Discrete Time Martingales

Week 4 **January 27, 2023**

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Origin is from horse-racing (betting system). The dictionary meaning of the word 'martingale' is the harness of a horse.
Let $\{Z_n\}_{n\geq 1}$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 4.0.1. *A sequence of random variables $\{Z_n\}_{n\geq 1}$ is said to be a Martingale if*

$$\mathbb{E}(Z_n | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) = z_{n-1} \quad \forall\ n \geq 2 \tag{4.1}$$

Things to understand- conditional expectation for discrete and conditional random variable [**?**].
Things we will explore-

1. Examples of $\{Z_n\}_{n\geq 1}$ that are martingales.

2. How different are martingales from iid sequences and markov chains?

3. How to interpret 4.1?

Example. $\{S_n\}_{n\geq 1}$ and $S_0 \equiv 0$.

$$X_i = \begin{cases} 1, & w.p \;\; 1/2 \\ -1, & w.p \;\; 1/2 \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i$$

Let $s_{n-1}, s_{n-2}, \ldots, s_1 \in \mathbb{Z}$ such that $\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) > 0$

$$\begin{aligned}
\mathbb{E}(S_n | S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) &= \sum_{k \in \mathbb{Z}} k \mathbb{P}(S_n = k | S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(S_n = k, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(S_{n-1} + X_n = k, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(X_n = k - s_{n-1}, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(X_n = k - s_{n-1}) \mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= (s_{n-1} + 1) \mathbb{P}(X_n = -1) + (s_{n-1} - 1) \mathbb{P}(X_n = 1) \\
&= (s_{n-1} + 1) \frac{1}{2} + (s_{n-1} - 1) \frac{1}{2} = s_{n-1}
\end{aligned}$$

Note that the summations here are "finite" sums.
As $s_{n-1}, \ldots, s_1 \in \mathbb{Z}$ were arbitrary, $\{S_n\}_{n \geq 1}$ is a martingale.

Example. $\{X_i\}_{i \geq 1}$ be an iid sequence on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z_n = \prod_{i=1}^n X_i$ and $\text{Range}(Z_n) \subset \mathbb{R} \;\; \forall \;\; n \geq 1$.
Let $z_{n-1}, \ldots, z_1 \in \mathbb{R}$ such that $\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) > 0$. Then

$$\begin{aligned}
\mathbb{E}(Z_n | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) &= \sum_{k \in Range(Z_n)} k \mathbb{P}(Z_n = k | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(Z_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(Z_{n-1} X_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(z_{n-1} X_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \mathbb{P}(Z_{n-1} X_n = k) \frac{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{u \in S^1, Range(X_n) = S^1} u z_{n-1} \mathbb{P}(X_n = u) \\
&= z_{n-1} \mathbb{E}[X_n] = z_{n-1}
\end{aligned}$$

Note that the sums here might be infinite. In the last step we assume $\mathbb{E}[X_i] = 1$. Now since $\{z_i\}_{i=1}^{n-1}$ were arbitrary, $\{Z_n\}_{n\geq 1}$ is a martingale.

Example.

$$X_i = \begin{cases} 2, & w.p \ \ 1/2 \\ 0, & w.p \ \ 1/2 \end{cases}$$

Then $\mathbb{E}(X_i) = 1$. Therefore, $Z_n = \prod_{i=1}^{n} X_i$ is a martingale. Range (Z_n)= $\{2^n, 0\}$. Note that the mean stays constant and

$$\mathbb{P}(Z_n = 0) = 1 - \frac{1}{2^n}$$

$$\mathbb{P}(Z_n = 2^n) = \frac{1}{2^n}$$

Intuition- The first equation shows that the martingale takes a very low value with very high probability and the second one shows that it takes a very large value with very low probability
Idea behind Markov Chains -

$$\text{“}X_n|X_{n-1}, \ldots, X_1\text{”} \overset{d}{=} X_n|X_{n-1}$$

Idea behind Martingales - Expected value of Z_n conditioned on the past depends only on Z_{n-1}. $\{Z_n\}_{n\geq 1}$ in law could depend on the entire past!

Week 5 **February 3, 2023**
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We define $f : D \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ where

$$f(z_1, z_2, \ldots, z_{n-1}) = \mathbf{E}[Z_n | Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \ldots, Z_1 = z_1]$$

Define $Y_n : \Omega \to \mathbb{R}$ where

$$Y_n(\omega) := f(Z_1(\omega), Z_2(\omega), \ldots, Z_{n-1}(\omega)) \tag{4.2}$$

You can check that $\{Y_n\}$ is a random variable.

Property 4.0.1. *Some properties of* $\{Y_n\}$

1. $A := \{Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \ldots, Z_1 = z_1\}$

$$\omega \in A \implies Y_n(\omega) = f(z_1, z_2, \ldots, z_{n-1})$$

2. $L := \{Y_n \leq c\} = \{f(Z_1, Z_2, \ldots, Z_{n-1}) \leq c\}$

$$L \in \mathcal{A}_{n-1} \equiv \textit{observable events upto } n-1$$

(4.2) $\iff$ $\{Y_n\}$ *has the above two properties*

If $\{Z_n\}$ *is martingale,* $Y_n = Z_{n-1}$

Lemma 4.0.1. *Let $\{Y_n\}_{n\geq 1}$ be martingale. Then,*

$$\forall\, 1 \leq i \leq n,\ \mathbf{E}[Z_n|Z_i, Z_{i-1}, \dots, Z_1] = Z_i$$

Proof. We fix i and prove by induction on n.
We look at n = i+1. By martingale property,

$$\mathbf{E}[Z_{i+1}|Z_i, Z_{i-1}, \dots, Z_1] = Z_i$$

Let $k > 0$ and the statement hold for $n = i + k$. We look at $n = i + k + 1$

$$\begin{aligned}
&\mathbf{E}[Z_{i+k+1}|Z_i, Z_{i-1}, \dots, Z_1] \\
&= \mathbf{E}[\mathbf{E}[Z_{i+k+1}|Z_{i+k}, Z_{i+k-1}, \dots, Z_1]|Z_i, Z_{i-1}, \dots, Z_1] \\
&= \mathbf{E}[Z_{i+k}|Z_i, Z_{i-1}, \dots, Z_1] \qquad \text{[using (4.0.1)]} \\
&= Z_i
\end{aligned}$$

where the last equality is obtained from the induction hypothesis □

The property used in the first equality is called the Tower property. We now formally state and prove the same.

Property 4.0.2 (Tower Property)**.**

$$\mathbf{E}[\mathbf{E}[X|Y, Z]|Y] = E[X|Y]$$

Proof.

$$\mathbf{E}[\mathbf{E}[X|Y, Z]|Y] = E[h(Y, Z)|Y] = k(Y)$$

Let $y \in \mathbb{R}$ such that $\mathbf{P}(Y = y) > 0$

$$\begin{aligned}
k(y) &= E[h(Y, Z)|Y] \\
&= \sum_{\substack{m\in\text{Range}(Y)\\ t\in\text{Range}(Z)}} h(m, t)\mathbf{P}(Y = m, Z = t|Y = y) \\
&= \sum_{t\in\text{Range}(Z)} h(y, t)\mathbf{P}(Z = t|Y = y) \\
&= \sum_{t\in\text{Range}(Z)} \sum_{k\in\text{Range}(X)} k\mathbf{P}(X = k|Y = y, Z = t)\mathbf{P}(Z = t|Y = y) \\
&= \sum_{t\in\text{Range}(Z)} \sum_{k\in\text{Range}(X)} k\frac{\mathbf{P}(X = k, Y = y, Z = t)}{\mathbf{P}(Y = y, Z = t)} \frac{\mathbf{P}(Z = t, Y = y)}{\mathbf{P}(Y = y)} \\
&= \sum_{k\in\text{Range}(X)} \sum_{t\in\text{Range}(Z)} k\frac{\mathbf{P}(X = k, Y = y, Z = t)}{\mathbf{P}(Y = y)} \\
&= \sum_{k\in\text{Range}(X)} k\frac{\mathbf{P}(X = k, Y = y)}{\mathbf{P}(Y = y)} \\
&= \mathbf{E}[X|Y = y]
\end{aligned}$$

□