Week 7

Isoperimetric Inequalities and Applications

LECTURER: SIVA ATHREYA SCRIBE: ATREYA CHOUDHURY

The focus of this chapter is to look at how the geometry of weighted graph affects the properties of the corresponding random walk on it.

Definition 7.0.1 (Isoperimetric Inequality). *Let* $A, B \subseteq V$, $\mu_E(A,B) = \sum_{x\in A}\sum_{y\in B}\mu_{xy}$ *and* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be an increasing function.*

$(\Gamma,\ \mu)$ *is said to satisfy the* $\psi-$*isoperimetric inequality if* $\exists\, c_0 > 0$ *such that*

$$\frac{\mu_E(A, V \smallsetminus A)}{\psi(\mu(A))} \geq \frac{1}{c_0} \qquad \forall A \subseteq V \text{ and } |A| < \infty$$

If a weighted graph satisfies the $\psi-$*isoperimetric inequality, we say it has the* I_ψ *property.*

A graph is said to have the property I_α *for* $\alpha \in [0,\infty)$ *when* $\psi(t) = t^{1-\frac{1}{\alpha}}$ *and said to have the property* I_∞ *when* $\psi(t) = t$

Example. $\mathbb{R}^d$. We look at $A = B(0,r)$

$$S_B \equiv \text{surface area of A} = c_d r^{d-1}$$
$$V_B \equiv \text{volume of A} = \widetilde{c_d} r^d$$
$$\therefore \frac{S_B}{V_B^{\frac{d-1}{d}}} \geq \frac{1}{c_0}$$

We can take $\psi(t) = t^{1-\frac{1}{d}}$

Show that Z^d has the I_d property for all such A such that $|A| < \infty$

Example. $\prod_2$, the binary tree has the I_∞ property with $c_0 = 3$

Observations. *If* $(\Gamma,\ \mu)$ *satisfies* $I_{\alpha+\delta}$*, then it satisfies* I_α

Definition 7.0.2 (Nash Inequality). $\alpha \in [1,\infty)$, $(\Gamma,\ \mu)$ *is said to have the property* N_α *if* $\forall f \in \mathbb{L}^1(V) \cap \mathbb{L}^2(V)$,

$$\mathcal{E}(f,f) \geq C_N \|f\|_1^{-\frac{4}{\alpha}} \|f\|_2^{2+\frac{4}{\alpha}}$$

Remark. 1. $(\Gamma,\ \mu)$ satisfies I_α for $\alpha \in [1, \infty) \implies (\Gamma,\ \mu)$ satisfies N_α

2. Z^d satisfies N_α

Theorem 7.0.1. *Let $\alpha \geq 1$. Then the following are equivalent*

1. $(\Gamma,\ \mu)$ *satisfies* N_α
2. $\exists\, C_H > 0$ *such that*

$$p_n(x,x) \leq \frac{C_H}{(n \vee 1)^{\frac{\alpha}{2}}} \qquad \forall\, n \geq 0 \text{ and } x \in V$$

3. $\exists\, C'_H > 0$ *such that*

$$p_n(x,y) \leq \frac{C'_H}{(n \vee 1)^{\frac{\alpha}{2}}} \qquad \forall\, n \geq 0 \text{ and } x, y \in V$$

Corollary 7.0.1. 1. *Suppose $(\Gamma,\ \mu)$ satisfies I_α. Then, $\exists\, C > 0$ such that*

$$p_n(x,y) \leq \frac{C}{(n \vee 1)^{\frac{\alpha}{2}}} \qquad \forall\, n \geq 0 \text{ and } x, y \in V$$

2. *Let V be infinite and $\mu_{xy} \geq c_0 > 0\ \ \forall\, x \sim y$. Then, $\exists\, C_1 > 0$ such that*

$$p_n(x,y) \leq \frac{C_1}{(n \vee 1)^{\frac{1}{2}}} \qquad \forall\, n \geq 0 \text{ and } x, y \in V$$

Remark. 1. $p_n(x,x) \equiv$ on-diagonal bounds

2. Theorem provides global upper bounds
3. part b of corollary 7.0.1 applied to $V = \mathbb{Z}$
   $\implies$ the shortest possible on-diagonal upper bounds with natural weights
4. Let $\Gamma = \mathbb{Z}^d$ have natural weights $\mu^{(0)}_{xy}$ and $\Gamma' = \mathbb{Z}^d$ have natural weights $\mu^{(1)}_{xy}$ such that $\mu^{(1)}_{xy} \geq c_0 \mu^{(0)}_{xy}$ Let $(\Gamma,\ \mu^0)$ satisfy N_d
   $\implies (\Gamma',\ \mu^1)$ satisfies N_d
   $\implies$ the upper bound of the theorem holds
5. $\Gamma = \mathbb{Z}^d \cup_{(0,\ldots,0)} \mathbb{Z}^d$
   $\implies \Gamma$ also satisfies N^d
6. 7.0.1 does not give us any information on upper bounds when we fix $n \geq 0$ and let $d(x,y)$ get large.

Theorem 7.0.2. *Let $(\Gamma,\ \mu)$ be a weighted graph. Then,*

$$p_n(x,y) \leq \frac{2}{\sqrt{\mu_x \mu_y}} e^{-\frac{d(x,y)^2}{2n}} \qquad \forall\, x, y \in V \text{ and } n \geq 1$$

Example. Consequences for $\mathbb{Z}^d$
We expect

$$p_n(x,y) \leq \frac{c_1}{n^{\frac{d}{2}}} e^{-c_2 \frac{d(x,y)^2}{n}}$$

$\mathbb{Z}^d$ satisfies $I_d \implies \mathbb{Z}^d$ satisfies $N_d \overset{7.0.1}{\implies} p_n(x,y) \leq \frac{c}{n^{\frac{d}{2}}} \qquad \forall\, x,y \in V$ and $n \geq 1$

$$\therefore\ p_n(x,y) \leq \frac{c}{n^{\frac{d}{2}}} \leq \frac{c}{n^{\frac{d}{2}}} e^{-\frac{d(x,y)^2}{n}} \qquad \text{when } d(x,y) \leq \sqrt{n}$$

When, $d(x,y) \geq \sqrt{2dn \log n}$,

$$p_n(x,y) \leq c_1 e^{-\frac{d(x,y)^2}{n}} = c_1 e^{-\frac{2c_2}{4}\frac{d(x,y)^2}{n}} e^{-\frac{2c_2}{4}\frac{d(x,y)^2}{n}} \leq \frac{\widetilde{c_1}}{n^{\frac{d}{2}}} e^{-\frac{c_2^2 d(x,y)^2}{n}}$$

Definition 7.0.3. $(\Gamma,\ \mu)$ *is said to have* ***polynomial volume growth*** *if* $\exists\, C_V$ *and* θ *such that*

$$\max\{|B(x,r)|\,,\ \mu(B(x,r))\} \leq C_V r^\theta \qquad \forall\, x \in V \ \text{ and } r \geq 1$$

Lemma 7.0.1. $(\Gamma,\ \mu)$ *has polynomial volume growth with index* θ*. Then,*

$$\mathbf{P}^x(d(x,X_n) > r) \leq c r^\theta e^{-\frac{r^2}{4n}}$$

This implies $\exists\, c_2 > 0$ *such that*

$$d(x,X_n) \leq c_2\sqrt{n \log n} \qquad \forall \text{ large } n \text{ w.p. } 1$$

Proof. We define $\mathcal{D}_k = B(x,2^k r) \setminus B(x,2^{k-1}r)$

$$\begin{aligned}
\mathbf{P}^x(d(x,X_n) > r) &\overset{Ex}{=} \sum_{k=1}^{\infty} \sum_{y \in \mathcal{D}_k} p_n(x,y)\mu_x \\
&\overset{Ex}{\leq} \sum_{k=1}^{\infty} \sum_{y \in \mathcal{D}_k} \frac{2}{\sqrt{\mu_x}} \sqrt{\mu_y} e^{-\frac{(2^{k-1}r)^2}{2n}} \\
&= \sum_{k=1}^{\infty} \frac{2}{\sqrt{\mu_x}} e^{-\frac{(2^{k-1}r)^2}{2n}} \sum_{y \in \mathcal{D}_k} \sqrt{\mu_y} \\
&\overset{Ex}{\leq} \sum_{k=1}^{\infty} \frac{2}{\sqrt{\mu_x}} e^{-\frac{(2^{k-1}r)^2}{2n}} \sqrt{|\mathcal{D}_k|}\sqrt{\mu(\mathcal{D}_k)} \\
&\overset{Ex}{\leq} \sum_{k=1}^{\infty} \frac{2}{\sqrt{\mu_x}} e^{-\frac{(2^{k-1}r)^2}{2n}} c(2^k r)^\theta
\end{aligned}$$

□