Discrete Time Martingales

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Origin is from horse-racing (betting system). The dictionary meaning of the word 'martingale' is the harness of a horse.
Let $\{Z_n\}_{n\geq 1}$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 4.0.1. *A sequence of random variables $\{Z_n\}_{n\geq 1}$ is said to be a Martingale if*

$$\mathbb{E}(Z_n | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) = z_{n-1} \quad \forall\ n \geq 2 \tag{4.1}$$

Things to understand- conditional expectation for discrete and conditional random variable [**?**].
Things we will explore-

1. Examples of $\{Z_n\}_{n\geq 1}$ that are martingales.

2. How different are martingales from iid sequences and markov chains?

3. How to interpret **??**?

Example. $\{S_n\}_{n\geq 1}$ and $S_0 \equiv 0$.

$$X_i = \begin{cases} 1, & w.p \ \ 1/2 \\ -1, & w.p \ \ 1/2 \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i$$

Let $s_{n-1}, s_{n-2}, \ldots, s_1 \in \mathbb{Z}$ such that $\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) > 0$

$$\begin{aligned}
\mathbb{E}(S_n | S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) &= \sum_{k \in \mathbb{Z}} k \mathbb{P}(S_n = k | S_{n-1} = s_{n-1}, \ldots, S_1 = s_1) \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(S_n = k, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(S_{n-1} + X_n = k, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(X_n = k - s_{n-1}, S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= \sum_{k \in \mathbb{Z}} k \frac{\mathbb{P}(X_n = k - s_{n-1})\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)}{\mathbb{P}(S_{n-1} = s_{n-1}, \ldots, S_1 = s_1)} \\
&= (s_{n-1} + 1)\mathbb{P}(X_n = -1) + (s_{n-1} - 1)\mathbb{P}(X_n = 1) \\
&= (s_{n-1} + 1)\frac{1}{2} + (s_{n-1} - 1)\frac{1}{2} = s_{n-1}
\end{aligned}$$

Note that the summations here are "finite" sums.
As $s_{n-1}, \ldots, s_1 \in \mathbb{Z}$ were arbitrary, $\{S_n\}_{n \geq 1}$ is a martingale.

Example. $\{X_i\}_{i \geq 1}$ be an iid sequence on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z_n = \prod_{i=1}^n X_i$ and Range$(Z_n) \subset \mathbb{R} \ \forall \ n \geq 1$.
Let $z_{n-1}, \ldots, z_1 \in \mathbb{R}$ such that $\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) > 0$. Then

$$\begin{aligned}
\mathbb{E}(Z_n | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) &= \sum_{k \in Range(Z_n)} k \mathbb{P}(Z_n = k | Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(Z_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(Z_{n-1} X_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \frac{\mathbb{P}(z_{n-1} X_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{k \in Range(Z_n)} k \mathbb{P}(Z_{n-1} X_n = k) \frac{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1)} \\
&= \sum_{u \in S^1, Range(X_n) = S^1} u z_{n-1} \mathbb{P}(X_n = u) \\
&= z_{n-1} \mathbb{E}[X_n] = z_{n-1}
\end{aligned}$$

Note that the sums here might be infinite. In the last step we assume $\mathbb{E}[X_i] = 1$. Now since $\{z_i\}_{i=1}^{n-1}$ were arbitrary, $\{Z_n\}_{n\geq 1}$ is a martingale.

Example.

$$X_i = \begin{cases} 2, & w.p \;\; 1/2 \\ 0, & w.p \;\; 1/2 \end{cases}$$

Then $\mathbb{E}(X_i) = 1$. Therefore, $Z_n = \prod_{i=1}^{n} X_i$ is a martingale. Range (Z_n)= $\{2^n, 0\}$. Note that the mean stays constant and

$$\mathbb{P}(Z_n = 0) = 1 - \frac{1}{2^n}$$

$$\mathbb{P}(Z_n = 2^n) = \frac{1}{2^n}$$

Intuition- The first equation shows that the martingale takes a very low value with very high probability and the second one shows that it takes a very large value with very low probability
Idea behind Markov Chains -

$$\text{“}X_n|X_{n-1}, \dots, X_1\text{”} \stackrel{d}{=} X_n|X_{n-1}$$

Idea behind Martingales - Expected value of Z_n conditioned on the past depends only on Z_{n-1}. $\{Z_n\}_{n\geq 1}$ in law could depend on the entire past!

Week 5 **February 3, 2023**

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We define $f : D \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$ where

$$f(z_1, z_2, \dots, z_{n-1}) = \mathbf{E}[Z_n | Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \dots, Z_1 = z_1]$$

Define $Y_n : \Omega \to \mathbb{R}$ where

$$Y_n(\omega) := f(Z_1(\omega), Z_2(\omega), \dots, Z_{n-1}(\omega)) \tag{4.2}$$

You can check that $\{Y_n\}$ is a random variable.

Property 4.0.1. *Some properties of* $\{Y_n\}$

1. $A := \{Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \dots, Z_1 = z_1\}$

$$\omega \in A \implies Y_n(\omega) = f(z_1, z_2, \dots, z_{n-1})$$

2. $L := \{Y_n \leq c\} = \{f(Z_1, Z_2, \dots, Z_{n-1}) \leq c\}$

$$L \in \mathcal{A}_{n-1} \equiv \textit{observable events upto } n-1$$

(??) $\iff$ $\{Y_n\}$ *has the above two properties*

If $\{Z_n\}$ *is martingale,* $Y_n = Z_{n-1}$

Lemma 4.0.1. *Let $\{Y_n\}_{n\geq 1}$ be martingale. Then,*

$$\forall\, 1 \leq i \leq n,\ \mathbf{E}[Z_n|Z_i, Z_{i-1}, \ldots, Z_1] = Z_i$$

Proof. We fix i and prove by induction on n.
We look at n = i+1. By martingale property,

$$\mathbf{E}[Z_{i+1}|Z_i, Z_{i-1}, \ldots, Z_1] = Z_i$$

Let $k > 0$ and the statement hold for $n = i + k$. We look at $n = i + k + 1$

$$\begin{aligned}
&\mathbf{E}[Z_{i+k+1}|Z_i, Z_{i-1}, \ldots, Z_1] \\
&= \mathbf{E}[\mathbf{E}[Z_{i+k+1}|Z_{i+k}, Z_{i+k-1}, \ldots, Z_1]|Z_i, Z_{i-1}, \ldots, Z_1] \\
&= \mathbf{E}[Z_{i+k}|Z_i, Z_{i-1}, \ldots, Z_1] \qquad \text{[using (??)]} \\
&= Z_i
\end{aligned}$$

where the last equality is obtained from the induction hypothesis □

The property used in the first equality is called the Tower property. We now formally state and prove the same.

Property 4.0.2 (Tower Property)**.**

$$\mathbf{E}[\mathbf{E}[X|Y,Z]|Y] = E[X|Y]$$

Proof.

$$\mathbf{E}[\mathbf{E}[X|Y,Z]|Y] = E[h(Y,Z)|Y] = k(Y)$$

Let $y \in \mathbb{R}$ such that $\mathbf{P}(Y = y) > 0$

$$\begin{aligned}
k(y) &= E[h(Y,Z)|Y] \\
&= \sum_{\substack{m\in\text{Range}(Y)\\ t\in\text{Range}(Z)}} h(m,t)\mathbf{P}(Y = m, Z = t|Y = y) \\
&= \sum_{t\in\text{Range}(Z)} h(y,t)\mathbf{P}(Z = t|Y = y) \\
&= \sum_{t\in\text{Range}(Z)} \sum_{k\in\text{Range}(X)} k\mathbf{P}(X = k|Y = y, Z = t)\mathbf{P}(Z = t|Y = y) \\
&= \sum_{t\in\text{Range}(Z)} \sum_{k\in\text{Range}(X)} k\frac{\mathbf{P}(X = k, Y = y, Z = t)}{\mathbf{P}(Y = y, Z = t)} \frac{\mathbf{P}(Z = t, Y = y)}{\mathbf{P}(Y = y)} \\
&= \sum_{k\in\text{Range}(X)} \sum_{t\in\text{Range}(Z)} k\frac{\mathbf{P}(X = k, Y = y, Z = t)}{\mathbf{P}(Y = y)} \\
&= \sum_{k\in\text{Range}(X)} k\frac{\mathbf{P}(X = k, Y = y)}{\mathbf{P}(Y = y)} \\
&= \mathbf{E}[X|Y = y]
\end{aligned}$$

□

$\{Z_n\}$ is a Martingale
$E[Z_n|Z_i, Z_{i-1}, ..., Z_1] = Z_i$ where $1 \leq i \leq n$
$E[Z_n] = E[Z_1]$

4.1 Stopping time and Stopped process

Definition 4.1.1. *Let $(\Omega, A, \mathbf{P})$ be a probability space on which $\{Z_n\}_{n\geq 1}$ is defined.*
A_k= events determined by $Z_1, Z_2, ..., Z_k$.
$T : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ is called a ***stopping time*** *for $\{Z_n\}_{n\geq 1}$ if $\{T = k\} \in A_k$, i.e. $\mathbf{1}_{T=k}$ = "function" of $Z_1, Z_2, ..., Z_k$.*

Definition 4.1.2. *for any stopping time T, we define the* ***stopped process****:*

$$Z_n^T(w) = Z_{n\wedge T(w)}(w) = \begin{cases} Z_n & \text{if } n < T \\ Z_T & \text{if } n \geq T \end{cases}$$

Theorem 4.1.1. *Given a sequence of random variables $\{Z_n\}_{n\geq 1}$ and $T : \Omega \longrightarrow \mathbb{N}\cup\{\infty\}$, a stopping time of $\{Z_n\}_{n\geq 1}$. Then $\{Z_n^T\}_{n\geq 1}$ is a martingale iff $\{Z_n\}_{n\geq 1}$ is a martingale*

Idea of the proof: $\mathbf{E}(Z_n^T|Z_{n-1}^T, ..., Z_1^T) = \mathbf{E}(Z_{n-1}^T)$
Take $Z_1 = z_1, ..., Z_{n-1} = z_{n-1} \rightarrow$ determine if T has happened by time n-1 or not
$\rightarrow$ if $T \geq n$, $Z_n^T = Z_n$
if $T < n$, $Z_n^T = z_{n-1}$ $\square$
Let $\{X_i\}, X, Y, Z$ be discrete random variables.

$$\mathbf{E}[Y|X = x_1] = \sum_{k\in Range(Y)} k\mathbf{P}(Y = k|X = x_1) \tag{4.3}$$

$$\mathbf{E}[Y|X_1 = x_1, ..., X_n = x_n] = \sum_{k\in Range(Y)} k\mathbf{P}(Y = k|X_1 = x_1, ..., X_n = x_n) \tag{4.4}$$

where $\mathbf{E}[Y|X_1 = x_1, ..., X_n = x_n] \equiv f(x_1, x_2, ..., x_n)$
$f : \prod_{i=i}^n Range(X_i) \rightarrow \mathbb{R}$

$$\mathbf{E}[Y|X_1, ..., X_n](\omega) = \sum_{x\in Range(X_i)} k\mathbf{E}(Y = k|X_1 = x_1, ..., X_n = x_n)\mathbf{1}_{(X_1=x_1,...,X_n=x_n)}(\omega) \tag{4.5}$$

where $\mathbf{E}[Y|X_1, ..., X_n] \equiv \mathbf{E}[Y|A_n]$, i.e. events observable by time n.

4.2 Tower Property

Let $A_n \subset A_m$, $n \leq m$ then $\mathbf{E}[E[Y|A_m]|A_n] = \mathbf{E}[Y|A_n]$

4.3 Markov property and Strong Markov Property

Property for $\{X_n\}$ random walk on (Γ, y).

$\Omega = \mathbb{V}^{\mathbb{Z}_+}$.
$X_n : \Omega \to \mathbb{V}$.
$X_n(\omega) = \omega(n)$.
A_n= events determined by $X_1, ..., X_n$.
$\mathbf{P}^x(X_0 = x_0, X_1 = x_1, ..., X_n = x_n) = \mathbf{1}_x(x_0) \prod_{i=0}^{n} P(x_{i-1}, x_i)$
$P(x, y) = \frac{\mu_{xy}}{\mu_y}$
$\xi \to$ random variable that is determinable by A_n i.e. $\xi = g(X_1, X_2, ..., X_n$ for some g.
$\forall k \geq 1$, $\theta_k : \Omega \to \mathbb{V}^{\mathbb{Z}_+}$, $\theta_k(\omega) = (\omega(k), \omega(k+1), ...)$
Let $\eta : \Omega \to \mathbb{R}$ be any random variable.
$\mathbf{E}[\xi"\eta \text{ after time } n"|A_n] = \mathbf{E}[\xi \mathbf{E}^{X_n}[\eta \text{ after time } n"]]$
Markov Property:

$$\mathbf{E}[(\xi) \times (\eta.\theta_n)|A_n] = \mathbf{E}[\xi \mathbf{E}^{X_n}[\eta]] \tag{4.6}$$

Strong Markov Property:
T is a stopping time of $\{X_n\}_{n\geq 1}$.
A_n ≡events determined by time T.
if ξ is determinable by time T, then

$$\mathbf{E}[(\xi) \times (\eta.\theta_T)|A_T] = \mathbf{E}[\xi \mathbf{E}^{X_T}[\eta]] \tag{4.7}$$