Week 4

Discrete Time Martingales

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Origin is from horse-racing (betting system). The dictionary meaning of the word 'martingale' is the harness of a horse.
Let $\{Z_n\}_{n\geq 1}$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 4.0.1. *A sequence of random variables $\{Z_n\}_{n\geq 1}$ is said to be a Martingale if*

$$\mathbb{E}(Z_n | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) = z_{n-1} \quad \forall\ n \geq 2 \tag{4.1}$$

Things to understand- conditional expectation for discrete and conditional r.v. Reference- Ch6 of Siva's book.
Things we will explore-

1. Examples of $\{Z_n\}_{n\geq 1}$ that are martingales.

2. How different are martingales from iid sequences and markov chains?

3. How to interpret 4.1?

Example. $\{S_n\}_{n\geq 1}$ and $S_0 \equiv 0$.

$$X_i = \begin{cases} 1, & w.p \ \ 1/2 \\ -1, & w.p \ \ 1/2 \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i$$

Let $s_{n-1}, s_{n-2}, \ldots, s_1 \in \mathbb{Z}$ such that $\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) > 0$

$$\begin{aligned}
\mathbb{E}(S_n | S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) &= \sum_{k \in \mathbb{Z}} k \mathbb{P}(S_n = k | S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(S_n = k, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(S_{n-1} + X_n = k, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(X_n = k - s_{n-1}, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(X_n = k - s_{n-1}) \mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= (s_{n-1} + 1) \mathbb{P}(X_n = -1) + (s_{n-1} - 1) \mathbb{P}(X_n = 1) \\
&= (s_{n-1} + 1) \frac{1}{2} + (s_{n-1} - 1) \frac{1}{2} = s_{n-1}
\end{aligned}$$

Note that the summations here are “finite” sums.
As $s_{n-1}, \ldots, s_1 \in \mathbb{Z}$ were arbitrary, $\{S_n\}_{n \geq 1}$ is a martingale.

Example. $\{X_i\}_{i \geq 1}$ be an iid sequence on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z_n = \prod_{i=1}^n X_i$ and Range$(Z_n) \subset \mathbb{R} \ \forall \ n \geq 1$.
Let $z_{n-1}, \ldots, z_1 \in \mathbb{R}$ such that $\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) > 0$. Then

$$\begin{aligned}
\mathbb{E}(Z_n | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) &= \sum_{k \in Range(Z_n)} k \mathbb{P}(Z_n = k | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(Z_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(Z_{n-1} X_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(z_{n-1} X_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \mathbb{P}(Z_{n-1} X_n = k) \frac{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{u \in S^1, Range(X_n) = S^1} u z_{n-1} \mathbb{P}(X_n = u) \\
&= z_{n-1} \mathbb{E}[X_n] = z_{n-1}
\end{aligned}$$

Note that the sums here might be infinite. In the last step we assume $\mathbb{E}[X_i] = 1$. Now since $\{z_i\}_{i=1}^{n-1}$ were arbitrary, $\{Z_n\}_{n\geq 1}$ is a martingale.

Example.

$$X_i = \begin{cases} 2, & w.p \;\; 1/2 \\ 0, & w.p \;\; 1/2 \end{cases}$$

Then $\mathbb{E}(X_i) = 1$. Therefore, $Z_n = \prod_{i=1}^{n} X_i$ is a martingale. Range (Z_n)= $\{2^n, 0\}$. Note that the mean stays constant and

$$\mathbb{P}(Z_n = 0) = 1 - \frac{1}{2^n}$$

$$\mathbb{P}(Z_n = 2^n) = \frac{1}{2^n}$$

Intuition- The first equation shows that the martingale takes a very low value with very high probability and the second one shows that it takes a very large value with very low probability
Idea behind Markov Chains -

$$\text{“}X_n | X_{n-1}, \ldots, X_1\text{”} \stackrel{d}{=} X_n | X_{n-1}$$

Idea behind Martingales - Expected value of Z_n conditioned on the past depends only on Z_{n-1}. $\{Z_n\}_{n\geq 1}$ in law could depend on the entire past!

Week 5

Discrete Time Martingales

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Definition 5.0.2. *A sequence of random variables $\{Z_n\}_{n\geq 1}$ with $\mathbb{E}[|Z_n|] < \infty$ is said to be martiangle if $\mathbb{E}[Z_n|Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, ..., Z_1 = z_1] = z_{n-1}$ all are discrete random variables. All $z_i's$ are continuous with appropriate joint deviation.*

Statement 4.0.1. Let us define $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f(z_1, z_2, ..., z_{n-1}) = \mathbb{E}[Z_n|Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, ..., Z_1 = z_1]$. Then set $Y_n^{(\omega)} = f(z_1^{(\omega)}, z_2^{(\omega)}, ..., z_{n-1}^{(\omega)})$. We can check that Y_n is a random variable.

Properties:

1. Take $A = Z_{n-1} = z_{n-1}, ..., Z_1 = z - 1\}$ for $z_i's \in \mathbb{R}$ where $1 \leq i \leq n$, then;

$$\omega \in A \Rightarrow Y_n^{(\omega)} = f(z_1, ..., z_{n-1})$$

2. Take $L = \{Y_n \leq c\} = \{f(z_1, ..., z_{n-1} \leq c\} \quad \forall c \in \mathbb{R} \Rightarrow L \in \mathcal{A}_{n-1} \equiv$ observable events upto $n - 1$.

$$\textit{Statement } 4.0.1. \iff Y_n \textit{ has properties } 1 \textit{ and } 2$$

Note that $Y_n = \mathbb{E}[Z_n|\mathcal{A}_{n-1}]$. If Z_n is martiangle then $Y_n = Z_{n-1}$.

Tower Property : X, Y, Z are discrete random variables. $\mathbb{P}(Y = y) > 0$ and $f(y) = \mathbb{E}[X|Y = y]$ then $\mathbb{E}[X|Y] = f(Y)$. $\mathbb{P}(Y = y, Z = z) > 0$, $h(y, z) = \mathbb{E}[X|Y = y, Z = z]$ then $\mathbb{E}[X|Y, Z] = h(Y, Z) \Longrightarrow \mathbb{E}[\mathbb{E}[X|Y, Z]|Y] = \mathbb{E}[X|Y]$.

$\mathbb{E}[\mathbb{E}[X|Y, Z]|Y] = \mathbb{E}[h(Y, Z)|Y] := k(Y)$, $\mathbb{E}[X|Y = y] := l(Y)$

Let $Y \in \mathbb{R}$, $\mathbb{P}(Y = y) > 0$,

$$\begin{aligned}
k(y) &= \mathbb{E}[h(Y, Z)|Y = y] \\
&= \sum_{\substack{m \in range(Y) \\ t \in Range(Z)}} h(m, t)\mathbb{P}(Y = m, Z = t|Y = y) \\
&= \sum_{t \in range(Z)} h(y, t)\mathbb{P}(Z = t, Y = y) \\
&= \sum_{t \in Range(Z)} (\sum_{c \in range(X)} c\mathbb{P}(X = x|Y = y, Z = t)\mathbb{P}(Z = t|Y = y)) \\
&= \sum_{t \in Range(Z)} \sum_{c \in range(X)} c\frac{\mathbb{P}(X = c, Y = y, Z = t)}{\mathbb{P}(Y = y, Z = t)} \frac{\mathbb{P}(Z = t, Y = y)}{\mathbb{P}(Y = y)} \\
&= \frac{1}{\mathbb{P}(Y = y)} \sum_{t \in Range(Z)} \sum_{c \in range(X)} c\mathbb{P}(X = c, Z = t, Y = y) \\
&= \sum_{c \in Range(X)} \frac{c\mathbb{P}(X = c, Y = y)}{\mathbb{P}(Y = y)} \\
&= \mathbb{E}[X|Y = y] \\
&= l(y)
\end{aligned}$$

$\implies k(Y) = l(Y)$
$\implies \mathbb{E}[\mathbb{E}[X|Y, Z]|Y] = \mathbb{E}[X|Y]$

Lemma 5.0.1. *Let $\{Z_n\}_{n \geq 1}$ be a martiangle, $1 \leq i \leq n$ then; $\mathbb{E}[Z_n|Z_i, Z_{i-1}, ..., Z_1] = Z_i$*

Proof. Fix $i \geq 1$. For $n = i + 1$,

$$\mathbb{E}[Z_{i+1}|Z_i, Z_{i-1}, ..., Z_1] = Z_i$$

Assume for $k \geq 1$, $n = i + k$,

$$\mathbb{E}[Z_{i+k}|Z_i, Z_{i-1}, ..., Z_1] = Z_i$$

Then for $n = i + k + 1$, by tower property;

$$\begin{aligned}
\mathbb{E}[Z_{i+k+1}|Z_i, Z_{i-1}, ..., Z_1] &= \mathbb{E}[\mathbb{E}[Z_{i+k+1}|Z_{i+k}, Z_{i+k-1}, ..., Z_1]|Z_i, Z_{i-1}, ..., Z_1] \\
&= \mathbb{E}[Z_{i+k}|Z_i, Z_{i-1}, ..., Z_1] \\
&= Z_i
\end{aligned}$$

$\square$