}^{2}\left[\frac{y^2}{\cancel{2}}\right]_0^b dx$$

$$= \int_{-a}^{a} -2b^2 \, dx$$

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$$-2b^2 [x]_{-a}^{a} = -2b^2 [a+a]$$
$$= -2b^2 [2a]$$
$$= -4ab^2 \quad (R.H.S)$$

$\therefore$ Hence (L.H.S = R.H.S)

(2) Evalute by Stokes theorem $\int_C e^x dx + 2y dy - dz$ where C is the curve $x^2+y^2=9$ & $z=2$

Sol By Stokes theorem $\oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\, \bar{F}\cdot \bar{n}\, ds$

$$\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$$
$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}, \quad d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$\bar{F}\cdot d\bar{r} = F_1 dx + F_2 dy + F_3 dz$$
$$= e^x dx + 2y dy - dz$$

$$F_1 = e^x, \quad F_2 = 2y, \quad F_3 = -1$$

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$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix}$$

$$\text{curl}\,\bar{F} = \bar{i}\left[\frac{\partial}{\partial y}(-1) - \frac{\partial}{\partial z}(2y)\right] - \bar{j}\left[\frac{\partial}{\partial x}(-1) - \frac{\partial}{\partial z}(e^x)\right] + \bar{k}\left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(e^x)\right]$$

$$\text{curl}\,\bar{F} = \bar{i}[0-0] - \bar{j}[0-0] + \bar{k}[0-0]$$

$$\text{curl}\,\bar{F} = 0.$$

$$\text{curl}\,\bar{F}\cdot\bar{n} = 0.$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_S \text{curl}\,\bar{F}\cdot\bar{n}\,ds = 0.$$

$$\therefore \oint e^x dx + 2y\,dy - dz = 0.$$

3. Evaluate using stokes theorem.

$\int \sin z\, dx - \cos x\, dy + \sin y\, dz$ where c is the boundary of the rectangle $0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$ [Hint: unit normal vector $\bar{n} = \bar{k}$]

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By stokes theorem $\oint_C \bar{F} \cdot d\bar{r} = \int_S \text{curl}\,\bar{F} \cdot \bar{n}\, ds$

$\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$\bar{F} \cdot d\bar{r} = F_1 dx + F_2 dy + F_3 dz$

$\bar{F} \cdot d\bar{r} = \sin z\, dx + \cos x\, dy + \sin y\, dz$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin z & -\cos x & \sin y \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(\sin y) - \frac{\partial}{\partial z}(-\cos x)\right] - \bar{j}\left[\frac{\partial}{\partial x}(\sin y) - \frac{\partial}{\partial z}(\sin z)\right]$$

$$+ \bar{k}\left[\frac{\partial}{\partial x}(-\cos x) - \frac{\partial}{\partial y}(\sin z)\right]$$

$\text{curl}\,\bar{F} = \bar{i}\,[\cos y - 0] - \bar{j}\,[0 - \cos z] + \bar{k}\,[\sin x]$

$\text{curl}\,\bar{F} \cdot \hat{n} = \left(\left[\cos y\,\bar{i} + \cos z\,\bar{j} + \sin x\,\bar{k}\right]\right) \cdot \bar{k}$

$\text{curl}\,\bar{F} \cdot \hat{n} = (\sin x\,\bar{k})\,\bar{k}$

$$\int_S \text{curl}\,\bar{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^{\pi} \sin x\, dy\, dx.$$

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$$= \int_0^1 \left[-\cos x\right]_0^{\pi} dx$$

$$= \int_0^1 -(\cos\pi - \cos 0)\, dx$$

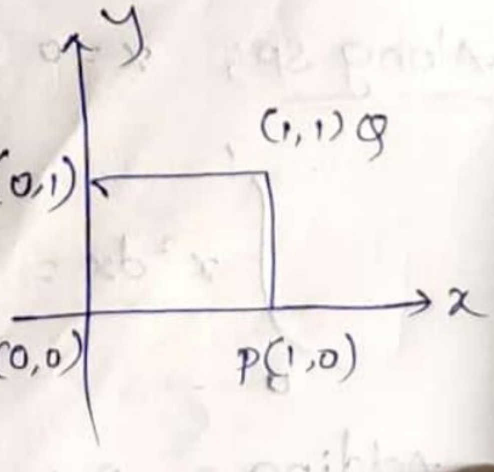
$$= \int_0^1 -[-1-1]\, dx$$

$$= \int_0^1 2\, dx = 2[x]_0^1 = 2 \text{ (R.H.S)}$$

$$\int \sin z\, dx - \cos x\, dy + \sin y\, dz = 2.$$

(4) Verify Stoke's theorem for $f = x^2\bar{i} - yz\bar{j} + \bar{k}$ integrated around the Square in the plane $z=0$, $x=0$, $y=0$, $x=1$, $y=1$.

$\bar{F} = x^2\bar{i} - yz\bar{j} + \bar{k}$



$\bar{F}\cdot d\bar{r} = x^2 dx - yz\,dy + dz$

where, xy plane, $z=0$, $dz=0$

$\bar{F}\cdot d\bar{r} = x^2 dx - yz\,dy$

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Along PQ :- $x = 1$; $y = 0$ to 1
$dx = 0$

$$\int_0^1 -yz\,dy = \frac{-y^2}{2} z = -\frac{3}{2} \text{ where } z = 0$$
$$= 0$$

Along QR ; $y = 1$
$dy = 0$; $x = 1$ to 0

$$\int_1^0 x^2 dx = \left[\frac{x^3}{3}\right]_1^0 = \left[\frac{x^3}{3}\right]_0^1 = -\frac{1}{3}$$

Along RS ; $x = 0$; $y = 1$ to 0
$dx = 0$

$$\int_0^1 -yz\,dy = \frac{-y^2}{2} z = \frac{-3}{2} \text{ where } z = 0$$
$$= 0$$

Along SP ; $x = 0$ to 1 ; $y = 0$
$dy = 0$

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Adding $= 0 + -\frac{1}{3} + 0 + \frac{1}{3} = 0$ (L.H.S)

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$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & -yz & 1 \end{vmatrix}$$

$$= \bar{i}\left[0 + \frac{yz^2}{2}\right] - \bar{j}\left[0-0\right] + \bar{k}\left[0-0\right].$$

$$\text{curl } \bar{F}.\hat{n} = \left(\bar{i}\left[\frac{yz^2}{2}\right] + 0\bar{j} + 0\bar{k}\right)\bar{k}.$$

$\hat{n} = \bar{k}$

$$\text{curl } \bar{F}\cdot\hat{n} = 0. \text{ (R.H.S)}$$

due to

xy plane

so, $\hat{n} = \hat{k}$.

$\therefore$ so L.H.S = R.H.S

$$\iint \text{curl } \bar{F}\cdot\hat{n}\, ds = 0.$$

$\therefore$ L.H.S = R.H.S.

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5-UNIT

23-04-20..

* Gauss Divergence theorem
(3)

let 'S' be the closed surface enclosing with volume V. If $\bar{F}$ is continuously differentiable vector point function then

$$\boxed{\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds}$$

where $\bar{n}$ is a outward drawn normal vector of any point of S.

* cartensian form

$$\iiint\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)dx\,dy\,dz$$

$$= \iint\left(f_1\cos\alpha+f_2\cos\beta+f_3\cos\gamma\right)ds$$

$$\left[\begin{array}{l}\because \cos\alpha\, ds = dy\, dz\\ \cos\beta\, ds = dx\, dz\\ \cos\gamma\, ds = dx\, dy\end{array}\right]$$

ds ↓ surface

dV ↓ Volume

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to find (L.H.S)

→ Evaluate triple integral $\int_S \bar{F}\cdot\bar{n}\,ds$,

$\bar{F} = xy\,\bar{i} + z^2\bar{j} + 2yz\bar{k}$ over the tr

bounded by $x=0, y=0, z=0$, & the

plane $x+y+z=1$ $\boxed{\int_V \operatorname{div}\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds.}$

divergent $\bar{F} = \operatorname{div}\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$f_1 = xy,\ f_2 = z^2,\ f_3 = 2yz.$

$\operatorname{div}\bar{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(2yz)$

$\operatorname{div}F = y + 0 + 2y = y + 2y = 3y$

$\operatorname{div}\bar{F} = 3y.$

$x+y+z=1$

$z = 1-x-y.$

$x+y = 1 \rightarrow (\because z=0)$ xy plane

$y = 1-x$

$x = 1 \rightarrow (\because y=0, z=0)$

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x limits 0 to 1

y limits 0 to 1 - x

z limits 0 to 1-x-y

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 3y \, dx \, dy \, dz$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 3y \, dz \, dy \, dx$$

$$\int_0^1 \int_0^{1-x} 3y [z]_0^{1-x-y} \, dy \, dz$$

$$\int_0^1 \int_0^{1-x} 3y [1-x-y-0] \, dy \, dx$$

$$\int_0^1 \int_0^{1-x} (3y - 3xy - 3y^2) \, dy \, dx$$

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$$\int_0^1 \left(3\left[\frac{y^2}{2}\right]_0^{1-x} - 3x\left[\frac{y^2}{2}\right]_0^{1-x} - 3\left[\frac{y^3}{3}\right]_0^{1-x}\right) dx$$

$$\int_0^1 \left[\frac{3[1-x]^2}{2} - \frac{3x[1-x]^2}{2} - \frac{3[1-x]^3}{3}\right] dx$$

$$3\int_0^1 \left[\frac{1+x^2-2x}{2} - \frac{x(1+x^2-2x)}{2} - \frac{(1-x^3-3x(1-x))}{3}\right] dx$$

$$(a-b)^3 = a^3 - b^3 - 3ab(a-b)$$

$$3\int_0^1 \left[\frac{1+x^2-2x-x-x^2+3x^2}{2} - \frac{(1-x^3-3x(1-x))}{3}\right] dx$$

$$3\int_0^1 \left[\frac{1-x^3-3x+3x^2}{2} - \frac{(1-x^3-3x(1-x))}{3}\right] dx$$

$$3\int_0^1 \left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3}\right] dx$$

$$3\int_0^1 \frac{(1-x)^3}{6_2} = \frac{1}{2}\left(\frac{(1-x)^4}{4(-1)}\right)_0^1 = \frac{1}{8}$$

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*10M

⟹ Verify Gauss divergent theorm for

$\bar{F} = (x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}$ taken

over the surface of the bounded by the

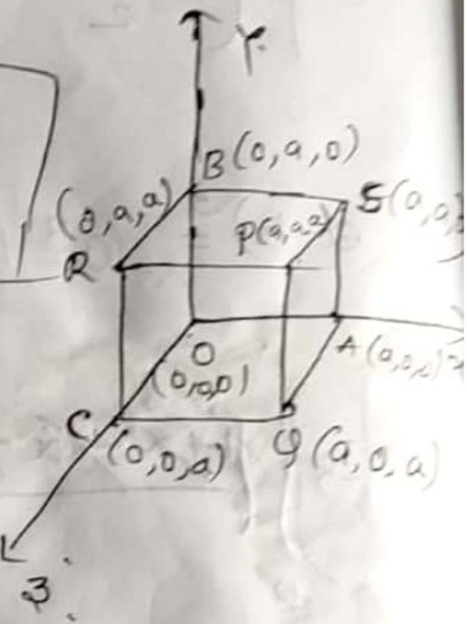
plane $x = y = z = a$ & "coordinate plane,"

$x=0, y=0, z=0$

limits

$x = 0$ to a, $y = 0 = a$, $z = a$

$$\boxed{\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F} \cdot \bar{n}\, ds}$$



O (0,0,0)
A (a,0,0)
B (0,a,0)
C (0,0,a)
P (a,a,a)
Q (a,0,a) S (a,a,0)
R (0,a,a)

Here the surfaces are ① OASB ⑥ BRPS
② PQCR
④ AQPS.
③ OBRC
⑤ OAQC

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L.H.S. $\int_V \operatorname{div} F \, dv$.

$$\operatorname{div} \bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\operatorname{div} \bar{F} = \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2 y) + \frac{\partial}{\partial z}(z)$$

$$\operatorname{div} \bar{F} = \cancel{3x^2} - \cancel{2x^2} + 1 = 3x^2 - 2x^2 + 1$$

$$\operatorname{div} \bar{F} = 1 + x^2$$

$$\int_0^a \int_0^a \int_0^a (1+x^2) \, dz \, dy \, dx$$

$$\int_0^a \int_0^a 1+x^2 \left(z\right]_0^a dy \, dx$$

$$\int_0^a \int_0^a 1+x^2 [a-0] \, dy \, dx$$

$$\int_0^a \int_0^a a(1+x^2) \, dy \, dx$$

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$$a\int_0^a\int_0^a (1+x^2)\,dy\,dx$$

$$a\int_0^a (1+x^2)\,[y]_0^a\,dx$$

$$a\int_0^a (1+x^2)\,[a-0]\,dx$$

$$a^2\int_0^a (1+x^2)\,dx$$

$$a^2\int_0^a (1+x^2)\,dx = a^2\left[x+\frac{x^3}{3}\right]_0^a$$

$$= a^2\left[a+\frac{a^3}{3}\right]$$

$$\boxed{\int_V \operatorname{div}\bar{F}\cdot dv = \frac{a^5}{3}+a^3} \quad (L.H.S) \quad (1)$$

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case:-1
OASB surface (xy plane $z=0$).

$$\int_S \bar{F}\cdot\bar{n}\, ds.$$

$O(0,0,0)$ $A(a,0,0)$, $S(a,a,0)$, $B(0,a,0)$

$\bar{n}=-\bar{k}$ $ds=dx\,dy$

$z=0 \Rightarrow dz=0$.

$$=\int_S \bar{F}\cdot\bar{n}\, ds =$$

$$=\int_S \left((x^3-yz)\bar{i}-2x^2y\bar{j}+z\bar{k}\right)(-\bar{k})\,dx\,dy.$$

$$=-\iint z\,dx\,dy.$$

$$=-\iint 0\,dx\,dy=0.$$ ——— (1)

Case:-2 $P(a,a,a)$ $Q(a,0,a)$ $C(0,0,a)$

PQCR surface (xy plane $z=a$) $R(0,a,a)$

$z=a$ $\bar{n}=\bar{k}$

$dz=0$

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$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint \left((x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right)\bar{k}\,dx\,dy$$

$$= \iint z\,dx\,dy$$

$$= \int_0^a\int_0^a a\,dx\,dy = a\int_0^a [x]_0^a\,dy$$

$$= a\int_0^a a\,dy$$

$$= a^2\,[y]_0^a = a^3 \quad \text{---} (3)$$

Case:- 3 OBRC (yz plane)

$n = -i$, $ds = d\sigma = dy\,dz$, $x = 0$

$dx = 0$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_S \left[(x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right](-\bar{i})\,dy\,dz$$

$$= \int_0^a\int_0^a (x^3 - yz)(-1)\,dy\,dz \qquad x = 0 \quad \text{SO}$$

$$= \int_0^a\int_0^a (-yz)(-1)\,dy\,dz$$

$$= \int_0^a\int_0^a yz\,dy\,dz$$

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$$= \int_0^a \left[\frac{y^2}{2}\right]_0^a z\,dz$$

$$= \int_0^a \left[\frac{a^2}{2}\right] z\,dz$$

$$= \frac{a^2}{2}\left[\frac{z^2}{2}\right]_0^a$$

$$= \frac{a^2}{2}\cdot\frac{a^2}{2} = \frac{a^4}{4} \quad (4)$$

Case:-4 ~~A~~OPS

$ds = dy\,dz$, $\bar{n} = \bar{i}$

$\boxed{x = a} \Rightarrow dx = 0$

$$\bar{F}\cdot\bar{n} = \left[(x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right]\bar{i}$$

$$\bar{F}\cdot\bar{n} = x^3 - yz$$

$$\bar{F}\cdot\bar{n} = a^3 - yz$$

$$\iint \bar{F}\cdot\bar{n} = \int_0^a\int_0^a a^3 - yz\,dy\,dz$$

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$$= \int_0^a \left[a^3 z - y\frac{z^2}{2}\right]_0^a dy$$

$$= \int_0^a \left(a^4 - y\frac{a^2}{2}\right) dy$$

$$= \left[a^4(y) - \left(\frac{y^2}{2}\right)\frac{a^2}{2}\right]_0^a$$

$$= a^5 - \frac{a^4}{4} \quad \text{--- } (5)$$

Case :- 5 OA&C. $\underline{y=0}$

$ds = dx\,dz \qquad n = -\bar{j}$

$\bar{F}\cdot\bar{n} = \left[(x^3 - yz)\bar{i} - 2x^2y\bar{j} + z\bar{k}\right](-\bar{j})$

$\bar{F}\cdot\bar{n} = 2x^2y = 0$ (where $y=0$)

$\bar{F}\cdot\bar{n} = 0.$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint 0\,dx\,dz = 0 \quad \text{--- } (6)$$

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case :- 6 BRPS

$$ds = dx\,dz \quad y = a$$

$$n = \bar{j}$$

$$\bar{F}\cdot\bar{n} = \left[(x^3 - yz)\bar{i} - 2x^2y\bar{j} + 2\bar{k}\right](\bar{j})$$

$$\bar{F}\cdot\bar{n} = -2x^2y \quad (\because y = a)$$

$$\bar{F}\cdot\bar{n} = -2x^2a$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_0^a\int_0^a -2x^2a\,dx\,dz$$

$$= -\int_0^a\int_0^a 2x^2a\,dz\,dx$$

$$= -\int_0^a 2ax^2\left[z\right]_0^a dx$$

$$= -\int_0^a 2ax^2[a]\,dx$$

$$= -2a^2\left[\frac{x^3}{3}\right]_0^a = -\frac{2a^5}{3} \text{ — } (7)$$

Add the all the points $(2) + (3) + (4) + (5) + (6) + (7)$

$$a^3 + \frac{8a^5}{3} \Leftarrow 0 + a^3 + \cancel{\frac{a^4}{4}} + a^5 - \cancel{\frac{a^4}{4}} + 0 - \frac{2a^5}{3}$$

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$\therefore$ Hence $L.H.S = R.H.S$

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