

Week 4

Energy and Variational Methods

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4.1 Transition densities and Laplacian

We can recall that $\mathbf{P}(X_1 = y) = \mathcal{P}(x, y) = \frac{\mu_{xy}}{\mu_x}$.

Definition 4.1.1. The transition density w.r.t weights μ of a random walk $\{X_n\}$ is given by:

$$p_n(x, y) = \frac{\mathbf{P}^x(X_n = y)}{\mu_y} := \frac{\mathcal{P}_n(x, y)}{\mu_y} \quad \mu \geq 1$$

$$p_0(x, y) = \frac{1_{\{x\}}(y)}{\mu_y} = \frac{1_{\{x\}}(y)}{\mu_x}.$$

$$p(x, y) \equiv p_1(x, y) = \frac{\mu_{xy}}{\mu_x \mu_y}$$

Lemma 4.1.1. Let p_n be the transition densities of $\{X_n\}_{n \geq 0}$

1. $p_{n+m}(x, y) = \sum_{z \in V} p_n(x, z) p_m(z, y) \mu_z$
2. $\forall x, y \in V, p_n(x, y) = p_n(y, x)$
3. $\forall x, y \in V, \sum_{z \in V} p_n(x, z) \mu_z = 1 = \sum_{z \in V} p_n(z, y) \mu_z$

Proof. 1.

$$\begin{aligned} p_{n+m}(x, y) &= \frac{\mathbf{P}^x(X_{n+m} = y)}{\mu_y} \\ &= \sum_{z \in V} \frac{\mathbf{P}^x(X_{n+m} = y, X_n = z)}{\mu_y} \\ &= \frac{1}{\mu_y} \sum_{z \in V} \sum_{0 \leq i < n+m, x_i \in V} 1_{\{x\}}(x_0) \prod_{i=0}^{n-1} \mathcal{P}(x_i, x_{i+1}) 1_{\{z\}}(x_n) \prod_{i=n}^{n+m} \mathcal{P}(x_i, x_{i+1}) 1_{\{y\}}(x_{n+m}) \\ &\stackrel{\text{H1}}{=} \frac{1}{\mu_y} \sum_{z \in V} \mathbf{P}^x(X_n = z) \mathbf{P}^z(X_m = y) \\ &= \frac{1}{\mu_y} \sum_{z \in V} p_n(x, z) \mu_z p_m(z, y) \mu_y \\ &= \sum_{z \in V} p_n(x, z) p_m(z, y) \mu_z \end{aligned}$$

2.

$$p_n(x, y) = \frac{\mathbf{P}^x(X_n = y)}{\mu_y} = \frac{\mathbf{P}^y(X_n = x)}{\mu_x} = p_n(y, x)$$

The second equality is obtained by applying the Detailed Balance equations.

3.

$$\begin{aligned} \sum_{z \in V} p_n(x, z) \mu_z &= \sum_{z \in V} \mathbf{P}^x(X_n = z) = 1 \\ \sum_{z \in V} p_n(z, y) \mu_z &= \sum_{z \in V} p_n(y, z) \mu_z = \sum_{z \in V} \mathbf{P}^y(X_n = z) = 1 \end{aligned}$$

□

4.2 Function Spaces

Definition 4.2.1.

$$\begin{aligned} C(V) &= \{f : V \rightarrow \mathbb{R}\} = \mathbb{R}^V \\ \text{Co}(V) &= \{f : V \rightarrow \mathbb{R}, f \neq 0 \text{ on finitely many points}\} \\ C_+(V) &= \{f : f \in C(V), f \geq 0\} \\ \text{Supp}(f) &= \{x : x \in V, f(x) \neq 0\} \end{aligned}$$

Definition 4.2.2. We define the **norm** of a function as the following

$$\begin{aligned} \forall p \in [1, \infty), \|f\|_p &= \left(\sum_{x \in V} |f(x)|^p \mu_x \right)^{\frac{1}{p}} \\ \|f\|_\infty &= \sup\{|f(x)| : x \in V\} \end{aligned}$$

f is said to be L^p on the graph with vertex set V and weights μ if and only if f is a function defined on the vertex set, V and its p -norm is finite everywhere.

$$f \in L^p(V, \mu) \iff f \in C(V) \text{ and } \|f\|_p < \infty$$

Definition 4.2.3. We define an inner product on the $L^2(V, \mu)$ space in the following way

$$\langle f, g \rangle = \sum_{x \in V} f(x)g(x)\mu_x$$

$$\begin{aligned} \mathbf{E}[f(X_n)] &= \sum_{x \in V} f(x) \mathbf{P}^x(X_n = x) \\ &= \sum_{x \in V} f(x) p_n(x, x) \mu_x \\ &= \langle f, p_n(x, \cdot) \rangle \end{aligned}$$

which brings us to define a new function

Definition 4.2.4. $\mathcal{P}_n : C(V) \rightarrow C(V)$ given by

$$\mathcal{P}_n f(x) = \sum_{z \in V} f(z) p_n(x, z) \mu_z = \langle f, p_n(x, \cdot) \rangle$$

where $\Delta : C(V) \rightarrow C(V)$ as an “operation” on $C(V)$ is

$$\Delta = P - I$$

We write $\mathcal{P}_1 f(x)$ as $\mathcal{P}f(x)$ and proceed to look at computations and lemmas involving $\mathcal{P}f$.

Lemma 4.2.1.

$$\forall x \in V, \mathcal{P}f(x) - f(x) = \Delta f(x)$$

Proof.

$$\begin{aligned} \mathcal{P}f(x) - f(x) &= \sum_{z \in V} f(z) p(x, z) \mu_z - f(x) \\ &\stackrel{*}{=} \sum_{z \in V} p(x, z) \mu_z (f(z) - f(x)) \\ &= \sum_{z \in V} \frac{\mu_{xz}}{\mu_x \mu_z} \mu_z (f(z) - f(x)) \\ &= \frac{1}{\mu_x} \sum_{z \in V} \mu_{xz} (f(z) - f(x)) \\ &= \Delta f(x) \end{aligned}$$

* is left as an exercise and can be proved using property 2 from (4.1.1) □

Corollary 4.2.1.

$$\Delta f = 0 \iff f(x) = \mathcal{P}f(x) = \mathbf{E}^x[f(X_1)]$$

Definition 4.2.5. We define a function $A : C(V) \rightarrow C(V)$ as

$$\|A\|_{p \rightarrow p} = \sup\{\|Af\|_p : \|f\|_p \leq 1\}$$

Proposition 4.2.1. 1. $\mathcal{P}1 = 1$

where $1(x) = 1 \forall x \in V$

2. $|\mathcal{P}f| \leq \mathcal{P}|f|$
where $f \in C(V)$

3. $\|\mathcal{P}\|_{p \rightarrow p} \leq 1$
 $\|\Delta\|_{p \rightarrow p} \leq 2$
where $p \in [1, \infty) \cup \{\infty\}$

Proof. 1.

$$\mathcal{P}1(x) = \sum_{z \in V} p(x, z) \mu_z = 1 = 1(x)$$

2.

$$\begin{aligned} |\mathcal{P}f(x)| &= \left| \sum_{z \in V} f(z) p(x, z) \mu_z \right| \\ &\leq \sum_{z \in V} |f(z)| p(x, z) \mu_z \\ &= \mathcal{P}|f|(x) \end{aligned}$$

3.

$$\begin{aligned} \|\mathcal{P}f\|_p^p &= \sum_{x \in V} |\mathcal{P}f(x)|^p \mu_x \\ &= \sum_{x \in V} \left| \sum_{z \in V} f(z) p(x, z) \mu_z \right|^p \mu_x \\ &\stackrel{*}{\leq} \sum_{x \in V} \left(\sum_{z \in V} |f(z)|^p p(x, z) \mu_z \right) \left(\sum_{z \in V} 1^q p(x, z) \mu_z \right) \mu_x \\ &= \sum_{x \in V} \left(\sum_{z \in V} |f(z)|^p p(x, z) \mu_z \right) \mu_x \\ &\stackrel{**}{=} \sum_{z \in V} |f(z)|^p \mu_z \\ &= \|f\|_p^p \\ \implies \|\mathcal{P}\|_{p \rightarrow p} &\leq 1 \end{aligned} \tag{4.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$

We leave the proofs of the following as exercises

*, which can be proved using Holder's inequality, ** and the $p = \infty$ case

$$\begin{aligned} \|\Delta f\|_p^p &= \|\mathcal{P}f - f\|_p^p \\ &\leq (\|\mathcal{P}f\|_p + \|f\|_p)^p \\ &\leq 2^{p-1} (\|\mathcal{P}f\|_p^p + \|f\|_p^p) \\ &\leq (2\|f\|_p)^p \quad [\because \|\mathcal{P}f\|_p \leq \|f\|_p] \\ \implies \|\Delta\|_{p \rightarrow p} &\leq 1 \end{aligned}$$

The final inequality is obtained from (4.1).

□

Proposition 4.2.2. \mathcal{P} is self-adjoint on $L^2(V, \mu)$

$$\forall f, g \in L^2(V, \mu), \langle \mathcal{P}f, g \rangle = \langle f, \mathcal{P}g \rangle$$

Proof.

$$\begin{aligned} \langle \mathcal{P}f, g \rangle &= \sum_{x \in V} \mathcal{P}f(x)g(x)\mu_x \\ &= \sum_{x \in V} \left(\sum_{z \in V} f(z)p(x, z)\mu_z \right) g(x)\mu_x \\ &\stackrel{Ex}{=} \sum_{z \in V} f(z)\mu_z \sum_{x \in V} p(z, x)g(x)\mu_x \\ &= \sum_{z \in V} f(z)\mathcal{P}g(z)\mu_z \\ &= \langle f, \mathcal{P}g \rangle \end{aligned}$$

□

4.3 Dirichlet forms

Definition 4.3.1. We define the quadratic form on $L^2(V, \mu)$, \mathcal{E} as

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y))(g(x) - g(y))\mu_{xy}$$

whenever the series converges absolutely.

Theorem 4.3.1 (Discrete Green's Theorem). $\forall f, g \in C(V)$,

$$\begin{aligned} \sum_{x \in V} \sum_{y \in V} |f(x) - f(y)| |g(x)| \mu_x &< \infty \\ \implies \mathcal{E}(f, g) &= -\langle \Delta f, g \rangle \end{aligned}$$

We present an application of (4.3.1)

Lemma 4.3.1. Let (Γ, μ) be a weighted graph such that $\mu(V) < \infty$. Then, (Γ, μ) is **recurrent**.

Proof. Fix $Z \in V$ Define $\Phi : V \rightarrow \mathbb{R}$ where $\Phi(x) := \mathbf{P}^x(\mathcal{T}_Z = \infty)$

1. Firstly observe that $\Phi(z) = \mathbf{P}^z(\mathcal{T}_Z = \infty) = 0$
2. $\forall n \geq 1, x \neq z$
 $\Phi_n(x) := \mathbf{P}^x(\mathcal{T}_Z = n) = \sum_{u \in V} \mathcal{P}(x, u)\Phi_{n-1}(u)$

This holds true from a simple logical argument. Starting from x , hitting z in n steps is equivalent to jumping from x to some vertex u and hitting z in $n - 1$ steps.

$$3. 1 - \Phi(x) = \mathbf{P}^x(\mathcal{T}_z < \infty) = \sum_{n=0}^{\infty} \mathbf{P}^x(\mathcal{T}_z = n) = \sum_{n=1}^{\infty} \Phi_n(x)$$

$$4. \Phi \equiv 0$$

$$\begin{aligned} \sum_{n=1}^k \Phi_n(x) &= \sum_{n=1}^k \sum_{u \in V} \mathcal{P}(x, u) \Phi_{n-1}(u) \\ \implies \sum_{n=1}^k \Phi_n(x) &= \sum_{u \in V} \mathcal{P}(x, u) \sum_{n=1}^k \Phi_{n-1}(u) \\ \implies \sum_{n=1}^{\infty} \Phi_n(x) &= \sum_{u \in V} \mathcal{P}(x, u) \sum_{n=1}^{\infty} \Phi_{n-1}(u) \\ \implies 1 - \Phi(x) &= \sum_{u \in V} \mathcal{P}(x, u) (1 - \Phi(u)) \\ \implies 1 - \Phi(x) &= \sum_{u \in V} p(x, u) (1 - \Phi(u)) \mu_u \\ \implies 1 - \Phi &= \mathcal{P}(1 - \Phi) \\ \implies \Delta(1 - \Phi) &= 0 \end{aligned}$$

Then, by theorem (4.3.1),

$$\begin{aligned} \mathcal{E}(1 - \Phi, 1 - \Phi) &= \langle \Delta(1 - \Phi), 1 - \Phi \rangle = 0 \\ \implies \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (\Phi(x) - \Phi(y))^2 \mu_{xy} &= 0 \\ \implies \Phi(x) &= \Phi(y) \quad \forall x, y \in V \\ \implies \Phi &\text{ is constant} \\ \implies \Phi &\equiv 0 \end{aligned}$$

The last equality holds as $\Phi(z) = 0$.

Since, $\Phi \equiv 0$ for arbitrary z , (Γ, μ) is recurrent.

□

To proof theorem (4.3.1), we start with some prerequisites.

Definition 4.3.2.

$$\begin{aligned} \mathcal{H}^2(V) &= \{f : f \in C(v), \mathcal{E}(f, f) < \infty\} \\ \|f\|_{\mathcal{H}^2} &= \sqrt{\mathcal{E}(f, f) + f^2(\rho)} \quad \text{for some fixed } \rho \in V \end{aligned}$$

Proposition 4.3.1. *Let (Γ, μ) be a graph satisfying properties, H1 and H2.*

1. $|f(x) - f(y)| \leq \frac{1}{\sqrt{\mu_{xy}}} \sqrt{\mathcal{E}(f, f)} \quad \forall x \sim y$
2. $\mathcal{E}(f, f) = 0 \iff f \text{ is constant}$

$$3. f \in L^2 \implies \mathcal{E}(f, f) \leq 2 \|f\|_2^2$$

Proof. 1. If $\mathcal{E}(f, f) = \infty$, then we are done

Let $\mathcal{E}(f, f)$ be finite

$$\begin{aligned} \mathcal{E}(f, f) &= \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y))^2 \mu_{xy} \\ &\geq (f(x) - f(y))^2 \mu_{xy} \quad \forall x, y \in V \end{aligned}$$

2. The forward direction is left as an exercise. The reverse direction follows from the definition.

3.

$$\begin{aligned} \mathcal{E}(f, f) &\leq \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y))^2 \mu_{xy} \\ &\leq \sum_{x \in V} \sum_{y \in V} (|f(x)|^2 + |f(y)|^2) \mu_{xy} \\ &\stackrel{Ex}{=} \sum_{x \in V} |f(x)|^2 \mu_x + \sum_{y \in V} |f(y)|^2 \mu_y \\ &= 2 \|f\|_2^2 \end{aligned}$$

The second last equality is left as an exercise. □

Proposition 4.3.2. *Let $f \in \mathcal{H}^2(V)$. Then,*

$$\|\Delta f\|_2^2 \leq 2\mathcal{E}(f, f)$$

Proof.

$$\begin{aligned} \|\Delta f\|_2^2 &= \sum_{x \in V} (\Delta f(x))^2 \mu_x \\ &= \sum_{x \in V} \left[\frac{1}{\mu_x} \sum_{y \in V} (f(y) - f(x))^2 \mu_{xy} \right]^2 \mu_x \\ &= \sum_{x \in V} \frac{1}{\mu_x} \left[\sum_{y \in V} (f(x) - f(y))^2 \mu_{xy} \right]^2 \\ &\stackrel{Ex}{\leq} \sum_{x \in V} \frac{1}{\mu_x} \left[\sum_{y \in V} (f(x) - f(y))^2 \mu_{xy} \right] \left[\sum_{y \in V} \mu_{xy} \right] \\ &= 2\mathcal{E}(f, f) \end{aligned}$$

The second last inequality is an exercise and can be shown using Cauchy-Schwarz inequality. □

Proof of Discrete Green's Theorem (4.3.1).

$$\begin{aligned}
\langle \Delta f, g \rangle &= \sum_{x \in V} \Delta f(x) g(x) \mu_x \\
&= \sum_{x \in V} \frac{1}{\mu_x} \sum_{y \in V} (f(y) - f(x)) \mu_{xy} g(x) \mu_x \\
&= - \sum_{x \in V} \sum_{y \in V} (f(x) - f(y)) \mu_{xy} g(x)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(f, g) &= \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y))(g(x) - g(y)) \mu_{xy} \\
&= \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y)) g(x) \mu_{xy} - \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y)) g(y) \mu_{xy} \\
&= -\frac{1}{2} \langle \Delta f, g \rangle - \frac{1}{2} \sum_{y \in V} \sum_{x \in V} (f(x) - f(y)) g(y) \mu_{xy} \\
&\stackrel{*}{=} -\frac{1}{2} \langle \Delta f, g \rangle - \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(y) - f(x)) g(x) \mu_{yx} \\
&= -\frac{1}{2} \langle \Delta f, g \rangle + \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y)) g(x) \mu_{xy} \\
&= -\frac{1}{2} \langle \Delta f, g \rangle - \frac{1}{2} \langle \Delta f, g \rangle \\
&= -\langle \Delta f, g \rangle
\end{aligned}$$

where $*$ is obtained by flipping the labels of x and y . □

Example.

Let $V = \mathbb{N}$ and μ be the usual weights.

Define $f, g : \mathbb{N} \rightarrow \mathbb{R}$ such that

$$\begin{aligned}
f(n) &:= \sum_{i=1}^n \frac{(-1)^i}{i} \\
g(n) &:= 1
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{E}(f, f) &= \frac{1}{2} \left[\sum_{k \geq 1} (f(k+1) - f(k))^2 + \sum_{k \geq 1} (f(k-1) - f(k))^2 \right] \\
&\leq \sum_{k \geq 2} \frac{1}{k^2} < \infty
\end{aligned}$$

$$\mathcal{E}(g, g) = 0$$

$$\mathcal{E}(f, g) = 0$$

$$\begin{aligned}\Delta f(n) &= \frac{1}{2}[f(n+1) + f(n-1) - 2f(n)] \\ &= \frac{(-1)^{n+1}}{2} \frac{2n+1}{n(n+1)} \\ \implies \langle \Delta f, g \rangle &= \frac{3}{4} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{2} \frac{2n+1}{n(n+1)} 2 \\ &= \frac{3}{4} - \frac{1}{2} \neq 0\end{aligned}$$

which contradicts the Discrete Green's Theorem ([4.3.1](#))