UNDERSTANDING RECURSION AND RECURRENCE RELATIONS IN DISCRETE MATHEMATICS

Introduction

Recursion and recurrence relations are fundamental concepts in discrete mathematics, often applied in computer science, physics, and finance to solve real-world problems involving sequences and iterative patterns. These tools help describe and solve problems where each step depends on previous outcomes, such as computing sequences or analyzing changes over time (Grimaldi, 2004). This assignment explores recursion, iteration, and characteristic root methods, and their applications in solving recurrence relations, particularly in the contexts of algorithms, temperature modeling, and investment forecasting.

Recursion and the Fibonacci Sequence

Recursion is a method of solving problems where a function calls itself with smaller inputs, ultimately reducing the problem to a base case. In the Fibonacci sequence, recursion defines each term as the sum of the two preceding terms:

$\mathbf{F(n) = F(n-1) + F(n-2)}$, with base cases $\mathbf{F\,(0) = 0}$ and $\mathbf{F\,(1) = 1}$.

This approach mirrors how problems are naturally broken down into smaller subproblems. While conceptually elegant, direct recursion can be inefficient due to repeated calculations. To address this, other methods like iteration and characteristic roots offer more efficient solutions for large values of n.

Iteration Method vs. Characteristic Root Method

The **iteration method** involves computing terms sequentially using previous values. For Fibonacci, the iterative approach avoids redundant calls by storing intermediate results:

- F (2) = F (1) + F (0) = 1 + 0 = 1
- F (3) = F (2) + F (1) = 1 + 1 = 2
- F (4) = F (3) + F (2) = 2 + 1 = 3

The **characteristic root method**, in contrast, converts the recurrence relation into a characteristic equation. For Fibonacci:

$\mathbf{F(n) = F(n-1) + F(n-2)}$

leads to the characteristic equation:

$\mathbf{r^2 - r - 1 = 0}$

Solving it yields the roots:

$\mathbf{r_1 = (1 + \sqrt{5})/2,\ r_2 = (1 - \sqrt{5})/2}$

The general solution is:

$\mathbf{F(n) = A(r_1)\ \hat{}n + B(r_2)\ \hat{}n}$,

where constants A and B are determined using initial conditions.

The characteristic root method is more efficient for closed-form solutions, particularly useful in complexity analysis of algorithms (Rosen, 2019). Therefore, it is superior for computing the Fibonacci sequence for large *n*.

Homogeneous vs. Non-Homogeneous Recurrence Relations

A **homogeneous recurrence relation** has all terms on the right-hand side expressed in terms of previous sequence values, with no external additions. A **non-homogeneous recurrence** includes an extra non-zero term independent of the sequence values.

Given:

$\mathbf{T(n) = 2T(n-1) + 3T(n-2) - 6}$

This is a **non-homogeneous recurrence relation** due to the constant **−6**.

To solve it:

1. Solve the homogeneous part:

$\mathbf{T_h(n) = 2T(n-1) + 3T(n-2)}$

Characteristic equation:

$\mathbf{r^2 - 2r - 3 = 0 \rightarrow (r - 3)(r + 1) = 0}$

So, $\mathbf{r = 3, -1}$

General solution:

$\mathbf{T_h(n) = A(3)^n + B(-1)^n}$

2. Find a particular solution (T_p):

Assume $T_p(n) = C$ (a constant)

Substitute into the original:

$C = 2C + 3C - 6 \rightarrow 0 = 4C - 6 \rightarrow C = 1.5$

3. Full solution:

$T(n) = A (3)^n + B (-1)^n + 1.5$

Using $T(0) = 20$:

$A(1) + B(1) + 1.5 = 20 \rightarrow A + B = 18.5$

$T(1) = 35$:

$A(3) + B(-1) + 1.5 = 35 \rightarrow 3A - B = 33.5$

Solving:

$A = 13, B = 5.5$

Final solution:

$T(n) = 13 (3)^n + 5.5 (-1)^n + 1.5$

Solving Investment Recurrence with Characteristic Roots

Given:

$I_n = 1.5I_{n-1} - 0.5I_{n-2}$, with **$I_0 = 10, I_1 = 5$**

1. Form the characteristic equation:

$r^2 - 1.5r + 0.5 = 0$

Solving:

$r = 1, r = 0.5$

General solution:

$\mathbf{I_n = A(1)^n + B(0.5)^n = A + B(0.5)^n}$

Using initial conditions:

I_0: $A + B = 10$

I_1: $A + 0.5B = 5$

Solving:

$B = 10, A = 0$

So, $\mathbf{I_n = 10 (0.5)^n}$

Verifying via Iteration

Using iteration:

- $\mathbf{I_2 = 1.5(5) - 0.5(10) = 7.5 - 5 = 2.5}$
- $\mathbf{I_3 = 1.5(2.5) - 0.5(5) = 3.75 - 2.5 = 1.25}$
- $\mathbf{I_4 = 1.5(1.25) - 0.5(2.5) = 1.875 - 1.25 = 0.625}$

From formula:

- $I_2 = 10 (0.5)^2 = 2.5$
- $I_3 = 10 (0.5)^3 = 1.25$
- $I_4 = 10 (0.5)^4 = 0.625$

Iteration results match the closed-form solution.

Conclusion

Recursion and recurrence relations offer powerful frameworks for modeling sequential dependencies. While recursion provides conceptual clarity, iteration and characteristic root methods enhance computational efficiency. Understanding the differences between homogeneous and non-homogeneous relations and choosing the right solving techniques for sequences, physical processes, or financial trends—is key to successful problem-solving in discrete mathematics.

References

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Grimaldi, R. P. (2004). *Discrete and Combinatorial Mathematics: An Applied Introduction* (5th ed.). Pearson Education. https://eu.pearson.com/products/9781292485768

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