INTRODUCTION

This assignment applies core graph-theoretic results to three practical planning problems: road-network design, exam scheduling, and supply-chain connectivity. I proved that the complete graph on five vertices K5 is nonplanar by contradiction using Euler's formula, analyzing a conflict graph for exam scheduling (determining its chromatic number and planarity), and answered two standard questions about spanning trees and binary trees.

QUESTION 1 — NONPLANARITY OF K5K_5 AND AN ALTERNATIVE PLANAR GRAPH

PROOF BY CONTRADICTION (USING EULER'S FORMULA).

Assume, for contradiction, that K5 (the complete graph on five vertices) is planar. Let V, E, and F denote the numbers of vertices, edges, and faces (including the outer face) of a planar embedding. Euler's formula for a connected planar graph state

$$V - E + F = 2.$$

For K5, we have V=5 and E= (5 2) =10. If K5 were planar, Euler's formula would give

$$F = 2 - V + E = 2 - 5 + 10 = 7.$$

In any simple planar embedding, every face boundary is a cycle of length at least 3, so summing face boundary lengths count each edge twice:

$$\sum faces\ (boundary\ length) = 2\ E.$$

Because each face has boundary length ≥3,

$$3F \leq \sum faces\ (boundary\ length) = 2\ E.$$

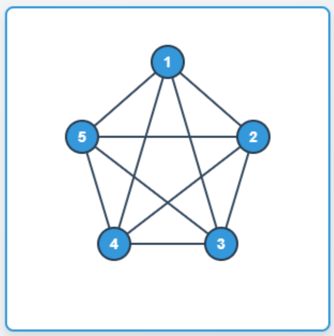
Substitute F=7 and E=10:

$$3 \cdot 7 \le 2 \cdot 10 \Longrightarrow 21 \le 20,$$

which is false. This contradiction shows the assumption that K5 is planar must be false. 2

Therefore, K5 is nonplanar.

K_5 - The Nonplanar Complete Graph: This graph has 5 vertices and 10 edges, with every vertex connected to every other vertex. As proven in the assignment, this graph cannot be drawn in a plane without edge crossings.



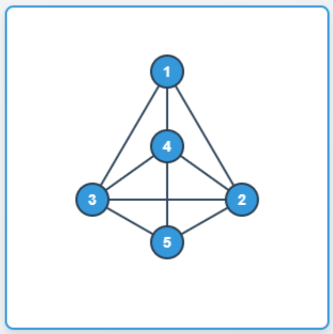
ALTERNATIVE PLANAR GRAPH THAT MAXIMIZES CONNECTIVITY WITHOUT CROSSINGS.

A planner who must avoid crossings while maximizing connectivity (i.e., the number of edges) on five vertices should choose a *maximal planar* graph on five vertices. For a simple planar graph with V ≥ 3, the edge bound is

$$E \le 3\,V - 6.$$

For V=5, this gives E ≤ 3·5 – 6 = 9. A concrete planar example with 9 edges is the graph formed by taking a planar drawing of K4 (four vertices forming a triangular embedding with one vertex inside connected to the three on the triangle) and then adding a fifth vertex inside one of the triangular faces and connecting it to the three vertices that define that face. The resulting graph is a triangulation (every face is a triangle) with 5 vertices and 9 edges; it is planar by construction and attains the upper bound E=9. This graph maximizes connectivity under the no-crossings constraint because no simple planar graph on five vertices can have more than 9 edges. 3

Maximal Planar Graph on 5 vertices: This graph achieves the maximum of 9 edges (3V-6 = 9) while remaining planar. Every face is a triangle, making it a triangulation.



QUESTION 2 — EXAM CONFLICT GRAPH: CHROMATIC NUMBER AND PLANARITY

GRAPH DESCRIPTION RECAP.

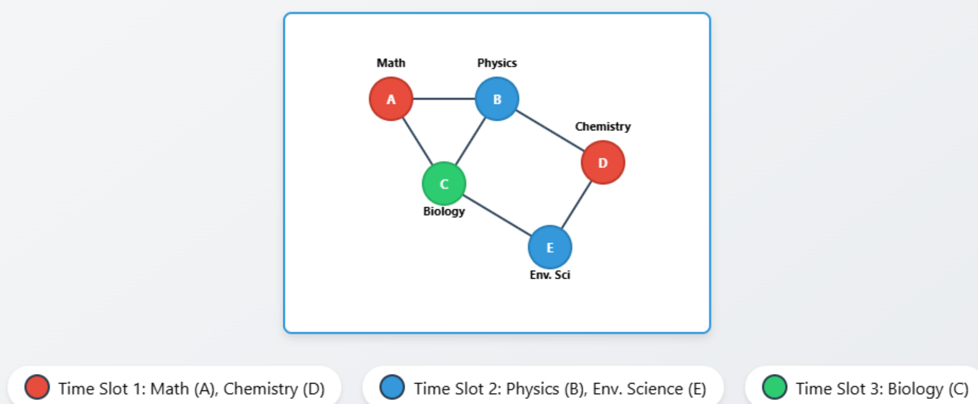
Vertices represent subjects: Math (A), Phys (B), Bio (C), Chem (D), Env (E). Edges indicate conflicts: ***AB, AC, BC, BD, CE, DE***. (This corresponds to a triangle between ***A, B, C***, an edge ***BD*** from ***B*** down to ***D***, an edge ***CE*** from ***C*** down to ***E***, and an edge ***DE*** between the two bottom vertices.)

CHROMATIC NUMBER.

A proper coloring requires that adjacent vertices have different colors. The subgraph induced by ***{A, B, C}*** is a triangle, so it needs 3 distinct colors. Assign ***A*** color 1, ***B*** color 2, and ***C*** color 3. Vertex ***D*** is adjacent to ***B*** but not to ***A*** or ***C***, so I can color ***D*** with color 1. Vertex ***E*** is adjacent to ***C*** and ***D***; since ***C*** uses color 3 and ***D*** uses color 1, I can give ***E*** color 2 (which is allowed because ***E*** is not adjacent to ***B***). Thus, three colors suffice, and because the triangle forces at least three, the chromatic number is

$$\chi(G) = 3.$$

Conflict Graph: Vertices represent subjects (Math, Physics, Biology, Chemistry, Environmental Science). Edges represent scheduling conflicts. The graph requires exactly 3 colors (time slots).



PLANARITY AND THE FOUR-COLOR THEOREM.

I can embed the given conflict graph in the plane without any edge crossings (place ***A*** at top, ***B*** and ***C*** left and right below it, and ***D*** and ***E*** below ***B*** and ***C*** respectively with straight-line edges as shown), so the graph is planar. The Four-Color Theorem states that every planar graph is vertex-colorable with at most four colors (and no planar graph ever requires more than four).

Therefore, any planar conflict graph representing exam overlaps requires at most four time slots; our conflict graph needs three, which is consistent with the theorem. The Four-Color Theorem is a fundamental result in planar graph theory (proved by Appel and Haken and subsequently simplified) and guarantees that four colors always suffice for planar scheduling-conflict graphs.

QUESTION 3 — SPANNING TREES AND A MAXIMALLY-LEAFY BINARY TREE

SPANNING TREE EDGES FOR 12 WAREHOUSES.

A spanning tree of a connected graph with n vertices always has exactly $\boldsymbol{n-1}$ edges (this follows from the definition of a tree: connected and acyclic). For 12 warehouses (vertices), any spanning tree will therefore have

$$\boldsymbol{12 - 1 = 11}$$

edges. The logistics company must therefore select 11 routes that connect all warehouses without forming cycles to obtain a minimal cost connected network.

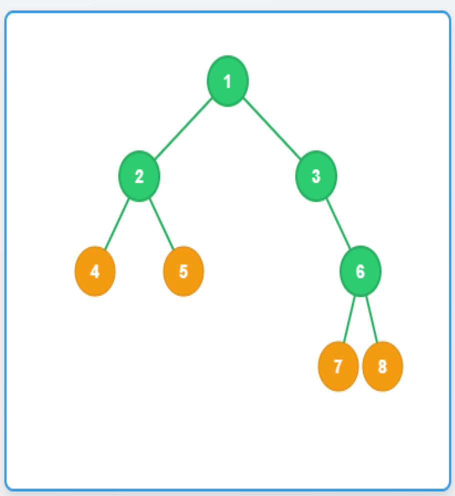
BINARY TREE WITH 8 VERTICES AND MAXIMUM LEAVES (CONSTRUCTION).

For rooted binary trees on n nodes, the maximum possible number of leaves equals $\boldsymbol{\lceil n / 2 \rceil}$. For $\boldsymbol{n=8}$, the maximum number of leaves is $\boldsymbol{\lceil 8 / 2 \rceil = 4}$. One concrete binary-tree layout with 8 labeled vertices (rooted at 1) and 4 leaves are:

```
        1
       / \
      2   3
     / \   \
    4   5   6
           / \
          7   8
```

Labels: 1(root), 2 and 3 are its children. Node 2 has two children (4 and 5), node 3 has one child 6, and node 6 has two children (7 and 8). Leaves are 4,5,7,8 (four leaves). This tree uses 8 vertices and attains the maximum leaf count for 8 nodes.

Binary Tree on 8 vertices: This tree achieves the maximum possible number of leaves (4) for a binary tree with 8 nodes. Leaf nodes are highlighted in orange.



Internal nodes (4 nodes)   Leaf nodes (4 leaves - maximum possible)

CONCLUSION

I used Euler's formula and the face-count inequality to show **K5** is nonplanar, proposed a maximal planar (triangulated) alternative with 9 edges on five vertices, determined the exam conflict graph needs three colors and is planar, invoked the Four-Color Theorem to bound colors for any planar conflict graph by four, and showed a spanning tree on 12 vertices requires 11 edges. Finally, I constructed a binary tree on 8 vertices achieving the maximal 4 leaves. These solutions use standard theorems and explicit constructions to justify each claim.

Key Theorems Illustrated:

- **Euler's Formula:** $V - E + F = 2$ for connected planar graphs
- **Planar Graph Edge Bound:** $E \leq 3V - 6$ for simple planar graphs with $V \geq 3$
- **Four-Color Theorem:** Every planar graph is 4-colorable
- **Spanning Tree Property:** n vertices require exactly n-1 edges
- **Binary Tree Leaves:** Maximum leaves = $\lceil n/2 \rceil$ for n nodes

REFERENCES

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