Week 10

Random Walk in random environment

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10.1 Random walk in random environment

10.1.1 Sinai's walk

Given $\bar{W}$,

$$\mathbf{P}_{\bar{W}}(X_{n+1} = z | X_n = y) = \begin{cases} W_z & \text{if } y = z+1 \\ 1 - W_z & \text{if } y = z-1 \\ 0 \text{otherwise} \end{cases}$$

$$\overrightarrow{\mathbf{E}}[X_1] = 1.\frac{3}{4} + (-1).\frac{1}{4} = \frac{1}{2}$$

$$\overleftarrow{\mathbf{E}}[X_1] = 1.\frac{1}{3} + (-1).\frac{2}{3} = \frac{-1}{3}.$$

$$\mathbf{E}[X_1] = p.\frac{1}{2} + (1-p)(\frac{-1}{3}) = \frac{1}{6}p - \frac{1}{3}.$$

$\{X_n\}_{n\geq 1}$ Recurrent. Moreover, $\frac{X_a}{(\log n)^2}$is the correct law of large numbers.

$\mathbf{P}^x(X_n \in A) = \mathbf{P}(\mathbf{P}^x_W(X_n \in A))$

Quenched distribution of walk under $\mathbf{P}_W(.)$
Walk is a Markov Chain and tools available.

Averaged: distribution of walk under as $\mathbf{P}$. Although it is not a Markov Chain but is homogenous.

$$\mathbf{E}^0[X_1] = \mathbf{E}[W_0]$$

$$\mathbf{P}(X_3 = 1 | X_1 = 1, X_2 = 0) = \frac{\mathbf{P}(X_3, X_1 = -1, X_2 = 0)}{\mathbf{P}(X_1 = 1, X_2 = 0)} = \frac{\mathbf{E}[W_0^2(1-W_0)]}{\mathbf{E}[W_0(1-W_0)]}.$$

$\Gamma = ((V, E), \{\mu_{xy}\}_{x\sim y})$ Random conducting network. $\mathbf{P}_W(X_n = | X_{n-1} = y) = \frac{\mu_{xy}}{\mu_x}$, $\{\mu_{xy}\}_{x\sim y}$ iid. collection and μ_x and μ_y are not independent.

10.1.2 Examples

Random walk in Galton Watson tree

Environment Choose a realization of Galton Watson tree.
Model1: Put natural weights.
Model2: Biased walk (β)
$x \in \tau$, has k-descendents $k \geq 0$ move to descendent with $\frac{\beta}{1+\beta k}$
move to parent with $\frac{1}{1+\beta k}$.

Random walk in Percolation clusters

Consider $\mathbb{Z}^d$. Now, each edge open with probability p and closed with probability $1-p$. If $p > p_c(d)$ then there exists a infinite connected component. The random walk in such a component is an example of random walk in random environment.

10.2 Recurrence/Transience

Theorem 10.2.1 (1975, Solomon)**.** *Under the assumptions,*

$$\{\omega_x\}_x \textit{is an i.i.d sequence} \tag{10.1}$$
$$\exists \epsilon > 0 \textit{ such that } \mathbf{P}(\omega \in (\epsilon, 1-\epsilon)) = 1 \tag{10.2}$$

the following holds,

$$\mathbf{E}[\log \rho_0] < 0 \Rightarrow \mathbf{P}(X_n \to \infty) = 1$$
$$\mathbf{E}[\log \rho_0] > 0 \Rightarrow \mathbf{P}(X_n \to -\infty) = 1$$
$$\mathbf{E}[\log \rho_0] = 0 \Rightarrow \mathbf{P}(\limsup_{n\to\infty} = \infty, \liminf_{n\to\infty} = -\infty) = 1$$

Theorem 10.2.2. *Under the assumptions, (10.1) and (10.2), the following holds.*

$$\textit{if } \mathbf{E}[\rho_0] < 1, \ \mathbf{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = \frac{1-\mathbf{E}[\rho_0]}{1+\mathbf{E}[\rho_0]}\right) = 1$$

$$\textit{if } \mathbf{E}\left[\frac{1}{\rho_0}\right] < 1, \ \mathbf{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = \frac{1-\mathbf{E}[\rho_0]}{1+\mathbf{E}[\rho_0]}\right) = 1$$

$$\textit{if } \frac{1}{\mathbf{E}[\rho_0]} \leq 1 \leq \mathbf{E}\left[\frac{1}{\rho_0}\right], \ \mathbf{P}\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 1$$

Lemma 10.2.1. *Suppose* $\limsup_{n\to\infty} X_n = \infty$, $T_k = \inf n \geq 0 \mid X_n = k$, *then*

$$\lim_{k\to\infty} \frac{T_k}{k} = c \in [1, \infty) \cup \{\infty\}$$

$$\lim_{n\to\infty} \frac{X_n}{n} = \begin{cases} \frac{1}{c} & \textit{if } c < \infty \\ 0 & \textit{if } c = \infty \end{cases}$$

Proof. $X_n^* = \max_{1\le k\le n} X_k$, $T_{X_n^*} \le n \le T_{X_n^*+1}$, we have,

$$\frac{T_{X_n^*}}{X_n^*} \le \frac{n}{X_n^*} < \frac{T_{X_n^*+1}}{T_{X_n^*+1}} \frac{X_n^*+1}{X_n^*}$$

$\Rightarrow \lim_{n\to\infty} \frac{X_n}{n} = \begin{cases} \frac{1}{c} & \text{if } c < \infty \\ 0 & \text{if } c = \infty \end{cases}$ Therefore,

$$\text{if } c = \infty, \lim_{n\to\infty} \frac{X_n}{n} = 0$$

$$\text{if } c < \infty, \frac{X_n^* - X_n}{n} \le \frac{n - T_{X_n^*}}{n}$$

$$\Rightarrow \limsup_{n\to\infty} \frac{X_n^* - X_n}{n} = \limsup_{n\to\infty} \left(1 - \frac{T_{X_n^*}}{n}\right) = 0$$

$\square$