Week 2

More on random walks

LECTURER: SIVA ATHREYA SCRIBE: SANCHAYAN BHOWAL, VENKAT TRIVIKRAM

Theorem 2.0.1. *Let $T : \Omega \to 0, 1, \ldots, N$ be a stopping time. Then,*

$$\mathbf{E}[S_T^2] = E[T].$$

Proof.

$$\begin{aligned}
S_T^2 &= \sum_{k=1}^{N} S_k^2 \mathbb{1}\{T = k\} \\
&= \sum_{k=1}^{N} (S_k^2 - S_{k-1}^2) \mathbb{1}\{T \geq k\} \\
&= \sum_{k=1}^{N} (X_k + S_{k-1})^2 - S_{k-1}^2 \mathbb{1}\{T \geq k\} \\
&= \sum_{k=1}^{N} (1 + 2X_k S_{k-1}) \mathbb{1}\{T \geq k\}.
\end{aligned}$$

Now, consider $V_k = S_{k-1} \mathbb{1}\{T \geq k\}$. Note that this is a bet sequence. Hence,

$$\begin{aligned}
\mathbf{E}[S_T^2] &= \mathbf{E}\left[\sum_{k=1}^{N} \mathbb{1}\{T \geq k\}\right] + 2\sum_{k=1}^{N} \mathbf{E}[X_k V_k] \\
&= \sum_{k=1}^{N} \mathbf{P}(T \geq k) + 0 \\
&= E[T].
\end{aligned}$$

□

2.1 Reflection Principle

Assume that $a \in \mathbb{Z}$ and $c > 0$. There is a bijection between the paths that cross $a + c$ and those that do not. This bijection is obtained by reflecting the part of the path crossing $a + c$ as shown in the Figure 2.1. So,

$$|S_n = a + c| = |\sigma_a \leq n \ \&\ S_n = a + c| = |\sigma_a \leq n \& S_n = a - c|$$

Now, we know that all the paths have equal probability. Hence, we get the following lemma.

Lemma 2.1.1. $\mathbf{P}(S_n = a + c) = \mathbf{P}(\sigma_a \leq n \ \&\ S_n = a - c)$ *where* $a \in \mathbb{Z}$ *and* $c > 0$*. This is also known as the* reflection principle*.*

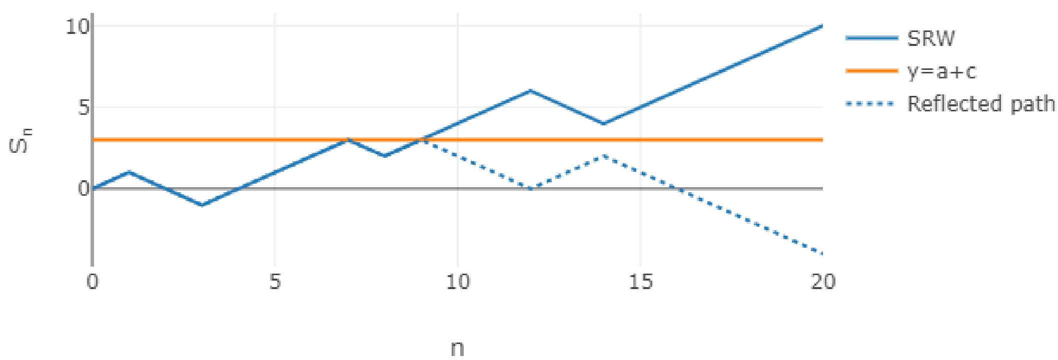

Figure 2.1: The figure shows that the bijection between the paths that cross a+c=3 and those that do not.

Theorem 2.1.1. $\mathbf{P}(\sigma_a \leq n) = \mathbf{P}(S_n \notin [-a, a))$ *where* $a \in \mathbb{Z}$
$\{0\}$.

Proof.

$$\begin{aligned}
\mathbf{P}(\sigma_a \le n) &= \mathbf{P}(\sigma_a \le n, \bigcup_{b\in\mathbb{Z}} S_n = b) \\
&= \sum_{b\in\mathbb{Z}} \mathbf{P}(\sigma_a \le n, S_n = b) \\
&= \sum_{b\in\mathbb{Z}, b\ge a} \mathbf{P}(\sigma_a \le n, S_n = b) + \sum_{b\in\mathbb{Z}, b<a} \mathbf{P}(\sigma_a \le n, S_n = b) \\
&= \sum_{b\in\mathbb{Z}, b\ge a} \mathbf{P}(S_n = b) + \sum_{b\in\mathbb{Z}, b<a} \mathbf{P}(S_n = 2a - b) \\
&= \mathbf{P}(S_n \ge a) + \mathbf{P}(S_n > a) \\
&= \mathbf{P}(S_n \ge a) + \mathbf{P}(S_n < -a) \\
&= \mathbf{P}(S_n \notin [-a, a))
\end{aligned}$$

□

Corollary 2.1.1. $\mathbf{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbf{P}(S_n = a-1) - \mathbf{P}(S_n = a+1)\right]$ *where* $a \in \mathbb{Z}$.

Proof. □

2.2 Arc-Sine Law

Let L denote the last time the random walk hits 0, i.e., $L = \max_{0\le n\le 2N} S_n = 0$, where N denotes the length of the walk.

Theorem 2.2.1.

$$\mathbf{P}(L = 2n) = \frac{1}{2^2 N}\binom{2n}{n}\binom{2N-2n}{N-n}.$$

Remark. By Stirling's approximation,

$$\mathbf{P}(L = 2n) \sim \frac{1}{\pi N}\frac{1}{\sqrt{\left(\frac{n}{N}\right)\left(1 - \frac{n}{N}\right)}}.$$

$$\begin{aligned}
\mathbf{P}(\frac{L}{2N} \le x) &= \mathbf{P}(L \le 2Nx) \\
&= \sum_{n=0}^{[2Nx]} \mathbf{P}(L = 2n) \\
&\sim \sum_{n=0}^{[2Nx]} \frac{1}{\pi N}\frac{1}{\sqrt{(x)(1-x)}} \\
&\sim \int_0^x \frac{dy}{pi\sqrt{y(1-y)}} \\
&= \frac{2}{\pi}\sin^{-1}(\sqrt{x}).
\end{aligned}$$

Proof of Theorem 2.2.1. Define $\tilde{\sigma_0} \inf\{n : S_n = 0, 0 < n \leq N\}$. Consider a path of length $2N$ with $L = 2n$. This path can be formed by a path which takes $S_2n = 0$ and followed by a path of length $2N - 2n$ with $\sigma_0 > 2N - 2n$. Hence, number of paths of length $2N$ with $L = 2n$ is the product of the number of paths of length $2n$ with $S_{2n} = 0$ and the number of paths of length $2N - 2n$ with $\sigma_0 > 2N - 2n$. Hence,

$$\mathbf{P}(L = 2n) = \mathbf{P}(S_{2n} = 0)\mathbf{P}(\tilde{\sigma_0} > 2N - 2n), \tag{2.1}$$

Now let us compute the distribution of $\tilde{\sigma_0}$.

$$\begin{aligned}
\mathbf{P}(\tilde{\sigma_0} > 2k) &= \mathbf{P}(S_1 \neq 0, \ldots, S_{2k} \neq 0) \\
&= 2\mathbf{P}(S_1 > 0, \ldots, S_{2k} > 0) \\
&= \frac{2}{2^{2k}}\{\text{No. of paths start at 0 and stay above -1 for } 2k-1 \text{ steps}\} \\
&= \frac{2}{2^{2k}}\{\text{No. of paths start at 0 and stay below 1 for } 2k-1 \text{ steps}\} \\
&= \mathbf{P}(\sigma_1 > 2k - 1) \\
&= 1 - \mathbf{P}(\sigma_1 \geq 2k - 1) \\
&= \mathbf{P}(S_{2k-1} = -1) + \mathbf{P}(S_{2k-1} = 0) \\
&= \mathbf{P}(S_{2k-1} = -1)
\end{aligned} \tag{2.2}$$

Using (2.1) and (2.2),

$$\begin{aligned}
\mathbf{P}(L = 2n) &= \mathbf{P}(S_{2n} = 0)\mathbf{P}(S_{2N-2n-1} = -1) \\
&= \mathbf{P}(S_{2n} = 0)\mathbf{P}(S_{2N-2n} = 0) \\
&= frac12^2N\binom{2n}{n}\binom{2N-2n}{N-n}.
\end{aligned}$$

The first step analysis of S_2n shows that, $\mathbf{P}(S_{2N-2n} = 0) = \frac{1}{2}\mathbf{P}(S_{2N-2n-1} = 1) + \frac{1}{2}\mathbf{P}(S_{2N-2n-1} = -1)$. Using the symmetry of the walk we know that $\mathbf{P}(S_{2N-2n-1} = 1) = \mathbf{P}(S_{2N-2n-1} = -1)$. This gives the second inequality. □

2.3 Exercises

1. Complete the proof of Reflection Principle (Lemma 2.1.1).
2. Find the distribution of $M_k = \max_{1 \leq k \leq n} S_k$.