

## **MODULE-I**

# **THEORY OF MATRICES AND HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS**

## Solution for linear systems

**Matrix :** A system of  $mn$  numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order  $m \times n$ .

$$\text{Eg: } \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{12} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m \times n} \text{ where } 1 \leq i \leq m, 1 \leq j \leq n.$$

### Some types of matrices:

1. **square matrix :** A square matrix A of order  $n \times n$  is sometimes called as a n- rowed matrix A (or) simply a square matrix of order n

$$\text{eg: } \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ is } 2^{\text{nd}} \text{ order matrix}$$

2. **Rectangular matrix:** A matrix which is not a square matrix is called a rectangular matrix,

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix} \text{ is a } 2 \times 3 \text{ matrix}$$

3. **Row matrix:** A matrix of order  $1 \times m$  is called a row matrix

$$\text{eg: } [1 \quad 2 \quad 3]_{1 \times 3}$$

4. **Column matrix:** A matrix of order  $n \times 1$  is called a column matrix

$$\text{Eg: } \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3 \times 1}$$

5. **Unit matrix:** if  $A = [a_{ij}]_{n \times n}$  such that  $a_{ij} = 1$  for  $i = j$  and  $a_{ij} = 0$  for  $i \neq j$ , then A is called a unit matrix.

$$\text{Eg: } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. **Zero matrix :** if  $A = [a_{ij}]_{m \times n}$  such that  $a_{ij} = 0 \quad \forall i \text{ and } j$  then A is called a zero matrix (or) null matrix

$$\text{Eg: } O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7. **Diagonal elements in a matrix:**  $A = [a_{ij}]_{n \times n}$ , the elements  $a_{ij}$  of A for which  $i = j$ . i.e.  $(a_{11}, a_{22}, \dots, a_{nn})$  are called the diagonal elements of A

$$\text{Eg: } A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ diagonal elements are } 1, 5, 9$$

Note: the line along which the diagonal elements lie is called the principle diagonal of A

- 8. Diagonal matrix:** A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

If  $d_1, d_2, \dots, d_n$  are diagonal elements of a diagonal matrix A, then A is written as  $A = \text{diag}(d_1, d_2, \dots, d_n)$

$$\text{E.g. : } A = \text{diag}(3, 1, -2) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- 9. Scalar matrix:** A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix.

$$\text{Eg : } A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- 10. Equal matrices :** Two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are said to be equal if and only if (i) A and B are of the same type (order) (ii)  $a_{ij} = b_{ij}$  for every i & j

- 11. The transpose of a matrix:** The matrix obtained from any given matrix A, by interchanging its rows and columns is called the transpose of A. It is denoted by  $A^1$  (or)  $A^T$ .

If  $A = [a_{ij}]_{m \times n}$  then the transpose of A is  $A^1 = [b_{ji}]_{n \times m}$ , where  $b_{ji} = a_{ij}$  Also  $(A^1)^1 = A$

Note:  $A^1$  and  $B^1$  be the transposes of A and B respectively, then

- (i)  $(A^1)^1 = A$
- (ii)  $(A+B)^1 = A^1 + B^1$
- (iii)  $(KA)^1 = KA^1$ , K is a scalar
- (iv)  $(AB)^1 = B^1 A^1$

- 12. The conjugate of a matrix:** The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by  $\bar{A}$

**Note:** if  $\bar{A}$  and  $\bar{B}$  be the conjugates of A and B respectively then,

- (i)  $\overline{(\bar{A})} = A$
- (ii)  $\overline{A+B} = \bar{A} + \bar{B}$
- (iii)  $\overline{(KA)} = \bar{K} \bar{A}$  (K is a any complex number)
- (iv)  $\overline{AB} = \bar{A} \bar{B}$

$$\text{Eg ; if } A = \begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}_{2 \times 3} \text{ then } \bar{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}_{2 \times 3}$$

**13. The conjugate Transpose of a matrix**

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by  $A^\theta$

Thus  $A^0 = (\bar{A^1})$  where  $A^1$  is the transpose of A. Now  $A = [a_{ij}]_{m \times n} \Rightarrow A^0 = [b_{ij}]_{n \times m}$ , where  $b_{ij} = \bar{a_{ji}}$   
 i.e. the  $(i,j)^{th}$  element of  $A^0$  conjugate complex of the  $(j, i)^{th}$  element of A.

Eg: if  $A = \begin{bmatrix} 5 & 3-i & -2i \\ 0 & 1+i & 4-i \end{bmatrix}_{2 \times 3}$

then  $A^0 = \begin{bmatrix} 5 & 0 \\ 3+i & 1-i \\ 2i & 4+i \end{bmatrix}_{3 \times 2}$

**14.**

**(i) Upper Triangular matrix:** A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix.

Eg:  $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$  is an upper triangular matrix

**(ii) Lower triangular matrix:** A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix. i.e,  $a_{ij}=0$  for  $i < j$

Eg:  $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$  is an Lower triangular matrix

**(iii) Triangular matrix:** A matrix is said to be triangular matrix it is either an upper triangular matrix or a lower triangular matrix

**15. Symmetric matrix:** A square matrix  $A = [a_{ij}]$  is said to be symmetric if  $a_{ij} = a_{ji}$  for every i and j

Thus A is a symmetric matrix if  $A^T = A$

Eg:  $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$  is a symmetric matrix

**16. Skew – Symmetric:** A square matrix  $A = [a_{ij}]$  is said to be skew – symmetric if  $a_{ij} = -a_{ji}$  for every i and j.

E.g. :  $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$  is a skew – symmetric matrix

Thus A is a skew – symmetric iff  $A = -A^1$  or  $-A = A^1$

**Note:** Every diagonal element of a skew – symmetric matrix is necessarily zero.

Since  $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

**17. Multiplication of a matrix by a scalar.**

Let 'A' be a matrix. The matrix obtain by multiplying every element of A by a scalar K, is called the product of A by K and is denoted by KA (or) AK

Thus:  $A = [a_{ij}]_{m \times n}$  then  $KA = [ka_{ij}]_{m \times n} = k[a_{ij}]_{m \times n}$

### 18. Sum of matrices:

Let  $A = [a_{ij}]_{m \times n}$ ,  $B = [b_{ij}]_{m \times n}$  be two matrices. The matrix  $C = [c_{ij}]_{m \times n}$  where  $c_{ij} = a_{ij} + b_{ij}$  is called the sum of the matrices A and B.

The sum of A and B is denoted by  $A+B$ .

Thus  $[a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$  and  $[a_{ij} + b_{ij}]_{m \times n} = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}$

**19. The difference of two matrices:** If A, B are two matrices of the same type then  $A+(-B)$  is taken as  $A - B$

**20. Matrix multiplication:** Let  $A = [a_{ik}]_{m \times n}$ ,  $B = [b_{kj}]_{n \times p}$  then the matrix  $C = [c_{ij}]_{m \times p}$  where  $c_{ij}$  is called the product of the matrices A and B in that order and we write  $C = AB$ .

The matrix A is called the pre-factor & B is called the post – factor

**Note:** If the number of columns of A is equal to the number of rows in B then the matrices are said to be conformable for multiplication in that order.

### 21. Positive integral powers of a square matrix:

Let A be a square matrix. Then  $A^2$  is defined  $A.A$

Now, by associative law  $A^3 = A^2.A = (AA)A$

$$= A(AA) = AA^2$$

Similarly we have  $A^{m-1}A = A A^{m-1} = A^m$  where m is a positive integer

**Note:**  $I^n = I$

$$O^n = O$$

**Note 1:** Multiplication of matrices is distributive w.r.t. addition of matrices.

$$\text{i.e., } A(B+C) = AB + AC$$

$$(B+C)A = BA + CA$$

**Note 2:** If A is a matrix of order  $m \times n$  then  $AI_n = I_n A = A$

**22. Trace of A square matrix :** Let  $A = [a_{ij}]_{n \times n}$  the trace of the square matrix A is defined as  $\sum_{i=1}^n a_{ii}$ . And is

denoted by 'tr A'

$$\text{Thus tr} A = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$$

$$\text{Eg : } A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ then tr} A = a + b + c$$

**Properties:** If A and B are square matrices of order n and  $\lambda$  is any scalar, then

$$(i) \quad \text{tr} (\lambda A) = \lambda \text{tr} A$$

$$(ii) \quad \text{tr} (A+B) = \text{tr} A + \text{tr} B$$

$$(iii) \quad \text{tr}(AB) = \text{tr}(BA)$$

**23. Idempotent matrix:** If A is a square matrix such that  $A^2 = A$  then 'A' is called idempotent matrix

**24. Nilpotent Matrix:** If A is a square matrix such that  $A^m = 0$  where m is a +ve integer then A is called nilpotent matrix.

Note: If m is least positive integer such that  $A^m = 0$  then A is called nilpotent of index m

**25. Involutary :** If A is a square matrix such that  $A^2 = I$  then A is called involutory matrix.

**26. Orthogonal Matrix:** A square matrix A is said to be orthogonal if  $AA^T = A^T A = I$

**Examples:**

1. Show that  $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$  is orthogonal.

Sol: Given  $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\text{Consider } A.A^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & -\cos \theta \sin \theta + \cos \theta \sin \theta \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & \cos^2 \theta + \sin^2 \theta \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$  A is orthogonal matrix.

2. Prove that the matrix  $\frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$  is orthogonal.

Sol: Given  $A = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$$\text{Then } A^T = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$\text{Consider } A.A^T = \frac{1}{9} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A.A^T = I$$

$$\text{Similarly } A^T.A = I$$

Hence A is orthogonal Matrix

3. Determine the values of a, b, c when  $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$  is orthogonal.

Sol: - For orthogonal matrix  $AA^T = I$

$$\text{So } AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$$

$$\Rightarrow \begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Solving } 2b^2 - c^2 = 0, a^2 - b^2 - c^2 = 0$$

$$\text{We get } c = \pm \sqrt{2}b \quad a^2 = b^2 + 2b^2 = 3b^2$$

$$\Rightarrow a = \pm \sqrt{3}b$$

From the diagonal elements of I

$$4b^2 + c^2 = 1 \Rightarrow 4b^2 + 2b^2 = 1 \quad (c^2 = 2b^2)$$

$$\Rightarrow b = \pm \frac{1}{\sqrt{6}}$$

$$a = \pm \sqrt{3}b$$

$$= \pm \frac{1}{\sqrt{2}}$$

$$b = \pm \frac{1}{\sqrt{6}}$$

$$c = \pm \sqrt{2}b$$

$$= \pm \frac{1}{\sqrt{3}}$$

## 27. Determinant of a square matrix:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n \times n} \quad \text{then } |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

## 28. Minors and cofactors of a square matrix

Let  $A = [a_{ij}]_{n \times n}$  be a square matrix when from  $A$  the elements of  $i^{\text{th}}$  row and  $j^{\text{th}}$  column are deleted the determinant of  $(n-1)$  rowed matrix  $[M_{ij}]$  is called the minor of  $a_{ij}$  of  $A$  and is denoted by  $|M_{ij}|$

The signed minor  $(-1)^{i+j} |M_{ij}|$  is called the cofactor of  $a_{ij}$  and is denoted by  $A_{ij}$ .

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{then}$$

$$|A| = a_{11} |M_{11}| + a_{12} |M_{12}| + a_{13} |M_{13}| \quad (\text{or}) \\ = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$$

E.g.: Find Determinant of  $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$  by using minors and co-factors.

$$\text{Sol: } A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$\det A = 1 \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1 \begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} \\ = 1(-12-12) - 1(-4-6) + 3(-4+6) \\ = -24+10+6 = -8$$

Similarly we find  $\det A$  by using co-factors also.

Note 1: If  $A$  is a square matrix of order  $n$  then  $|KA| = K^n |A|$ , where  $k$  is a scalar.

Note 2: If  $A$  is a square matrix of order  $n$ , then  $|A| = |A^T|$



**29. Inverse of a Matrix:** Let A be any square matrix, then a matrix B, if exists such that  $AB = BA = I$  then B is called inverse of A and is denoted by  $A^{-1}$ .

Note:1  $(A^{-1})^{-1} = A$

Note 2:  $I^{-1} = I$

**30. Adjoint of a matrix:**

Let A be a square matrix of order n. The transpose of the matrix got from A by replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by adj A.

Note: For any scalar k,  $\text{adj}(kA) = k^{n-1} \text{adj } A$

**Note:** The necessary and sufficient condition for a square matrix to possess inverse is that  $|A| \neq 0$

Note: if  $|A| \neq 0$  then  $A^{-1} = \frac{1}{|A|} (\text{adj } A)$

**31. Singular and Non-singular Matrices:**

A square matrix A is said to be singular if  $|A| = 0$ .

If  $|A| \neq 0$  then 'A' is said to be non-singular.

**Note:** 1. only non-singular matrices possess inverses.

2. The product of non-singular matrices is also non-singular.

**Theorem :** If A, B are invertible matrices of the same order, then

(i).  $(AB)^{-1} = B^{-1}A^{-1}$

(ii).  $(A^{-1})^{-1} = (A^{-1})^1$

Proof: (i). we have  $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$   
 $= B^{-1}(I)B$   
 $= B^{-1}B$   
 $= I$

$(AB)^{-1} = B^{-1}A^{-1}$

(ii).  $A^{-1}A = AA^{-1} = I$

Consider  $A^{-1}A = I$

$\Rightarrow (A^{-1}A)^1 = I^1$

$\Rightarrow A^{-1} \cdot (A^{-1})^1 = I$

$\Rightarrow (A^{-1})^{-1} = (A^{-1})^1$

**Unitary matrix:**

A square matrix A such that  $(\overline{A})^T = A^{-1}$

i.e  $(\overline{A})^T A = A(\overline{A})^T = I$

If  $A^{\theta} A = I$  then A is called Unitary matrix

## PROBLEMS

1) If  $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$  then show that

A is Hermitian and iA is skew-Hermitian.

Sol: Given  $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$  then

$$\bar{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix} \text{ And } (\bar{A})^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$$

$$\therefore A = (\bar{A})^T \text{ Hence A is Hermitian matrix.}$$

Let  $B = iA$

i.e  $B = \begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$  then

$$\bar{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$(\bar{B})^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1) \begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$$

$$\therefore (\bar{B})^T = -B$$

$\therefore B = iA$  is a skew Hermitian matrix.

2) If A and B are Hermitian matrices, prove that  $AB-BA$  is a skew-Hermitian matrix.

Sol: Given A and B are Hermitian matrices

$$\therefore (\bar{A})^T = A \text{ And } (\bar{B})^T = B \text{ ----- (1)}$$

$$\text{Now } \overline{(AB-BA)}^T = (\overline{AB-BA})^T$$

$$= (\overline{AB} - \overline{BA})^T$$

$$= (\overline{AB})^T - (\overline{BA})^T = (\bar{B})^T (\bar{A})^T - (\bar{A})^T (\bar{B})^T$$

$$= BA - AB \text{ (By (1))}$$

$$= -(AB - BA)$$

Hence  $AB - BA$  is a skew-Hermitian matrix.

**3) Show that  $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$  is unitary if and only if  $a^2+b^2+c^2+d^2=1$**

Sol: Given  $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then  $\bar{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence  $A^\theta = (\bar{A})^T = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$$\begin{aligned} \therefore AA^\theta &= \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix} \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix} \\ &= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix} \end{aligned}$$

$$\therefore AA^\theta = I \text{ if and only if } a^2+b^2+c^2+d^2=1$$

**4) Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew-Hermitian matrix.**

Sol. Let  $A$  be any square matrix

$$\begin{aligned} \text{Now } (A + A^\theta)^\theta &= A^\theta + (A^\theta)^\theta \\ &= A^\theta + A \end{aligned}$$

$$(A + A^\theta)^\theta = A + A^\theta \Rightarrow A + A^\theta \text{ is a Hermitian matrix.}$$

$$\therefore \frac{1}{2}(A + A^\theta) \text{ is also a Hermitian matrix}$$

$$\begin{aligned} \text{Now } (A - A^\theta)^\theta &= A^\theta - (A^\theta)^\theta \\ &= A^\theta - A = -(A - A^\theta)^\theta \end{aligned}$$

Hence  $A - A^\theta$  is a skew-Hermitian matrix

•  $\frac{1}{2}(A - A^\theta)$  is also a skew-Hermitian matrix

### Uniqueness:

Let  $A = R + S$  be another such representation of  $A$

Where  $R$  is Hermitian and

$S$  is skew-Hermitian

$$\text{Then } A^\theta = (R + S)^\theta$$

$$= R^\theta + S^\theta$$

$$= R - S \quad (\because R^\theta = R, S^\theta = -S)$$

$$\therefore R = \frac{1}{2}(A + A^\theta) = P \text{ and } S = \frac{1}{2}(A - A^\theta) = Q$$

Hence  $P = R$  and  $Q = S$

Thus the representation is unique.

**5) Given that  $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ , show that  $(I - A)(I + A)^{-1}$  is a unitary matrix.**

$$\text{Sol: we have } I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \text{ And}$$

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (I + A)^{-1} = \frac{1}{1 - (4i^2 - 1)} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\text{Let } B = (I - A)(I + A)^{-1}$$

$$B = \frac{1}{6} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ (-1-2i)(1-2i)+1 & 1 \end{bmatrix}$$

$$B = \frac{1}{6} \begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

$$\text{Now } \bar{B} = \frac{1}{6} \begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix} \text{ and } (\bar{B})^T = \frac{1}{6} \begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$B(\bar{B})^T = \frac{1}{36} \begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$= \frac{1}{36} \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$(\bar{B})^T = B^{-1}$$

i.e., B is unitary matrix.

$\therefore (I-A)(I+A)^{-1}$  is a unitary matrix.

**6) Show that the inverse of a unitary matrix is unitary.**

Sol: Let A be a unitary matrix. Then  $AA^\theta = I$

$$\text{i.e. } (AA^\theta)^{-1} = I^{-1}$$

$$\Rightarrow (A^\theta)^{-1} A^{-1} = I$$

$$\Rightarrow (A^{-1})^\theta A^{-1} = I$$

Thus  $A^{-1}$  is unitary.

## Problems

**1). Express the matrix A as sum of symmetric and skew – symmetric matrices. Where**

$$A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$$

$$\text{Sol: Given } A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$$

$$\text{Then } A^T = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$$

Matrix A can be written as  $A = \frac{1}{2} (A+A^T) + \frac{1}{2} (A-A^T)$

$$\begin{aligned} \Rightarrow P = \frac{1}{2} (A+A^T) &= \frac{1}{2} \left\{ \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & +2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix} \right\} \\ &= \frac{1}{2} \begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 7 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix} \end{aligned}$$

$$Q = \frac{1}{2} (A-A^T)$$

$$\begin{aligned} &= \frac{1}{2} \left\{ \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix} \right\} = \frac{1}{2} \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix}_s \end{aligned}$$

$A = P+Q$  where 'P' is symmetric matrix  
'Q' is skew-symmetric matrix.

**Sub – Matrix:** Any matrix obtained by deleting some rows or columns or both of a given matrix is called is sub matrix.

E.g.: Let  $A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$ . Then  $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2 \times 3}$  is a sub matrix of A obtained by deleting first row and 4<sup>th</sup> column of A.

**Minor of a Matrix:** Let A be an m x n matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is 't' then its determinant is called a minor of order is 't'.

$$\text{Eg: } A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{4 \times 3} \text{ be a matrix}$$

$$\rightarrow B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \text{ is a sub-matrix of order '2'}$$

$$|B| = 2-3 = -1 \text{ is a minor of order '2'}$$

$$\rightarrow C = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix} \text{ is a sub-matrix of order '3'}$$

$$\begin{aligned} \det C &= 2(7-12) - 1(21-10) + (18-5) \\ &= 2(-5) - 1(11) + 1(13) \\ &= -10 - 11 + 13 = -8 \text{ is a minor of order '3'} \end{aligned}$$

#### **\*Rank of a Matrix:**

Let A be m x n matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-zero matrix, we say that r is the rank of A if

- (i) Every  $(r+1)^{\text{th}}$  order minor of A is '0' (zero) &
- (ii) At least one  $r^{\text{th}}$  order minor of A which is not zero.

**Note:** 1. It is denoted by  $\rho(A)$

2. Rank of a matrix is unique.

3. Every matrix will have a rank.

4. If A is a matrix of order mxn,

$$\text{Rank of } A \leq \min(m, n)$$

5. If  $\rho(A) = r$  then every minor of A of order  $r+1$ , or more is zero.

6. Rank of the Identity matrix  $I_n$  is n.

7. If A is a matrix of order n and A is non-singular then  $\rho(A) = n$

#### **Important Note:**

- 1. The rank of a matrix is  $\leq r$  if all minors of  $(r+1)^{\text{th}}$  order are zero.
- 2. The rank of a matrix is  $\geq r$ , if there is at least one minor of order 'r' which is not equal to zero.

#### **PROBLEMS**

1. Find the rank of the given matrix  $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix A =

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

$$\begin{aligned} \rightarrow \det A &= 1(48-40)-2(36-28)+3(30-28) \\ &= 8-16+6 = -2 \neq 0 \end{aligned}$$

We have minor of order 3

$$\rho(A) = 3$$

2. Find the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$$

Sol: Given the matrix is of order 3x4

$$\text{Its Rank} \leq \min(3,4) = 3$$

Highest order of the minor will be 3.

Let us consider the minor

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$$

Determinant of minor is  $1(-49)-2(-56)+3(35-48)$

$$= -49+112-39 = 24 \neq 0.$$

Hence rank of the given matrix is '3'.

**\* Elementary Transformations on a Matrix:**

i). Interchange of  $i^{\text{th}}$  row and  $j^{\text{th}}$  row is denoted by  $R_i \leftrightarrow R_j$

(ii). If  $i^{\text{th}}$  row is multiplied with  $k$  then it is denoted by  $R_i \rightarrow K R_i$

(iii). If all the elements of  $i^{\text{th}}$  row are multiplied with  $k$  and added to the corresponding elements of  $j^{\text{th}}$  row then it is denoted by  $R_j \rightarrow R_j + K R_i$

**Note:** 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$$c_i \leftrightarrow c_j, \quad c_i \rightarrow k c_j \quad c_j \rightarrow c_j + k c_i$$

2. The elementary operations on a matrix do not change its rank.

**Equivalence of Matrices:** If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.

It is denoted as  $B \sim A$ .

**Note :** 1. If A and B are two equivalent matrices, then  $\text{rank } A = \text{rank } B$ .



### Echelon form of a matrix:

A matrix is said to be in Echelon form, if

- (i). Zero rows, if any exists, they should be below the non-zero row.
- (ii). the first non-zero entry in each non-zero row is equal to '1'.
- (iii). the number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

**Note:** 1. the number of non-zero rows in echelon form of A is the rank of 'A'.

2. The rank of the transpose of a matrix is the same as that of original matrix.

3. The condition (ii) is optional.

E.g.: 1. 
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
 is a row echelon form.

3. 
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
 is a row echelon form.

## PROBLEMS

1. Find the rank of the matrix  $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$  by reducing it to Echelon form.

sol: Given  $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A.

$$A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix} \quad R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 \rightarrow R_2/7, R_3 \rightarrow R_3/9$$

$$\sim \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

This is the Echelon form of matrix A.

The rank of a matrix A.

= Number of non – zero rows =2

**2. For what values of k the matrix**  $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$  **has rank '3'.**

Sol: The given matrix is of the order 4x4

If its rank is 3  $\Rightarrow \det A = 0$

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

Applying  $R_2 \rightarrow 4R_2 - R_1$ ,  $R_3 \rightarrow 4R_3 - kR_1$ ,  $R_4 \rightarrow 4R_4 - 9R_1$

$$\text{We get } A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$$

Since Rank A = 3  $\Rightarrow \det A = 0$

$$\Rightarrow 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$$

$$\Rightarrow 1[(8-4k)3] - 1(8-4k)(4k+27) = 0$$

$$\Rightarrow (8-4k)(3-4k-27) = 0$$

$$\Rightarrow (8-4k)(-24-4k) = 0$$

$$\Rightarrow (2-k)(6+k) = 0$$

$$\Rightarrow k = 2 \text{ or } k = -6$$

**Normal Form:**

Every  $m \times n$  matrix of rank  $r$  can be reduced to the form  $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$  (or)  $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$  (or)  $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$  by

a finite number of elementary transformations, where  $I_r$  is the  $r$  – rowed unit matrix.

Note: 1. If  $A$  is an  $m \times n$  matrix of rank  $r$ , there exists non-singular matrices  $P$  and  $Q$  such that  $PAQ =$

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

**Normal form another name is “canonical form”**

e.g.: By reducing the matrix  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$  into normal form, find its rank.

Sol: Given  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix} R_3 \rightarrow R_3 / -2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix} R_3 \rightarrow R_3 + R_2$$

$$A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{array}{l} c_2 \rightarrow c_2 - 2c_1, c_3 \rightarrow c_3 - 3c_1, c_4 \rightarrow c_4 - 4c_1 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix} \begin{array}{l} c_3 \rightarrow 3c_3 - 2c_2, c_4 \rightarrow 3c_4 - 5c_2 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} c_2 \rightarrow c_2 / -3, c_4 \rightarrow c_4 / 18 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad c_4 \leftrightarrow c_3$$

This is in normal form  $[I_3 \ 0]$

Hence Rank of A is '3'.

### Gauss – Jordan method

- The inverse of a matrix by elementary Transformations: (Gauss – Jordan method)
  1. suppose A is a non-singular matrix of order 'n' then we write  $A = I_n A$
  2. Now we apply elementary row-operations only to the matrix A and the pre-factor  $I_n$  of the R.H.S
  3. We will do this till we get  $I_n = BA$  then obviously B is the inverse of A.

Find the inverse of the matrix A using elementary operations where  $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Sol:

$$\text{Given } A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

We can write  $A = I_3 A$

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying  $R_3 \rightarrow 2R_3 - R_2$ , we get

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying  $R_1 \rightarrow R_1 - 3R_2$ , we get

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying  $R_1 \rightarrow R_1 + 5R_3$ ,  $R_2 \rightarrow R_2 - 3R_3$ , we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying  $R_2 \rightarrow R_2/2$ , we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} A \Rightarrow I_3 = BA$$

B is the inverse of A.

$$\begin{aligned} &= 1(10+6)-2(15-1)+3(-18+2) \\ &= 16+32-48=0 \end{aligned}$$

# HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

## LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

**Definition:** An equation of the form  $\frac{d^n y}{dx^n} + P_1(x) \frac{d^{n-1} y}{dx^{n-1}} + P_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1}(x) \frac{dy}{dx} + P_n(x) y = Q(x)$  is called a linear differential equation of order n.

Where  $P_1(x), P_2(x), P_3(x), \dots, P_n(x)$  and  $Q(x)$  (functions of x) continuous is called a linear differential equation of order n.

### **LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS**

Def: An equation of the form  $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = Q(x)$  where  $P_1, P_2, P_3, \dots, P_n$ , are real constants and  $Q(x)$  is a continuous function of x is called an linear differential equation of order 'n' with constant coefficients.

Note:

- Operator  $D = \frac{d}{dx}$  ;  $D^2 = \frac{d^2}{dx^2}$  ; .....  $D^n = \frac{d^n}{dx^n}$   
 $Dy = \frac{dy}{dx}$  ;  $D^2 y = \frac{d^2 y}{dx^2}$  ; .....  $D^n y = \frac{d^n y}{dx^n}$
- Operator  $\frac{1}{D} Q = \int Q$  i.e  $D^{-1} Q$  is called the integral of Q.

### **To find the general solution of $f(D).y = 0$ :**

Where  $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_n$  is a polynomial in D.

Now consider the auxiliary equation :  $f(m) = 0$

i.e  $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2} + \dots + P_n = 0$

where  $p_1, p_2, p_3, \dots, p_n$  are real constants.

Let the roots of  $f(m) = 0$  be  $m_1, m_2, m_3, \dots, m_n$ .

Depending on the nature of the roots we write the complementary function as follows:

**Consider the following table**

S.No	Roots of A.E $f(m) = 0$	Complementary function(C.F)
1.	$m_1, m_2, \dots, m_n$ are real and distinct.	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$
2.	$m_1, m_2, \dots, m_n$ are and two roots are equal i.e., $m_1, m_2$ are equal and real(i.e repeated twice) & the rest are real and different.	$y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
3.	$m_1, m_2, \dots, m_n$ are real and three roots are equal i.e., $m_1, m_2, m_3$ are equal and real(i.e repeated thrice) & the rest are real and different.	$y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
4.	Two roots of A.E are complex say $\alpha + i\beta$ and $\alpha - i\beta$ and rest are real and distinct.	$y_c = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
5.	If $\alpha \pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$
6.	If $\alpha \pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x} [(c_1 + c_2 x + c_3 x^2) \cos \beta x + (c_4 + c_5 x + c_6 x^2) \sin \beta x] + c_7 e^{m_7 x} + \dots + c_n e^{m_n x}$
7.	If roots of A.E. irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x} [c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

Solve the following Differential equations :

1. Solve  $\frac{d^3 y}{dx^3} - 3 \frac{dy}{dx} + 2y = 0$

**Sol:** Given equation is of the form  $f(D).y = 0$

Where  $f(D) = (D^3 - 3D + 2) y = 0$

Now consider the auxiliary equation  $f(m) = 0$

$$f(m) = m^3 - 3m + 2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$$

$$\Rightarrow m = 1, 1, -2$$

Since  $m_1$  and  $m_2$  are equal and  $m_3$  is -2

We have  $y_c = (c_1 + c_2 x) e^x + c_3 e^{-2x}$

2. Solve  $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

Sol: Given  $f(D) = (D^4 - 2D^3 - 3D^2 + 4D + 4) y = 0$

$$\Rightarrow \text{A. equation } f(m) = (m^4 - 2m^3 - 3m^2 + 4m + 4) = 0$$

$$\Rightarrow (m+1)^2 (m-2)^2 = 0$$

$$\Rightarrow m = -1, -1, 2, 2$$

$$\Rightarrow y_c = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}$$

### 3. Solve $(D^4 + 8D^2 + 16)y = 0$

Sol: Given  $f(D) = (D^4 + 8D^2 + 16)y = 0$

Auxiliary equation  $f(m) = (m^4 + 8m^2 + 16) = 0$

$$\Rightarrow (m^2 + 4)^2 = 0$$

$$\Rightarrow (m+2i)^2 (m-2i)^2 = 0$$

$$\Rightarrow m = 2i, 2i, -2i, -2i$$

$$Y_c = e^{0x} [(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$$

### 4. Solve $y^{11} + 6y^1 + 9y = 0$ ; $y(0) = -4$ , $y^1(0) = 14$

Sol: Given equation is  $y^{11} + 6y^1 + 9y = 0$

$$f(D)y = 0 \Rightarrow (D^2 + 6D + 9)y = 0$$

$$\text{Auxiliary equation } f(m) = 0 \Rightarrow (m^2 + 6m + 9) = 0$$

$$\Rightarrow m = -3, -3$$

$$y_c = (c_1 + c_2 x)e^{-3x} \text{ -----} > (1)$$

$$\text{Differentiate of (1) w.r.to } x \Rightarrow y^1 = (c_1 + c_2 x)(-3e^{-3x}) + c_2(e^{-3x})$$

$$\text{Given } y_1(0) = 14 \Rightarrow c_1 = -4 \text{ \& } c_2 = 2$$

$$\text{Hence we get } y = (-4 + 2x)(e^{-3x})$$

### 5. Solve $4y^{111} + 4y^{11} + y^1 = 0$

Sol: Given equation is  $4y^{111} + 4y^{11} + y^1 = 0$

That is  $(4D^3 + 4D^2 + D)y = 0$

Auxiliary equation  $f(m) = 0$

$$4m^3 + 4m^2 + m = 0$$

$$m(4m^2 + 4m + 1) = 0$$

$$m(2m + 1)^2 = 0$$

$$m = 0, -1/2, -1/2$$

$$y = c_1 + (c_2 + c_3 x)e^{-x/2}$$

### 6. Solve $(D^2 - 3D + 4)y = 0$

Sol: Given equation  $(D^2 - 3D + 4)y = 0$

A.E.  $f(m) = 0$

$$m^2 - 3m + 4 = 0$$

$$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}$$

$$\alpha \pm i\beta = \frac{3 \pm i\sqrt{7}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$y = e^{\frac{3}{2}x} (c_1 \cos \frac{\sqrt{7}}{2}x + c_2 \sin \frac{\sqrt{7}}{2}x)$$

### General solution of $f(D)y = Q(x)$

Is given by  $y = y_c + y_p$

$$\text{i.e. } y = C.F + P.I$$



Where the P.I consists of no arbitrary constants and P.I of  $f(D) y = Q(x)$

Is evaluated as  $P.I = \frac{1}{f(D)} \cdot Q(x)$

Depending on the type of function of  $Q(x)$ .

P.I is evaluated as follows:

**1. P.I of  $f(D) y = Q(x)$  where  $Q(x) = e^{ax}$  for  $(a) \neq 0$**

$$\text{Case 1: } P.I = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

Provided  $f(a) \neq 0$

Case 2: If  $f(a) = 0$  then the above method fails. Then

$$\text{if } f(D) = (D-a)^k \phi(D)$$

(i.e 'a' is a repeated root k times).

$$\text{Then } P.I = \frac{1}{\phi(a)} e^{ax} \cdot \frac{1}{k!} x^k \text{ provided } \phi(a) \neq 0$$

**2. P.I of  $f(D) y = Q(x)$  where  $Q(x) = \sin ax$  or  $Q(x) = \cos ax$  where 'a' is constant then  $P.I = \frac{1}{f(D)} \cdot Q(x)$ .**

$$\text{Case 1: In } f(D) \text{ put } D^2 = -a^2 \Rightarrow f(-a^2) \neq 0 \text{ then } P.I = \frac{\sin ax}{f(-a^2)}$$

Case 2: If  $f(-a^2) = 0$  then  $D^2 + a^2$  is a factor of  $\phi(D^2)$  and hence it is a factor of  $f(D)$ . Then let  $f(D) = (D^2 + a^2) \cdot \Phi(D^2)$ .

$$\text{Then } \frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2 + a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)} \frac{\sin ax}{D^2 + a^2} = \frac{1}{\Phi(-a^2)} \frac{-x \cos ax}{2a}$$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2 + a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)} \frac{\cos ax}{D^2 + a^2} = \frac{1}{\Phi(-a^2)} \frac{x \sin ax}{2a}$$

**3. P.I for  $f(D) y = Q(x)$  where  $Q(x) = x^k$  where k is a positive integer  $f(D)$  can be express as  $f(D) = [1 \pm \phi(D)]$**

$$\text{Express } \frac{1}{f(D)} = \frac{1}{1 \pm \phi(D)} = [1 \pm \phi(D)]^{-1}$$

$$\text{Hence } P.I = \frac{1}{1 \pm \phi(D)} Q(x).$$

$$= [1 \pm \phi(D)]^{-1} \cdot x^k$$

**4. P.I of  $f(D) y = Q(x)$  when  $Q(x) = e^{ax} V$  where 'a' is a constant and V is function of x. where  $V = \sin ax$  or  $\cos ax$  or  $x^k$**

$$\text{Then } P.I = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} e^{ax} V$$

$$= e^{ax} \left[ \frac{1}{f(D+a)} (V) \right]$$

&  $\frac{1}{f(D+a)}$  V is evaluated depending on V.

5. P.I of  $f(D)$   $y = Q(x)$  when  $Q(x) = x^m V$  where V is a function of x.

$$\begin{aligned} \text{Then P.I} &= \frac{1}{f(D)} Q(x) \\ &= \frac{1}{f(D)} x^m V \\ &= \left[ x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} V \end{aligned}$$

6. i. P.I. of  $f(D)y=Q(x)$  where  $Q(x)=x^m v$  where v is a function of x.

$$\begin{aligned} \text{Then P.I.} &= \frac{1}{f(D)} \times Q(x) = \frac{1}{f(D)} x^m v = I.P. \text{ of } \frac{1}{f(D)} x^m (\cos ax + i \sin ax) \\ &= I.P. \text{ of } \frac{1}{f(D)} x^m e^{iax} \end{aligned}$$

$$\text{ii. P.I.} = \frac{1}{f(D)} x^m \cos ax = R.P. \text{ of } \frac{1}{f(D)} x^m e^{iax}$$

### Formulae

1.  $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$
2.  $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$
3.  $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$
4.  $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$
5.  $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \dots$
6.  $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \dots$

### I. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS:

1. Find the Particular integral of  $f(D)$   $y = e^{ax}$  when  $f(a) \neq 0$
2. Solve the D.E  $(D^2 + 5D + 6) y = e^x$
3. Solve  $y^{(4)} + 4y'' + 4y = 4e^{3x}$ ;  $y(0) = -1$ ,  $y'(0) = 3$
4. Solve  $y^{(4)} + 4y'' + 4y = 4\cos x + 3\sin x$ ,  $y(0) = 1$ ,  $y'(0) = 0$
5. Solve  $(D^2 + 9) y = \cos 3x$

6. Solve  $y^{11} + 2y^{11} - y^1 - 2y = 1 - 4x^3$

7. Solve the D.E  $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$

8. Solve the D.E  $(D^3 - 4D^2 - D + 4)y = e^{3x} \cos 2x$

9. Solve  $(D^2 - 4D + 4)y = x^2 \sin x + e^{2x} + 3$

10. Apply the method of variation parameters to solve  $\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$

11. Solve  $\frac{dx}{dt} = 3x + 2y$ ,  $\frac{dy}{dt} = 5x + 3y = 0$

12. Solve  $(D^2 + D - 3)y = x^2 e^{-3x}$

13. Solve  $(D^2 - D - 2)y = 3e^{2x}$ ,  $y(0) = 0$ ,  $y'(0) = -2$

## SOLUTIONS:

### 1) Particular integral of $f(D)y = e^{ax}$ when $f(a) \neq 0$

Working rule:

Case (i):

In  $f(D)$ , put  $D=a$  and Particular integral will be calculated.

$$\text{Particular integral} = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \text{ provided } f(a) \neq 0$$

Case (ii) :

If  $f(a) = 0$ , then above method fails. Now proceed as below.

$$\text{If } f(D) = (D-a)^k \phi(D)$$

i.e. 'a' is a repeated root k times, then

$$\text{Particular integral} = \frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!} \text{ provided } \phi(a) \neq 0$$

### 2. Solve the Differential equation $(D^2 + 5D + 6)y = e^x$

Sol : Given equation is  $(D^2 + 5D + 6)y = e^x$

Here  $Q(x) = e^x$

Auxiliary equation is  $f(m) = m^2 + 5m + 6 = 0$

$$m^2 + 3m + 2m + 6 = 0$$

$$m(m+3) + 2(m+3) = 0$$

$$m = -2 \text{ or } m = -3$$

The roots are real and distinct

$$\text{C.F} = y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

$$\text{Particular Integral} = y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2 + 5D + 6} e^x = \frac{1}{(D+2)(D+3)} e^x$$

Put  $D = 1$  in  $f(D)$

$$\text{P.I.} = \frac{1}{(3)(4)} e^x$$

$$\text{Particular Integral} = y_p = \frac{1}{12} \cdot e^x$$

General solution is  $y = y_c + y_p$

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12}$$

**3) Solve  $y'' - 4y' + 3y = 4e^{3x}$ ,  $y(0) = -1$ ,  $y'(0) = 3$**

Sol : Given equation is  $y'' - 4y' + 3y = 4e^{3x}$

$$\text{i.e. } \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = 4e^{3x}$$

it can be expressed as

$$D^2 y - 4Dy + 3y = 4e^{3x}$$

$$(D^2 - 4D + 3)y = 4e^{3x}$$

Here  $Q(x) = 4e^{3x}$ ;  $f(D) = D^2 - 4D + 3$

Auxiliary equation is  $f(m) = m^2 - 4m + 3 = 0$

$$m^2 - 3m - m + 3 = 0$$

$$m(m-3) - 1(m-3) = 0 \Rightarrow m = 3 \text{ or } 1$$

The roots are real and distinct.

$$\text{C.F.} = y_c = c_1 e^{3x} + c_2 e^x \rightarrow (2)$$

$$\text{P.I.} = y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= y_p = \frac{1}{D^2 - 4D + 3} \cdot 4e^{3x}$$

$$= y_p = \frac{1}{(D-1)(D-3)} \cdot 4e^{3x}$$

Put  $D = 3$

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2} \frac{e^{3x}}{(D-3)} = 2 \frac{x^1}{1!} e^{3x} = 2xe^{3x}$$

General solution is  $y = y_c + y_p$

$$y = c_1 e^{3x} + c_2 e^x + 2xe^{3x} \rightarrow (3)$$

$$y^1 = 3c_1 e^{3x} + c_2 e^x + 2e^{3x} + 6xe^{3x} \longrightarrow (4)$$

By data,  $y(0) = -1$ ,  $y^1(0) = 3$

From (3),  $-1 = c_1 + c_2 \longrightarrow (5)$

From (4),  $3 = 3c_1 + c_2 + 2$

$$3c_1 + c_2 = 1 \longrightarrow (6)$$

Solving (5) and (6) we get  $c_1 = 1$  and  $c_2 = -2$

$$y = -2e^x + (1+2x)e^{3x}$$

**(4). Solve  $y^{11} + 4y^1 + 4y = 4\cos x + 3\sin x$ ,  $y(0) = 0$ ,  $y^1(0) = 0$**

Sol: Given differential equation in operator form

$$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$$

A.E is  $m^2 + 4m + 4 = 0$

$$(m+2)^2 = 0 \quad \text{then } m = -2, -2$$

∴ C.F is  $y_c = (c_1 + c_2 x)e^{-2x}$

P.I is  $y_p = \frac{4\cos x + 3\sin x}{(D^2 + 4D + 4)} \quad \text{put } D^2 = -1$

$$y_p = \frac{4\cos x + 3\sin x}{(4D + 3)} = \frac{(4D - 3)(4\cos x + 3\sin x)}{(4D - 3)(4D + 3)}$$

$$= \frac{(4D - 3)(4\cos x + 3\sin x)}{16D^2 - 9}$$

Put  $D^2 = -1$

$$\therefore y_p = \frac{(4D - 3)(4\cos x + 3\sin x)}{-16 - 9}$$

$$= \frac{-16\sin x + 12\cos x - 12\cos x - 9\sin x}{-25} = \frac{-25\sin x}{-25} = \sin x$$

∴ General equation is  $y = y_c + y_p$

$$y = (c_1 + c_2 x)e^{-2x} + \sin x \longrightarrow (1)$$

By given data,  $y(0) = 0$  ∴  $c_1 = 0$  and

Diff (1) w.r.t.  $y^1 = (c_1 + c_2 x)(-2)e^{-2x} + e^{-2x}(c_2) + \cos x \longrightarrow (2)$

given  $y^1(0) = 0$

$$(2) \Rightarrow -2c_1 + c_2 + 1 = 0 \quad \therefore c_2 = -1$$

### 5. Solve $(D^2+9)y = \cos 3x$

Sol: Given equation is  $(D^2+9)y = \cos 3x$

A.E is  $m^2+9 = 0$

$$\therefore m = \pm 3i$$

$$y_c = C.F = c_1 \cos 3x + c_2 \sin 3x$$

$$y_c = P.I = \frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$$

$$= \frac{x}{2(3)} \sin 3x = \frac{x}{6} \sin 3x$$

General equation is  $y = y_c + y_p$

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{6} \sin 3x$$

### 6. Solve $y^{111} + 2y^{11} - y^1 - 2y = 1 - 4x^3$

Sol: Given equation can be written as

$$(D^3 + 2D^2 - D - 2)y = 1 - 4x^3$$

$$\text{A.E is } (m^3 + 2m^2 - m - 2) = 0$$

$$(m^2 - 1)(m+2) = 0$$

$$m^2 = 1 \text{ or } m = -2$$

$$m = 1, -1, -2$$

$$C.F = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$$

$$P.I = \frac{1}{(D^3 + 2D^2 - D - 2)} (1 - 4x^3)$$

$$= \frac{-1}{2 \left[ 1 - \frac{(D^3 + 2D^2 - D)}{2} \right]} (1 - 4x^3)$$

$$= \frac{-1}{2} \left[ 1 - \frac{(D^3 + 2D^2 - D)}{2} \right]^{-1} (1 - 4x^3)$$

$$\frac{-1}{2} \left[ 1 + \frac{(D^3 + 2D^2 - D)}{2} + \frac{(D^3 + 2D^2 - D)^2}{4} + \frac{(D^3 + 2D^2 - D)^3}{8} + \dots \right] (1 - 4x^3)$$

$$= \frac{-1}{2} \left[ 1 + \frac{1}{2}(D^3 + 2D^2 - D) + \frac{1}{4}(D^2 - 4D^3) + \frac{1}{8}(-D^3) \right] (1 - 4x^3)$$

$$= \frac{-1}{2} \left[ 1 - \frac{5}{8}(D^3) + \frac{5}{4}(D^2) - \frac{1}{2}D \right] (1 - 4x^3)$$

$$= \frac{-1}{2} \left[ (1 - 4x^3) - \frac{5}{8}(-24) + \frac{5}{4}(-24x) - \frac{1}{2}(-12x^2) \right]$$

$$= \frac{-1}{2} [-4x^3 + 6x^2 - 30x + 16] =$$

$$= [2x^3 - 3x^2 + 15x - 8]$$

The general solution is

$$y = C.F + P.I$$

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x} + [2x^3 - 3x^2 + 15x - 8]$$

7. Solve  $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$

Given equation is

$$(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$$

$$\text{A.E is } (m^3 - 7m^2 + 14m - 8) = 0$$

$$(m-1)(m-2)(m-4) = 0$$

Then  $m = 1, 2, 4$

$$\text{C.F} = c_1 e^x + c_2 e^{2x} + c_3 e^{4x}$$

$$\text{P.I} = \frac{e^x \cos 2x}{(D^3 - 7D^2 + 14D - 8)}$$

$$= e^x \cdot \frac{1}{(D+1)^3 - 7(D+1)^2 + 14(D+1) - 8} \cdot \cos 2x$$

$$\left[ \because P.I = \frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} v \right]$$

$$= e^x \cdot \frac{1}{(D^3 - 4D^2 + 3D)} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{(-4D + 3D + 16)} \cdot \cos 2x \text{ (Replacing } D^2 \text{ with } -2^2)$$

$$= e^x \cdot \frac{1}{(16 - D)} \cdot \cos 2x$$

$$= e^x \cdot \frac{16 + D}{(16 - D)(16 + D)} \cdot \cos 2x$$

$$= e^x \cdot \frac{16 + D}{256 - D^2} \cdot \cos 2x$$

$$= e^x \cdot \frac{16 + D}{256 - (-4)} \cdot \cos 2x$$

$$= \frac{e^x}{260} (16 \cos 2x - 2 \sin 2x)$$

$$= \frac{2e^x}{260} (8 \cos 2x - \sin 2x)$$

$$= \frac{e^x}{130} (8 \cos 2x - \sin 2x)$$

General solution is  $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{4x} + \frac{e^x}{130} (8 \cos 2x - \sin 2x)$$

8. Solve  $(D^2 - 4D + 4)y = x^2 \sin x + e^{2x} + 3$

Sol: Given  $(D^2 - 4D + 4)y = x^2 \sin x + e^{2x} + 3$

$$\text{A.E is } (m^2 - 4m + 4) = 0$$

$$(m - 2)^2 = 0 \text{ then } m = 2, 2$$

$$\text{C.F.} = (c_1 + c_2 x)e^{2x}$$

$$\text{P.I} = \frac{x^2 \sin x + e^{2x} + 3}{(D - 2)^2} = \frac{1}{(D - 2)^2} (x^2 \sin x) + \frac{1}{(D - 2)^2} e^{2x} + \frac{1}{(D - 2)^2} (3)$$

$$\text{Now } \frac{1}{(D - 2)^2} (x^2 \sin x) = \frac{1}{(D - 2)^2} (x^2) \quad (\text{I.P of } e^{ix})$$

$$= \text{I.P of } \frac{1}{(D - 2)^2} (x^2) (e^{ix})$$

$$= \text{I.P of } (e^{ix}) \cdot \frac{1}{(D + i - 2)^2} (x^2)$$

On simplification, we get



$$\frac{1}{(D-2)^2} (x^2 \sin x) = \frac{1}{625} [(220x+244)\cos x + (40x+33)\sin x]$$

$$\text{and } \frac{1}{(D-2)^2} (e^{2x}) = \frac{x^2}{2} (e^{2x}),$$

$$= \frac{3}{4}$$

$$P.I = \frac{1}{625} [(220x+244)\cos x + (40x+33)\sin x] + \frac{x^2}{2} (e^{2x}) + \frac{3}{4}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{625} [(220x+244)\cos x + (40x+33)\sin x] + \frac{x^2}{2} (e^{2x}) + \frac{3}{4}$$

**Variation of Parameters :**

**Working Rule :**

1. Reduce the given equation of the form  $\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R$
2. Find C.F.
3. Take P.I.  $y_p = Au + Bv$  where  $A = -\int \frac{vRdx}{uv^1 - vu^1}$  and  $B = \int \frac{uRdx}{uv^1 - vu^1}$
4. Write the G.S. of the given equation  $y = y_c + y_p$

**9. Apply the method of variation of parameters to solve  $\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$**

**Sol:** Given equation in the operator form is  $(D^2 + 1)y = \operatorname{cosec} x$ -----(1)

$$\text{A.E is } (m^2 + 1) = 0$$

$$\therefore m = \pm i$$

The roots are complex conjugate numbers.

$$\therefore \text{C.F. is } y_c = c_1 \cos x + c_2 \sin x$$

Let  $y_p = A \cos x + B \sin x$  be P.I. of (1)

$$u \frac{dv}{dx} - v \frac{du}{dx} = \cos^2 x + \sin^2 x = 1$$

A and B are given by

$$A = -\int \frac{vRdx}{uv^1 - vu^1} = -\int \frac{\sin x \operatorname{cosec} x}{1} dx = -\int dx = -x$$

$$B = \int \frac{uRdx}{uv^1 - vu^1} = \int \cos x \cdot \operatorname{cosec} x dx = \int \cot x dx = \log(\sin x)$$

$$\therefore y_p = -x \cos x + \sin x \cdot \log(\sin x)$$

General solution is  $y = y_c + y_p$ .

$$y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \cdot \log(\sin x)$$

10. Solve  $(4D^2 - 4D + 1)y = 100$

Sol: A.E is  $(4m^2 - 4m + 1) = 0$

$$(2m - 1)^2 = 0 \text{ then } m = \frac{1}{2}, \frac{1}{2}$$

$$\text{C.F} = (c_1 + c_2 x) e^{\frac{x}{2}}$$

$$\text{P.I} = \frac{100}{(4D^2 - 4D + 1)} = \frac{100 e^{0 \cdot x}}{(2D - 1)^2} = \frac{100}{(0 - 1)^2} = 100$$

Hence the general solution is  $y = \text{C.F} + \text{P.I}$

$$y = (c_1 + c_2 x) e^{\frac{x}{2}}$$

## **MODULE-II**

# **LINEAR TRANSFORMATIONS AND DOUBLE INTEGRALS**

## Cayley - Hamilton Theorem:

### Statement:

Every square matrix satisfies its own characteristic equation

#### PROBLEMS

1. Show that the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$  satisfies its characteristic equation Hence find  $A^{-1}$

Sol: Characteristic equation of A is  $\det (A-\lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 1 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \quad C_2 \rightarrow C_2 + C_3$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley - Hamilton theorem, we have  $A^3 - A^2 + A - I = 0$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with  $A^{-1}$  we get  $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2) Using Cayley - Hamilton Theorem find the inverse and  $A^4$  of the matrix  $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let  $A = \begin{bmatrix} 7 & 2 & -2 \\ 6 & 1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by  $|A-\lambda I|=0$

$$\text{i.e., } \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem we have  $A^3 - 5A^2 + 7A - 3I = 0 \dots (1)$

Multiply with  $A^{-1}$  we get

$$A^{-1} = \frac{1}{3} [A^2 - 5A + 7I]$$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

Multiply (1) with  $A$ , we get

$$A^4 - 5A^3 + 7A^2 - 3A = 0$$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

### **Problem**

Verify Cayley – Hamilton Theorem for  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ . Hence find  $A^{-1}$ .

### **Linear dependence and independence of Vectors:**

Show that the vectors  $(1,2,3)$ ,  $(3,-2,1)$ ,  $(1,-6,-5)$  form a linearly dependent set.

$$\text{Sol. The Given Vector } X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} X_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} X_3 = \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$$

The Vectors  $X_1, X_2, X_3$  form a square matrix.

$$\text{Let } A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$$

$$\text{Then } |A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{vmatrix}$$

$$= 1(10+6)-2(15-1)+3(-18+2) \\ = 16+32-48=0$$

The given vectors are linearly dependent  $\because |A|=0$

Show that the Vector  $X_1=(2,2,1), X_2=(1,4,-1)$  and  $X_3=(4,6,-3)$  are linearly independent.

Sol. Given Vectors  $X_1=(2,-2,1), X_2=(1,4,-1)$  and  $X_3=(4,6,-3)$  The Vectors  $X_1, X_2, X_3$  form a square matrix.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\text{Then } |A| = \begin{vmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{vmatrix}$$

$$= 2(-12+6)+2(-3+4)+1(6-16) \\ = -20 \neq 0$$

$\therefore$  The given vectors are linearly independent

$\therefore |A| \neq 0$

## **Eigen Values & Eigen Vectors**

**Def: Characteristic vector of a matrix:**

Let  $A = [a_{ij}]$  be an  $n \times n$  matrix. A non-zero vector  $X$  is said to be a Characteristic Vector of  $A$  if there exists a scalar such that  $AX = \lambda X$ .

Note: If  $AX = \lambda X$  ( $X \neq 0$ ), then we say ' $\lambda$ ' is the Eigen value (or) characteristic root of ' $A$ '.

Eg: Let  $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$   $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ = 1 \cdot X$$

Here Characteristic vector of A is  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and Characteristic root of A is "1".

**Note:** We notice that an Eigen value of a square matrix A can be 0. But a zero vector cannot be an Eigen vector of A.

### Method of finding the Eigen vectors of a matrix.

Let  $A = [a_{ij}]$  be a  $n \times n$  matrix. Let X be an Eigen vector of A corresponding to the Eigen value  $\lambda$ .

Then by definition  $AX = \lambda X$ .

$$\Rightarrow AX = \lambda X$$

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow (A - \lambda I)X = 0 \text{ ----- (1)}$$

This is a homogeneous system of n equations in n unknowns. Will have a non-zero solution X if and only

$$|A - \lambda I| = 0$$

- $A - \lambda I$  is called characteristic matrix of A
- $|A - \lambda I|$  is a polynomial in  $\lambda$  of degree n and is called the characteristic polynomial of A
- $|A - \lambda I| = 0$  is called the characteristic equation

Solving characteristic equation of A, we get the roots,  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ , these are called the characteristic roots or Eigen values of the matrix.

- Corresponding to each one of these n Eigen values, we can find the characteristic vectors.

### Procedure to find Eigen values and Eigen vectors

Let  $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$  be a given matrix

Characteristic matrix of A is  $A - \lambda I$

$$i.e., A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix}$$

Then the characteristic polynomial is  $|A - \lambda I|$

$$say \phi(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

The characteristic equation is  $|A - \lambda I| = 0$  we solve the  $\phi(\lambda) = |A - \lambda I| = 0$ , we get  $n$  roots, these are called

Eigen values or latent values or proper values.

Let each one of these Eigen values say  $\lambda$  their Eigen vector  $X$  corresponding the given value  $\lambda$  is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and determining the non-trivial solution.}$$

### **PROBLEMS**

**Find the Eigen values and the corresponding Eigen vectors of  $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$**

*sol:* Let  $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Characteristic matrix  $= A - \lambda I$

$$= \begin{bmatrix} 8 - \lambda & -4 \\ 2 & 2 - \lambda \end{bmatrix}$$

Characteristic equation of  $A$  is  $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8 - \lambda & -4 \\ 2 & 2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (8 - \lambda)(2 - \lambda) + 8 = 0$$

$$\Rightarrow 16 + \lambda^2 - 10\lambda + 8 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 24 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = 6, 4 \text{ are eigen values of } A$$

Consider system  $\begin{bmatrix} 8 - \lambda & -4 \\ 2 & 2 - \lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

Eigen vector corresponding to  $\lambda = 4$

Put  $\lambda = 4$  in the above system, we get

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 4x_1 - 4x_2 = 0 \text{ --- (1)}$$

$$2x_1 - 2x_2 = 0 \text{ --- (2)}$$

from (1) and (2) we have  $x_1 = x_2$

Let  $x_1 = \alpha$

Eigen vector is  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is a Eigen vector of matrix  $A$ , corresponding eigen value  $\lambda = 4$



Eigen Vector corresponding to  $\lambda = 6$

put  $\lambda = 6$  in the above system, we get

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x_1 - 4x_2 = 0 \text{ --- (1)}$$

$$2x_1 - 4x_2 = 0 \text{ --- (2)}$$

from (1) and (2) we have  $x_1 = 2x_2$

Say  $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$$\text{Eigen vector} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is eigen vector of matrix A corresponding eigen value  $\lambda = 6$

**Find the eigen values and the corresponding eigen vectors of matrix**  $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$$\text{Sol: Let } A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The characteristic equation is  $|A - \lambda I| = 0$

$$\text{i.e. } |A - \lambda I| = \begin{vmatrix} 2 - \lambda & 0 & 1 \\ 0 & 2 - \lambda & 0 \\ 1 & 0 & 2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (2 - \lambda)(2 - \lambda)^2 - 0 + [-(2 - \lambda)] = 0$$

$$\Rightarrow (2 - \lambda)^3 - (\lambda - 2) = 0$$

$$\Rightarrow \lambda - 2 [-(\lambda - 2)^2 - 1] = 0$$

$$\Rightarrow \lambda - 2 [-\lambda^2 + 4\lambda - 3] = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 3)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

The eigen values of A is 1, 2, 3.

For finding eigen vector the system is  $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 2 - \lambda & 0 & 1 \\ 0 & 2 - \lambda & 0 \\ 1 & 0 & 2 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to  $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$x_2 = 0$$

$$x_1 + x_3 = 0$$

$$x_1 = -x_3, x_2 = 0$$

$$\text{say } x_3 = \alpha$$

$$x_1 = -\alpha \quad x_2 = 0, \quad x_3 = \alpha$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ is Eigen vector}$$

*Eigen vector corresponding to  $\lambda = 2$*

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

*Here  $x_1 = 0$  and  $x_3 = 0$  and we can take any arbitrary value  $x_2$  i.e  $x_2 = \alpha$  (say)*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Eigen vector is } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

*Eigen vector corresponding to  $\lambda = 3$*

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_3 = 0$$

$$-x_2 = 0$$

$$x_1 - x_3 = 0$$

*here by solving we get  $x_1 = x_3, x_2 = 0$  say  $x_3 = \alpha$*

$$x_1 = \alpha, \quad x_2 = 0, \quad x_3 = \alpha$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Eigen vector is } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

### Properties of Eigen Values:

**Theorem 1:** The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Example: if  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 2 & -1 & 1 \end{bmatrix}$  then trace =  $1+2+1=4$  and determinant = 15

**Theorem 2:** If  $\lambda$  is an Eigen value of A corresponding to the Eigen vector X, then  $\lambda^n$  is Eigen value  $A^n$  corresponding to the Eigen vector X.

Example: if  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  then Eigen values of  $A^3$  are 1, 8, 1

**Theorem 3:** A Square matrix A and its transpose  $A^T$  have the same Eigen values.

Example: if  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  then Eigen values of  $A^T$  are 1, 2, 1.

**Theorem 4:** If A and B are n-rowed square matrices and If A is invertible show that  $A^{-1}B$  and  $B A^{-1}$  have same Eigen values.

**Theorem 5:** If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the Eigen values of a matrix A then  $k\lambda_1, k\lambda_2, \dots, k\lambda_n$  are the Eigen value of the matrix KA, where K is a non-zero scalar.

Example:

If 1, 2, 3 are eigen values of A then eigen values of 3A are 3, 6, 9

**Theorem 6:** If  $\lambda$  is an Eigen values of the matrix A then  $\lambda + K$  is an Eigen value of the matrix  $A + KI$

Example:

If 1, 2, 3 are eigen values of A then eigen values of 3+A are 4, 5, 6

**Theorem 7:** If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the Eigen values of A, then  $\lambda_1 - K, \lambda_2 - K, \dots, \lambda_n - K$ , are the eigen values of the matrix  $(A - KI)$ , where K is a non-zero scalar

Example:

If 1, 2, 3 are eigen values of A then eigen values of 3-A are 2, 1, 0

**Theorem 8:** If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the Eigen values of A, find the Eigen values of the matrix  $(A - \lambda I)^2$

**Theorem 9:** If  $\lambda$  is an Eigen value of a non-singular matrix A corresponding to the Eigen vector X, then  $\lambda^{-1}$  is an Eigen value of  $A^{-1}$  and corresponding Eigen vector X itself.

**Theorem 10:** If

$\lambda$  is an eigen value of a non – singular matrix A, then  $\frac{[A]}{\lambda}$  is an eigen value of the matrix Adj A

**Theorem 11:** If  $\lambda$  is an eigen value of an orthogonal matrix then  $\frac{1}{\lambda}$  is also an eigen value

**Theorem 12:** If  $\lambda$  is Eigen value of A then prove that the Eigen value of  $B = a_0A^2 + a_1A + a_2I$  is  $a_0\lambda^2 + a_1\lambda + a_2$

**Theorem 14:** Suppose that A and P be square matrices of order n such that P is non singular. Then A and  $P^{-1}AP$  have the same Eigen values.

**Corollary 1:** If A and B are square matrices such that A is non-singular, then  $A^{-1}B$  and  $BA^{-1}$  have the same Eigen values.

**Corollary 2:** If A and B are non-singular matrices of the same order, then AB and BA have the same Eigen

**Theorem 15:** The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

**Theorem 16:** The Eigen values of a real symmetric matrix are always real.

**Theorem 17:** For a real symmetric matrix, the Eigen vectors corresponding to two distinct Eigen values are orthogonal.

### PROBLEMS

Find the Eigen values and Eigen vectors of the matrix A and its inverse, where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{Sol: Given } A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

The characteristic equation of A is given by  $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & 3 & 4 \\ 0 & 2 - \lambda & 5 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (1 - \lambda)[(2 - \lambda)(3 - \lambda)] = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

Characteristic roots are 1, 2, 3

Characteristic vector for  $\lambda = 1$

$$\text{For } \lambda = 1, \text{ becomes } \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_2 + 4x_3 = 0$$

$$x_2 + 5x_3 = 0$$

$$2x_3 = 0$$

$$x_2 = 0, x_3 = 0 \text{ and } x_1 = \alpha$$

$$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is the solution where } \alpha \text{ is arbitrary constant}$$

$$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is the Eigen vector corresponding to } \lambda = 1$$

*Characteristic vector for  $\lambda = 2$*

$$\text{For } \lambda = 2, \text{ becomes } \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0$$

$$5x_3 = 0 \Rightarrow x_3 = 0$$

$$-x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$$

$$\text{Let } x_2 = k$$

$$x_1 = 3k$$

$$X = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

is the solution

$$\therefore X = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \text{ is the Eigen vector corresponding to } \lambda = 2$$

*Characteristic vector for  $\lambda = 3$*

$$\text{For } \lambda = 3, \text{ becomes } \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + 3x_2 + 4x_3 = 0$$

$$-x_2 + 5x_3 = 0$$

$$\text{Say } x_3 = K \Rightarrow x_2 = 5K$$

$$x_1 = \frac{19}{2}K$$

$$A^{-1} = \frac{1}{K} \begin{bmatrix} \frac{19}{2}K \\ 5K \\ K \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix} \text{ is the Eigen vector corresponding to } \lambda = 3$$

Eigen values of  $A^{-1}$  are  $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

$\Rightarrow$  Eigen values of  $A^{-1}$  are  $1, \frac{1}{2}, \frac{1}{3}$

We know Eigen vectors of  $A^{-1}$  are same as Eigen vectors of  $A$ .

Find the Eigen values of  $3A^3 + 5A^2 - 6A + 2I$  where  $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Sol: The characteristic equation of  $A$  is  $|A - \lambda I| = 0$

$$\text{i.e., } \begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow [(1-\lambda)(3-\lambda)(-2-\lambda) - 0] = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda)(2+\lambda) = 0 \quad \lambda = 1, 3, -2$$

Eigen values of  $A$  are  $1, 3, -2$

We know that if  $\lambda$  is an eigen value of  $A$  and  $f(A)$  is a polynomial in  $A$ , then the eigen value of  $f(A)$  is  $f(\lambda)$

$$\text{Let } f(A) = 3A^3 + 5A^2 - 6A + 2I$$

Then eigen values of  $f(A)$  are  $f(1), f(3)$  and  $f(-2)$

$$f(1) = 3(1)^3 + 5(1)^2 - 6(1) + 2(1) = 4$$

$$f(3) = 3(3)^3 + 5(3)^2 - 6(3) + 2(1) = 110$$

$$f(-2) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2(1) = 10$$

Eigen values of  $3A^3 + 5A^2 - 6A + 2I$  are  $4, 110, 10$

1) Find the eigen values of  $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

$$\text{Sol: we have } A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$$

$$\text{So } \bar{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$$

$$\Rightarrow \bar{A} = -A^T$$

Thus A is a skew-Hermitian matrix.

$\therefore$  The characteristic equation of A is  $|A - \lambda I| = 0$

$$\Rightarrow A^T = \begin{vmatrix} 3i - \lambda & -2 + i \\ -2 + i & -i - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$$

$\Rightarrow \lambda = 4i, -2i$  are the Eigen values of A

2) Find the Eigen values of  $A = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$$\text{Now } \bar{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix} \text{ and}$$

$$(\bar{A})^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$$

$$\text{We can see that } \bar{A}^T . A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus A is a unitary matrix

$\therefore$  The characteristic equation is  $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}i - \lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i - \lambda \end{vmatrix} = 0$$

$$\text{Which gives } \lambda = \frac{\sqrt{3}}{2} + i\frac{1}{2} \text{ and } \frac{-\sqrt{3}}{2} + \frac{1}{2}i \text{ and}$$

$$\lambda = 1/2\sqrt{3} + 1/2i$$

Hence above  $\lambda$  values are Eigen values of A.

### Diagonalization of a matrix:

**Theorem:** If a square matrix A of order n has n linearly independent eigen vectors  $(X_1, X_2, \dots, X_n)$  corresponding to the n eigen values  $\lambda_1, \lambda_2, \dots, \lambda_n$  respectively then a matrix P can be found such that

$P^{-1}AP$  is a diagonal matrix.

Proof: Given that  $(X_1, X_2, \dots, X_n)$  be eigen vectors of A corresponding to the eigen values  $\lambda_1, \lambda_2, \dots, \lambda_n$  respectively and

these eigen vectors are linearly independent Define  $P = (X_1, X_2, \dots, X_n)$

Since the n columns of P are linearly independent  $|P| \neq 0$

Hence  $P^{-1}$  exists

Consider  $AP = A[X_1, X_2, \dots, X_n]$

$$= [AX_1, AX_2, \dots, AX_n]$$

$$= [\lambda_1 X_1, \lambda_2 X_2, \dots, \lambda_n X_n]$$

$$[X_1, X_2, \dots, X_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$= PD$$

Where  $D = \text{diag} (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$

$$AP = PD$$

$$P^{-1}(AP) = P^{-1}(PD) \Rightarrow P^{-1}AP = (P^{-1}P)D$$

$$\Rightarrow P^{-1}AP = (I)D$$

$$= D$$

$$= \text{diag} (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$$

Hence the theorem is proved.

### Modal and Spectral matrices:

The matrix P in the above result which diagonalizable the square matrix A is called modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Note 1: If  $X_1, X_2, \dots, X_n$  are not linearly independent this result is not true.

2: Suppose A is a real symmetric matrix with n pair wise distinct Eigen values  $\lambda_1, \lambda_2, \dots, \lambda_n$  then the corresponding Eigen vectors  $X_1, X_2, \dots, X_n$  are pair wise orthogonal.

Hence if  $P = (e_1, e_2, \dots, e_n)$

Where  $e_1 = (X_1 / \|X_1\|)$ ,  $e_2 = (X_2 / \|X_2\|)$ , ....  $e_n = (X_n) / \|X_n\|$

then P will be an orthogonal matrix.

$$\text{i.e, } P^T P = P P^T = I$$

$$\text{Hence } P^{-1} = P^T$$

$$P^{-1}AP = D \Rightarrow P^T AP = D$$

### Calculation of powers of a matrix:



We can obtain the power of a matrix by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that  $D = P^{-1}AP$

$$D^2 = (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A(PP^{-1})AP$$

$$= P^{-1}A^2P \quad (\text{since } PP^{-1} = I)$$

$$\text{Similarly } D^3 = P^{-1}A^3P$$

$$\text{In general } D^n = P^{-1}A^nP \dots \dots (1)$$

To obtain  $A^n$ , Pre-multiply (1) by P and post multiply by  $P^{-1}$

$$\text{Then } PD^nP^{-1} = P(P^{-1}A^nP)P^{-1}$$

$$= (PP^{-1})A^n(PP^{-1}) = A^n \Rightarrow A^n = PD^nP^{-1}$$

$$\text{Hence } A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0 \dots & 0 \\ 0 & \lambda_2^n & 0 \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda_n^n \end{bmatrix} P^{-1}$$

### PROBLEMS

Determine the modal matrix P of  $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ . Verify that  $P^{-1}AP$  is a diagonal matrix.

Sol: The characteristic equation of A is  $|A - \lambda I| = 0$

$$\text{i.e., } \begin{vmatrix} -2 - \lambda & 2 & -3 \\ 2 & 1 - \lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\text{which gives } (\lambda - 5)(\lambda + 3)^2 = 0$$

Thus the eigen values are  $\lambda=5$ ,  $\lambda=-3$  and  $\lambda=-3$

$$\text{when } \lambda=5 \Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By solving above we get } X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Similarly, for the given eigen value  $\lambda=-3$  we can have two linearly independent eigen vectors  $X_2 =$

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$P = (X_1 \ X_2 \ X_3)$$

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \text{modal matrix of } A$$

$$\text{Now } \det P = 1(-1) - 2(2) + 3(0 - 1) = -8$$

$$\begin{aligned}
 P^{-1} &= \frac{\text{adj } P}{\det P} = -\frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix} \\
 &= -\frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \\
 &= -\frac{1}{8} \begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix} \\
 P^{-1}AP &= -\frac{1}{8} \begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}
 \end{aligned}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \text{diag}(5, -3, -3)$$

Hence  $P^{-1}AP$  is a diagonal matrix.

Find a matrix P which transform the matrix A =

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \text{ to diagonal form. Hence calculate } A^4$$

Sol: Characteristic equation of A is given by  $|A-\lambda I| = 0$

$$\text{i.e., } \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-2] - 0 - 1[2-2(2-\lambda)] = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, \lambda = 2, \lambda = 3$$

Thus the eigen values of A are 1, 2, 3

If  $x_1, x_2, x_3$  be the components of an Eigen vector corresponding to the Eigen value  $\lambda$ , we have

$$[A-\lambda I]X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For  $\lambda = 1$ , eigen vectors are given by

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e., } 0.x_1+0.x_2+0.x_3=0 \text{ and } x_1+x_2+x_3=0$$

$$x_3=0 \text{ and } x_1+x_2+x_3=0$$

$$x_3=0, x_1=-x_2$$

$$x_1=1, x_2=-1, x_3=0$$

Eigen vector is  $[1, -1, 0]^T$

Also every non-zero multiple of this vector is an Eigen vector corresponding to  $\lambda=1$

For  $\lambda=2, \lambda=3$  we can obtain Eigen vector  $[-2, 1, 2]^T$  and  $[-1, 1, 2]^T$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

The Matrix P is called modal matrix of A

$$P^{-1} = -\frac{1}{2} \begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$\text{Now } P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D \text{ (say)}$$

$$A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0 & -1 & \frac{-1}{2} \\ -1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

### Double Integral :

I. When  $y_1, y_2$  are functions of  $x$  and  $x_1$  and  $x_2$  are constants.  $f(x, y)$  is first integrated w.r.t  $y$  keeping 'x' fixed between limits  $y_1, y_2$  and then the resulting expression is integrated w.r.t 'x' within the limits  $x_1, x_2$  i.e.,

$$\iint_R f(x, y) dx dy = \int_{x=x_1}^{x=x_2} \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y) dy dx$$

II. When  $x_1, x_2$  are functions of  $y$  and  $y_1, y_2$  are constants,  $f(x, y)$  is first integrated w.r.t 'x' keeping 'y' fixed, with in the limits  $x_1, x_2$  and then resulting expression is integrated w.r.t 'y' between the limits  $y_1, y_2$  i.e.,

$$\iint_R f(x, y) dx dy = \int_{y=y_1}^{y=y_2} \int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x, y) dx dy$$

III. When  $x_1, x_2, y_1, y_2$  are all constants. Then

$$\iint_R f(x, y) dx dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) dx dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dy dx$$

### Problems

1. Evaluate  $\int_1^2 \int_1^3 xy^2 dx dy$

$$\text{Sol. } \int_1^2 \left[ \int_1^3 xy^2 dx \right] dy$$

$$= \int_1^2 \left[ y^2 \cdot \frac{x^2}{2} \right]_1^3 dy = \int_1^2 \frac{y^2}{2} dy [9-1]$$

$$= \frac{8}{2} \int_1^2 y^2 dy = 4 \int_1^2 y^2 dy$$

$$= 4 \left[ \frac{y^3}{3} \right]_1^2 = \frac{4}{3} [8-1] = \frac{4 \cdot 7}{3} = \frac{28}{3}$$

2. Evaluate  $\int_0^2 \int_0^x y dy dx$

$$\text{Sol. } \int_{x=0}^2 \int_{y=0}^x y dy dx = \int_{x=0}^2 \left[ \int_{y=0}^x y dy \right] dx$$

$$= \int_{x=0}^2 \left[ \frac{y^2}{2} \right]_0^x dx = \int_{x=0}^2 \frac{1}{2} (x^2 - 0) dx = \frac{1}{2} \int_{x=0}^2 x^2 dx = \frac{1}{2} \left[ \frac{x^3}{3} \right]_0^2 = \frac{1}{6} (8-0) = \frac{8}{6} = \frac{4}{3}$$

3. Evaluate  $\int_0^5 \int_0^{x^2} x(x^2 + y^2) dx dy$

Sol.

$$\int_{x=0}^5 \int_{y=0}^{x^2} x(x^2 + y^2) dy dx = \int_{x=0}^5 \left[ x^3 y + \frac{xy^3}{3} \right]_{y=0}^{x^2} dx$$

$$= \int_{x=0}^5 \left[ x^3 \cdot x^2 + \frac{x(x^2)^3}{3} \right] dx = \int_{x=0}^5 \left( x^5 + \frac{x^7}{3} \right) dx = \left[ \frac{x^6}{6} + \frac{1}{3} \cdot \frac{x^8}{8} \right]_0^5 = \frac{5^6}{6} + \frac{5^8}{24}$$

4. Evaluate  $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$

$$\begin{aligned} \text{Sol: } \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} &= \int_{x=0}^1 \left[ \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2} dy \right] dx \\ &= \int_{x=0}^1 \left[ \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2 + y^2} dy \right] dx = \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} \left[ \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_{y=0}^{\sqrt{1+x^2}} dx \quad \left[ \because \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(x/a) \right] \\ &= \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} [\tan^{-1} 1 - \tan^{-1} 0] dx \quad \text{or} \quad \frac{\pi}{4} (\sinh^{-1} x)_0^1 = \frac{\pi}{4} (\sinh^{-1} 1) \\ &= \frac{\pi}{4} \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} dx = \frac{\pi}{4} \left[ \log(x + \sqrt{x^2+1}) \right]_{x=0}^1 \end{aligned}$$

$$= \frac{\pi}{4} \log(1 + \sqrt{2})$$

5. Evaluate  $\int_0^4 \int_0^{x^2} e^{y/x} dydx$

Answer:  $3e^4 - 7$

6. Evaluate  $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dx dy$

Answer:  $3/35$

7. Evaluate  $\int_0^2 \int_0^x e^{(x+y)} dydx$

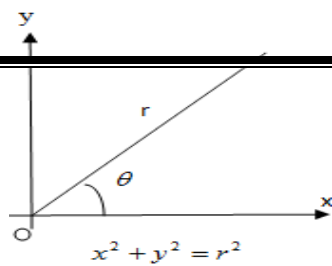
Ans:  $\frac{e^4 - e^2}{2}$

8. Evaluate  $\int_0^{\frac{\pi}{2}} \int_{-1}^1 x^2 y^2 dx dy$

Ans:  $\frac{\pi^3}{36}$

9. Evaluate  $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$

Sol:  $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^\infty e^{-y^2} \left[ \int_0^\infty e^{-x^2} dx \right] dy$



$$= \int_0^{\infty} e^{-y^2} \frac{\sqrt{\pi}}{2} dy \quad \because \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{2} \int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

Alter:

$$\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta \quad (\because x^2 + y^2 = r^2)$$

(changing to polar coordinates taking  $x = r \cos \theta$ ,  $y = r \sin \theta$ )

$$= \int_0^{\frac{\pi}{2}} \left[ \frac{e^{-r^2}}{-2} \right]_0^{\infty} d\theta = \int_0^{\frac{\pi}{2}} \left[ \frac{0-1}{-2} \right] d\theta$$

$$= \frac{1}{2} (\theta)_0^{\frac{\pi}{2}} = \frac{1}{2} (\frac{\pi}{2} - 0)$$

$$= \frac{\pi}{4}$$

10. Evaluate  $\iint_R xy(x+y) dx dy$  over the region R bounded by  $y=x^2$  and  $y=x$

Sol:  $y=x^2$  is a parabola through (0, 0) symmetric about y-axis  $y=x$  is a straight line through (0,0) with slope 1.

Let us find their points of intersection solving  $y=x^2$ ,  $y=x$  we get  $x^2=x \Rightarrow x=0,1$  Hence  $y=0, 1$

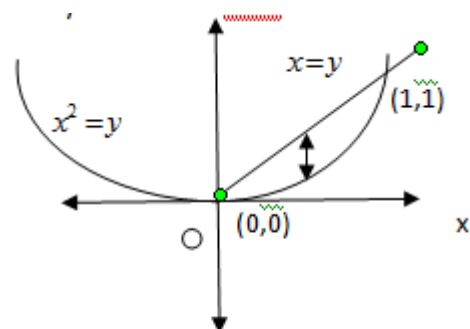
$\therefore$  The point of intersection of the curves are (0,0), (1,1)

Consider  $\iint_R xy(x+y) dx dy$

For the evaluation of the integral, we first integrate w.r.t 'y' from  $y=x^2$  to  $y=x$  and then w.r.t. 'x' from  $x=0$  to  $x=1$

$$\int_{x=0}^1 \left[ \int_{y=x^2}^x xy(x+y) dy \right] dx = \int_{x=0}^1 \left[ \int_{y=x^2}^x (x^2 y + xy^2) dy \right] dx$$

$$= \int_{x=0}^1 \left( x^2 \frac{y^2}{2} + \frac{xy^3}{3} \right)_{y=x^2}^x dx$$



$$= \int_{x=0}^1 \left( \frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \int_{x=0}^1 \left( \frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \left( \frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24} \right)_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28-12-7}{168} = \frac{28-19}{168} = \frac{9}{168} = \frac{3}{56}$$

11. Evaluate  $\iint_R xy dx dy$  where R is the region bounded by x-axis and  $x=2a$  and the curve  $x^2=4ay$ .

Sol. The line  $x=2a$  and the parabola  $x^2=4ay$  intersect at  $B(2a,a)$

✚The given integral =  $\iint_R xy dx dy$

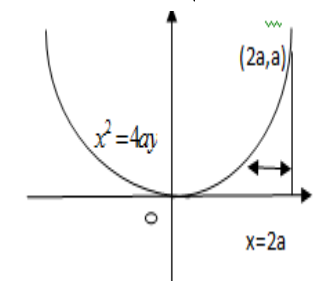
Let us fix 'y'

For a fixed 'y', x varies from  $2\sqrt{ay}$  to  $2a$ . Then y varies from 0 to a.

Hence the given integral can also be written as

$$\int_{y=0}^a \int_{x=2\sqrt{ay}}^{x=2a} xy dx dy = \int_{y=0}^a \left[ \int_{x=2\sqrt{ay}}^{x=2a} x dx \right] y dy$$

$$= \int_{y=0}^a \left[ \frac{x^2}{2} \right]_{x=2\sqrt{ay}}^{2a} y dy$$



$$= \int_{y=0}^a [2a^2 - 2ay] y dy$$

$$= \left[ \frac{2a^2 y^2}{2} - \frac{2ay^3}{3} \right]_0^a = a^4 - \frac{2a^4}{3} = \frac{3a^4 - 2a^4}{3} = \frac{a^4}{3}$$

12. Evaluate  $\int_0^1 \int_0^{\pi/2} r \sin \theta d\theta dr$

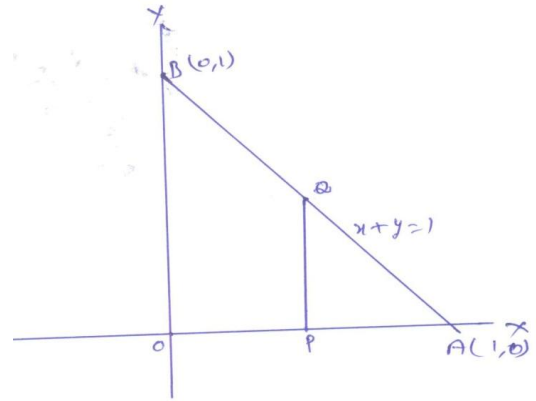
Sol.  $\int_{r=0}^1 r \left[ \int_{\theta=0}^{\pi/2} \sin \theta d\theta \right] dr$

$$= \int_{r=0}^1 r (-\cos \theta)_{\theta=0}^{\pi/2} dr$$

$$= \int_{r=0}^1 -r (\cos \pi/2 - \cos 0) dr$$

$$= \int_{r=0}^1 -r (0-1) dr = \int_0^1 r dr = \left( \frac{r^2}{2} \right)_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

13. Evaluate  $\iint_R (x^2 + y^2) dx dy$  in the positive quadrant



For

Which  $x + y \leq 1$

$$\begin{aligned} \text{Sol. } \iint_R (x^2 + y^2) dx dy &= \int_{x=0}^1 dx \int_{y=0}^{y=1-x} (x^2 + y^2) dy \\ &= \int_{x=0}^1 \left( x^2 y + \frac{y^3}{3} \right)_0^{1-x} dx \\ &= \int_{x=0}^1 \left( x^2 - x^3 + \frac{1}{3}(1-x)^3 \right) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} - \frac{1}{12}(1-x)^4 \right]_0^1 \\ &= \frac{1}{3} - \frac{1}{4} - 0 + \frac{1}{12} = \frac{1}{6} \end{aligned}$$

14. Evaluate  $\iint_R (x^2 + y^2) dx dy$  over the area bounded by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol. Given ellipse is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

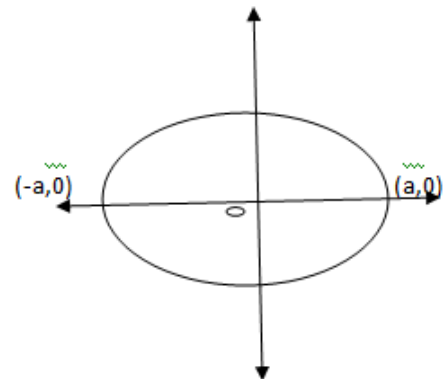
$$\text{i.e., } \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{1}{a^2}(a^2 - x^2) \text{ (or) } y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

Hence the region of integration R can be expressed as

$$-a \leq x \leq a, \frac{-b}{a} \sqrt{a^2 - x^2} \leq y \leq \frac{b}{a} \sqrt{a^2 - x^2}$$

$$\therefore \iint_R (x^2 + y^2) dx dy = \int_{x=-a}^a \int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2) dy dx$$



$$\begin{aligned} &= 2 \int_{x=-a}^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2) dy dx = 2 \int_{-a}^a \left( x^2 y + \frac{y^3}{3} \right)_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx \\ &= 2 \int_{-a}^a \left[ x^2 \cdot \frac{b}{a} \sqrt{a^2 - x^2} + \frac{b^3}{3a^3} (a^2 - x^2)^{3/2} \right] dx \end{aligned}$$



$$= 4 \int_0^a \left[ \frac{b}{a} \sqrt{a^2 - x^2} + \frac{b^3}{3a^3} (a^2 - x^2)^{3/2} \right] dx$$

Changing to polar coordinates

putting  $x = a \sin \theta$

$$dx = a \cos \theta d\theta$$

$$\frac{x}{a} = \sin \theta \Rightarrow \theta = \sin^{-1} \frac{x}{a}$$

$$x \rightarrow 0, \theta \rightarrow 0$$

$$x \rightarrow a, \theta \rightarrow \frac{\pi}{2}$$

$$= 4 \int_0^{\pi/2} \left[ \frac{b}{a} \cdot a^2 \sin^2 \theta \cdot a \cos \theta + \frac{b^3}{3a^3} \cdot a^3 \cos^3 \theta \right] a \cos \theta d\theta$$

$$= 4 \int_0^{\pi/2} \left[ a^3 b \sin^2 \theta \cos^2 \theta + \frac{ab^3}{3} \cos^4 \theta \right] d\theta = 4 \left[ a^3 b \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{ab^3}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$\left[ \because \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \dots \dots \dots \frac{1}{m} \cdot \frac{\pi}{2} \right]$$

$$= \frac{4\pi}{16} (a^3 b + ab^3) = \frac{\pi ab}{4} (a^2 + b^2)$$

### Double integrals in polar co-ordinates:

1. Evaluate  $\int_0^{\pi/4} \int_0^{a \sin \theta} \frac{r dr d\theta}{\sqrt{a^2 - r^2}}$

Sol.  $\int_0^{\pi/4} \int_0^{a \sin \theta} \frac{r dr d\theta}{\sqrt{a^2 - r^2}} = \int_0^{\pi/4} \left\{ \int_0^{a \sin \theta} \frac{r}{\sqrt{a^2 - r^2}} dr \right\} d\theta = -\frac{1}{2} \int_0^{\pi/4} \left\{ \int_0^{a \sin \theta} \frac{-2r}{\sqrt{a^2 - r^2}} dr \right\} d\theta$

$$= -\frac{1}{2} \int_0^{\pi/4} 2 \left( \sqrt{a^2 - r^2} \right)_0^{a \sin \theta} d\theta = (-1) \int_0^{\pi/4} 2 \left[ \sqrt{a^2 - a^2 \sin^2 \theta} - \sqrt{a^2 - 0} \right] d\theta$$

$$= (-a) \int_0^{\pi/4} (\cos \theta - 1) d\theta = (-a) (\sin \theta - \theta)_0^{\pi/4}$$

$$= (-a) \left[ \left[ \sin \frac{\pi}{4} - \frac{\pi}{4} \right] - (0 - 0) \right]$$

$$= (-a) \left[ \frac{1}{\sqrt{2}} - \frac{\pi}{4} \right] = 2 \left[ \frac{\pi}{4} - \frac{1}{\sqrt{2}} \right]$$

2. Evaluate  $\int_0^\pi \int_0^{a \sin \theta} r \, dr \, d\theta$  Ans:  $\frac{a^2 \pi}{4}$

3. Evaluate  $\int_0^\infty \int_0^{\pi/2} e^{-r^2} r \, d\theta \, dr$  Ans:  $\frac{\pi}{4}$

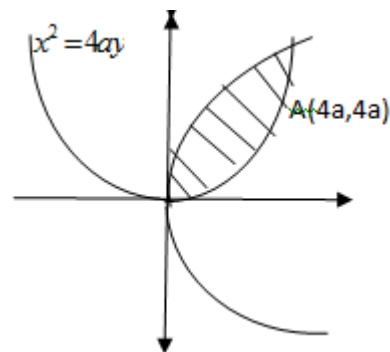
4. Evaluate  $\int_0^\pi \int_0^{a(1+\cos \theta)} r \, dr \, d\theta$  Ans:  $\frac{3\pi a^2}{4}$

### Change of order of Integration:

1. Change the order of Integration and evaluate  $\int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy \, dx$

Sol. In the given integral for a fixed  $x$ ,  $y$  varies from  $\frac{x^2}{4a}$  to  $2\sqrt{ax}$  and then  $x$  varies from 0 to  $4a$ . Let us draw

the curves  $y = \frac{x^2}{4a}$  and  $y = 2\sqrt{ax}$



The region of integration is the shaded region in diagram.

The given integral is  $= \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy \, dx$

Changing the order of integration, we must fix  $y$  first, for a fixed  $y$ ,  $x$  varies from  $\frac{y^2}{4a}$  to  $\sqrt{4ay}$  and then  $y$  varies from 0 to  $4a$ .

Hence the integral is equal to

$$\begin{aligned} \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} dx \, dy &= \int_{y=0}^{4a} \left[ \int_{x=y^2/4a}^{2\sqrt{ay}} dx \right] dy \\ &= \int_{y=0}^{4a} \left[ x \right]_{x=y^2/4a}^{2\sqrt{ay}} dy = \int_{y=0}^{4a} \left[ 2\sqrt{ay} - \frac{y^2}{4a} \right] dy \\ &= \left[ 2\sqrt{a} \cdot \frac{y^{3/2}}{3/2} - \frac{1}{4a} \cdot \frac{y^3}{3} \right]_0^{4a} \end{aligned}$$

$$= \frac{4}{3} \sqrt{a} \cdot 4a \sqrt{4a} - \frac{1}{12a} \cdot 64a^3$$

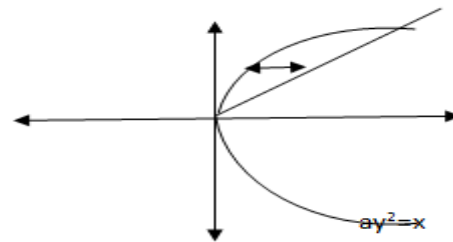
$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

2. Change the order of integration and evaluate  $= \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) dx dy$

Sol. In the given integral for a fixed x, y varies from  $\frac{x}{a}$  to  $\sqrt{\frac{x}{a}}$  and then x varies from 0 to a

Hence we shall draw the curves  $y = \frac{x}{a}$  and  $y = \sqrt{\frac{x}{a}}$

i.e.  $ay = x$  and



and  $ay^2 = x$

we get  $ay = ay^2$

$$\Rightarrow ay - ay^2 = 0$$

$$\Rightarrow ay(1 - y) = 0$$

$$\Rightarrow y = 0, y = 1$$

If  $y=0$ ,  $x=0$  if  $y=1$ ,  $x=a$

The shaded region is the region of integration. The given integral is  $\int_{x=0}^a \int_{y=x/a}^{\sqrt{x/a}} (x^2 + y^2) dx dy$

Changing the order of integration, we must fix y first. For a fixed y, x varies from  $ay^2$  to ay and then y varies from 0 to 1.

Hence the given integral, after change of the order of integration becomes

$$\int_{y=0}^1 \int_{x=ay^2}^{ay} (x^2 + y^2) dx dy$$

$$= \int_{y=0}^1 \left[ \int_{x=ay^2}^{ay} (x^2 + y^2) dx \right] dy$$

$$= \int_{y=0}^1 \left( \frac{x^3}{3} + xy^2 \right)_{x=ay^2}^{ay} dy$$

$$= \int_{y=0}^1 \left( \frac{a^3 y^3}{3} + ay^3 - \frac{a^3 y^6}{3} - ay^4 \right) dy$$

$$= \left( \frac{a^3 y^4}{12} + \frac{ay^4}{4} - \frac{a^3 y^7}{21} - \frac{ay^5}{5} \right)_{y=0}^1$$

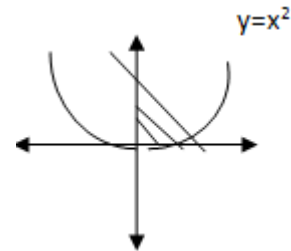
$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} = \frac{a^3}{28} + \frac{a}{20}$$

3. Change the order of integration in  $\int_0^1 \int_{x^2}^{2-x} xy dx dy$  and hence evaluate the double integral.

Sol. In the given integral for a fixed  $x$ ,  $y$  varies from  $x^2$  to  $2-x$  and then  $x$  varies from 0 to 1. Hence we shall draw the curves  $y=x^2$  and  $y=2-x$

The line  $y=2-x$  passes through (0,2), (2,0)

Solving  $y=x^2$ ,  $y=2-x$



Then we get  $x^2 = 2-x$

$$\Rightarrow x^2 + x - 2 = 0$$

$$\Rightarrow x^2 + 2x - x - 2 = 0$$

$$\Rightarrow x(x+2) - 1(x+2) = 0$$

$$\Rightarrow (x-1)(x+2) = 0$$

$$\Rightarrow x = 1, -2$$

$$\text{If } x = 1, y = 1$$

$$\text{If } x = -2, y = 4$$

Hence the points of intersection of the curves are  $(-2,4)$   $(1,1)$

The Shaded region in the diagram is the region of intersection.

Changing the order of integration, we must fix  $y$ , for the region within OACO for a fixed  $y$ ,  $x$  varies from 0 to  $\sqrt{y}$

Then  $y$  varies from 0 to 1

For the region within CABC, for a fixed  $y$ ,  $x$  varies from 0 to  $2-y$ , then  $y$  varies from 1 to 2

$$\text{Hence } \int_0^1 \int_{x^2}^{2-x} xy dy dx = \iint_{OACO} xy dx dy + \iint_{CABC} xy dx dy$$

$$= \int_{y=0}^1 \left[ \int_{x=0}^{\sqrt{y}} x dx \right] y dy + \int_{y=1}^2 \left[ \int_{x=0}^{2-y} x dx \right] y dy$$

$$= \int_{y=0}^1 \left( \frac{x^2}{2} \right)_{x=0}^{\sqrt{y}} y dy + \int_{y=1}^2 \left( \frac{x^2}{2} \right)_{x=0}^{2-y} y dy$$

$$\begin{aligned}
& \int_{y=0}^1 \frac{y}{2} dy + \int_{y=1}^2 \frac{(2-y)^2}{2} dy \\
&= \frac{1}{2} \int_{y=0}^1 y^2 dy + \frac{1}{2} \int_{y=1}^2 (4y - 4y^2 + y^3) dy \\
&= \frac{1}{2} \cdot \left( \frac{y^3}{3} \right)_0^1 + \frac{1}{2} \cdot \left[ \frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4} \right]_1^2 \\
&= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \left[ 2 \cdot 4 - 2 \cdot 1 - \frac{4}{3}(8-1) + \frac{1}{4}(16-1) \right] \\
&= \frac{1}{6} + \frac{1}{2} \left[ 6 - \frac{28}{3} + \frac{15}{4} \right] = \frac{1}{6} + \frac{1}{2} \left[ \frac{72 - 112 + 45}{12} \right] = \frac{1}{6} + \frac{1}{2} \left[ \frac{5}{12} \right] = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8}
\end{aligned}$$

4. Changing the order of integration  $\int_0^a \int_{x^2/a}^{2a-x} xy^2 dy dx$

5. Change of the order of integration  $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dx dy$     Ans:  $\frac{\pi}{16}$

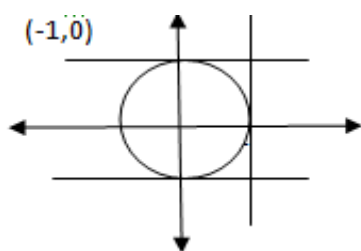
Hint : Now limits are  $y = 0$  to  $1$  and  $x = 0$  to  $\sqrt{1-y^2}$

put  $y = \sin \theta$

$\sqrt{1-y^2} = \cos \theta$

$dy = \cos \theta d\theta$

$$\begin{aligned}
&= \int_0^1 y^2 \sqrt{1-y^2} dy \\
&= \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta = \int_0^{\pi/2} \sin^2 \theta d\theta - \int_0^{\pi/2} \sin^4 \theta d\theta \\
&= \frac{1}{2} \left( \frac{\pi}{2} \right) - \frac{3}{4} \cdot \frac{1}{2} \left( \frac{\pi}{2} \right) = \frac{\pi}{16}
\end{aligned}$$



### Change of variables:

The variables  $x, y$  in  $\iint_R f(x, y) dx dy$  are changed to  $u, v$  with the help of the relations  $x = f_1(u, v), y = f_2(u, v)$

then the double integral is transferred into

$$\iint_{R^1} f[f_1(u, v), f_2(u, v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Where  $R^1$  is the region in the  $uv$  plane, corresponding to the region  $R$  in the  $xy$ -plane.

### Changing from Cartesian to polar co-ordinates

$$x = r \cos \theta, y = r \sin \theta$$

$$\frac{\partial}{\partial r} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial r} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$\frac{\partial}{\partial \theta} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix}$$

$$= r(\cos^2 \theta + \sin^2 \theta) = r \therefore \iint_R f(x, y) dx dy = \iint_{R_1} f(r \cos \theta, r \sin \theta) r dr d\theta$$

**Note :** In polar form  $dx dy$  is replaced by  $r dr d\theta$

### Problems:

1. Evaluate the integral by changing to polar co-ordinates  $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$

Sol. The limits of  $x$  and  $y$  are both from  $0$  to  $\infty$ .

$\therefore$  The region is in the first quadrant where  $r$  varies from  $0$  to  $\infty$  and  $\theta$  varies from  $0$  to  $\pi/2$

Substituting  $x = r \cos \theta, y = r \sin \theta$  and  $dx dy = r dr d\theta$

$$\text{Hence } \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^\infty e^{-r^2} r dr d\theta$$

$$\text{Put } r^2 = t$$

$$\Rightarrow 2r dr = dt$$

$$\Rightarrow r dr = \frac{dt}{2}$$

$$\text{Where } r=0 \Rightarrow t=0 \text{ and } r=\infty \Rightarrow t=\infty$$

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_{\theta=0}^{\pi/2} \int_{t=0}^\infty \frac{1}{2} e^{-t} dt d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} (e^{-t})_0^\infty d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} (0-1) d\theta \Rightarrow \frac{1}{2} (\theta)_0^{\pi/2} = \frac{1}{2} \pi/2 = \pi/4$$

2. Evaluate the integral by changing to polar co-ordinates  $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) dx dy$

$$\text{Sol. The limits for } x \text{ are } x=0 \text{ to } x = \sqrt{a^2 - y^2}$$

$$\Rightarrow x^2 + y^2 = a^2$$

$\therefore$  The given region is the first quadrant of the circle.

By changing to polar co-ordinates

$$x = r \cos \theta, y = r \sin \theta, dx dy = r dr d\theta$$

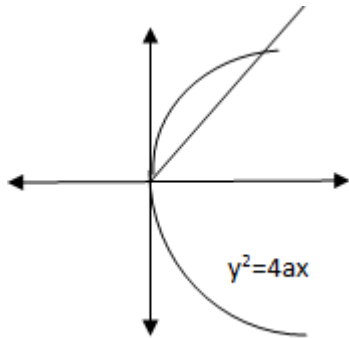
Here ' $r$ ' varies from  $0$  to  $a$  and ' $\theta$ ' varies from  $0$  to  $\pi/2$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) dx dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^a r^2 r dr d\theta = \int_0^{\pi/2} \left( \frac{r^4}{4} \right)_0^a d\theta = \frac{a^4}{4} (\theta)_0^{\pi/2}$$

$$= \frac{\pi}{8} a^4$$

3. Show that  $\int_0^{4a} \int_{y^2/4a}^y \frac{x^2 - y^2}{x^2 + y^2} dx dy = 8a^2 \left( \frac{\pi}{2} - \frac{5}{3} \right)$

Sol. The region of integration is given by  $x = \frac{y^2}{4a}$ ,  $x = y$  and  $y=0$ ,  $y=4a$



i.e., The region is bounded by the parabola  $y^2=4ax$  and the straight line  $x=y$ .

Let  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then  $dx dy = r dr d\theta$

The limits for  $r$  are  $r=0$  at  $O$  and for  $P$  on the parabola

$$r^2 \sin^2 \theta = 4a(r \cos \theta) \Rightarrow r = \frac{4a \cos \theta}{\sin^2 \theta}$$

For the line  $y=x$ , slope  $m=1$  i.e.,  $\tan \theta = 1$ ,  $\theta = \pi/4$

The limits for  $\theta$ :  $\pi/4 \rightarrow \pi/2$

Also  $x^2 - y^2 = r^2 (\cos^2 \theta - \sin^2 \theta)$  and  $x^2 + y^2 = r^2$

$$\therefore \int_0^{4a} \int_{y^2/4a}^y \frac{x^2 - y^2}{x^2 + y^2} dx dy = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{4a \cos \theta / \sin^2 \theta} (\cos^2 \theta - \sin^2 \theta) r dr d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2 \theta - \sin^2 \theta) \left( \frac{r^2}{2} \right)_0^{4a \cos \theta / \sin^2 \theta} d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cos^2 \theta - \sin^2 \theta) \frac{\cos^2 \theta}{\sin^4 \theta} d\theta = 8a^2 \int_{\pi/4}^{\pi/2} (\cos^4 \theta - \cot^2 \theta) d\theta = 8a^2 \left[ \frac{3\pi - 8}{12} + \frac{\pi}{4} - 1 \right] = 8a^2 \left( \frac{\pi}{2} - \frac{5}{3} \right)$$

## Area and Double Integrals

If a region  $R$  is bounded below by  $y = g_1(x)$  and above by  $y = g_2(x)$ , and  $a \leq x \leq b$ , then the area is given by

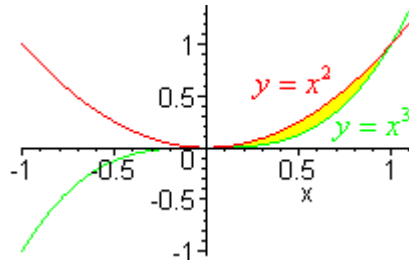
**Example**

$$\text{Area} = \int_a^b \int_{g_1(x)}^{g_2(x)} dy dx$$

Set up the double integral for the area between the curves  $y = x^2$  and  $y = x^3$ . Then use a computer or calculator to evaluate this integral.

**Solution**

The picture below shows the region



We set up the integral

$$\int_0^1 \int_{x^3}^{x^2} dy dx$$

A computer gives the answer of  $1/12$ .



## **MODULE III**

# **FUNCTIONS OF SINGLE VARIABLES AND TRIPLE INTEGRALS**

# MEAN VALUE THEOREMS

## I Rolle's Theorem:

Let  $f(x)$  be a function such that

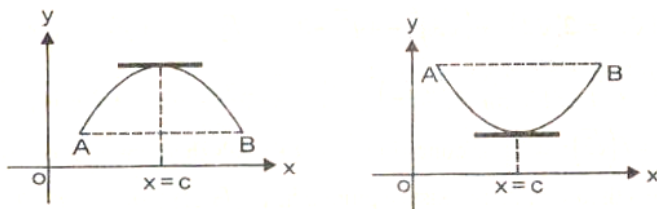
- (i). It is continuous in closed interval  $[a,b]$
- (ii). It is differentiable in open interval  $(a,b)$  and
- (iii).  $f(a) = f(b)$ .

Then there exists at least one point ' $c$ ' in  $(a,b)$  such that

$$f'(c) = 0.$$

## Geometrical Interpretation of Rolle's Theorem:

Let  $f : [a,b] \rightarrow R$  be a function satisfying the three conditions of Rolle's Theorem. Then the graph.



1.  $y=f(x)$  in a continuous curve in  $[a,b]$ .
2. There exist a unique tangent line at every point  $x=c$ , where  $a < c < b$
3. The ordinates  $f(a)$ ,  $f(b)$  at the end points A,B are equal so that the points A and B are equidistant from the X-axis.
4. By Rolle's Theorem, There is at least one point  $x=c$  between A and B on the curve at which the tangent line is parallel to the x-axis and also it is parallel to chord of the curve.

### 1. Verify Rolle's theorem for the function $f(x) = \sin x / e^x$ or $e^{-x} \sin x$ in $[0, \pi]$

Sol: i) Since  $\sin x$  and  $e^x$  are both continuous functions in  $[0, \pi]$ .

Therefore,  $\sin x / e^x$  is also continuous in  $[0, \pi]$ .

ii) Since  $\sin x$  and  $e^x$  be derivable in  $(0, \pi)$ , then  $f$  is also derivable in  $(0, \pi)$ .

iii)  $f(0) = \sin 0 / e^0 = 0$  and  $f(\pi) = \sin \pi / e^\pi = 0$

$$\therefore f(0) = f(\pi)$$

Thus all three conditions of Rolle's Theorem are satisfied.

$\therefore$  There exists  $c \in (0, \pi)$  such that  $f'(c) = 0$

$$\text{Now } f'(x) = \frac{e^x \cos x - \sin x e^x}{(e^x)^2} = \frac{\cos x - \sin x}{e^x}$$

$$f'(c) = 0 \Rightarrow \frac{\cos c - \sin c}{e^c} = 0$$

$$\cos c = \sin c \Rightarrow \tan c = 1$$

$$c = \pi/4 \in (0, \pi)$$

Hence Rolle's theorem is verified.

**2. Verify Rolle's theorem for the functions  $\log \left( \frac{x^2 + ab}{x(a+b)} \right)$  in  $[a, b]$ ,  $a > 0$ ,  $b > 0$ ,**

$$\text{Sol: Let } f(x) = \log \left( \frac{x^2 + ab}{x(a+b)} \right)$$

$$= \log(x^2 + ab) - \log x - \log(a+b)$$

(i). Since  $f(x)$  is a composite function of continuous functions in  $[a, b]$ , it is continuous in  $[a, b]$ .

$$(ii). f'(x) = \frac{1}{x^2 + ab} \cdot 2x - \frac{1}{x} = \frac{x^2 - ab}{x(x^2 + ab)}$$

$f'(x)$  exists for all  $x \in (a, b)$

$$(iii). f(a) = \log \left[ \frac{a^2 + ab}{a^2 + ab} \right] = \log 1 = 0$$

$$f(b) = \log \left[ \frac{b^2 + ab}{b^2 + ab} \right] = \log 1 = 0$$

$$f(a) = f(b)$$

Thus  $f(x)$  satisfies all the three conditions of Rolle's Theorem.

$$\text{So, } \exists c \in (a, b) \Rightarrow f'(c) = 0,$$

$$f'(c) = 0 \Rightarrow \frac{c^2 - ab}{c(c^2 + ab)} = 0 \Rightarrow c^2 = ab$$

$$\Rightarrow c = \sqrt{ab} \in (a, b)$$

Hence Rolle's theorem verified.

**3. Verify whether Rolle's Theorem can be applied to the following functions in the intervals.**

**i)  $f(x) = \tan x$  in  $[0, \pi]$  and ii)  $f(x) = 1/x^2$  in  $[-1, 1]$**

(i)  $f(x)$  is discontinuous at  $x = \pi/2$  as it is not defined there. Thus condition (i) of Rolle's Theorem is not satisfied. Hence we cannot apply Rolle's Theorem here.

$\therefore$  Rolle's theorem cannot be applicable to  $f(x) = \tan x$  in  $[0, \pi]$ .

(ii).  $f(x) = 1/x^2$  in  $[-1, 1]$

$f(x)$  is discontinuous at  $x = 0$ . Hence Rolle's Theorem cannot be applied.

**4. Verify Rolle's theorem for the function  $f(x) = (x-a)^m(x-b)^n$  where  $m, n$  are positive integers in  $[a, b]$ .**

Sol: (i). Since every polynomial is continuous for all values,  $f(x)$  is also continuous in  $[a, b]$ .

$$(ii) \quad f(x) = (x-a)^m(x-b)^n$$

$$\begin{aligned} f'(x) &= m(x-a)^{m-1}(x-b)^n + (x-a)^m \cdot n(x-b)^{n-1} \\ &= (x-a)^{m-1}(x-b)^{n-1} [m(x-b) + n(x-a)] \end{aligned}$$

$$=(x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)]$$

Which exists

Thus  $f(x)$  is derivable in  $(a,b)$

(iii)  $f(a) = 0$  and  $f(b) = 0$

$$\therefore f(a) = f(b)$$

Thus three conditions of Rolle's theorem are satisfied.

$\therefore$  There exists  $c \in (a,b)$  such that  $f'(c) = 0$

$$(c-a)^{m-1}(c-b)^{n-1}[(m+n)c-(mb+na)] = 0$$

$$\Rightarrow (m+n)c - (mb+na) = 0 \Rightarrow (m+n)c = mb+na$$

$$\Rightarrow c = \frac{mb+na}{m+n} \in (a,b)$$

$\therefore$  Rolle's Theorem verified.

**5. Using Rolle's Theorem, show that  $g(x) = 8x^3 - 6x^2 - 2x + 1$  has a zero between 0 and 1.**

Sol:  $g(x) = 8x^3 - 6x^2 - 2x + 1$  being a polynomial, it is continuous on  $[0,1]$  and differentiable on  $(0,1)$

$$\text{Now } g(0) = 1 \text{ and } g(1) = 8 - 6 - 2 + 1 = 1$$

$$\text{Also } g(0) = g(1)$$

Hence, all the conditions of Rolle's theorem are satisfied on  $[0,1]$ .

Therefore, there exists a number  $c \in (0,1)$  such that  $g'(c) = 0$ .

$$\text{Now } g'(x) = 24x^2 - 12x - 2$$

$$\therefore g'(c) = 0 \Rightarrow 24c^2 - 12c - 2 = 0$$

$$\Rightarrow c = \frac{3 \pm \sqrt{21}}{12} \text{ i.e. } c = 0.63 \text{ or } -0.132$$

only the value  $c = 0.63$  lies in  $(0,1)$

Thus there exists at least one root between 0 and 1.

**6. Verify Rolle's theorem for  $f(x) = x^{2/3} - 2x^{1/3}$  in the interval  $(0,8)$ .**

$$\text{Sol: Given } f(x) = x^{2/3} - 2x^{1/3}$$

$f(x)$  is continuous in  $[0,8]$

$$f'(x) = \frac{2}{3} \cdot \frac{1}{x^{1/3}} - \frac{2}{3} \cdot \frac{1}{x^{2/3}} = \frac{2}{3} \left( \frac{1}{x^{1/3}} - \frac{1}{x^{2/3}} \right)$$

Which exists for all  $x$  in the interval  $(0,8)$

$\therefore f$  is derivable  $(0,8)$ .

$$\text{Now } f(0) = 0 \text{ and } f(8) = (8)^{2/3} - 2(8)^{1/3} = 4 - 4 = 0$$

$$\text{i.e., } f(0) = f(8)$$

Thus all the three conditions of Rolle's Theorem are satisfied.

$\therefore$  There exists at least one value of  $c$  in  $(0,8)$  such that  $f'(c) = 0$

$$\text{ie. } \frac{1}{c^3} - \frac{1}{c^3} = 0 \Rightarrow c = 1 \in (0,8)$$

Hence Rolle's Theorem is verified.

### 7. Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3,0]$ .

Sol: - (i). Since  $x(x+3)$  being a polynomial is continuous for all values of  $x$  and  $e^{-x/2}$  is also continuous for all  $x$ , their product  $x(x+3)e^{-x/2} = f(x)$  is also continuous for every value of  $x$  and in particular  $f(x)$  is continuous in the  $[-3,0]$ .

$$(ii). \text{ we have } f'(x) = x(x+3)\left(-\frac{1}{2}e^{-x/2}\right) + (2x+3)e^{-x/2}$$

$$= e^{-x/2} \left[ 2x+3 - \frac{x^2 + 3x}{2} \right]$$

$$= e^{-x/2} [6+x-x^2/2]$$

Since  $f'(x)$  doesnot become infinite or indeterminate at any point of the interval  $(-3,0)$ .

$f(x)$  is derivable in  $(-3,0)$

$$(iii) \quad \text{Also we have } f(-3) = 0 \text{ and } f(0) = 0$$

$$\therefore f(-3) = f(0)$$

Thus  $f(x)$  satisfies all the three conditions of Rolle's theorem in the interval  $[-3,0]$ .

Hence there exist at least one value  $c$  of  $x$  in the interval  $(-3,0)$  such that  $f'(c) = 0$

$$\text{i.e., } \frac{1}{2} e^{-c/2} (6+c-c^2) = 0 \Rightarrow 6+c-c^2 = 0 \quad (e^{-c/2} \neq 0 \text{ for any } c)$$

$$\Rightarrow c^2 + c - 6 = 0 \Rightarrow (c-3)(c+2) = 0$$

$$c = 3, -2$$

Clearly, the value  $c = -2$  lies within the  $(-3,0)$  which verifies Rolle's theorem.

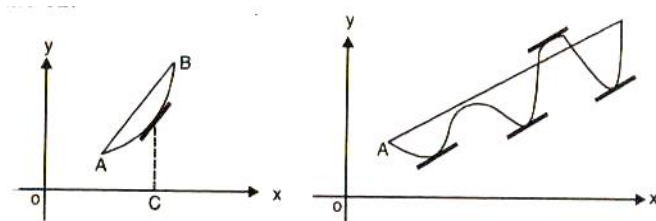
### II. Lagrange's mean value Theorem

Let  $f(x)$  be a function such that (i) it is continuous in closed interval  $[a,b]$  & (ii) differentiable in  $(a,b)$ . Then  $\exists$  at least one point  $c$  in  $(a,b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

### Geometrical Interpretation of Lagrange's Mean Value theorem:

Let  $f : [a,b] \rightarrow R$  be a function satisfying the two conditions of Lagrange's theorem. Then the graph.



1.  $y=f(x)$  is continuous curve in  $[a,b]$

2. At every point  $x=c$ , when  $a < c < b$ , on the curve  $y=f(x)$ , there is unique tangent to the curve. By Lagrange's

theorem there exists at least one point  $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

Geometrically there exist at least one point  $c$  on the curve between  $A$  and  $B$  such that the tangent line is parallel to the chord  $\overleftrightarrow{AB}$

### 1. Verify Lagrange's Mean value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

Sol: Let  $f(x) = x^3 - x^2 - 5x + 3$  is a polynomial in  $x$ .

$\therefore$  It is continuous & derivable for every value of  $x$ .

In particular,  $f(x)$  is continuous  $[0,4]$  & derivable in  $(0,4)$

Hence by Lagrange's Mean value theorem  $\exists c \in (0,4) \ni$

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\text{i.e., } 3c^2 - 2c - 5 = \frac{f(4) - f(0)}{4} \dots\dots\dots(1)$$

$$\text{Now } f(4) = 4^3 - 4^2 - 5 \cdot 4 + 3 = 64 - 16 - 20 - 3 = 67 - 36 = 31 \text{ \& } f(0) = 3$$

$$\frac{f(4) - f(0)}{4} = \frac{(31 - 3)}{4} = 7$$

From equation (1), we have

$$3c^2 - 2c - 5 = 7 \Rightarrow 3c^2 - 2c - 12 = 0$$

$$c = \frac{2 \pm \sqrt{4 + 144}}{6} = \frac{2 \pm \sqrt{148}}{6} = \frac{1 \pm \sqrt{37}}{3}$$

We see that  $\frac{1 + \sqrt{37}}{3}$  lies in open interval  $(0,4)$  & thus Lagrange's Mean value theorem is verified.

### 2. Verify Lagrange's Mean value theorem for $f(x) = \log_e x$ in $[1,e]$

Sol: -  $f(x) = \log_e x$

This function is continuous in closed interval  $[1,e]$  & derivable in  $(1,e)$ . Hence L.M.V.T is applicable here. By this theorem,  $\exists$  a point  $c$  in open interval  $(1,e)$  such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1} = \frac{1 - 0}{e - 1} = \frac{1}{e - 1}$$

$$\text{But } f'(c) = \frac{1}{e - 1} \Rightarrow \frac{1}{c} = \frac{1}{e - 1}$$

$$\therefore c = e - 1$$

Note that  $(e-1)$  is in the interval  $(1,e)$ .

Hence Lagrange's mean value theorem is verified.

3. Give an example of a function that is continuous on  $[-1, 1]$  and for which mean value theorem does not hold with explanations.

Sol:- The function  $f(x) = |x|$  is continuous on  $[-1, 1]$

But Lagrange Mean value theorem is not applicable for the function  $f(x)$  as its derivative does not exist in  $(-1, 1)$  at  $x=0$ .

4. If  $a < b$ , P.T  $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$  using Lagrange's Mean value theorem. Deduce the following.

$$\text{i). } \frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

$$\text{ii). } \frac{5\pi + 4}{20} < \tan^{-1} 2 < \frac{\pi + 2}{4}$$

Sol: consider  $f(x) = \tan^{-1} x$  in  $[a, b]$  for  $0 < a < b < 1$

Since  $f(x)$  is continuous in closed interval  $[a, b]$  & derivable in open interval  $(a, b)$ .

We can apply Lagrange's Mean value theorem here.

Hence there exists a point  $c$  in  $(a, b)$  ∃

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\text{Here } f'(x) = \frac{1}{1+x^2} \text{ \& hence } f'(c) = \frac{1}{1+c^2}$$

Thus  $\exists c, a < c < b$  ∃

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a} \text{ ----- (1)}$$

We have  $1+a^2 < 1+c^2 < 1+b^2$

$$\therefore \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \text{ ..... (2)}$$

From (1) and (2), we have

$$\frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2}$$

or

$$\frac{b-a}{1+a^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+b^2} \text{ .....(3)}$$

Hence the result

**Deductions: -**

$$\text{(i) We have } \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

Take  $h = \frac{4}{5}$  &  $a=1$  we get

$$\begin{aligned} \frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1^2} \implies \frac{\frac{4-3}{3}}{\frac{25}{9}} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{4-3}{\frac{3}{2}} \\ \implies \frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6} \end{aligned}$$

(ii) Taking  $b=2$  and  $a=1$ , we get

$$\begin{aligned} \frac{2-1}{1+2^2} < \tan^{-1}2 - \tan^{-1}1 < \frac{2-1}{1+1^2} \implies \frac{1}{5} < \tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2} \\ \implies \frac{1}{5} + \frac{\pi}{4} < \tan^{-1}2 < \frac{2+\pi}{4} \\ \implies \frac{4+5\pi}{20} < \tan^{-1}2 < \frac{2+\pi}{4} \end{aligned}$$

**5. Show that for any  $x > 0$ ,  $1 + x < e^x < 1 + xe^x$ .**

Sol: - Let  $f(x) = e^x$  defined on  $[0, x]$ . Then  $f(x)$  is continuous on  $[0, x]$  & derivable on  $(0, x)$ .

By Lagrange's Mean value theorem  $\exists$  a real number  $c \in (0, x)$  such that

$$\begin{aligned} \frac{f(x) - f(0)}{x - 0} &= f'(c) \\ \implies \frac{e^x - e^0}{x - 0} &= e^c \implies \frac{e^x - 1}{x} = e^c \dots\dots\dots(1) \end{aligned}$$

Note that  $0 < c < x \implies e^0 < e^c < e^x$  ( $e^x$  is an increasing function)

$$\implies 1 < \frac{e^x - 1}{x} < e^x \text{ From (1)}$$

$$\implies x < e^x - 1 < xe^x$$

$$\implies 1 + x < e^x < 1 + xe^x.$$

**6. Calculate approximately  $\sqrt[5]{245}$  by using L.M.V.T.**

Sol:- Let  $f(x) = \sqrt[5]{x} = x^{1/5}$  &  $a=243$ ,  $b=245$

Then  $f'(x) = \frac{1}{5} x^{-4/5}$  &  $f'(c) = \frac{1}{5} c^{-4/5}$

By L.M.V.T, we have



$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{f(245) - f(243)}{245 - 243} = \frac{1}{5} c^{-\frac{4}{5}}$$

$$\Rightarrow f(245) = f(243) + \frac{2}{5} c^{-4/5}$$

$$\Rightarrow c \text{ lies b/w } 243 \text{ \& } 245 \text{ take } c = 243$$

$$\Rightarrow \sqrt[5]{245} = (243)^{1/5} + \frac{2}{5}(243)^{-4/5} = (3^5)^{\frac{1}{5}} + \frac{2}{5}(3^5)^{-\frac{4}{5}}$$

$$= 3 + (2/5)(1/81) = 3 + 2/405 = 3.0049$$

**7. Find the region in which  $f(x) = 1 - 4x - x^2$  is increasing & the region in which it is decreasing using M.V.T.**

Sol: - Given  $f(x) = 1 - 4x - x^2$

$f(x)$  being a polynomial function is continuous on  $[a, b]$  & differentiable on  $(a, b) \forall a, b \in \mathbb{R}$

$\therefore f$  satisfies the conditions of L.M.V.T on every interval on the real line.

$$f'(x) = -4 - 2x = -2(2 + x) \forall x \in \mathbb{R}$$

$$f'(x) = 0 \text{ if } x = -2$$

$$\text{for } x < -2, f'(x) > 0 \text{ \& for } x > -2, f'(x) < 0$$

Hence  $f(x)$  is strictly increasing on  $(-\infty, -2)$  & strictly decreasing on  $(-2, \infty)$

**8. Using Mean value theorem prove that  $\tan x > x$  in  $0 < x < \pi/2$**

Sol:- Consider  $f(x) = \tan x$  in  $[\xi, x]$  where  $0 < \xi < x < \pi/2$

Apply L.M.V.T to  $f(x)$

$\exists$  a points  $c$  such that  $0 < \xi < c < x < \pi/2$  such that

$$\frac{\tan x - \tan \xi}{x - \xi} = \sec^2 c \implies$$

$$\tan x - \tan \xi = (x - \xi) \sec^2 c$$

$$\text{Take } \xi \rightarrow 0 + 0 \text{ then } \tan x = x \sec^2 c$$

$$\text{But } \sec^2 c > 1.$$

Hence  $\tan x > x$

**9. If  $f'(x) = 0$  Through out an interval  $[a, b]$ , prove using M.V.T  $f(x)$  is a constant in that interval.**

Sol:- Let  $f(x)$  be function defined in  $[a, b]$  & let  $f'(x) = 0 \forall x$  in  $[a, b]$ .

Then  $f'(t)$  is defined & continuous in  $[a, x]$  where  $a \leq x \leq b$ .

&  $f'(t)$  exist in open interval  $(a, x)$ .

By L.M.V.T  $\exists$  a point  $c$  in open interval  $(a, x) \ni$

$$\frac{f(x) - f(a)}{x - a} = f'(c)$$

But it is given that  $f'(c) = 0$

$$\therefore f(x) - f(a) = 0$$

$$\therefore f(x) = f(a) \quad \forall x$$

Hence  $f(x)$  is constant.

## 10 Using mean value theorem

$$\text{i) } x > \log(1+x) > \frac{x}{1+x} \quad x > 0$$

$$\text{ii) } \pi/6 + (\sqrt{3}/15) < \sin^{-1}(0.6) < \pi/6 + (1/6)$$

$$\text{iii) } 1+x < e^x < 1+xe^x \quad \forall x > 0$$

$$\text{iv) } \frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2} \quad \text{where } 0 < u < v \text{ hence deduce}$$

$$\text{a) } \pi/4 + (3/25) < \tan^{-1}(4/3) < \pi/4 + (1/6)$$

## III. Cauchy's Mean Value Theorem

If  $f: [a, b] \rightarrow \mathbb{R}$ ,  $g: [a, b] \rightarrow \mathbb{R} \ni$  (i)  $f, g$  are continuous on  $[a, b]$  (ii)  $f, g$  are differentiable on  $(a, b)$

(iii)  $g'(x) \neq 0 \forall x \in (a, b)$ , then

$$\exists \text{ a point } c \in (a, b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

### 1. Find $c$ of Cauchy's mean value theorem for

$$f(x) = \sqrt{x} \quad \& \quad g(x) = \frac{1}{\sqrt{x}} \quad \text{in } [a, b] \text{ where } 0 < a < b$$

Sol: - Clearly  $f, g$  are continuous on  $[a, b] \subseteq \mathbb{R}^+$

$$\text{We have } f'(x) = \frac{1}{2\sqrt{x}} \text{ and } g'(x) = \frac{-1}{2x\sqrt{x}} \quad \text{which exists on } (a, b)$$

$$\therefore f, g \text{ are differentiable on } (a, b) \subseteq \mathbb{R}^+$$

Also  $g'(x) \neq 0, \forall x \in (a, b) \subseteq \mathbb{R}^+$

Conditions of Cauchy's Mean value theorem are satisfied on  $(a, b)$  so  $\exists c \in (a, b) \ni$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}}{\frac{1}{2c\sqrt{c}}} = \frac{\frac{1}{-1}}{\frac{-1}{2\sqrt{ab}}} \implies \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{a} - \sqrt{b}}{\sqrt{ab}}} = \frac{-2c\sqrt{c}}{2\sqrt{c}} \implies \sqrt{ab} = c$$

Since  $a, b > 0$ ,  $\sqrt{ab}$  is their geometric mean and we have  $a < \sqrt{ab} < b$

$c \in (a, b)$  which verifies Cauchy's mean value theorem.

## 2. Verify Cauchy's Mean value theorem for $f(x) = e^x$ & $g(x) = e^{-x}$ in $[3,7]$ &

find the value of c.

Sol: We are given  $f(x) = e^x$  &  $g(x) = e^{-x}$

$f(x)$  &  $g(x)$  are continuous and derivable for all values of x.

$\Rightarrow f$  &  $g$  are continuous in  $[3,7]$

$\Rightarrow f$  &  $g$  are derivable on  $(3,7)$

Also  $g'(x) = e^{-x} \neq 0 \forall x \in (3,7)$

Thus  $f$  &  $g$  satisfies the conditions of Cauchy's mean value theorem.

Consequently,  $\exists$  a point  $c \in (3,7)$  such that

$$\frac{f(7) - f(3)}{g(7) - g(3)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{e^7 - e^3}{e^{-7} - e^{-3}} = \frac{e^c}{-e^{-c}} \Rightarrow \frac{e^7 - e^3}{\frac{1}{e^7} - \frac{1}{e^3}} = -e^{2c}$$

$$\Rightarrow -e^{7+3} = -e^{2c}$$

$$\Rightarrow 2c = 10$$

$$\Rightarrow c = 5 \in (3,7)$$

Hence C.M.T. is verified

### Triple integrals:

If  $x_1, x_2$  are constants.  $y_1, y_2$  are functions of  $x$  and  $z_1, z_2$  are functions of  $x$  and  $y$ , then  $f(x, y, z)$  is first integrated w.r.t. 'z' between the limits  $z_1$  and  $z_2$  keeping  $x$  and  $y$  fixed. The resulting expression is integrated w.r.t. 'y' between the limits  $y_1$  and  $y_2$  keeping  $x$  constant. The resulting expression is integrated w.r.t. 'x' from  $x_1$  to  $x_2$

$$\text{i.e. } \iiint_V f(x, y, z) dx dy dz = \int_{x=a}^b \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x, y, z) dz dy dx$$

### Problems

1. Evaluate  $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dx dy dz$

Sol. 
$$\begin{aligned} & \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz dx dy dz \\ &= \int_{x=0}^1 dx \int_{y=0}^{\sqrt{1-x^2}} dy \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz dz \\ &= \int_{x=0}^1 dx \int_{y=0}^{\sqrt{1-x^2}} xy \left( \frac{z^2}{2} \right)_{z=0}^{\sqrt{1-x^2-y^2}} dy \\ &= \frac{1}{2} \int_{x=0}^1 dx \int_{y=0}^{\sqrt{1-x^2}} xy (1-x^2-y^2) dy \\ &= \frac{1}{2} \int_{x=0}^1 dx \int_{y=0}^{\sqrt{1-x^2}} x [(1-x^2)y - y^3] dy \\ &= \frac{1}{2} \int_{x=0}^1 x \left[ (1-x^2) \frac{y^2}{2} - \frac{y^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \frac{1}{2} \int_{x=0}^1 x \left[ \frac{y^2}{2} - \frac{x^2 y^2}{2} - \frac{y^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \frac{1}{8} \int_{x=0}^1 x [2(1-x^2) - 2x^2(1-x^2) - (1-x^2)^2] dx \\ &= \frac{1}{8} \int_{x=0}^1 (x - 2x^3 + x^5) dx = \frac{1}{8} \left[ \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right]_0^1 \\ &= \frac{1}{8} \left( \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48} \end{aligned}$$

2. Evaluate  $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dx dy dz$

$$\text{Sol: } \int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dx dy dz$$

$$= \int_{-1}^1 \int_0^z \left[ xy + \frac{y^2}{2} + zy \right]_{x-z}^{x+z} dx dz$$

$$= \int_{-1}^1 \int_0^z x(x+z) - x(x-z) + \left[ \frac{x+z}{2} \right]^2 - \left[ \frac{x-z}{2} \right]^2 + z(x+z) - z(x-z) dx dz$$

$$= \int_{-1}^1 \int_0^z \left[ 2z(x+z) + \frac{1}{2} 4xz \right] dx dz$$

$$= 2 \int_{-1}^1 \left[ z \cdot \frac{x^2}{2} + z^2 x + z \cdot \frac{x^2}{2} \right]_0^z dz = 2 \int_{-1}^1 \left[ \frac{z^3}{2} + z^3 + \frac{z^3}{2} \right] dz = 4 \cdot \left( \frac{z^4}{4} \right)_{-1}^1 = 0$$

**Module-IV**

**FUNCTIONS OF SEVERAL VARIABLES AND  
EXTREMA OF A FUNCTION**

## Partial Differentiation

**Partial differential coefficients :** The Partial differential coefficient of  $f(x,y)$  with respect to  $x$  is the ordinary differential coefficient of  $f(x,y)$  when  $y$  is regarded as a constant. It is written as

$$\frac{\partial f}{\partial x} \text{ or } \partial f / \partial x \text{ or } D_x f$$

$$\text{Thus } \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Again, the partial differential coefficient  $\partial f / \partial y$  of  $f(x,y)$  with respect to  $y$  is the ordinary differential coefficient of  $f(x,y)$  when  $x$  is regarded as a constant.

$$\text{Thus } \frac{\partial f}{\partial y} = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Similarly, if  $f$  is a function of the  $n$  variables  $x_1, x_2, \dots, x_n$ , the partial differential coefficient of  $f$  with respect to  $x_1$  is the ordinary differential coefficient of  $f$  when all the variables except  $x_1$  are regarded as constants and is written as  $\partial f / \partial x_1$ .

$\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  are also denoted by  $f_x$  and  $f_y$  respectively.

The partial differential coefficients of  $f_x$  and  $f_y$  are  $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

or  $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ , respectively.

It should be specially noted that  $\frac{\partial^2 f}{\partial y \partial x}$  means  $\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)$  and  $\frac{\partial^2 f}{\partial x \partial y}$  means  $\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right)$ .

The student will be able to convince himself that in all ordinary cases

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

## PROBLEMS

**Example 1 :** If  $u = \log (x^3 + y^3 + z^3 - 3xyz)$  show that

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = -\frac{9}{(x + y + z)^2}$$

**Solution :** The given relation is

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

Differentiate it w.r.t.  $x$  partially, we get

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\text{similarly } \frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\text{and } \frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - yz - zx - xy)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - yz - xz - xy)}{(x + y + z)(x^2 + y^2 + z^2 - yz - xz - xy)}$$

$$= \frac{3}{x + y + z}$$

$$\text{Now } \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right)$$

$$= \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right)$$

$$= \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{3}{x + y + z} \right)$$

$$= 3 \left[ \frac{\partial}{\partial x} \left( \frac{1}{x + y + z} \right) + \frac{\partial}{\partial y} \left( \frac{1}{x + y + z} \right) + \frac{\partial}{\partial z} \left( \frac{1}{x + y + z} \right) \right]$$

$$= 3 \left[ -\frac{1}{(x + y + z)^2} + \frac{-1}{(x + y + z)^2} + \frac{-1}{(x + y + z)^2} \right]$$

$$= 3 \left[ \frac{-3}{(x + y + z)^2} \right]$$

$$= -\frac{9}{(x + y + z)^2} \text{ Hence Proved.}$$



## Example 2:

If  $u = u(y - z, z - x, x - y)$  Prove that  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

**Solution :** Here given  $u = u(y - z, z - x, x - y)$

Let  $X = y - z$ ,  $Y = z - x$  and  $Z = x - y$ .....(i)

Then  $u = u(X, Y, Z)$ , where  $X, Y, Z$  are function of  $x, y$  and  $z$ .

Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \frac{\partial Z}{\partial x} \dots\dots\dots(ii)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z} \frac{\partial Z}{\partial y} \dots\dots\dots(iii)$$

$$\text{and } \frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \frac{\partial Z}{\partial z} \dots\dots\dots(iv)$$

with the help of (i), equations (ii), (iii) and (iv) gives.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} 0 + \frac{\partial u}{\partial Y} (-1) + \frac{\partial u}{\partial Z} (1) = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} \dots\dots\dots(v)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} .1 + \frac{\partial u}{\partial Y} .0 + \frac{\partial u}{\partial Z} (-1) = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \dots\dots\dots(vi)$$

$$\text{and } \frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} (-1) + \frac{\partial u}{\partial Y} (1) + \frac{\partial u}{\partial Z} (0) = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} \dots\dots\dots(vii)$$

Adding (v), (vi) and (vii) we get  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ . Hence Proved.

**Example 3:** If  $z$  is a function of  $x$  and  $y$  and  $x = e^u + e^{-v}$ ,  $y = e^{-u} - e^v$ ,

Prove that  $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$

**Solution :** Here  $z$  is a function of  $x$  and  $y$ , where  $x$  and  $y$  are functions of  $u$  and  $v$ .

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \dots\dots\dots(i)$$

$$\text{and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \dots\dots\dots(ii)$$

Also given that

$$x = e^u + e^{-v} \text{ and } y = e^{-u} - e^v$$

$$\therefore \frac{\partial x}{\partial u} = e^u, \frac{\partial x}{\partial v} = -e^{-v}, \frac{\partial y}{\partial u} = -e^{-u}, \frac{\partial y}{\partial v} = -e^v$$

$\therefore$  From (i) we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} (e^u) + \frac{\partial z}{\partial y} (-e^{-u}) \dots\dots\dots(iii)$$

and from (ii) we get

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (-e^{-v}) + \frac{\partial z}{\partial y} (-e^v) \dots\dots\dots(iv)$$

Subtracting (iv) from (iii) we get

$$\begin{aligned} \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} &= (e^u + e^{-v}) \frac{\partial z}{\partial x} - (e^{-u} - e^v) \frac{\partial z}{\partial y} \\ &= x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}. \end{aligned} \quad \text{Hence Proved.}$$

**\*\* Maximum & Minimum for function of a single Variable:**

To find the Maxima & Minima of  $f(x)$  we use the following procedure.

- (i) Find  $f'(x)$  and equate it to zero
- (ii) Solve the above equation we get  $x_0, x_1$  as roots.
- (iii) Then find  $f''(x)$ .

If  $f''(x)_{(x=x_0)} > 0$ , then  $f(x)$  is minimum at  $x_0$

If  $f''(x)_{(x=x_0)} < 0$ ,  $f(x)$  is maximum at  $x_0$ . Similarly we do this for other stationary points.

**PROBLEMS:**

1. Find the max & min of the function  $f(x) = x^5 - 3x^4 + 5$  ('08 S-1)

Sol: Given  $f(x) = x^5 - 3x^4 + 5$

$$f'(x) = 5x^4 - 12x^3$$

for maxima or minima  $f'(x) = 0$

$$5x^4 - 12x^3 = 0$$

$$x = 0, x = 12/5$$

$$f''(x) = 20x^3 - 36x^2$$

At  $x = 0 \Rightarrow f''(x) = 0$ . So  $f$  is neither maximum nor minimum at  $x = 0$

$$\text{At } x = (12/5) \Rightarrow f''(x) = 20(12/5)^3 - 36(12/5)$$

$$= 144(48-36)/25 = 1728/25 > 0$$

So  $f(x)$  is minimum at  $x = 12/5$

The minimum value is  $f(12/5) = (12/5)^5 - 3(12/5)^4 + 5$

**\*\* Maxima & Minima for functions of two Variables:**

**Working procedure:**

1. Find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ . Equate each to zero. Solve these equations for x & y we get the pair of

values  $(a_1, b_1)$   $(a_2, b_2)$   $(a_3, b_3)$  .....

2. Find  $l = \frac{\partial^2 f}{\partial x^2}$ ,  $m = \frac{\partial^2 f}{\partial x \partial y}$ ,  $n = \frac{\partial^2 f}{\partial y^2}$

3. i. If  $ln - m^2 > 0$  and  $l < 0$  at  $(a_1, b_1)$  then  $f(x, y)$  is maximum at  $(a_1, b_1)$  and maximum value is  $f(a_1, b_1)$

- ii. If  $ln - m^2 > 0$  and  $l > 0$  at  $(a_1, b_1)$  then  $f(x, y)$  is minimum at  $(a_1, b_1)$  and minimum value is  $f(a_1, b_1)$ .

- iii. If  $ln - m^2 < 0$  and at  $(a_1, b_1)$  then  $f(x, y)$  is neither maximum nor minimum at  $(a_1, b_1)$ . In this case  $(a_1, b_1)$  is saddle point.

- iv. If  $ln - m^2 = 0$  and at  $(a_1, b_1)$ , no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

### **PROBLEMS:**

**Locate the stationary points & examine their nature of the following functions.**

$$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2, \quad (x > 0, y > 0)$$

Sol: Given  $u(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima  $\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$

$$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0 \quad \text{-----> (1)}$$

$$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0 \quad \text{-----> (2)}$$

Adding (1) & (2),

$$x^3 + y^3 = 0$$

$$\Rightarrow x = -y \quad \text{-----> (3)}$$

$$(1) \Rightarrow x^3 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$$

Hence (3)  $\Rightarrow y = 0, -\sqrt{2}, \sqrt{2}$

$$l = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4, m = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = 4 \text{ \& } n = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$$

$$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$$

At  $(-\sqrt{2}, \sqrt{2})$ ,  $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$  and  $l = 20 > 0$

The function has minimum value at  $(-\sqrt{2}, \sqrt{2})$

$$\text{At } (0,0), \ln - m^2 = (0-4)(0-4) - 16 = 0$$

$(0,0)$  is not a extreme value.

**Investigate the maxima & minima, if any, of the function  $f(x) = x^3y^2(1-x-y)$ .**

$$\text{Sol: Given } f(x) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 \quad \frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$$

$$\text{For maxima \& minima } \frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0 \text{ -----} > (1)$$

$$\Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0 \text{ -----} > (2)$$

$$\text{From (1) \& (2) } 4x + 3y - 3 = 0$$

$$2x + 3y - 2 = 0$$

$$2x = 1 \Rightarrow x = 1/2$$

$$4(1/2) + 3y - 3 = 0 \Rightarrow 3y = 3 - 2, y = (1/3)$$

$$1 = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$\left( \frac{\partial^2 f}{\partial x^2} \right)_{(1/2, 1/3)} = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3 = 1/3 - 1/3 - 1/9 = -1/9$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = 6x^2y - 8x^3y - 9x^2y^2$$

$$\left( \frac{\partial^2 f}{\partial x \partial y} \right)_{(1/2, 1/3)} = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^3 = \frac{6-4-3}{12} = \frac{-1}{12}$$

$$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

$$\left( \frac{\partial^2 f}{\partial y^2} \right)_{(1/2, 1/3)} = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$$

$$\ln - m^2 = (-1/9)(-1/8) - (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0 \text{ and } 1 = \frac{-1}{9} < 0$$

The function has a maximum value at  $(1/2, 1/3)$

$$\therefore \text{Maximum value is } f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{8} \times \frac{1}{9}\right) \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{72} \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$$

**Find three positive numbers whose sum is 100 and whose product is maximum.**

Sol: Let  $x, y, z$  be three +ve numbers.

$$\text{Then } x + y + z = 100$$

$$\Rightarrow z = 100 - x - y$$

$$\text{Let } f(x, y) = xyz = xy(100 - x - y) = 100xy - x^2y - xy^2$$

$$\text{For maxima or minima } \frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = 100y - 2xy - y^2 = 0 \Rightarrow y(100 - 2x - y) = 0 \text{ -----} > (1)$$

$$\frac{\partial f}{\partial y} = 100x - x^2 - 2xy = 0 \Rightarrow x(100 - x - 2y) = 0 \text{ -----} > (2)$$

**From (1) & (2)**

$$100 - 2x - y = 0$$

$$200 - 2x - 4y = 0$$

$$\text{-----}$$

$$-100 + 3y = 0 \Rightarrow 3y = 100 \Rightarrow y = 100/3$$

$$100 - x - (200/3) = 0 \Rightarrow x = 100/3$$

$$l = \frac{\partial^2 f}{\partial x^2} = -2y$$

$$\left( \frac{\partial^2 f}{\partial x^2} \right) (100/3, 100/3) = -200/3$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = 100 - 2x - 2y$$

$$\left( \frac{\partial^2 f}{\partial x \partial y} \right) (100/3, 100/3) = 100 - (200/3) - (200/3) = -(100/3)$$

$$n = \frac{\partial^2 f}{\partial y^2} = -2x$$

$$\left( \frac{\partial^2 f}{\partial y^2} \right) (100/3, 100/3) = -200/3$$

$$\ln -m^2 = (-200/3) (-200/3) - (-100/3)^2 = (100)^2 / 3$$

The function has a maximum value at  $(100/3, 100/3)$

$$\text{i.e. at } x = 100/3, y = 100/3 \quad \therefore z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$$

The required numbers are  $x = 100/3, y = 100/3, z = 100/3$

**Find the maxima & minima of the function  $f(x) = 2(x^2 - y^2) - x^4 + y^4$**

$$\text{Sol: Given } f(x) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$$

$$\text{For maxima & minima } \frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = 4x - 4x^3 = 0 \Rightarrow 4x(1 - x^2) = 0 \Rightarrow x = 0, x = \pm 1$$

$$\frac{\partial f}{\partial y} = -4y + 4y^3 = 0 \Rightarrow -4y(1 - y^2) = 0 \Rightarrow y = 0, y = \pm 1$$

$$l = \left( \frac{\partial^2 f}{\partial x^2} \right) = 4 - 12x^2$$

$$m = \left( \frac{\partial^2 f}{\partial x \partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = 0$$

$$n = \left( \frac{\partial^2 f}{\partial y^2} \right) = -4 + 12y^2$$

$$\begin{aligned} \text{we have } \ln - m^2 &= (4 - 12x^2)(-4 + 12y^2) - 0 \\ &= -16 + 48x^2 + 48y^2 - 144x^2y^2 \\ &= 48x^2 + 48y^2 - 144x^2y^2 - 16 \end{aligned}$$

i) At  $(0, \pm 1)$

$$\ln - m^2 = 0 + 48 - 0 - 16 = 32 > 0$$

$$l = 4 - 0 = 4 > 0$$

f has minimum value at  $(0, \pm 1)$

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$$

$$f(0, \pm 1) = 0 - 2 - 0 + 1 = -1$$

The minimum value is '-1'.

ii) At  $(\pm 1, 0)$

$$\ln - m^2 = 48 + 0 - 0 - 16 = 32 > 0$$

$$l = 4 - 12 = -8 < 0$$

f has maximum value at  $(\pm 1, 0)$

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$$

$$f(\pm 1, 0) = 2 - 0 - 1 + 0 = 1$$

The maximum value is '1'.

iii) At  $(0, 0), (\pm 1, \pm 1)$

$$\ln - m^2 < 0$$

$$l = 4 - 12x^2$$

$(0, 0)$  &  $(\pm 1, \pm 1)$  are saddle points.

f has no max & min values at  $(0, 0), (\pm 1, \pm 1)$ .

**\*Extremum** : A function which have a maximum or minimum or both is called 'extremum'

**\*Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.

**\*Stationary points** : - To get stationary points we solve the equations  $\frac{\partial f}{\partial x} = 0$  and

$$\frac{\partial f}{\partial y} = 0 \text{ i.e the pairs } (a_1, b_1), (a_2, b_2) \dots \dots \dots \text{ are called}$$

Stationary.

**\*Maxima & Minima for a function with constant condition :Lagranges Method**

Suppose  $f(x, y, z) = 0$  -----(1)

$$\phi(x, y, z) = 0 \text{ ----- (2)}$$

$F(x, y, z) = f(x, y, z) + \gamma \phi(x, y, z)$  where  $\gamma$  is called Lagrange's constant.

$$1. \quad \frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \phi}{\partial x} = 0 \text{ ----- (3)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \gamma \frac{\partial \phi}{\partial y} = 0 \text{ ----- (4)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \gamma \frac{\partial \phi}{\partial z} = 0 \text{ ----- (5)}$$

2. Solving the equations (2) (3) (4) & (5) we get the stationary point  $(x, y, z)$ .

3. Substitute the value of  $x, y, z$  in equation (1) we get the extremum

### **Problem:**

**Find the minimum value of  $x^2 + y^2 + z^2$ , given  $x + y + z = 3a$**

Sol:  $u = x^2 + y^2 + z^2$

$$\phi = x + y + z - 3a = 0$$

Using Lagrange's function

$$F(x, y, z) = u(x, y, z) + \gamma \phi(x, y, z)$$

For maxima or minima

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \phi}{\partial x} = 2x + \gamma = 0 \text{ ----- (1)}$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \phi}{\partial y} = 2y + \gamma = 0 \text{ ----- (2)}$$

$$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \phi}{\partial z} = 2z + \gamma = 0 \text{ ----- (3)}$$

**From** (1) , (2) & (3)

$$\gamma = -2x = -2y = -2z$$

$$x = y = z$$

$$\phi = x + x + x - 3a = 0 \quad x = a$$

$$x = y = z = a$$

$$\text{Minimum value of } u = a^2 + a^2 + a^2 = 3a^2$$



**MODULE-III**  
**HIGHER ORDER LINEAR**  
**DIFFERENTIAL EQUATIONS AND**  
**THEIR APPLICATIONS**

## **LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER**

**Definition:** An equation of the form  $\frac{d^n y}{dx^n} + P_1(x) \cdot \frac{d^{n-1} y}{dx^{n-1}} + P_2(x) \cdot \frac{d^{n-2} y}{dx^{n-2}} + \dots +$

$P_n(x) \cdot y = Q(x)$  Where  $P_1(x), P_2(x), P_3(x), \dots, P_n(x)$  and  $Q(x)$  (functions of  $x$ ) continuous is called a linear differential equation of order  $n$ .

## **LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS**

Def: An equation of the form  $\frac{d^n y}{dx^n} + P_1 \cdot \frac{d^{n-1} y}{dx^{n-1}} + P_2 \cdot \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n \cdot y = Q(x)$  where  $P_1, P_2, P_3, \dots, P_n$ , are real constants and  $Q(x)$  is a continuous function of  $x$  is called an linear differential equation of order '  $n$ ' with constant coefficients.

Note:

$$7. \text{ Operator } D = \frac{d}{dx} ; D^2 = \frac{d^2}{dx^2} ; \dots \dots \dots D^n = \frac{d^n}{dx^n}$$

$$Dy = \frac{dy}{dx} ; D^2 y = \frac{d^2 y}{dx^2} ; \dots \dots \dots D^n y = \frac{d^n y}{dx^n}$$

$$8. \text{ Operator } \frac{1}{D} Q = \int Q \text{ i.e. } D^{-1} Q \text{ is called the integral of } Q.$$

### **To find the general solution of $f(D).y = 0$ :**

Where  $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_n$  is a polynomial in  $D$ .

Now consider the auxiliary equation :  $f(m) = 0$

$$\text{i.e. } f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2} + \dots + P_n = 0$$

where  $p_1, p_2, p_3, \dots, p_n$  are real constants.

Let the roots of  $f(m) = 0$  be  $m_1, m_2, m_3, \dots, m_n$ .

Depending on the nature of the roots we write the complementary function

as follows:

**Consider the following table**

S.No	Roots of A.E $f(m) = 0$	Complementary function(C.F)
1.	$m_1, m_2, \dots, m_n$ are real and distinct.	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$
2.	$m_1, m_2, \dots, m_n$ are and two roots are equal i.e., $m_1, m_2$ are equal and real(i.e repeated twice) & the rest are real and different.	$y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
3.	$m_1, m_2, \dots, m_n$ are real and three roots are equal i.e., $m_1, m_2, m_3$ are equal and real(i.e repeated thrice) & the rest are real and different.	$y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
4.	Two roots of A.E are complex say $\alpha + i\beta$ and $\alpha - i\beta$ and rest are real and distinct.	$y_c = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
5.	If $\alpha \pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$
6.	If $\alpha \pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x} [(c_1 + c_2 x + c_3 x^2) \cos \beta x + (c_4 + c_5 x + c_6 x^2) \sin \beta x] + c_7 e^{m_7 x} + \dots + c_n e^{m_n x}$
7.	If roots of A.E. irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x} [c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

7. Solve  $\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

**Sol:** Given equation is of the form  $f(D).y = 0$

Where  $f(D) = (D^2 - 3D + 2) y = 0$

Now consider the auxiliary equation  $f(m) = 0$

$$f(m) = m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$$

$$\Rightarrow m = 1, 1, -2$$

Since  $m_1$  and  $m_2$  are equal and  $m_3$  is -2

We have  $y_c = (c_1 + c_2 x)e^x + c_3 e^{-2x}$

8. Solve  $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

**Sol:** Given  $f(D) = (D^4 - 2D^3 - 3D^2 + 4D + 4) y = 0$

$$\Rightarrow \text{A. equation } f(m) = (m^4 - 2m^3 - 3m^2 + 4m + 4) = 0$$

$$\Rightarrow (m+1)^2 (m-2)^2 = 0$$

$$\Rightarrow m = -1, -1, 2, 2$$

$$\Rightarrow y_c = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}$$

9. Solve  $(D^4 + 8D^2 + 16)y = 0$

**Sol:** Given  $f(D) = (D^4 + 8D^2 + 16) y = 0$

Auxiliary equation  $f(m) = (m^4 + 8m^2 + 16) = 0$

$$\Rightarrow (m^2 + 4)^2 = 0$$

$$\Rightarrow (m+2i)^2 (m-2i)^2 = 0$$

$$\Rightarrow m = 2i, 2i, -2i, -2i$$

$$Y_c = e^{0x} [(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$$

10. Solve  $y^{11} + 6y^1 + 9y = 0$  ;  $y(0) = -4$  ,  $y^1(0) = 14$

**Sol:** Given equation is  $y^{11} + 6y^1 + 9y = 0$

$$f(D) y = 0 \Rightarrow (D^2 + 6D + 9) y = 0$$

Auxiliary equation  $f(m) = 0 \Rightarrow (m^2 + 6m + 9) = 0$

$$\Rightarrow m = -3, -3$$

$$y_c = (c_1 + c_2 x)e^{-3x} \text{ -----} (1)$$

Differentiate of (1) w.r.to x  $\Rightarrow y^1 = (c_1 + c_2 x)(-3e^{-3x}) + c_2(e^{-3x})$

Given  $y_1(0) = 14 \Rightarrow c_1 = -4$  &  $c_2 = 2$

Hence we get  $y = (-4 + 2x)(e^{-3x})$

11. Solve  $4y^{111} + 4y^{11} + y^1 = 0$

**Sol:** Given equation is  $4y^{111} + 4y^{11} + y^1 = 0$

That is  $(4D^3 + 4D^2 + D)y = 0$

Auxiliary equation  $f(m) = 0$

$$4m^3 + 4m^2 + m = 0$$

$$m(4m^2 + 4m + 1) = 0$$

$$m(2m + 1)^2 = 0$$

$$m = 0, -1/2, -1/2$$

$$y = c_1 + (c_2 + c_3 x) e^{-x/2}$$

## 12. Solve $(D^2 - 3D + 4)y = 0$

Sol: Given equation  $(D^2 - 3D + 4)y = 0$

A.E.  $f(m) = 0$

$$m^2 - 3m + 4 = 0$$

$$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}$$

$$\alpha \pm i\beta = \frac{3 \pm i\sqrt{7}}{2} = \frac{3}{2} \pm i \frac{\sqrt{7}}{2}$$

$$y = e^{\frac{3}{2}x} (c_1 \cos \frac{\sqrt{7}}{2}x + c_2 \sin \frac{\sqrt{7}}{2}x)$$

## General solution of $f(D)y = Q(x)$

Is given by  $y = y_c + y_p$

$$\text{i.e. } y = C.F + P.I$$

Where the P.I consists of no arbitrary constants and P.I of  $f(D)y = Q(x)$

$$\text{Is evaluated as } P.I = \frac{1}{f(D)} \cdot Q(x)$$

Depending on the type of function of  $Q(x)$ .

P.I is evaluated as follows:

### 1. P.I of $f(D)y = Q(x)$ where $Q(x) = e^{ax}$ for $(a) \neq 0$

$$\text{Case 1: } P.I = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

Provided  $f(a) \neq 0$

Case 2: If  $f(a) = 0$  then the above method fails. Then

$$\text{if } f(D) = (D-a)^k \Phi(D)$$

(i.e. 'a' is a repeated root k times).

$$\text{Then } P.I = \frac{1}{\Phi(a)} e^{ax} \cdot \frac{1}{k!} x^k \text{ provided } \Phi(a) \neq 0$$

### 2. P.I of $f(D)y = Q(x)$ where $Q(x) = \sin ax$ or $Q(x) = \cos ax$ where 'a' is constant then $P.I =$

$$\frac{1}{f(D)} \cdot Q(x).$$

$$\text{Case 1: In } f(D) \text{ put } D^2 = -a^2 \ni f(-a^2) \neq 0 \text{ then } P.I = \frac{\sin ax}{f(-a^2)}$$

Case 2: If  $f(-a^2) = 0$  then  $D^2 + a^2$  is a factor of  $\Phi(D^2)$  and hence it is a factor of  $f(D)$ . Then let  $f(D) = (D^2 + a^2) \cdot \Phi(D^2)$ .

$$\text{Then } \frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2 + a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)} \frac{\sin ax}{D^2 + a^2} = \frac{1}{\Phi(-a^2)} \frac{-x \cos ax}{2a}$$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D+a)\Phi(D)} = \frac{1}{\Phi(-a)} \frac{\cos ax}{D+a} = \frac{1}{\Phi(-a)} \frac{x \sin ax}{2a}$$

9. P.I for  $f(D) y = Q(x)$  where  $Q(x) = x^k$  where  $k$  is a positive integer  $f(D)$  can be express as  
 $f(D) = [1 \pm \phi(D)]$

$$\text{Express } \frac{1}{f(D)} = \frac{1}{1 \pm \phi(D)} = [1 \pm \phi(D)]^{-1}$$

$$\begin{aligned} \text{Hence P.I} &= \frac{1}{1 \pm \phi(D)} Q(x). \\ &= [1 \pm \phi(D)]^{-1} \cdot x^k \end{aligned}$$

10. P.I of  $f(D) y = Q(x)$  when  $Q(x) = e^{ax} V$  where 'a' is a constant and  $V$  is function of  $x$ .  
 where  $V = \sin ax$  or  $\cos ax$  or  $x^k$

$$\begin{aligned} \text{Then P.I} &= \frac{1}{f(D)} Q(x) \\ &= \frac{1}{f(D)} e^{ax} V \\ &= e^{ax} \left[ \frac{1}{f(D+a)} (V) \right] \end{aligned}$$

&  $\frac{1}{f(D+a)} V$  is evaluated depending on  $V$ .

11. P.I of  $f(D) y = Q(x)$  when  $Q(x) = x V$  where  $V$  is a function of  $x$ .

$$\begin{aligned} \text{Then P.I} &= \frac{1}{f(D)} Q(x) \\ &= \frac{1}{f(D)} x V \\ &= \left[ x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} V \end{aligned}$$

12. i. P.I. of  $f(D)y=Q(x)$  where  $Q(x)=x^m v$  where  $v$  is a function of  $x$ .

$$\begin{aligned} \text{Then P.I.} &= \frac{1}{f(D)} \times Q(x) = \frac{1}{f(D)} x^m v = I.P.of \frac{1}{f(D)} x^m (\cos ax + i \sin ax) \\ &= I.P.of \frac{1}{f(D)} x^m e^{iax} \end{aligned}$$

$$\text{ii. P.I.} = \frac{1}{f(D)} x^m \cos ax = R.P.of \frac{1}{f(D)} x^m e^{iax}$$

### Formulae

$$7. \frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$8. \frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$$9. \frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$$

$$10. \frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$$

$$11. \frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \dots$$

$$12. \frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \dots$$

## II. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS:

14. Find the Particular integral of  $f(D) y = e^{ax}$  when  $f(a) \neq 0$

15. Solve the D.E  $(D^2 + 5D + 6) y = e^x$

16. Solve  $y^{11} + 4y^1 + 4y = 4e^{3x}$ ;  $y(0) = -1$ ,  $y^1(0) = 3$

17. Solve  $y^{11} + 4y^1 + 4y = 4\cos x + 3\sin x$ ,  $y(0) = 1$ ,  $y^1(0) = 0$

18. Solve  $(D^2 + 9) y = \cos 3x$

19. Solve  $y^{11} + 2y^{11} - y^1 - 2y = 1 - 4x^3$

20. Solve the D.E  $(D^3 - 7D^2 + 14D - 8) y = e^x \cos 2x$

21. Solve the D.E  $(D^3 - 4D^2 - D + 4) y = e^{3x} \cos 2x$

22. Solve  $(D^2 - 4D + 4) y = x^2 \sin x + e^{2x} + 3$

23. Apply the method of variation parameters to solve  $\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$

24. Solve  $\frac{dx}{dt} = 3x + 2y$ ,  $\frac{dy}{dt} + 5x + 3y = 0$

25. Solve  $(D^2 + D - 3) y = x^2 e^{-3x}$

26. Solve  $(D^2 - D - 2) y = 3e^{2x}$ ,  $y(0) = 0$ ,  $y^1(0) = -2$

### SOLUTIONS:

#### 2) Particular integral of $f(D) y = e^{ax}$ when $f(a) \neq 0$

Working rule:

Case (i):

In  $f(D)$ , put  $D=a$  and Particular integral will be calculated.

Particular integral =  $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$  provided  $f(a) \neq 0$

Case (ii) :

If  $f(a) = 0$ , then above method fails. Now proceed as below.

If  $f(D) = (D-a)^k \phi(D)$

i.e. 'a' is a repeated root k times, then

Particular integral =  $\frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$  provided  $\phi(a) \neq 0$

#### 3. Solve the Differential equation $(D^2 + 5D + 6)y = e^x$

Sol : Given equation is  $(D^2 + 5D + 6)y = e^x$

Here  $Q(x) = e^x$

Auxiliary equation is  $f(m) = m^2 + 5m + 6 = 0$

$$m^2 + 3m + 2m + 6 = 0$$

$$m(m+3) + 2(m+3) = 0$$

$$m = -2 \text{ or } m = -3$$

The roots are real and distinct

$$C.F = y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

$$\text{Particular Integral} = y_p = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2 + 5D + 6} e^x = \frac{1}{(D+2)(D+3)} e^x$$

Put  $D = 1$  in  $f(D)$

$$P.I. = \frac{1}{(3)(4)} e^x$$

$$\text{Particular Integral} = y_p = \frac{1}{12} \cdot e^x$$

General solution is  $y = y_c + y_p$

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12}$$

**3) Solve  $y'' - 4y' + 3y = 4e^{3x}$ ,  $y(0) = -1$ ,  $y'(0) = 3$**

Sol : Given equation is  $y'' - 4y' + 3y = 4e^{3x}$

$$\text{i.e. } \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = 4e^{3x}$$

it can be expressed as

$$D^2 y - 4Dy + 3y = 4e^{3x}$$

$$(D^2 - 4D + 3)y = 4e^{3x}$$

$$\text{Here } Q(x) = 4e^{3x}; f(D) = D^2 - 4D + 3$$

Auxiliary equation is  $f(m) = m^2 - 4m + 3 = 0$

$$m^2 - 3m - m + 3 = 0$$

$$m(m-3) - 1(m-3) = 0 \Rightarrow m = 3 \text{ or } 1$$

The roots are real and distinct.

$$C.F = y_c = c_1 e^{3x} + c_2 e^x \rightarrow (2)$$

$$P.I. = y_p = \frac{1}{f(D)} \cdot Q(x)$$



$$= y_p = \frac{1}{(D-1)(D-3)} \cdot 4e^{3x}$$

$$= y_p = \frac{1}{(D-1)(D-3)} \cdot 4e^{3x}$$

Put  $D=3$

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2} \frac{e^{3x}}{(D-3)} = 2 \frac{x^1}{1!} e^{3x} = 2xe^{3x}$$

General solution is  $y=y_c+y_p$

$$y=c_1e^{3x}+c_2e^x+2xe^{3x} \longrightarrow (3)$$

Equation (3) differentiating with respect to 'x'

$$y^1=3c_1e^{3x}+c_2e^x+2e^{3x}+6xe^{3x} \longrightarrow (4)$$

By data,  $y(0) = -1$ ,  $y^1(0)=3$

$$\text{From (3), } -1=c_1+c_2 \longrightarrow (5)$$

$$\text{From (4), } 3=3c_1+c_2+2$$

$$3c_1+c_2=1 \longrightarrow (6)$$

Solving (5) and (6) we get  $c_1=1$  and  $c_2 = -2$

$$y=-2e^x+(1+2x)e^{3x}$$

**(4). Solve  $y^{11}+4y^1+4y=4\cos x + 3\sin x$ ,  $y(0) = 0$ ,  $y^1(0) = 0$**

Sol: Given differential equation in operator form

$$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$$

$$\text{A.E is } m^2+4m+4 = 0$$

$$(m+2)^2=0 \quad \text{then } m=-2, -2$$

$$\therefore \text{ C.F is } y_c = (c_1 + c_2x)e^{-2x}$$

$$\text{P.I is } y_p = \frac{4\cos x + 3\sin x}{(D^2 + 4D + 4)} \quad \text{put } D^2 = -1$$

$$y_p = \frac{4\cos x + 3\sin x}{(4D + 3)} = \frac{(4D-3)(4\cos x + 3\sin x)}{(4D-3)(4D+3)}$$

$$= \frac{(4D-3)(4\cos x + 3\sin x)}{16D^2 - 9}$$

$$\text{Put } D^2 = -1$$

$$\therefore y_p = \frac{(4D-3)(4\cos x + 3\sin x)}{-16-9}$$

$$= \frac{-16\sin x + 12\cos x - 12\cos x - 9\sin x}{-25} = \frac{-25\sin x}{-25} = \sin x$$

∴ General equation is  $y = y_c + y_p$

$$y = (c_1 + c_2x)e^{-2x} + \sin x \quad \text{----- (1)}$$

By given data,  $y(0) = 0$  ∴  $c_1 = 0$  and

$$\text{Diff (1) w.r.t. } y' = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2) + \cos x \quad \text{----- (2)}$$

$$\text{given } y'(0) = 0$$

$$(2) \Rightarrow -2c_1 + c_2 + 1 = 0 \quad \therefore c_2 = -1$$

∴ Required solution is  $y = -xe^{-2x} + \sin x$

### 5. Solve $(D^2+9)y = \cos 3x$

Sol: Given equation is  $(D^2+9)y = \cos 3x$

$$\text{A.E is } m^2+9 = 0$$

$$\therefore m = \pm 3i$$

$$y_c = \text{C.F} = c_1 \cos 3x + c_2 \sin 3x$$

$$y_c = \text{P.I} = \frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$$

$$= \frac{x}{2(3)} \sin 3x = \frac{x}{6} \sin 3x$$

General equation is  $y = y_c + y_p$

$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{6} \sin 3x$$

### 6. Solve $y^{11} + 2y^{11} - y' - 2y = 1 - 4x^3$

Sol: Given equation can be written as

$$(D^3 + 2D^2 - D - 2)y = 1 - 4x^3$$

$$\text{A.E is } (m^3 + 2m^2 - m - 2) = 0$$

$$(m^2 - 1)(m+2) = 0$$

$$m^2 = 1 \text{ or } m = -2$$

$$m = 1, -1, -2$$

$$\text{C.F} = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$$

$$\begin{aligned} \text{P.I} &= \frac{1}{(D^3 + 2D^2 - D - 2)} (1 - 4x^3) \\ &= \frac{-1}{2 \left[ 1 - \frac{(D^3 + 2D^2 - D)}{2} \right]} (1 - 4x^3) \end{aligned}$$

$$\begin{aligned} &= \frac{-1}{2} \left[ 1 - \frac{(D^3 + 2D^2 - D)}{2} \right]^{-1} (1 - 4x^3) \\ &= \frac{-1}{2} \left[ 1 + \frac{(D^3 + 2D^2 - D)}{2} + \frac{(D^3 + 2D^2 - D)^2}{4} + \frac{(D^3 + 2D^2 - D)^3}{8} + \dots \right] (1 - 4x^3) \\ &= \frac{-1}{2} \left[ 1 + \frac{1}{2} (D^3 + 2D^2 - D) + \frac{1}{4} (D^2 - 4D^3) + \frac{1}{8} (-D^3) \right] (1 - 4x^3) \\ &= \frac{-1}{2} \left[ 1 - \frac{5}{8} (D^3) + \frac{5}{4} (D^2) - \frac{1}{2} D \right] (1 - 4x^3) \\ &= \frac{-1}{2} \left[ (1 - 4x^3) - \frac{5}{8} (-24) + \frac{5}{4} (-24x) - \frac{1}{2} (-12x^2) \right] \\ &= \frac{-1}{2} [-4x^3 + 6x^2 - 30x + 16] = \\ &= [2x^3 - 3x^2 + 15x - 8] \end{aligned}$$

The general solution is

$$y = \text{C.F} + \text{P.I}$$

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x} + [2x^3 - 3x^2 + 15x - 8]$$

7. Solve  $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$

Given equation is

$$(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$$

A.E is  $(m^3 - 7m^2 + 14m - 8) = 0$

$$(m-1)(m-2)(m-4) = 0$$

Then  $m = 1, 2, 4$

$$C.F = c_1 e^x + c_2 e^{2x} + c_3 e^{4x}$$

$$P.I = \frac{e^x \cos 2x}{(D^3 - 7D^2 + 14D - 8)}$$

$$= e^x \cdot \frac{1}{(D+1)^3 - 7(D+1)^2 + 14(D+1) - 8} \cdot \cos 2x$$

$$\left[ \because P.I = \frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} v \right]$$

$$= e^x \cdot \frac{1}{(D^3 - 4D^2 + 3D)} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{(-4D + 3D + 16)} \cdot \cos 2x \text{ (Replacing } D^2 \text{ with } -2^2)$$

$$= e^x \cdot \frac{1}{(16 - D)} \cdot \cos 2x$$

$$= e^x \cdot \frac{16 + D}{(16 - D)(16 + D)} \cdot \cos 2x$$

$$= e^x \cdot \frac{16 + D}{256 - D^2} \cdot \cos 2x$$

$$= e^x \cdot \frac{16 + D}{256 - (-4)} \cdot \cos 2x$$

$$= \frac{e^x}{260} (16 \cos 2x - 2 \sin 2x)$$

$$= \frac{2e^x}{260} (8 \cos 2x - \sin 2x)$$

$$= \frac{e^x}{130} (8 \cos 2x - \sin 2x)$$

General solution is  $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{4x} + \frac{e^x}{130} (8 \cos 2x - \sin 2x)$$

**8. Solve  $(D^2 - 4D + 4)y = x^2 \sin x + e^{2x} + 3$**

Sol: Given  $(D^2 - 4D + 4)y = x^2 \sin x + e^{2x} + 3$

A.E is  $(m^2 - 4m + 4) = 0$

$(m - 2)^2 = 0$  then  $m=2,2$

C.F. =  $(c_1 + c_2x)e^{2x}$

P.I =  $\frac{x^2 \sin x + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2} (x^2 \sin x) + \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} (3)$

Now  $\frac{1}{(D-2)^2} (x^2 \sin x) = \frac{1}{(D-2)^2} (x^2) \quad (\text{I.P of } e^{ix})$

$= \text{I.P of } \frac{1}{(D-2)^2} (x^2) (e^{ix})$

$= \text{I.P of } (e^{ix}) \cdot \frac{1}{(D+i-2)^2} (x^2)$

On simplification, we get

$\frac{1}{(D+i-2)^2} (x^2 \sin x) = \frac{1}{625} [(220x+244)\cos x + (40x+33)\sin x]$

and  $\frac{1}{(D-2)^2} (e^{2x}) = \frac{x^2}{2} (e^{2x}),$

$= \frac{3}{4}$

P.I =  $\frac{1}{625} [(220x+244)\cos x + (40x+33)\sin x] + \frac{x^2}{2} (e^{2x}) + \frac{3}{4}$

$y = y_c + y_p$

$y = (c_1 + c_2x)e^{2x} + \frac{1}{625} [(220x+244)\cos x + (40x+33)\sin x] + \frac{x^2}{2} (e^{2x}) + \frac{3}{4}$

**Variation of Parameters :**

**Working Rule :**

5. Reduce the given equation of the form  $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R$

6. Find C.F.

7. Take P.I.  $y_p = Au + Bv$  where  $A = -\int \frac{vRdx}{uv^1 - vu^1}$  and  $B = \int \frac{uRdx}{uv^1 - vu^1}$

8. Write the G.S. of the given equation  $y = y_c + y_p$

**9. Apply the method of variation of parameters to solve  $\frac{d^2y}{dx^2} + y = \text{cosec} x$**

**Sol:** Given equation in the operator form is  $(D^2 + 1)y = \text{cosec} x$ -----(1)

A.E is  $(m^2 + 1) = 0$

$\therefore m = \pm i$

The roots are complex conjugate numbers.

∴ C.F. is  $y_c = c_1 \cos x + c_2 \sin x$

Let  $y_p = A \cos x + B \sin x$  be P.I. of (1)

$$u \frac{dv}{dx} - v \frac{du}{dx} = \cos^2 x + \sin^2 x = 1$$

A and B are given by

$$A = - \int \frac{v R dx}{uv^1 - vu^1} = - \int \frac{\sin x \operatorname{cosec} x}{1} dx = - \int dx = -x$$

$$B = \int \frac{u R dx}{uv^1 - vu^1} = \int \cos x \cdot \operatorname{cosec} x dx = \int \cot x dx = \log(\sin x)$$

$$\therefore y_p = -x \cos x + \sin x \cdot \log(\sin x)$$

∴ General solution is  $y = y_c + y_p$ .

$$y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \cdot \log(\sin x)$$

**10. Solve  $(4D^2 - 4D + 1)y = 100$**

Sol: A.E is  $(4m^2 - 4m + 1) = 0$

$$(2m - 1)^2 = 0 \text{ then } m = \frac{1}{2}, \frac{1}{2}$$

$$\text{C.F.} = (c_1 + c_2 x) e^{\frac{x}{2}}$$

$$\text{P.I.} = \frac{100}{(4D^2 - 4D + 1)} = \frac{100 e^{0 \cdot x}}{(2D - 1)^2} = \frac{100}{(0 - 1)^2} = 100$$

Hence the general solution is  $y = \text{C.F.} + \text{P.I.}$

$$y = (c_1 + c_2 x) e^{\frac{x}{2}} + 100$$

### Applications of Differential Equations:

11. The differential equation satisfying a beam uniformly loaded ( $w$  kg/meter) with one end fixed and the second end subjected to tensile force  $p$  is given by

$$EI \frac{d^2 y}{dx^2} = py - \frac{1}{2} wx^2$$

Show that the elastic curve for the beam with conditions  $y=0 = \frac{dy}{dx}$  at  $x=0$  is given by  $y = \frac{w}{n^2 p}$

$$(1 - \cosh nx) + \frac{wx^2}{2p} \text{ where } n^2 = \frac{p}{EI}$$

Sol: The given differential equation can be written as

$$\frac{d^2 y}{dx^2} - \frac{p}{EI} y = \frac{-1}{2EI} wx^2 \text{ (or)}$$

$$\frac{d^2 y}{dx^2} - n^2 y = \frac{-w}{2EI} x^2 \text{ (or)}$$

$$(D^2 - n^2)y = \frac{-w}{2EI} x^2 \text{ -----(1)}$$

The auxiliary equation is  $(m^2 - n^2) = 0 \Rightarrow m = n$  and  $m = -n$

$$\therefore \text{C.F} = y_c = c_1 e^{nx} + c_2 e^{-nx}$$

$$\begin{aligned} \text{P.I} &= \frac{1}{(D^2 - n^2)} \left( \frac{-w}{2EI} x^2 \right) \\ &= \frac{w}{2EI} \left( \frac{1}{(n^2 - D^2)} x^2 \right) \\ &= \frac{w}{2EI} \left( \frac{1}{\left( n^2 \left( 1 - \frac{D^2}{n^2} \right) \right)} x^2 \right) \\ &= \frac{w}{2EI \cdot n^2} \left( 1 - \frac{D^2}{n^2} \right)^{-1} \cdot x^2 \\ &= \frac{w}{2EI \cdot n^2} \left( 1 + \frac{D^2}{n^2} + \dots \right) \cdot x^2 \\ &= \frac{w}{2EI \cdot n^2} \left( x^2 + \frac{2}{n^2} \right) \end{aligned}$$

$\therefore$  The general solution of equation (1) is given by  $y = \text{C.F} + \text{P.I}$

$$y = c_1 e^{nx} + c_2 e^{-nx} + \frac{w}{2EI \cdot n^2} \left( x^2 + \frac{2}{n^2} \right)$$

12. A condenser of capacity 'C' discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation  $L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$ . Given that  $L = 0.25\text{H}$ ,  $R = 250\text{ohms}$ ,  $c = 2 \times 10^{-6}\text{farads}$ , and that when  $t = 0$ , charge q is 0.002 coulombs and the current  $\frac{dq}{dt} = 0$ , obtain the value of 'q' in terms of t.

Sol:

Given differential equation is

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0 \text{ or } \frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{q}{LC} = 0 \text{ -----(1)}$$

Substituting the given values in (1), we get

$$\frac{d^2 q}{dt^2} + \frac{250}{0.25} \frac{dq}{dt} + \frac{q}{0.25 \times 2 \times 10^{-6}} = 0 \quad \text{or}$$

$$\frac{d^2 q}{dt^2} + 1000 \frac{dq}{dt} + 2 \times 10^6 q = 0 \quad \text{or}$$

$$(D^2 + 1000D + 2 \times 10^6)q = 0$$

$$\text{Its A.E is } m^2 + 1000m + 2 \times 10^6 = 0$$

$$\therefore m = \frac{-1000 \pm \sqrt{10^6 - 8 \times 10^6}}{2} = \frac{-1000 \pm 1000\sqrt{7}i}{2}$$

$$= -500 \pm 1323i$$

$$\text{Thus the solution is } q = e^{-500t} (c_1 \cos 1323t + c_2 \sin 1323t)$$

$$\text{When } t=0, q=0.002 \quad \text{since } c_1 = 0.002$$

$$\text{Now } \frac{dq}{dt} = -500e^{-500t} (c_1 \cos 1323t + c_2 \sin 1323t) + e^{-500t} \times 1323 (-c_1 \sin 1323t + c_2 \cos 1323t)$$

$$\text{When } t = 0, \frac{dq}{dt} = 0$$

$$\text{There fore } c_2 = 0.0008$$

$$\text{Hence the required solution is } q = e^{-500t} (0.002 \cos 1323t + 0.0008 \sin 1323t)$$

**13.** A particle is executing S.H.M, with amplitude 5 meters and time 4 seconds. Find the time required by the particle in passing between points which are at distances 4 and 2 meters from the Centre of force and are on the same side of it.

$$\text{Sol: The equation of S.H.M is } \frac{d^2 x}{dt^2} = -\mu^2 x \text{-----(1)}$$

$$\text{Give time period} = \frac{2\pi}{\mu} = 4$$

$$\mu = \frac{\pi}{2}$$

$$\text{We have the solution of (1) is } x = a \cos \mu t$$



$$a = 5, \mu = \frac{\pi}{2}$$

$$x = 5 \cos \frac{\pi}{2} t \text{-----(2)}$$

Let the times when the particle is at distances of 4 meters and 2 meters from the centre of motion respectively be  $t_1$  sec and  $t_2$  sec

$$\therefore t_1 = \frac{2}{\pi} \cos^{-1} \left( \frac{4}{5} \right) \quad \text{since } [4 = 5 \cos \left( \frac{\pi}{2} t_1 \right)]$$

$$\text{and } t_2 = \frac{2}{\pi} \cos^{-1} \left( \frac{2}{5} \right) \quad \text{since } [2 = 5 \cos \left( \frac{\pi}{2} t_2 \right)]$$

time required in passing through these points

$$t_2 - t_1 = \frac{2}{\pi} \left[ \cos^{-1} \left( \frac{2}{5} \right) - \cos^{-1} \left( \frac{4}{5} \right) \right] = 0.33 \text{ sec}$$

differentiating (2) w.r.to 't'

$$\frac{dx}{dt} = \frac{-5\pi}{2} \sin \frac{\pi}{2} t$$

$$= \frac{-5\pi}{2} \sqrt{1 - \frac{x^2}{25}}$$

$$\frac{dx}{dt} = \frac{-\pi}{2} \sqrt{25 - x^2}$$

$$\text{When } x=4 \text{ meters } v = \frac{\pi}{2} \sqrt{5^2 - 4^2} = 4.71 \text{ m/sec}$$

$$\text{When } x=2 \text{ meters } v = \frac{\pi}{2} \sqrt{21} \text{ m/sec}$$

**14.** A body weighing 10kgs is hung from a spring. A pull of 20kgs will stretch the spring to 10cms. The body is pulled down to 20cms below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds the maximum velocity and the period of oscillation.

Sol: Let O be the fixed end and A be the other end of the spring. Since load of 20kg attached to A stretches the spring by 0.1m.

Let e(AB) be the elongation produced by the mass 'm' hanging in equilibrium.

If 'k' be the restoring force per unit stretch of the spring due to elasticity, then for the equilibrium at B

$$Mg = T = ke$$

$$20 = T_0 = k * 0.1$$

$$K = 200 \text{ kg/m}$$

Let B be the equilibrium position when 10kg weight is

$$10 = T_B = k * AB \Rightarrow AB = \frac{10}{200} = 0.05\text{m}$$

Now the weight is pulled down to c, where BC=0.2. After any time t of its release from c, let the weight be at p, where BP=x.

Then the tension T = k \* AP

$$= 200(0.05+x) = 10 + 200x$$

∴ The equation of motion of the body is

$$\frac{w}{g} \frac{d^2 x}{dt^2} = w - T \quad \text{where } g = 9.8\text{m/sec}^2$$

$$= \frac{10}{9.8} \frac{d^2 x}{dt^2}$$

$$= 10 - (10+200x)$$

$$\Rightarrow \frac{d^2 x}{dt^2} = -\mu^2 x \quad \text{where } \mu = 14$$

This shows that the motion of the body is simple harmonic about B as centre and the period of oscillation =

$$\frac{2\pi}{\mu} = 0.45\text{sec}$$

Also the amplitude of motion being B C=0.2m, the displacement of the body from B at time t is given by x = 0.2cosct

$$X = 0.2\text{cosct} = 0.2\cos 14t \text{ m.}$$

$$\text{Maximum velocity} = \mu (\text{amplitude}) = 14 * 0.2 = 2.8\text{m/sec}$$

**MODULE -IV**  
**Multiple Integrals**

## Multiple Integrals

### Double Integral :

I. When  $y_1, y_2$  are functions of  $x$  and  $x_1$  and  $x_2$  are constants.  $f(x, y)$  is first integrated w.r.t  $y$  keeping 'x' fixed between limits  $y_1, y_2$  and then the resulting expression is integrated w.r.t 'x' within the limits  $x_1, x_2$  i.e.,

$$\iint_R f(x, y) dx dy = \int_{x=x_1}^{x=x_2} \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x, y) dy dx$$

II. When  $x_1, x_2$  are functions of  $y$  and  $y_1, y_2$  are constants,  $f(x, y)$  is first integrated w.r.t 'x' keeping 'y' fixed, within the limits  $x_1, x_2$  and then the resulting expression is integrated w.r.t 'y' between the limits  $y_1, y_2$  i.e.,

$$\iint_R f(x, y) dx dy = \int_{y=y_1}^{y=y_2} \int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x, y) dx dy$$

III. When  $x_1, x_2, y_1, y_2$  are all constants. Then

$$\iint_R f(x, y) dx dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) dx dy = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dy dx$$

### Problems

1. Evaluate  $\int_1^2 \int_1^3 xy^2 dx dy$

$$\text{Sol.} \quad \int_1^2 \left[ \int_1^3 xy^2 dx \right] dy$$

$$= \int_1^2 \left[ y^2 \cdot \frac{x^2}{2} \right]_1^3 dy = \int_1^2 \frac{y^2}{2} dy [9-1]$$

$$= \frac{8}{2} \int_1^2 y^2 dy = 4 \int_1^2 y^2 dy$$

$$= 4 \left[ \frac{y^3}{3} \right]_1^2 = \frac{4}{3} [8-1] = \frac{4 \cdot 7}{3} = \frac{28}{3}$$

2. Evaluate  $\int_0^2 \int_0^x y dy dx$

Sol.  $\int_{x=0}^2 \int_{y=0}^x y dy dx = \int_{x=0}^2 \left[ \int_{y=0}^x y dy \right] dx$

$$= \int_{x=0}^2 \left[ \frac{y^2}{2} \right]_0^x dx = \int_{x=0}^2 \frac{1}{2} (x^2 - 0) dx = \frac{1}{2} \int_{x=0}^2 x^2 dx = \frac{1}{2} \left[ \frac{x^3}{3} \right]_0^2 = \frac{1}{6} (8-0) = \frac{8}{6} = \frac{4}{3}$$

3. Evaluate  $\int_0^5 \int_0^{x^2} x(x^2 + y^2) dx dy$

Sol.

$$\int_{x=0}^5 \int_{y=0}^{x^2} x(x^2 + y^2) dy dx = \int_{x=0}^5 \left[ x^3 y + \frac{xy^3}{3} \right]_{y=0}^{x^2} dx$$

$$= \int_{x=0}^5 \left[ x^3 \cdot x^2 + \frac{x(x^2)^3}{3} \right] dx = \int_{x=0}^5 \left( x^5 + \frac{x^7}{3} \right) dx = \left[ \frac{x^6}{6} + \frac{1}{3} \cdot \frac{x^8}{8} \right]_0^5 = \frac{5^6}{6} + \frac{5^8}{24}$$

4. Evaluate  $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2}$

Sol:  $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2} = \int_{x=0}^1 \left[ \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2} dy \right] dx$

$$= \int_{x=0}^1 \left[ \int_{y=0}^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2} dy \right] dx = \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} \left[ \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_{y=0}^{\sqrt{1+x^2}} dx \quad \left[ \because \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \right]$$

$$= \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} \left[ \tan^{-1} 1 - \tan^{-1} 0 \right] dx \quad \text{or} \quad \frac{\pi}{4} (\sinh^{-1} x)_0^1 = \frac{\pi}{4} (\sinh^{-1} 1)$$

$$= \frac{\pi}{4} \int_{x=0}^1 \frac{1}{\sqrt{1+x^2}} dx = \frac{\pi}{4} \left[ \log(x + \sqrt{x^2+1}) \right]_{x=0}^1$$

$$= \frac{\pi}{4} \log(1 + \sqrt{2})$$

5. Evaluate  $\int_0^4 \int_0^{x^2} e^{y/x} dy dx$

Answer:  $3e^4 - 7$

6. Evaluate  $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dx dy$

Answer:  $3/35$

7. Evaluate  $\int_0^2 \int_0^x e^{(x+y)} dy dx$

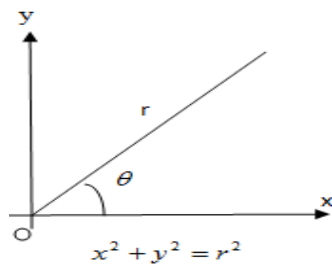
Ans:  $\frac{e^4 - e^2}{2}$

8. Evaluate  $\int_0^{\frac{\pi}{2}} \int_{-1}^1 x^2 y^2 dx dy$

Ans:  $\frac{\pi^3}{36}$

9. Evaluate  $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$

Sol:  $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^\infty e^{-y^2} \left[ \int_0^\infty e^{-x^2} dx \right] dy$



$$= \int_0^\infty e^{-y^2} \frac{\sqrt{\pi}}{2} dy \quad \because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

Alter:

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^\infty e^{-r^2} r dr d\theta \quad (\because x^2 + y^2 = r^2)$$

(changing to polar coordinates taking  $x = r \cos \theta$ ,  $y = r \sin \theta$ )

$$\begin{aligned}
 &= \int_0^{\pi/2} \left[ e^{-r^2} \right]_0^{\infty} d\theta - \int_0^{\pi/2} \left[ 0-1 \right] d\theta \\
 &= \int_0^{\pi/2} [-2] d\theta \\
 &= \frac{1}{2} (\theta)_0^{\pi/2} = \frac{1}{2} \left( \frac{\pi}{2} - 0 \right) \\
 &= \frac{\pi}{4}
 \end{aligned}$$

10. Evaluate  $\iint_R xy(x+y) dx dy$  over the region R bounded by  $y=x^2$  and  $y=x$

Sol:  $y=x^2$  is a parabola through (0, 0) symmetric about y-axis  $y=x$  is a straight line through (0,0) with slope 1.

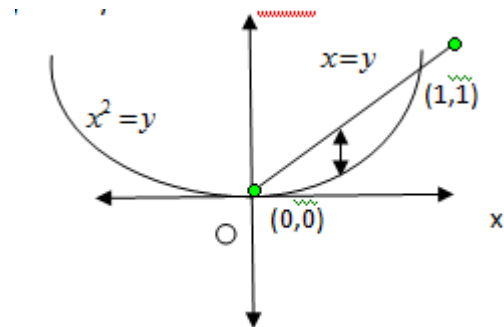
Let us find their points of intersection solving  $y=x^2$ ,  $y=x$  we get  $x^2=x \Rightarrow x=0,1$  Hence  $y=0, 1$

$\therefore$  The point of intersection of the curves are (0,0), (1,1)

Consider  $\iint_R xy(x+y) dx dy$

For the evaluation of the integral, we first integrate w.r.t 'y' from  $y=x^2$  to  $y=x$  and then w.r.t. 'x' from  $x=0$  to  $x=1$

$$\int_{x=0}^1 \left[ \int_{y=x^2}^x xy(x+y) dy \right] dx = \int_{x=0}^1 \left[ \int_{y=x^2}^x (x^2 y + xy^2) dy \right] dx$$



$$\begin{aligned}
 &= \int_{x=0}^1 \left( x^2 \frac{y^2}{2} + \frac{xy^3}{3} \right)_{y=x^2}^x dx \\
 &= \int_{x=0}^1 \left( \frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx \\
 &= \int_{x=0}^1 \left( \frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx \\
 &= \left( \frac{5}{6} \cdot \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24} \right)_0^1 \\
 &= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28-12-7}{168} = \frac{28-19}{168} = \frac{9}{168} = \frac{3}{56}
 \end{aligned}$$

11. Evaluate  $\iint_R xy dx dy$  where R is the region bounded by x-axis and  $x=2a$  and the curve  $x^2=4ay$ .

Sol. The line  $x=2a$  and the parabola  $x^2=4ay$  intersect at B(2a,a)

• The given integral =  $\iint_R xy dx dy$

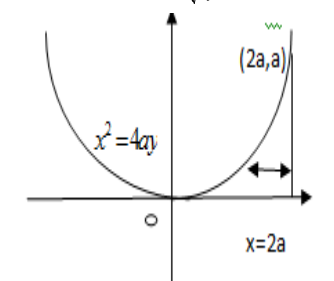
Let us fix 'y'

For a fixed 'y', x varies from  $2\sqrt{ay}$  to 2a. Then y varies from 0 to a.

Hence the given integral can also be written as

$$\int_{y=0}^a \int_{x=2\sqrt{ay}}^{x=2a} xy \, dx \, dy = \int_{y=0}^a \left[ \int_{x=2\sqrt{ay}}^{x=2a} x \, dx \right] y \, dy$$

$$= \int_{y=0}^a \left[ \frac{x^2}{2} \right]_{x=2\sqrt{ay}}^{x=2a} y \, dy$$



$$= \int_{y=0}^a [2a^2 - 2ay] y \, dy$$

$$= \left[ \frac{2a^2 y^2}{2} - \frac{2ay^3}{3} \right]_0^a = a^4 - \frac{2a^4}{3} = \frac{3a^4 - 2a^4}{3} = \frac{a^4}{3}$$

12. Evaluate  $\int_0^1 \int_0^{\pi/2} r \sin \theta \, d\theta \, dr$

Sol.  $\int_{r=0}^1 r \left[ \int_{\theta=0}^{\pi/2} \sin \theta \, d\theta \right] dr$

$$= \int_{r=0}^1 r (-\cos \theta)_{\theta=0}^{\pi/2} dr$$

$$= \int_{r=0}^1 -r (\cos \pi/2 - \cos 0) dr$$

$$= \int_{r=0}^1 -r (0 - 1) dr = \int_0^1 r dr = \left[ \frac{r^2}{2} \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

14. Evaluate  $\iint_R (x^2 + y^2) \, dx \, dy$  in the positive quadrant

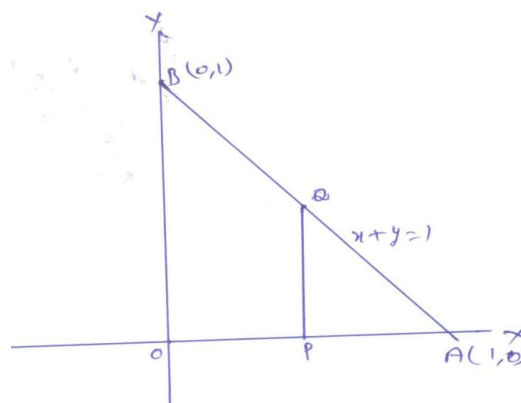
For

Which  $x + y \leq 1$

Sol.  $\iint_R (x^2 + y^2) \, dx \, dy = \int_{x=0}^1 dx \int_{y=0}^{y=1-x} (x^2 + y^2) \, dy$

$$= \int_{x=0}^1 \left( x^2 y + \frac{y^3}{3} \right)_0^{1-x} dx$$

$$= \int_{x=0}^1 \left( x^2 - x^3 + \frac{1}{3}(1-x)^3 \right) dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} - \frac{1}{12}(1-x)^4 \right]_0^1$$





$$= \frac{1}{5} - \frac{1}{4} - 0 + \frac{1}{12} = \frac{1}{6}$$

14. Evaluate  $\iint_R (x^2 + y^2) dx dy$  over the area bounded by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol. Given ellipse is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

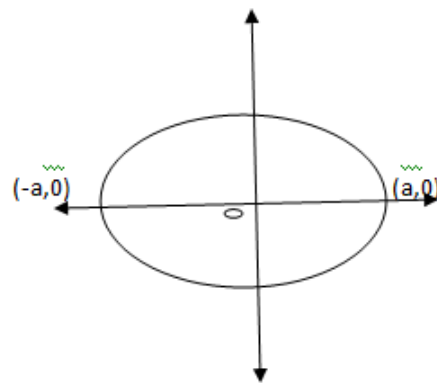
$$\text{i.e., } \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{1}{a^2} (a^2 - x^2) \text{ (or) } y^2 = \frac{b^2}{a^2} (a^2 - x^2)$$

$$\therefore y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

Hence the region of integration R can be expressed as

$$-a \leq x \leq a, \frac{-b}{a} \sqrt{a^2 - x^2} \leq y \leq \frac{b}{a} \sqrt{a^2 - x^2}$$

$$\therefore \iint_R (x^2 + y^2) dx dy = \int_{x=-a}^a \int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2) dy dx$$



$$= 2 \int_{x=-a}^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2) dy dx = 2 \int_{-a}^a \left( x^2 y + \frac{y^3}{3} \right) \Big|_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2 \int_{-a}^a \left[ x^2 \cdot \frac{b}{a} \sqrt{a^2 - x^2} + \frac{b^3}{3a^3} (a^2 - x^2)^{3/2} \right] dx$$

$$= 4 \int_0^a \left[ \frac{b}{a} x^2 \sqrt{a^2 - x^2} + \frac{b^3}{3a^3} (a^2 - x^2)^{3/2} \right] dx$$

Changing to polar coordinates

putting  $x = a \sin \theta$

$$dx = a \cos \theta d\theta$$

$$\frac{x}{a} = \sin \theta \Rightarrow \theta = \sin^{-1} \frac{x}{a}$$

$$x \rightarrow 0, \theta \rightarrow 0$$

$$x \rightarrow a, \theta \rightarrow \frac{\pi}{2}$$

$$= 4 \int_0^{\pi/2} \left[ \frac{a^3 b \sin^2 \theta \cos^2 \theta}{3} + \frac{ab^3 \cos^4 \theta}{4} \right] d\theta = 4 \left[ \frac{a^3 b}{4 \cdot 2 \cdot 2} \cdot \frac{\pi}{2} + \frac{ab^3}{3 \cdot 4 \cdot 2 \cdot 2} \cdot \frac{\pi}{2} \right]$$

$$\left[ \because \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \dots \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= \frac{4\pi}{16} (a^3 b + ab^3) = \frac{\pi ab}{4} (a^2 + b^2)$$

### Double integrals in polar co-ordinates:

5. Evaluate  $\int_0^{\pi/4} \int_0^{a \sin \theta} \frac{r dr d\theta}{\sqrt{a^2 - r^2}}$

Sol.  $\int_0^{\pi/4} \int_0^{a \sin \theta} \frac{r dr d\theta}{\sqrt{a^2 - r^2}} = \int_0^{\pi/4} \left\{ \int_0^{a \sin \theta} \frac{r}{\sqrt{a^2 - r^2}} dr \right\} d\theta = -\frac{1}{2} \int_0^{\pi/4} \left\{ \int_0^{a \sin \theta} \frac{-2r}{\sqrt{a^2 - r^2}} dr \right\} d\theta$

$$= -\frac{1}{2} \int_0^{\pi/4} 2 \left( \sqrt{a^2 - r^2} \right)_0^{a \sin \theta} d\theta = (-1) \int_0^{\pi/4} 2 \left[ \sqrt{a^2 - a^2 \sin^2 \theta} - \sqrt{a^2 - 0} \right] d\theta$$

$$= (-a) \int_0^{\pi/4} (\cos \theta - 1) d\theta = (-a) (\sin \theta - \theta)_0^{\pi/4}$$

$$= (-a) \left[ \left[ \sin \frac{\pi}{4} - \frac{\pi}{4} \right] - (0 - 0) \right]$$

$$= (-a) \left[ \frac{1}{\sqrt{2}} - \frac{\pi}{4} \right] = 2 \left[ \frac{\pi}{4} - \frac{1}{\sqrt{2}} \right]$$

6. Evaluate  $\int_0^{\pi} \int_0^{a \sin \theta} r dr d\theta$       Ans:  $\frac{a^2 \pi}{4}$

7. Evaluate  $\int_0^{\infty} \int_0^{\pi/2} e^{-r^2} r d\theta dr$       Ans:  $\frac{\pi}{4}$

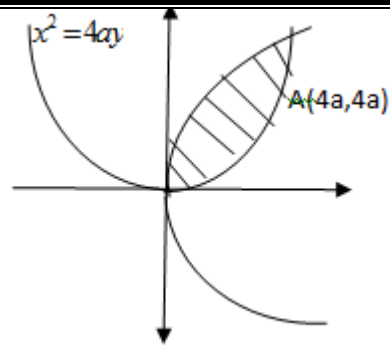
8. Evaluate  $\int_0^{\pi} \int_0^{a(1+\cos \theta)} r dr d\theta$       Ans:  $\frac{3\pi a^2}{4}$

### Change of order of Integration:

4. Change the order of Integration and evaluate  $\int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy dx$

Sol. In the given integral for a fixed x, y varies from  $\frac{x^2}{4a}$  to  $2\sqrt{ax}$  and then x varies from 0 to 4a. Let us draw

the curves  $y = \frac{x^2}{4a}$  and  $y = 2\sqrt{ax}$



the region of integration is the shaded region in diagram.

The given integral is 
$$= \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy dx$$

Changing the order of integration, we must fix  $y$  first, for a fixed  $y$ ,  $x$  varies from  $\frac{y^2}{4a}$  to  $\sqrt{4ay}$  and then  $y$  varies from 0 to  $4a$ .

Hence the integral is equal to

$$\begin{aligned} \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} dx dy &= \int_{y=0}^{4a} \left[ \int_{x=y^2/4a}^{2\sqrt{ay}} dx \right] dy \\ &= \int_{y=0}^{4a} \left[ x \right]_{x=y^2/4a}^{2\sqrt{ay}} dy = \int_{y=0}^{4a} \left[ 2\sqrt{ay} - \frac{y^2}{4a} \right] dy \\ &= \left[ 2\sqrt{a} \cdot \frac{y^{3/2}}{3/2} - \frac{1}{4a} \cdot \frac{y^3}{3} \right]_0^{4a} \\ &= \frac{4}{3} \cdot \sqrt{a} \cdot 4a \sqrt{4a} - \frac{1}{12a} \cdot 64a^3 \end{aligned}$$

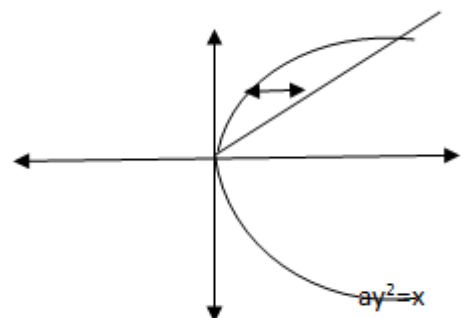
$$= \frac{32}{3} a^2 - \frac{16}{3} a^2 = \frac{16}{3} a^2$$

5. Change the order of integration and evaluate 
$$= \int_0^a \int_{x/a}^{\sqrt{y/a}} (x^2 + y^2) dx dy$$

Sol. In the given integral for a fixed  $x$ ,  $y$  varies from  $\frac{x}{a}$  to  $\sqrt{\frac{x}{a}}$  and then  $x$  varies from 0 to  $a$

Hence we shall draw the curves  $y = \frac{x}{a}$  and  $y = \sqrt{\frac{x}{a}}$

i.e.



$ay^2=x$  and  $ay^2=x$

we get  $ay = ay^2$

$$\Rightarrow ay - ay^2 = 0$$

$$\Rightarrow ay(1 - y) = 0$$

$$\Rightarrow y = 0, y = 1$$

If  $y=0$ ,  $x=0$  if  $y=1$ ,  $x=a$

The shaded region is the region of integration. The given integral is  $\int_{x=0}^a \int_{y=x/a}^{\sqrt{x/a}} (x^2 + y^2) dx dy$

Changing the order of integration, we must fix  $y$  first. For a fixed  $y$ ,  $x$  varies from  $ay^2$  to  $ay$  and then  $y$  varies from 0 to 1.

Hence the given integral, after change of the order of integration becomes

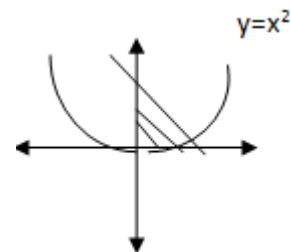
$$\begin{aligned} & \int_{y=0}^1 \int_{x=ay^2}^{ay} (x^2 + y^2) dx dy \\ &= \int_{y=0}^1 \left[ \int_{x=ay^2}^{ay} (x^2 + y^2) dx \right] dy \\ &= \int_{y=0}^1 \left( \frac{x^3}{3} + xy^2 \right)_{x=ay^2}^{ay} dy \\ &= \int_{y=0}^1 \left( \frac{a^3 y^3}{3} + ay^3 - \frac{a^3 y^6}{3} - ay^4 \right) dy \\ &= \left( \frac{a^3 y^4}{12} + \frac{ay^4}{4} - \frac{a^3 y^7}{21} - \frac{ay^5}{5} \right)_{y=0}^1 \\ &= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} = \frac{a^3}{28} + \frac{a}{20} \end{aligned}$$

3. Change the order of integration in  $\int_0^1 \int_{x^2}^{2-x} xy dx dy$  and hence evaluate the double integral.

Sol. In the given integral for a fixed  $x$ ,  $y$  varies from  $x^2$  to  $2-x$  and then  $x$  varies from 0 to 1. Hence we shall draw the curves  $y=x^2$  and  $y=2-x$

The line  $y=2-x$  passes through  $(0,2)$ ,  $(2,0)$

Solving  $y=x^2$ ,  $y=2-x$



Then we get  $x^2 = 2 - x$

$$\Rightarrow x^2 + x - 2 = 0$$

$$\Rightarrow x^2 + 2x - x - 2 = 0$$

$$\Rightarrow x(x+2) - 1(x+2) = 0$$

$$\Rightarrow (x-1)(x+2) = 0$$

$$\Rightarrow x = 1, -2$$

$$\text{If } x = 1, y = 1$$

$$\text{If } x = -2, y = 4$$

Hence the points of intersection of the curves are  $(-2, 4)$   $(1, 1)$

The Shaded region in the diagram is the region of intersection.

Changing the order of integration, we must fix  $y$ , for the region within OACO for a fixed  $y$ ,  $x$  varies from 0 to  $\sqrt{y}$

Then  $y$  varies from 0 to 1

For the region within CABC, for a fixed  $y$ ,  $x$  varies from 0 to  $2-y$ , then  $y$  varies from 1 to 2

$$\begin{aligned} \text{Hence } \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx &= \iint_{OACO} xy \, dx \, dy + \iint_{CABC} xy \, dx \, dy \\ &= \int_{y=0}^1 \left[ \int_{x=0}^{\sqrt{y}} x \, dx \right] y \, dy + \int_{y=1}^2 \left[ \int_{x=0}^{2-y} x \, dx \right] y \, dy \\ &= \int_{y=0}^1 \left( \frac{x^2}{2} \right)_{x=0}^{\sqrt{y}} y \, dy + \int_{y=1}^2 \left( \frac{x^2}{2} \right)_{x=0}^{2-y} y \, dy \\ &= \int_{y=0}^1 \frac{y}{2} \cdot y \, dy + \int_{y=1}^2 \frac{(2-y)^2}{2} y \, dy \\ &= \frac{1}{2} \int_{y=0}^1 y^2 \, dy + \frac{1}{2} \int_{y=1}^2 (4y - 4y^2 + y^3) \, dy \\ &= \frac{1}{2} \cdot \left( \frac{y^3}{3} \right)_0^1 + \frac{1}{2} \cdot \left[ \frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4} \right]_1^2 \\ &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \left[ 2 \cdot 4 - 2 \cdot 1 - \frac{4}{3}(8-1) + \frac{1}{4}(16-1) \right] \\ &= \frac{1}{6} + \frac{1}{2} \left[ 6 - \frac{28}{3} + \frac{15}{4} \right] = \frac{1}{6} + \frac{1}{2} \left[ \frac{72-112+45}{12} \right] = \frac{1}{6} + \frac{1}{2} \left[ \frac{5}{12} \right] = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8} \end{aligned}$$

4. Changing the order of integration  $\int_0^a \int_{x^2/a}^{2a-x} xy^2 \, dy \, dx$

5. Change of the order of integration  $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dx \, dy$     *Ans* :  $\frac{\pi}{16}$

Hint : Now limits are  $y = 0$  to  $1$  and  $x = 0$  to  $\sqrt{1-y^2}$

$$\text{put } y = \sin \theta$$

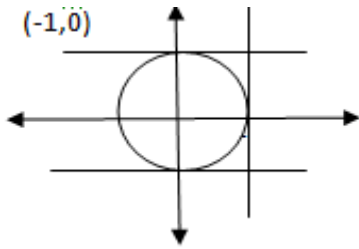
$$\sqrt{1-y^2} = \cos \theta$$

$$dy = \cos \theta \, d\theta$$

$$= \int_{-1}^1 y^2 \sqrt{1-y^2} dy$$

$$= \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta = \int_0^{\pi/2} \sin^2 \theta d\theta - \int_0^{\pi/2} \sin^4 \theta d\theta$$

$$= \frac{1}{2} \left( \frac{\pi}{2} \right) - \frac{3}{4} \cdot \frac{1}{2} \left( \frac{\pi}{2} \right) = \frac{\pi}{16}$$



### Change of variables:

The variables  $x, y$  in  $\iint_R f(x, y) dx dy$  are changed to  $u, v$  with the help of the relations  $x = f_1(u, v), y = f_2(u, v)$

then the double integral is transferred into

$$\iint_{R^1} f[f_1(u, v), f_2(u, v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Where  $R^1$  is the region in the  $uv$  plane, corresponding to the region  $R$  in the  $xy$ -plane.

### Changing from Cartesian to polar co-ordinates

$$x = r \cos \theta, y = r \sin \theta$$

$$\frac{\partial \left( \begin{matrix} x \\ y \end{matrix} \right)}{\partial \left( \begin{matrix} r \\ \theta \end{matrix} \right)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r(\cos^2 \theta + \sin^2 \theta) = r \therefore \iint_R f(x, y) dx dy = \iint_{R_1} f(r \cos \theta, r \sin \theta) r dr d\theta$$

**Note :** In polar form  $dx dy$  is replaced by  $r dr d\theta$

### Problems:

1. Evaluate the integral by changing to polar co-ordinates  $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$

Sol. The limits of  $x$  and  $y$  are both from 0 to  $\infty$ .

$\therefore$  The region is in the first quadrant where  $r$  varies from 0 to  $\infty$  and  $\theta$  varies from 0 to  $\frac{\pi}{2}$

Substituting  $x = r \cos \theta, y = r \sin \theta$  and  $dx dy = r dr d\theta$

$$\text{Hence } \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^\infty e^{-r^2} r dr d\theta$$

Put  $r^2 = t$

$$\Rightarrow 2rdr = dt$$

$$\Rightarrow r dr = \frac{dt}{2}$$

Where  $r = 0 \Rightarrow t = 0$  and  $r = \infty \Rightarrow t = \infty$

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_{\theta=0}^{\pi/2} \int_{t=0}^\infty \frac{1}{2} e^{-t} dt d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} (e^{-t})_0^\infty d\theta$$

$$= \frac{-1}{2} \int_0^{\pi/2} (0-1) d\theta \Rightarrow \frac{1}{2} (\theta)_0^{\pi/2} = \frac{1}{2} \pi/2 = \pi/4$$

2. Evaluate the integral by changing to polar co-ordinates  $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) dx dy$

Sol. The limits for x are  $x=0$  to  $x = \sqrt{a^2 - y^2}$   
 $\Rightarrow x^2 + y^2 = a^2$

$\therefore$  The given region is the first quadrant of the circle.

By changing to polar co-ordinates

$$x = r \cos \theta, y = r \sin \theta, dx dy = r dr d\theta$$

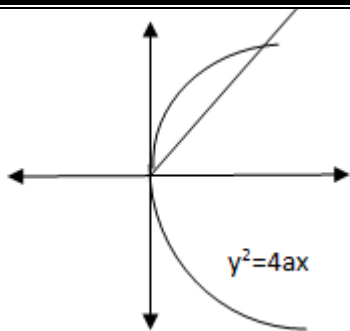
Here 'r' varies from 0 to a and ' $\theta$ ' varies from 0 to  $\pi/2$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2) dx dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^a r^2 r dr d\theta = \int_0^{\pi/2} \left( \frac{r^4}{4} \right)_0^a d\theta = \frac{a^4}{4} (\theta)_0^{\pi/2}$$

$$= \pi/8 a^4$$

6. Show that  $\int_0^{4a} \int_{y^2/4a}^y \frac{x^2 - y^2}{x^2 + y^2} dx dy = 8a^2 \left( \frac{\pi}{2} - \frac{5}{3} \right)$

Sol. The region of integration is given by  $x = \frac{y^2}{4a}$ ,  $x = y$  and  $y=0$ ,  $y=4a$



i.e., The region is bounded by the parabola  $y^2=4ax$  and the straight line  $x=y$ .

Let  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then  $dx dy = r dr d\theta$

The limits for  $r$  are  $r=0$  at  $O$  and for  $P$  on the parabola

$$r^2 \sin^2 \theta = 4a(r \cos \theta) \Rightarrow r = \frac{4a \cos \theta}{\sin^2 \theta}$$

For the line  $y=x$ , slope  $m=1$  i.e.,  $\tan \theta = 1$ ,  $\theta = \pi/4$

The limits for  $\theta: \pi/4 \rightarrow \pi/2$

Also  $x^2 - y^2 = r^2 (\cos^2 \theta - \sin^2 \theta)$  and  $x^2 + y^2 = r^2$

$$\therefore \int_0^{4a} \int_{y^2/4a}^y \frac{x^2 - y^2}{x^2 + y^2} dx dy = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{4a \cos \theta / \sin^2 \theta} (\cos^2 \theta - \sin^2 \theta) r dr d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2 \theta - \sin^2 \theta) \left( \frac{r^2}{2} \right)_0^{4a \cos \theta / \sin^2 \theta} d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cos^2 \theta - \sin^2 \theta) \frac{\cos^2 \theta}{\sin^4 \theta} d\theta = 8a^2 \int_{\pi/4}^{\pi/2} (\cos^4 \theta - \cot^2 \theta) d\theta = 8a^2 \left[ \frac{3\pi - 8}{12} + \frac{\pi}{4} - 1 \right] = 8a^2 \left( \frac{\pi}{2} - \frac{5}{3} \right).$$



### Triple integrals:

If  $x_1, x_2$  are constants.  $y_1, y_2$  are functions of  $x$  and  $z_1, z_2$  are functions of  $x$  and  $y$ , then  $f(x, y, z)$  is first integrated w.r.t. 'z' between the limits  $z_1$  and  $z_2$  keeping  $x$  and  $y$  fixed. The resulting expression is integrated w.r.t. 'y' between the limits  $y_1$  and  $y_2$  keeping  $x$  constant. The resulting expression is integrated w.r.t. 'x' from  $x_1$  to  $x_2$

$$\iiint_V f(x, y, z) dx dy dz = \int_{x=a}^b \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x, y, z) dz dy dx$$

### Problems

3. Evaluate  $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dx dy dz$

Sol.  $\int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz dx dy dz$

$$\begin{aligned} &= \int_{x=0}^1 dx \int_{y=0}^{\sqrt{1-x^2}} dy \int_{z=0}^{\sqrt{1-x^2-y^2}} xyz dz \\ &= \int_{x=0}^1 dx \int_{y=0}^{\sqrt{1-x^2}} xy \left( \frac{z^2}{2} \right)_{z=0}^{\sqrt{1-x^2-y^2}} dy \\ &= \frac{1}{2} \int_{x=0}^1 dx \int_{y=0}^{\sqrt{1-x^2}} xy (1-x^2-y^2) dy \\ &= \frac{1}{2} \int_{x=0}^1 dx \int_{y=0}^{\sqrt{1-x^2}} x [(1-x^2)y - y^3] dy \\ &= \frac{1}{2} \int_{x=0}^1 x \left[ (1-x^2) \frac{y^2}{2} - \frac{y^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \frac{1}{2} \int_{x=0}^1 x \left[ \frac{y^2}{2} - \frac{x^2 y^2}{2} - \frac{y^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx \\ &= \frac{1}{8} \int_{x=0}^1 x \left[ 2(1-x^2) - 2x^2(1-x^2) - (1-x^2)^2 \right] dx \\ &= \frac{1}{8} \int_{x=0}^1 (x - 2x^3 + x^5) dx = \frac{1}{8} \left[ \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right]_0^1 \end{aligned}$$

$$= \frac{1}{8} \left( \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48}$$

4. Evaluate  $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dx dy dz$

$$\text{Sol: } \int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dx dy dz$$

$$= \int_{-1}^1 \int_0^z \left[ xy + \frac{y^2}{2} + zy \right]_{x-z}^{x+z} dx dz$$

$$= \int_{-1}^1 \int_0^z x(x+z) - x(x-z) + \left[ \frac{x+z}{2} \right]^2 - \left[ \frac{x-z}{2} \right]^2 + z(x+z) - z(x-z) dx dz$$

$$= \int_{-1}^1 \int_0^z \left[ 2z(x+z) + \frac{1}{2} 4xz \right] dx dz$$

$$= 2 \int_{-1}^1 \left[ z \cdot \frac{x^2}{2} + z^2 x + z \cdot \frac{x^2}{2} \right]_0^z dz = 2 \int_{-1}^1 \left[ \frac{z^3}{2} + z^3 + \frac{z^3}{2} \right] dz = 4 \cdot \left( \frac{z^4}{4} \right)_{-1}^1 = 0$$

# MODULE-V

## VECTOR CALCULUS

## Vector Calculus and Vector Operators

### INTRODUCTION

In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introduced.

### DIFFERENTIATION OF A VECTOR FUNCTION

Let  $S$  be a set of real numbers. Corresponding to each scalar  $t \in S$ , let there be associated a unique vector  $\vec{f}$ . Then  $\vec{f}$  is said to be a vector (vector valued) function.  $S$  is called the domain of  $\vec{f}$ . We write  $\vec{f} = \vec{f}(t)$ .

Let  $\vec{i}, \vec{j}, \vec{k}$  be three mutually perpendicular unit vectors in three dimensional space. We can write  $\vec{f} = \vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ , where  $f_1(t), f_2(t), f_3(t)$  are real valued functions (which are called components of  $\vec{f}$ ). (we shall assume that  $\vec{i}, \vec{j}, \vec{k}$  are constant vectors).

#### 1. Derivative:

Let  $\vec{f}$  be a vector function on an interval  $I$  and  $a \in I$ . Then  $\lim_{t \rightarrow a} \frac{\vec{f}(t) - \vec{f}(a)}{t - a}$ , if exists, is called the derivative of  $\vec{f}$  at  $a$  and is denoted by  $\vec{f}'(a)$  or  $\left(\frac{d\vec{f}}{dt}\right)$  at  $t = a$ . We also say that  $\vec{f}$  is differentiable at  $t = a$  if  $\vec{f}'(a)$  exists.

#### 2. Higher order derivatives

Let  $\vec{f}$  be differentiable on an interval  $I$  and  $\vec{f}' = \frac{d\vec{f}}{dt}$  be the derivative of  $\vec{f}$ . If  $\lim_{t \rightarrow a} \frac{\vec{f}'(t) - \vec{f}'(a)}{t - a}$  exists for every  $a \in I_1 \subset I$ . It is denoted by  $\vec{f}'' = \frac{d^2\vec{f}}{dt^2}$ .

Similarly we can define  $\vec{f}'''(t)$  etc.

#### **We now state some properties of differentiable functions (without proof)**

(1) Derivative of a constant vector is  $\vec{0}$ .

If  $\vec{a}$  and  $\vec{b}$  are differentiable vector functions, then

$$(2). \frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$$

$$(3). \frac{d}{dt}(\bar{a} \cdot \bar{b}) = \frac{d\bar{a}}{dt} \cdot \bar{b} + \bar{a} \cdot \frac{d\bar{b}}{dt}$$

$$(4). \frac{d}{dt}(\bar{a} \times \bar{b}) = \frac{d\bar{a}}{dt} \times \bar{b} + \bar{a} \times \frac{d\bar{b}}{dt}$$

(5). If  $\bar{f}$  is a differentiable vector function and  $\phi$  is a scalar differential function, then

$$\frac{d}{dt}(\phi \bar{f}) = \phi \frac{d\bar{f}}{dt} + \frac{d\phi}{dt} \bar{f}$$

(6). If  $\bar{f} = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$  where  $f_1(t), f_2(t), f_3(t)$  are cartesian components of the vector  $\bar{f}$ , then  $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$

(7). The necessary and sufficient condition for  $\bar{f}(t)$  to be constant vector function is  $\frac{d\bar{f}}{dt} = \bar{0}$ .

### 3. Partial Derivatives

Partial differentiation for vector valued functions can be introduced as was done in the case of functions of real variables. Let  $\bar{f}$  be a vector function of scalar variables  $p, q, t$ . Then we write  $\bar{f} = \bar{f}(p, q, t)$ . Treating  $t$  as a variable and  $p, q$  as constants, we define

$$Lt_{\delta t \rightarrow 0} \frac{\bar{f}(p, q, t + \delta t) - \bar{f}(p, q, t)}{\delta t}$$

if exists, as partial derivative of  $\bar{f}$  w.r.t.  $t$  and is denote by  $\frac{\partial \bar{f}}{\partial t}$

Similarly, we can define  $\frac{\partial \bar{f}}{\partial p}, \frac{\partial \bar{f}}{\partial q}$  also. The following are some useful results on partial differentiation.

### 4. Properties

$$1) \frac{\partial}{\partial t}(\phi \bar{a}) = \frac{\partial \phi}{\partial t} \bar{a} + \phi \frac{\partial \bar{a}}{\partial t}$$

$$2). \text{ If } \lambda \text{ is a constant, then } \frac{\partial}{\partial t}(\lambda \bar{a}) = \lambda \frac{\partial \bar{a}}{\partial t}$$

$$3). \text{ If } \bar{c} \text{ is a constant vector, then } \frac{\partial}{\partial t}(\phi \bar{c}) = \bar{c} \frac{\partial \phi}{\partial t}$$

$$4). \frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial \bar{a}}{\partial t} \pm \frac{\partial \bar{b}}{\partial t}$$

$$5). \frac{\partial}{\partial t}(\bar{a} \cdot \bar{b}) = \frac{\partial \bar{a}}{\partial t} \cdot \bar{b} + \bar{a} \cdot \frac{\partial \bar{b}}{\partial t}$$

$$6). \frac{\partial}{\partial t}(\bar{a} \times \bar{b}) = \frac{\partial \bar{a}}{\partial t} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial t}$$

7). Let  $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ , where  $f_1, f_2, f_3$  are differential scalar functions of more than one variable,

Then  $\frac{\partial \bar{f}}{\partial t} = \bar{i} \frac{\partial f_1}{\partial t} + \bar{j} \frac{\partial f_2}{\partial t} + \bar{k} \frac{\partial f_3}{\partial t}$  (treating  $\bar{i}, \bar{j}, \bar{k}$  as fixed directions)

Let  $\bar{f} = \bar{f}(p, q, t)$ . Then  $\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial t} \right)$ ,  $\frac{\partial^2 f}{\partial p \partial t} = \frac{\partial}{\partial p} \left( \frac{\partial f}{\partial t} \right)$  etc.

**6. Scalar and vector point functions:** Consider a region in three dimensional space. To each point  $p(x, y, z)$ , suppose we associate a unique real number (called scalar) say  $\phi$ . This  $\phi(x, y, z)$  is called a scalar point function. Scalar point function defined on the region. Similarly if to each point  $p(x, y, z)$  we associate a unique vector  $\bar{f}(x, y, z)$ ,  $\bar{f}$  is called a **vector point function**.

### Examples:

For example take a heated solid. At each point  $p(x, y, z)$  of the solid, there will be temperature  $T(x, y, z)$ . This  $T$  is a scalar point function.

Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position  $p(x, y, z)$  in space, it will be having some speed, say,  $v$ . This **speed**  $v$  is a scalar point function.

Consider a particle moving in space. At each point  $P$  on its path, the particle will be having a velocity  $\bar{v}$  which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

In a magnetic field, at any point  $P(x, y, z)$  there will be a magnetic force  $\bar{f}(x, y, z)$ . This is called magnetic force field. This is also an example of a vector point function.

## 7. Tangent vector to a curve in space.

Consider an interval  $[a, b]$ .

Let  $x = x(t), y = y(t), z = z(t)$  be continuous and derivable for  $a \leq t \leq b$ .

Then the set of all points  $(x(t), y(t), z(t))$  is called a curve in a space.

Let  $A = (x(a), y(a), z(a))$  and  $B = (x(b), y(b), z(b))$ . These  $A, B$  are called the end points of the curve. If  $A = B$ , the curve is said to be a closed curve.

Let  $P$  and  $Q$  be two neighbouring points on the curve.

Let  $\overline{OP} = \bar{r}(t), \overline{OQ} = \bar{r}(t + \delta t) = \bar{r} + \delta \bar{r}$ . Then  $\delta \bar{r} = \overline{OQ} - \overline{OP} = \overline{PQ}$

Then  $\frac{\delta \bar{r}}{\delta t}$  is along the vector  $\overline{PQ}$ . As  $Q \rightarrow P$ ,  $\overline{PQ}$  and hence  $\frac{\delta \bar{r}}{\delta t}$  tends to be along the tangent to the

curve at  $P$ .

Hence  $\lim_{\delta t \rightarrow 0} \frac{\delta \bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$  will be a tangent vector to the curve at  $P$ . (This  $\frac{d\bar{r}}{dt}$  may not be a unit vector)

Suppose arc length  $AP = s$ . If we take the parameter as the arc length parameter, we can observe that  $\frac{d\bar{r}}{ds}$  is unit tangent vector at  $P$  to the curve.

## VECTOR DIFFERENTIAL OPERATOR

Def. The vector differential operator  $\nabla$  (read as del) is defined as

$\nabla \equiv \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z}$ . This operator possesses properties analogous to those of ordinary vectors as well as

differentiation operator. We will define now some quantities known as “gradient”, “divergence” and “curl”

involving this operator  $\nabla$ . We must note that this operator has no meaning by itself unless it operates on some function suitably.

### **GRADIENT OF A SCALAR POINT FUNCTION**

Let  $\phi(x, y, z)$  be a scalar point function of position defined in some region of space. Then the vector function  $\bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$  is known as the gradient of  $\phi$  or  $\nabla \phi$

$$\nabla \phi = \left( \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

#### **Properties:**

- (1) If  $f$  and  $g$  are two scalar functions then  $\text{grad}(f \pm g) = \text{grad } f \pm \text{grad } g$
- (2) The necessary and sufficient condition for a scalar point function to be constant is that  $\nabla f = \bar{0}$
- (3)  $\text{grad}(fg) = f(\text{grad } g) + g(\text{grad } f)$
- (4) If  $c$  is a constant,  $\text{grad}(cf) = c(\text{grad } f)$
- (5)  $\text{grad} \left( \frac{f}{g} \right) = \frac{g(\text{grad } f) - f(\text{grad } g)}{g^2}, (g \neq 0)$
- (6) Let  $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ . Then  $d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$  if  $\phi$  is any scalar point function, then

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \left( \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z} \right) (dx\bar{i} + dy\bar{j} + dz\bar{k}) = \nabla \phi \cdot d\bar{r}$$

### **DIRECTIONAL DERIVATIVE**

Let  $\phi(x, y, z)$  be a scalar function defined throughout some region of space. Let this function have a value  $\phi$  at a point  $P$  whose position vector referred to the origin  $O$  is  $\overline{OP} = \bar{r}$ . Let  $\phi + \Delta\phi$  be the value of the function at neighboring point  $Q$ . If  $\overline{OQ} = \bar{r} + \Delta\bar{r}$ . Let  $\Delta r$  be the length of  $\Delta\bar{r}$

$$\frac{\Delta\phi}{\Delta r}$$

gives a measure of the rate at which  $\phi$  change when we move from  $P$  to  $Q$ . The limiting value of

$\frac{\Delta\phi}{\Delta r}$  as  $\Delta r \rightarrow 0$  is called the derivative of  $\phi$  in the direction of  $\overline{PQ}$  or simply directional derivative of  $\phi$  at  $P$  and is denoted by  $d\phi/dr$ .

**Theorem 1:** The directional derivative of a scalar point function  $\phi$  at a point  $P(x, y, z)$  in the direction of a unit vector  $\bar{e}$  is equal to  $\bar{e} \cdot \text{grad } \phi = \bar{e} \cdot \nabla \phi$ .

Level Surface

If a surface  $\phi(x,y,z)=c$  be drawn through any point  $P(\bar{r})$ , such that at each point on it, function has the same value as at  $P$ , then such a surface is called a level surface of the function  $\phi$  through  $P$ .

e.g. : equipotential or isothermal surface.

Theorem 2:  $\nabla\phi$  at any point is a vector normal to the level surface  $\phi(x,y,z)=c$  through that point, where  $c$  is a constant.

### The physical interpretation of $\nabla\phi$

The gradient of a scalar function  $\phi(x,y,z)$  at a point  $P(x,y,z)$  is a vector along the normal to the level surface  $\phi(x,y,z)=c$  at  $P$  and is in increasing direction. Its magnitude is equal to the greatest rate of increase of  $\phi$ . Greatest value of directional derivative of  $\bar{\Phi}$  at a point  $P = |\text{grad } \phi|$  at that point.

### SOLVED PROBLEMS

1: If  $a=x+y+z$ ,  $b=x^2+y^2+z^2$ ,  $c=xy+yz+zx$ , prove that  $[\text{grad } a, \text{grad } b, \text{grad } c] = 0$ .

Sol:- Given  $a=x+y+z$

$$\text{There fore } \frac{\partial a}{\partial x} = 1, \frac{\partial a}{\partial y} = 1, \frac{\partial a}{\partial z} = 1$$

$$\text{Grad } a = \nabla a = \sum \bar{i} \frac{\partial a}{\partial x} = \bar{i} + \bar{j} + \bar{k}$$

$$\text{Given } b=x^2+y^2+z^2$$

$$\text{Therefore } \frac{\partial b}{\partial x} = 2x, \frac{\partial b}{\partial y} = 2y, \frac{\partial b}{\partial z} = 2z$$

$$\text{Grad } b = \nabla b = \bar{i} \frac{\partial b}{\partial x} + \bar{j} \frac{\partial b}{\partial y} + \bar{k} \frac{\partial b}{\partial z} = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\text{Again } c = xy+yz+zx$$

$$\text{Therefore } \frac{\partial c}{\partial x} = y+z, \frac{\partial c}{\partial y} = z+x, \frac{\partial c}{\partial z} = y+x$$

$$\text{Grad } c = \bar{i} \frac{\partial c}{\partial x} + \bar{j} \frac{\partial c}{\partial y} + \bar{k} \frac{\partial c}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$$

$$[\text{grad } a, \text{grad } b, \text{grad } c] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0, (\text{on simplification})$$

$$[\text{grad } a, \text{grad } b, \text{grad } c] = 0$$

2: Show that  $\nabla[f(r)] = \frac{f'(r)}{r} \bar{r}$  where  $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ .

Sol:- Since  $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ , we have  $r^2 = x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get



$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}. \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned}\nabla[f(r)] &= \left( \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) f(r) = \sum \bar{i} f^1(r) \frac{\partial r}{\partial x} = \sum \bar{i} f^1(r) \frac{x}{r} \\ &= \frac{f^1(r)}{r} \sum \bar{i} x = \frac{f^1(r)}{r} \cdot \bar{r}\end{aligned}$$

Note : From the above result,  $\nabla(\log r) = \frac{1}{r^2} \bar{r}$

**3:** Prove that  $\nabla(r^n) = nr^{n-2} \bar{r}$ .

Sol:- Let  $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$  and  $r = |\bar{r}|$ . Then we have  $r^2 = x^2 + y^2 + z^2$ . Differentiating w.r.t. x partially, we have

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}. \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla(r^n) = \sum \bar{i} \frac{\partial}{\partial x} (r^n) = \sum \bar{i} nr^{n-1} \frac{\partial r}{\partial x} = \sum \bar{i} nr^{n-1} \frac{x}{r} = nr^{n-2} \sum \bar{i} x = nr^{n-2} (\bar{r})$$

Note : From the above result, we can have

$$(1). \nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}, \text{ taking } n = -1 \quad (2) \text{ grad } r = \frac{\bar{r}}{r}, \text{ taking } n = 1$$

**4:** Find the directional derivative of  $f = xy + yz + zx$  in the direction of vector  $\bar{i} + 2\bar{j} + 2\bar{k}$  at the point (1,2,0).

Sol:- Given  $f = xy + yz + zx$ .

$$\text{Grad } f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z} = (y + z)\bar{i} + (z + x)\bar{j} + (x + y)\bar{k}$$

If  $\bar{e}$  is the unit vector in the direction of the vector  $\bar{i} + 2\bar{j} + 2\bar{k}$ , then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of  $f$  along the given direction =  $\bar{e} \cdot \nabla f$

$$= \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})[(y + z)\bar{i} + (z + x)\bar{j} + (x + y)\bar{k}] \text{ at } (1, 2, 0)$$

$$= \frac{1}{3}[(y + z) + 2(z + x) + 2(x + y)](1, 2, 0) = \frac{10}{3}$$

**5:** Find the directional derivative of the function  $xy^2 + yz^2 + zx^2$  along the tangent to the curve  $x = t, y = t^2, z = t^3$  at the point (1,1,1).

Sol: - Here  $f = xy^2 + yz^2 + zx^2$

$$\nabla f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z} = (y^2 + 2xz)\bar{i} + (z^2 + 2xy)\bar{j} + (x^2 + 2yz)\bar{k}$$

$$\text{At } (1, 1, 1), \nabla f = 3\bar{i} + 3\bar{j} + 3\bar{k}$$

Let  $\bar{r}$  be the position vector of any point on the curve  $x = t, y = t^2, z = t^3$ . Then

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i} + 2t\bar{j} + 3t^2\bar{k} = (\bar{i} + 2\bar{j} + 3\bar{k}) \text{ at } (1, 1, 1)$$

We know that  $\frac{\partial \vec{r}}{\partial t}$  is the vector along the tangent to the curve.

$$\text{Unit vector along the tangent} = \vec{e} = \frac{\vec{i} + 2\vec{j} + 3\vec{k}}{\sqrt{1+2^2+3^2}} = \frac{\vec{i} + 2\vec{j} + 3\vec{k}}{\sqrt{14}}$$

$$\text{Directional derivative along the tangent} = \nabla f \cdot \vec{e}$$

$$= \frac{1}{\sqrt{14}} (\vec{i} + 2\vec{j} + 3\vec{k}) \cdot 3(\vec{i} + \vec{j} + \vec{k}) = \frac{3}{\sqrt{14}} (1+2+3) = \frac{18}{\sqrt{14}}$$

**6:** Find the directional derivative of the function  $f = x^2 - y^2 + 2z^2$  at the point  $P = (1, 2, 3)$  in the direction of the line  $\overline{PQ}$  where  $Q = (5, 0, 4)$ .

Sol:- The position vectors of P and Q with respect to the origin are  $\overline{OP} = \vec{i} + 2\vec{j} + 3\vec{k}$  and

$$\overline{OQ} = 5\vec{i} + 4\vec{k}$$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\vec{i} - 2\vec{j} + \vec{k}$$

Let  $\vec{e}$  be the unit vector in the direction of  $\overline{PQ}$ . Then  $\vec{e} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}}$

$$\text{grad } f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z} = 2x\vec{i} - 2y\vec{j} + 4z\vec{k}$$

The directional derivative of  $f$  at P (1,2,3) in the direction of  $\overline{PQ} = \vec{e} \cdot \nabla f$

$$= \frac{1}{\sqrt{21}} (4\vec{i} - 2\vec{j} + \vec{k}) \cdot (2x\vec{i} - 2y\vec{j} + 4z\vec{k}) = \frac{1}{\sqrt{21}} (8x - 4y + 4z)_{at(1,2,3)} = \frac{1}{\sqrt{21}} (28)$$

**7:** Find the greatest value of the directional derivative of the function  $f = x^2 y z^3$  at (2,1,-1).

Sol: we have

$$\text{grad } f = \vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z} = 2xyz^3 \vec{i} + x^2 z^3 \vec{j} + 3x^2 y z^2 \vec{k} = -4\vec{i} - 4\vec{j} + 12\vec{k} \text{ at } (2,1,-1).$$

$$\text{Greatest value of the directional derivative of } f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$$

**8:** Find the directional derivative of  $xyz^2 + xz$  at (1, 1, 1) in a direction of the normal to the surface  $3xy^2 + y = z$  at (0,1,1).

Sol:- Let  $f(x, y, z) \equiv 3xy^2 + y - z = 0$

Let us find the unit normal  $\vec{e}$  to this surface at (0,1,1). Then

$$\frac{\partial f}{\partial x} = 3y^2, \frac{\partial f}{\partial y} = 6xy + 1, \frac{\partial f}{\partial z} = -1.$$

$$\nabla f = 3y^2 \vec{i} + (6xy + 1) \vec{j} - \vec{k}$$

$$(\nabla f)_{(0,1,1)} = 3\vec{i} + \vec{j} - \vec{k} = \vec{n}$$

$$\vec{e} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\vec{i} + \vec{j} - \vec{k}}{\sqrt{9+1+1}} = \frac{3\vec{i} + \vec{j} - \vec{k}}{\sqrt{11}}$$

Let  $g(x, y, z) = xyz^2 + xz$ , then

$$\frac{\partial g}{\partial x} = yz^2 + z, \quad \frac{\partial g}{\partial y} = xz^2, \quad \frac{\partial g}{\partial z} = 2xz + x$$

$$\nabla g = (yz^2 + z)\mathbf{i} + xz^2\mathbf{j} + (2xz + x)\mathbf{k}$$

$$\text{And } [\nabla g]_{(1,1,1)} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

Directional derivative of the given function in the direction of  $\bar{e}$  at  $(1,1,1) = \nabla g \cdot \bar{e}$

$$= (2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot \left( \frac{3\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{11}} \right) = \frac{6 + 1 - 3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

**9:** Find the directional derivative of  $2xy + z^2$  at  $(1, -1, 3)$  in the direction of  $\bar{i} + 2\bar{j} + 3\bar{k}$ .

$$\text{Sol: Let } f = 2xy + z^2 \text{ then } \frac{\partial f}{\partial x} = 2y, \quad \frac{\partial f}{\partial y} = 2x, \quad \frac{\partial f}{\partial z} = 2z.$$

$$\text{grad } f = \sum \bar{i} \frac{\partial f}{\partial x} = 2y\bar{i} + 2x\bar{j} + 2z\bar{k} \text{ and } (\text{grad } f) \text{ at } (1, -1, 3) = -2\bar{i} + 2\bar{j} + 6\bar{k}$$

$$\text{given vector is } \bar{a} = \bar{i} + 2\bar{j} + 3\bar{k} \Rightarrow |\bar{a}| = \sqrt{1 + 4 + 9} = \sqrt{14}$$

Directional derivative of  $f$  in the direction of  $\bar{a}$  is

$$\frac{\bar{a} \cdot \nabla f}{|\bar{a}|} = \frac{(\bar{i} + 2\bar{j} + 3\bar{k}) \cdot (-2\bar{i} + 2\bar{j} + 6\bar{k})}{\sqrt{14}} = \frac{-2 + 4 + 18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

**10:** Find the directional derivative of  $\phi = x^2yz + 4xz^2$  at  $(1, -2, -1)$  in the direction  $2\bar{i} - \bar{j} - 2\bar{k}$ .

$$\text{Sol:- Given } \phi = x^2yz + 4xz^2$$

$$\frac{\partial \phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial \phi}{\partial y} = x^2z, \quad \frac{\partial \phi}{\partial z} = x^2y + 8xz.$$

$$\text{Hence } \nabla \phi = \sum \bar{i} \frac{\partial \phi}{\partial x} = \bar{i}(2xyz + 4z^2) + \bar{j}x^2z + \bar{k}(x^2y + 8xz)$$

$$\nabla \phi \text{ at } (1, -2, -1) = \mathbf{i}(4 + 4) + \mathbf{j}(-1) + \mathbf{k}(-2 - 8) = 8\mathbf{i} - \mathbf{j} - 10\mathbf{k}.$$

The unit vector in the direction  $2\bar{i} - \bar{j} - 2\bar{k}$  is

$$\bar{a} = \frac{2\bar{i} - \bar{j} - 2\bar{k}}{\sqrt{4 + 1 + 4}} = \frac{1}{3}(2\bar{i} - \bar{j} - 2\bar{k})$$

Required directional derivative along the given direction =  $\nabla \phi \cdot \bar{a}$

$$= (8\mathbf{i} - \mathbf{j} - 10\mathbf{k}) \cdot \frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

$$= \frac{1}{3}(16 + 1 + 20) = 37/3.$$

**11:** If the temperature at any point in space is given by  $t = xy + yz + zx$ , find the direction in which temperature changes most rapidly with distance from the point  $(1, 1, 1)$  and determine the maximum rate of change.

**Sol:-** The greatest rate of increase of  $t$  at any point is given in magnitude and direction by  $\nabla t$ .

$$\text{We have } \nabla t = \left( \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) (xy + yz + zx)$$

$$= \bar{i}(y + z) + \bar{j}(z + x) + \bar{k}(x + y) = 2\bar{i} + 2\bar{j} + 2\bar{k} \text{ at } (1, 1, 1)$$

$$\text{Magnitude of this vector is } \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12} = 2\sqrt{3}$$

Hence at the point  $(1, 1, 1)$  the temperature changes most rapidly in the direction given by the

vector  $2\bar{i} + 2\bar{j} + 2\bar{k}$  and greatest rate of increase =  $2\sqrt{3}$ .

**12:** Find the directional derivative of  $\phi(x,y,z) = x^2yz + 4xz^2$  at the point  $(1,-2,-1)$  in the direction of the normal to the surface  $f(x,y,z) = x \log z - y^2$  at  $(-1,2,1)$ .

Sol:- Given  $\phi(x,y,z) = x^2yz + 4xz^2$  at  $(1,-2,-1)$  and  $f(x,y,z) = x \log z - y^2$  at  $(-1,2,1)$

$$\begin{aligned}\text{Now } \nabla \phi &= \frac{\partial \phi}{\partial x} \bar{i} + \frac{\partial \phi}{\partial y} \bar{j} + \frac{\partial \phi}{\partial z} \bar{k} \\ &= (2xyz + 4z^2) \bar{i} + (x^2 z) \bar{j} + (x^2 y + 8xz) \bar{k}\end{aligned}$$

$$\begin{aligned}(\nabla \phi)_{(1,-2,-1)} &= [2(1)(-2)(-1) + 4(-1)^2] \bar{i} + [(1)^2(-1)] \bar{j} + [(1^2)(-2) + 8(-1)] \bar{k} \text{ --- (1)} \\ &= 8\bar{i} - \bar{j} - 10\bar{k}\end{aligned}$$

Unit normal to the surface  $f(x,y,z) = x \log z - y^2$  is  $\frac{\nabla f}{|\nabla f|}$

$$\text{Now } \nabla f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z} = \log z \bar{i} + (-2y) \bar{j} + \frac{x}{z} \bar{k}$$

$$\text{At } (-1,2,1), \nabla f = \log(1) \bar{i} - 2(2) \bar{j} + \frac{-1}{1} \bar{k} = -4\bar{j} - \bar{k}$$

$$\frac{\nabla f}{|\nabla f|} = \frac{-4\bar{j} - \bar{k}}{\sqrt{16+1}} = \frac{-4\bar{j} - \bar{k}}{\sqrt{17}}$$

$$\text{Directional derivative} = \nabla \phi \cdot \frac{\nabla f}{|\nabla f|}$$

$$= (8\bar{i} - \bar{j} - 10\bar{k}) \cdot \frac{-4\bar{j} - \bar{k}}{\sqrt{17}} = \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}$$

**13:** Find a unit normal vector to the given surface  $x^2y + 2xz = 4$  at the point  $(2,-2,3)$ .

Sol:- Let the given surface be  $f = x^2y + 2xz - 4$

On differentiating,

$$\frac{\partial f}{\partial x} = 2xy + 2z, \quad \frac{\partial f}{\partial y} = x^2, \quad \frac{\partial f}{\partial z} = 2x.$$

$$\text{grad } f = \sum \bar{i} \frac{\partial f}{\partial x} = \bar{i}(2xy + 2z) + \bar{j}x^2 + 2x\bar{k}$$

$$(\text{grad } f) \text{ at } (2,-2,3) = \bar{i}(-8+6) + 4\bar{j} + 4\bar{k} = 2\bar{i} + 4\bar{j} + 4\bar{k}$$

grad (f) is the normal vector to the given surface at the given point.

$$\text{Hence the required unit normal vector } \frac{\nabla f}{|\nabla f|} = \frac{2(-\bar{i} + 2\bar{j} + 2\bar{k})}{2\sqrt{1+2^2+2^2}} = \frac{-\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

**14:** Evaluate the angle between the normal to the surface  $xy = z^2$  at the points  $(4,1,2)$  and  $(3,3,-3)$ .

Sol:- Given surface is  $f(x,y,z) = xy - z^2$

Let  $\bar{n}_1$  and  $\bar{n}_2$  be the normal to this surface at  $(4,1,2)$  and  $(3,3,-3)$  respectively.

Differentiating partially, we get

$$\frac{\partial f}{\partial x} = y, \frac{\partial f}{\partial y} = x, \frac{\partial f}{\partial z} = -2z.$$

$$\text{grad } f = y\bar{i} + x\bar{j} - 2z\bar{k}$$

$$\bar{n}_1 = (\text{grad } f) \text{ at } (4,1,2) = \bar{i} + 4\bar{j} - 4\bar{k}$$

$$\bar{n}_2 = (\text{grad } f) \text{ at } (3,3,-3) = 3\bar{i} + 3\bar{j} + 6\bar{k}$$

Let  $\theta$  be the angle between the two normal.

$$\begin{aligned} \cos \theta &= \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|} = \frac{(i + 4j - 4k) \cdot (3i + 3j + 6k)}{\sqrt{1+16+16} \cdot \sqrt{9+9+36}} \\ &= \frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}} \end{aligned}$$

**15:** Find a unit normal vector to the surface  $x^2+y^2+2z^2 = 26$  at the point  $(2, 2, 3)$ .

Sol:- Let the given surface be  $f(x,y,z) \equiv x^2+y^2+2z^2 - 26=0$ . Then

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 4z.$$

$$\text{grad } f = \sum \bar{i} \frac{\partial f}{\partial x} = 2xi+2yj+4zk$$

$$\text{Normal vector at } (2,2,3) = [\nabla f]_{(2,2,3)} = 4\bar{i} + 4\bar{j} + 12\bar{k}$$

$$\text{Unit normal vector} = \frac{\nabla f}{|\nabla f|} = \frac{4(\bar{i} + \bar{j} + 3\bar{k})}{4\sqrt{11}} = \frac{\bar{i} + \bar{j} + 3\bar{k}}{\sqrt{11}}$$

**16:** Find the values of  $a$  and  $b$  so that the surfaces  $ax^2-byz = (a+2)x$  and  $4x^2y+z^3 = 4$  may intersect orthogonally at the point  $(1, -1, 2)$ .

(or) Find the constants  $a$  and  $b$  so that surface  $ax^2-byz=(a+2)x$  will be orthogonal to  $4x^2y+z^3=4$  at the point  $(1, -1, 2)$ .

**Sol:-** Let the given surfaces be  $f(x,y,z) = ax^2-byz - (a+2)x$ ------(1)

$$\text{And } g(x,y,z) = 4x^2y+z^3 - 4$$
------(2)

Given the two surfaces meet at the point  $(1, -1, 2)$ .

Substituting the point in (1), we get

$$a+2b-(a+2) = 0 \Rightarrow b=1$$

$$\text{Now } \frac{\partial f}{\partial x} = 2ax - (a+2), \frac{\partial f}{\partial y} = -bz \text{ and } \frac{\partial f}{\partial z} = -by.$$

$$\nabla f = \sum \bar{i} \frac{\partial f}{\partial x} = [(2ax-(a+2))\bar{i} - bz\bar{j} + b\bar{k}] = (a-2)\bar{i} - 2b\bar{j} + b\bar{k}$$

$$= (a-2)\bar{i} - 2\bar{j} + \bar{k} = \bar{n}_1, \text{ normal vector to surface 1.}$$

Also  $\frac{\partial g}{\partial x} = 8xy, \frac{\partial g}{\partial y} = 4x^2, \frac{\partial g}{\partial z} = 3z^2.$

$$\nabla g = \sum \bar{i} \frac{\partial g}{\partial x} = 8xy\bar{i} + 4x^2\bar{j} + 3z^2\bar{k}$$

$$(\nabla g)_{(1,-1,2)} = -8\bar{i} + 4\bar{j} + 12\bar{k} = \bar{n}_2, \text{ normal vector to surface 2.}$$

Given the surfaces  $f(x,y,z)$ ,  $g(x,y,z)$  are orthogonal at the point  $(1,-1,2)$ .

$$[\bar{\nabla} f][\bar{\nabla} g] = 0 \Rightarrow ((a-2)\bar{i} - 2\bar{j} + \bar{k}) \cdot (-8\bar{i} + 4\bar{j} + 12\bar{k}) = 0$$

$$\Rightarrow -8a + 16 - 8 + 12 \Rightarrow a = 5/2$$

Hence  $a = 5/2$  and  $b = 1$ .

**17:** Find a unit normal vector to the surface  $z = x^2 + y^2$  at  $(-1, -2, 5)$

Sol:- Let the given surface be  $f = x^2 + y^2 - z$

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = -1.$$

$$\text{grad } f = \nabla f = \sum \bar{i} \frac{\partial f}{\partial x} = 2x\bar{i} + 2y\bar{j} - \bar{k}$$

$$(\nabla f) \text{ at } (-1, -2, 5) = -2\bar{i} - 4\bar{j} - \bar{k}$$

$\nabla f$  is the normal vector to the given surface.

Hence the required unit normal vector =  $\frac{\nabla f}{|\nabla f|} =$

$$\frac{-2\bar{i} - 4\bar{j} - \bar{k}}{\sqrt{(-2)^2 + (-4)^2 + (-1)^2}} = \frac{-2\bar{i} - 4\bar{j} - \bar{k}}{\sqrt{21}} = -\frac{1}{\sqrt{21}}(2\bar{i} + 4\bar{j} + \bar{k})$$

**18:** Find the angle of intersection of the spheres  $x^2 + y^2 + z^2 = 29$  and  $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$  at the point  $(4, -3, 2)$ .

Sol:- Let  $f = x^2 + y^2 + z^2 - 29$  and  $g = x^2 + y^2 + z^2 + 4x - 6y - 8z - 47$

$$\text{Then grad } f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z} = 2x\bar{i} + 2y\bar{j} + 2z\bar{k} \text{ and}$$

$$\text{grad } g = (2x + 4)\bar{i} + (2y - 6)\bar{j} + (2z - 8)\bar{k}$$

The angle between two surfaces at a point is the angle between the normal to the surfaces at that point.

$$\text{Let } \bar{n}_1 = (\text{grad } f) \text{ at } (4, -3, 2) = 8\bar{i} - 6\bar{j} + 4\bar{k}$$

$$\bar{n}_2 = (\text{grad } g) \text{ at } (4, -3, 2) = 12\bar{i} - 12\bar{j} - 4\bar{k}$$

The vectors  $\bar{n}_1$  and  $\bar{n}_2$  are along the normal to the two surfaces at  $(4, -3, 2)$ . Let  $\theta$  be the angle between the surfaces. Then

$$\cos \theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|} = \frac{152}{\sqrt{116} \sqrt{304}}$$

**19:** Find the angle between the surfaces  $x^2+y^2+z^2=9$ , and  $z = x^2+y^2-3$  at point  $(2,-1,2)$ .

Sol:- Let  $\phi_1 = x^2+y^2+z^2-9=0$  and  $\phi_2 = x^2+y^2-z-3=0$  be the given surfaces. Then

$$\nabla\phi_1 = 2xi+2yj+2zk \text{ and } \nabla\phi_2 = 2xi+2yj-k$$

Let  $\bar{n}_1 = \nabla\phi_1$  at  $(2,-1,2) = 4i-2j+4k$  and

$$\bar{n}_2 = \nabla\phi_2 \text{ at } (2,-1,2) = 4i-2j-k$$

The vectors  $\bar{n}_1$  and  $\bar{n}_2$  are along the normals to the two surfaces at the point  $(2,-1,2)$ . Let  $\theta$  be the angle between the surfaces. Then

$$\begin{aligned} \cos \theta &= \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| |\bar{n}_2|} = \frac{(4i-2j+4k) \cdot (4i-2j-k)}{\sqrt{16+4+16} \cdot \sqrt{16+4+16}} = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}} \\ \therefore \theta &= \cos^{-1} \left( \frac{8}{3\sqrt{21}} \right). \end{aligned}$$

**20:** If  $\bar{a}$  is constant vector then prove that  $\text{grad} (\bar{a} \cdot \bar{r}) = \bar{a}$

Sol: Let  $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ , where  $a_1, a_2, a_3$  are constants.

$$\bar{a} \cdot \bar{r} = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = a_1x + a_2y + a_3z$$

$$\frac{\partial}{\partial x}(\bar{a} \cdot \bar{r}) = a_1, \frac{\partial}{\partial y}(\bar{a} \cdot \bar{r}) = a_2, \frac{\partial}{\partial z}(\bar{a} \cdot \bar{r}) = a_3$$

$$\text{grad} (\bar{a} \cdot \bar{r}) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

**21:** If  $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$ , find  $\phi$ .

$$\text{Sol:- We know that } \nabla\phi = \bar{i} \frac{\partial\phi}{\partial x} + \bar{j} \frac{\partial\phi}{\partial y} + \bar{k} \frac{\partial\phi}{\partial z}$$

$$\text{Given that } \nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$$

$$\text{Comparing the corresponding coefficients, we have } \frac{\partial\phi}{\partial x} = yz, \frac{\partial\phi}{\partial y} = zx, \frac{\partial\phi}{\partial z} = xy$$

Integrating partially w.r.t.  $x, y, z$ , respectively, we get

$$\phi = xyz + \text{a constant independent of } x.$$

$$\phi = xyz + \text{a constant independent of } y.$$

$$\phi = xyz + \text{a constant independent of } z.$$

Here a possible form of  $\phi$  is  $\phi = xyz + \text{a constant}$ .

### DIVERGENCE OF A VECTOR

Let  $\bar{f}$  be any continuously differentiable vector point function. Then  $\bar{i} \cdot \frac{\partial\bar{f}}{\partial x} + \bar{j} \cdot \frac{\partial\bar{f}}{\partial y} + \bar{k} \cdot \frac{\partial\bar{f}}{\partial z}$  is called the divergence of  $\bar{f}$  and is written as  $\text{div } \bar{f}$ .

$$\text{i.e., } \text{div } \bar{f} = \bar{i} \cdot \frac{\partial\bar{f}}{\partial x} + \bar{j} \cdot \frac{\partial\bar{f}}{\partial y} + \bar{k} \cdot \frac{\partial\bar{f}}{\partial z} = \left( \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) \cdot \bar{f}$$

Hence we can write  $\text{div } \vec{f}$  as

$$\text{div } \vec{f} = \nabla \cdot \vec{f}$$

This is a scalar point function.

**Theorem 1:** If the vector  $\vec{f} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$ , then  $\text{div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Prof: Given  $\vec{f} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$

$$\frac{\partial \vec{f}}{\partial x} = \vec{i} \frac{\partial f_1}{\partial x} + \vec{j} \frac{\partial f_2}{\partial x} + \vec{k} \frac{\partial f_3}{\partial x}$$

Also  $\vec{i} \cdot \frac{\partial \vec{f}}{\partial x} = \frac{\partial f_1}{\partial x}$ . Similarly  $\vec{j} \cdot \frac{\partial \vec{f}}{\partial y} = \frac{\partial f_2}{\partial y}$  and  $\vec{k} \cdot \frac{\partial \vec{f}}{\partial z} = \frac{\partial f_3}{\partial z}$

$$\text{We have } \text{div } \vec{f} = \sum \vec{i} \cdot \left( \frac{\partial \vec{f}}{\partial x} \right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Note : If  $\vec{f}$  is a constant vector then  $\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$  are zeros.

$\therefore \text{div } \vec{f} = 0$  for a constant vector  $\vec{f}$ .

**Theorem 2:**  $\text{div} (\vec{f} \pm \vec{g}) = \text{div } \vec{f} \pm \text{div } \vec{g}$

Proof:  $\text{div} (\vec{f} \pm \vec{g}) = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{f} \pm \vec{g}) = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{f}) \pm \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{g}) = \text{div } \vec{f} \pm \text{div } \vec{g}$ .

Note: If  $\phi$  is a scalar function and  $\vec{f}$  is a vector function, then

$$\begin{aligned} \text{(i). } (\vec{a} \cdot \nabla) \phi &= \left[ \vec{a} \cdot \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \right] \phi \\ &= \left[ (\vec{a} \cdot \vec{i}) \frac{\partial}{\partial x} + (\vec{a} \cdot \vec{j}) \frac{\partial}{\partial y} + (\vec{a} \cdot \vec{k}) \frac{\partial}{\partial z} \right] \phi \\ &= \left[ (\vec{a} \cdot \vec{i}) \frac{\partial \phi}{\partial x} + (\vec{a} \cdot \vec{j}) \frac{\partial \phi}{\partial y} + (\vec{a} \cdot \vec{k}) \frac{\partial \phi}{\partial z} \right] \\ &= \sum (\vec{a} \cdot \vec{i}) \frac{\partial \phi}{\partial x} \text{. and} \end{aligned}$$

$$\text{(ii). } (\vec{a} \cdot \nabla) \vec{f} = \sum (\vec{a} \cdot \vec{i}) \frac{\partial \vec{f}}{\partial x} \text{. by proceeding as in (i) [simply replace } \phi \text{ by } \vec{f} \text{ in (i)].}$$

## SOLENOIDAL VECTOR

A vector point function  $\vec{f}$  is said to be solenoidal if  $\text{div } \vec{f} = 0$ .

### Physical interpretation of divergence:

Depending upon  $\vec{f}$  in a physical problem, we can interpret  $\text{div } \vec{f}$  ( $= \nabla \cdot \vec{f}$ ).



Suppose  $\vec{F}(x,y,z,t)$  is the velocity of a fluid at a point  $(x,y,z)$  and time 't'. Though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of  $\vec{F}$  measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Similar meanings are to be understood with respect to divergence of vectors  $\vec{f}$  from other branches. A detailed elementary interpretation can be seen in standard books on fluid dynamics, electricity and magnetism etc.

### SOLVED PROBLEMS

**1:** If  $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$  find  $\text{div } \vec{f}$  at  $(1, -1, 1)$ .

**Sol:-** Given  $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ .

$$\text{Then div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) = y^2 + 2x^2z - 6yz$$

$$(\text{div } \vec{f}) \text{ at } (1, -1, 1) = 1 + 2 + 6 = 9$$

**2:** Find  $\text{div } \vec{f}$  when  $\text{grad}(x^3 + y^3 + z^3 - 3xyz)$

**Sol:-** Let  $\phi = x^3 + y^3 + z^3 - 3xyz$ .

$$\text{Then } \frac{\partial \phi}{\partial x} = 3x^2 - 3yz, \quad \frac{\partial \phi}{\partial y} = 3y^2 - 3zx, \quad \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$$

$$\text{grad } \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}]$$

$$\begin{aligned} \text{div } \vec{f} &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)] + \frac{\partial}{\partial y}[3(y^2 - zx)] + \frac{\partial}{\partial z}[3(z^2 - xy)] \\ &= 3(2x) + 3(2y) + 3(2z) = 6(x + y + z) \end{aligned}$$

**3:** If  $\vec{f} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + pz)\vec{k}$  is solenoid, find  $P$ .

**Sol:-** Let  $\vec{f} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + pz)\vec{k} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$

$$\text{We have } \frac{\partial f_1}{\partial x} = 1, \quad \frac{\partial f_2}{\partial y} = 1, \quad \frac{\partial f_3}{\partial z} = p$$

$$\text{div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 1 + 1 + p = 2 + p$$

$$\text{since } \vec{f} \text{ is solenoid, we have } \text{div } \vec{f} = 0 \Rightarrow 2 + p = 0 \Rightarrow p = -2$$

**4:** Find  $\text{div } \vec{f} = r^n \vec{r}$ . Find  $n$  if it is solenoid?

**Sol:** Given  $\vec{f} = r^n \vec{r}$ . where  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $r = |\vec{r}|$

We have  $r^2 = x^2 + y^2 + z^2$

Differentiating partially w.r.t.  $x$ , we get

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

$$\text{Similarly } \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\vec{f} = r^n (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\begin{aligned} \text{div } \vec{f} &= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) \\ &= nr^{n-1} \frac{\partial r}{\partial x} x + r^n + nr^{n-1} \frac{\partial r}{\partial y} y + r^n + nr^{n-1} \frac{\partial r}{\partial z} z + r^n \end{aligned}$$

$$= nr^{n-1} \left[ \frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} \right] + 3r^n = nr^{n-1} \left( \frac{r^2}{r} \right) + 3r^n = nr^n + 3r^n = (n+3)r^n$$

Let  $\vec{f} = r^n \vec{r}$  be solenoid. Then  $\text{div } \vec{f} = 0$

$$(n+3)r^n = 0 \Rightarrow n = -3$$

**5:** Evaluate  $\nabla \cdot \left( \frac{\vec{r}}{r^3} \right)$  where  $\vec{r} = xi + yj + zk$  and  $r = |\vec{r}|$ .

Sol:- We have

$$\vec{r} = xi + yj + zk \text{ and } r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \frac{\vec{r}}{r^3} = \vec{r} \cdot r^{-3} = r^{-3}xi + r^{-3}yj + r^{-3}zk = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$\text{Hence } \nabla \cdot \left( \frac{\vec{r}}{r^3} \right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{We have } f_1 = r^{-3}x \Rightarrow \frac{\partial f_1}{\partial x} = r^{-3} \cdot 1 + x(-3)r^{-4} \cdot \frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial f_1}{\partial x} = r^{-3} - 3xr^{-4} \frac{x}{r} = r^{-3} - 3x^2r^{-5}$$

$$\nabla \cdot \left( \frac{\vec{r}}{r^3} \right) = \sum \frac{\partial f_1}{\partial x} = 3r^{-3} - 3r^{-5} \sum x^2$$

$$= 3r^{-3} - 3r^{-5}r^2 = 3r^{-3} - 3r^{-3} = 0$$

**6:** Find  $\text{div } \vec{r}$  where  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Sol:- We have  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$

$$\operatorname{div} \bar{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

## CURL OF A VECTOR

**Def:** Let  $\bar{f}$  be any continuously differentiable vector point function. Then the vector function defined by

$\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$  is called curl of  $\bar{f}$  and is denoted by  $\operatorname{curl} \bar{f}$  or  $(\nabla \times \bar{f})$ .

$$\operatorname{Curl} \bar{f} = \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z} = \sum \left( \bar{i} \times \frac{\partial \bar{f}}{\partial x} \right)$$

**Theorem 1:** If  $\bar{f}$  is differentiable vector point function given by  $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$  then  $\operatorname{curl} \bar{f} =$

$$\left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \bar{i} + \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \bar{j} + \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \bar{k}$$

$$\begin{aligned} \text{Proof : } \operatorname{curl} \bar{f} &= \sum \bar{i} \times \frac{\partial}{\partial x}(\bar{f}) = \sum \bar{i} \times \frac{\partial}{\partial x}(f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}) = \sum \left( \frac{\partial f_2}{\partial x} \bar{k} - \frac{\partial f_3}{\partial x} \bar{j} \right) \\ &= \left( \frac{\partial f_2}{\partial x} \bar{k} - \frac{\partial f_3}{\partial x} \bar{j} \right) + \left( \frac{\partial f_3}{\partial y} \bar{i} - \frac{\partial f_1}{\partial y} \bar{k} \right) + \left( \frac{\partial f_1}{\partial z} \bar{j} - \frac{\partial f_2}{\partial z} \bar{i} \right) \\ &= \bar{i} \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) + \bar{j} \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) + \bar{k} \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \end{aligned}$$

**Note: (1)** the above expression for  $\operatorname{curl} \bar{f}$  can be remembered easily through the representation.

$$\operatorname{curl} \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \times \bar{f}$$

**Note (2) :** If  $\bar{f}$  is a constant vector then  $\operatorname{curl} \bar{f} = \bar{0}$ .

**Theorem 2:**  $\operatorname{curl} (\bar{a} \pm \bar{b}) = \operatorname{curl} \bar{a} \pm \operatorname{curl} \bar{b}$

$$\begin{aligned} \text{Proof: } \operatorname{curl} (\bar{a} \pm \bar{b}) &= \sum \bar{i} \times \frac{\partial}{\partial x} (\bar{a} \pm \bar{b}) \\ &= \sum \bar{i} \times \left( \frac{\partial \bar{a}}{\partial x} \pm \frac{\partial \bar{b}}{\partial x} \right) = \sum \bar{i} \times \frac{\partial \bar{a}}{\partial x} \pm \sum \bar{i} \times \frac{\partial \bar{b}}{\partial x} \\ &= \operatorname{curl} \bar{a} \pm \operatorname{curl} \bar{b} \end{aligned}$$

### 1. Physical Interpretation of curl

If  $\vec{\omega}$  is the angular velocity of a rigid body rotating about a fixed axis and  $\vec{v}$  is the velocity of any point

P(x,y,z) on the body, then  $\vec{\omega} = \frac{1}{2} \text{curl } \vec{v}$ . Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word “curl of a vector”.

## 2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e  $\text{curl } \vec{v} = \vec{0}$  is said to be Irrotational.

Def: A vector  $\vec{f}$  is said to be Irrotational if  $\text{curl } \vec{f} = \vec{0}$ .

If  $\vec{f}$  is Irrotational, there will always exist a scalar function  $\phi(x,y,z)$  such that  $\vec{f} = \text{grad } \phi$ . This  $\phi$  is called scalar potential of  $\vec{f}$ .

It is easy to prove that, if  $\vec{f} = \text{grad } \phi$ , then  $\text{curl } \vec{f} = 0$ .

Hence  $\nabla \times \vec{f} = 0 \Leftrightarrow$  there exists a scalar function  $\phi$  such that  $\vec{f} = \nabla \phi$ .

This idea is useful when we study the “work done by a force” later.

### SOLVED PROBLEMS

1: If  $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$  find  $\text{curl } \vec{f}$  at the point (1,-1,1).

Sol:- Let  $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ . Then

$$\begin{aligned} \text{curl } \vec{f} = \nabla \times \vec{f} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix} \\ &= \vec{i} \left( \frac{\partial}{\partial y} (-3yz^2) - \frac{\partial}{\partial z} (2x^2yz) \right) + \vec{j} \left( \frac{\partial}{\partial z} (xy^2) - \frac{\partial}{\partial x} (-3yz^2) \right) + \vec{k} \left( \frac{\partial}{\partial x} (2x^2yz) - \frac{\partial}{\partial y} (xy^2) \right) \\ &= \vec{i} (-3z^2 - 2x^2z) + \vec{j} (0 - 0) + \vec{k} (4xyz - 2xy) = -(3z^2 + 2x^2z)\vec{i} + (4xyz - 2xy)\vec{k} \\ &= \text{curl } \vec{f} \text{ at } (1,-1,1) = -\vec{i} - 2\vec{k}. \end{aligned}$$

2: Find  $\text{curl } \vec{f}$  where  $\vec{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol:- Let  $\phi = x^3+y^3+z^3-3xyz$ . Then

$$\text{grad } \phi = \sum \vec{i} \frac{\partial \phi}{\partial x} = 3(x^2 - yz)\vec{i} + 3(y^2 - zx)\vec{j} + 3(z^2 - xy)\vec{k}$$

$$\begin{aligned} \text{curl grad } \phi &= \nabla \times \text{grad } \phi = 3 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} \\ &= 3[\vec{i}(-x+x) - \vec{j}(-y+y) + \vec{k}(-z+z)] = \vec{0} \\ \therefore \text{curl } \vec{f} &= \vec{0}. \end{aligned}$$

Note: We can prove in general that  $\text{curl}(\text{grad } \phi) = \vec{0}$ . (i.e)  $\text{grad } \phi$  is always irrotational.

**3:** Prove that if  $\vec{r}$  is the position vector of a point in space, then  $r^n \vec{r}$  is Irrotational. (or) Show that

$$\text{curl}(\vec{r}^n \vec{r}) = \vec{0}$$

Sol:- Let  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  and  $r = |\vec{r}| \quad \therefore r^2 = x^2 + y^2 + z^2$ .

Differentiating partially w.r.t. 'x', we get

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly  $\frac{\partial r}{\partial y} = \frac{y}{r}$ , and  $\frac{\partial r}{\partial z} = \frac{z}{r}$  We have  $r^n \vec{r} = r^n (x\vec{i} + y\vec{j} + z\vec{k})$

$$\begin{aligned} \nabla \times (r^n \vec{r}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix} \\ &= \vec{i} \left( \frac{\partial}{\partial y} (r^n z) - \frac{\partial}{\partial z} (r^n y) \right) + \vec{j} \left( \frac{\partial}{\partial z} (r^n x) - \frac{\partial}{\partial x} (r^n z) \right) + \vec{k} \left( \frac{\partial}{\partial x} (r^n y) - \frac{\partial}{\partial y} (r^n x) \right) \\ &= \sum \vec{i} \left\{ znr^{n-1} \frac{\partial r}{\partial y} - ynr^{n-1} \frac{\partial r}{\partial z} \right\} = nr^{n-1} \sum \vec{i} \left\{ z \left( \frac{y}{r} \right) - y \left( \frac{z}{r} \right) \right\} \\ &= nr^{n-2} [(zy - yz)\vec{i} + (xz - zx)\vec{j} + (xy - yx)\vec{k}] \\ &= nr^{n-2} [0\vec{i} + 0\vec{j} + 0\vec{k}] = nr^{n-2} [\vec{0}] = \vec{0} \end{aligned}$$

Hence  $r^n \vec{r}$  is Irrotational.

**4:** Prove that  $\text{curl } \vec{r} = \vec{0}$

Sol:- Let  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\text{curl } \vec{r} = \sum \vec{i} \times \frac{\partial}{\partial x} (\vec{r}) = \sum (\vec{i} \times \vec{i}) = \vec{0} + \vec{0} + \vec{0} = \vec{0}$$

$\therefore \vec{r}$  is Irrotational vector.

**5:** If  $\vec{a}$  is a constant vector, prove that  $\text{curl} \left( \frac{\vec{a} \times \vec{r}}{r^3} \right) = -\frac{\vec{a}}{r^3} + \frac{3\vec{r}}{r^5} (\vec{a} \cdot \vec{r})$ .

Sol:- We have  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\frac{\partial \vec{r}}{\partial x} = \vec{i}, \frac{\partial \vec{r}}{\partial y} = \vec{j}, \frac{\partial \vec{r}}{\partial z} = \vec{k}$$

If  $|\vec{r}| = r$  then  $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{curl} \left( \frac{\vec{a} \times \vec{r}}{r^3} \right) = \sum \vec{i} \times \frac{\partial}{\partial x} \left( \frac{\vec{a} \times \vec{r}}{r^3} \right)$$

$$\text{Now } \frac{\partial}{\partial x} \left( \frac{\bar{a} \times \bar{r}}{r^3} \right) = \bar{a} \times \frac{\partial}{\partial x} \left( \frac{\bar{r}}{r^3} \right) = \bar{a} \times \left[ \frac{1}{r^3} \frac{\partial \bar{r}}{\partial x} - \frac{3}{r^4} \frac{\partial r}{\partial x} \bar{r} \right]$$

$$= \bar{a} \times \left[ \frac{1}{r^3} \bar{i} - \frac{3}{r^5} x \bar{r} \right] = \frac{\bar{a} \times \bar{i}}{r^3} - \frac{3x(\bar{a} \times \bar{r})}{r^5}.$$

$$\therefore i \times \frac{\partial}{\partial x} \left( \frac{\bar{a} \times \bar{r}}{r^3} \right) = \bar{i} \times \left[ \frac{\bar{a} \times \bar{i}}{r^3} - \frac{3x}{r^5} (\bar{a} \times \bar{r}) \right] = \frac{\bar{i} \times (\bar{a} \times \bar{i})}{r^3} - \frac{3x}{r^5} \bar{i} \times (\bar{a} \times \bar{r})$$

$$= \frac{(\bar{i} \cdot \bar{i})\bar{a} - (\bar{i} \cdot \bar{a})\bar{i}}{r^3} - \frac{3x}{r^5} [(\bar{i} \cdot \bar{r})\bar{a} - (\bar{i} \cdot \bar{a})\bar{r}]$$

Let  $\bar{a} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$ . Then  $\bar{i} \cdot \bar{a} = a_1$ , etc.

$$\therefore i \times \frac{\partial}{\partial x} \left( \frac{\bar{a} \times \bar{r}}{r^3} \right) = \sum \frac{(\bar{a} - a_1 \bar{i})}{r^3} - \frac{3x}{r^3} (x\bar{a} - a_1 \bar{r})$$

$$\therefore \sum i \times \frac{\partial}{\partial x} \left( \frac{\bar{a} \times \bar{r}}{r^3} \right) = \sum \frac{\bar{a} - a_1 \bar{i}}{r^3} - \frac{3}{r^5} \sum (x^2 \bar{a} - a_1 x \bar{r})$$

$$= \frac{3\bar{a} - \bar{a}}{r^3} - \frac{3\bar{a}}{r^5} (r^2) + \frac{3\bar{r}}{r^5} (a_1 x + a_2 y + a_3 z)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5} (\bar{r} \cdot \bar{a}) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5} (\bar{r} \cdot \bar{a})$$

**6:** Show that the vector  $(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$  is irrotational and find its scalar potential.

Sol: let  $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum \bar{i}(-x + x) = \bar{0}$$

$\therefore \bar{f}$  is Irrotational. Then there exists  $\phi$  such that  $\bar{f} = \nabla \phi$ .

$$\Rightarrow \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$$

Comparing components, we get

$$\frac{\partial \phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int (x^2 - yz) dx = \frac{x^3}{3} - xyz + f_1(y, z) \dots (1)$$

$$\frac{\partial \phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z, x) \dots (2)$$

$$\frac{\partial \phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x, y) \dots (3)$$

From (1), (2), (3),  $\phi = \frac{x^3 + y^3 + z^3}{3} - xyz$

$$\therefore \phi = \frac{1}{3}(x^3 + y^3 + z^3) - xyz + \text{constant}$$

Which is the required scalar potential.

**7:** Find constants a, b and c if the vector  $\vec{f} = (2x + 3y + az)\vec{i} + (bx + 2y + 3z)\vec{j} + (2x + cy + 3z)\vec{k}$  is Irrotational.

Sol:- Given  $\vec{f} = (2x + 3y + az)\vec{i} + (bx + 2y + 3z)\vec{j} + (2x + cy + 3z)\vec{k}$

$$\text{Curl } \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + 3y + az & bx + 2y + 3z & 2x + cy + 3z \end{vmatrix} = (c-3)\vec{i} - (2-a)\vec{j} + (b-3)\vec{k}$$

If the vector is Irrotational then  $\text{curl } \vec{f} = \vec{0}$

$$\therefore 2-a=0 \Rightarrow a=2, b-3=0 \Rightarrow b=3, c-3=0 \Rightarrow c=3$$

**8:** If  $f(r)$  is differentiable, show that  $\text{curl} \{ \vec{r} f(r) \} = \vec{0}$  where  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ .

Sol:  $r = \vec{r} = \sqrt{x^2 + y^2 + z^2}$

$$\therefore r^2 = x^2 + y^2 + z^2$$

$$\Rightarrow 2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \text{ similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{curl} \{ \vec{r} f(r) \} = \text{curl} \{ f(r) (x\vec{i} + y\vec{j} + z\vec{k}) \} = \text{curl} (x.f(r)\vec{i} + y.f(r)\vec{j} + z.f(r)\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \vec{i} \left[ \frac{\partial}{\partial y} [zf(r)] - \frac{\partial}{\partial z} [yf(r)] \right]$$

$$\sum \vec{i} \left[ zf^1(r) \frac{\partial r}{\partial y} - yf^1(r) \frac{\partial r}{\partial z} \right] = \sum \vec{i} \left[ zf^1(r) \frac{y}{r} - yf^1(r) \frac{z}{r} \right]$$

$$= \vec{0}.$$

9: If  $\vec{A}$  is irrotational vector, evaluate  $\text{div}(\vec{A} \times \vec{r})$  where  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ .

**Sol:** We have  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Given  $\vec{A}$  is an irrotational vector

$$\nabla \times \vec{A} = \vec{0}$$

$$\begin{aligned}\text{div}(\vec{A} \times \vec{r}) &= \nabla \cdot (\vec{A} \times \vec{r}) \\ &= \vec{r} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{r}) \\ &= \vec{r} \cdot (\vec{0}) - \vec{A} \cdot (\nabla \times \vec{r}) \quad [\text{using (1)}] \\ &= -\vec{A} \cdot (\nabla \times \vec{r}) \dots (2)\end{aligned}$$

$$\text{Now } \nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i} \left( \frac{\partial}{\partial y} z - \frac{\partial}{\partial z} y \right) - \vec{j} \left( \frac{\partial}{\partial x} z - \frac{\partial}{\partial z} x \right) + \vec{k} \left( \frac{\partial}{\partial x} y - \frac{\partial}{\partial y} x \right) = \vec{0}$$

$$\therefore \vec{A} \cdot (\nabla \times \vec{r}) = 0 \dots (3)$$

$\therefore$

Hence  $\text{div}(\vec{A} \times \vec{r}) = 0$ . [using (2) and (3)]

10: Find constants a, b, c so that the vector  $\vec{A} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$  is Irrotational. Also find  $\phi$  such that  $\vec{A} = \nabla\phi$ .

**Sol:** Given vector is  $\vec{A} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$

Vector  $\vec{A}$  is Irrotational  $\Rightarrow \text{curl } \vec{A} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow (c+1)\vec{i} + (a-4)\vec{j} + (b-2)\vec{k} = \vec{0}$$

$$\Rightarrow (c+1)\vec{i} + (a-4)\vec{j} + (b-2)\vec{k} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Comparing both sides,

$$c+1=0, a-4=0, b-2=0$$

$$c = -1, a=4, b=2$$

Now  $\vec{A} = (x + 2y + 4z)\vec{i} + (2x - 3y - z)\vec{j} + (4x - y + 2z)\vec{k}$ , on substituting the values of a, b, c

we have  $\vec{A} = \nabla\phi$ .

$$\Rightarrow \vec{A} = (x + 2y + 4z)\vec{i} + (2x - 3y - z)\vec{j} + (4x - y + 2z)\vec{k} = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$



Comparing both sides, we have

$$\frac{\partial \phi}{\partial x} = x + 2y + 4z \Rightarrow \phi = x^2/2 + 2xy + 4zx + f_1(y, z)$$

$$\frac{\partial \phi}{\partial y} = 2x - 3y - z \Rightarrow \phi = 2xy - 3y^2/2 - yz + f_2(z, x)$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z \Rightarrow \phi = 4xz - yz + z^2 + f_3(x, y)$$

Hence  $\phi = x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

**11:** If  $\omega$  is a constant vector, evaluate curl  $V$  where  $V = \omega \times \bar{r}$ .

$$\begin{aligned} \text{Sol: } \text{curl}(\omega \times \bar{r}) &= \sum \bar{i} \times \frac{\partial}{\partial x} (\omega \times \bar{r}) = \sum \bar{i} \times \left[ \frac{\partial \omega}{\partial x} \times \bar{r} + \omega \times \frac{\partial \bar{r}}{\partial x} \right] \\ &= \sum \bar{i} \times [\bar{0} + \omega \times \bar{i}] \quad [\because \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}] \\ &= \sum \bar{i} \times (\omega \times \bar{i}) = \sum [(\bar{i} \cdot \bar{i})\omega - (\bar{i} \cdot \omega)\bar{i}] = \sum \omega - \sum (\bar{i} \cdot \omega)\bar{i} = 3\omega - \omega = 2\omega \end{aligned}$$

### Assignments

1. If  $\bar{f} = e^{x+y+z}(\bar{i} + \bar{j} + \bar{k})$  find curl  $\bar{f}$ .
2. Prove that  $\bar{f} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$  is irrotational.
3. Prove that  $\nabla \cdot (\bar{a} \times \bar{f}) = -\bar{a} \cdot \text{curl } \bar{f}$  where  $\bar{a}$  is a constant vector.
4. Prove that  $\text{curl}(\bar{a} \times \bar{r}) = 2\bar{a}$  where  $\bar{a}$  is a constant vector.
5. If  $\bar{f} = x^2 y \bar{i} - 2zx \bar{j} + 2yz \bar{k}$  find (i) curl  $\bar{f}$  (ii) curl curl  $\bar{f}$ .

## OPERATORS

### Vector differential operator $\nabla$

The operator  $\nabla = \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z}$  is defined such that  $\nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$  where  $\phi$  is a scalar point function.

Note: If  $\phi$  is a scalar point function then  $\nabla \phi = \text{grad } \phi = \sum \bar{i} \frac{\partial \phi}{\partial x}$

(2) Scalar differential operator  $\bar{a} \cdot \nabla$

The operator  $\bar{a} \cdot \nabla = (\bar{a} \cdot \bar{i}) \frac{\partial \phi}{\partial x} + (\bar{a} \cdot \bar{j}) \frac{\partial \phi}{\partial y} + (\bar{a} \cdot \bar{k}) \frac{\partial \phi}{\partial z}$  is defined such that

$$(\bar{a} \cdot \nabla) \phi = (\bar{a} \cdot \bar{i}) \frac{\partial \phi}{\partial x} + (\bar{a} \cdot \bar{j}) \frac{\partial \phi}{\partial y} + (\bar{a} \cdot \bar{k}) \frac{\partial \phi}{\partial z}$$

$$\text{And } (\bar{a} \cdot \nabla) \bar{f} = (\bar{a} \cdot \bar{i}) \frac{\partial \bar{f}}{\partial x} + (\bar{a} \cdot \bar{j}) \frac{\partial \bar{f}}{\partial y} + (\bar{a} \cdot \bar{k}) \frac{\partial \bar{f}}{\partial z}$$

(3). Vector differential operator  $\bar{a} \times \nabla$

The operator  $\bar{a} \times \nabla = (\bar{a} \times \bar{i}) \frac{\partial}{\partial x} + (\bar{a} \times \bar{j}) \frac{\partial}{\partial y} + (\bar{a} \times \bar{k}) \frac{\partial}{\partial z}$  is defined such that

$$(i). (\bar{a} \times \nabla)\phi = (\bar{a} \times \bar{i}) \frac{\partial \phi}{\partial x} + (\bar{a} \times \bar{j}) \frac{\partial \phi}{\partial y} + (\bar{a} \times \bar{k}) \frac{\partial \phi}{\partial z}$$

$$(ii). (\bar{a} \times \nabla) \cdot \bar{f} = (\bar{a} \times \bar{i}) \cdot \frac{\partial \bar{f}}{\partial x} + (\bar{a} \times \bar{j}) \cdot \frac{\partial \bar{f}}{\partial y} + (\bar{a} \times \bar{k}) \cdot \frac{\partial \bar{f}}{\partial z}$$

$$(iii). (\bar{a} \times \nabla) \times \bar{f} = (\bar{a} \times \bar{i}) \times \frac{\partial \bar{f}}{\partial x} + (\bar{a} \times \bar{j}) \times \frac{\partial \bar{f}}{\partial y} + (\bar{a} \times \bar{k}) \times \frac{\partial \bar{f}}{\partial z}$$

(4). Scalar differential operator  $\nabla$ .

The operator  $\nabla = \bar{i} \cdot \frac{\partial}{\partial x} + \bar{j} \cdot \frac{\partial}{\partial y} + \bar{k} \cdot \frac{\partial}{\partial z}$  is defined such that  $\nabla \cdot \bar{f} = \bar{i} \cdot \frac{\partial \bar{f}}{\partial x} + \bar{j} \cdot \frac{\partial \bar{f}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{f}}{\partial z}$

Note:  $\nabla \cdot \bar{f}$  is defined as  $\text{div } \bar{f}$ . It is a scalar point function.

(5). Vector differential operator  $\nabla \times$

The operator  $\nabla \times = \bar{i} \times \frac{\partial}{\partial x} + \bar{j} \times \frac{\partial}{\partial y} + \bar{k} \times \frac{\partial}{\partial z}$  is defined such that

$$\nabla \times \bar{f} = \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$$

Note :  $\nabla \times \bar{f}$  is defined as  $\text{curl } \bar{f}$ . It is a vector point function.

(6). Laplacian Operator  $\nabla^2$

$$\nabla \cdot \nabla \phi = \sum \bar{i} \cdot \frac{\partial}{\partial x} \left( \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z} \right) = \sum \frac{\partial^2 \phi}{\partial x^2} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi = \nabla^2 \phi$$

Thus the operator  $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$  is called Laplacian operator.

Note : (i).  $\nabla^2 \phi = \nabla \cdot (\nabla \phi) = \text{div}(\text{grad } \phi)$

(ii). If  $\nabla^2 \phi = 0$  then  $\phi$  is said to satisfy Laplacian equation. This  $\phi$  is called a harmonic function.

### SOLVED PROBLEMS

**1:** Prove that  $\text{div}(\text{grad } r^m) = m(m+1)r^{m-2}$  (or)  $\nabla^2(r^m) = m(m+1)r^{m-2}$  (or)  $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let  $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$  and  $r = |\bar{r}|$  then  $r^2 = x^2 + y^2 + z^2$ .

Differentiating w.r.t. 'x' partially, we get  $2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$ .

Similarly  $\frac{\partial r}{\partial y} = \frac{y}{r}$  and  $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{Now } \text{grad}(r^m) = \sum \bar{i} \frac{\partial}{\partial x} (r^m) = \sum \bar{i} m r^{m-1} \frac{\partial r}{\partial x} = \sum \bar{i} m r^{m-1} \frac{x}{r} = \sum \bar{i} m r^{m-2} x$$

$$\begin{aligned}
 \therefore \operatorname{div}(\operatorname{grad} r^m) &= \sum \frac{\partial}{\partial x} [m r^{m-2} x] = m \sum \left[ (m-2) r^{m-3} \frac{\partial r}{\partial x} x + r^{m-2} \right] \\
 &= m \sum [(m-2) r^{m-4} x^2 + r^{m-2}] = m [(m-2) r^{m-4} \sum x^2 + \sum r^{m-2}] \\
 &= m [(m-2) r^{m-4} (r^2) + 3 r^{m-2}] \\
 &= m [(m-2) r^{m-2} + 3 r^{m-2}] = m [(m-2+3) r^{m-2}] = m(m+1) r^{m-2}.
 \end{aligned}$$

$$\text{Hence } \nabla^2(r^m) = m(m+1)r^{m-2}$$

**2:** Show that  $\nabla^2[f(r)] = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr} = f^{11}(r) + \frac{2}{r} f^1(r)$  where  $r = |\vec{r}|$ .

$$\text{Sol: } \operatorname{grad} [f(r)] = \nabla f(r) = \sum i \frac{\partial}{\partial x} [f(r)] = \sum i f^1(r) \frac{\partial r}{\partial x} = \sum i f^1(r) \frac{x}{r}$$

$$\begin{aligned}
 \therefore \operatorname{div} [\operatorname{grad} f(r)] &= \nabla^2[f(r)] = \nabla \cdot \nabla f(r) = \sum \frac{\partial}{\partial x} \left[ f^1(r) \frac{x}{r} \right] \\
 &= \sum \frac{r \frac{\partial}{\partial x} [f^1(r)x] - f^1(r)x \frac{\partial}{\partial x} (r)}{r^2} \\
 &= \sum \frac{r \left( f^{11}(r) \frac{\partial r}{\partial x} x + f^1(r) \right) - f^1(r)x \left( \frac{x}{r} \right)}{r^2}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum \frac{r f^{11}(r) \frac{x}{r} x + r f^1(r) - f^1(r)x \left( \frac{x}{r} \right)}{r^2} \\
 &= \frac{\sum r f^{11}(r) \frac{x}{r} x + r f^1(r) - x^2}{r^2} \cdot \frac{f^1(r)}{r} \\
 &= \frac{f^{11}(r)}{r^2} \sum x^2 + \frac{1}{r} \sum f^1(r) - \frac{1}{r^3} f^1(r) \sum x^2 \\
 &= \frac{f^{11}(r)}{r^2} (r^2) + \frac{3}{r} f^1(r) - \frac{1}{r^3} f^1(r) r^2 \\
 &= f^{11}(r) + \frac{2}{r} f^1(r)
 \end{aligned}$$

**3:** If  $\phi$  satisfies Laplacian equation, show that  $\nabla \phi$  is both solenoidal and irrotational.

Sol: Given  $\nabla^2 \phi = 0 \Rightarrow \operatorname{div}(\operatorname{grad} \phi) = 0 \Rightarrow \operatorname{grad} \phi$  is solenoidal

We know that  $\operatorname{curl}(\operatorname{grad} \phi) = \vec{0} \Rightarrow \operatorname{grad} \phi$  is always irrotational.

2. Show that (i)  $(\vec{a} \cdot \nabla)\phi = \vec{a} \cdot \nabla \phi$  (ii)  $(\vec{a} \cdot \nabla)\vec{r} = \vec{a}$ .

Sol: (i). Let  $\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$ . Then

$$\vec{a} \cdot \nabla = (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \cdot \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$$

$$\therefore (\bar{a} \cdot \nabla)\phi = a_1 \frac{\partial \phi}{\partial x} + a_2 \frac{\partial \phi}{\partial y} + a_3 \frac{\partial \phi}{\partial z}$$

Hence  $(\bar{a} \cdot \nabla)\phi = \bar{a} \cdot \nabla \phi$

$$(ii). \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\therefore \frac{\partial \bar{r}}{\partial x} = \bar{i}, \frac{\partial \bar{r}}{\partial y} = \bar{j}, \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

$$(\bar{a} \cdot \nabla)\bar{r} = \sum a_1 \frac{\partial}{\partial x}(\bar{r}) = \sum a_1 \frac{\partial \bar{r}}{\partial x} = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k} = \bar{a}$$

**5:** Prove that (i)  $(\bar{f} \times \nabla) \cdot \bar{r} = 0$  (ii).  $(\bar{f} \times \nabla) \times \bar{r} = -2\bar{f}$

$$\text{Sol: (i) } (\bar{f} \times \nabla) \cdot \bar{r} = \sum (\bar{f} \times \bar{i}) \cdot \frac{\partial \bar{r}}{\partial x} = \sum (\bar{f} \times \bar{i}) \cdot \bar{i} = 0$$

$$(ii) (\bar{f} \times \nabla) = (\bar{f} \times \bar{i}) \frac{\partial}{\partial x} + (\bar{f} \times \bar{j}) \frac{\partial}{\partial y} + (\bar{f} \times \bar{k}) \frac{\partial}{\partial z}$$

$$(\bar{f} \times \nabla) \times \bar{r} = (\bar{f} \times \bar{i}) \times \frac{\partial \bar{r}}{\partial x} + (\bar{f} \times \bar{j}) \times \frac{\partial \bar{r}}{\partial y} + (\bar{f} \times \bar{k}) \times \frac{\partial \bar{r}}{\partial z} = \sum (\bar{f} \times \bar{i}) \times \bar{i} = \sum [(\bar{f} \cdot \bar{i})\bar{i} - \bar{f}]$$

$$= (\bar{f} \cdot \bar{i})\bar{i} + (\bar{f} \cdot \bar{j})\bar{j} + (\bar{f} \cdot \bar{k})\bar{k} - 3\bar{f} = \bar{f} - 3\bar{f} = -2\bar{f}$$

**6:** Find  $\text{div } \bar{F}$ , where  $\bar{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

**Sol:** Let  $\phi = x^3 + y^3 + z^3 - 3xyz$ . Then

$$\bar{F} = \text{grad } \phi$$

$$= \sum i \frac{\partial \phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\bar{j} + 3(z^2 - xy)\bar{k} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k} \text{ (say)}$$

$$\therefore \text{div } \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 6x + 6y + 6z = 6(x + y + z)$$

$$\text{i.e. } \text{div}[\text{grad}(x^3 + y^3 + z^3 - 3xyz)] = \nabla^2(x^3 + y^3 + z^3 - 3xyz) = 6(x + y + z).$$

**7:** If  $f = (x^2 + y^2 + z^2)^{-n}$  then find  $\text{div grad } f$  and determine  $n$  if  $\text{div grad } f = 0$ .

**Sol:** Let  $f = (x^2 + y^2 + z^2)^{-n}$  and  $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = |\bar{r}| \Rightarrow r^2 = x^2 + y^2 + z^2$$

$$\Rightarrow f(r) = (r^2)^{-n} = r^{-2n}$$

$$\therefore f^1(r) = -2n r^{-2n-1}$$

$$\text{and } f^{11}(r) = (-2n)(-2n-1)r^{-2n-2} = 2n(2n+1)r^{-2n-2}$$

$$\text{We have } \text{div grad } f = \nabla^2 f(r) = f^{11}(r) + \frac{2}{r} f^1(r) = (2n)(2n+1)r^{-2n-2} - 4n r^{-2n-2}$$

$$= r^{-2n-2}[2n(2n+1-2)] = (2n)(2n-1)r^{-2n-2}$$

If  $\text{div grad } f(r)$  is zero, we get  $n = 0$  or  $n = \frac{1}{2}$ .

8: Prove that  $\nabla \times \left( \frac{\bar{A} \times \bar{r}}{r^n} \right) = \frac{(2-n)\bar{A}}{r^n} + \frac{n(\bar{r} \cdot \bar{A})\bar{r}}{r^{n+2}}$ .

Sol: We have  $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$  and  $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\therefore \frac{\partial \bar{r}}{\partial x} = \bar{i}, \frac{\partial \bar{r}}{\partial y} = \bar{j}, \frac{\partial \bar{r}}{\partial z} = \bar{k} \text{ and}$$

$$r^2 = x^2 + y^2 + z^2 \dots (1)$$

Diff. (1) partially,

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \text{ similarly } \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla \times \left( \frac{\bar{A} \times \bar{r}}{r^n} \right) = \sum \bar{i} \times \frac{\partial}{\partial x} \left( \frac{(\bar{A} \times \bar{r})}{r^n} \right)$$

$$\begin{aligned} \text{Now } \frac{\partial}{\partial x} \left( \frac{(\bar{A} \times \bar{r})}{r^n} \right) &= \bar{A} \times \frac{\partial}{\partial x} \left( \frac{\bar{r}}{r^n} \right) = \bar{A} \times \left[ \frac{r^n \bar{i} - \bar{r} n r^{n-1}}{r^{2n}} \right] \frac{\partial r}{\partial x} \\ &= \bar{A} \times \left[ \frac{r^n \bar{i} - n r^{n-2} x \bar{r}}{r^{2n}} \right] = \bar{A} \times \left[ \frac{1}{r^n} \bar{i} - \frac{n}{r^{n+2}} x \bar{r} \right] \\ &= \frac{\bar{A} \times \bar{i}}{r^n} - \frac{n}{r^{n+2}} x (\bar{A} \times \bar{r}) \\ \therefore \bar{i} \times \frac{\partial}{\partial x} \left( \frac{(\bar{A} \times \bar{r})}{r^n} \right) &= \frac{\bar{i} \times (\bar{A} \times \bar{i})}{r^n} - \frac{n x}{r^{n+2}} \bar{i} \times (\bar{A} \times \bar{r}) \\ &= \frac{(\bar{i} \cdot \bar{i}) \bar{A} - (\bar{i} \cdot \bar{A}) \bar{i}}{r^n} - \frac{n x}{r^{n+2}} [(\bar{i} \cdot \bar{r}) \bar{A} - (\bar{i} \cdot \bar{A}) \bar{r}] \end{aligned}$$

Let  $A_1 \bar{i} + A_2 \bar{j} + A_3 \bar{k}$ . Then  $\bar{i} \cdot \bar{A} = A_1$

$$\therefore \bar{i} \times \frac{\partial}{\partial x} \left( \frac{(\bar{A} \times \bar{r})}{r^n} \right) = \left( \frac{\bar{A} - A_1 \bar{i}}{r^n} \right) - \frac{n x}{r^{n+2}} [x \bar{A} - A_1 \bar{r}]$$

$$\begin{aligned} \text{and } \sum \bar{i} \times \frac{\partial}{\partial x} \left( \frac{(\bar{A} \times \bar{r})}{r^n} \right) &= \sum \left( \frac{\bar{A} - A_1 \bar{i}}{r^n} \right) - \frac{n x}{r^{n+2}} [x \bar{A} - A_1 \bar{r}] \\ &= \frac{3\bar{A} - \bar{A}}{r^n} - \frac{n}{r^{n+2}} [r^2 \bar{A}] + \frac{n \bar{r}}{r^{n+2}} (A_1 x + A_2 y + A_3 z) \\ &= \frac{2\bar{A}}{r^n} - \frac{n}{r^n} \bar{A} + \frac{n \bar{r}}{r^{n+2}} (\bar{A} \cdot \bar{r}) = \frac{(2-n)\bar{A}}{r^n} + \frac{n \bar{r}}{r^{n+2}} (\bar{A} \cdot \bar{r}) \end{aligned}$$

Hence the result.

### VECTOR IDENTITIES

**Theorem 1:** If  $\bar{a}$  is a differentiable function and  $\phi$  is a differentiable scalar function, then prove that  $\text{div}(\phi \bar{a}) = (\text{grad } \phi) \cdot \bar{a} + \phi \text{div } \bar{a}$  or  $\nabla \cdot (\phi \bar{a}) = (\nabla \phi) \cdot \bar{a} + \phi (\nabla \cdot \bar{a})$

Proof:  $\text{div}(\phi \bar{a}) = \nabla \cdot (\phi \bar{a}) = \sum_i \frac{\partial}{\partial x_i} (\phi \bar{a}_i)$

$$= \sum_i \bar{a}_i \left( \frac{\partial \phi}{\partial x_i} + \phi \frac{\partial}{\partial x_i} \right) = \sum_i \left( \bar{a}_i \frac{\partial \phi}{\partial x_i} \right) + \sum_i \left( \phi \frac{\partial \bar{a}_i}{\partial x_i} \right)$$

$$= \sum_i \left( \bar{a}_i \frac{\partial \phi}{\partial x_i} \right) + \phi \left( \sum_i \frac{\partial \bar{a}_i}{\partial x_i} \right) = (\nabla \phi) \cdot \bar{a} + \phi (\nabla \cdot \bar{a})$$

**Theorem 2:** Prove that  $\text{curl}(\phi \bar{a}) = (\text{grad } \phi) \times \bar{a} + \phi \text{curl } \bar{a}$

Proof :  $\text{curl}(\phi \bar{a}) = \nabla \times (\phi \bar{a}) = \sum_i \frac{\partial}{\partial x_i} (\phi \bar{a})$

$$= \sum_i \bar{a}_i \times \left( \frac{\partial \phi}{\partial x_i} + \phi \frac{\partial}{\partial x_i} \right) = \sum_i \left( \bar{a}_i \times \frac{\partial \phi}{\partial x_i} \right) + \sum_i \left( \phi \times \frac{\partial \bar{a}_i}{\partial x_i} \right)$$

$$= \nabla \phi \times \bar{a} + (\nabla \times \bar{a}) \phi = (\text{grad } \phi) \times \bar{a} + \phi \text{curl } \bar{a}$$

**Theorem 3:** Prove that  $\text{grad}(\bar{a} \cdot \bar{b}) = (\bar{b} \cdot \nabla) \bar{a} + (\bar{a} \cdot \nabla) \bar{b} + \bar{b} \times \text{curl } \bar{a} + \bar{a} \times \text{curl } \bar{b}$

**Proof:** Consider

$$\bar{a} \times \text{curl}(\bar{b}) = \bar{a} \times (\nabla \times \bar{b}) = \sum_i \bar{a}_i \times \left( \frac{\partial \bar{b}}{\partial x_i} \right)$$

$$= \sum_i \bar{a}_i \times \left( \bar{b} \times \frac{\partial}{\partial x_i} \right)$$

$$= \sum_i \left\{ \left( \bar{a}_i \cdot \frac{\partial \bar{b}}{\partial x_i} \right) \bar{b} - (\bar{a}_i \cdot \bar{b}) \frac{\partial \bar{b}}{\partial x_i} \right\} = \sum_i \bar{b} \left\{ \bar{a}_i \cdot \frac{\partial}{\partial x_i} \right\} - \left\{ \bar{a}_i \cdot \sum_i \frac{\partial \bar{b}}{\partial x_i} \right\} \bar{b}$$

$$\therefore \bar{a} \times \text{curl } \bar{b} = \sum_i \bar{b} \left( \bar{a}_i \cdot \frac{\partial}{\partial x_i} \right) - (\bar{a} \cdot \nabla) \bar{b} \dots (1)$$

$$\text{Similarly, } \bar{b} \times \text{curl } \bar{a} = \sum_i \bar{a} \left( \bar{b}_i \cdot \frac{\partial}{\partial x_i} \right) - (\bar{b} \cdot \nabla) \bar{a} \dots (2)$$

(1)+(2) gives

$$\bar{a} \times \text{curl } \bar{b} + \bar{b} \times \text{curl } \bar{a} = \sum_i \bar{b} \left( \bar{a}_i \cdot \frac{\partial}{\partial x_i} \right) - (\bar{a} \cdot \nabla) \bar{b} + \sum_i \bar{a} \left( \bar{b}_i \cdot \frac{\partial}{\partial x_i} \right) - (\bar{b} \cdot \nabla) \bar{a}$$

$$\Rightarrow \bar{a} \times \text{curl } \bar{b} + \bar{b} \times \text{curl } \bar{a} + (\bar{a} \cdot \nabla) \bar{b} + (\bar{b} \cdot \nabla) \bar{a} = \sum_i \bar{b} \left( \bar{a}_i \cdot \frac{\partial}{\partial x_i} \right) + \bar{a} \left( \bar{b}_i \cdot \frac{\partial}{\partial x_i} \right)$$

$$= \sum_i \bar{b} \frac{\partial}{\partial x_i} (\bar{a}_i \bar{b})$$

$$= \nabla (\bar{a} \cdot \bar{b}) = \text{grad}(\bar{a} \cdot \bar{b})$$

**Theorem 4:** Prove that  $\text{div}(\bar{a} \times \bar{b}) = \bar{b} \cdot \text{curl} \bar{a} - \bar{a} \cdot \text{curl} \bar{b}$

$$\begin{aligned} \text{Proof: } \text{div}(\bar{a} \times \bar{b}) &= \sum \bar{i} \cdot \frac{\partial}{\partial x} (\bar{a} \times \bar{b}) = \sum \bar{i} \cdot \left( \frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right) \\ &= \sum \bar{i} \cdot \left( \frac{\partial \bar{a}}{\partial x} \times \bar{b} \right) + \sum \bar{i} \cdot \left( \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right) = \sum \left( \bar{i} \times \frac{\partial \bar{a}}{\partial x} \right) \cdot \bar{b} - \sum \left( \bar{i} \times \frac{\partial \bar{b}}{\partial x} \right) \cdot \bar{a} \\ &= (\nabla \times \bar{a}) \cdot \bar{b} - (\nabla \times \bar{b}) \cdot \bar{a} = \bar{b} \cdot \text{curl} \bar{a} - \bar{a} \cdot \text{curl} \bar{b} \end{aligned}$$

**Theorem 5 :** Prove that  $\text{curl}(\bar{a} \times \bar{b}) = \bar{a} \text{div} \bar{b} - \bar{b} \text{div} \bar{a} + (\bar{b} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{b}$

$$\begin{aligned} \text{Proof : } \text{curl}(\bar{a} \times \bar{b}) &= \sum \bar{i} \times \frac{\partial}{\partial x} (\bar{a} \times \bar{b}) = \sum \bar{i} \times \left[ \frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right] \\ &= \sum \bar{i} \times \left( \frac{\partial \bar{a}}{\partial x} \times \bar{b} \right) + \sum \bar{i} \times \left( \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right) \\ &= \sum \left\{ (\bar{i} \cdot \bar{b}) \frac{\partial \bar{a}}{\partial x} - \left( \bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) \bar{b} \right\} + \sum \left\{ \left( \bar{i} \cdot \frac{\partial \bar{b}}{\partial x} \right) \bar{a} - (\bar{i} \cdot \bar{a}) \frac{\partial \bar{b}}{\partial x} \right\} \\ &= \sum (\bar{b} \cdot \bar{i}) \frac{\partial \bar{a}}{\partial x} - \sum \left( \bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) \bar{b} + \sum \left( \bar{i} \cdot \frac{\partial \bar{b}}{\partial x} \right) \bar{a} - \left( \bar{a} \cdot \sum \bar{i} \frac{\partial}{\partial x} \right) \bar{b} \\ &= (\bar{b} \cdot \nabla) \bar{a} - (\nabla \cdot \bar{a}) \bar{b} + (\nabla \cdot \bar{b}) \bar{a} - (\bar{a} \cdot \nabla) \bar{b} \\ &= (\nabla \cdot \bar{b}) \bar{a} - (\nabla \cdot \bar{a}) \bar{b} + (\bar{b} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{b} \\ &= \bar{a} \text{div} \bar{b} - \bar{b} \text{div} \bar{a} + (\bar{b} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{b} \end{aligned}$$

**Theorem 6:** Prove that  $\text{curl grad } \phi = 0$ .

Proof: Let  $\phi$  be any scalar point function. Then

$$\text{grad } \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$\begin{aligned} \text{curl}(\text{grad} \phi) &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} \\ &= \bar{i} \left( \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right) - \bar{j} \left( \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right) - \bar{k} \left( \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right) = \bar{0} \end{aligned}$$

Note : Since  $\text{Curl}(\text{grad} \phi) = \bar{0}$ , we have  $\text{grad } \phi$  is always irrotational.

7. Prove that  $\text{div curl } \bar{f} = 0$

Proof : Let  $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$

$$\therefore \text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \bar{i} - \left( \frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \bar{j} + \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \bar{k}$$

$$\therefore \text{div } \text{curl } \bar{f} = \nabla \cdot (\nabla \times \bar{f}) = \frac{\partial}{\partial x} \left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} = 0$$

Note : Since  $\text{div}(\text{curl } \bar{f}) = 0$ , we have  $\text{curl } \bar{f}$  is always solenoidal.

**Theorem 8:** If  $f$  and  $g$  are two scalar point functions, prove that  $\text{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$

**Sol:** Let  $f$  and  $g$  be two scalar point functions. Then

$$\nabla g = \bar{i} \frac{\partial g}{\partial x} + \bar{j} \frac{\partial g}{\partial y} + \bar{k} \frac{\partial g}{\partial z}$$

Now  $f \nabla g = \bar{i} f \frac{\partial g}{\partial x} + \bar{j} f \frac{\partial g}{\partial y} + \bar{k} f \frac{\partial g}{\partial z}$

$$\therefore \nabla \cdot (f \nabla g) = \frac{\partial}{\partial x} \left( f \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial y} \left( f \frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial z} \left( f \frac{\partial g}{\partial z} \right)$$

$$= f \left( \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right) + \left( \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial g}{\partial z} \right)$$

$$= f \nabla^2 g + \left( \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z} \right) \cdot \left( \bar{i} \frac{\partial g}{\partial x} + \bar{j} \frac{\partial g}{\partial y} + \bar{k} \frac{\partial g}{\partial z} \right)$$

$$= f \nabla^2 g + \nabla f \cdot \nabla g$$

**Theorem 9:** Prove that  $\nabla \times (\nabla \times \bar{a}) = \nabla (\nabla \cdot \bar{a}) - \nabla^2 \bar{a}$ .

Proof:  $\nabla \times (\nabla \times \bar{a}) = \sum \bar{i} \times \frac{\partial}{\partial x} (\nabla \times \bar{a})$

Now  $\bar{i} \times \frac{\partial}{\partial x} (\nabla \times \bar{a}) = \bar{i} \times \frac{\partial}{\partial x} \left( \bar{i} \times \frac{\partial \bar{a}}{\partial x} + \bar{j} \times \frac{\partial \bar{a}}{\partial y} + \bar{k} \times \frac{\partial \bar{a}}{\partial z} \right)$

$$= \bar{i} \times \left( \bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2} + \bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y} + \bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z} \right)$$

$$= \bar{i} \times \left( \bar{i} \times \frac{\partial^2 \bar{a}}{\partial x^2} \right) + \bar{i} \times \left( \bar{j} \times \frac{\partial^2 \bar{a}}{\partial x \partial y} \right) + \bar{i} \times \left( \bar{k} \times \frac{\partial^2 \bar{a}}{\partial x \partial z} \right)$$



$$\begin{aligned}
& \left( \bar{i} \cdot \frac{\partial^2 \bar{a}}{\partial x^2} \right) \bar{i} + \frac{\partial^2 \bar{a}}{\partial x^2} + \left( \bar{j} \cdot \frac{\partial^2 \bar{a}}{\partial x \partial y} \right) \bar{j} + \left( \bar{k} \cdot \frac{\partial^2 \bar{a}}{\partial x \partial z} \right) \bar{k} + \left[ \bar{i} \cdot \bar{j} + \bar{j} \cdot \bar{i} + \bar{j} \cdot \bar{k} + \bar{k} \cdot \bar{j} + \bar{k} \cdot \bar{i} + \bar{i} \cdot \bar{k} \right] \frac{\partial^2 \bar{a}}{\partial x^2} \\
&= \bar{i} \frac{\partial}{\partial x} \left( \bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) + \bar{j} \frac{\partial}{\partial y} \left( \bar{j} \cdot \frac{\partial \bar{a}}{\partial y} \right) + \bar{k} \frac{\partial}{\partial z} \left( \bar{k} \cdot \frac{\partial \bar{a}}{\partial z} \right) - \frac{\partial^2 \bar{a}}{\partial x^2} = \nabla \left( \bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) - \frac{\partial^2 \bar{a}}{\partial x^2} \\
&\therefore \sum \bar{i} \times \frac{\partial}{\partial x} (\nabla \times \bar{a}) = \nabla \sum \bar{i} \cdot \frac{\partial \bar{a}}{\partial x} - \sum \frac{\partial^2 \bar{a}}{\partial x^2} = \nabla (\nabla \cdot \bar{a}) - \left( \frac{\partial^2 \bar{a}}{\partial x^2} + \frac{\partial^2 \bar{a}}{\partial y^2} + \frac{\partial^2 \bar{a}}{\partial z^2} \right)
\end{aligned}$$

$$\therefore \nabla \times (\nabla \times \bar{a}) = \nabla (\nabla \cdot \bar{a}) - \nabla^2 \bar{a}$$

$$\text{i.e., } \text{curl curl } \bar{a} = \text{grad div } \bar{a} - \nabla^2 \bar{a}$$

### SOLVED PROBLEMS

1: Prove that  $(\nabla f \times \nabla g)$  is solenoidal.

Sol: We know that  $\text{div}(\bar{a} \times \bar{b}) = \bar{b} \cdot \text{curl } \bar{a} - \bar{a} \cdot \text{curl } \bar{b}$

Take  $\bar{a} = \nabla f$  and  $\bar{b} = \nabla g$

Then  $\text{div}(\nabla f \times \nabla g) = \nabla g \cdot \text{curl}(\nabla f) - \nabla f \cdot \text{curl}(\nabla g) = 0 \left[ \because \text{curl}(\nabla f) = \bar{0} = \text{curl}(\nabla g) \right]$

$\therefore \nabla f \times \nabla g$  is solenoidal.

2. Prove that (i)  $\text{div}\{(\bar{r} \times \bar{a}) \times \bar{b}\} = -2(\bar{b} \cdot \bar{a})$  (ii)  $\text{curl}\{(\bar{r} \cdot \bar{a}) \times \bar{b}\} = \bar{b} \times \bar{a}$  where  $\bar{a}$  and  $\bar{b}$  are constant vectors.

Sol: (i)

$$\begin{aligned}
\text{div}\{(\bar{r} \times \bar{a}) \times \bar{b}\} &= \text{div}[(\bar{r} \cdot \bar{b})\bar{a} - (\bar{a} \cdot \bar{b})\bar{r}] \\
&= \text{div}(\bar{r} \cdot \bar{b})\bar{a} - (\bar{a} \cdot \bar{b})\text{div } \bar{r} \\
&= \left[ (\bar{r} \cdot \bar{b})\text{div } \bar{a} + \bar{a} \cdot \text{grad}(\bar{r} \cdot \bar{b}) \right] - \left[ (\bar{a} \cdot \bar{b})\text{div } \bar{r} + \bar{r} \cdot \text{grad}(\bar{a} \cdot \bar{b}) \right]
\end{aligned}$$

We have  $\text{div } \bar{a} = 0, \text{div } \bar{r} = 3, \text{grad}(\bar{a} \cdot \bar{b}) = 0$

$$\begin{aligned}
\text{div}\{(\bar{r} \times \bar{a}) \times \bar{b}\} &= 0 + \bar{a} \cdot \text{grad}(\bar{r} \cdot \bar{b}) - 3(\bar{a} \cdot \bar{b}) \\
&= \bar{a} \cdot \sum \frac{\partial}{\partial x} (\bar{r} \cdot \bar{b}) - 3(\bar{a} \cdot \bar{b}) \\
&= \bar{a} \cdot \sum \bar{i} \frac{\partial \bar{r}}{\partial x} \cdot \bar{b} - 3(\bar{a} \cdot \bar{b}) \\
&= \bar{a} \cdot \sum \bar{i} (\bar{i} \cdot \bar{b}) - 3(\bar{a} \cdot \bar{b}) \\
&= \bar{a} \cdot \bar{b} - 3(\bar{a} \cdot \bar{b}) = -2(\bar{a} \cdot \bar{b}) \\
&= -2(\bar{b} \cdot \bar{a})
\end{aligned}$$

$$(ii) \text{curl}\{(\bar{r} \times \bar{a}) \times \bar{b}\} = \text{curl}\left[(\bar{r} \cdot \bar{b})\bar{a} - (\bar{a} \cdot \bar{b})\bar{r}\right]$$

$$= \text{curl}(\bar{r} \cdot \bar{b})\bar{a} - \text{curl}(\bar{a} \cdot \bar{b})\bar{r}$$

$$= (\bar{r} \cdot \bar{b})\text{curl}\bar{a} + \text{grad}(\bar{r} \cdot \bar{b}) \times \bar{a}$$

$$= \bar{0} + \nabla(\bar{r} \cdot \bar{b}) \times \bar{a} \quad (\because \text{curl}\bar{a} = \bar{0})$$

$$= \bar{b} \times \bar{a} \quad \text{Since } \text{grad}(\bar{r} \cdot \bar{b}) = \bar{b}$$

$$\mathbf{3: Prove that} \quad \nabla \left[ \nabla \cdot \frac{\bar{r}}{r} \right] = \frac{-2}{r^3} \bar{r}.$$

$$\text{Sol: We have } \nabla \cdot \left( \frac{\bar{r}}{r} \right) = \sum i \cdot \frac{\partial}{\partial x} \left( \frac{\bar{r}}{r} \right)$$

$$= \sum i \cdot \left[ \frac{1}{r} \frac{\partial \bar{r}}{\partial x} + \bar{r} \left( \frac{-1}{r^2} \right) \left( \frac{x}{r} \right) \right] = \sum i \cdot \left( \frac{1}{r} i - \frac{\bar{r}}{r^3} x \right)$$

$$= \frac{1}{r} \sum i \cdot i - \frac{1}{r^3} r^2 = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\therefore \nabla \left[ \nabla \cdot \left( \frac{\bar{r}}{r} \right) \right] = \sum i \left( \frac{\partial}{\partial x} \left( \frac{2}{r} \right) \right) = \sum i \left( \frac{-2}{r^2} \right) \left( \frac{x}{r} \right) = \frac{-2}{r^3} \sum xi = \frac{-2\bar{r}}{r^3}.$$

$$\mathbf{4: Find} (A \times \nabla) \phi, \text{ if } A = yz^2 \bar{i} - 3xz^2 \bar{j} + 2xyz \bar{k} \text{ and } \phi = xyz.$$

Sol : We have

$$A \times \nabla = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ yz^2 & -3xz^2 & 2xyz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \bar{i} \left[ \frac{\partial}{\partial x} (-3xz^2) - \frac{\partial}{\partial y} (2xyz) \right] - \bar{j} \left[ \frac{\partial}{\partial z} (yz^2) - \frac{\partial}{\partial x} (2xyz) \right] + \bar{k} \left[ \frac{\partial}{\partial y} (yz^2) - \frac{\partial}{\partial x} (-3xz^2) \right]$$

$$= \bar{i} (-6xz - 2xz) - \bar{j} (2yz - 2yz) + \bar{k} (z^2 + 3z^2) = -8xz \bar{i} - 0 \bar{j} + 4z^2 \bar{k}$$

$$\therefore (A \times \nabla) \phi = (-8xz \bar{i} + 4z^2 \bar{k}) xyz = -8x^2 y z^2 \bar{i} + 4x y z^3 \bar{k}$$

### Vector Integration

**Line integral:-** (i)  $\int_c \bar{F} \cdot d\bar{r}$  is called Line integral of  $\bar{F}$  along c

**Note :** Work done by  $\bar{F}$  along a curve c is  $\int_c \bar{F} \cdot d\bar{r}$

### PROBLEMS

1. If  $\vec{F} = (x^2 - 27)\vec{i} - 6yz\vec{j} + 8xz^2\vec{k}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  from the point (0,0,0) to the point (1,1,1) along the

Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

**Solution:** Given  $\vec{F} = (x^2 - 27)\vec{i} - 6yz\vec{j} + 8xz^2\vec{k}$

Now  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\therefore \vec{F} \cdot d\vec{r} = (x^2 - 27)dx - (6yz)dy + 8xz^2dz$$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y = 0 = z and dy = dz = 0. Also x changes from 0 to 1.

$$\therefore \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^1 (x^2 - 27)dx = \left[ \frac{x^3}{3} - 27x \right]_0^1 = \frac{1}{3} - 27 = -\frac{80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x = 1, z = 0  $\Rightarrow$  dx = 0, dz = 0. y changes from 0 to 1.

$$\therefore \int_{AB} \vec{F} \cdot d\vec{r} = \int_{y=0}^1 (-6yz)dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

x = 1 = y  $\Rightarrow$  dx = dy = 0 and z changes from 0 to 1.

$$\therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_{z=0}^1 8xz^2dz = \int_{z=0}^1 8xz^2dz = \left[ \frac{8z^3}{3} \right]_0^1 = \frac{8}{3}$$

$$(i) + (ii) + (iii) \Rightarrow \int_C \vec{F} \cdot d\vec{r} = \frac{88}{3}$$

2. If  $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$ , evaluate  $\int_C \vec{F} \cdot d\vec{r}$  Along the curve C in xy-plane  $y = x^3$  from (1,1) to (2,8).

**Solution :** Given  $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$ , -----(1)

Along the curve  $y = x^3$ ,  $dy = 3x^2 dx$

$$\therefore \vec{F} = (5x^4 - 6x^2)\vec{i} + (2x^3 - 4x)\vec{j}, \text{ [Putting } y = x^3 \text{ in (1)]}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} = dx\vec{i} + 3x^2dx\vec{j}$$

$$\therefore \vec{F} \cdot d\vec{r} = [(5x^4 - 6x^2)\vec{i} + (2x^3 - 4x)\vec{j}] \cdot [dx\vec{i} + 3x^2dx\vec{j}]$$

$$= (5x^4 - 6x^2)dx + (2x^3 - 4x)3x^2dx$$

$$= (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$$

$$\text{Hence } \int_{y=x^3} \vec{F} \cdot d\vec{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$$

$$\left( \frac{x^6}{6} + \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{4} \right) - \left( \frac{x^6}{6} + \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{4} \right)^2$$

$$= 16(4+2-3-1) - (1+1-3-2) = 32+3 = 35$$

3. Find the work done by the force  $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ , when it moves a particle along the arc of the curve  $\vec{r} = \cos t \vec{i} + \sin t \vec{j} - t \vec{k}$  from  $t = 0$  to  $t = 2\pi$

**Solution :** Given force  $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$  and the arc is  $\vec{r} = \cos t \vec{i} + \sin t \vec{j} - t \vec{k}$

i.e.,  $x = \cos t, y = \sin t, z = -t$

$$\therefore d\vec{r} = (-\sin t \vec{i} + \cos t \vec{j} - \vec{k}) dt$$

$$\therefore \vec{F} \cdot d\vec{r} = (-t \vec{i} + \cos t \vec{j} + \sin t \vec{k}) \cdot (-\sin t \vec{i} + \cos t \vec{j} - \vec{k}) dt = (t \sin t + \cos^2 t - \sin t) dt$$

$$\text{Hence work done} = \int_0^{2\pi} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (t \sin t + \cos^2 t - \sin t) dt$$

$$= \left[ t(-\cos t) \right]_0^{2\pi} - \int_0^{2\pi} (-\sin t) dt + \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt - \int_0^{2\pi} \sin t dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2} \left( t + \frac{\sin 2t}{2} \right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2} (2\pi) + (1-1) = -2\pi + \pi = -\pi$$

### PROBLEMS

1. Evaluate  $\int \vec{F} \cdot n dS$  where  $\vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$  and  $S$  is the surface  $x^2 + y^2 = 16$  included in the first octant between  $z = 0$  and  $z = 5$ .

Sol. The surface  $S$  is  $x^2 + y^2 = 16$  included in the first octant between  $z = 0$  and  $z = 5$ .

$$\text{Let } \phi = x^2 + y^2 = 16$$

$$\text{Then } \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} = 2x\vec{i} + 2y\vec{j}$$

$$\therefore \text{unit normal } \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{x\vec{i} + y\vec{j}}{4} \quad (\because x^2 + y^2 = 16)$$

Let  $R$  be the projection of  $S$  on  $yz$ -plane

$$\text{Then } \int_S \vec{F} \cdot n dS = \iint_R \vec{F} \cdot \vec{n} \frac{dydz}{|\vec{n} \cdot \vec{i}|} \dots\dots\dots *$$

$$\text{Given } \vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$$

$$\vec{F} \cdot \vec{n} = \frac{1}{4}(yz + xy)$$

and  $\vec{n} \cdot \vec{i} = \frac{x}{4}$

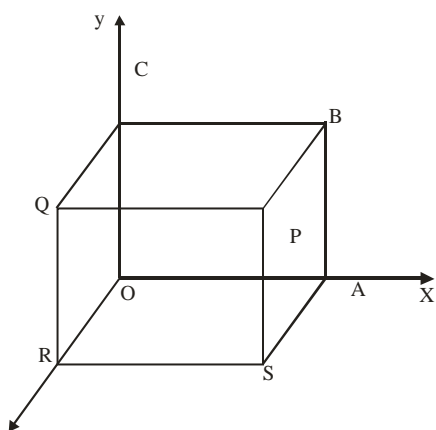
In yz-plane,  $x = 0$ ,  $y = 4$

In first octant,  $y$  varies from 0 to 4 and  $z$  varies from 0 to 5.

$$\begin{aligned} \int_S \vec{F} \cdot \vec{n} dS &= \int_{y=0}^4 \int_{z=0}^5 \left( \frac{xz + xy}{4} \right) dy dz \Big|_{\frac{x}{4}} \\ &= \int_{y=0}^4 \int_{z=0}^5 (y + z) dz dy \\ &= 90. \end{aligned}$$

2. If  $\vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$ , evaluate  $\int_S \vec{F} \cdot \vec{n} dS$  where  $S$  is the surface of the cube bounded by  $x = 0$ ,  $x = a$ ,  $y = 0$ ,  $y = a$ ,  $z = 0$ ,  $z = a$ .

Sol. Given that  $S$  is the surface of the  $x = 0$ ,  $x = a$ ,  $y = 0$ ,  $y = a$ ,  $z = 0$ ,  $z = a$ , and  $\vec{F} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$  we need to evaluate  $\int_S \vec{F} \cdot \vec{n} dS$ .



(i) For OABC

Eqn is  $z = 0$  and  $dS = dx dy$

$$\vec{n} = -\vec{k}$$

$$\int_{S_1} \vec{F} \cdot \vec{n} dS = - \int_{x=0}^a \int_{y=0}^a (yz) dx dy = 0$$

(ii) For PQRS

$$\vec{n} = \vec{k}$$

$$\int_{S_2} \vec{F} \cdot \vec{n} dS = \int_{x=0}^a \left( \int_{y=0}^a y(a) dy \right) dx = \frac{a^4}{2}$$

**(iii) For OCQR**

Eqn is  $x = 0$ , and  $\vec{n} = -\vec{i}$ ,  $dS = dy dz$

$$\int_{S_3} \vec{F} \cdot \vec{n} dS = \int_{y=0}^a \int_{z=0}^a 4xz dy dz = 0$$

**(iv) For ABPS**

Eqn is  $x = a$ , and  $\vec{n} = \vec{i}$ ,  $dS = dy dz$

$$\int_{S_4} \vec{F} \cdot \vec{n} dS = \int_{y=0}^a \left( \int_{z=0}^a 4az dz \right) dy = 2a^4$$

**(v) For OASR**

Eqn is  $y = 0$ , and  $\vec{n} = -\vec{j}$ ,  $dS = dx dz$

$$\int_{S_5} \vec{F} \cdot \vec{n} dS = \int_{x=0}^a \int_{z=0}^a y^2 dz dx = 0$$

**(vi) For PBCQ**

Eqn is  $y = a$ , and  $\vec{n} = \vec{j}$ ,  $dS = dx dz$

$$\int_{S_6} \vec{F} \cdot \vec{n} dS = - \int_{x=0}^a \int_{z=0}^a y^2 dz dx = 0$$

From (i) – (vi) we get

$$\int_{S_6} \vec{F} \cdot \vec{n} dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a^4 = \frac{3a^4}{2}$$

## VOLUME INTEGRALS

Let  $V$  be the volume bounded by a surface  $\vec{r} = \vec{f}(u, v)$ . Let  $\vec{F}(\vec{r})$  be a vector point function define over  $V$ .

Divide  $V$  into  $m$  sub-regions of volumes  $\delta V_1, \delta V_2, \dots, \delta V_p, \dots, \delta V_m$

Let  $\vec{r}_i$  be a point in  $\delta V$ . Then form the sum  $I_m = \sum_{i=1}^m \vec{F}(\vec{r}_i) \delta V$ . Let  $m \rightarrow \infty$  in such a way that  $\delta V$

shrinks to a point,. The limit of  $I_m$  if it exists, is called the volume integral of  $\vec{F}(\vec{r})$  in the region V is denoted by  $\int_V \vec{F}(\vec{r}) dV$  or  $\int_V \vec{F} dV$ .

**Cartesian form** : Let  $\vec{F}(\vec{r}) = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$  where  $F_1, F_2, F_3$  are functions of  $x, y, z$ . We know that  $dV = dx dy dz$ . The volume integral given by

$$\int_V \vec{F} dV = \int \int \int (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) dx dy dz = \vec{i} \int \int \int F_1 dx dy dz + \vec{j} \int \int \int F_2 dx dy dz + \vec{k} \int \int \int F_3 dx dy dz$$

**Example 1 :** If  $\vec{F} = 2xz\vec{i} - x\vec{j} + y^2\vec{k}$  evaluate  $\int_V \vec{F} dv$  where  $V$  is the region bounded by the

surfaces  $x=0, x=2, y=0, y=6, z=x^2, z=4$ .

**Solution :** Given  $\vec{F} = 2xz\vec{i} - x\vec{j} + y^2\vec{k}$ .  $\therefore$  The volume integral is

$$\begin{aligned}\int_V \vec{F} dv &= \iiint_V (2xz\vec{i} - x\vec{j} + y^2\vec{k}) dx dy dz \\&= \vec{i} \int_{x=0}^2 \int_{y=0}^6 \int_{z=x^2}^4 2xz dx dy dz - \vec{j} \int_{x=0}^2 \int_{y=0}^6 \int_{z=x^2}^4 x dx dy dz + \vec{k} \int_{x=0}^2 \int_{y=0}^6 \int_{z=x^2}^4 y^2 dx dy dz \\&= \vec{i} \int_{x=0}^2 \int_{y=0}^6 [xz^2]_{x^2}^4 dx dy - \vec{j} \int_{x=0}^2 \int_{y=0}^6 (xz)_{x^2}^4 dx dy + \vec{k} \int_{x=0}^2 \int_{y=0}^6 y^2 (z)_{x^2}^4 dx dy \\&= \vec{i} \int_{x=0}^2 \int_{y=0}^6 x(16 - x^4) dx dy - \vec{j} \int_{x=0}^2 \int_{y=0}^6 x(4 - x^2) dx dy - \vec{k} \int_{x=0}^2 \int_{y=0}^6 y^2 (x^2 - 4) dx dy \\&= \vec{i} \int_{x=0}^2 (16x - x^5)(y)_0^6 dx - \vec{j} \int_{x=0}^2 (4x - x^3)(y)_0^6 dx - \vec{k} \int_{x=0}^2 (x^2 - 4) \left( \frac{y^3}{3} \right)_0^6 dx \\&= \vec{i} \left( 8x^2 - \frac{x^6}{6} \right)_0^2 (6) - \vec{j} \left( 2x^2 - \frac{x^4}{4} \right)_0^2 (6) - \vec{k} \left( 4x - \frac{x^3}{3} \right)_0^2 \left( \frac{216}{3} \right) \\&= 128\vec{i} - 24\vec{j} - 384\vec{k}\end{aligned}$$

## **Vector Integral Theorems**

### **Introduction**

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

- (i)  $\int_S \vec{F} \cdot \vec{n} ds$  into a volume integral where  $S$  is a closed surface.
- (ii)  $\int_C \vec{F} \cdot d\vec{r}$  into a double integral over a region in a plane when  $C$  is a closed curve in the plane and.
- (iii)  $\int_S (\nabla \times \vec{A}) \cdot \vec{n} ds$  into a line integral around the boundary of an open two sided surface.



## I. GAUSS'S DIVERGENCE THEOREM

### (Transformation between surface integral and volume integral)

Let  $S$  be a closed surface enclosing a volume  $V$ . If  $\vec{F}$  is a continuously differentiable vector point function, then

$$\int_V \text{div } \vec{F} dv = \int_S \vec{F} \cdot \vec{n} dS$$

When  $\vec{n}$  is the outward drawn normal vector at any point of  $S$ .

### SOLVED PROBLEMS

1) Verify Gauss Divergence theorem for  $\vec{F} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + z\vec{k}$  taken over the surface of the cube bounded by the planes  $x = y = z = a$  and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int_S \vec{F} \cdot \vec{n} dS = \int_V \text{div } \vec{F} dv$$

$$RHS = \int_0^a \int_0^a \int_0^a (3x^2 - 2x^2 + 1) dx dy dz = \int_0^a \int_0^a \int_0^a (x^2 + 1) dx dy dz = \int_0^a \int_0^a \left( \frac{x^3}{3} + x \right)_0^a dy dz$$

$$\int_0^a \int_0^a \left[ \frac{a^3}{3} + a \right] dy dz = \int_0^a \left[ \frac{a^3}{3} + a \right] (y)_0^a dz = \left( \frac{a^3}{3} + a \right) a \int_0^a dz = \left( \frac{a^3}{3} + a \right) (a^2) = \frac{a^5}{3} + a^3 \dots (1)$$

Verification: We will calculate the value of  $\int_S \vec{F} \cdot \vec{n} dS$  over the six faces of the cube.

(i) For  $S_1 = PQAS$ ; unit outward drawn normal  $\vec{n} = \vec{i}$

$x=a$ ;  $ds=dy dz$ ;  $0 \leq y \leq a$ ,  $0 \leq z \leq a$

$$\therefore \vec{F} \cdot \vec{n} = x^3 - yz = a^3 - yz \text{ since } x = a$$

$$\therefore \int_{S_1} \vec{F} \cdot \vec{n} dS = \int_{z=0}^a \int_{y=0}^a (a^3 - yz) dy dz$$

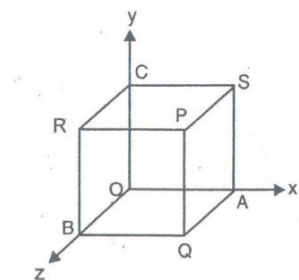
$$= \int_{z=0}^a \left[ a^3 y - \frac{y^2}{2} z \right]_{y=0}^a dz$$

$$= \int_{z=0}^a \left( a^4 - \frac{a^2}{2} z \right) dz$$

$$= a^5 - \frac{a^4}{4} \dots (2)$$

(ii) For  $S_2 = OCRB$ ; unit outward drawn normal  $\vec{n} = -\vec{i}$

$x=0$ ;  $ds=dy dz$ ;  $0 \leq y \leq a$ ,  $0 \leq z \leq a$



$$\vec{F} \cdot \vec{n} = -(x^3 - yz) = yz \text{ since } x = 0$$

$$\begin{aligned} \iint_{S_3} \vec{F} \cdot \vec{n} dS &= \int_{z=0}^a \int_{y=0}^a yz \, dy \, dz = \int_{z=0}^a \left[ \frac{y^2}{2} \right]_{y=0}^a dz \\ &= \frac{a^2}{2} \int_{z=0}^a z \, dz = \frac{a^4}{4} \dots (3) \end{aligned}$$

- (iii) For  $S_3 = \text{RBQP}$ ;  $z = a$ ;  $ds = dx dy$ ;  $\vec{n} = \vec{k}$

$$0 \leq x \leq a, 0 \leq y \leq a$$

$$\vec{F} \cdot \vec{n} = z = a \text{ since } z = a$$

$$\therefore \iint_{S_3} \vec{F} \cdot \vec{n} dS = \int_{y=0}^a \int_{x=0}^a a \, dx \, dy = a^3 \dots (4)$$

- (iv) For  $S_4 = \text{OASC}$ ;  $z = 0$ ;  $\vec{n} = -\vec{k}$ ,  $ds = dx dy$ ;

$$0 \leq x \leq a, 0 \leq y \leq a$$

$$\vec{F} \cdot \vec{n} = -z = 0 \text{ since } z = 0$$

$$\iint_{S_4} \vec{F} \cdot \vec{n} dS = 0 \dots (5)$$

- (v) For  $S_5 = \text{PSCR}$ ;  $y = a$ ;  $\vec{n} = \vec{j}$ ,  $ds = dz dx$ ;

$$0 \leq x \leq a, 0 \leq z \leq a$$

$$\vec{F} \cdot \vec{n} = -2x^2 y = -2ax^2 \text{ since } y = a$$

$$\iint_{S_5} \vec{F} \cdot \vec{n} dS = \int_{x=0}^a \int_{z=0}^a (-2ax^2) \, dz \, dx$$

$$\int_{x=0}^a (-2ax^2 z) \Big|_{z=0}^a \, dx$$

$$= -2a^2 \left( \frac{x^3}{3} \right) \Big|_0^a = \frac{-2a^5}{3} \dots (6)$$

- (vi) For  $S_6 = \text{OBQA}$ ;  $y = 0$ ;  $\vec{n} = -\vec{j}$ ,  $ds = dz dx$ ;

$$0 \leq x \leq a, 0 \leq z \leq a$$

$$\vec{F} \cdot \vec{n} = 2x^2 y = 0 \text{ since } y = 0$$

$$\iint_{S_6} \vec{F} \cdot \vec{n} dS = 0 \iint_S \vec{F} \cdot \vec{n} dS = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$\frac{a^5}{3} + a^3 \int_V \int_V \int_V \bar{r} \cdot \bar{r} \, dV = \frac{4\pi a^5}{3} \quad (2)$$

Hence Gauss Divergence theorem is verified

2. Compute  $\int (ax^2 + by^2 + cz^2) dS$  over the surface of the sphere  $x^2 + y^2 + z^2 = 1$

Sol: By divergence theorem  $\int_S \bar{F} \cdot \bar{n} dS = \int_V \nabla \cdot \bar{F} \, dV$

Given  $\bar{F} \cdot \bar{n} = ax^2 + by^2 + cz^2$ . Let  $\phi = x^2 + y^2 + z^2 - 1$

$\therefore$  Normal vector  $\bar{n}$  to the surface  $\phi$  is

$$\nabla \phi = \left( \bar{i} \frac{\partial}{\partial x} + \bar{j} \frac{\partial}{\partial y} + \bar{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - 1) = 2(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\therefore \text{Unit normal vector} = \bar{n} = \frac{2(x\bar{i} + y\bar{j} + z\bar{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\bar{i} + y\bar{j} + z\bar{k} \quad \text{Since } x^2 + y^2 + z^2 = 1$$

$$\therefore \bar{F} \cdot \bar{n} = \bar{F} \cdot (x\bar{i} + y\bar{j} + z\bar{k}) = (ax^2 + by^2 + cz^2) = (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k})$$

$$\text{i.e., } \bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k} \quad \nabla \cdot \bar{F} = a + b + c$$

Hence by Gauss Divergence theorem,

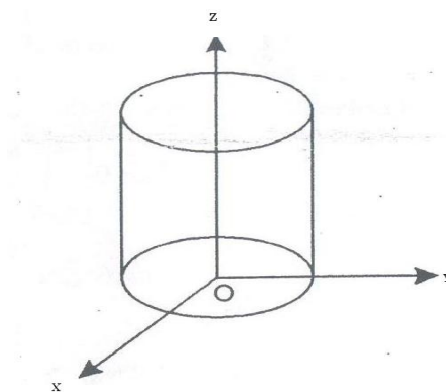
$$\int_S (ax^2 + by^2 + cz^2) dS = \int_V (a + b + c) dV = (a + b + c)V = \frac{4\pi}{3}(a + b + c)$$

$\left[ \text{Since } V = \frac{4\pi}{3} \text{ is the volume of the sphere of unit radius} \right]$

By transforming into triple integral, evaluate  $\int \int x^3 \, dy \, dz + x^2 y \, dz \, dx + x^2 \, dx \, dy$  where S is the closed surface consisting of the cylinder  $x^2 + y^2 = a^2$  and the circular discs  $z = 0, z = b$ .

Sol: Here  $F_1 = x^3, F_2 = x^2 y, F_3 = x^2 z$  and  $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$

$$\frac{\partial F_1}{\partial x} = 3x^2, \frac{\partial F_2}{\partial y} = x^2, \frac{\partial F_3}{\partial z} = x^2$$



$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$

By Gauss Divergence theorem,

$$\iiint F_1 dydz + F_2 dzdx + F_3 dxdy = \iiint \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$$\therefore \iiint_s (x^3 dydz + x^2 ydzdx + x^2 z dxdy) = \iiint 5x^2 dxdydz$$

$$= 5 \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^b x^2 dxdydz$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_{z=0}^b x^2 dxdydz \text{ [Integrand is even function]}$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2(z)_0^b dxdy = 20b \int_{x=0}^a \int_0^{\sqrt{a^2-x^2}} x^2 dxdy$$

$$= 20b \int_{x=0}^a x^2(y)_0^{\sqrt{a^2-x^2}} dx = 20b \int_0^a x^2 \sqrt{a^2-x^2} dx$$

$$= 20b \int_0^{\frac{\pi}{2}} a^2 \sin^2 \theta \sqrt{a^2 - a^2 \sin^2 \theta} (a \cos \theta d\theta)$$

$$[\text{Put } x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta \text{ when } x = a \Rightarrow \theta = \frac{\pi}{2} \text{ and } x = 0 \Rightarrow \theta = 0]$$

$$= 20a^4 b \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^2 \theta d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} (2 \sin \theta \cos \theta)^2 d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2} d\theta$$

$$= \frac{5a^4 b}{2} \left[ \theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{5a^4 b}{2} \left[ \frac{\pi}{2} \right] = \frac{5}{4} \pi a^4 b$$

**3:** Applying Gauss divergence theorem, Prove that  $\int \vec{r} \cdot \vec{n} dS = 3V$  or  $\int \vec{r} \cdot d\vec{s} = 3V$

Sol: Let  $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$  we know that  $\text{div } \vec{r} = 3$

By Gauss divergence theorem,  $\int_V \vec{F} \cdot \vec{n} dS = \int_V \text{div } \vec{F} dv$

$$\text{Take } \vec{F} = \vec{r} \Rightarrow \int_S \vec{r} \cdot \vec{n} dS = \int_V 3 dv = 3V. \text{ Hence the result}$$

**4:** Show that  $\int_S (ax\vec{i} + by\vec{j} + cz\vec{k}) \cdot \vec{n} dS = \frac{4\pi}{3}(a+b+c)$ , where S is the surface of the sphere  $x^2+y^2+z^2=1$ .

Sol: Take  $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = a + b + c$$

By Gauss divergence theorem,  $\int_S \vec{F} \cdot \vec{n} dS = \int_V \vec{\nabla} \cdot \vec{F} dV = (a + b + c) \int_V dV = (a + b + c)V$

We have  $V = \frac{4}{3}\pi r^3$  for the sphere. Here  $r = 1$

$$\therefore \int_S \vec{F} \cdot \vec{n} dS = (a + b + c) \frac{4\pi}{3}$$

5: Using Divergence theorem, evaluate

$$\int \int_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy), \text{ where } S: x^2 + y^2 + z^2 = a^2$$

Sol: We have by Gauss divergence theorem,  $\int_S \vec{F} \cdot \vec{n} dS = \int_V \text{div} \vec{F} dV$

L.H.S can be written as  $\int (F_1 \, dy \, dz + F_2 \, dz \, dx + F_3 \, dx \, dy)$  in Cartesian form

Comparing with the given expression, we have  $F_1 = x, F_2 = y, F_3 = z$

$$\text{Then } \text{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3$$

$$\therefore \int_V \text{div} \vec{F} dV = \int_V 3 dV = 3V$$

Here V is the volume of the sphere with radius a.

$$\therefore V = \frac{4}{3}\pi a^3$$

$$\text{Hence } \int \int (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy) = 4\pi a^3$$

1: Apply divergence theorem to evaluate  $\int \int_S (x + z) \, dy \, dz + (y + z) \, dz \, dx + (x + y) \, dx \, dy$  S is the surface of

the sphere  $x^2 + y^2 + z^2 = 4$

Sol: Given  $\int \int_S (x + z) \, dy \, dz + (y + z) \, dz \, dx + (x + y) \, dx \, dy$

Here  $F_1 = x + z, F_2 = y + z, F_3 = x + y$

$$\frac{\partial F_1}{\partial x} = 1, \frac{\partial F_2}{\partial y} = 1, \frac{\partial F_3}{\partial z} = 0 \text{ and } \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 1 + 0 = 2$$

By Gauss Divergence theorem,

$$\int \int_S F_1 \, dy \, dz + F_2 \, dz \, dx + F_3 \, dx \, dy = \int \int \int_V \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \, dy \, dz$$

$$= \int \int \int_V 2 \, dx \, dy \, dz = 2 \int_V dV = 2V$$

$$= 2 \left[ \frac{4}{3}\pi (2)^3 \right] = \frac{64\pi}{3} \text{ [for the sphere, radius = 2]}$$

2: Evaluate  $\int_S \vec{F} \cdot \vec{n} ds$ , if  $\vec{F} = xy\vec{i} + z^2\vec{j} + 2yz\vec{k}$  over the tetrahedron bounded by  $x=0, y=0, z=0$  and the plane  $x+y+z=1$ .

Sol: Given  $\vec{F} = xy\vec{i} + z^2\vec{j} + 2yz\vec{k}$ , then  $\text{div. } \vec{F} = y+2y = 3y$

$$\begin{aligned}\therefore \int_S \vec{F} \cdot \vec{n} dS &= \int_V \text{div} \vec{F} dv = \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 3y dx dy dz \\&= 3 \int_{x=0}^1 \int_{y=0}^{1-x} y[z]_0^{1-x-y} dx dy = 3 \int_{x=0}^1 \int_{y=0}^{1-x} y(1-x-y) dx dy \\&= 3 \int_{x=0}^1 \left[ \frac{y^2}{2} - \frac{xy^2}{2} - \frac{y^3}{3} \right]_0^{1-x} dx = 3 \int_0^1 \left[ \frac{(1-x)^2}{2} - \frac{x(1-x)^2}{2} - \frac{(1-x)^3}{3} \right] dx \\&= 3 \int_0^1 \left[ \frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \right] dx = 3 \int_0^1 \frac{(1-x)^3}{6} dx = \frac{3}{6} \left[ \frac{-(1-x)^4}{4} \right]_0^1 = \frac{1}{8}\end{aligned}$$

3: Use divergence theorem to evaluate  $\int_S \vec{F} \cdot d\vec{S}$  where  $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$  and  $S$  is the surface of the sphere

$$x^2 + y^2 + z^2 = r^2$$

Sol: We have

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2 + y^2 + z^2)$$

∴ By divergence theorem,

$$\vec{\nabla} \cdot \vec{F} dV = \int_V \int_V \int_V \vec{\nabla} \cdot \vec{F} dV = \iiint_V 3(x^2 + y^2 + z^2) dx dy dz$$

$$= 3 \int_{r=0}^a \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 (r^2 \sin \theta dr d\theta d\phi)$$

[Changing into spherical polar coordinates  $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$ ]

$$\int_S \vec{F} \cdot d\vec{S} = 3 \int_{r=0}^a \int_{\theta=0}^{\pi} r^4 \sin \theta \left[ \int_{\phi=0}^{2\pi} d\phi \right] dr d\theta$$

$$= 3 \int_{r=0}^a \int_{\theta=0}^{\pi} r^4 \sin \theta (2\pi - 0) dr d\theta = 6\pi \int_{r=0}^a r^4 \left[ \int_0^{\pi} \sin \theta d\theta \right] dr$$

$$= 6\pi \int_{r=0}^a r^4 (-\cos \theta)_0^{\pi} dr = -6\pi \int_0^a r^4 (\cos \pi - \cos 0) dr$$

$$= 12\pi \int_0^a r^4 dr = 12\pi \left[ \frac{r^5}{5} \right]_0^a = \frac{12\pi a^5}{5}$$

4: Use divergence theorem to evaluate  $\int_S \vec{F} \cdot d\vec{S}$  where  $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$  and  $S$  is the surface bounded by the region  $x^2 + y^2 = 4, z=0$  and  $z=3$ .

Sol: We have

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

By divergence theorem,

$$\begin{aligned} \int \int_S \vec{F} \cdot d\vec{S} &= \int \int \int_V \vec{F} \cdot \nabla dV \\ &= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^3 (4 - 4y + 2z) dx dy dz \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(4 - 4y)z + z^2]_0^3 dx dy \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [12(1 - y) + 9] dx dy \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dx dy \\ &= \int_{-2}^2 \left[ \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21 dy - 12 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y dy \right] dx \\ &= \int_{-2}^2 \left[ 21 \times 2 \int_0^{\sqrt{4-x^2}} dy - 12(0) \right] dx \end{aligned}$$

[Since the integrands in first integral is even and in 2<sup>nd</sup> integral it is an odd function]

$$\begin{aligned} &= 42 \int_{-2}^2 (y)_0^{\sqrt{4-x^2}} dx \\ &= 42 \int_{-2}^2 \sqrt{4-x^2} dx = 42 \times 2 \int_0^2 \sqrt{4-x^2} dx \\ &= 84 \left[ \frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2 \\ &= 84 \left[ 0 + 2 \cdot \frac{\pi}{2} - 0 \right] = 84\pi \end{aligned}$$

5: Verify divergence theorem for  $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$  over the surface S of the solid cut off by the plane  $x+y+z=a$  in the first octant.

**Sol:** By Gauss theorem,  $\int_S \vec{F} \cdot \vec{n} dS = \int_V \text{div} \vec{F} dv$

Let  $\phi = x + y + z - a$  be the given plane then

$$\frac{\partial \phi}{\partial x} = 1, \frac{\partial \phi}{\partial y} = 1, \frac{\partial \phi}{\partial z} = 1$$

$$\therefore \text{grad} \phi = \sum \vec{i} \frac{\partial \phi}{\partial x} = \vec{i} + \vec{j} + \vec{k}$$

$$\text{Unit normal} = \frac{\text{grad} \phi}{|\text{grad} \phi|} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

Let R be the projection of S on xy-plane

Then the equation of the given plane will be  $x+y=a \Rightarrow y=a-x$

Also when  $y=0$ ,  $x=a$

$$\begin{aligned} \therefore \int_S \vec{F} \cdot \vec{n} dS &= \int_R \frac{\vec{F} \cdot \vec{n} dx dy}{|\vec{n} \cdot \vec{k}|} \\ &= \int_{x=0}^a \int_{y=0}^{a-x} \frac{x^2 + y^2 + z^2}{\frac{1}{\sqrt{3}}} dx dy = \int_0^a \int_{y=0}^{a-x} [x^2 + y^2 + (a-x-y)^2] dx dy \quad [\text{since } x+y+z=a] \\ &= \int_0^a \int_0^{a-x} [2x^2 + 2y^2 - 2ax + 2xy - 2ay + a^2] dx dy \\ &= \int_{x=0}^a \left[ 2x^2 y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2 y \right]_0^{a-x} dx \\ &= \int_{x=0}^a [2x^2(a-x) + \frac{2}{3}(a-x)^3 + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x)] dx \end{aligned}$$

$$\therefore \int_S \vec{F} \cdot \vec{n} dS = \int_0^a \left( -\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3 \right) dx = \frac{a^4}{4}, \text{ on simplification...}(1)$$

Given  $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$$\therefore \text{div} \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

$$\text{Now } \iiint \text{div} \vec{F} \cdot dv = 2 \int_{x=0}^a \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} (x+y+z) dx dy dz$$



$$\begin{aligned}
&= 2 \int_{x=0}^a \int_{y=0}^{a-x} \left[ z(x+y) + \frac{z^2}{2} \right]_0^{a-x-y} dx dy \\
&= 2 \int_{x=0}^a \int_{y=0}^{a-x} (a-x-y) \left[ x+y + \frac{a-x-y}{2} \right] dx dy \\
&= \int_{x=0}^a \int_{y=0}^{a-x} (a-x-y)[a+x+y] dx dy \\
&= \int_0^a \int_0^{a-x} [a^2 - (x+y)^2] dy dx = \int_0^a \int_0^{a-x} (a^2 - x^2 - y^2 - 2xy) dx dy \\
&= \int_0^a \left[ a^2 y - x^2 y - \frac{y^3}{3} - xy^2 \right]_0^{a-x} dx \\
&= \int_0^a (a-x)(2a^2 - x^2 - ax) dx = \frac{a^4}{4} \dots \dots (2)
\end{aligned}$$

Hence from (1) and (2), the Gauss Divergence theorem is verified.

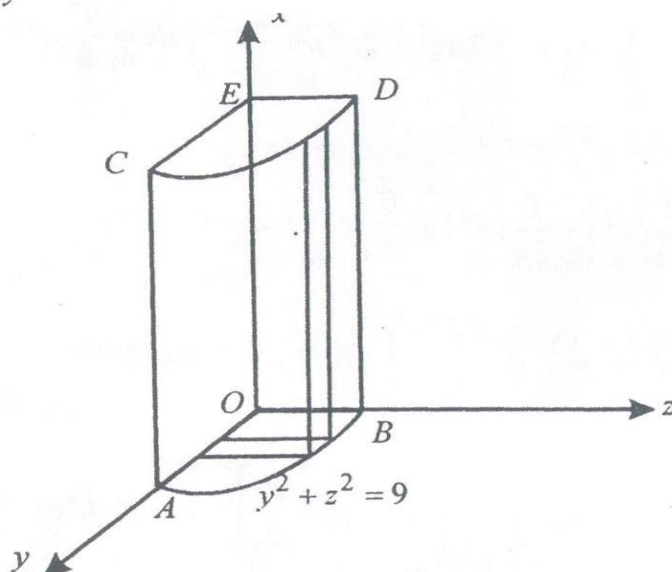
6: Verify divergence theorem for  $2x^2\bar{y}\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$  taken over the region of first octant of the cylinder  $y^2+z^2=9$  and  $x=2$ .

(or) Evaluate  $\int_S \bar{F} \cdot \bar{n} dS$ , where  $\bar{F} = 2x^2\bar{y}\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$  and  $S$  is the closed surface of the region in the first

octant bounded by the cylinder  $y^2+z^2=9$  and the planes  $x=0, x=2, y=0, z=0$

Sol: Let  $\bar{F} = 2x^2\bar{y}\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$

$$\therefore \nabla \cdot \bar{F} = \frac{\partial}{\partial x}(2x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2) = 4xy - 2y + 8xz$$



$$\begin{aligned}
\iiint_V \vec{V} \cdot \vec{F} dv &= \int_{x=0}^2 \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz) dz dy dx \\
&= \int_0^2 \int_0^3 \left[ (4xy - 2y)z + 8x \frac{z^2}{2} \right]_{z=0}^{\sqrt{9-y^2}} dy dx \\
&= \int_0^2 \int_0^3 \left[ (4xy - 2y)\sqrt{9-y^2} + 4x(9-y^2) \right] dy dx \\
&= \int_0^2 \int_0^3 [(1-2x)(-2y)\sqrt{9-y^2} + 4x(9-y^2)] dy dx \\
&= \int_0^2 \left\{ \left[ (1-2x) \frac{(9-y^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^3 + 4x \left( 9y - \frac{y^3}{3} \right)_0^3 \right\} dx \\
&= \int_0^2 \left\{ \frac{2}{3} (1-2x)[0-27] + 4x[27-9] \right\} dx = \int_0^2 [-18(1-2x) + 72x] dx \\
&= \left[ -18(x-x^2) + 72 \frac{x^2}{2} \right]_0^2 = -18(2-4) + 36(4) = 36 + 144 = 180 \dots (1)
\end{aligned}$$

Now we shall calculate  $\int_S \vec{F} \cdot \vec{n} ds$  for all the five faces.

$$\int_S \vec{F} \cdot \vec{n} dS = \int_{S_1} \vec{F} \cdot \vec{n} dS + \int_{S_2} \vec{F} \cdot \vec{n} dS + \dots + \int_{S_5} \vec{F} \cdot \vec{n} dS$$

Where  $S_1$  is the face OAB,  $S_2$  is the face CED,  $S_3$  is the face OBDE,  $S_4$  is the face OACE and  $S_5$  is the curved surface ABDC.

(i) On  $S_1 : x=0, \vec{n} = -i \therefore \vec{F} \cdot \vec{n} = 0$  Hence  $\int_{S_1} \vec{F} \cdot \vec{n} dS$

(ii) On  $S_2 : x=2, \vec{n} = i \therefore \vec{F} \cdot \vec{n} = 8y$

$$\therefore \int_{S_2} \vec{F} \cdot \vec{n} dS = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y dy dz = \int_0^3 8 \left( \frac{y^2}{2} \right)_0^{\sqrt{9-z^2}} dz$$

$$= 4 \int_0^3 (9 - z^2) dz = 4 \left( 9z - \frac{z^3}{3} \right)_0^3 = 4(27 - 9) = 72$$

(iii) On  $S_3: y=0, \bar{n} = -\bar{j}$ .  $\therefore \bar{F} \cdot \bar{n} = 0$  Hence  $\int_{S_3} \bar{F} \cdot \bar{n} ds$

(iv) On  $S_4: z=0, \bar{n} = -\bar{k}$ .  $\bar{F} \cdot \bar{n} = 0$ . Hence  $\int_{S_4} \bar{F} \cdot \bar{n} ds = 0$

$$(v) \text{ On } S_5: y^2 + z^2 = 9, \bar{n} = \frac{\nabla(y^2 + z^2)}{|\nabla(y^2 + z^2)|} = \frac{2y\bar{j} + 2z\bar{k}}{\sqrt{4y^2 + 4z^2}} = \frac{y\bar{j} + z\bar{k}}{\sqrt{4 \times 9}} = \frac{y\bar{j} + z\bar{k}}{3}$$

$$\bar{F} \cdot \bar{n} = \frac{-y^3 + 4xz^3}{3} \text{ and } \bar{n} \cdot \bar{k} = \frac{z}{3} = \frac{1}{3} \sqrt{9 - y^2}$$

Hence  $\int_{S_5} \bar{F} \cdot \bar{n} ds = \int \int_R \bar{F} \cdot \bar{n} \frac{dx dy}{|\bar{n} \cdot \bar{k}|}$  Where  $R$  is the projection of  $S_5$  on  $xy$ -plane.

$$\begin{aligned} &= \int \int_R \frac{4xz^3 - y^3}{\sqrt{9 - y^2}} dx dy = \int_{x=0}^2 \int_{y=0}^3 [4x(9 - y^2) - y^3(9 - y^2)^{-\frac{1}{2}}] dy dx \\ &= \int_0^2 72x dx - 18 \int_0^2 dx = 72 \left( \frac{x^2}{2} \right)_0^2 - 18(x)_0^2 = 144 - 36 = 108 \end{aligned}$$

$$\text{Thus } \int_S \bar{F} \cdot \bar{n} ds = 0 + 72 + 0 + 0 + 108 = 180 \dots \dots (2)$$

Hence the Divergence theorem is verified from the equality of (1) and (2).

**7:** Use Divergence theorem to evaluate  $\int \int (x\bar{i} + y\bar{j} + z^2\bar{k}) \cdot \bar{n} ds$ . Where  $S$  is the surface bounded by the cone  $x^2 + y^2 = z^2$  in the plane  $z = 4$ .

Sol: Given  $\int \int (x\bar{i} + y\bar{j} + z^2\bar{k}) \cdot \bar{n} ds$  Where  $S$  is the surface bounded by the cone  $x^2 + y^2 = z^2$  in the plane  $z = 4$ .

$$\text{Let } \bar{F} = x\bar{i} + y\bar{j} + z^2\bar{k}$$

By Gauss Divergence theorem, we have

$$\int \int (x\bar{i} + y\bar{j} + z^2\bar{k}) \cdot \bar{n} ds = \int \int \int_V \nabla \cdot \bar{F} dv$$

$$\text{Now } \nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2) = 1 + 1 + 2z = 2(1 + z)$$

On the cone,  $x^2 + y^2 = z^2$  and  $z=4 \Rightarrow x^2 + y^2 = 16$

The limits are  $z = 0$  to  $4$ ,  $y = 0$  to  $\sqrt{16 - x^2}$ ,  $x = 0$  to  $4$ .

$$\begin{aligned}\iiint_V \vec{V} \cdot \vec{F} \, dv &= \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^4 2(1+z) \, dx \, dy \, dz \\ &= 2 \int_0^4 \int_0^{\sqrt{16-x^2}} \left\{ [z]_0^4 + \left[ \frac{z^2}{2} \right]_0^4 \right\} \, dx \, dy \\ &= 24 \int_0^4 \sqrt{16-x^2} \, dx = 24 \int_0^{\frac{\pi}{2}} \sqrt{16-16\sin^2 \theta} \cdot 4 \cos \theta \, d\theta\end{aligned}$$

[put  $x = 4 \sin \theta \Rightarrow dx = 4 \cos \theta \, d\theta$ . Also  $x=0 \Rightarrow \theta=0$  and  $x=4 \Rightarrow \theta = \frac{\pi}{2}$ ]

$$\therefore \iiint_V \nabla \cdot \vec{F} \, dv = 96 \times 4 \int_0^{\frac{\pi}{2}} 4 \sqrt{1-\sin^2 \theta} \cos \theta \, d\theta = 96 \times 4 \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$$

$$\begin{aligned}\iiint_V \vec{V} \cdot \vec{F} \, dv &= 96 \times 4 \int_0^{\frac{\pi}{2}} 4 \sqrt{1-\sin^2 \theta} \cos \theta \, d\theta = 96 \times 4 \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \\ &= 96 \times 4 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} \, d\theta = 96 \times 4 \int_0^{\frac{\pi}{2}} \left[ \frac{1}{2} + \frac{\cos 2\theta}{2} \right] \, d\theta \\ &= 384 \left[ \frac{1}{2} \theta + \frac{1}{2} \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} = 96\pi\end{aligned}$$

**8: Use Gauss Divergence theorem to evaluate  $\int \int_S (yz^2 \vec{i} + zx^2 \vec{j} + 2z^2 \vec{k}) \cdot d\vec{s}$ , where  $S$  is the closed surface bounded by the  $xy$ -plane and the upper half of the sphere  $x^2+y^2+z^2=a^2$  above this plane.**

Sol: Divergence theorem states that

$$\int \int_S \vec{F} \cdot d\vec{s} = \int \int \int_V \vec{V} \cdot \vec{F} \, dv$$

$$\text{Here } \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2) = 4z$$

$$\therefore \int \int_S \vec{F} \cdot d\vec{s} = \int \int \int_V 4z \, dx \, dy \, dz$$

Introducing spherical polar coordinates  $x = r \sin \theta \cos \phi$ ,  $y = r \sin \theta \sin \phi$ ,

$$z = r \cos \theta \text{ then } dx dy dz = r^2 dr d\theta d\phi$$

$$\therefore \int \int \int_s \bar{F} \cdot ds = 4 \int_{r=0}^a \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (r \cos \theta) (r^2 \sin \theta dr d\theta d\phi)$$

$$= 4 \int_{r=0}^a \int_{\theta=0}^{\pi} r^3 \sin \theta \cos \theta \left[ \int_{\phi=0}^{2\pi} d\phi \right] dr d\theta$$

$$= 4 \int_{r=0}^a \int_{\theta=0}^{\pi} r^3 \sin \theta \cos \theta (2\pi - 0) dr d\theta$$

$$= 4\pi \int_{r=0}^a r^3 \left[ \int_0^{\pi} \sin 2\theta d\theta \right] dr = 4\pi \int_{r=0}^a r^3 \left( -\frac{\cos 2\theta}{2} \right)_0^{\pi} dr$$

$$= (-2\pi) \int_0^a r^3 (1 - 1) dr = 0$$

9: Verify Gauss divergence theorem for  $\bar{F} = x^3 \bar{i} + y^3 \bar{j} + z^3 \bar{k}$  taken over the cube bounded by  $x = 0, x = a, y = 0, y = a, z = 0, z = a$ .

Sol: We have  $\bar{F} = x^3 \bar{i} + y^3 \bar{j} + z^3 \bar{k}$

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

$$\int \int \int_V \bar{V} \cdot \bar{F} dv = \int \int \int_V (3x^2 + 3y^2 + 3z^2) dx dy dz$$

$$= 3 \int_{z=0}^a \int_{y=0}^a \int_{x=0}^a (x^2 + y^2 + z^2) dx dy dz$$

$$= 3 \int_{z=0}^a \int_{y=0}^a \left( \frac{x^3}{3} + xy^2 + z^2 x \right)_0^a dy dz$$

$$= 3 \int_{z=0}^a \int_{y=0}^a \left( \frac{a^3}{3} + ay^2 + az^2 \right) dy dz$$

$$= 3 \int_{z=0}^a \left( \frac{a^3}{3} y + a \frac{y^3}{3} + az^2 y \right)_0^a dz$$

$$= 3 \int_0^a \left( \frac{a^4}{3} + \frac{a^4}{3} + a^2 z^2 \right) dz = 3 \int_0^a \left( \frac{2}{3} a^4 + a^2 z^2 \right) dz$$

$$= 3 \left( \frac{2}{3} a^4 z + a^2 \cdot \frac{z^3}{3} \right)_0^a = 3 \left( \frac{2}{3} a^5 + \frac{1}{3} a^5 \right)$$

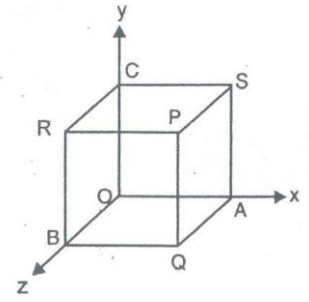
$$= 3a^5$$

To evaluate the surface integral divide the closed surface  $S$  of the cube into 6 parts.

i.e.,  $S_1$  : The face DEFA ;  $S_4$  : The face OBDC

$S_2$  : The face AGCO ;  $S_5$  : The face GCDE

$S_3$  : The face AGEF ;  $S_6$  : The face AFBO



$$\int_S \vec{F} \cdot \vec{n} ds = \int_{S_1} \vec{F} \cdot \vec{n} ds + \int_{S_2} \vec{F} \cdot \vec{n} ds + \dots + \int_{S_6} \vec{F} \cdot \vec{n} ds$$

On  $S_1$ , we have  $\vec{n} = \vec{i}, x = a$

$$\therefore \int_{S_1} \vec{F} \cdot \vec{n} ds = \int_{z=0}^a \int_{y=0}^a (a^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}) \cdot \vec{i} dy dz$$

$$\int_{S_1} \vec{F} \cdot \vec{n} ds = \int_{z=0}^a \int_{y=0}^a (a^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}) \cdot \vec{i} dy dz$$

$$= \int_{z=0}^a \int_{y=0}^a a^3 dy dz = a^3 \int_0^a (y)_0^a dz$$

$$= a^4 (z)_0^a = a^5$$

On  $S_2$ , we have  $\vec{n} = -\vec{i}, x = 0$

$$\int_{S_2} \vec{F} \cdot \vec{n} ds = \int_{z=0}^a \int_{y=0}^a (y^3 \vec{j} + z^3 \vec{k}) \cdot (-\vec{i}) dy dz = 0$$

On  $S_3$ , we have  $\vec{n} = \vec{j}, y = a$

$$\int_{S_3} \vec{F} \cdot \vec{n} ds = \int_{z=0}^a \int_{x=0}^a (x^3 \vec{i} + a^3 \vec{j} + z^3 \vec{k}) \cdot \vec{j} dx dz = a^3 \int_{z=0}^a \int_{x=0}^a dx dz = a^3 \int_0^a adz = a^4 (z)_0^a$$

$$= a^5$$

On  $S_4$ , we have  $\vec{n} = -\vec{j}, y = 0$

$$\int_{S_4} \vec{F} \cdot \vec{n} ds = \int_{z=0}^a \int_{x=0}^a (x^3 \vec{i} + z^3 \vec{k}) \cdot (-\vec{j}) dx dz = 0$$

On  $S_5$ , we have  $\vec{n} = \vec{k}, z = a$

$$\int_{S_5} \vec{F} \cdot \vec{n} ds = \int_{y=0}^a \int_{x=0}^a (x^3 \vec{i} + y^3 \vec{j} + a^3 \vec{k}) \cdot \vec{k} dx dy$$

$$= \int_{y=0}^a \int_{x=0}^a a^3 dx dy = a^3 \int_0^a (x)_0^a dy = a^4 (y)_0^a = a^5$$

On  $S_6$ , we have  $\vec{n} = -\vec{k}, z = 0$

$$\int_{S_6} \vec{F} \cdot \vec{n} ds = \int_{y=0}^a \int_{x=0}^a (x^3 \vec{i} + y^3 \vec{j}) \cdot (-\vec{k}) dx dy = 0$$

$$\text{Thus } \int_S \vec{F} \cdot \vec{n} ds = a^5 + 0 + a^5 + 0 + a^5 + 0 = 3a^5$$

$$\text{Hence } \int_S \vec{F} \cdot \vec{n} ds = \int_V \vec{\nabla} \cdot \vec{F} dv$$

$\therefore$  The Gauss divergence theorem is verified.

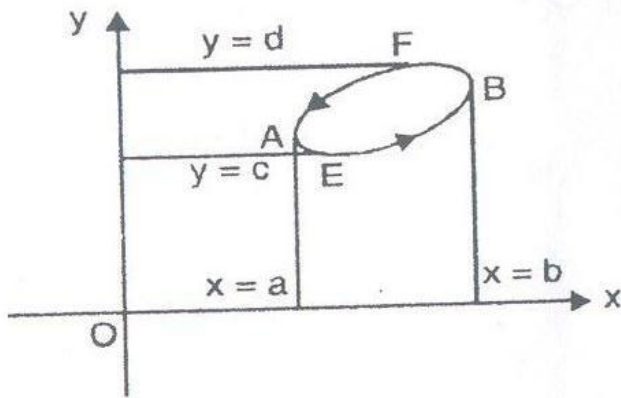
## II. GREEN'S THEOREM IN A PLANE

**(Transformation Between Line Integral and Surface Integral ) [JNTU 2001S].**

If  $S$  is Closed region in  $xy$  plane bounded by a simple closed curve  $C$  and if  $M$  and  $N$  are continuous functions of  $x$  and  $y$  having continuous derivatives in  $R$ , then

$$\oint_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

Where  $C$  is traversed in the positive(anti clock-wise) direction

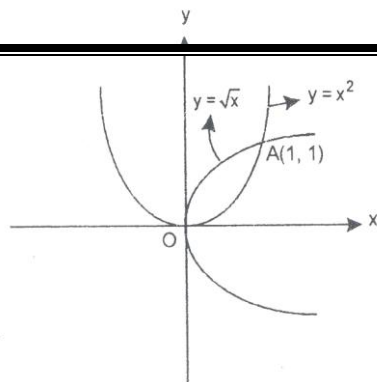


### SOLVED PROBLEMS

1. Verify Green's theorem in plane for  $\oint (3x^2 - 8y^2) dx + (4y - 6xy) dy$  where  $C$  is the region bounded by  $y = \sqrt{x}$  and  $y = x^2$ .

**Solution:** Let  $M = 3x^2 - 8y^2$  and  $N = 4y - 6xy$ . Then

$$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$$



We have by Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

$$\text{Now } \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_R (16y - 6y) dxdy$$

$$\begin{aligned} &= 10 \iint_R y dxdy = 10 \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} y dy dx = 10 \int_{x=0}^1 \left( \frac{y^2}{2} \right)_{x^2}^{\sqrt{x}} dx \\ &= 5 \int_0^1 (x - x^4) dx = 5 \left( \frac{x^2}{2} - \frac{x^5}{5} \right)_0^1 = 5 \left( \frac{1}{2} - \frac{1}{5} \right) = \frac{3}{2} \quad \dots(1) \end{aligned}$$

Verification:

We can write the line integral along c

= [line integral along  $y = x^2$  (from O to A)] + [line integral along  $y^2 = x$  (from A to O)]

=  $I_1 + I_2$  (say)

$$\begin{aligned} \text{Now } I_1 &= \int_{x=0}^1 \{ [3x^2 - 8(x^2)^2] dx + [4x^2 - 6x(x^2)] 2x dx \} \left[ \because y = x^2 \Rightarrow \frac{dy}{dx} = 2x \right] \\ &= \int_0^1 (3x^3 + 8x^3 - 20x^4) dx = -1 \end{aligned}$$

$$\text{And } I_2 = \int_1^0 \left[ (3x^2 - 8x) dx + \left( 4\sqrt{x} - 6x^{3/2} \right) \frac{1}{2\sqrt{x}} dx \right] = \int_1^0 (3x^2 - 11x + 2) dx = \frac{5}{2}$$

$$\therefore I_1 + I_2 = -1 + 5/2 = 3/2.$$

$$\text{From (1) and (2), we have } \oint_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

hence the verification of the Green's theorem.

2. Evaluate by Green's theorem  $\int_C (y - \sin x) dx + \cos x dy$  where C is the triangle enclosed by the lines  $y=0$ ,  $x=\frac{\pi}{2}$ ,  $\pi y = 2x$ .

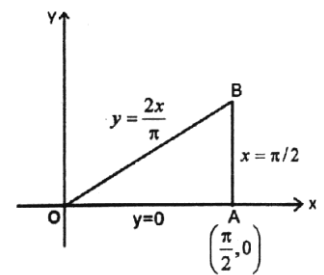
**Solution:** Let  $M = y - \sin x$  and  $N = \cos x$  Then



$$\frac{\partial M}{\partial y}=1 \text{ and } \frac{\partial N}{\partial x}=-\sin x$$

$$\therefore \text{By Green's theorem } \oint_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

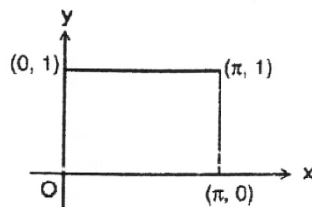
$$\begin{aligned} \Rightarrow \int_c (y - \sin x) dx + \cos x dy &= \iint_R (-1 - \sin x) dx dy \\ &= - \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (1 + \sin x) dx dy \\ &= - \int_{x=0}^{\pi/2} (\sin x + 1) [y]_0^{2x/\pi} dx \\ &= \frac{-2}{\pi} \int_{x=0}^{\pi/2} x(\sin x + 1) dx \end{aligned}$$



$$\begin{aligned} &= \frac{-2}{\pi} \left[ x(-\cos x + x) \right]_0^{\pi/2} - \int_0^{\pi/2} 1(-\cos x + x) dx \\ &= \frac{-2}{\pi} \left[ x(-\cos x + x) + \sin x - \frac{x^2}{2} \right]_0^{\pi/2} \\ &= \frac{-2}{\pi} \left[ -x \cos x + \frac{x^2}{2} + \sin x \right]_0^{\pi/2} = \frac{-2}{\pi} \left[ \frac{\pi^2}{8} + 1 \right] = - \left( \frac{\pi}{4} + \frac{2}{\pi} \right) \end{aligned}$$

3. Evaluate by Green's theorem for  $\oint_C (x^2 - \cosh y) dx + (y + \sin x) dy$  where C is the rectangle with vertices  $(0,0), (\pi, 0), (\pi, 1), (0,1)$ .

**Solution:** Let  $M = x^2 - \cosh y, N = y + \sin x$



$$\therefore \frac{\partial M}{\partial y} = -\sinh y \text{ and } \frac{\partial N}{\partial x} = \cos x$$

$$\text{By Green's theorem, } \oint_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

$$\begin{aligned} \Rightarrow \oint_C (x^2 - \cosh y) dx + (y + \sin x) dy &= \iint_R (\cos x + \sinh y) dx dy \\ &= \oint_C (x^2 - \cosh y) dx + (y + \sin x) dy = \int_0^{\pi} \int_0^1 (\cos x + \sinh y) dx dy \\ &= \int_{x=0}^{\pi} \int_{y=0}^1 (\cos x + \sinh y) dy dx = \int_{x=0}^{\pi} (y \cos x + \cosh y) \Big|_0^1 dx \\ &= \int_{x=0}^{\pi} (\cos x + \cosh 1 - 1) dx \\ &= \pi(\cosh 1 - 1) \end{aligned}$$

4. A Vector field is given by  $\vec{F} = (\sin y)\vec{i} + x(1 + \cos y)\vec{j}$

Evaluate the line integral over the circular path  $x^2 + y^2 = a^2$ ,  $z=0$

(i) Directly (ii) By using Green's theorem

**Solution:** (i) Using the line integral

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_1 dx + F_2 dy = \oint_C \sin y dx + x(1 + \cos y) dy$$

$$= \oint_C \sin y dx + x \cos y dy + x dy = \oint_C d(x \sin y) + x dy$$

Given Circle is  $x^2 + y^2 = a^2$ . Take  $x = a \cos \theta$  and  $y = a \sin \theta$  so that  $dx = -a \sin \theta d\theta$  and  $dy = a \cos \theta d\theta$  and  $\theta = 0 \rightarrow 2\pi$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} d[a \cos \theta \sin(a \sin \theta)] + \int_0^{2\pi} a(\cos \theta) a \cos \theta d\theta$$

$$= [a \cos \theta \sin(a \sin \theta)]_0^{2\pi} + 4a^2 \int_0^{2\pi} \cos^2 \theta d\theta$$

$$= 0 + 4a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi a^2$$

(ii) Using Green's theorem

Let  $M = \sin y$  and  $N = x(1 + \cos y)$ . Then

$$\frac{\partial M}{\partial y} = \cos y \text{ and } \frac{\partial N}{\partial x} = (1 + \cos y)$$

By Green's theorem,

$$\oint_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\therefore \oint_C \sin y dx + x(1 + \cos y) dy = \iint_R (-\cos y + 1 + \cos y) dx dy = \iint_R dx dy$$

$$= \iint_R dA = A = \pi a^2 (\because \text{area of circle} = \pi a^2)$$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

5. Show that area bounded by a simple closed curve C is given by  $\frac{1}{2} \oint_C x dy - y dx$  and hence find the area of

(i) The ellipse  $x = a \cos \theta$ ,  $y = b \sin \theta$  (i.e)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii) The Circle  $x = a \cos \theta$ ,  $y = a \sin \theta$  (i.e)  $x^2 + y^2 = a^2$

**Solution:** We have by Green's theorem  $\oint_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

Here  $M = -y$  and  $N = x$  so that  $\frac{\partial M}{\partial y} = -1$  and  $\frac{\partial N}{\partial x} = 1$

$$\oint_C x dy - y dx = 2 \iint_R dx dy = 2A \text{ where } A \text{ is the area of the surface.}$$

$$\therefore \frac{1}{2} \oint_C x dy - y dx = A$$

(i) For the ellipse  $x = a \cos \theta$  and  $y = b \sin \theta$  and  $\theta = 0 \rightarrow 2\pi$

$$\therefore \text{Area, } A = \frac{1}{2} \oint_C x dy - y dx = \frac{1}{2} \int_0^{2\pi} [(a \cos \theta)(b \cos \theta) - (b \sin \theta)(-a \sin \theta)] d\theta$$

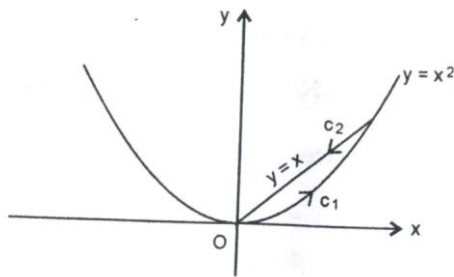
$$= \frac{1}{2} ab \int_0^{2\pi} (\cos^2 \theta + \sin^2 \theta) d\theta = \frac{1}{2} ab (\theta)_0^{2\pi} = \frac{ab}{2} (2\pi - 0) = \pi ab$$

(ii) Put  $a=b$  to get area of the circle  $A=\pi a^2$

**6:** Verify Green's theorem for  $\int_C [(xy + y^2)dx + x^2 dy]$ , where  $C$  is bounded by  $y=x$  and  $y=x^2$

**Solution:** By Green's theorem, we have  $\oint_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

Here  $M=xy + y^2$  and  $N=x^2$



The line  $y=x$  and the parabola  $y=x^2$  intersect at  $O(0,0)$  and  $A(1,1)$

$$\text{Now } \oint_C Mdx + Ndy = \int_{C_1} Mdx + Ndy + \int_{C_2} Mdx + Ndy \dots (1)$$

Along  $C_1$  (i.e.  $y = x^2$ ), the line integral is

$$\begin{aligned} \int_{C_1} Mdx + Ndy &= \int_{C_1} [x(x^2) + x^4]dx + x^2 d(x^2) = \int_0^1 (x^3 + x^4 + 2x^3)dx = \int_0^1 (3x^3 + x^4)dx \\ &= \left( 3 \cdot \frac{x^4}{4} + \frac{x^5}{5} \right)_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \end{aligned} \dots (2)$$

Along  $C_2$  (i.e.  $y = x$ ) from  $(1,1)$  to  $(0,0)$ , the line integral is

$$\begin{aligned} \int_{C_2} Mdx + Ndy &= \int_{C_2} (x \cdot x + x^2)dx + x^2 dx \quad [\because dy = dx] \\ &= \int_{C_2} 3x^2 dx = 3 \int_1^0 x^2 dx = 3 \left( \frac{x^3}{3} \right)_1^0 = (x^3)_1^0 = 0 - 1 = -1 \end{aligned} \dots (3)$$

From (1), (2) and (3), we have

$$\int_C Mdx + Ndy = \frac{19}{20} - 1 = \frac{-1}{20} \dots (4)$$

Now

$$\begin{aligned} \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \iint_R (2x - x - 2y) dx dy \\ &= \int_0^1 [(x^2 - x^2) - (x^3 - x^4)] dx = \int_0^1 (x^4 - x^3) dx \end{aligned}$$

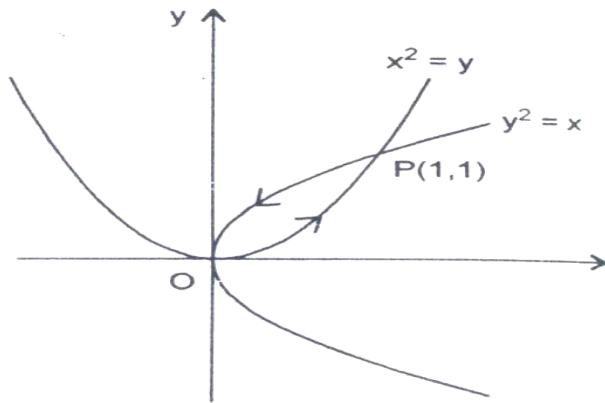
$$= \left( \frac{x^5}{5} + \frac{x^4}{4} \right)_0^1 = \frac{1}{5} + \frac{1}{4} = \frac{-1}{20} \quad \dots(5)$$

From (4) and (5), We have  $\oint_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

Hence the verification of the Green's theorem.

13. Using Green's theorem evaluate  $\int_C (2xy - x^2)dx + (x^2 + y^2)dy$ , Where "C" is the closed curve of the region bounded by  $y = x^2$  and  $y^2 = x$

**Solution:**



The two parabolas  $y^2 = x$  and  $y = x^2$  are intersecting at  $O(0,0)$ , and  $P(1,1)$

Here  $M = 2xy - x^2$  and  $N = x^2 + y^2$

$$\therefore \frac{\partial M}{\partial y} = 2x \text{ and } \frac{\partial N}{\partial x} = 2x$$

$$\text{Hence } \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2x = 0$$

$$\text{By Green's theorem } \int_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

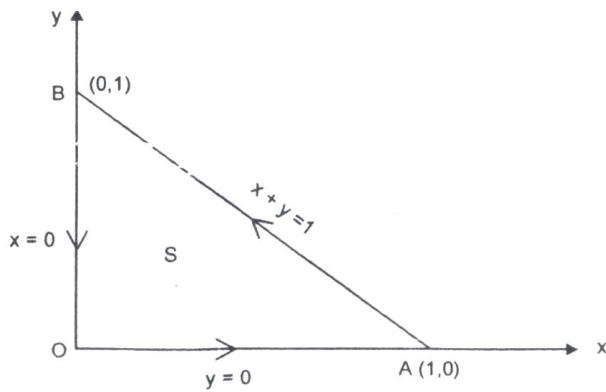
$$\text{i.e., } \int_C (2xy - x^2)dx + (x^2 + y^2)dy = \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} (0) dx dy = 0$$

8. Verify Green's theorem for  $\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$  where c is the region bounded by  $x=0$ ,  $y=0$  and  $x+y=1$ .

**Solution :** By Green's theorem, we have

$$\int_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Here  $M = 3x^2 - 8y^2$  and  $N = 4y - 6xy$



$$\therefore \frac{\partial M}{\partial y} = -16y \text{ and } \frac{\partial N}{\partial x} = -6y$$

$$\text{Now } \int_c Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy \dots (1)$$

Along OA,  $y=0 \therefore dy = 0$

$$\int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = \left( \frac{x^3}{3} \right)_0^1 = 1$$

Along AB,  $x+y=1 \therefore dy = -dx$  and  $x=1-y$  and  $y$  varies from 0 to 1.

$$\begin{aligned} \int_{AB} Mdx + Ndy &= \int_0^1 [3(y-1)^2 - 8y^2](-dy) + [4y + 6y(y-1)]dy \\ &= \int_0^1 (-5y^2 - 6y + 3)(-dy) + (6y^2 - 2y)dy \\ &= \int_0^1 (11y^2 + 4y - 3)dy = \left( 11 \frac{y^3}{3} + 4 \frac{y^2}{2} - 3y \right)_0^1 \\ &= \frac{11}{3} + 2 - 3 = \frac{8}{3} \end{aligned}$$

Along BO,  $x=0 \therefore dx = 0$  and limits of  $y$  are from 1 to 0

$$\int_{BO} Mdx + Ndy = \int_1^0 4y dy = \left( 4 \frac{y^2}{2} \right)_1^0 = (2y^2)_1^0 = -2$$

$$\text{from (1), we have } \int_c Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

$$\begin{aligned} \text{Now } \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \int_{x=0}^1 \int_{y=0}^{1-x} (-6y + 16y) dx dy \\ &= 10 \int_{x=0}^1 \left[ \int_{y=0}^{1-x} y dy \right] dx = 10 \int_0^1 \left( \frac{y^2}{2} \right)_0^{1-x} dx \\ &= 5 \int_0^1 (1-x)^2 dx = 5 \left[ \frac{(1-x)^3}{-3} \right]_0^1 \\ &= -\frac{5}{3} [(1-1)^3 - (1-0)^3] = \frac{5}{3} \end{aligned}$$

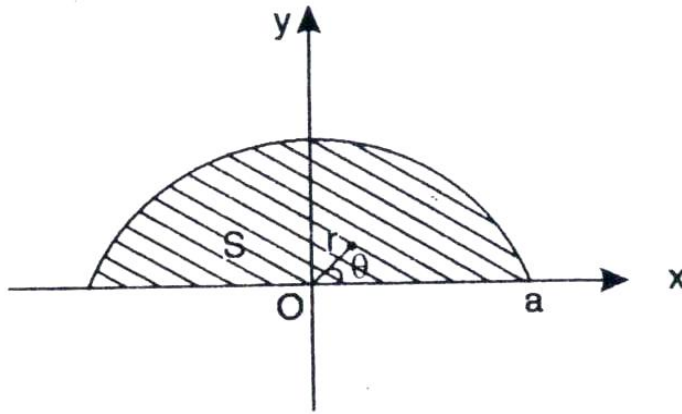
$$\text{From (2) and (3), we have } \int_c Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Hence the verification of the Green's Theorem.

9. Apply Green's theorem to evaluate  $\oint_c (2x^2 - y^2)dx + (x^2 + y^2)dy$ , where  $c$  is the boundary of the area enclosed by the x-axis and upper half of the circle  $x^2 + y^2 = a^2$

**Solution :** Let  $M=2x^2 - y^2$  and  $N=x^2 + y^2$  Then

$$\frac{\partial M}{\partial y} = -2y \text{ and } \frac{\partial N}{\partial x} = 2x$$



**Figure**

$$\therefore \text{ By Green's Theorem, } \int_c Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$\oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = \iint_R (2x + 2y) dxdy$$

$$= 2 \iint_R (x + y) dy$$

$$= 2 \int_0^a \int_0^\pi r(\cos \theta + \sin \theta) \cdot r d\theta dr$$

[Changing to polar coordinates  $(r, \theta)$ ,  $r$  varies from 0 to  $a$  and  $\theta$  varies from 0 to  $\pi$ ]

$$\therefore \oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = 2 \int_0^a r^2 dr \int_0^\pi (\cos \theta + \sin \theta) d\theta$$

$$= 2 \cdot \frac{a^3}{3} (1 + 1) = \frac{4a^3}{3}$$

10. Find the area of the Folium of Descartes  $x^3 + y^3 = 3axy$  ( $a > 0$ ) using Green's Theorem.

**Solution:** from Green's theorem, we have

$$\int Pdx + Qdy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

$$\text{By Green's theorem, Area} = \frac{1}{2} \oint (xdy - ydx)$$

Considering the loop of folium Descartes ( $a > 0$ )

$$\text{Let } x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}, \text{ Then } dx = \left[ \frac{d}{dt} \left( \frac{3at}{1+t^3} \right) \right] dt \text{ and } dy = \left[ \frac{d}{dt} \left( \frac{3at^2}{1+t^3} \right) \right] dt$$

The point of intersection of the loop is  $\left( \frac{3a}{2}, \frac{3a}{2} \right) \Rightarrow t = 1$

Along OA, t varies from 0 to 1.

$$\begin{aligned} \therefore \frac{1}{2} \oint (x dy - y dx) &= \frac{1}{2} \int_0^1 \left( \frac{3at}{1+t^3} \right) \left[ \frac{d}{dt} \left( \frac{3at^2}{1+t^3} \right) \right] dt - \left( \frac{3at^2}{1+t^3} \right) \left[ \frac{d}{dt} \left( \frac{3at}{1+t^3} \right) \right] dt \\ &= \frac{1}{2} \int_0^1 \left\{ \frac{3at}{1+t^3} \left[ \frac{3at(2-t^3)}{(1+t^3)^2} \right] - \frac{3at^2}{1+t^3} \left[ \frac{3a(1-2t^3)}{(1+t^3)^2} \right] \right\} dt \\ &= \frac{9a^2}{2} \int_0^1 \left[ \frac{t^2(2-t^3)}{(1+t^3)^3} - \frac{t^2(1-2t^3)}{(1+t^3)^3} \right] dt = \frac{9a^2}{2} \int_0^1 \frac{2t^2 - t^5 - t^2 + 2t^5}{(1+t^3)^3} dt \end{aligned}$$

$$\begin{aligned} &= \frac{9a^2}{2} \int_0^1 \frac{t^2 + t^5}{(1+t^3)^3} dt = \frac{9a^2}{2} \int_0^1 \frac{t^2(1+t^3)}{(1+t^3)^3} dt \\ &= \frac{9a^2}{2} \int_0^1 \frac{t^2}{(1+t^3)^2} dt \quad [\text{Put } 1+t^3 = x \Rightarrow 3t^2 dt = dx] \end{aligned}$$

L.L. :  $x=1$ , U.L.:  $x=2$

$$= \frac{9a^2}{2} \int_1^2 \frac{t^2}{x^2} \cdot \frac{dx}{3t^2} = \frac{9a^2}{6} \int_1^2 \frac{1}{x^2} dx = \frac{3a^2}{4} \text{ sq. units } (a > 0).$$

11: Verify Green's theorem in the plane for  $\int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy$

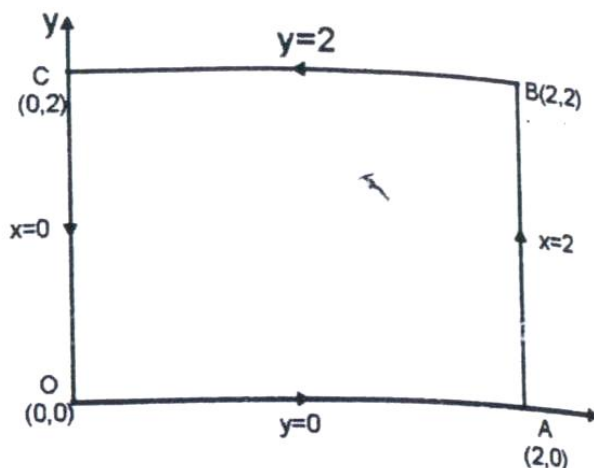
Where C is square with vertices (0,0), (2,0), (2,2), (0,2).

**Solution:** The Cartesian form of Green's theorem in the plane is

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Here  $M = x^2 - xy^3$  and  $N = y^2 - 2xy$

$$\therefore \frac{\partial M}{\partial y} = -3xy^2 \text{ and } \frac{\partial N}{\partial x} = -2y$$



To Evaluate  $\int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy$ , we shall take C in four different segments viz (i) along OA(y=0) (ii) along AB(x=2) (iii) along BC(y=2) (iv) along CO(x=0).

**(i) Along OA(y=0)**

$$\int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3} \quad \dots(1)$$

**(ii) Along AB(x=2)**

$$\begin{aligned} \int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy &= \int_0^2 (y^2 - 4y) dy \quad [\because x = 2, dx = 0] \\ &= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8 \left(-\frac{2}{3}\right) = -\frac{16}{3} \quad \dots(2) \end{aligned}$$

**(iii) Along BC(y=2)**

$$\begin{aligned} \int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy &= \int_2^0 (x^2 - 8x) dx \quad [\because y = 2, dy = 0] \\ &= \left(\frac{x^3}{3} - 4x^2\right)_2^0 = -\left(\frac{8}{3} - 16\right) = \frac{40}{3} \quad \dots(3) \end{aligned}$$

**(iv) Along CO(x=0)**

$$\int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_2^0 y^2 dx \quad [\because x = 0, dx = 0] = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3} \quad \dots(4)$$

Adding(1),(2),(3) and (4), we get

$$\int_C (x^2 - xy^3) dx + (y^2 - 2xy) dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8 \quad \dots(5)$$

**Evaluation of**  $\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

Here x ranges from 0 to 2 and y ranges from 0 to 2.

$$\begin{aligned} \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \int_0^2 \int_0^2 (-2y + 3xy^2) dx dy \\ &= \int_0^2 \left( -2xy + \frac{3x^2}{2} y^2 \right)_0^2 dy \\ &= \int_0^2 (-4y + 6y^2) dy = \left( -2y^2 + 2y^3 \right)_0^2 \\ &= -8 + 16 = 8 \quad \dots(6) \end{aligned}$$

From (5) and (6), we have

$$\int_C Mdx + Ndy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Hence the Green's theorem is verified.



### III. STOKES'S THEOREM

(Transformation between Line Integral and Surface Integral)

[JNTU 2000]

Let S be a open surface bounded by a closed, non intersecting curve C. If  $\vec{F}$  is any differentieable vector point function then  $\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \vec{n} ds$  where c is traversed in the positive direction and  $\vec{n}$  is unit outward drawn normal at any point of the surface.

Prove by Stokes theorem,  $\text{Curl grad } \phi = \vec{0}$

**Solution:** Let S be the surface enclosed by a simple closed curve C.

$\therefore$  By Stokes theorem

$$\begin{aligned} \int_S (\text{curl grad } \phi) \cdot \vec{n} ds &= \int_S (\nabla \times \nabla \phi) \cdot \vec{n} ds = \oint_C \nabla \phi \cdot d\vec{r} = \oint_C \nabla \phi \cdot d\vec{r} \\ &= \oint_C \left( \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right) \cdot (\vec{i} dx + \vec{j} dy + \vec{k} dz) \\ &= \oint_C \left( \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) = \int d\phi = [\phi]_P \text{ where P is any point on C.} \end{aligned}$$

$$\therefore \int \text{curl grad } \phi \cdot \vec{n} ds = 0 \Rightarrow \text{curl grad } \phi = \vec{0}$$

$$\text{prove that } \int_S \phi \text{curl } \vec{f} \cdot d\vec{S} = \int_C \phi \vec{f} \cdot d\vec{r} - \int_S \text{curl grad } \phi \times \vec{f} \cdot d\vec{S}$$

**Solution:** Applying Stokes theorem to the function  $\phi \vec{f}$

$$\int_C \phi \vec{f} \cdot d\vec{r} = \int_S \text{curl}(\phi \vec{f}) \cdot \vec{n} ds = \int_S (\text{grad } \phi \times \vec{f} + \phi \text{curl } \vec{f}) \cdot \vec{n} ds$$

3: Prove that  $\oint_C \phi \nabla f \cdot d\vec{r} = 0$ .

Solution: By Stokes Theorem,

$$\oint_C (f \nabla f) \cdot d\vec{r} = \int_S \text{curl} f \nabla f \cdot \vec{n} ds = \int_S [f \text{curl } \nabla f + \nabla f \times \nabla f] \cdot \vec{n} ds$$

$$= \int \vec{0} \cdot \vec{n} ds = 0 \quad [\because \text{curl } \nabla f = \vec{0} \text{ and } \nabla f \times \nabla f = \vec{0}]$$

$$\text{Prove that } \oint_C f \nabla g \cdot d\vec{r} = \int_S (\nabla f \times \nabla g) \cdot \vec{n} ds$$

**Solution:** By Stokes Theorem,

$$\oint_C (f \nabla g \cdot d\vec{r}) = \int_S [\nabla \times (f \nabla g)] \cdot \vec{n} ds = \int_S [\nabla f \times \nabla g + f \text{curl grad } g] \cdot \vec{n} ds$$

$$= \int [\nabla f \times \nabla g] \cdot \vec{n} ds \quad [\because \text{curl}(\text{grad } g) = \vec{0}]$$

1. Verify Stokes theorem for  $\vec{F} = -y^3 \vec{i} + x^3 \vec{j}$ , Where S is the circular disc

$$x^2 + y^2 \leq 1, z = 0.$$

**Solution:** Given that  $\vec{F} = -y^3\vec{i} + x^3\vec{j}$ . The boundary of C of S is a circle in xy plane.

$x^2 + y^2 \leq 1, z = 0$ . We use the parametric co-ordinates  $x = \cos\theta, y = \sin\theta, z = 0, 0 \leq \theta \leq 2\pi$ ;

$dx = -\sin\theta d\theta$  and  $dy = \cos\theta d\theta$

$$\begin{aligned}\therefore \oint_C \vec{F} \cdot d\vec{r} &= \int_C F_1 dx + F_2 dy + F_3 dz = \int_C -y^3 dx + x^3 dy \\ &= \int_0^{2\pi} [-\sin^3\theta(-\sin\theta) + \cos^3\theta\cos\theta] d\theta = \int_0^{2\pi} (\cos^4\theta + \sin^4\theta) d\theta \\ &= \int_0^{2\pi} (1 - 2\sin^2\theta \cos^2\theta) d\theta = \int_0^{2\pi} d\theta - \frac{1}{2} \int_0^{2\pi} (2\sin\theta \cos\theta)^2 d\theta\end{aligned}$$

$$\begin{aligned}&= \int_0^{2\pi} d\theta - \frac{1}{2} \int_0^{2\pi} \sin^2 2\theta d\theta = (2\pi - 0) - \frac{1}{4} \int_0^{2\pi} (1 - \cos 4\theta) d\theta \\ &= 2\pi + \left[ -\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta \right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}\end{aligned}$$

$$\text{Now } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \vec{k}(3x^2 + 3y^2)$$

$$\therefore \int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = 3 \int_S (x^2 + y^2) \vec{k} \cdot \vec{n} ds$$

We have  $(\vec{k} \cdot \vec{n}) ds = dxdy$  and R is the region on xy-plane

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot \vec{n} ds = 3 \iint_R (x^2 + y^2) dx dy$$

Put  $x = r \cos\theta, y = r \sin\theta$ .  $dxdy = r dr d\theta$

r is varying from 0 to 1 and  $0 \leq \theta \leq 2\pi$ .

$$\therefore \int (\nabla \times \vec{F}) \cdot \vec{n} ds = 3 \int_{\theta=0}^{2\pi} \int_{r=0}^1 r^2 \cdot r dr d\theta = \frac{3\pi}{2}$$

L.H.S=R.H.S.Hence the theorem is verified.

2.If  $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$ , evaluate  $\int_S (\nabla \times \vec{F}) \cdot \vec{n} ds$ . Where S is the surface of sphere

$x^2 + y^2 + z^2 = a^2$ , above the xy-plane.

**Solution:** Given  $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$ .

By Stokes Theorem,

$$\int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = \int_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz = \int_C y dx + (x - 2xz) dy - xy dz$$

Above the xy plane the sphere is  $x^2 + y^2 + z^2 = a^2, z = 0$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C y dx + x dy.$$

Put  $x = a \cos \theta, y = a \sin \theta$  so that  $dx = -a \sin \theta d\theta, dy = a \cos \theta d\theta$  and  $\theta = 0 \rightarrow 2\pi$

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (a \sin \theta)(-a \sin \theta) d\theta + (a \cos \theta)(a \cos \theta) d\theta \\ &= a^2 \int_0^{2\pi} \cos 2\theta d\theta = a^2 \left[ \frac{\sin 2\theta}{2} \right]_0^{2\pi} = \frac{a^2}{2} (0) = 0\end{aligned}$$

Verify Stokes theorem for  $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$  over the upper half surface of the sphere  $x^2 + y^2 + z^2 = 1$  bounded by the projection of the xy-plane.

**Solution:** The boundary C of S is a circle in xy plane i.e  $x^2 + y^2 = 1, z=0$

The parametric equations are  $x = \cos\theta, y = \sin\theta, \theta = 0 \rightarrow 2\pi$

$$\therefore dx = -\sin\theta d\theta, dy = \cos\theta d\theta$$

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_C \vec{F}_1 dx + \vec{F}_2 dy + \vec{F}_3 dz = \int_C (2x - y)dx - yz^2 dy - y^2 z dz \\ &= \int_C (2x - y)dx \text{ (since } z = 0 \text{ and } dz = 0) \\ &= -\int_0^{2\pi} (2\cos\theta - \sin\theta) \sin\theta d\theta = \int_0^{2\pi} \sin^2\theta d\theta - \int_0^{2\pi} \sin 2\theta d\theta \\ &= \int_{\theta=0}^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta - \int_0^{2\pi} \sin 2\theta d\theta = \left[ \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{2}\cos 2\theta \right]_0^{2\pi} \\ &= \frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}(\cos 4\pi - \cos 0) = \pi\end{aligned}$$

$$\text{Again } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = \vec{i}(-2yz + 2yz) - \vec{j}(0 - 0) + \vec{k}(0 + 1) = \vec{k}$$

$$\therefore \int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = \int_S \vec{k} \cdot \vec{n} ds = \int_R dx dy$$

Where R is the projection of S on xy plane and  $\vec{k} \cdot \vec{n} ds = dx dy$

$$\begin{aligned}\text{Now } \int_R dx dy &= 4 \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} dy dx = 4 \int_{x=0}^1 \sqrt{1-x^2} dx = 4 \left[ \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right]_0^1 \\ &= 4 \left[ \frac{1}{2} \sin^{-1} 1 \right] = 2 \frac{\pi}{2} = \pi\end{aligned}$$

$\therefore$  The Stokes theorem is verified.

3. Verify Stokes theorem for the function  $\vec{F} = x^2 \vec{i} + xy \vec{j}$  integrated round the square in the plan  $z=0$  whose sides are along the lines  $x=0, y=0, x=a, y=a$ .

**Solution:** Given  $\vec{F} = x^2 \vec{i} + xy \vec{j}$

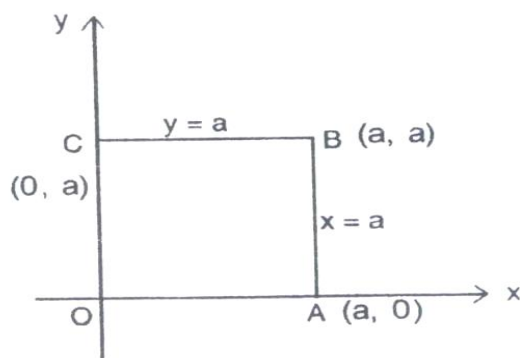


Fig. 13

By Stokes Theorem,  $\int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = \int_C \vec{F} \cdot d\vec{r}$

$$\text{Now } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = \vec{k}y$$

$$\text{L.H.S.} = \int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = \int_S y(\vec{n} \cdot \vec{k}) ds = \int_S y dx dy$$

$\therefore \vec{n} \cdot \vec{k} \cdot ds = dx dy$  and R is the region bounded for the square.

$$\therefore \int_S (\nabla \times \vec{F}) \cdot \vec{n} ds = \int_0^a \int_0^a y dy dx = \frac{a^3}{2}$$

$$\text{R.H.S.} = \int_C \vec{F} \cdot d\vec{r} = \int_C (x^2 dx + xy dy)$$

$$\text{But } \int \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

(i) Along OA:  $y=0, z=0, dy=0, dz=0$

$$\therefore \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2 dx = \frac{a^3}{3}$$

(ii) Along AB:  $x=a, z=0, dx=0, dz=0$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^a ay dy = \frac{1}{2} a^3$$

(iii) Along BC:  $y=a, z=0, dy=0, dz=0$

$$\therefore \int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 0 dx = -\frac{1}{3} a^3$$

(iv) Along CO:  $x=0, z=0, dx=0, dz=0$

$$\therefore \int_{CO} \vec{F} \cdot d\vec{r} = \int_a^0 0 dy = 0$$

$$\text{Adding } \int_C \vec{F} \cdot d\vec{r} = \frac{1}{3} a^3 + \frac{1}{2} a^3 + \frac{1}{3} a^3 + 0 = \frac{1}{2} a^3$$

Hence the verification.

4. Apply Stokes theorem, to evaluate  $\oint_C (y dx + z dy + x dz)$  where c is the curve of intersection of the

sphere  $x^2 + y^2 + z^2 = a^2$  and  $x+z=a$ .

**Solution :** The intersection of the sphere  $x^2 + y^2 + z^2 = a^2$  and the plane  $x+z=a$  is a circle in the plane  $x+z=a$  with AB as diameter.

$$\text{Equation of the plane is } x+z=a \Rightarrow \frac{x}{a} + \frac{z}{a} = 1$$

$$\therefore OA = OB = a \text{ i.e., } A = (a, 0, 0) \text{ and } B = (0, 0, a)$$

$$\therefore \text{Length of the diameter AB} = \sqrt{a^2 + a^2 + 0} = a\sqrt{2}$$

$$\text{Radius of the circle, } r = \frac{a}{\sqrt{2}}$$

$$\text{Let } \vec{F} \cdot d\vec{r} = y dx + z dy + x dz \Rightarrow \vec{F} \cdot d\vec{r} = \vec{F} \cdot (\vec{i} dx + \vec{j} dy + \vec{k} dz) = y dx + z dy + x dz$$

$$\Rightarrow \vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$$

$$\therefore \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\vec{i} + \vec{j} + \vec{k})$$

Let  $\vec{n}$  be the unit normal to this surface.  $\vec{n} = \frac{\nabla S}{|\nabla S|}$

Then  $s = x + z = a$ ,  $\nabla S = \vec{i} + \vec{k} \therefore \vec{n} = \frac{\nabla S}{|\nabla S|} = \frac{\vec{i} + \vec{k}}{\sqrt{2}}$

Hence  $\oint_C \vec{F} \cdot d\vec{r} = \int \text{curl } \vec{F} \cdot \vec{n} ds$  (by Stokes Theorem)

$$\begin{aligned} &= -\int (\vec{i} + \vec{j} + \vec{k}) \cdot \left(\frac{\vec{i} + \vec{k}}{\sqrt{2}}\right) ds = -\int \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) ds \\ &= -\sqrt{2} \int_S ds = -\sqrt{2} S = -\sqrt{2} \left(\frac{\pi a^2}{2}\right) = \frac{\pi a^2}{\sqrt{2}} \end{aligned}$$

5. Apply the Stoke's theorem and show that  $\int_S \int \text{curl } \vec{F} \cdot \vec{n} d\vec{s} = 0$  where  $\vec{F}$  is any vector and  $S = x^2 + y^2 + z^2 = 1$

Solution: Cut the surface if the Sphere  $x^2 + y^2 + z^2 = 1$  by any plane, Let  $S_1$  and  $S_2$  denotes its upper and lower portions a C, be the common curve bounding both these portions.

$$\therefore \int_S \text{curl } \vec{F} \cdot d\vec{s} = \int_{S_1} \vec{F} \cdot d\vec{s} + \int_{S_2} \vec{F} \cdot d\vec{s}$$

Applying Stoke's theorem,

$$\int_S \text{curl } \vec{F} \cdot d\vec{s} = \int_{S_1} \vec{F} \cdot d\vec{R} + \int_{S_2} \vec{F} \cdot d\vec{R} = 0$$

The 2<sup>nd</sup> integral  $\text{curl } \vec{F} \cdot d\vec{s}$  is negative because it is traversed in opposite direction to first integral.

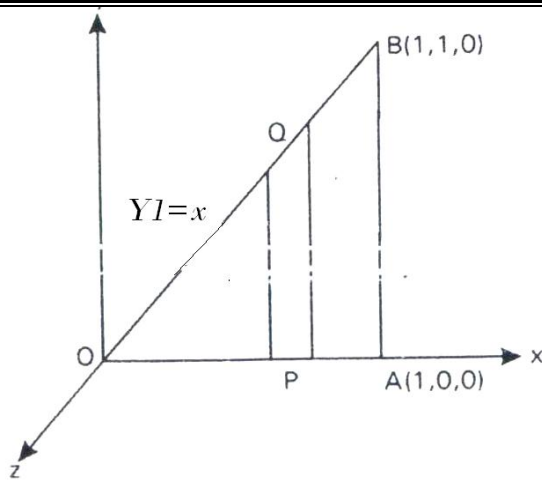
The above result is true for any closed surface S.

6. Evaluate by Stokes theorem  $\oint_C (x+y)dx + (2x-z)dy + (y+z)dz$  where C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

**Solution:** Let  $\vec{F} \cdot d\vec{r} = \vec{F} \cdot (\vec{i}dx + \vec{j}dy + \vec{k}dz) = (x+y)dx + (2x-z)dy + (y+z)dz$

Then  $\vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

By Stokes theorem,  $\oint_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl } \vec{F} \cdot \vec{n} ds$



Where S is the surface of the triangle OAB which lies in the xy plane. Since the z Co-ordinates of O,A and B Are zero. Therefore  $\vec{n} = \vec{k}$ . Equation of OA is y=0 and that of OB, y=x in the xy plane.

$$\therefore \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\vec{i} + \vec{k}$$

$$\therefore \text{curl } \vec{F} \cdot \vec{n} ds = \text{curl } \vec{F} \cdot \vec{k} dx dy = dx dy$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \iint_S dx dy = \iint_S dA = A = \text{area of the } \Delta OAB$$

$$= \frac{1}{2} \text{OA} \times \text{AB} = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

**7:** Use Stoke's theorem to evaluate  $\iint_S \text{curl } \vec{F} \cdot \vec{n} dS$  over the surface of the paraboloid

$$z + x^2 + y^2 = 1, z \geq 0 \text{ where } \vec{F} = y\vec{i} + z\vec{j} + x\vec{k}.$$

**Solution :** By Stoke's theorem

$$\begin{aligned} \int_S \text{curl } \vec{F} \cdot d\vec{s} &= \oint_C \vec{F} \cdot d\vec{r} = \int_C (y\vec{i} + z\vec{j} + x\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}) \\ &= \int_C y dx \quad (\text{Since } z=0, dz=0) \dots\dots(1) \end{aligned}$$

Where C is the circle  $x^2 + y^2 = 1$

The parametric equations of the circle are  $x = \cos\theta, y = \sin\theta$

$$\therefore dx = -\sin\theta d\theta$$

Hence (1) becomes

$$\int_S \text{curl } \vec{F} \cdot d\vec{s} = \int_{\theta=0}^{2\pi} \sin\theta (-\sin\theta) d\theta = - \int_{\theta=0}^{2\pi} \sin^2 \theta d\theta = -4 \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = -4 \times \frac{1}{2} \times \frac{\pi}{2} = -\pi$$

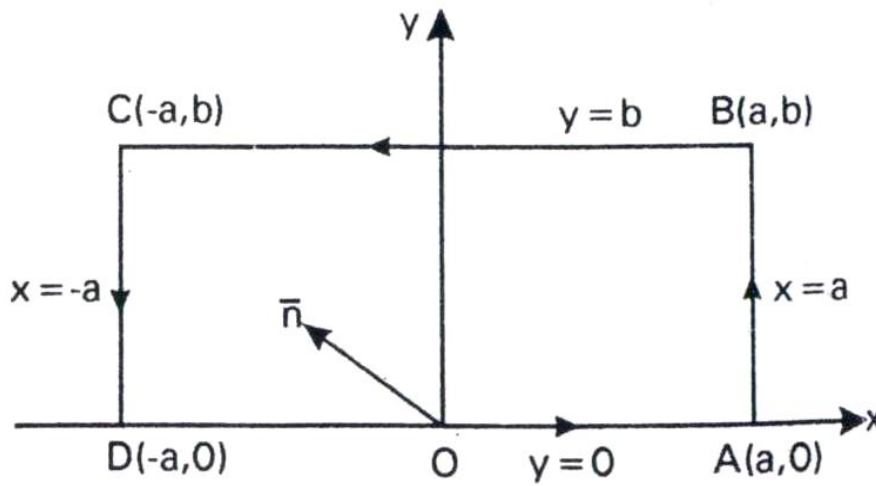
**8:** Verify Stoke's theorem for  $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$  taken round the rectangle bounded by the lines  $x = \pm a, y = 0, y = b$ .

**Solution:** Let ABCD be the rectangle whose vertices are  $(a,0), (a,b), (-a,b)$  and  $(-a,0)$ .

Equations of AB, BC, CD and DA are  $x=a, y=b, x=-a$  and  $y=0$ .

We have to prove that  $\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \vec{n} ds$

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \oint_C \{(x^2 + y^2)\vec{i} - 2xy\vec{j}\} \cdot \{\vec{i}dx + \vec{j}dy\} \\ &= \oint_C (x^2 + y^2) dx - 2xydy \\ &= \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} \dots (1)\end{aligned}$$



(i) Along AB,  $x=a, dx=0$

$$\text{from (1), } \int_{AB} = \int_{y=0}^b -2ay dy = -2a \left[ \frac{y^2}{2} \right]_0^b = -ab^2$$

(ii) Along BC,  $y=b, dy=0$

$$\text{from (1), } \int_{BC} = \int_{x=a}^{-a} (x^2 + b^2) dx = \left[ \frac{x^3}{3} + b^2x \right]_{x=a}^{-a} = \frac{-2a^3}{3} - 2ab^2$$

(iii) Along CD,  $x=-a, dx=0$

$$\text{from (1), } \int_{CD} = \int_{y=b}^0 2ay dy = 2a \left[ \frac{y^2}{2} \right]_{y=b}^0 = -ab^2$$

(iv) Along DA,  $y=0, dy=0$

$$\text{from (1), } \int_{DA} = \int_{x=-a}^a x^2 dx = \left[ \frac{x^3}{3} \right]_{x=-a}^a = \frac{2a^3}{3}$$

(i)+(ii)+(iii)+(iv) gives

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \dots (2)$$

Consider  $\int_S \text{curl } \vec{F} \cdot \vec{n} dS$

Vector Perpendicular to the xy-plane is  $\vec{n} = \vec{k}$

$$\therefore \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + y^2) & -2xy & 0 \end{vmatrix} = 4y\vec{k}$$

Since the rectangle lies in the xy plane,

$$\vec{n} = \vec{k} \text{ and } ds = dx dy$$

$$\begin{aligned} \int_S \text{curl } \vec{F} \cdot \vec{n} dS &= \int_S -4y\vec{k} \cdot \vec{k} dx dy = \int_{x=-a}^a \int_{y=0}^b -4y dx dy \\ &= \int_{y=0}^b \int_{x=-a}^a -4y dx dy = 4 \int_{y=0}^b y \left[ x \right]_{-a}^a dy = -4 \int_{y=0}^b 2ay dy \\ &= -4a \left[ y^2 \right]_{y=0}^b = -4ab^2 \end{aligned} \quad \dots(3)$$

Hence from (2) and (3), the Stoke's theorem is verified.

**9:** Verify Stoke's theorem for  $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$  where S is the surface of the cube  $x=0, y=0, z=0, x=2, y=2, z=2$  above the xy plane.

Solution: Given  $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$  where S is the surface of the cube.

$x=0, y=0, z=0, x=2, y=2, z=2$  above the xy plane.

By Stoke's theorem, we have  $\int \text{curl } \vec{F} \cdot \vec{n} ds = \int \vec{F} \cdot d\vec{r}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix} = \vec{i}(0 + y) - \vec{j}(-z + 1) + \vec{k}(0 - 1) = y\vec{i} - (1 - z)\vec{j} - \vec{k}$$

$$\therefore \nabla \times \vec{F} \cdot \vec{n} = \nabla \times \vec{F} \cdot \vec{k} = (y\vec{i} - (1 - z)\vec{j} - \vec{k}) \cdot \vec{k} = -1$$

$$\therefore \int \nabla \times \vec{F} \cdot \vec{n} \cdot ds = \int_0^2 \int_0^2 -1 dx dy \quad (\because z = 0, dz = 0) = -4$$

.....(1)

**To find  $\int \vec{F} \cdot d\vec{r}$**

$$\begin{aligned} \int \vec{F} \cdot d\vec{r} &= \int \left( (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k} \right) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}) \\ &= \int [(y - z + 2)dx + (yz + 4)dy - (xz)dz] \end{aligned}$$

Sis the surface of the cube above the xy-plane

$$\therefore z = 0 \Rightarrow dz = 0$$

$$\therefore \int \vec{F} \cdot d\vec{r} = \int (y + 2)dx + \int 4dy$$

Along  $\overline{OA}, y = 0, z = 0, dy = 0, dz = 0, x \text{ change from } 0 \text{ to } 2.$

$$\int_0^2 2dx = 2[x]_0^2 = 4 \quad \dots\dots\dots(2)$$

Along  $\overline{BC}, y = 2, z = 0, dy = 0, dz = 0, x \text{ change from } 2 \text{ to } 0.$

$$\int_2^0 4dx = 4[x]_2^0 = -8 \quad \dots\dots\dots(3)$$

Along  $\overline{AB}, x = 2, z = 0, dx = 0, dz = 0, y \text{ change from } 0 \text{ to } 2.$



$$\int \vec{F} \cdot d\vec{r} = \int_0^2 4dy = [4y]_0^2 = 8 \quad \dots(4)$$

Along  $\overline{CO}$ ,  $x = 0, z = 0, dx = 0, dz = 0, y$  change from 2 to 0.

$$\int_2^0 4dy = -8 \quad \dots(5)$$

Above the surface When  $z=2$

$$\text{Along } \overline{O'A'}, \int_0^2 \vec{F} \cdot d\vec{r} = 0 \quad \dots(6)$$

Along  $\overline{A'B'}$ ,  $x = 2, z = 2, dx = 0, dz = 0, y$  changes from 0 to 2

$$\int_0^2 \vec{F} \cdot d\vec{r} = \int_0^2 (2y+4)dy = 2 \left[ \frac{y^2}{2} \right]_0^2 + 4[y]_0^2 = 4 + 8 = 12 \quad \dots(7)$$

Along  $\overline{B'C'}$ ,  $y = 2, z = 2, dy = 0, dz = 0, x$  changes from 2 to 0

$$\int_0^2 \vec{F} \cdot d\vec{r} = 0 \quad \dots(8)$$

Along  $\overline{C'D'}$ ,  $x = 0, z = 2, dx = 0, dz = 0, y$  changes from 2 to 0.

$$\int_2^0 (2y+4)dy = 2 \left[ \frac{y^2}{2} \right]_2^0 + 4[y]_2^0 = -12 \quad \dots(9)$$

(2)+(3)+(4)+(5)+(6)+(7)+(8)+(9) gives

$$\int_C \vec{F} \cdot d\vec{r} = 4 - 8 + 8 - 8 + 0 + 12 + 0 - 12 = -4 \quad \dots(10)$$

By Stokes theorem, We have

$$\int \vec{F} \cdot d\vec{r} = \int \text{curl } \vec{F} \cdot \vec{n} ds = -4$$

Hence Stoke's theorem is verified.

10. Verify the Stoke's theorem for  $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$  and surface is the part of the sphere  $x^2 + y^2 + z^2 = 1$  above the  $xy$  plane.

Solution: Given  $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$  over the surface  $x^2 + y^2 + z^2 = 1$  is  $xy$  plane.

We have to prove  $\int_C \vec{F} \cdot d\vec{r} = \int_S \text{Curl } \vec{F} \cdot \vec{n} ds$

$$\vec{F} \cdot d\vec{r} = (y\vec{i} + z\vec{j} + x\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k}) = ydx + zdy + xdz$$

$$\int_C (ydx + zdy + xdz) = \int ydx \quad (\text{in } xy \text{ plane } z = 0, dz = 0)$$

$$\text{Let } x = \cos\theta, y = \sin\theta \Rightarrow dx = -\sin\theta d\theta, dy = \cos\theta d\theta$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C y \cdot dx = \int_0^{2\pi} \sin\theta (-\sin\theta) d\theta \quad [\because x^2 + y^2 = 1, z = 0]$$

$$= \int_0^{2\pi} \sin\theta (-\sin\theta) d\theta = -4 \int_0^{\pi/2} \sin^2\theta d\theta$$

$$= -4 \int_0^{\pi/2} \frac{1 - \cos 2\theta}{2} d\theta = -4 \left[ \left( \frac{1}{2} \cdot \frac{\pi}{2} \right) - \frac{1}{4} (\sin \pi) \right]$$

$$= -4 \left[ \left( \frac{1}{2} \cdot \frac{\pi}{2} \right) - 0 \right] = -4 \left[ \frac{\pi}{4} \right] = -\pi \quad \dots(1)$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\vec{i} + \vec{j} + \vec{k})$$

$$\text{Unit normal vector } \bar{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = x\bar{i} + y\bar{j} + z\bar{k}$$

Substituting the spherical polar coordinates, we get

$$\bar{n} = \sin\theta \cos\phi \bar{i} + \sin\theta \sin\phi \bar{j} + \cos\theta \bar{k}$$

$$\therefore \text{Curl } \bar{F} \cdot \bar{n} = -(\sin\theta \cos\phi + \sin\theta \sin\phi + \cos\theta)$$

$$\begin{aligned} \iint \text{curl } \bar{F} \cdot \bar{n} ds &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (\sin\theta \cos\phi + \sin\theta \sin\phi + \cos\theta) \sin\theta d\theta d\phi \\ &= - \int_0^{\pi/2} [\sin\theta \sin\phi - \sin\theta \cos\phi + \phi \cos\theta]_0^{2\pi} \sin\theta d\theta \\ &= -2\pi \int_0^{\pi/2} \cos\theta \sin\theta d\theta = -\pi \int_0^{\pi/2} \sin 2\theta d\theta = (-\pi) \left[ \frac{-\cos 2\theta}{2} \right]_0^{\pi/2} \\ &= \frac{\pi}{2} (-1 - 1) = -\pi \quad \dots(2) \end{aligned}$$

From (1) and (2), we have

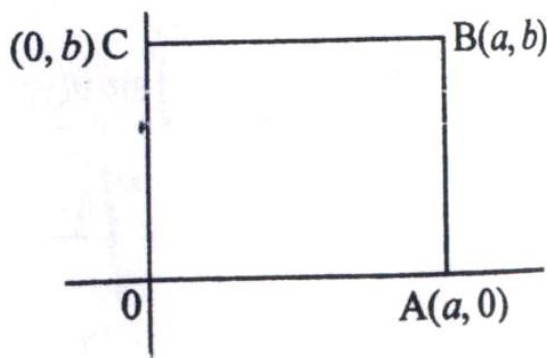
$$\int_C \bar{F} \cdot d\bar{r} = \iint_S \text{Curl } \bar{F} \cdot \bar{n} ds = -\pi$$

$\therefore$  Stoke's theorem is verified.

**11:** Verify Stoke's theorem for  $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$  over the box bounded by the planes

$x=0, x=a, y=0, y=b$ .

**Solution :**



Stokes theorem states that  $\int_C \bar{F} \cdot d\bar{r} = \iint_S \text{Curl } \bar{F} \cdot \bar{n} ds$

$$\text{Given } \bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$$

$$\text{Curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = \bar{i}(0,0) - \bar{j}(0,0) + \bar{k}(2y+2y) = 4y\bar{k}$$

$$\text{R.H.S} = \iint_S \text{Curl } \bar{F} \cdot \bar{n} ds = \iint_S 4y(\bar{k} \cdot \bar{n}) ds$$

Let R be the region bounded by the rectangle

$$(\vec{k} \cdot \vec{n}) ds = dx dy$$

$$\int_s \text{Curl } \vec{F} \cdot \vec{n} ds = \int_{x=0}^a \int_{y=0}^b 4y dx dy = \int_{x=0}^a \left[ 4 \frac{y^2}{2} \right]_0^b dx = 2b^2 \int_{x=0}^a 1 dx \\ = 2b^2(x)_0^a = 2ab^2$$

To Calculate L.H.S

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2) dx + 2xy dy$$

Let  $O=(0,0), A=(a, 0), B=(a,b)$  and

$C=(0,b)$  are the vertices of the rectangle.

(i) Along the line OA

$y=0; dy=0$ ,  $x$  ranges from 0 to  $a$ .

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_{x=0}^a x^2 dx = \left[ \frac{x^3}{3} \right]_0^a = \frac{a^3}{3}$$

(ii) Along the line AB

$x=a; dx=0$ ,  $y$  ranges from 0 to  $b$ .

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{y=0}^b (2xy) dy = \left[ 2a \frac{y^2}{2} \right]_0^b = ab^2$$

(iii) Along the line BC

$y=b; dy=0$ ,  $x$  ranges from  $a$  to 0

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_{x=a}^0 (x^2 - y^2) dx = \left[ \frac{x^3}{3} - b^2 x \right]_a^0 = 0 - \left( \frac{a^3}{3} - b^2 a \right) \\ = ab^2 - \frac{a^3}{3}$$

(iv) Along the line CO

$x=0, dx=0$ ,  $y$  changes from  $b$  to 0

$$\int_C \vec{F} \cdot d\vec{r} = \int_{y=b}^0 2xy dy = 0$$

Adding these four values

$$\int_{CO} \vec{F} \cdot d\vec{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2$$

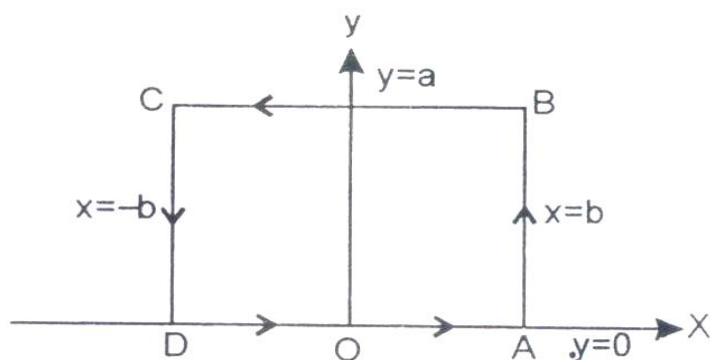
$$\text{L.H.S} = \text{R.H.S}$$

Hence the verification of the stoke's theorem.

**12:** Verify Stoke's theorem for  $\vec{F} = y^2 \vec{i} - 2xy \vec{j}$  taken round the rectangle bounded by

$x = \pm b, y=0, y=a$ .

**Solution:**



$$\text{Curl } \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix} = -4y\vec{k}$$

For the given surface S,  $\vec{n} = \vec{k}$

$$\therefore (\text{Curl } \vec{F}) \cdot \vec{n} = -4y$$

$$\text{Now } \iint_S (\text{Curl } \vec{F}) \cdot \vec{n} dS = \iint_S -4y dx dy$$

$$= \int_{y=0}^a \left[ \int_{x=-b}^b -4y dx \right] dy$$

$$= \int_0^a \left[ -4xy \right]_{-b}^b dy$$

$$= \int_0^a -8by dy = \left[ -4by^2 \right]_0^a = -4a^2b \dots\dots\dots(1)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{DA} + \int_{AB} + \int_{BC} + \int_{CD}$$

$$\int \vec{F} \cdot d\vec{r} = y^2 dx - 2xy dy$$

$$\text{Along DA, } y=0, dy=0 \Rightarrow \int_{DA} \vec{F} \cdot d\vec{r} = 0 \quad (\because \vec{F} \cdot d\vec{r} = 0)$$

Along AB,  $x=b, dx=0$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{y=0}^a -2by dy = \left[ -by^2 \right]_0^a = -a^2b$$

Along BC,  $y=a, dy=0$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_b^{-b} a^2 dx = -2a^2b$$

Along CD,  $x=-b, dx=0$

$$\int_{CD} \vec{F} \cdot d\vec{r} = \int_a^0 2by dy = \left[ -by^2 \right]_a^0 = -a^2b$$

$$\int_C \vec{F} \cdot d\vec{r} = 0 - a^2b - 2a^2b - a^2b = -4a^2b \dots\dots\dots(2)$$

From (1),(2)  $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{Curl } \vec{F}) \cdot \vec{n} dS$

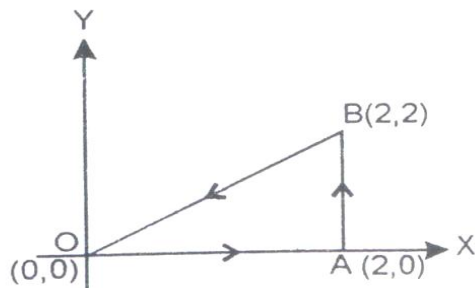
Hence the theorem is verified.

**13:** Using Stroke's theorem evaluate the integral  $\int_C \vec{F} \cdot d\vec{r}$  where

$\vec{F} = 2y^2 \vec{i} + 3x^2 \vec{j} - (2x+z) \vec{k}$  and C is the boundary of the triangle whose vertices are (0,0,0), (2,0,0), (2,2,0).

**Solution:**

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & -2x-z \end{vmatrix} = 2\vec{j} + (6x-4y) \vec{k}$$



Since the z-coordinate of each vertex of the triangle is zero, the triangle lies in the xy-plane.

$$\therefore \vec{n} = \vec{k}$$

$$\therefore (\text{Curl } \vec{F}) \cdot \vec{n} = 6x - 4y$$

Consider the triangle in xy-plane.

Equation of the straight line OB is  $y=x$ .

By Stroke's theorem

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_S (\text{curl } \vec{F}) \cdot \vec{n} ds \\ &= \int_{x=0}^2 \int_{y=0}^x (6x - 4y) dx dy = \int_{x=0}^2 \left[ \int_{y=0}^x (6x - 4y) dy \right] dx \\ &= \int_{x=0}^2 \left[ 6xy - 2y^2 \right]_0^x dx = \int_0^2 (6x^2 - 2x^2) dx \\ &= 4 \left[ \frac{x^3}{3} \right]_0^2 = \frac{32}{3} \end{aligned}$$

