Kerr Black holes

Boyer–Lindquist coordinates

ボイヤー・リンキスト（Boyer-Lindquist）座標による表現

$$x = \sqrt{r^2 + a^2}\,\sin(\theta)\cos(\varphi)$$

$$y = \sqrt{r^2 + a^2}\,\sin(\theta)\sin(\varphi)$$

$$z = r\cos(\theta)$$

$$R \equiv x^2 + y^2 + z^2$$

$$u \equiv R - a^2$$

$$r = \sqrt{\frac{u + \sqrt{u^2 + (2az)^2}}{2}}$$

$$\theta = \cos^{-1}(\frac{z}{r})$$

$$\varphi = \mathrm{atan2}(\frac{y}{r}, \frac{x}{r})$$

事象の地平線半径（radius of black hole's event horizon）

$$r_\rho = 1 + \sqrt{1 - a^2}$$

膠着円盤（accretion disc）

$$r_{ad} = 3 + Z_2 - \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}$$

$$Z_1 \equiv 1 + \sqrt[3]{1 - a^2}\left(\sqrt[3]{1 + a} + \sqrt[3]{1 - a}\right)$$

$$Z_2 \equiv \sqrt{3a^2 + Z_1^2}$$

Kerr metric (G=c=M=1)

$$ds = -\frac{\Delta}{\Sigma}(dt - a\sin^2(\theta)d\phi)^2 + \frac{\sin^2(\theta)}{\Sigma}((r^2 + a^2)d\phi - adt)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$\Delta \equiv r^2 - 2Mr + a^2$$

$$\Sigma \equiv r^2 + a^2\cos^2(\theta)$$

$$s \equiv \Sigma - 2r$$

$$E \equiv \sqrt{s\left(\frac{\left(\frac{dr}{d\tau}\right)^2}{\Delta} + \left(\frac{d\theta}{d\tau}\right)^2\right) + \Delta\sin^2(\theta)\left(\frac{d\varphi}{d\tau}\right)^2}$$

$$L \equiv \frac{\left(\Sigma\Delta\frac{d\varphi}{d\tau} - 2arE\right)\sin^2(\theta)}{sE}$$

$$\kappa \equiv \left(\frac{d\theta}{d\tau}\Sigma\right)^2 + a^2\sin^2(\theta) + \frac{L^2}{\sin^2(\theta)}$$

Null 測地線の微分方程式

$$\frac{dr}{d\tau} = \frac{\Delta}{\Sigma} p_r$$

$$\frac{d\theta}{d\tau} = \frac{1}{\Sigma} p_\theta$$

$$\frac{d\varphi}{d\tau} = \frac{2ar + (\Sigma - 2r)L / \sin^2(\theta)}{\Delta\Sigma}$$

$$\frac{dt}{d\tau} = \frac{1 + (2r(r^2 + a^2) - 2arL)}{\Delta\Sigma}$$

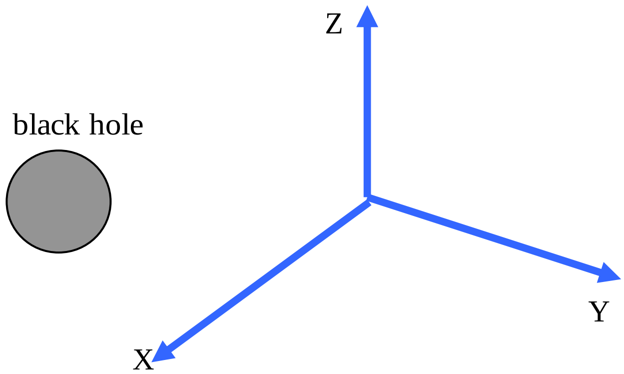
$$\frac{dp_r}{d\tau} = \frac{(r-1)(-\kappa) + 2r(r^2 + a^2) - 2aL}{\Delta\Sigma} - \frac{2 p_r{}^2 (r-1)}{\Sigma}$$

$$\frac{dp_\theta}{d\tau} = \frac{\sin(\theta)\cos(\theta)\left(\frac{L^2}{\kappa^2} - a^2\right)}{\Sigma}$$

$$\frac{dp_\varphi}{d\tau} = 0$$

$$\frac{dp_t}{d\tau} = 0$$

Initial Conditions



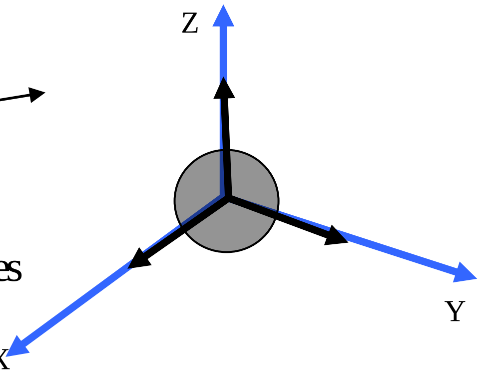
Coordinate transformation

$$RayDir = (r_s, \theta_s, \varphi_s)$$

Spherical coordinates

$$RayPosition = (r, \theta, \varphi)$$

Boyer-Lindquist coordinates



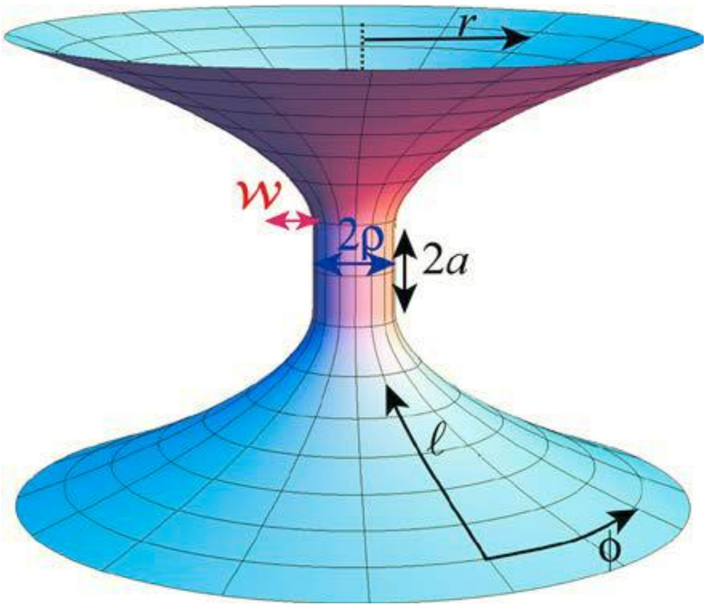
$$R \equiv \sqrt{a^2 + r^2}$$

$$\frac{dr}{d\tau} = -\frac{rR\sin(\theta)\sin(\theta_s)\cos(\varphi - \varphi_s) + R^2\cos(\theta)\cos(\varphi - \varphi_s)}{\Sigma}$$

$$\frac{d\theta}{d\tau} = \frac{r\sin(\theta)\cos(\theta_s) - R\cos(\theta)\sin(\theta_s)\cos(\varphi - \varphi_s)}{\Sigma}$$

$$\frac{d\varphi}{d\tau} = \frac{\sin(\theta_s)\sin(\varphi - \varphi_s)}{R\sin(\theta)}$$

Wormhole 3-parameters(W, a, ρ)



$$\frac{W}{M} = 1.42953 = -\log(\sec(\frac{\pi}{2\sqrt{2}})) + \frac{\pi}{2\sqrt{2}}\tan(2\sqrt{2})$$

$$M = \frac{W}{1.42953}$$

$$x \equiv \frac{2|l| - a}{\pi M}$$

$$r(l) = \begin{cases} \rho + M\left(x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) \right) & |l| > a \\ \rho & |l| \le a \end{cases}$$

$$\frac{dr}{dl} = \begin{cases} 2\tan^{-1}(\frac{-a+|l|}{\pi M})sign(l) & |l| > a \\ 0 & |l| \le a \end{cases}$$

Direction of the light ray($\theta_{cs}, \varphi_{cs}$)

$$N_x = \sin(\theta_{cs})\cos(\varphi_{cs})$$
$$N_y = \sin(\theta_{cs})\sin(\varphi_{cs})$$
$$N_z = \cos(\theta_{cs})$$

Initial camera position(l_c, θ_c, φ_c)

$$n_l = -N_x$$
$$n_\varphi = -N_y$$
$$n_\theta = N_z$$

光線運動量

$$p_l Init = n_l$$
$$p_\theta Init = r\ n_\theta$$
$$p_\varphi Init = r\ \sin(\theta_c) n_\varphi$$

(A9d)

$$b \equiv r(l_c)\sin(\theta_c) n_\varphi$$
$$B^2 \equiv r(l_c)^2 \left(n_\theta{}^2 + n_\varphi{}^2\right)$$

Null 測地線の微分方程式 A7a～A7e

$$\frac{dl}{d\tau} = p_l$$

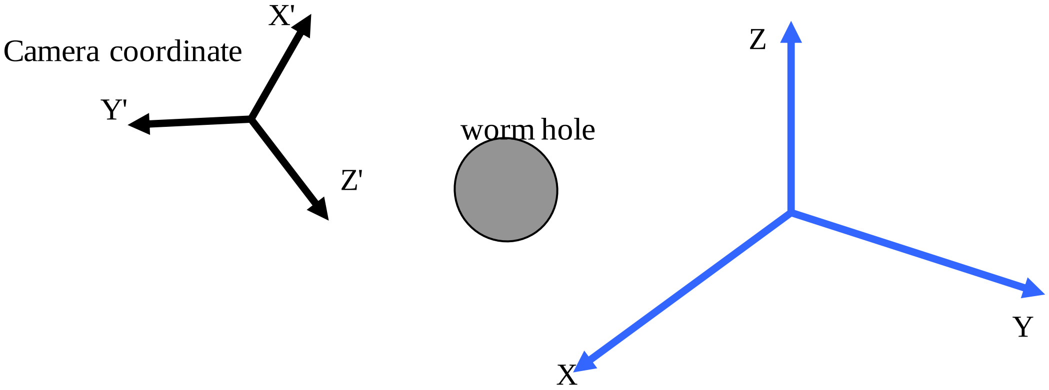
$$\frac{d\theta}{d\tau} = \frac{p_\theta}{r^2}$$

$$\frac{d\varphi}{d\tau} = \frac{b}{r^2 \sin^2(\theta)}$$

$$\frac{dp_l}{d\tau} = B^2 \frac{\left(\frac{dr}{dl}\right)}{r^3}$$

$$\frac{dp_\theta}{d\tau} = \frac{b^2 \cos(\theta)}{r^2 \sin^3(\theta)}$$

Initial Conditions



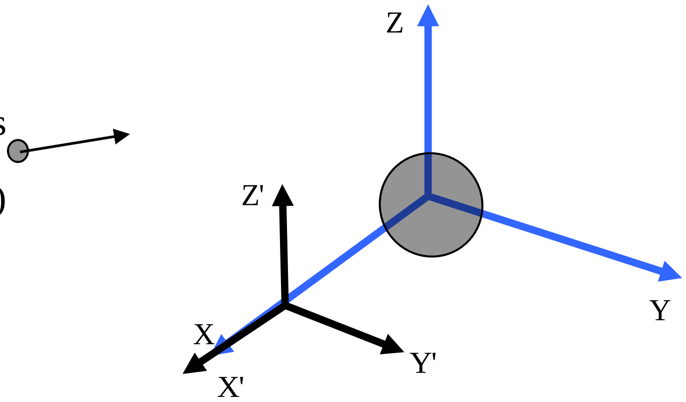
Coordinate transformation

$RayDir = (r_s, \theta_s, \varphi_s)$

Spherical coordinates

$RayPosition = (r, \theta, \varphi)$

Spherical coordinates



$$X = camera\_positon - position$$

$$Y = normalize\big((0,0,1) \times X\big)$$

$$Y.length = 0 -> Y = normalize\big((0,1,0) \times X\big)$$

$$Z = normalize\big(X \times X\big)$$

$$M = \begin{pmatrix} X_X & Y_x & Z_x & 0 \\ X_y & Y_y & Z_y & 0 \\ X_z & Y_z & Z_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$(r, \theta, \varphi) = \big(M\ RayPosition(x, y, z)\big)$ Spherical coordinates

$$l = l_c$$

$$\theta = \theta_c$$

$$\varphi = \varphi_c$$

$$p_l = p_l Init$$

$$p_\theta = p_\theta Init$$

$$r(l) = \begin{cases} \rho + M\left(x\tan^{-1}(x) - \dfrac{1}{2}\ln(1+x^2)\right) & |l| > a \\ \rho & |l| \le a \end{cases}$$

の逆関数の計算にはニュートンラプソン法を使う。

$$f(x) = \rho + M\left(x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2)\right)$$

$$\frac{df(x)}{dx} = M\,\tan^{-1}(x)$$

$$x_{n+1} = x_n - \frac{f(x)}{\left(\dfrac{df(x)}{dx}\right)} = \frac{\rho + M\left(x\tan^{-1}(x) - \dfrac{1}{2}\ln(1+x^2)\right)}{M\,\tan^{-1}(x)}$$

で x_n 求めて $x \equiv \dfrac{2|l| - a}{\pi M}$ であるので

$$\pm l = x_n \frac{\pi M}{2} + a$$

として計算される（符号±はこの計算では決まらない）。

Gravitational Lensing by Spinning Black Holes in Astrophysics, and in the Movie Interstellar
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A PUBLIC GPU-BASED CODE FOR GENERAL-RELATIVISTIC RADIATIVE TRANSFER IN KERR SPACETIME
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Visualizing Interstellar's Wormhole

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