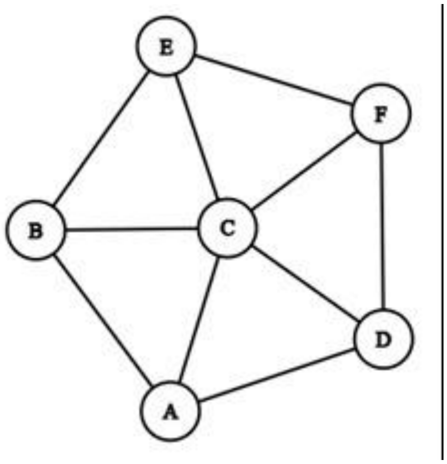
EULER AND HAMILTONIAN PATHS AND CIRCUITS, AND GRAPH OPTIMIZATION TECHNIQUES

Introduction

Graph theory provides a foundational approach for solving real-world problems involving networks, routing, and optimization. Cities, delivery services, and communication systems can be efficiently modeled using vertices and edges (Bondy & Murty, 2025). This assignment examines two practical scenarios: Graphville's road infrastructure and Swift Deliver's route optimization. Using Eulerian and Hamiltonian paths and circuits, and shortest path algorithms, this report explores solutions to optimize travel routes and fuel efficiency.

Scenario 1: Graphville's Road Network



The image shows Graphville's road network, represented by vertices **A, B, C, D, E, F** connected through various edges. This scenario requires analysis of Eulerian and Hamiltonian paths and circuits.

a. Eulerian Path and Circuit

A **Eulerian circuit** exists in a graph if every vertex has an **even degree** (even number of edges connected to it). A **Eulerian path** exists if exactly **two vertices have an odd degree**, and the graph is connected.

- **Degrees of vertices:**
  - A: 2
  - B: 3 (odd)
  - C: 5 (odd)
  - D: 3 (odd)
  - E: 3 (odd)
  - F: 2

There are **four vertices with odd degrees (B, C, D, E)**.

- **Conclusion:** No **Eulerian path** or **Eulerian circuit** exists because more than two vertices have odd degrees.
- **Modification for Eulerian Circuit:** To create an Eulerian circuit, the degrees of **B, C, D, E** must be made even. Adding edges like:
  - Connect B to F.
  - Connect D to E.
  - This would make B and D even-degree vertices, reducing the number of odd-degree vertices to two.

- Further adding C to F would make C even as well.
- Finally, adding an edge between E and F would make E even-degree.

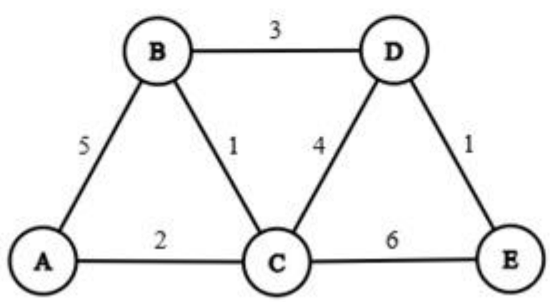
With these modifications, all vertices would have even degrees, enabling a Eulerian circuit.

b. Hamiltonian Path and Circuit

A **Hamiltonian path** visits each vertex exactly once. A **Hamiltonian circuit** returns to the starting vertex after visiting all others exactly once.

- In the current graph, a **Hamiltonian circuit exists**.
- Example Hamiltonian circuit: **A → B → E → F → D → C → A**.
- **Conclusion:** A Hamiltonian circuit exists without modifications, making it suitable for tourist route planning.

Scenario 2: Swift Deliver's Route Optimization



The image shows a weighted graph where:

- Nodes: A, B, C, D, E
- Weights represent fuel consumption.

a. Shortest Route from A to B, D, E

Using Dijkstra's algorithm to minimize fuel consumption:

1. **From A:**
    - $A \rightarrow C = 2$
    - $A \rightarrow B = 5$ (direct but larger)
2. From **C:**
    - $C \rightarrow B = 1$ (better than direct $A \rightarrow B$)
    - $C \rightarrow D = 4$
    - $C \rightarrow E = 6$
3. **Best path to B:**
    - $A \rightarrow C \rightarrow B = 2 + 1 = 3$ (better than direct $A \rightarrow B = 5$)
4. From **B:**
    - $B \rightarrow D = 3$ (to reach D)
5. From **D:**
    - $D \rightarrow E = 1$

- **Total path:** $A \rightarrow C \rightarrow B \rightarrow D \rightarrow E$
    - $A \rightarrow C = 2$
    - $C \rightarrow B = 1$

- $B \rightarrow D = 3$
- $D \rightarrow E = 1$
- **Total fuel consumption = 2 + 1 + 3 + 1 = 7**

- **Conclusion:** The shortest route is **A → C → B → D → E** with a total fuel consumption of **7**.

b. Effect of Increasing C-D by 50%

- Original $C \rightarrow D$ weight = 4
- New weight: $4 + (0.5 \times 4) = 6$

Recalculate best path:

1. From **A:**
   - $A \rightarrow C = 2$
   - $A \rightarrow B = 5$
2. From **C:**
   - $C \rightarrow B = 1$
   - $C \rightarrow D = 6$ (now equal to $C \rightarrow E$)
3. **Best path to B:**
   - $A \rightarrow C \rightarrow B = 2 + 1 = 3$
4. From **B:**
   - $B \rightarrow D = 3$ (still best route to D)

5. From **D:**
    - D → E = 1

- **Total path remains:** A → C → B → D → E
    - A → C = 2
    - C → B = 1
    - B → D = 3
    - D → E = 1
    - **Total fuel consumption = 7**

- **Conclusion:** Increasing C → D by 50% does not change the shortest route. A → C → B → D → E remains optimal with total fuel consumption of **7**.

Conclusion

Graph theory applications, specifically Eulerian and Hamiltonian paths, provide valuable insights into infrastructure and route optimization. Graphville's network lacks a Eulerian path or circuit but supports a Hamiltonian circuit, making it ideal for tourism routing with minor adjustments (Cormen, Leiserson, Rivest, & Stein, 2009). In the Swift Deliver scenario, Dijkstra's algorithm successfully identifies the most fuel-efficient route, even after edge weight modifications. These techniques are vital in network design, logistics, and operational efficiency, proving the importance of discrete mathematics in real-world problem-solving.

References

Bondy, J. A., & Murty, U. S. R. (2008). *Graph theory* (Vol. 244). Springer Science & Business Media. https://link.springer.com/book/9781846289699

Cormen, T. H., Leiserson, C. E., Rivest, R. L., & Stein, C. (2009). *Introduction to algorithms* (3rd ed.). MIT Press. https://mitpress.mit.edu/9780262533058/introduction-to-algorithms/

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