Week 2

More on random walks

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Theorem 2.0.1. *Let $T : \Omega \to 0, 1, \ldots, N$ be a stopping time. Then,*

$$\mathbf{E}[S_T^2] = E[T].$$

Proof.

$$\begin{aligned} S_T^2 &= \sum_{k=1}^{N} S_k^2 \mathbb{1}\{T = k\} \\ &= \sum_{k=1}^{N} (S_k^2 - S_{k-1}^2) \mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} (X_k + S_{k-1})^2 - S_{k-1}^2 \mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} (1 + 2X_k S_{k-1}) \mathbb{1}\{T \geq k\} \end{aligned}$$

Now, consider $V_k = S_{k-1} \mathbb{1}\{T \geq k\}$. Note that this is a bet sequence. Hence,

$$\begin{aligned} \mathbf{E}[S_T^2] &= \mathbf{E}\left[\sum_{k=1}^{N} \mathbb{1}\{T \geq k\}\right] + 2 \sum_{k=1}^{N} \mathbf{E}[X_k V_k] \\ &= \sum_{k=1}^{N} \mathbf{P}(T \geq k) + 0 \\ &= E[T] \end{aligned}$$

□

2.1 Reflection Principle