Week 2

More on random walks

LECTURER: SIVA ATHREYA SCRIBE: SANCHAYAN BHOWAL, VENKAT TRIVIKRAM

Theorem 2.0.1. *Let $T : \Omega \to 0, 1, \ldots, N$ be a stopping time. Then,*

$$\mathbf{E}[S_T^2] = E[T].$$

Proof.

$$\begin{aligned} S_T^2 &= \sum_{k=1}^{N} S_k^2 \mathbb{1}\{T = k\} \\ &= \sum_{k=1}^{N} (S_k^2 - S_{k-1}^2) \mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} (X_k + S_{k-1})^2 - S_{k-1}^2 \mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} (1 + 2X_k S_{k-1}) \mathbb{1}\{T \geq k\}. \end{aligned}$$

Now, consider $V_k = S_{k-1}\mathbb{1}\{T \geq k\}$. Note that this is a bet sequence. Hence,

$$\begin{aligned} \mathbf{E}[S_T^2] &= \mathbf{E}\left[\sum_{k=1}^{N} \mathbb{1}\{T \geq k\}\right] + 2\sum_{k=1}^{N} \mathbf{E}[X_k V_k] \\ &= \sum_{k=1}^{N} \mathbf{P}(T \geq k) + 0 \\ &= E[T]. \end{aligned}$$

□

2.1 Reflection Principle

Assume that $a \in \mathbb{Z}$ and $c > 0$. There is a bijection between the paths that cross $a + c$ and those that do not. This bijection is obtained by reflecting the part of the path crossing $a + c$ as shown in the Figure **??**. So,

$$|S_n = a + c| = |\sigma_a \leq n \;\&\; S_n = a + c| = |\sigma_a \leq n \& S_n = a - c|$$

Now, we know that all the paths have equal probability. Hence, we get the following lemma.

Lemma 2.1.1. $\mathbf{P}(S_n = a + c) = \mathbf{P}(\sigma_a \le n \;\&\; S_n = a - c)$ *where* $a \in \mathbb{Z}$ *and* $c > 0$. *This is also known as the* reflection principle.

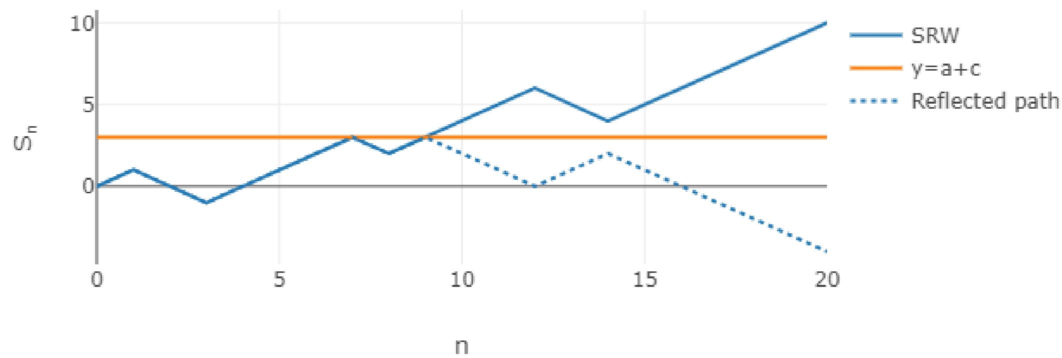

Figure 2.1: The figure shows that the bijection between the paths that cross a+c=3 and those that do not.

Theorem 2.1.1. $\mathbf{P}(\sigma_a \le n) = \mathbf{P}(S_n \notin [-a, a))$ *where* $a \in \mathbb{Z}$
$\{0\}$.

Proof.

$$\begin{aligned}
\mathbf{P}(\sigma_a \le n) &= \mathbf{P}(\sigma_a \le n, \bigcup_{b \in \mathbb{Z}} S_n = b) \\
&= \sum_{b \in \mathbb{Z}} \mathbf{P}(\sigma_a \le n, S_n = b) \\
&= \sum_{b \in \mathbb{Z}, b \ge a} \mathbf{P}(\sigma_a \le n, S_n = b) + \sum_{b \in \mathbb{Z}, b < a} \mathbf{P}(\sigma_a \le n, S_n = b) \\
&= \sum_{b \in \mathbb{Z}, b \ge a} \mathbf{P}(S_n = b) + \sum_{b \in \mathbb{Z}, b < a} \mathbf{P}(S_n = 2a - b) \\
&= \mathbf{P}(S_n \ge a) + \mathbf{P}(S_n > a) \\
&= \mathbf{P}(S_n \ge a) + \mathbf{P}(S_n < -a) \\
&= \mathbf{P}(S_n \notin [-a, a))
\end{aligned}$$

□

Corollary 2.1.1. $\mathbf{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbf{P}(S_{n-1} = a - 1) - \mathbf{P}(S_{n-1} = a + 1)\right]$ *where* $a \in \mathbb{Z}$.

Proof. □

2.2 Arc-Sine Law

Let L denote the last time the random walk hits 0, i.e., $L = \max_{0 \le n \le 2N} S_n = 0$, where N denotes the length of the walk.

Theorem 2.2.1.

$$\mathbf{P}(L = 2n) = \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}.$$

Remark. By Stirling's approximation,

$$\mathbf{P}(L = 2n) \sim \frac{1}{\pi N} \frac{1}{\sqrt{\left(\frac{n}{N}\right)\left(1 - \frac{n}{N}\right)}}.$$

$$\begin{aligned}
\mathbf{P}\left(\frac{L}{2N} \le x\right) &= \mathbf{P}(L \le 2Nx) \\
&= \sum_{n=0}^{[2Nx]} \mathbf{P}(L = 2n) \\
&\sim \sum_{n=0}^{[2Nx]} \frac{1}{\pi N} \frac{1}{\sqrt{(x)(1-x)}} \\
&\sim \int_0^x \frac{dy}{pi\sqrt{y(1-y)}} \\
&= \frac{2}{\pi} \sin^{-1}(\sqrt{x}).
\end{aligned}$$

Proof of Theorem **??**. Define $\tilde{\sigma_0} \inf\{n : S_n = 0, 0 < n \le N\}$. Consider a path of length $2N$ with $L = 2n$. This path can be formed by a path which takes $S_2 n = 0$ and followed by a path of length $2N - 2n$ with $\sigma_0 > 2N - 2n$. Hence, number of paths of length $2N$ with $L = 2n$ is the product of the number of paths of length $2n$ with $S_{2n} = 0$ and the number of paths of length $2N - 2n$ with $\sigma_0 > 2N - 2n$. Hence,

$$\mathbf{P}(L = 2n) = \mathbf{P}(S_{2n} = 0)\mathbf{P}(\tilde{\sigma_0} > 2N - 2n), \tag{2.1}$$

Now let us compute the distribution of $\tilde{\sigma_0}$.

$$\begin{aligned}
\mathbf{P}(\tilde{\sigma_0} > 2k) &= \mathbf{P}(S_1 \ne 0, \ldots, S_{2k} \ne 0) \\
&= 2\mathbf{P}(S_1 > 0, \ldots, S_{2k} > 0) \\
&= \frac{2}{2^{2k}} \{\text{No. of paths start at 0 and stay above -1 for } 2k-1 \text{ steps}\} \\
&= \frac{2}{2^{2k}} \{\text{No. of paths start at 0 and stay below 1 for } 2k-1 \text{ steps}\} \qquad (2.2) \\
&= \mathbf{P}(\sigma_1 > 2k - 1) \\
&= 1 - \mathbf{P}(\sigma_1 \ge 2k - 1) \\
&= \mathbf{P}(S_{2k-1} = -1) + \mathbf{P}(S_{2k-1} = 0) \\
&= \mathbf{P}(S_{2k-1} = -1)
\end{aligned}$$

Using (**??**) and (**??**),

$$\begin{aligned}\mathbf{P}(L=2n) &= \mathbf{P}(S_{2n}=0)\mathbf{P}(S_{2N-2n-1}=-1)\\ &= \mathbf{P}(S_{2n}=0)\mathbf{P}(S_{2N-2n}=0)\\ &= \frac{1}{2^{2N}}\binom{2n}{n}\binom{2N-2n}{N-n}.\end{aligned}$$

The first step analysis of S_{2n} shows that, $\mathbf{P}(S_{2N-2n}=0)=\frac{1}{2}\mathbf{P}(S_{2N-2n-1}=1)+\frac{1}{2}\mathbf{P}(S_{2N-2n-1}=-1)$. Using the symmetry of the walk we know that $\mathbf{P}(S_{2N-2n-1}=1)=\mathbf{P}(S_{2N-2n-1}=-1)$. This gives the second inequality. □

2.3 SRW of length N in $\mathbb{Z}^d$

2.3.1 Notations and notions in higher dimension

- $e_i \in \mathbb{Z}^d$, $\forall\, i \in \{1,2,\cdots,d\}$, defined as the vector of length d with all entries zeroes except i^{th} being 1.

$$e_i = (0,0,\cdots,\underbrace{1}_{i^{th}},0,\cdots,0)$$

- For $x \in \mathbb{Z}^d$,

$$x=\sum_{i=1}^{d} x_i e_i,\; x_i \in \mathbb{Z} \qquad\qquad \|x\| = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}$$

- $\Omega_N = \{(\omega_1,\omega_2,\cdots,\omega_N) \mid \omega_i \in \mathbb{Z}^d, \|\omega_i\|=1\,\forall\, 1\le i\le N\}$
- We have, for $1 \le k,n \le N$

$$X_k:\Omega_N\to\mathbb{Z}^d,\; X_k(\omega)=\omega_k \qquad\qquad S_n:\Omega_N\to\mathbb{Z}^d,\; S_n(\omega)=\sum_{k=1}^{n} X_k(\omega)$$

  with $S_0(\omega)=0$. We can consider S_n as a d-dimensional vector given by $S_n=\left(S_n^{(1)},S_n^{(2)},\cdots S_n^{(d)}\right)$, where each $S_n^{(i)}$ is a random walk on $\mathbb{Z}$.

- The probability function $\mathbf{P}^N$, given by,

$$\mathbf{P}^N:\mathcal{P}(\Omega_N)\to[0,1], \quad \mathbf{P}(A)=\frac{|A|}{(2d)^N}\,\forall\, A\subseteq\Omega_N$$

2.3.2 Infinite length random walk

On extending $N\to\infty$, we preserve something called as "consistency". First, let us define, for $0<N<M$,

$$\pi_N:\Omega_M\to\Omega_N,\;\; \pi_N(\omega_1,\omega_2,\cdots,\omega_M)=(\omega_1,\omega_2,\cdots,\omega_N)$$

Under $(\Omega_N, \mathcal{P}(\Omega_N), \mathbf{P}^N)$ and $(\Omega_M, \mathcal{P}(\Omega_M), \mathbf{P}^M)$, if we observe the walk till time $n < N$ the probability of evenets concerning the walk should be same under $\mathbf{P}^N$ or $\mathbf{P}^M$. For any event $\{\tilde{\omega} \in \Omega_N\}$, there exists a corresponding same event namely $\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}$. We have,

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \frac{1}{(2d)^N} \qquad \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}) = \frac{(2d)^{M-N}}{(2d)^M} = \frac{1}{(2d)^N}$$

So, we say the sequence of probability spaces $(\Omega_1, \mathbf{P}^1), (\Omega_2, \mathbf{P}^2), \cdots, (\Omega_N, \mathbf{P}^N)$ satisfies the consistency condition

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \frac{1}{(2d)^N} = \frac{(2d)^{M-N}}{(2d)^M} = \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}),\ 0 < N < M,\ \tilde{\omega} \in \Omega_N$$

We define the space of infinite sequences,

$$\Omega_\infty = \{\omega = (\omega_k)k \geq 1 \mid \omega_k \in \mathbb{Z}^d,\ \|\omega_k\| = 1\}$$

$$\mathcal{A}_\infty\,(\equiv \mathcal{P}(\Omega_\infty))\text{ denotes the class of events observable “for ever”}$$

For $N \in \mathbb{N}$,

$$\pi_N : \Omega_\infty \to \Omega_N,\ \ \pi_N(\omega) = (\omega_1, \omega_2, \cdots, \omega_N)$$

Theorem 2.3.1 (**Kolmogorov Consistency Theorem**)**.** There exists a unique probability measure on $(\Omega_\infty, \mathcal{A}_\infty)$ such that $\forall\, N \geq 1$, $\forall\, \tilde{\omega} \in \Omega_N$,

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}) = \frac{1}{(2d)^N}$$

Now, we can define,

$$X_k : \Omega_\infty \to \mathbb{Z}^d,\ X_k(\omega) = \omega_k \qquad S_n = \sum_{k=1}^{n} X_k\ \forall\, n \geq 1$$

under $\mathbf{P}$, $\{S_n\}_{n\geq 1}$ is a simple random walk starting at $S_0 = 0$.

Definition 2.3.1. $A \subseteq \Omega_\infty$ is said to be **observable** by time n if A is a union of the events of the form

$$\{\omega \in \Omega_\infty : \omega_i = o_i,\ 1 \leq i \leq N\} \text{ with } o_i \in \mathbb{Z}^d,\ \|o_i\| = 1$$

For, $k \in \mathbb{N}_0$, $\mathcal{A}_k$ denotes the set of all events in Ω_∞ observable by time k.

Definition 2.3.2. $T : \Omega_\infty \to \mathbb{N} \cup \{\infty\} \cup \{0\}$ is a **stopping time** if

$$\text{for any }\ k \in \mathbb{N}_0,\ \{T = k\} \in \mathcal{A}_k$$

For example, $\sigma_a = \min\{n \geq 0 \mid S_n = a\}$ is a stopping time.

2.3.3 Speed of the walk

Definition 2.3.3. For, $S_n = \sum_{k=1}^{n} X_k$, we define **speed of the walk** as

$$\text{Speed} \;=\; \frac{S_n}{n} = \frac{1}{n}\sum_{k=1}^{n} X_k$$

We have, $X_k = \left(X_k^{(1)}, X_k^{(2)}, \cdots, X_k^{(d)}\right)$, $\{X_k\}_{k\geq 1}$ which is an i.i.d sequence of random variables with

$$\mathbf{P}(X_k = e_i) = \frac{1}{2d} = \mathbf{P}(X_k = -e_i)$$

$\Rightarrow$ $\mathbf{E}[X_k] = 0$ and $\mathbf{E}[\|X_k\|] = 1$ $(\leq \infty)$

Theorem 2.3.2 (**Strong law of large numbers**)**.** For simple random walk on $\mathbb{Z}^d$,

$$\frac{S_n}{n} \to 0 \text{ with probability 1 under } (\Omega_\infty, \mathcal{A}_\infty, \mathbf{P})$$

2.3.4 Typical position of the walk

For $d = 1$,

$$\frac{S_n - (n)(0)}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$$
$$\Rightarrow \quad \sqrt{n}\left(\frac{S_n}{n}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

For $d > 1$, $\mu \in \mathbb{R}^d$ and a positive definite matrix $\Sigma \in \mathbb{R}^{d\times d}$, we have d-dimensional normal distribution as,

$$\Phi_{d,\mu,\Sigma}(y) = \frac{1}{(2\pi)^{d/2}}\frac{1}{\det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^t\Sigma^{-1}(x-\mu)\right)$$

$$\mathbf{P}\left(\frac{S_n}{\sqrt{n}} \in \prod_{i=1}^{d}[a_i, b_i]\right) \xrightarrow[n\to\infty]{} \int\limits_{\prod_{i=1}^{d}[a_i,b_i]} \Phi_{d,0,\Sigma^d}(y)\,dy$$

where, $\mu = 0$, $\Sigma^d = \operatorname{diag}\left(\frac{1}{d}, \cdots, \frac{1}{d}\right)$

2.3.5 Large deviation principle

From the CLT, we have that

$$\mathbf{P}(\|S_n\| > a\sqrt{n}) \xrightarrow[n\to\infty]{} \int\limits_{\|x\|>a} \Phi_{d,0,\Sigma^d}(y)\,dy$$

We consider the events of the form $\{\|S_n\| > an\}$, $a \in [0, \infty)$, which are "rare" in the sense that their probabiility tends to 0 as $n \to \infty$. On formal application of CLT shows that probability of these rare events are exponentially small.

Theorem 2.3.3 (Cramer's theorem). For, $a > 0$,

$$\lim_{n\to\infty} \frac{\log(\mathbf{P}(\|S_n\| > an))}{n} = -I(a)$$

where,

$$I(a) = \begin{cases} \log 2 + \frac{1+a}{2}\log\frac{1+a}{2} + \frac{1-a}{2}\log\frac{1-a}{2}, & \text{for } a \in [-1, 1] \\ \infty, & \text{otherwise} \end{cases}$$

It can be vaguely interpreted as, $\mathbf{P}(\|S_n\| > na) \sim e^{-nI(a)}$

2.4 Exercises

1. Complete the proof of Reflection Principle (Lemma **??**).
2. Find the distribution of $M_k = \max_{1\leq k\leq n} S_k$.
3. Show that $\mathbf{E}[\|X_k\|] = 1$.