Week 6

More on random walks

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Harmonic Functions:

Let $\Gamma = (V, E, \mu)$ be weighted graph and let $A \subseteq V$, $\bar{A} = A \bigcup \partial A$. Then $f : \bar{A} \to \mathbb{R}$ is said to be:
Harmonic if $\Delta f = 0,\ i.e.\ (Pf = f)$,
Super Harmonic if $\Delta f \leq 0,\ i.e.\ (Pf \leq f)$,
Sub Harmonic if $\Delta f \geq 0,\ i.e.\ (Pf \geq f)$.

Examples:

1. For $x, y \in V$ and $A \subseteq V$,

$$g_A(x,y) = \sum_{n=0}^{\infty} p_n^A(x,y) = \frac{\mathbb{E}[L^y_{\tau_A}]}{\mu_y}.$$

$$\Delta g_A(x,y) = -\frac{1}{\mu_x}\mathbf{1}_{\{\mathbf{x}\}}(\mathbf{y}) = \begin{cases} -\frac{1}{\mu_x}, & \text{if } y = x \\ 0, & \text{if } y \neq x. \end{cases}$$

Therefore, $g_A(x, \cdot)$ is Harmonic in $A \backslash x$ and Super Harmonic in A.

2. For $z \in V$ and $x \neq z$,$\phi(x) = \mathbb{P}^x(T_z = \infty)$ is Harmonic in $V \backslash z$.

Theorem 6.0.1. ***(Foster's Criteria/Lyapunov Function)****: Let $A \subseteq V$ be a finite set. Then (Γ, μ) is recurrent iff there exists a function h, which is:*
non negative, Super Harmonic on $V \backslash A$ and $|\{x : h(x) < M\}| < \infty\ \forall\ M > 0$.

Proof:

$(\Rightarrow)$

WLOG, we take $A = \{\rho\}$. Suppose $\exists h : V \to [0, \infty)$ such that h is super harmonic on $V \backslash \{\rho\}$ and $|\{x : h(x) < M\}| < \infty\ \forall\ M > 0$.
$T_\rho = min\{n \geq 0 | X_n = \rho\}$, $\{X_n\}_{n \geq 0}$ is a random walk on (Γ, μ). Let $Y_n = h(X_{n \bigcap T_\rho})$ and let $\mathcal{A}_n$ be the observable events upto time n. So for $X_0 = x$,

$$\begin{aligned}
\mathbb{E}^x[Y_n | \mathcal{A}_{n-1}] &= \mathbb{E}^x[h(X_{n \bigcap T_\rho}) | \mathcal{A}_{n-1}] \\
&= \mathbb{E}^{X_{n-1}}[h(X_{n \bigcap T_\rho})],\ \text{(SMP)} \\
&= Ph(X_{n-1 \bigcap T_\rho}), \\
&\leq h(X_{n-1 \bigcap T_\rho}),\ \text{(super harmonic)}, \\
&= Y_{n-1}.
\end{aligned}$$

Super Martingle: Let $\{Z_n\}_{n\geq 1}$ be a sequence of random variables such that $\mathbb{E}[Z_n] < \infty$. Then $\{Z_n\}_{n\geq 1}$ is a super Martingle if $\mathbb{E}[Z_n|Z_{n-1}, Z_{n-2}, ...Z_1] \leq z_{n-1}$.

Theorem 6.0.2. ***(Martingle Convergence Theorem):*** *Let $Y_n \geq 0$ be Super Martingle, $\exists Y \equiv Y_\infty$ such that $Y_n \to Y_\infty$ wp 1 and $\mathbb{E}[Y_\infty] \leq \mathbb{E}[Y_0]$ as $n \to \infty$.*

So, in our case, $Y_0 = h(X_{0 \bigcap T_\rho}) = h(X_0) < \infty$.
Therefore, from the above theorem, $Y_\infty < \infty$ wp 1.
Now, suppose (Γ, μ) is Transient. Then $\exists x \in V\backslash\{\rho\}$ such that $\mathbb{P}^x(T_{\rho)} < 1$.
Let $C_n = \{y \in V\backslash\{\rho\}|h(y) \geq n\}\ \forall\ n \geq 1$. Then $|C_n^c| < \infty$.
$N = \{T_\rho = \infty\} \bigcap \{\exists n_k \geq 1 : X_{n_k} \in C_k\}$. Let $w \in N$ and n_k be as given by N.

$$Y_{n_k} = h(X_{n_k \bigcap T_\rho}),$$
$$\Rightarrow Y_{n_k} \geq k,$$
$$\Rightarrow N \subseteq \{Y_\infty = \infty\},$$

$\Rightarrow \mathbb{P}(Y_\infty = \infty) > 0$, which contradicts Martingle Convergence Theorem.
$\Rightarrow \{X_n\}_{n\geq 0}$ can not be Transient.
Hence, $\{X_n\}_{n\geq 0}$ is Recurrent.

($\Leftarrow$)

Suppose (Γ, μ) is Recurrent.
Let $B(\rho, n) = \{x \in V|d(x, \rho) \leq n\}$ and let $h_n : V \to [0, 1]$ such that

$$h_n(x) = \mathbb{P}^x(\tau_{B(\rho,n)} < T_\rho);\ \tau_{B(\rho,n)} = T_{B(\rho,n)^c}.$$
$$(SMP) \Rightarrow Ph_n = h_n,\ x \neq \rho,$$
$$\Rightarrow \Delta h_n = 0,\ \forall\ x \neq \rho.$$

In particular, $h_n(\cdot)$ is super harmonic in $V\backslash\{\rho\}$.

$$\lim_{n\to\infty} h_n(x) = 0,$$
$$h_n(x) = 1\ \forall\ x \in B(\rho, n)^c.$$

$\exists\{n_k\}_{k\geq 1}$ such that $h_{n_k}(x) \leq \frac{1}{2^k}\ \forall\ x \in B(\rho, n)$.
Let $x \in V$, $\sum\limits_{k=1}^{\infty} h_{n_k}(x)$ (Ex: $h(\cdot) < \infty$).
(I) $h \geq 0$.

(II) $x \in V \backslash \{\rho\}$,

$$\begin{aligned} Ph(x) &= \sum_{x \sim y} \frac{\mu_{xy}}{\mu_x} h(y), \\ &= \sum_{x \sim y} \frac{\mu_{xy}}{\mu_x} \sum_{k=1}^{\infty} h_{n_k}(x), \\ &= \sum_{k=1}^{\infty} \sum_{x \sim y} \frac{\mu_{xy}}{\mu_x} h_{n_k}(x), \\ &= \sum_{k=1}^{\infty} Ph_{n_k}(x), \\ &= \sum_{k=1}^{\infty} h_{n_k}(x) = h(x). \end{aligned}$$

(III) Let $M > 0$ and $U = \{x \in V | h(x) < M\}$.

$$\forall j \geq h_{n_j}(x) = 1 \ \forall \ x \in B(\rho, n_j)^c.$$

Claim: $U \subseteq B(\rho, n_m)^c$.
Proof: Given $M > 0$, $\exists n_m$ such that
$\forall j \geq h_{n_j}(x) = 1 \ \forall \ x \in B(\rho, n_m)^c$, for $1 \leq j \leq M$.

$$\sum_{j=1}^{m} h_{n_j}(x) = M \ \forall \ x \in B(\rho, n_m)^c.$$

$$\Rightarrow h(x) \geq M \ \forall \ x \in B(\rho, n_m)^c.$$

Theorem (Maximum Principle):
$A \subset V$, connected, $h : V \to \mathbb{R}$ such that $\Delta h \geq 0$ on A

a) If $\exists x \in A$ such that $h(x) = \max_{z \in A \cup \partial A} h(z)$ then h is constant on $\overline{A}$

b) $|A| < \infty$, $\ h(z) = \max_{z \in \partial A} \ h(z)$

Proof
a) $B = \{y \in \overline{A} \mid h(y) = h(x)\}$
$B \neq \emptyset$ as $x \in B$
$y \in B \cap A$, $z \sim y \Rightarrow z_0 \in \overline{A} \Rightarrow z_0 \in B$
$h(z_0) \leq \max_{u \in \overline{A}} h(u)$

$y \in B \cap A$ and $z \sim y$ then
$z_0 \in \overline{A} \Rightarrow h(z_0) \leq \max_{u \in \overline{A}} \ h(u) = h(x) = h(y)$

But $\Delta h(y) \geq 0$
$\frac{1}{\mu_y} \Sigma_{z \sim y} \ \ \mu_{zy}(h(y) - h(z)) \geq 0$

Hence if $z \sim y \Rightarrow h(z) = h(y)$
Inductively, as A is connected, we have that $B = \overline{A}$

b) $|A| < \infty \Rightarrow \exists x_0 \in \overline{A}$, $h(x) = \max\limits_{z \in A} \ h(z)$

If $x_0 \in \partial A \Rightarrow \max\limits_{u \in \partial A} \ h(u) = \max\limits_{z \in \overline{A}} \ h(z)$

If $x_0 \in A \Rightarrow$ (a) h is constant on $\overline{A}$ and $\max\limits_{u \in \partial A} \ h(u) = \max\limits_{x \in \overline{A}} \ h(x)$

Liouville Property: (Γ, μ) is said to have the Liouville Property if all bounded harmonic functions are constant.

Strong Liouville Property: A graph is said to have the strong Liouville Property if all positive harmonic functions are constant.

Theorem: Let (Γ, μ) be recurrent. Any positive superharmonic function is constant (in particular, (Γ, μ) has the strong Liouville Property).

Notes for continuation of Martingales

Let $\{Z_n\}$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.
Let $E[Z_n] < \infty$

If $E[Z_n | Z_{n-1}, Z_{n-2}, ..., Z_1] = Z_{n-1}$, then $\{Z_n\}_{n \geq 1}$ is a martingale. $\{h(.)$ is bounded, harmonic, X_n r.v on (Γ, μ), $\{h(X_n)\}_{n \geq 1}$ $\}$

If $E[Z_n | Z_{n-1}, Z_{n-2}, ..., Z_1] \geq Z_{n-1}$, then $\{Z_n\}_{n \geq 1}$ is a submartingale. $\{h(.)$ is bounded, subharmonic, X_n r.v on (Γ, μ), $\{h(X_n)\}_{n \geq 1}$ $\}$

If $E[Z_n | Z_{n-1}, Z_{n-2}, ..., Z_1] \leq Z_{n-1}$, then $\{Z_n\}_{n \geq 1}$ is a supermartingale. $\{h(.)$ is bounded, superbharmonic, X_n r.v on (Γ, μ), $\{h(X_n)\}_{n \geq 1}$ $\}$

Jensen's Inequality

Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\forall a \in \mathbb{R}, \exists c \in \mathbb{R}$ such that $f(x) \geq f(a) + c(x - a)$
Then f is said to be a convex function.

Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Then Jensen's Inequality states that $E[f(X)] \geq f(E[X])$ wherever both expectations are well defined.

Theorem (Kolmogorov's Maximal Inequality): Let $\{Z_n\}_{n \geq 1}$ be a non-negative submartingale. Then $\mathbb{P}(\max\limits_{1 \leq i \leq m} \ Z_i \geq a) \leq \frac{E[Z_m]}{a} \ \ \forall a > 0, \forall m \geq 1$

Proof:
Let $m \in \mathbb{N}$ and $a > 0$ be given.
$J = min.\{min.\{n \geq 1 | Z_n > a\}, m\}$
$\tilde{J} = min.\{n \geq 1 | Z_n > a\}$
It is left as an exercise to show that J is a bounded stopping time. $Z_J \geq a \Leftrightarrow Z_n \geq a$ for some $n \leq m$

$\mathbb{P}(\max_{1 \leq i \leq m} Z_i \geq a) = \mathbb{P}(Z_J \geq a) \leq \frac{E[Z_J]}{a}$

This follows from the Markov Inequality which can be applied here since $Z_i \geq 0$

$Z_J = \sum_{k=1}^m Z_k \mathbf{1}_{J=k} + Z_m \mathbf{1}_{\tilde{J}>m}$

$E[Z_J] = \sum_{k=1}^m E[Z_k \mathbf{1}_{J=k}] + E[Z_m \mathbf{1}_{\tilde{J}>m}]$
Since $\{Z_k\}_{k \geq 1}$ is a submartingale, we have that the above is less than or equal to
$\sum_{k=1}^m E[E[Z_m|\mathcal{A}_k]\mathbf{1}_{J=k}] + E[Z_m \mathbf{1}_{\tilde{J}>m}]$
The above is equal to
$\sum_{k=1}^m E[E[Z_m \mathbf{1}_{J=k}|\mathcal{A}_k]] + E[Z_m \mathbf{1}_{\tilde{J}>m}]$ (Since $E[XY|\mathcal{A}_Y] = YE[X|\mathcal{A}_Y]$)
$= \sum_{k=1}^m E[Z_m \mathbf{1}_{J=k}] + E[Z_m \mathbf{1}_{\tilde{J}>m}]$
$= E[Z_m(\sum_{k=1}^m \mathbf{1}_{J=k} + \mathbf{1}_{\tilde{J}>m}]$
$= E[Z_m] = 0$

Corollary: Let $\{Z_m\}_{m \geq 1}$ be a martingale.

1) $E[Z_m^2] < \infty \ \ \forall m \geq 1, \mathbb{P}(\max_{1 \leq i \leq m} |Z_i| \geq a) \leq \frac{E[|Z_m|^2]}{a^2}$

2) $Z_m \geq 0, \mathbb{P}(\sup_{n \geq 1} Z_n > a) \leq \frac{E[Z_1]}{a}$

Proof
For 1) Let $Y_n = Z_n^2$. Then we apply Kolmogorov's Maximal Inequality. The proof of 2) will be done later.

Theorem (Martingale Convergence):
Let $\{Z_n\}_{n \geq 1}$ be a martingale and $\sup_{n \geq 1} E[Z_n^2] < \infty$. Then $\exists \ Z$ such that $Z_n \to Z$ w.p. 1 as $n \to \infty$

Proof
$f(x) = x^2$ is a convex function. Hence, Jensen's inequality applies to conditional expectation.
$E[Z_n^2 | Z_{n-1}^2, ..., Z_1^2] \geq Z_{n-1}^2 \ \ \forall n \geq 2$

From the Tower Property, it follows that
$E[Z_n^2 | Z_i^2, ..., Z_1^2] \geq Z_i^2$ for $1 \leq i \leq n$

Thus, $E[Z_n^2] \geq E[Z_i^2] \ \ \forall 1 \leq i \leq n$

In particular, $E[Z_n^2] \geq E[Z_{n-1}^2]$ and $\sup\limits_{n\geq 1} \; E[Z_n^2] < \infty$

Thus $\exists \;\; \alpha > 0$ such that $E[Z_n^2] \to \alpha$ as $n \to \infty$

Let $k \geq 1$, $Y_m = Z_{k+m} - Z_k \;\; \forall m \geq 1$
Exercise: $\{Y_m\}_{m\geq 1}$ is also a martingale.

From the Tower Property, we have that $E[Z_{k+m}Z_k] = E[E[Z_{k+m}Z_k|\mathcal{A}_k]]$ which is equal to $E[Z_k E[Z_{k+m}|\mathcal{A}_k]]$ which is equal to $E[Z_k^2]$ since $\{Z_n\}_{n\geq 1}$ is a martingale and Z_k is observable in A_k.

Hence $E[Y_m^2] = E[Z_{k+m}^2] - E[Z_k^2]$

By Corollary 1,
$\mathbb{P}(\max\limits_{1\leq i\leq m} \; |Y_i| > a) \leq \frac{E[Z_m^2]}{a^2} = \frac{E[Z_{k+m}^2]-E[Z_k^2]}{a^2}$

$\mathbb{P}(\max\limits_{1\leq i\leq m} \; |Y_i| > a) = \mathbb{P}(\max\limits_{1\leq i\leq m} \; |Z_{i+k} - Z_k| > b)$

Letting m go to 0 on both sides, we get
$\mathbb{P}(\cup_{i\geq 1} \;\; |Z_{i+k} - Z_k| > a) \leq \frac{\alpha - E[Z_k^2]}{a^2}$

(Exercise:
i) $0 \leq \mathbb{P}(\sup\limits_{i\geq 1} \; |Z_{i+k} - Z_k| > a) \leq \frac{\alpha - E[Z_k^2]}{a^2}$

ii) $\lim\limits_{k\to\infty} \; \mathbb{P}(\cup_{i\geq 1} \;\; |Z_{i+k} - Z_k| > a) = 0 \;\; \forall a)$

From ii) above and the Borel Cantelli Lemma, we have that if $E = \{\{Z_k\}_{k\geq 1}$ is a Cauchy sequence $\}$ then $\mathbb{P}(E) = 1$.
$\Rightarrow \exists \;\; Z$ such that $Z_n \to Z$ w.p. 1 as $n \to \infty$.