Lecture 1: Finite length random walks on $\mathbb{Z}$

LECTURER: SIVA ATHREYA SCRIBE: SRIVATSA B, VENKAT TRIVIKRAM[†]

1.1 Definitions

Random walks serve as very useful models in many applications. They are simple to state and understand, yet they lead to lots of intractable questions.

Notation. $\mathbb{N} = \{k \in \mathbb{Z} : k \geq 1\}$ *and* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

We now proceed to construct what is called a "simple random walk" on $\mathbb{Z}$ of finite length $N \in \mathbb{N}$. The sample space Ω_N and the event space $\mathcal{F}_N$ are described below.

$$\Omega_N := \{(\omega_1, \ldots, \omega_N) : \omega_i \in \{-1, 1\} \ \forall \ 1 \leq i \leq N\}$$

$$\mathcal{F}_N := \{A : A \subseteq \Omega_N\}$$

The probability function $\mathbb{P}_N : \Omega_N \to [0, 1]$ is defined as

$$\mathbb{P}_N(A) := |A| \ 2^{-N}$$

We also define random variables X_k and S_k on Ω_N for $1 \leq k \leq N$ as

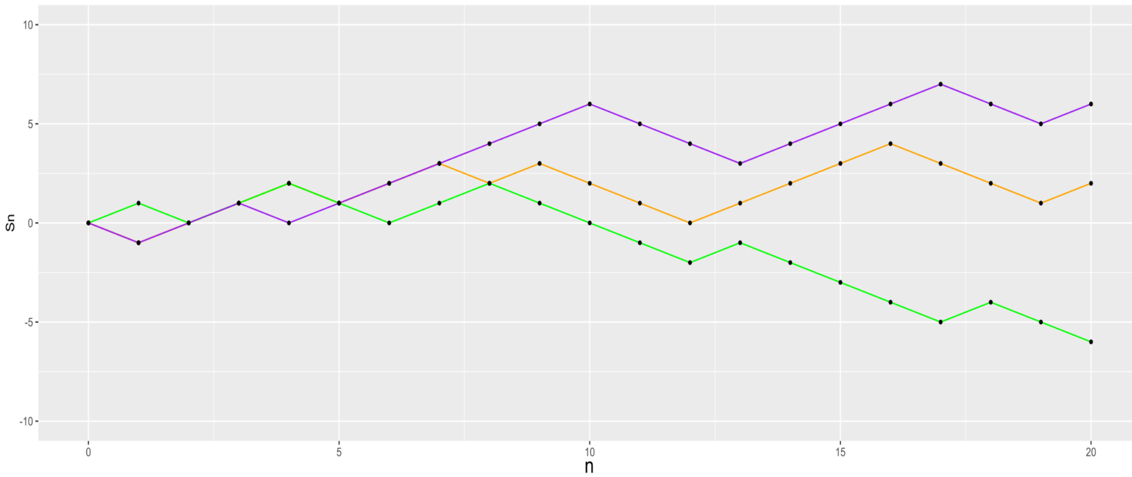
$$X_k : \Omega_N \to \{-1, 1\} \ ; \ X_k(\omega) := \omega_k$$

$$S_k : \Omega_N \to \mathbb{Z} \ ; \ S_k(\omega) := \sum_{i=1}^{k} X_k(\omega) \ ; \ S_0(\omega) := 0 \text{ for all } \omega \in \Omega_N$$

Definition 1.1.1. *Fix $N \in \mathbb{N}$. The sequence of random variables $\{S_k\}_{k=1}^{N}$ on $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ is called a (symmetric) simple random walk on $\mathbb{Z}$, of finite length N, starting at 0.*

[†] added illustrations

Figure 1.1: Three possible trajectories for $(S_n)_{n=0}^N$



In what follows, we suppress the subscript N while referring to the probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$, and we assume that $N \in \mathbb{N}$ is fixed.

Observations.

(a) $\{X_k\}_{k=1}^N$ are iid, i.e. independent and identically distributed.

Proof.

$$\begin{aligned}\mathbb{P}(X_k = 1) = \mathbb{P}(\{\omega \in \Omega : \omega_k = 1\}) &= 2^{-N}|\{\omega \in \Omega : \omega_k = 1\}| \\ &= 2^{-N}2^{N-1} \\ &= \frac{1}{2} \\ &= \mathbb{P}(X_k = -1)\end{aligned}$$

So $\{X_k\}_{k=1}^N$ are identically distributed. Independence is left as an exercise. □

(b) (Independent increments) For $1 \leq k_1 \leq k_2 \leq \ldots \leq N$, $\{S_{k_i} - S_{k_{i-1}} : 1 \leq i \leq N\}$ are independent random variables.

Proof. Observe that, for $1 \leq k < l \leq N$, we have $S_l - S_k = \sum_{i=k+1}^{l} X_i$. Therefore, if $1 \leq a < b \leq c < d \leq N$, we see that $S_b - S_a$ and $S_d - S_c$ are functions of disjoint sets of independent random variables, and hence the claim is true. □

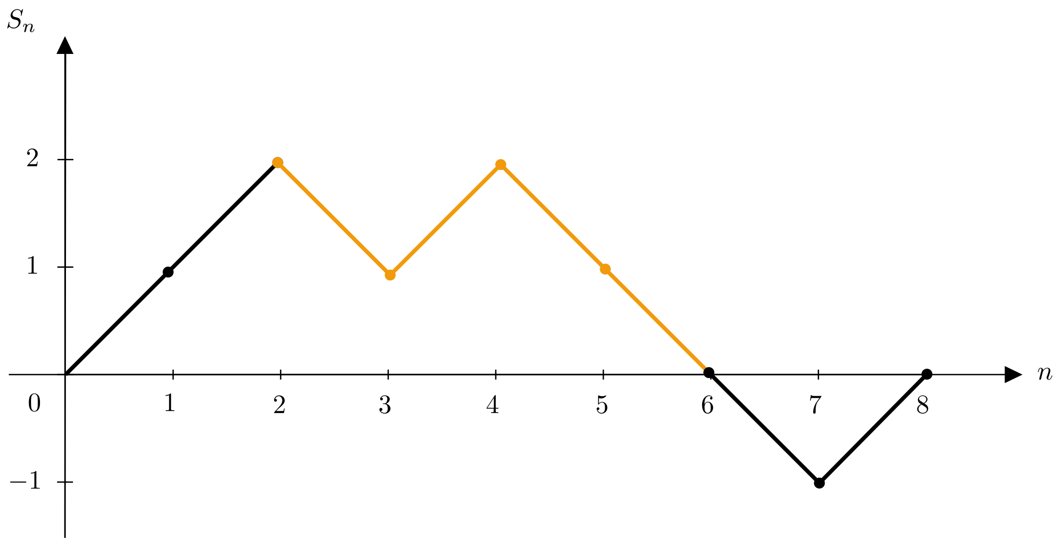


Figure 1.2: Independent (colored) increments in a simple random walk

(c) (Stationary in increments) For $1 \leq k < m \leq N$, $\mathbb{P}(S_m - S_k = \alpha) = \mathbb{P}(S_{m-k} = \alpha)$ for every $\alpha \in \mathbb{Z}$.

Proof. We use the fact that $\{X_i\}_{i=1}^N$ are identically distributed in the following argument.

$$\mathbb{P}(S_m - S_k = \alpha) = \mathbb{P}\left(\sum_{i=k+1}^{m} X_i = \alpha\right) = \mathbb{P}\left(\sum_{i=1}^{m-k} X_i = \alpha\right) = \mathbb{P}(S_{m-k} = \alpha)$$

□

(d) (Markov Property) For $\alpha_i \in \mathbb{Z},\ 1 \leq i \leq N$ and $0 \leq n \leq N$,

$$\mathbb{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}, \ldots, S_1 = \alpha_1) = \mathbb{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}),$$

assuming (of course) that the conditional probabilities are well defined.

Proof. Left as an exercise. □

(e) (Conditional Law) For $1 \leq k < m \leq N$, $\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b - a)$.

Proof. Left as an exercise. □

(f) (Moments) For $1 \leq k \leq N$, we have $\mathbb{E}[X_k] = \mathbb{E}[S_k] = 0$ and $\mathrm{Var}[S_k] = k$.

Proof. By definition of expected value, $\mathbb{E}[X_k] = 1(1/2) - 1(1/2) = 0$. By linearity of expected values, $\mathbb{E}[S_k] = \sum_{i=1}^{k} \mathbb{E}[X_i] = 0$.

Since $\mathbb{E}[S_k] = 0$, $\mathrm{Var}[S_k] = \mathbb{E}[(\sum_{i=1}^{k} X_i)^2] = \sum_{i=1}^{k} \mathbb{E}[X_k^2] = k$. As an exercise, show that $\mathbb{E}[(\sum_{i=1}^{k} X_i)^2] = \sum_{i=1}^{k} \mathbb{E}[X_k^2]$. $\square$

(g) (Distribution of S_n) For $x \in \{-n, -n+2, \ldots, n-2, n\}$, we have

$$\mathbb{P}(S_n = x) = \mathbb{P}(S_n = -x) = \binom{n}{\frac{n+x}{2}} 2^{-n}$$

Proof. We only provide a sketch of the proof, which is left as an exercise. For $0 \leq j \leq N$, $\{S_n = 2j - n\} = \{S_n = j - (n-j)\}$. So there must be a total of j steps to the right and $n - j$ steps to the left. Therefore

$$\mathbb{P}(S_n = 2j - n) = 2^{-N} \ |\{\omega \in \Omega : \cdots\}| = 2^{-n}\binom{n}{j}$$

$\square$

(h) (Mode) The mode of the above distribution is achieved in the middle, i.e. at $x = 0$ and at $x = 1, -1$ for S_{2n} and S_{2n-1} respectively.

Proof.

$$\mathbb{P}(S_{2n} = 0) = \mathbb{P}(S_{2n-1} = 1) = \binom{2n}{n} 2^{-2n}$$

$\square$

(i) (Stirling's formula) Using Stirling's approximation, for large n, we have

$$\binom{2n}{n} = \frac{2n!}{n!n!} \sim \frac{(2n)^{2n} e^{-2n}\sqrt{4\pi n}}{n^{2n} e^{-2n}\sqrt{2\pi n}\sqrt{2\pi n}} \sim \frac{2^{2n}}{\sqrt{\pi n}} \tag{$*$}$$

Therefore,

$$\mathbb{P}(S_{2n} = 0) = \binom{2n}{n}\frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}} \quad \text{as} \quad n \to \infty$$

This approximation, although correct, has a caveat - we chose to keep N fixed, but as $n \to \infty$, we must also let $N \to \infty$, and this requires subtler arguments. A few consequences of this approximation are mentioned in the exercises.

1.2 Stopping times

Motivation for this section comes from the classic Gambler's ruin problem. We can interpret a simple random walk as a fair game between two players, where in round k, a player wins the amount X_k. Then S_n denotes the capital of one player over the other after n rounds.

We would like to answer the following question - "Is it possible to stop the game in a favorite moment, i.e., can clever stopping lead to a positive expected gain?". In other words, can we design a $T(\omega)$ for every $\omega \in \Omega$ such that $\mathbb{E}[S_T] > 0$? Of course, the decision to stop may only depend on the trajectory until that time: no "insider knowledge" about the future of the trajectory is permitted.

To formalize this setting, we make the following definition.

Definition 1.2.1. *An event $A \subseteq \Omega$ is said to be observable by time n if it is a (possibly empty) union of basic / elementary events of the form*

$$\{\omega \in \Omega : \omega_1 = o_1, \ldots, \omega_n = o_n\}$$

where $o_i \in \{-1, 1\}$ for $1 \leq i \leq n$.

We also define $\mathcal{A}_0 = \{\phi, \Omega\}$ and set

$$\mathcal{A}_n := \{A \in \mathcal{F} : A \text{ is observable by time } n\}.$$

Immediately, we observe that

$$\{\phi, \Omega\} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_{N-1} \subseteq \mathcal{A}_N = \mathcal{F}$$

As an easy exercise, verify that each $\mathcal{A}_n$ is closed with respect to taking complement, union and intersection. Such a sequence $\{\mathcal{A}_i\}_{i=0}^{N}$ is called a *filtration*.

Definition 1.2.2. *A function $T : \Omega \to \{0, 1, \ldots, N\} \cup \{\infty\}$ is called a stopping time if for each $0 \leq n \leq N$,*

$$\{T = n\} = \{\omega \in \Omega : T(\omega) = n\} \in \mathcal{A}_n$$

Example. For $a \in \mathbb{Z}$, let $\sigma_a = \inf\{n : S_n = a, 0 \leq n \leq N\}$ denote the *first* hitting time of a. As an exercise, show that σ_a is a stopping time.

Example. For $a \in \mathbb{Z}$, let $L_a = \max\{n : S_n = a, 0 \leq n \leq N\}$ denote the *last* hitting time of a. As an exercise, show that L_a is NOT a stopping time.

Theorem 1. *Let $T : \Omega \to \{0, 1, \ldots, N\}$ be a stopping time. Then*

$$\mathbb{E}[S_T] = 0$$

where $S_T : \Omega \to \mathbb{Z}$ maps $\omega \mapsto S_{T(\omega)}(\omega)$.

Proof.

$$
\begin{aligned}
S_T = \sum_{k=1}^{N} S_k \mathbb{1}\{T = k\} &= \sum_{k=1}^{N} S_k(\mathbb{1}\{T \geq k\} - \mathbb{1}\{T \geq k+1\}) \\
&= \sum_{k=1}^{N} (S_k - S_{k-1})\mathbb{1}\{T \geq k\} \\
&= \sum_{k=1}^{N} X_k \mathbb{1}\{T \geq k\}
\end{aligned}
$$

where we take $\mathbb{1}\{T \geq N+1\} = 0$. Now, we can write $\mathbb{E}[S_T]$ as

$$
\mathbb{E}[S_T] = \sum_{k=1}^{N} \mathbb{E}[X_k \mathbb{1}\{T \geq k\}] \tag{$\dagger$}
$$

Observe that for $1 \leq k \leq N$, we have

$$
X_k \mathbb{1}\{T \geq k\} = \begin{cases} 1, & \text{for } X_k = 1,\ T \geq k \\ -1, & \text{for } X_k = -1,\ T \geq k \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\mathbb{E}[X_k \mathbb{1}\{T \geq k\}] = \mathbb{P}(X_k = 1, T \geq k) - \mathbb{P}(X_k = -1, T \geq k) \tag{$\dagger\dagger$}
$$

Now,

$$
\{T \geq k\} = \{T < k\}^c = \left(\bigcup_{l=0}^{k-1} \{T = l\} \right)^c \in \mathcal{A}_{k-1}
$$

Using the fact that $\{T \geq k\} \in \mathcal{A}_{k-1}$, one can show that (details left as an exercise)

$$
\mathbb{P}(X_k = 1, T \geq k) = \mathbb{P}(X_k = -1, T \geq k) = \frac{1}{2}\mathbb{P}(T \geq k)
$$

Substituting the above values in ($\dagger$) and ($\dagger\dagger$), we finally have

$$
\mathbb{E}[S_T] = 0
$$

$\square$

As an exercise, compute $\mathrm{Var}[S_T]$.

Definition 1.2.3. *A bet sequence / game system is a sequence of random variables $V_k : \Omega \to \mathbb{R}$ such that*

$$
\{V_k = c\} \in \mathcal{A}_{k-1} \textit{ for every } c \in \mathbb{R} \textit{ and } 1 \leq k \leq N
$$

Theorem 2. *Let $\{V_k\}_{k=1}^{N}$ be a bet sequence. Then*

$$
\mathbb{E}[S_N^V] = 0 \quad \textit{where} \quad S_N^V = \sum_{k=1}^{N} V_k X_k
$$

In this setting, S_N^V is interpreted as the "total gain".

Proof. Since Ω is finite, we may write

$$\text{Range}(V_k) = \{c_i^k : 1 \le i \le m_k\} \text{ where } c_i^k \in \mathbb{R}$$

$$V_k = \sum_{i=1}^{m_k} c_i^k \mathbb{1}\{V_k = c_i^k\}$$

Now, since $\mathbb{E}[X_k] = 0$, and since $X_k \perp \mathbb{1}\{V_k = c_i^k\}$, we get

$$\begin{aligned}
\mathbb{E}[S_N^V] = \sum_{k=1}^{N} \mathbb{E}[V_k X_k] &= \sum_{k=1}^{N} \mathbb{E}\left[X_k \sum_{i=1}^{m_k} c_i^k \mathbb{1}\{V_k = c_i^k\}\right] \\
&= \sum_{k=1}^{N} \sum_{i=1}^{m_k} c_i^k \mathbb{E}[X_k \mathbb{1}\{V_k = c_i^k\}] \\
&= \sum_{k=1}^{N} \sum_{i=1}^{m_k} c_i^k \mathbb{E}[X_k] \mathbb{P}(V_k = c_i^k) \\
&= 0
\end{aligned}$$

□

1.3 Exercises

1. Show that $\{X_k\}_{k=1}^N$ are independent.
2. Show that $\{S_n\}_{n=0}^N$ satisfies the Markov property.
3. For $1 \le k < m \le N$, show that $\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b - a)$.
4. Show that $\mathbb{E}[S_n^2] = \sum_{i=1}^n \mathbb{E}[X_i^2]$.
5. (a) Show that for any $a, b \in \mathbb{R}$,

   $$\mathbb{P}(a \le S_n < b) \le (b - a)\ \mathbb{P}(S_n \in \{-1, 0, 1\}).$$

   (b) Using (a), conclude that

   $$\mathbb{P}(a \le S_n < b) \to 0 \quad \text{as} \quad n \to \infty.$$

   Thus, we observe that the walk exits any finite interval as $n \to \infty$.
6. Verify that each $\mathcal{A}_n$, $0 \le n \le N$, is closed with respect to taking complement, union and intersection.
7. For $a \in \mathbb{Z}$, let $\sigma_a = \inf\{n : S_n = a, 0 \le n \le N\}$. Show that σ_a is a stopping time.
8. For $a \in \mathbb{Z}$, let $L_a = \max\{n : S_n = a, 0 \le n \le N\}$. Show that L_a is not a stopping time.
9. Let $T : \Omega \to \{0, 1, \ldots, N\}$ be a stopping time. Compute $\text{Var}[S_T]$.
10. Show that X_k and $\mathbb{1}\{T \ge k\}$ are independent.