Topics in Applied Stochastic Processes

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Week 1 **January 13, 2023**

Finite length random walks on $\mathbb{Z}$

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1.1 Definitions

Random walks serve as very useful models in many applications. They are simple to state and understand, yet they lead to lots of intractable questions.

Notation. $\mathbb{N} = \{k \in \mathbb{Z} : k \geq 1\}$ *and* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

We now proceed to construct what is called a "simple random walk" on $\mathbb{Z}$ of finite length $N \in \mathbb{N}$. The sample space Ω_N and the event space $\mathcal{F}_N$ are described below.

$$\Omega_N := \{(\omega_1, \dots, \omega_N) : \omega_i \in \{-1, 1\} \ \forall \ 1 \leq i \leq N\}$$

$$\mathcal{F}_N := \{A : A \subseteq \Omega_N\}$$

The probability function $\mathbf{P}_N : \Omega_N \to [0, 1]$ is defined as

$$\mathbf{P}_N(A) := |A| \ 2^{-N}$$

We also define random variables X_k and S_k on Ω_N for $1 \leq k \leq N$ as

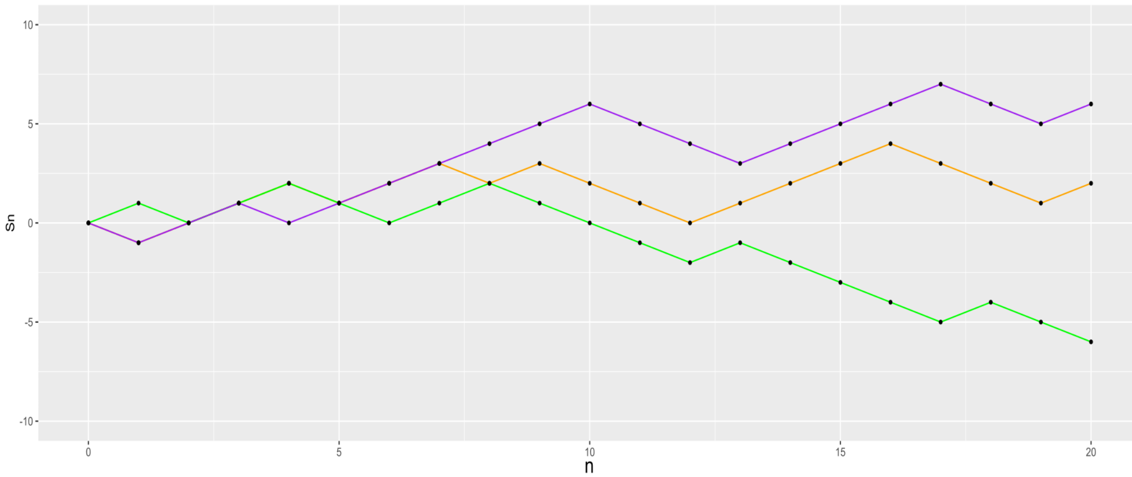
$$X_k : \Omega_N \to \{-1, 1\} \ ; \ X_k(\omega) := \omega_k$$

$$S_k : \Omega_N \to \mathbb{Z} \ ; \ S_k(\omega) := \sum_{i=1}^{k} X_k(\omega) \ ; \ S_0(\omega) := 0 \text{ for all } \omega \in \Omega_N$$

Definition 1.1.1. *Fix $N \in \mathbb{N}$. The sequence of random variables $\{S_k\}_{k=1}^{N}$ on $(\Omega_N, \mathcal{F}_N, \mathbf{P}_N)$ is called a (symmetric) simple random walk on $\mathbb{Z}$, of finite length N, starting at 0.*

† added illustrations

Figure 1.1: Three possible trajectories for $(S_n)_{n=0}^N$



In what follows, we suppress the subscript N while referring to the probability space $(\Omega_N, \mathcal{F}_N, \mathbf{P}_N)$, and we assume that $N \in \mathbb{N}$ is fixed.

Observations.

(a) $\{X_k\}_{k=1}^N$ are iid, i.e. independent and identically distributed.

Proof.

$$\begin{aligned} \mathbf{P}(X_k = 1) = \mathbf{P}(\{\omega \in \Omega : \omega_k = 1\}) &= 2^{-N} |\{\omega \in \Omega : \omega_k = 1\}| \\ &= 2^{-N} 2^{N-1} \\ &= \frac{1}{2} \\ &= \mathbf{P}(X_k = -1) \end{aligned}$$

So $\{X_k\}_{k=1}^N$ are identically distributed. Independence is left as an exercise. □

(b) (Independent increments) For $1 \leq k_1 \leq k_2 \leq \ldots \leq N$, $\{S_{k_i} - S_{k_{i-1}} : 1 \leq i \leq N\}$ are independent random variables.

Proof. Observe that, for $1 \leq k < l \leq N$, we have $S_l - S_k = \sum_{i=k+1}^{l} X_i$. Therefore, if $1 \leq a < b \leq c < d \leq N$, we see that $S_b - S_a$ and $S_d - S_c$ are functions of disjoint sets of independent random variables, and hence the claim is true. □

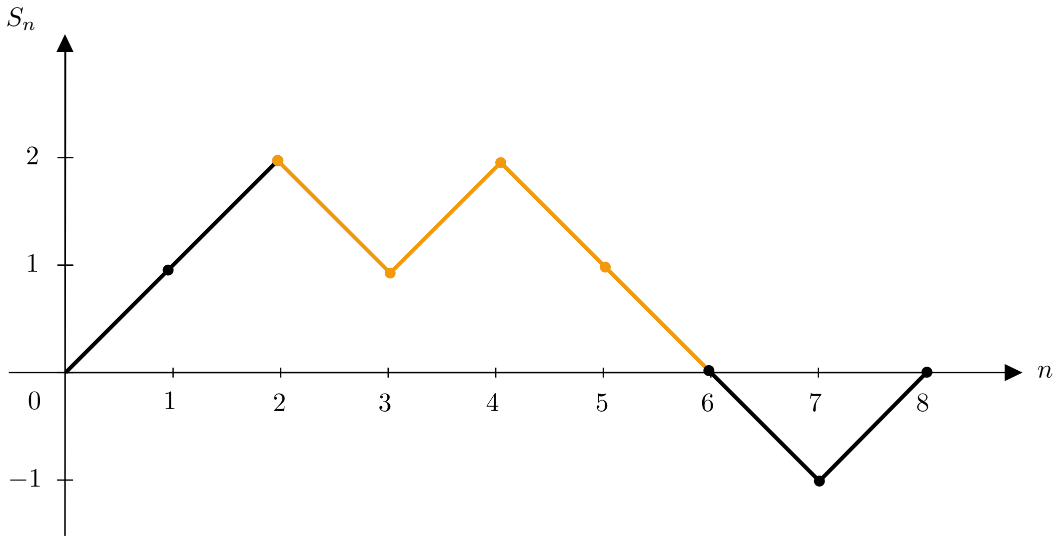


Figure 1.2: Independent (colored) increments in a simple random walk

(c) (Stationary in increments) For $1 \leq k < m \leq N$, $\mathbf{P}(S_m - S_k = \alpha) = \mathbf{P}(S_{m-k} = \alpha)$ for every $\alpha \in \mathbb{Z}$.

Proof. We use the fact that $\{X_i\}_{i=1}^N$ are identically distributed in the following argument.

$$\mathbf{P}(S_m - S_k = \alpha) = \mathbf{P}\left(\sum_{i=k+1}^{m} X_i = \alpha\right) = \mathbf{P}\left(\sum_{i=1}^{m-k} X_i = \alpha\right) = \mathbf{P}(S_{m-k} = \alpha)$$

□

(d) (Markov Property) For $\alpha_i \in \mathbb{Z},\ 1 \leq i \leq N$ and $0 \leq n \leq N$,

$$\mathbf{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}, \ldots, S_1 = \alpha_1) = \mathbf{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}),$$

assuming (of course) that the conditional probabilities are well defined.

Proof. Left as an exercise. □

(e) (Conditional Law) For $1 \leq k < m \leq N$, $\mathbf{P}(S_m = b \mid S_k = a) = \mathbf{P}(S_{m-k} = b - a)$.

Proof. Left as an exercise. □

(f) (Moments) For $1 \leq k \leq N$, we have $\mathbf{E}[X_k] = \mathbf{E}[S_k] = 0$ and $\mathrm{Var}[S_k] = k$.

Proof. By definition of expected value, $\mathbf{E}[X_k] = 1(1/2) - 1(1/2) = 0$. By linearity of expected values, $\mathbf{E}[S_k] = \sum_{i=1}^{k} \mathbf{E}[X_i] = 0$.

Since $\mathbf{E}[S_k] = 0$, $\mathrm{Var}[S_k] = \mathbf{E}[(\sum_{i=1}^{k} X_i)^2] = \sum_{i=1}^{k} \mathbf{E}[X_k^2] = k$. As an exercise, show that $\mathbf{E}[(\sum_{i=1}^{k} X_i)^2] = \sum_{i=1}^{k} \mathbf{E}[X_k^2]$. □

(g) (Distribution of S_n) For $x \in \{-n, -n+2, \ldots, n-2, n\}$, we have

$$\mathbf{P}(S_n = x) = \mathbf{P}(S_n = -x) = \binom{n}{\frac{n+x}{2}} 2^{-n}$$

Proof. We only provide a sketch of the proof, which is left as an exercise. For $0 \leq j \leq N$, $\{S_n = 2j - n\} = \{S_n = j - (n-j)\}$. So there must be a total of j steps to the right and $n - j$ steps to the left. Therefore

$$\mathbf{P}(S_n = 2j - n) = 2^{-N} \ |\{\omega \in \Omega : \cdots\}| = 2^{-n} \binom{n}{j}$$

□

(h) (Mode) The mode of the above distribution is achieved in the middle, i.e. at $x = 0$ and at $x = 1, -1$ for S_{2n} and S_{2n-1} respectively.

Proof.

$$\mathbf{P}(S_{2n} = 0) = \mathbf{P}(S_{2n-1} = 1) = \binom{2n}{n} 2^{-2n}$$

□

(i) (Stirling's formula) Using Stirling's approximation, for large n, we have

$$\binom{2n}{n} = \frac{2n!}{n!n!} \sim \frac{(2n)^{2n} e^{-2n} \sqrt{4\pi n}}{n^{2n} e^{-2n} \sqrt{2\pi n} \sqrt{2\pi n}} \sim \frac{2^{2n}}{\sqrt{\pi n}} \qquad (*)$$

Therefore,

$$\mathbf{P}(S_{2n} = 0) = \binom{2n}{n} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}} \quad \text{as} \quad n \to \infty$$

This approximation, although correct, has a caveat - we chose to keep N fixed, but as $n \to \infty$, we must also let $N \to \infty$, and this requires subtler arguments. A few consequences of this approximation are mentioned in the exercises.

1.2 Stopping times

Motivation for this section comes from the classic Gambler's ruin problem. We can interpret a simple random walk as a fair game between two players, where in round k, a player wins the amount X_k. Then S_n denotes the capital of one player over the other after n rounds.

We would like to answer the following question - "Is it possible to stop the game in a favorite moment, i.e., can clever stopping lead to a positive expected gain?". In other words, can we design a $T(\omega)$ for every $\omega \in \Omega$ such that $\mathbf{E}[S_T] > 0$? Of course, the decision to stop may only depend on the trajectory until that time: no "insider knowledge" about the future of the trajectory is permitted.

To formalize this setting, we make the following definition.

Definition 1.2.1. *An event $A \subseteq \Omega$ is said to be observable by time n if it is a (possibly empty) union of basic / elementary events of the form*

$$\{\omega \in \Omega : \omega_1 = o_1, \ldots, \omega_n = o_n\}$$

where $o_i \in \{-1, 1\}$ for $1 \leq i \leq n$.

We also define $\mathcal{A}_0 = \{\phi, \Omega\}$ and set

$$\mathcal{A}_n := \{A \in \mathcal{F} : A \text{ is observable by time } n\}.$$

Immediately, we observe that

$$\{\phi, \Omega\} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_{N-1} \subseteq \mathcal{A}_N = \mathcal{F}$$

As an easy exercise, verify that each $\mathcal{A}_n$ is closed with respect to taking complement, union and intersection. Such a sequence $\{\mathcal{A}_i\}_{i=0}^N$ is called a *filtration.*

Definition 1.2.2. *A function $T : \Omega \to \{0, 1, \ldots, N\} \cup \{\infty\}$ is called a stopping time if for each $0 \leq n \leq N$,*

$$\{T = n\} = \{\omega \in \Omega : T(\omega) = n\} \in \mathcal{A}_n$$

Example. For $a \in \mathbb{Z}$, let $\sigma_a = \inf\{n : S_n = a, 0 \leq n \leq N\}$ denote the *first* hitting time of a. As an exercise, show that σ_a is a stopping time.

Example. For $a \in \mathbb{Z}$, let $L_a = \max\{n : S_n = a, 0 \leq n \leq N\}$ denote the *last* hitting time of a. As an exercise, show that L_a is NOT a stopping time.

Theorem 1.2.1. *Let $T : \Omega \to \{0, 1, \ldots, N\}$ be a stopping time. Then*

$$\mathbf{E}[S_T] = 0$$

where $S_T : \Omega \to \mathbb{Z}$ maps $\omega \mapsto S_{T(\omega)}(\omega)$.

Proof.

$$
\begin{aligned}
S_T = \sum_{k=1}^{N} S_k \mathbb{1}\{T = k\} &= \sum_{k=1}^{N} S_k(\mathbb{1}\{T \geq k\} - \mathbb{1}\{T \geq k+1\}) \\
&= \sum_{k=1}^{N} (S_k - S_{k-1})\mathbb{1}\{T \geq k\} \\
&= \sum_{k=1}^{N} X_k \mathbb{1}\{T \geq k\}
\end{aligned}
$$

where we take $\mathbb{1}\{T \geq N+1\} = 0$. Now, we can write $\mathbf{E}[S_T]$ as

$$
\mathbf{E}[S_T] = \sum_{k=1}^{N} \mathbf{E}[X_k \mathbb{1}\{T \geq k\}] \tag{$\dagger$}
$$

Observe that for $1 \leq k \leq N$, we have

$$
X_k \mathbb{1}\{T \geq k\} = \begin{cases} 1, & \text{for } X_k = 1,\ T \geq k \\ -1, & \text{for } X_k = -1,\ T \geq k \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\mathbf{E}[X_k \mathbb{1}\{T \geq k\}] = \mathbf{P}(X_k = 1, T \geq k) - \mathbf{P}(X_k = -1, T \geq k) \tag{$\dagger\dagger$}
$$

Now,

$$
\{T \geq k\} = \{T < k\}^c = \left(\bigcup_{l=0}^{k-1} \{T = l\} \right)^c \in \mathcal{A}_{k-1}
$$

Using the fact that $\{T \geq k\} \in \mathcal{A}_{k-1}$, one can show that (details left as an exercise)

$$
\mathbf{P}(X_k = 1, T \geq k) = \mathbf{P}(X_k = -1, T \geq k) = \frac{1}{2}\mathbf{P}(T \geq k)
$$

Substituting the above values in ($\dagger$) and ($\dagger\dagger$), we finally have

$$
\mathbf{E}[S_T] = 0
$$

□

As an exercise, compute $\mathrm{Var}[S_T]$.

Definition 1.2.3. *A bet sequence / game system is a sequence of random variables $V_k : \Omega \to \mathbb{R}$ such that*

$$
\{V_k = c\} \in \mathcal{A}_{k-1} \textit{ for every } c \in \mathbb{R} \textit{ and } 1 \leq k \leq N
$$

Theorem 1.2.2. *Let $\{V_k\}_{k=1}^{N}$ be a bet sequence. Then*

$$
\mathbf{E}[S_N^V] = 0 \quad \textit{where} \quad S_N^V = \sum_{k=1}^{N} V_k X_k
$$

In this setting, S_N^V is interpreted as the "total gain".

Proof. Since Ω is finite, we may write

$$\text{Range}(V_k) = \{c_i^k : 1 \le i \le m_k\} \text{ where } c_i^k \in \mathbb{R}$$

$$V_k = \sum_{i=1}^{m_k} c_i^k \mathbb{1}\{V_k = c_i^k\}$$

Now, since $\mathbf{E}[X_k] = 0$, and since $X_k \perp \mathbb{1}\{V_k = c_i^k\}$, we get

$$\begin{aligned}
\mathbf{E}[S_N^V] = \sum_{k=1}^{N} \mathbf{E}[V_k X_k] &= \sum_{k=1}^{N} \mathbf{E}\left[X_k \sum_{i=1}^{m_k} c_i^k \mathbb{1}\{V_k = c_i^k\}\right] \\
&= \sum_{k=1}^{N} \sum_{i=1}^{m_k} c_i^k \mathbf{E}[X_k \mathbb{1}\{V_k = c_i^k\}] \\
&= \sum_{k=1}^{N} \sum_{i=1}^{m_k} c_i^k \mathbf{E}[X_k] \mathbf{P}(V_k = c_i^k) \\
&= 0
\end{aligned}$$

□

1.3 Exercises

1. Show that $\{X_k\}_{k=1}^N$ are independent.
2. Show that $\{S_n\}_{n=0}^N$ satisfies the Markov property.
3. For $1 \le k < m \le N$, show that $\mathbf{P}(S_m = b \mid S_k = a) = \mathbf{P}(S_{m-k} = b - a)$.
4. Show that $\mathbf{E}[S_n^2] = \sum_{i=1}^n \mathbf{E}[X_i^2]$.
5. (a) Show that for any $a, b \in \mathbb{R}$,

   $$\mathbf{P}(a \le S_n < b) \le (b - a)\ \mathbf{P}(S_n \in \{-1, 0, 1\}).$$

   (b) Using (a), conclude that

   $$\mathbf{P}(a \le S_n < b) \to 0 \quad \text{as} \quad n \to \infty.$$

   Thus, we observe that the walk exits any finite interval as $n \to \infty$.
6. Verify that each $\mathcal{A}_n$, $0 \le n \le N$, is closed with respect to taking complement, union and intersection.
7. For $a \in \mathbb{Z}$, let $\sigma_a = \inf\{n : S_n = a, 0 \le n \le N\}$. Show that σ_a is a stopping time.
8. For $a \in \mathbb{Z}$, let $L_a = \max\{n : S_n = a, 0 \le n \le N\}$. Show that L_a is not a stopping time.
9. Let $T : \Omega \to \{0, 1, \ldots, N\}$ be a stopping time. Compute $\text{Var}[S_T]$.
10. Show that X_k and $\mathbb{1}\{T \ge k\}$ are independent.

Week 2 **January 20, 2023**

More on random walks

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Theorem 2.0.1. *Let $T : \Omega \to 0, 1, \ldots, N$ be a stopping time. Then,*

$$\mathbf{E}[S_T^2] = E[T].$$

Proof.

$$\begin{aligned} S_T^2 &= \sum_{k=1}^{N} S_k^2 \mathbb{1}\{T = k\} \\ &= \sum_{k=1}^{N} (S_k^2 - S_{k-1}^2) \mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} (X_k + S_{k-1})^2 - S_{k-1}^2 \mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} (1 + 2X_k S_{k-1}) \mathbb{1}\{T \geq k\}. \end{aligned}$$

Now, consider $V_k = S_{k-1}\mathbb{1}\{T \geq k\}$. Note that this is a bet sequence. Hence,

$$\begin{aligned} \mathbf{E}[S_T^2] &= \mathbf{E}\left[\sum_{k=1}^{N} \mathbb{1}\{T \geq k\}\right] + 2\sum_{k=1}^{N} \mathbf{E}[X_k V_k] \\ &= \sum_{k=1}^{N} \mathbf{P}(T \geq k) + 0 \\ &= E[T]. \end{aligned}$$

□

2.1 Reflection Principle

Assume that $a \in \mathbb{Z}$ and $c > 0$. There is a bijection between the paths that cross $a + c$ and those that do not. This bijection is obtained by reflecting the part of the path crossing $a + c$ as shown in the Figure 2.1. So,

$$|S_n = a + c| = |\sigma_a \leq n \;\&\; S_n = a + c| = |\sigma_a \leq n \& S_n = a - c|$$

Now, we know that all the paths have equal probability. Hence, we get the following lemma.

Lemma 2.1.1. $\mathbf{P}(S_n = a + c) = \mathbf{P}(\sigma_a \leq n \;\&\; S_n = a - c)$ *where* $a \in \mathbb{Z}$ *and* $c > 0$*. This is also known as the* reflection principle*.*

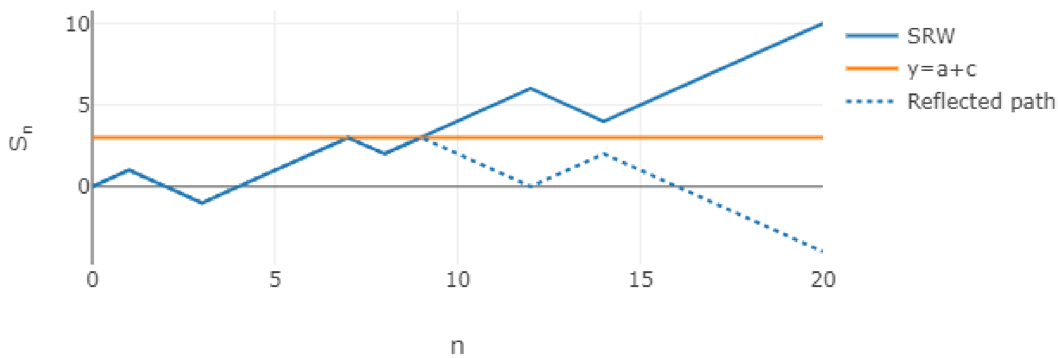

Figure 2.3: The figure shows that the bijection between the paths that cross a+c=3 and those that do not.

Theorem 2.1.1. $\mathbf{P}(\sigma_a \leq n) = \mathbf{P}(S_n \notin [-a, a))$ *where* $a \in \mathbb{Z}$
$\{0\}$*.*

Proof.

$$\begin{aligned}
\mathbf{P}(\sigma_a \leq n) &= \mathbf{P}(\sigma_a \leq n, \bigcup_{b \in \mathbb{Z}} S_n = b) \\
&= \sum_{b \in \mathbb{Z}} \mathbf{P}(\sigma_a \leq n, S_n = b) \\
&= \sum_{b \in \mathbb{Z}, b \geq a} \mathbf{P}(\sigma_a \leq n, S_n = b) + \sum_{b \in \mathbb{Z}, b < a} \mathbf{P}(\sigma_a \leq n, S_n = b) \\
&= \sum_{b \in \mathbb{Z}, b \geq a} \mathbf{P}(S_n = b) + \sum_{b \in \mathbb{Z}, b < a} \mathbf{P}(S_n = 2a - b) \\
&= \mathbf{P}(S_n \geq a) + \mathbf{P}(S_n > a) \\
&= \mathbf{P}(S_n \geq a) + \mathbf{P}(S_n < -a) \\
&= \mathbf{P}(S_n \notin [-a, a))
\end{aligned}$$

□

Corollary 2.1.1. $\mathbf{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbf{P}(S_{n-1} = a - 1) - \mathbf{P}(S_{n-1} = a + 1)\right]$ *where* $a \in \mathbb{Z}$*.*

Proof. □

2.2 Arc-Sine Law

Let L denote the last time the random walk hits 0, i.e., $L = \max_{0 \leq n \leq 2N} S_n = 0$, where N denotes the length of the walk.

Theorem 2.2.1.

$$\mathbf{P}(L = 2n) = \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}.$$

Remark. By Stirling's approximation,

$$\mathbf{P}(L = 2n) \sim \frac{1}{\pi N} \frac{1}{\sqrt{\left(\frac{n}{N}\right)\left(1 - \frac{n}{N}\right)}}.$$

$$\begin{aligned}
\mathbf{P}\left(\frac{L}{2N} \leq x\right) &= \mathbf{P}(L \leq 2Nx) \\
&= \sum_{n=0}^{[2Nx]} \mathbf{P}(L = 2n) \\
&\sim \sum_{n=0}^{[2Nx]} \frac{1}{\pi N} \frac{1}{\sqrt{(x)(1-x)}} \\
&\sim \int_0^x \frac{dy}{pi\sqrt{y(1-y)}} \\
&= \frac{2}{\pi} \sin^{-1}(\sqrt{x}).
\end{aligned}$$

Proof of Theorem 2.2.1. Define $\tilde{\sigma_0} \inf\{n : S_n = 0, 0 < n \leq N\}$. Consider a path of length $2N$ with $L = 2n$. This path can be formed by a path which takes $S_2 n = 0$ and followed by a path of length $2N - 2n$ with $\sigma_0 > 2N - 2n$. Hence, number of paths of length $2N$ with $L = 2n$ is the product of the number of paths of length $2n$ with $S_{2n} = 0$ and the number of paths of length $2N - 2n$ with $\sigma_0 > 2N - 2n$. Hence,

$$\mathbf{P}(L = 2n) = \mathbf{P}(S_{2n} = 0)\mathbf{P}(\tilde{\sigma_0} > 2N - 2n), \tag{2.1}$$

Now let us compute the distribution of $\tilde{\sigma_0}$.

$$\begin{aligned}
\mathbf{P}(\tilde{\sigma_0} > 2k) &= \mathbf{P}(S_1 \neq 0, \ldots, S_{2k} \neq 0) \\
&= 2\mathbf{P}(S_1 > 0, \ldots, S_{2k} > 0) \\
&= \frac{2}{2^{2k}}\{\text{No. of paths start at 0 and stay above -1 for } 2k-1 \text{ steps}\} \\
&= \frac{2}{2^{2k}}\{\text{No. of paths start at 0 and stay below 1 for } 2k-1 \text{ steps}\} \qquad (2.2) \\
&= \mathbf{P}(\sigma_1 > 2k-1) \\
&= 1 - \mathbf{P}(\sigma_1 \geq 2k-1) \\
&= \mathbf{P}(S_{2k-1} = -1) + \mathbf{P}(S_{2k-1} = 0) \\
&= \mathbf{P}(S_{2k-1} = -1)
\end{aligned}$$

Using (2.1) and (2.2),

$$\begin{aligned}\mathbf{P}(L=2n) &= \mathbf{P}(S_{2n}=0)\mathbf{P}(S_{2N-2n-1}=-1)\\ &= \mathbf{P}(S_{2n}=0)\mathbf{P}(S_{2N-2n}=0)\\ &= \frac{1}{2^{2N}}\binom{2n}{n}\binom{2N-2n}{N-n}.\end{aligned}$$

The first step analysis of S_{2n} shows that, $\mathbf{P}(S_{2N-2n}=0) = \frac{1}{2}\mathbf{P}(S_{2N-2n-1}=1) + \frac{1}{2}\mathbf{P}(S_{2N-2n-1}=-1)$. Using the symmetry of the walk we know that $\mathbf{P}(S_{2N-2n-1}=1) = \mathbf{P}(S_{2N-2n-1}=-1)$. This gives the second inequality. $\square$

2.3 SRW of length N in $\mathbb{Z}^d$

2.3.1 Notations and notions in higher dimension

- $e_i \in \mathbb{Z}^d$, $\forall\, i \in \{1, 2, \cdots, d\}$, defined as the vector of length d with all entries zeroes except i^{th} being 1.

$$e_i = (0, 0, \cdots, \underbrace{1}_{i^{th}}, 0, \cdots, 0)$$

- For $x \in \mathbb{Z}^d$,

$$x = \sum_{i=1}^{d} x_i e_i,\ x_i \in \mathbb{Z} \qquad\qquad \|x\| = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}$$

- $\Omega_N = \{(\omega_1, \omega_2, \cdots, \omega_N) \mid \omega_i \in \mathbb{Z}^d, \|\omega_i\| = 1\, \forall\, 1 \leq i \leq N\}$
- We have, for $1 \leq k, n \leq N$

$$X_k : \Omega_N \to \mathbb{Z}^d,\ X_k(\omega) = \omega_k \qquad\qquad S_n : \Omega_N \to \mathbb{Z}^d,\ S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$$

  with $S_0(\omega) = 0$. We can consider S_n as a d-dimensional vector given by $S_n = \left(S_n^{(1)}, S_n^{(2)}, \cdots S_n^{(d)}\right)$, where each $S_n^{(i)}$ is a random walk on $\mathbb{Z}$.

- The probability function $\mathbf{P}^N$, given by,

$$\mathbf{P}^N : \mathcal{P}(\Omega_N) \to [0,1], \quad \mathbf{P}(A) = \frac{|A|}{(2d)^N}\, \forall\, A \subseteq \Omega_N$$

2.3.2 Infinite length random walk

On extending $N \to \infty$, we preserve something called as “consistency”. First, let us define, for $0 < N < M$,

$$\pi_N : \Omega_M \to \Omega_N,\ \ \pi_N(\omega_1, \omega_2, \cdots, \omega_M) = (\omega_1, \omega_2, \cdots, \omega_N)$$

Under $(\Omega_N, \mathcal{P}(\Omega_N), \mathbf{P}^N)$ and $(\Omega_M, \mathcal{P}(\Omega_M), \mathbf{P}^M)$, if we observe the walk till time $n < N$ the probability of evenets concerning the walk should be same under $\mathbf{P}^N$ or $\mathbf{P}^M$. For any event $\{\tilde{\omega} \in \Omega_N\}$, there exists a corresponding same event namely $\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}$. We have,

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \frac{1}{(2d)^N} \qquad \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}) = \frac{(2d)^{M-N}}{(2d)^M} = \frac{1}{(2d)^N}$$

So, we say the sequence of probability spaces $(\Omega_1, \mathbf{P}^1), (\Omega_2, \mathbf{P}^2), \cdots, (\Omega_N, \mathbf{P}^N)$ satisfies the consistency condition

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \frac{1}{(2d)^N} = \frac{(2d)^{M-N}}{(2d)^M} = \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}),\ 0 < N < M,\ \tilde{\omega} \in \Omega_N$$

We define the space of infinite sequences,

$$\Omega_\infty = \{\omega = (\omega_k)k \geq 1 \mid \omega_k \in \mathbb{Z}^d,\ \|\omega_k\| = 1\}$$

$$\mathcal{A}_\infty\ (\equiv \mathcal{P}(\Omega_\infty))\ \text{denotes the class of events observable “for ever”}$$

For $N \in \mathbb{N}$,

$$\pi_N : \Omega_\infty \to \Omega_N,\ \ \pi_N(\omega) = (\omega_1, \omega_2, \cdots, \omega_N)$$

Theorem 2.3.1 (Kolmogorov Consistency Theorem). There exists a unique probability measure on $(\Omega_\infty, \mathcal{A}_\infty)$ such that $\forall\, N \geq 1$, $\forall\, \tilde{\omega} \in \Omega_N$,

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}) = \frac{1}{(2d)^N}$$

Now, we can define,

$$X_k : \Omega_\infty \to \mathbb{Z}^d,\ \ X_k(\omega) = \omega_k \qquad S_n = \sum_{k=1}^{n} X_k\ \forall\, n \geq 1$$

under $\mathbf{P}$, $\{S_n\}_{n\geq 1}$ is a simple random walk starting at $S_0 = 0$.

Definition 2.3.1. $A \subseteq \Omega_\infty$ is said to be **observable** by time n if A is a union of the events of the form

$$\{\omega \in \Omega_\infty : \omega_i = o_i,\ 1 \leq i \leq N\} \text{ with } o_i \in \mathbb{Z}^d,\ \|o_i\| = 1$$

For, $k \in \mathbb{N}_0$, $\mathcal{A}_k$ denotes the set of all events in Ω_∞ observable by time k.

Definition 2.3.2. $T : \Omega_\infty \to \mathbb{N} \cup \{\infty\} \cup \{0\}$ is a **stopping time** if

$$\text{for any }\ k \in \mathbb{N}_0,\ \{T = k\} \in \mathcal{A}_k$$

For example, $\sigma_a = \min\{n \geq 0 \mid S_n = a\}$ is a stopping time.

2.3.3 Speed of the walk

Definition 2.3.3. For, $S_n = \sum_{k=1}^{n} X_k$, we define **speed of the walk** as

$$\text{Speed} \ = \ \frac{S_n}{n} = \frac{1}{n}\sum_{k=1}^{n} X_k$$

We have, $X_k = \left(X_k^{(1)}, X_k^{(2)}, \cdots, X_k^{(d)}\right)$, $\{X_k\}_{k\geq 1}$ which is an i.i.d sequence of random variables with

$$\mathbf{P}(X_k = e_i) = \frac{1}{2d} = \mathbf{P}(X_k = -e_i)$$

$\Rightarrow$ $\mathbf{E}[X_k] = 0$ and $\mathbf{E}[\|X_k\|] = 1$ $(\leq \infty)$

Theorem 2.3.2 (**Strong law of large numbers**)**.** For simple random walk on $\mathbb{Z}^d$,

$$\frac{S_n}{n} \to 0 \text{ with probability 1 under } (\Omega_\infty, \mathcal{A}_\infty, \mathbf{P})$$

2.3.4 Typical position of the walk

For $d = 1$,

$$\frac{S_n - (n)(0)}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$$
$$\Rightarrow \ \sqrt{n}\left(\frac{S_n}{n}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

For $d > 1$, $\mu \in \mathbb{R}^d$ and a positive definite matrix $\Sigma \in \mathbb{R}^{d\times d}$, we have d-dimensional normal distribution as,

$$\Phi_{d,\mu,\Sigma}(y) = \frac{1}{(2\pi)^{d/2}}\frac{1}{\det(\Sigma)^{1/2}}\exp\left(-\frac{1}{2}(x-\mu)^t\Sigma^{-1}(x-\mu)\right)$$

$$\mathbf{P}\left(\frac{S_n}{\sqrt{n}} \in \prod_{i=1}^{d}[a_i, b_i]\right) \xrightarrow[n\to\infty]{} \int\limits_{\prod_{i=1}^{d}[a_i,b_i]} \Phi_{d,0,\Sigma^d}(y)\,dy$$

where, $\mu = 0$, $\Sigma^d = \text{diag}\,(\frac{1}{d}, \cdots, \frac{1}{d})$

2.3.5 Large deviation principle

From the CLT, we have that

$$\mathbf{P}(\|S_n\| > a\sqrt{n}) \xrightarrow[n\to\infty]{} \int\limits_{\|x\|>a} \Phi_{d,0,\Sigma^d}(y)\,dy$$

We consider the events of the form $\{\|S_n\| > an\}$, $a \in [0, \infty)$, which are "rare" in the sense that their probabiility tends to 0 as $n \to \infty$. On formal application of CLT shows that probability of these rare events are exponentially small.

Theorem 2.3.3 (Cramer's theorem). For, $a > 0$,

$$\lim_{n\to\infty} \frac{\log(\mathbf{P}(\|S_n\| > an))}{n} = -I(a)$$

where,

$$I(a) = \begin{cases} \log 2 + \frac{1+a}{2}\log\frac{1+a}{2} + \frac{1-a}{2}\log\frac{1-a}{2}, & \text{for } a \in [-1,1] \\ \infty, & \text{otherwise} \end{cases}$$

It can be vaguely interpreted as, $\mathbf{P}(\|S_n\| > na) \sim e^{-nI(a)}$

2.4 Exercises

1. Complete the proof of Reflection Principle (Lemma 2.1.1).
2. Find the distribution of $M_k = \max_{1\leq k\leq n} S_k$.
3. Show that $\mathbf{E}[\|X_k\|] = 1$.

Week 3 **January 27, 2023**

Random Walks on Graphs

LECTURER: SIVA ATHREYA SCRIBE: ABHITI MISHRA, DEVESH BAJAJ

3.1 Introduction

- A random walk on a graph is basically a reversible Markov chain on the graph.
- many results of random walks will hold true for general markov chains but we will not go into it
- we will study some of the geometric properties of the Graph which translate to different properties of the Random walks

$\Gamma = (V, E)$
$V \equiv$ Vertex set = finite or countably infinite set.
$E \equiv$ Edge set $= E \subset \mathcal{P}(V) = \{\{x, y\} : |x, y \in V, x \neq y\}$.
(No self loops, No multiple edges)

1. $x \in V; y \in V$ is a neighbour of x in $\{x, y\} \in E$ $(x \sim y)$
2. A path $\gamma \in \Gamma$ is any sequence $\{x_i\}_{i=0}^{n}$ such that $x_{i-1} \sim x_i$ in Γ for some $n \geq 1, x_i \in V, 1 \leq i \leq n$
   •γ is a loop if $x_0 = x_n$
   •γ is self avoiding if $x_i \neq x_j$ $\forall$ $i \neq j$.
3. "chemical metric" $d : V \times V \longrightarrow [0, \infty) \bigcup \{\infty\}$
   $d(x, x) = 0$,
   $$d(x, y) = \begin{cases} \text{length of smallest path from x to y} \\ \infty \text{ if no path exixts} \end{cases}$$
4. Γ is connected if $d(x, y) < \infty, \forall x, y \in V$ **(H1 property)**
5. Γ is locally finite if $\forall x \in V$,
   $N(x) = \{y \in V | y \sim x\} \Rightarrow |N(x)| < \infty$ **(H2 property)**
6. we say Γ has a bounded geometry if $\sup_{x \in V} |N(x)| < \infty$ **(H3 property)**

Definition 3.1.1. $\forall$ $x, y \in V$, *we assume that thre is a weight* μ_{xy} *such that:*

1. $\mu_{xy} = \mu_{yx}$
2. $\mu_{xy} \geq 0$
3. *if* $x \neq y$ *then,* $\mu_{xy} > 0 \Leftrightarrow x \sim y$

we will call (Γ, μ) *a weighted graph.*
Using property 3 above, $E = \{\{x, y\} | x, y \in V, \mu_{xy} > 0, x \neq y\}$

Definition 3.1.2. (Γ, μ) *has bounded weights if* $\exists\ C_1, C_2 > 0$ *such that* $C_1 < \mu_{xy} \leq C_2\ \forall\ x, y \in V, x \neq y$. *This is called the* ***(H4 Property)***.

Definition 3.1.3. (Γ, μ) *has controlled weights if* $\exists\ c > 0$ *such that* $\frac{\mu_{xy}}{\mu_x} \geq c^{-1}\ \forall\ x, y \in V, x \neq y$. *This is called the* ***(H5 Property)***.

Define for $x \in V$: $\mu_x = \sum_{y \sim x} \mu_{xy}$

Definition 3.1.4. *Natural weights:*

$$\mu_{xy} = \begin{cases} 1 \text{ if } x \sim y \\ 0 \text{ otherwise} \end{cases}$$

Lemma 3.1.1. *Suppose* (Γ, μ) *is a weighted graph then,*

1. *(H3), (H5) holds.*

2. $\forall x \in V, n > 0\ , B(x, n) = \{y \in V | d(x, y) \leq n\}$ *(balls are not exponentially large)*

3. $\forall x \in V, n \geq 0, \mu(B(x, n)) = \sum_{y \in B(x,n)} \mu_y \leq 2\mu_x(c_2)^n$ *(Balls have bounded weights)*

Proof. 1. Take $x \in V$.

$$\begin{aligned} N(x) &= c \sum_{y \in V} \frac{1}{c} 1_{\{x \sim y\}} \\ &\leq c \sum_{y \in V} \frac{\mu_{xy}}{\mu_x} 1_{\{x \sim y\}} \\ &= c \frac{1}{\mu_x} \sum_{y \in V} \mu_{xy} = c \end{aligned}$$

2. $S(x, n) = \{y \in V | d(x, y) = n\}$

$$|S(x, n)| \leq c |S(x, n-1)|\ \ \forall\ n \geq 1$$

Arguing inductively,

$$\begin{aligned}|B(x,n)| &= \sum_{k=0}^{n} |S(x,k)| \\ &\leq \sum_{k=0}^{n} c^k \\ &= \frac{c^{n+1}-1}{c-1} \leq 2c^n\end{aligned}$$

3. $n = 1$.

$$\begin{aligned}\mu(B(x,1)) &= \mu_x + \sum_{y\sim x} \mu_y \\ &\leq c\sum_{y\sim x} \mu_{xy} + \mu_x \\ &= c\mu_x + \mu_x\end{aligned}$$

Second step follows from the H5 assumption.
We also note

$$\mu(B(x,2)) = \sum_{y\in B(x,2)} \mu_y = \mu(B(x,1)) + \sum_{y\sim x}\sum_{z\sim y} \mu_z$$

Therefore

$$\begin{aligned}\mu(B(x,2)) &\leq \mu_x + c\mu_x + \sum_{y\sim x} c \sum_{z\sim y} \mu_{zy} \\ &= \mu_x + c\mu_x c \sum_{y\sim x} \mu_y \\ &\leq \mu_x + c\mu_x + c^2\mu_x\end{aligned}$$

$\square$

Example. $V = \mathbb{Z}^d$. Take $x, y \in V, |x-y| = \sum_{i=1}^{d} |x_i - y_i|$
$E = \{(x,y) | |x-y| = 1\}$. $\mu_{xy} = 1$ whenever $(x,y) \in E$. $N(x) = 2d \ \ \forall x \in V$
$|B(x,n)| \sim n^d \leq 2c^n \ \ \forall c \geq 2$.
Example. Rooted Binary Tree- Let the root be $B_0 = \{\rho\}$.
$\forall\ n \geq 1, B_n = \{0,1\}^n$

$$V = \cup_{n=1}^{\infty} B_n \cup \{\rho\}$$

For $x \in B_n, n \geq 2, x = (x_1, \ldots, x_n), x_i \in \{0,1\}$.
Let the parent of x be- $\alpha(x) = (x_1, \ldots, x_{n-1})$
For $n = 1, x \in B_1, \alpha(x) = \rho$

$$E = \{(x, \alpha(x)) |\ x \in V, x \notin B_0\}$$

$$|N(\rho)| = 2, |N(x)| = 3 \ \ \forall\ x \notin B_0$$

Canopy Tree

$$\bar{V} = \{x \in V | x = (x_1, \ldots, x_n) \text{ and } x_i = 0 \ \ \forall\, 1 \leq i \leq n \text{ forsome } n \geq 1\} \cup \{\rho\}$$

$f(x)$ is the element in $\bar{V}$ closest to x.
V_{canopy} is a subset of V such that-

$$V_{canopy} = \{x \in V |\ d(x, f(x)) \leq d(\rho, f(x))\}$$

Observe that in the canopy tree, there is only one self-avoiding path to infinity, but the size of the balls $B(\rho, n)$ still grows exponentially. It shows that one does not need too many paths to infinity for the size of your graph to grow exponentially. Denoted by $\mathbb{T}^2_{canopy}$

3.2 Random Walks on Weighted Graphs

(This section will be done as a discrete time reversible Markov Chain)
Formally, X_n jumps from $x \sim y_i$ with probability proportional to μ_{xy_i}. It stays at x with probability proportional to μ_{xx}.

Our graph is denoted by $\Gamma = (V, E)$. We assume there are no isolated edges that is $\{\mu_x \neq 0 \ \ \forall x \in V\}$. Also assume H(1) and H(2).

$$\Omega = \{f : \mathbb{N} \cup \{0\} \to V\} \equiv V^{\mathbb{N} \cup \{0\}}$$

$\forall\ n \geq 0$, $X_n : \Omega \to V$ where $X_n(\omega) = \omega(n)$
Let $\mathcal{A}_n \equiv$ observable events upto time n (all events that can be derived from $X_1, \ldots, X_n$). This will be a filtration.

$$\mathcal{F} \equiv \cup_{n \geq 1}\ \mathcal{A}_n$$

Set $\mathcal{P}(x, y) = \frac{\mu_{xy}}{\mu_x}\ \ \forall x, y \in V$.
$\forall x\ \in V$, there exists a unique $\mathcal{P}^x(.)$ on $(\Omega, \mathcal{F})$.
(Existence can be shown using Kolmogorov consistency theorem).
$\forall\ n \geq 1$

$$\mathbb{P}^x(X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = 1_{\{x\}}(x_0) \prod_{i=1}^{n} P(x_n, x_{n-1})$$

$$\begin{aligned}
\mathbb{P}^x(X_1 = y) &= \mathbb{P}^x(X_1 = y, \cup_{z \in V} X_0 = z) \\
&= \sum_{z \in V} \mathbb{P}^x(X_1 = y, X_0 = z) \\
&= \sum_{z \in V} \mathcal{P}(y, z) 1_{\{x\}}(z) \\
&= \mathcal{P}(y, x)
\end{aligned}$$

One-step transition probability-

$$\mathbb{P}(X_n = y | X_{n-1} = z) = \frac{\mathbb{P}(X_n = y, X_{n-1} = z)}{\mathbb{P}(X_{n-1} = z)} = \mathcal{P}(y, z)$$

The last equality is left as an exercise.
Reversibility-

$$\mu_x \mathcal{P}(x, y) = \mu_x \frac{\mu_{xy}}{\mu_x} = \mu yx = \mu_y \mathcal{P}(y, x)$$

$(X_n, \mathcal{P})$ markov chain is symmetric with repsect to $\{\mu_x\}_{x \in V}$

Lemma 3.2.1. *Let $x_0, \ldots, x_n \in V$*

$$\mu_{x_0} \mathbb{P}^{x_0}(X_n = x_n, \ldots, X_0 = x_0) = \mu_{x_n} \mathbb{P}^{x_n}(X_n = x_0, \ldots, X_0 = x_n)$$

The above shows the reversibility of the markov chain wrt μ.

Proof.

$$\begin{aligned}
\mu_{x_0} \mathbb{P}^{x_0}(X_n = x_n, \ldots, X_0 = x_0) &= \mu_{x_0} \prod_{i=1}^{n} \mathcal{P}(x_i, x_{i-1}) \\
&= \mu_{x_0} \prod_{i=1}^{n} \frac{\mu_{x_i, x_{i-1}}}{\mu_{x_{i-1}}} \\
&= \mu_{x_n} \prod_{i=1}^{n} \frac{\mu_{x_{n-i}, x_{n-i+1}}}{\mu_{x_{n-i+1}}} \\
&= \mu_{x_n} \mathbb{P}^{x_n}(X_n = x_0, \ldots, X_0 = x_n)
\end{aligned}$$

□

Remark. If $\mu(V) = \sum_{x \in V} \mu_x = 1$ and $\mu(A) = \sum_{x \in A}$, then μ is the reversible distribution for $\{X_n\}_{n \geq 0}$ that is

$$\mu_x \mathcal{P}(x, y) = \mu_y \mathcal{P}(y, x)$$

Hence $\{\mu_x\}_{x \in V}$ is the stationary distribution.

Definition 3.2.1. *$A \subseteq V$. The hitting time of A be given by*

$$T_A = \min\{n \geq 0 | X_n \in A\}$$

By convention, $T_A = \infty$ iff X_n does not visit A.

Definition 3.2.2. *The return time of A is defined as -*

$$T_A^+ = \min\{n \geq 1 | X_n \in A\}$$

Note that $X_0 \notin A \implies T_A^+ = T_A$

Definition 3.2.3. *The exit time of A is-*

$$\tau_A = T_{A^c}$$

Theorem 3.2.1. *Let Γ be H(1) and H(2) and $|V| = \infty$. Then TFAE-*

1. $\exists\ x \in V$ *such that* $\mathbb{P}^x(\tau_x^+ < \infty) < 1$

2. $\forall\ x \in V$, $\mathbb{P}^x(\tau_x^+ < \infty) < 1$

3. $\forall\ x \in V$, $\sum_{n=0}^{\infty} \mathbb{P}^x(X_n = x) < \infty$

4. $\forall\ x, y \in V$, $\mathbb{P}^x(\tau_y < \infty) < 1$

5. $\mathbb{P}^x(\sum_{n\geq 0} 1_{\{X_n = x\}} < \infty) = 1 \quad \forall x, y \in V$

If the above is satisfied, the Markov Chain is transient.

Theorem 3.2.2. *Let Γ be H(1) and H(2) and $|V| = \infty$. Then TFAE-*

1. $\exists\ x \in V$ *such that* $\mathbb{P}^x(\tau_x^+ < \infty) = 1$

2. $\forall\ x \in V$, $\mathbb{P}^x(\tau_x^+ < \infty) = 1$

3. $\forall\ x \in V$, $\sum_{n=0}^{\infty} \mathbb{P}^x(X_n = x) = \infty$

4. $\forall\ x, y \in V$, $\mathbb{P}^x(\tau_y < \infty) = 1$

5. $\mathbb{P}^x(\sum_{n\geq 0} 1_{\{X_n = x\}} = \infty) = 1 \quad \forall x, y \in V$

If the above is satisfied, the Markov Chain is recurrent.

Definition 3.2.4. *If $\{X_n\}_{n\geq 0}$ random walk on (Γ, μ) satisfies*

1. *any statement of theorem 1.6, the graph (Γ, μ) is transient.*

2. *any statement of theorem 1.7, the graph (Γ, μ) is recurrent.*

3.3 Exercises

1. Show that $H_3, H_4 \Rightarrow H_5$

2. When is (Γ, μ) transient or recurrent?
   Partial answer- When $|V| < \infty$, (Γ, μ) is recurrent.

3. **Kesten Problem-** G is a finitely generated group with generating set A. Look at the Cayley graph of G. Which groups provide transient graphs?

Week 4 **January 27, 2023**

Discrete Time Martingales

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Origin is from horse-racing (betting system). The dictionary meaning of the word 'martingale' is the harness of a horse.
Let $\{Z_n\}_{n\geq 1}$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 4.0.1. *A sequence of random variables $\{Z_n\}_{n\geq 1}$ is said to be a Martingale if*

$$\mathbb{E}(Z_n | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) = z_{n-1} \quad \forall\ n \geq 2 \tag{4.3}$$

Things to understand- conditional expectation for discrete and conditional r.v. Reference- Ch6 of Siva's book.
Things we will explore-

1. Examples of $\{Z_n\}_{n\geq 1}$ that are martingales.

2. How different are martingales from iid sequences and markov chains?

3. How to interpret 4.3?

Example. $\{S_n\}_{n\geq 1}$ and $S_0 \equiv 0$.

$$X_i = \begin{cases} 1, & w.p\ \ 1/2 \\ -1, & w.p\ \ 1/2 \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i$$

Let $s_{n-1}, s_{n-2}, \ldots, s_1 \in \mathbb{Z}$ such that $\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) > 0$

$$\begin{aligned}
\mathbb{E}(S_n | S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) &= \sum_{k \in \mathbb{Z}} k \mathbb{P}(S_n = k | S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(S_n = k, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(S_{n-1} + X_n = k, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(X_n = k - s_{n-1}, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(X_n = k - s_{n-1}) \mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= (s_{n-1} + 1) \mathbb{P}(X_n = -1) + (s_{n-1} - 1) \mathbb{P}(X_n = 1) \\
&= (s_{n-1} + 1) \frac{1}{2} + (s_{n-1} - 1) \frac{1}{2} = s_{n-1}
\end{aligned}$$

Note that the summations here are "finite" sums.
As $s_{n-1}, \ldots, s_1 \in \mathbb{Z}$ were arbitrary, $\{S_n\}_{n \geq 1}$ is a martingale.

Example. $\{X_i\}_{i \geq 1}$ be an iid sequence on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z_n = \prod_{i=1}^n X_i$ and $\text{Range}(Z_n) \subset \mathbb{R} \ \forall \ n \geq 1$.
Let $z_{n-1}, \ldots, z_1 \in \mathbb{R}$ such that $\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) > 0$. Then

$$\begin{aligned}
\mathbb{E}(Z_n | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) &= \sum_{k \in Range(Z_n)} k \mathbb{P}(Z_n = k | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(Z_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(Z_{n-1} X_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(z_{n-1} X_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \mathbb{P}(Z_{n-1} X_n = k) \frac{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{u \in S^1, Range(X_n) = S^1} u z_{n-1} \mathbb{P}(X_n = u) \\
&= z_{n-1} \mathbb{E}[X_n] = z_{n-1}
\end{aligned}$$

Note that the sums here might be infinite. In the last step we assume $\mathbb{E}[X_i] = 1$. Now since $\{z_i\}_{i=1}^{n-1}$ were arbitrary, $\{Z_n\}_{n\geq 1}$ is a martingale.

Example.

$$X_i = \begin{cases} 2, & w.p \;\; 1/2 \\ 0, & w.p \;\; 1/2 \end{cases}$$

Then $\mathbb{E}(X_i) = 1$. Therefore, $Z_n = \prod_{i=1}^{n} X_i$ is a martingale. Range (Z_n)= $\{2^n, 0\}$. Note that the mean stays constant and

$$\mathbb{P}(Z_n = 0) = 1 - \frac{1}{2^n}$$

$$\mathbb{P}(Z_n = 2^n) = \frac{1}{2^n}$$

Intuition- The first equation shows that the martingale takes a very low value with very high probability and the second one shows that it takes a very large value with very low probability
Idea behind Markov Chains -

$$\text{“}X_n | X_{n-1}, \ldots, X_1\text{”} \stackrel{d}{=} X_n | X_{n-1}$$

Idea behind Martingales - Expected value of Z_n conditioned on the past depends only on Z_{n-1}. $\{Z_n\}_{n\geq 1}$ in law could depend on the entire past!