Topics in Applied Stochastic Processes

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Contents

Week 1 **January 13, 2023**

Finite length random walks on $\mathbb{Z}$

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1.1 Definitions

Random walks serve as very useful models in many applications. They are simple to state and understand, yet they lead to lots of intractable questions.

Notation. $\mathbb{N} = \{k \in \mathbb{Z} : k \geq 1\}$ *and* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

We now proceed to construct what is called a "simple random walk" on $\mathbb{Z}$ of finite length $N \in \mathbb{N}$. The sample space Ω_N and the event space $\mathcal{F}_N$ are described below.

$$\Omega_N := \{(\omega_1, \dots, \omega_N) : \omega_i \in \{-1, 1\} \ \forall \ 1 \leq i \leq N\}$$

$$\mathcal{F}_N := \{A : A \subseteq \Omega_N\}$$

The probability function $\mathbf{P}_N : \Omega_N \to [0, 1]$ is defined as

$$\mathbf{P}_N(A) := |A| \ 2^{-N}$$

We also define random variables X_k and S_k on Ω_N for $1 \leq k \leq N$ as

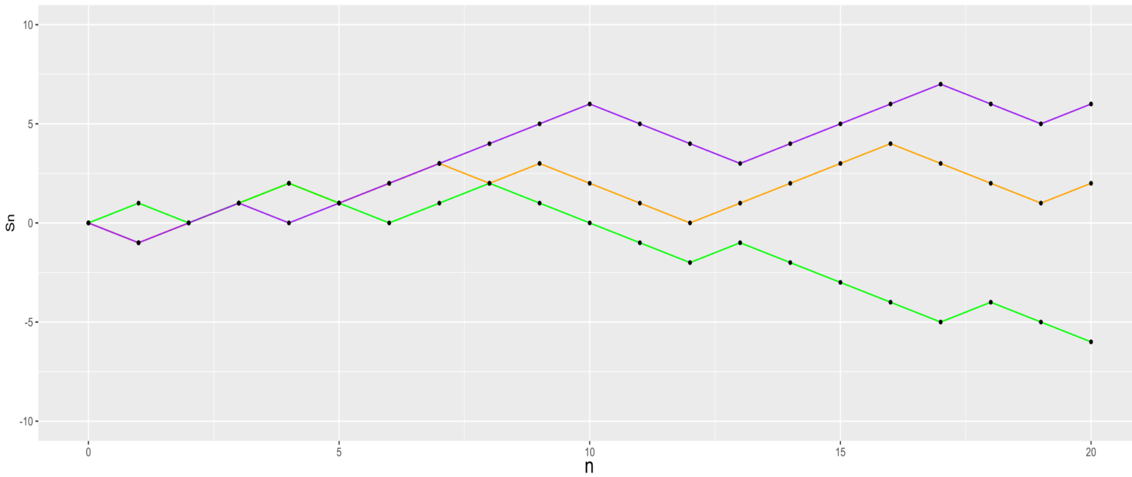
$$X_k : \Omega_N \to \{-1, 1\} \ ; \ X_k(\omega) := \omega_k$$

$$S_k : \Omega_N \to \mathbb{Z} \ ; \ S_k(\omega) := \sum_{i=1}^{k} X_k(\omega) \ ; \ S_0(\omega) := 0 \text{ for all } \omega \in \Omega_N$$

Definition 1.1.1. *Fix $N \in \mathbb{N}$. The sequence of random variables $\{S_k\}_{k=1}^{N}$ on $(\Omega_N, \mathcal{F}_N, \mathbf{P}_N)$ is called a (symmetric) simple random walk on $\mathbb{Z}$, of finite length N, starting at 0.*

[†] added illustrations

Figure 1.1: Three possible trajectories for $(S_n)_{n=0}^N$



In what follows, we suppress the subscript N while referring to the probability space $(\Omega_N, \mathcal{F}_N, \mathbf{P}_N)$, and we assume that $N \in \mathbb{N}$ is fixed.

Observations.

(a) $\{X_k\}_{k=1}^N$ are iid, i.e. independent and identically distributed.

Proof.

$$\begin{aligned}\mathbf{P}(X_k = 1) = \mathbf{P}(\{\omega \in \Omega : \omega_k = 1\}) &= 2^{-N}|\{\omega \in \Omega : \omega_k = 1\}| \\ &= 2^{-N}2^{N-1} \\ &= \frac{1}{2} \\ &= \mathbf{P}(X_k = -1)\end{aligned}$$

So $\{X_k\}_{k=1}^N$ are identically distributed. Independence is left as an exercise. □

(b) (Independent increments) For $1 \leq k_1 \leq k_2 \leq \ldots \leq N$, $\{S_{k_i} - S_{k_{i-1}} : 1 \leq i \leq N\}$ are independent random variables.

Proof. Observe that, for $1 \leq k < l \leq N$, we have $S_l - S_k = \sum_{i=k+1}^{l} X_i$. Therefore, if $1 \leq a < b \leq c < d \leq N$, we see that $S_b - S_a$ and $S_d - S_c$ are functions of disjoint sets of independent random variables, and hence the claim is true. □

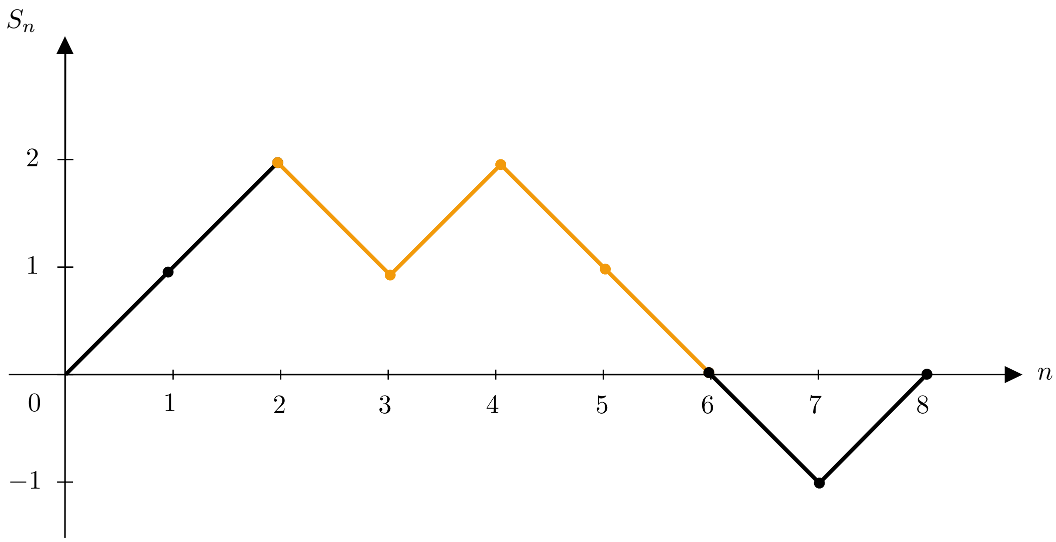


Figure 1.2: Independent (colored) increments in a simple random walk

(c) (Stationary in increments) For $1 \leq k < m \leq N$, $\mathbf{P}(S_m - S_k = \alpha) = \mathbf{P}(S_{m-k} = \alpha)$ for every $\alpha \in \mathbb{Z}$.

Proof. We use the fact that $\{X_i\}_{i=1}^N$ are identically distributed in the following argument.

$$\mathbf{P}(S_m - S_k = \alpha) = \mathbf{P}\left(\sum_{i=k+1}^{m} X_i = \alpha\right) = \mathbf{P}\left(\sum_{i=1}^{m-k} X_i = \alpha\right) = \mathbf{P}(S_{m-k} = \alpha)$$

□

(d) (Markov Property) For $\alpha_i \in \mathbb{Z},\ 1 \leq i \leq N$ and $0 \leq n \leq N$,

$$\mathbf{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}, \ldots, S_1 = \alpha_1) = \mathbf{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}),$$

assuming (of course) that the conditional probabilities are well defined.

Proof. Left as an exercise. □

(e) (Conditional Law) For $1 \leq k < m \leq N$, $\mathbf{P}(S_m = b \mid S_k = a) = \mathbf{P}(S_{m-k} = b - a)$.

Proof. Left as an exercise. □

(f) (Moments) For $1 \leq k \leq N$, we have $\mathbf{E}[X_k] = \mathbf{E}[S_k] = 0$ and $\mathrm{Var}[S_k] = k$.

Proof. By definition of expected value, $\mathbf{E}[X_k] = 1(1/2) - 1(1/2) = 0$. By linearity of expected values, $\mathbf{E}[S_k] = \sum_{i=1}^{k} \mathbf{E}[X_i] = 0$.

Since $\mathbf{E}[S_k] = 0$, $\mathrm{Var}[S_k] = \mathbf{E}[(\sum_{i=1}^{k} X_i)^2] = \sum_{i=1}^{k} \mathbf{E}[X_k^2] = k$. As an exercise, show that $\mathbf{E}[(\sum_{i=1}^{k} X_i)^2] = \sum_{i=1}^{k} \mathbf{E}[X_k^2]$. $\square$

(g) (Distribution of S_n) For $x \in \{-n, -n+2, \ldots, n-2, n\}$, we have

$$\mathbf{P}(S_n = x) = \mathbf{P}(S_n = -x) = \binom{n}{\frac{n+x}{2}} 2^{-n}$$

Proof. We only provide a sketch of the proof, which is left as an exercise. For $0 \le j \le N$, $\{S_n = 2j - n\} = \{S_n = j - (n-j)\}$. So there must be a total of j steps to the right and $n - j$ steps to the left. Therefore

$$\mathbf{P}(S_n = 2j - n) = 2^{-N} \ |\{\omega \in \Omega : \cdots\}| = 2^{-n} \binom{n}{j}$$

$\square$

(h) (Mode) The mode of the above distribution is achieved in the middle, i.e. at $x = 0$ and at $x = 1, -1$ for S_{2n} and S_{2n-1} respectively.

Proof.

$$\mathbf{P}(S_{2n} = 0) = \mathbf{P}(S_{2n-1} = 1) = \binom{2n}{n} 2^{-2n}$$

$\square$

(i) (Stirling's formula) Using Stirling's approximation, for large n, we have

$$\binom{2n}{n} = \frac{2n!}{n!n!} \sim \frac{(2n)^{2n} e^{-2n} \sqrt{4\pi n}}{n^{2n} e^{-2n} \sqrt{2\pi n} \sqrt{2\pi n}} \sim \frac{2^{2n}}{\sqrt{\pi n}} \tag{$*$}$$

Therefore,

$$\mathbf{P}(S_{2n} = 0) = \binom{2n}{n} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}} \quad \text{as} \quad n \to \infty$$

This approximation, although correct, has a caveat - we chose to keep N fixed, but as $n \to \infty$, we must also let $N \to \infty$, and this requires subtler arguments. A few consequences of this approximation are mentioned in the exercises.

1.2 Stopping times

Motivation for this section comes from the classic Gambler's ruin problem. We can interpret a simple random walk as a fair game between two players, where in round k, a player wins the amount X_k. Then S_n denotes the capital of one player over the other after n rounds.

We would like to answer the following question - "Is it possible to stop the game in a favorite moment, i.e., can clever stopping lead to a positive expected gain?". In other words, can we design a $T(\omega)$ for every $\omega \in \Omega$ such that $\mathbf{E}[S_T] > 0$? Of course, the decision to stop may only depend on the trajectory until that time: no "insider knowledge" about the future of the trajectory is permitted.

To formalize this setting, we make the following definition.

Definition 1.2.1. *An event $A \subseteq \Omega$ is said to be observable by time n if it is a (possibly empty) union of basic / elementary events of the form*

$$\{\omega \in \Omega : \omega_1 = o_1, \ldots, \omega_n = o_n\}$$

where $o_i \in \{-1, 1\}$ for $1 \leq i \leq n$.

We also define $\mathcal{A}_0 = \{\phi, \Omega\}$ and set

$$\mathcal{A}_n := \{A \in \mathcal{F} : A \text{ is observable by time } n\}.$$

Immediately, we observe that

$$\{\phi, \Omega\} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_{N-1} \subseteq \mathcal{A}_N = \mathcal{F}$$

As an easy exercise, verify that each $\mathcal{A}_n$ is closed with respect to taking complement, union and intersection. Such a sequence $\{\mathcal{A}_i\}_{i=0}^N$ is called a *filtration*.

Definition 1.2.2. *A function $T : \Omega \to \{0, 1, \ldots, N\} \cup \{\infty\}$ is called a stopping time if for each $0 \leq n \leq N$,*

$$\{T = n\} = \{\omega \in \Omega : T(\omega) = n\} \in \mathcal{A}_n$$

Example. For $a \in \mathbb{Z}$, let $\sigma_a = \inf\{n : S_n = a, 0 \leq n \leq N\}$ denote the *first* hitting time of a. As an exercise, show that σ_a is a stopping time.

Example. For $a \in \mathbb{Z}$, let $L_a = \max\{n : S_n = a, 0 \leq n \leq N\}$ denote the *last* hitting time of a. As an exercise, show that L_a is NOT a stopping time.

Theorem 1.2.1. *Let $T : \Omega \to \{0, 1, \ldots, N\}$ be a stopping time. Then*

$$\mathbf{E}[S_T] = 0$$

where $S_T : \Omega \to \mathbb{Z}$ maps $\omega \mapsto S_{T(\omega)}(\omega)$.

Proof.

$$\begin{aligned} S_T = \sum_{k=1}^{N} S_k \mathbb{1}\{T = k\} &= \sum_{k=1}^{N} S_k(\mathbb{1}\{T \geq k\} - \mathbb{1}\{T \geq k+1\}) \\ &= \sum_{k=1}^{N} (S_k - S_{k-1})\mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} X_k \mathbb{1}\{T \geq k\} \end{aligned}$$

where we take $\mathbb{1}\{T \geq N+1\} = 0$. Now, we can write $\mathbf{E}[S_T]$ as

$$\mathbf{E}[S_T] = \sum_{k=1}^{N} \mathbf{E}[X_k \mathbb{1}\{T \geq k\}] \tag{$\dagger$}$$

Observe that for $1 \leq k \leq N$, we have

$$X_k \mathbb{1}\{T \geq k\} = \begin{cases} 1, & \text{for } X_k = 1,\ T \geq k \\ -1, & \text{for } X_k = -1,\ T \geq k \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{E}[X_k \mathbb{1}\{T \geq k\}] = \mathbf{P}(X_k = 1, T \geq k) - \mathbf{P}(X_k = -1, T \geq k) \tag{$\dagger\dagger$}$$

Now,

$$\{T \geq k\} = \{T < k\}^c = \left(\bigcup_{l=0}^{k-1} \{T = l\}\right)^c \in \mathcal{A}_{k-1}$$

Using the fact that $\{T \geq k\} \in \mathcal{A}_{k-1}$, one can show that (details left as an exercise)

$$\mathbf{P}(X_k = 1, T \geq k) = \mathbf{P}(X_k = -1, T \geq k) = \frac{1}{2}\mathbf{P}(T \geq k)$$

Substituting the above values in (†) and (††), we finally have

$$\mathbf{E}[S_T] = 0$$

□

As an exercise, compute $\mathrm{Var}[S_T]$.

Definition 1.2.3. *A bet sequence / game system is a sequence of random variables $V_k : \Omega \to \mathbb{R}$ such that*

$$\{V_k = c\} \in \mathcal{A}_{k-1} \textit{ for every } c \in \mathbb{R} \textit{ and } 1 \leq k \leq N$$

Theorem 1.2.2. *Let $\{V_k\}_{k=1}^{N}$ be a bet sequence. Then*

$$\mathbf{E}[S_N^V] = 0 \quad \textit{where} \quad S_N^V = \sum_{k=1}^{N} V_k X_k$$

In this setting, S_N^V is interpreted as the "total gain".

Proof. Since Ω is finite, we may write

$$\text{Range}(V_k) = \{c_i^k : 1 \le i \le m_k\} \text{ where } c_i^k \in \mathbb{R}$$

$$V_k = \sum_{i=1}^{m_k} c_i^k \mathbb{1}\{V_k = c_i^k\}$$

Now, since $\mathbf{E}[X_k] = 0$, and since $X_k \perp \mathbb{1}\{V_k = c_i^k\}$, we get

$$\begin{aligned}
\mathbf{E}[S_N^V] = \sum_{k=1}^{N} \mathbf{E}[V_k X_k] &= \sum_{k=1}^{N} \mathbf{E}\left[X_k \sum_{i=1}^{m_k} c_i^k \mathbb{1}\{V_k = c_i^k\}\right] \\
&= \sum_{k=1}^{N} \sum_{i=1}^{m_k} c_i^k \mathbf{E}[X_k \mathbb{1}\{V_k = c_i^k\}] \\
&= \sum_{k=1}^{N} \sum_{i=1}^{m_k} c_i^k \mathbf{E}[X_k] \mathbf{P}(V_k = c_i^k) \\
&= 0
\end{aligned}$$

□

1.3 Exercises

1. Show that $\{X_k\}_{k=1}^N$ are independent.
2. Show that $\{S_n\}_{n=0}^N$ satisfies the Markov property.
3. For $1 \le k < m \le N$, show that $\mathbf{P}(S_m = b \mid S_k = a) = \mathbf{P}(S_{m-k} = b - a)$.
4. Show that $\mathbf{E}[S_n^2] = \sum_{i=1}^n \mathbf{E}[X_i^2]$.
5. (a) Show that for any $a, b \in \mathbb{R}$,

   $$\mathbf{P}(a \le S_n < b) \le (b-a)\ \mathbf{P}(S_n \in \{-1, 0, 1\}).$$

   (b) Using (a), conclude that

   $$\mathbf{P}(a \le S_n < b) \to 0 \quad \text{as} \quad n \to \infty.$$

   Thus, we observe that the walk exits any finite interval as $n \to \infty$.
6. Verify that each $\mathcal{A}_n$, $0 \le n \le N$, is closed with respect to taking complement, union and intersection.
7. For $a \in \mathbb{Z}$, let $\sigma_a = \inf\{n : S_n = a, 0 \le n \le N\}$. Show that σ_a is a stopping time.
8. For $a \in \mathbb{Z}$, let $L_a = \max\{n : S_n = a, 0 \le n \le N\}$. Show that L_a is not a stopping time.
9. Let $T : \Omega \to \{0, 1, \ldots, N\}$ be a stopping time. Compute $\text{Var}[S_T]$.
10. Show that X_k and $\mathbb{1}\{T \ge k\}$ are independent.

Week 2 **January 20, 2023**

More on random walks

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Theorem 2.0.1. *Let $T : \Omega \to 0, 1, \ldots, N$ be a stopping time. Then,*

$$\mathbf{E}[S_T^2] = E[T].$$

Proof.

$$\begin{aligned} S_T^2 &= \sum_{k=1}^{N} S_k^2 \mathbb{1}\{T = k\} \\ &= \sum_{k=1}^{N} (S_k^2 - S_{k-1}^2) \mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} (X_k + S_{k-1})^2 - S_{k-1}^2 \mathbb{1}\{T \geq k\} \\ &= \sum_{k=1}^{N} (1 + 2X_k S_{k-1}) \mathbb{1}\{T \geq k\}. \end{aligned}$$

Now, consider $V_k = S_{k-1}\mathbb{1}\{T \geq k\}$. Note that this is a bet sequence. Hence,

$$\begin{aligned} \mathbf{E}[S_T^2] &= \mathbf{E}\left[\sum_{k=1}^{N} \mathbb{1}\{T \geq k\}\right] + 2\sum_{k=1}^{N} \mathbf{E}[X_k V_k] \\ &= \sum_{k=1}^{N} \mathbf{P}(T \geq k) + 0 \\ &= E[T]. \end{aligned}$$

□

2.1 Reflection Principle

Assume that $a \in \mathbb{Z}$ and $c > 0$. There is a bijection between the paths that cross $a + c$ and those that do not. This bijection is obtained by reflecting the part of the path crossing $a + c$ as shown in the Figure 2.1. So,

$$|S_n = a + c| = |\sigma_a \leq n \;\&\; S_n = a + c| = |\sigma_a \leq n \& S_n = a - c|$$

Now, we know that all the paths have equal probability. Hence, we get the following lemma.

Lemma 2.1.1. $\mathbf{P}(S_n = a + c) = \mathbf{P}(\sigma_a \le n \ \& \ S_n = a - c)$ *where* $a \in \mathbb{Z}$ *and* $c > 0$*. This is also known as the* reflection principle*.*

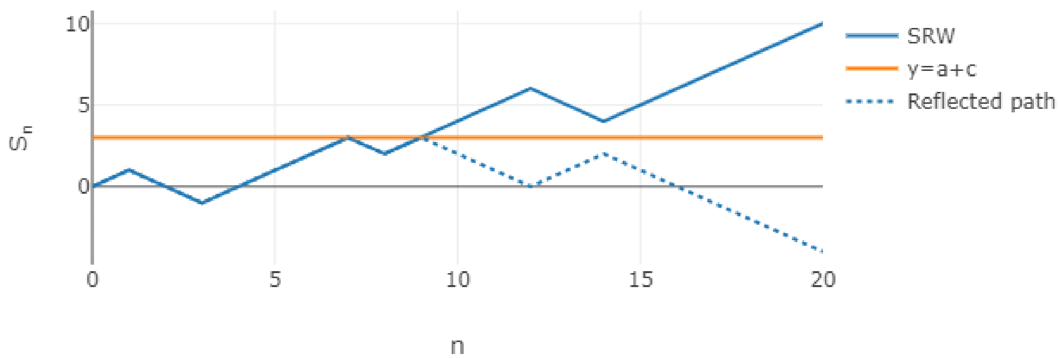

Figure 2.3: The figure shows that the bijection between the paths that cross a+c=3 and those that do not.

Theorem 2.1.1. $\mathbf{P}(\sigma_a \le n) = \mathbf{P}(S_n \notin [-a, a))$ *where* $a \in \mathbb{Z}$
$\{0\}$.

Proof.

$$\begin{aligned}
\mathbf{P}(\sigma_a \le n) &= \mathbf{P}(\sigma_a \le n, \bigcup_{b\in\mathbb{Z}} S_n = b) \\
&= \sum_{b\in\mathbb{Z}} \mathbf{P}(\sigma_a \le n, S_n = b) \\
&= \sum_{b\in\mathbb{Z}, b\ge a} \mathbf{P}(\sigma_a \le n, S_n = b) + \sum_{b\in\mathbb{Z}, b<a} \mathbf{P}(\sigma_a \le n, S_n = b) \\
&= \sum_{b\in\mathbb{Z}, b\ge a} \mathbf{P}(S_n = b) + \sum_{b\in\mathbb{Z}, b<a} \mathbf{P}(S_n = 2a - b) \\
&= \mathbf{P}(S_n \ge a) + \mathbf{P}(S_n > a) \\
&= \mathbf{P}(S_n \ge a) + \mathbf{P}(S_n < -a) \\
&= \mathbf{P}(S_n \notin [-a, a))
\end{aligned}$$

□

Corollary 2.1.1. $\mathbf{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbf{P}(S_{n-1} = a - 1) - \mathbf{P}(S_{n-1} = a + 1)\right]$ *where* $a \in \mathbb{Z}$*.*

Proof. □

2.2 Arc-Sine Law

Let L denote the last time the random walk hits 0, i.e., $L = \max_{0 \le n \le 2N} S_n = 0$, where N denotes the length of the walk.

Theorem 2.2.1.
$$\mathbf{P}(L = 2n) = \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}.$$

Remark. By Stirling's approximation,
$$\mathbf{P}(L = 2n) \sim \frac{1}{\pi N} \frac{1}{\sqrt{\left(\frac{n}{N}\right)\left(1 - \frac{n}{N}\right)}}.$$

$$\begin{aligned}
\mathbf{P}\left(\frac{L}{2N} \le x\right) &= \mathbf{P}(L \le 2Nx) \\
&= \sum_{n=0}^{[2Nx]} \mathbf{P}(L = 2n) \\
&\sim \sum_{n=0}^{[2Nx]} \frac{1}{\pi N} \frac{1}{\sqrt{(x)(1-x)}} \\
&\sim \int_0^x \frac{dy}{pi\sqrt{y(1-y)}} \\
&= \frac{2}{\pi} \sin^{-1}(\sqrt{x}).
\end{aligned}$$

Proof of Theorem 2.2.1. Define $\tilde{\sigma_0} \inf\{n : S_n = 0, 0 < n \le N\}$. Consider a path of length $2N$ with $L = 2n$. This path can be formed by a path which takes $S_2 n = 0$ and followed by a path of length $2N - 2n$ with $\sigma_0 > 2N - 2n$. Hence, number of paths of length $2N$ with $L = 2n$ is the product of the number of paths of length $2n$ with $S_{2n} = 0$ and the number of paths of length $2N - 2n$ with $\sigma_0 > 2N - 2n$. Hence,
$$\mathbf{P}(L = 2n) = \mathbf{P}(S_{2n} = 0)\mathbf{P}(\tilde{\sigma_0} > 2N - 2n), \tag{2.1}$$

Now let us compute the distribution of $\tilde{\sigma_0}$.

$$\begin{aligned}
\mathbf{P}(\tilde{\sigma_0} > 2k) &= \mathbf{P}(S_1 \ne 0, \ldots, S_{2k} \ne 0) \\
&= 2\mathbf{P}(S_1 > 0, \ldots, S_{2k} > 0) \\
&= \frac{2}{2^{2k}}\{\text{No. of paths start at 0 and stay above -1 for } 2k-1 \text{ steps}\} \\
&= \frac{2}{2^{2k}}\{\text{No. of paths start at 0 and stay below 1 for } 2k-1 \text{ steps}\} \qquad (2.2) \\
&= \mathbf{P}(\sigma_1 > 2k-1) \\
&= 1 - \mathbf{P}(\sigma_1 \ge 2k-1) \\
&= \mathbf{P}(S_{2k-1} = -1) + \mathbf{P}(S_{2k-1} = 0) \\
&= \mathbf{P}(S_{2k-1} = -1)
\end{aligned}$$

Using (2.1) and (2.2),

$$\begin{aligned}\mathbf{P}(L=2n) &= \mathbf{P}(S_{2n}=0)\mathbf{P}(S_{2N-2n-1}=-1)\\ &= \mathbf{P}(S_{2n}=0)\mathbf{P}(S_{2N-2n}=0)\\ &= \frac{1}{2^{2N}}\binom{2n}{n}\binom{2N-2n}{N-n}.\end{aligned}$$

The first step analysis of S_{2n} shows that, $\mathbf{P}(S_{2N-2n}=0)=\frac{1}{2}\mathbf{P}(S_{2N-2n-1}=1)+\frac{1}{2}\mathbf{P}(S_{2N-2n-1}=-1)$. Using the symmetry of the walk we know that $\mathbf{P}(S_{2N-2n-1}=1)=\mathbf{P}(S_{2N-2n-1}=-1)$. This gives the second inequality. $\square$

2.3 SRW of length N in $\mathbb{Z}^d$

2.3.1 Notations and notions in higher dimension

- $e_i \in \mathbb{Z}^d$, $\forall\, i \in \{1,2,\cdots,d\}$, defined as the vector of length d with all entries zeroes except i^{th} being 1.

$$e_i = (0,0,\cdots,\underbrace{1}_{i^{th}},0,\cdots,0)$$

- For $x \in \mathbb{Z}^d$,

$$x=\sum_{i=1}^{d} x_i e_i,\; x_i \in \mathbb{Z} \qquad\qquad \|x\| = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}$$

- $\Omega_N = \{(\omega_1,\omega_2,\cdots,\omega_N) \mid \omega_i \in \mathbb{Z}^d, \|\omega_i\|=1\,\forall\, 1\le i\le N\}$
- We have, for $1\le k,n\le N$

$$X_k:\Omega_N\to\mathbb{Z}^d,\; X_k(\omega)=\omega_k \qquad\qquad S_n:\Omega_N\to\mathbb{Z}^d,\; S_n(\omega)=\sum_{k=1}^{n}X_k(\omega)$$

  with $S_0(\omega)=0$. We can consider S_n as a d-dimensional vector given by $S_n=\left(S_n^{(1)},S_n^{(2)},\cdots S_n^{(d)}\right)$, where each $S_n^{(i)}$ is a random walk on $\mathbb{Z}$.

- The probability function $\mathbf{P}^N$, given by,

$$\mathbf{P}^N:\mathcal{P}(\Omega_N)\to[0,1],\quad \mathbf{P}(A)=\frac{|A|}{(2d)^N}\,\forall\, A\subseteq\Omega_N$$

2.3.2 Infinite length random walk

On extending $N\to\infty$, we preserve something called as "consistency". First, let us define, for $0<N<M$,

$$\pi_N:\Omega_M\to\Omega_N,\;\; \pi_N(\omega_1,\omega_2,\cdots,\omega_M)=(\omega_1,\omega_2,\cdots,\omega_N)$$

Under $(\Omega_N, \mathcal{P}(\Omega_N), \mathbf{P}^N)$ and $(\Omega_M, \mathcal{P}(\Omega_M), \mathbf{P}^M)$, if we observe the walk till time $n < N$ the probability of evenets concerning the walk should be same under $\mathbf{P}^N$ or $\mathbf{P}^M$. For any event $\{\tilde{\omega} \in \Omega_N\}$, there exists a corresponding same event namely $\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}$. We have,

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \frac{1}{(2d)^N} \qquad \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}) = \frac{(2d)^{M-N}}{(2d)^M} = \frac{1}{(2d)^N}$$

So, we say the sequence of probability spaces $(\Omega_1, \mathbf{P}^1), (\Omega_2, \mathbf{P}^2), \cdots, (\Omega_N, \mathbf{P}^N)$ satisfies the consistency condition

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \frac{1}{(2d)^N} = \frac{(2d)^{M-N}}{(2d)^M} = \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}),\ 0 < N < M,\ \tilde{\omega} \in \Omega_N$$

We define the space of infinite sequences,

$$\Omega_\infty = \{\omega = (\omega_k)k \geq 1 \mid \omega_k \in \mathbb{Z}^d,\ \|\omega_k\| = 1\}$$

$$\mathcal{A}_\infty\,(\equiv \mathcal{P}(\Omega_\infty))\,\text{denotes the class of events observable “for ever”}$$

For $N \in \mathbb{N}$,

$$\pi_N : \Omega_\infty \to \Omega_N,\ \ \pi_N(\omega) = (\omega_1, \omega_2, \cdots, \omega_N)$$

Theorem 2.3.1 (**Kolmogorov Consistency Theorem**)**.** There exists a unique probability measure on $(\Omega_\infty, \mathcal{A}_\infty)$ such that $\forall\, N \geq 1$, $\forall\, \tilde{\omega} \in \Omega_N$,

$$\mathbf{P}^N(\{\tilde{\omega}\}) = \mathbf{P}^M(\{\omega \in \Omega_M : \pi_N(\omega) = \tilde{\omega}\}) = \frac{1}{(2d)^N}$$

Now, we can define,

$$X_k : \Omega_\infty \to \mathbb{Z}^d,\ X_k(\omega) = \omega_k \qquad S_n = \sum_{k=1}^{n} X_k\ \forall\, n \geq 1$$

under $\mathbf{P}$, $\{S_n\}_{n\geq 1}$ is a simple random walk starting at $S_0 = 0$.

Definition 2.3.1. $A \subseteq \Omega_\infty$ is said to be **observable** by time n if A is a union of the events of the form

$$\{\omega \in \Omega_\infty : \omega_i = o_i,\ 1 \leq i \leq N\} \text{ with } o_i \in \mathbb{Z}^d,\ \|o_i\| = 1$$

For, $k \in \mathbb{N}_0$, $\mathcal{A}_k$ denotes the set of all events in Ω_∞ observable by time k.

Definition 2.3.2. $T : \Omega_\infty \to \mathbb{N} \cup \{\infty\} \cup \{0\}$ is a **stopping time** if

$$\text{for any }\ k \in \mathbb{N}_0,\ \{T = k\} \in \mathcal{A}_k$$

For example, $\sigma_a = \min\{n \geq 0 \mid S_n = a\}$ is a stopping time.

2.3.3 Speed of the walk

Definition 2.3.3. For, $S_n = \sum_{k=1}^{n} X_k$, we define **speed of the walk** as

$$\text{Speed} \;=\; \frac{S_n}{n} = \frac{1}{n}\sum_{k=1}^{n} X_k$$

We have, $X_k = \left(X_k^{(1)}, X_k^{(2)}, \cdots, X_k^{(d)}\right)$, $\{X_k\}_{k\geq 1}$ which is an i.i.d sequence of random variables with

$$\mathbf{P}(X_k = e_i) = \frac{1}{2d} = \mathbf{P}(X_k = -e_i)$$

$\Rightarrow$ $\mathbf{E}[X_k] = 0$ and $\mathbf{E}[\|X_k\|] = 1$ $(\leq \infty)$

Theorem 2.3.2 (**Strong law of large numbers**)**.** For simple random walk on $\mathbb{Z}^d$,

$$\frac{S_n}{n} \to 0 \text{ with probability 1 under } (\Omega_\infty, \mathcal{A}_\infty, \mathbf{P})$$

2.3.4 Typical position of the walk

For $d = 1$,

$$\frac{S_n - (n)(0)}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$$
$$\Rightarrow \;\; \sqrt{n}\left(\frac{S_n}{n}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

For $d > 1$, $\mu \in \mathbb{R}^d$ and a positive definite matrix $\Sigma \in \mathbb{R}^{d\times d}$, we have d-dimensional normal distribution as,

$$\Phi_{d,\mu,\Sigma}(y) = \frac{1}{(2\pi)^{d/2}} \frac{1}{\det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^t \Sigma^{-1}(x-\mu)\right)$$

$$\mathbf{P}\left(\frac{S_n}{\sqrt{n}} \in \prod_{i=1}^{d}[a_i, b_i]\right) \xrightarrow[n\to\infty]{} \int\limits_{\prod_{i=1}^{d}[a_i,b_i]} \Phi_{d,0,\Sigma^d}(y)\,dy$$

where, $\mu = 0$, $\Sigma^d = \operatorname{diag}\left(\frac{1}{d}, \cdots, \frac{1}{d}\right)$

2.3.5 Large deviation principle

From the CLT, we have that

$$\mathbf{P}(\|S_n\| > a\sqrt{n}) \xrightarrow[n\to\infty]{} \int\limits_{\|x\|>a} \Phi_{d,0,\Sigma^d}(y)\,dy$$

We consider the events of the form $\{\|S_n\| > an\}$, $a \in [0, \infty)$, which are "rare" in the sense that their probabiility tends to 0 as $n \to \infty$. On formal application of CLT shows that probability of these rare events are exponentially small.

Theorem 2.3.3 (Cramer's theorem). For, $a > 0$,

$$\lim_{n \to \infty} \frac{\log(\mathbf{P}(\|S_n\| > an))}{n} = -I(a)$$

where,

$$I(a) = \begin{cases} \log 2 + \frac{1+a}{2} \log \frac{1+a}{2} + \frac{1-a}{2} \log \frac{1-a}{2}, & \text{for } a \in [-1, 1] \\ \infty, & \text{otherwise} \end{cases}$$

It can be vaguely interpreted as, $\mathbf{P}(\|S_n\| > na) \sim e^{-nI(a)}$

2.4 Exercises

1. Complete the proof of Reflection Principle (Lemma 2.1.1).
2. Find the distribution of $M_k = \max_{1 \leq k \leq n} S_k$.
3. Show that $\mathbf{E}[\|X_k\|] = 1$.