Week 4

Chapter 4

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4.1 Transition densities and Laplacian

We can recall that $\mathbf{P}(X_1 = y) = \mathcal{P}(x,y) = \frac{\mu_{xy}}{\mu_x}$.

Definition 4.1.1. *The transition density w.r.t weights μ of a random walk $\{X_n\}$ is given by:*

$$p_n(x,y) = \frac{\mathbf{P}^x(X_n = y)}{\mu_y} := \frac{\mathcal{P}_n(x,y)}{\mu_y} \qquad \mu \geq 1$$

$p_0(x,y) = \frac{1_{\{x\}}(y)}{\mu_y} = \frac{1_{\{x\}}(y)}{\mu_x}$.
$p(x,y) \equiv p_1(x,y) = \frac{\mu_{xy}}{\mu_x \mu_y}$

Lemma 4.1.1. *Let p_n be the transition densities of $\{X_n\}_{n\geq 0}$*

1. $p_{n+m}(x,y) = \sum_{z\in V} p_n(x,z)p_m(z,y)\mu_z$
2. $\forall\ x,y \in V$, $p_n(x,y) = p_n(y,x)$
3. $\forall\ x,y \in V$, $\sum_{z\in V} p_n(x,z)\mu_z = 1 = \sum_{z\in V} p_n(z,y)\mu_z$

Proof. 1.

$$\begin{aligned}
p_{n+m}(x,y) &= \frac{\mathbf{P}^x(X_{n+m} = y)}{\mu_y} \\
&= \sum_{z\in V} \frac{\mathbf{P}^x(X_{n+m} = y, X_n = z)}{\mu_y} \\
&= \frac{1}{\mu_y} \sum_{z\in V} \sum_{o\leq i<n+m, x_i\in V} 1_{\{x\}}(x_0) \prod_{i=0}^{n-1} \mathcal{P}(x_i, x_{i+1}) 1_{\{z\}}(x_n) \prod_{i=n}^{n+m} \mathcal{P}(x_i, x_{i+1}) 1_{\{y\}}(x_{n+m}) \\
&\overset{\text{H1}}{=} \frac{1}{\mu_y} \sum_{z\in V} \mathbf{P}^x(X_n = z)\mathbf{P}^z(X_m = y) \\
&= \frac{1}{\mu_y} \sum_{z\in V} p_n(x,z)\mu_z\ p_m(z,y)\mu_y \\
&= \sum_{z\in V} p_n(x,z)p_m(z,y)\mu_z
\end{aligned}$$

2.

$$p_n(x,y) = \frac{\mathbf{P}^x(X_n = y)}{\mu_y} = \frac{\mathbf{P}^y(X_n = x)}{\mu_x} = p_n(y,x)$$

The second equality is obtained by applying the Detailed Balance equations.

3.

$$\sum_{z\in V} p_n(x,z)\mu_z = \sum_{z\in V} \mathbf{P}^x(X_n = z) = 1$$
$$\sum_{z\in V} p_n(z,y)\mu_z = \sum_{z\in V} p_n(y,z)\mu_z = \sum_{z\in V} \mathbf{P}^y(X_n = z) = 1$$

□

4.2 Function Spaces

Definition 4.2.1.

$$C(V) = \{f : V \to \mathbb{R}\} = \mathbb{R}^V$$
$$\mathrm{Co}(V) = \{f : V \to \mathbb{R},\ f \neq 0 \textit{ on finitely many points}\}$$
$$C_+(V) = \{f : f \in C(V),\ f \geq 0\}$$
$$\mathrm{Supp}(f) = \{x : x \in V,\ f(x) \neq 0\}$$

Definition 4.2.2. *We define the* ***norm*** *of a function as the following*

$$\forall p \in [1,\ \infty),\ \|f\|_p = (\sum_{x\in V} |f(x)|^p \mu_x)^{\frac{1}{p}}$$
$$\|f\|_\infty = \sup\{|f(x)| : x \in V\}$$

f is said to be L^p on the graph with vertex set V and weights μ if and only if f is a function defined on the vertex set, V and its p-norm is finite everywhere.

$$f \in L^p(V,\mu) \iff f \in C(V) \text{ and } \|f\|_p < \infty$$

Definition 4.2.3. *We define an inner product on the $L^2(V,\ \mu)$ space in the following way*

$$\langle f,\ g\rangle = \sum_{x\in V} f(x)g(x)\mu_x$$

$$\begin{aligned}\mathbf{E}[f(X_n)] &= \sum_{x\in V} f(z)\mathbf{P}^x(X_n = z)\\ &= \sum_{x\in V} f(z)p_n(x,z)\mu_z\\ &= \langle f,\ p_n(x,\ .)\rangle\end{aligned}$$

which brings us to define a new function

Definition 4.2.4. *$\mathcal{P}_n : C(V) \to C(V)$ given by*

$$\mathcal{P}_n f(x) = \sum_{x \in V} f(z) p_n(x, z) \mu_z = \langle f,\ p_n(x,\ .) \rangle$$

where $\Delta : C(V) \to C(V)$ as an "operation" on C(V) is

$$\Delta = P - I$$

We write $\mathcal{P}_1 f(x)$ as $\mathcal{P}f(x)$ and proceed to look at computations and lemmas involving $\mathcal{P}f$.

Lemma 4.2.1.

$$\forall x \in V,\ \mathcal{P}f(x) - f(x) = \Delta f(x)$$

Proof.

$$\begin{aligned}
\mathcal{P}f(x) - f(x) &= \sum_{x \in V} f(z) p(x, z) \mu_z - f(x) \\
&\overset{*}{=} \sum_{x \in V} p(x, z) \mu_z (f(z) - f(x)) \\
&= \sum_{x \in V} \frac{\mu_{xz}}{\mu_x \mu_z} \mu_z (f(z) - f(x)) \\
&= \frac{1}{\mu_x} \sum_{x \in V} \mu_{xz} (f(z) - f(x)) \\
&= \Delta f(x)
\end{aligned}$$

* is left as an exercise and can be proved using property 2 from (4.1.1) ☐

Corollary 4.2.1.

$$\Delta f = 0 \iff f(x) = \mathcal{P}f(x) = \mathbf{E}^x[f(X_1)]$$

Definition 4.2.5. *We define a function $A : C(V) \to C(V)$ as*

$$\|A\|_{p \to p} = \sup\{\|Af\|_p : \|f\|_p \leq 1\}$$

Proposition 4.2.1.
1. *$\mathcal{P}1 = 1$*
   where $1(x) = 1\ \forall x \in V$
2. *$|\mathcal{P}f| \leq \mathcal{P}|f|$*
   where $f \in C(V)$
3. *$\|\mathcal{P}\|_{p \to p} \leq 1$*
   $\|\Delta\|_{p \to p} \leq 2$
   where $p \in [1, \infty) \cup \{\infty\}$

Proof. 1.

$$\mathcal{P}1(x) = \sum_{x \in V} p(x,z)\mu_z = 1 = 1(x)$$

2.

$$\begin{aligned} |\mathcal{P}f(x)| &= \left| \sum_{x \in V} f(z)p(x,z)\mu_z \right| \\ &\leq \sum_{x \in V} |f(z)|\, p(x,z)\mu_z \\ &= \mathcal{P}|f|(x) \end{aligned}$$

3.

$$\begin{aligned} \|\mathcal{P}f\|_p^p &= \sum_{x \in V} |\mathcal{P}f(x)|^p \mu_x \\ &= \sum_{x \in V} \left| \sum_{z \in V} f(z)p(x,z)\mu_z \right|^p \mu_x \\ &\overset{*}{\leq} \sum_{x \in V} \left(\sum_{z \in V} |f(z)|^p p(x,z)\mu_z \right) \left(\sum_{z \in V} 1^q p(x,z)\mu_z \right) \mu_x \\ &= \sum_{x \in V} \left(\sum_{z \in V} |f(z)|^p p(x,z)\mu_z \right) \mu_x \\ &\overset{**}{=} \sum_{z \in V} |f(z)|^p \mu_z \\ &= \|f\|_p \\ \implies \|\mathcal{P}\|_{p \to p} &\leq 1 \end{aligned} \tag{4.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$

We leave the proofs of the following as exercises
*, which can be proved using Holder's inequality, ** and the $p = \infty$ case

$$\begin{aligned} \|\Delta f\|_p^p &= \|\mathcal{P}f - f\|_p^p \\ &\leq (\|\mathcal{P}f\|_p + \|f)\|_p)^p \\ &\leq 2^{p-1}(\|\mathcal{P}f\|_p^p + \|f\|_p^p) \\ &\leq (2\,\|f\|_p)^p \qquad\qquad [\because \|\mathcal{P}f\|_p \leq \|f\|_p] \\ \implies \|\Delta\|_{p \to p} &\leq 1 \end{aligned}$$

The final inequality is obtained from (4.1).

∎

Proposition 4.2.2. *$\mathcal{P}$ is self-adjoint on $L^2(V,\mu)$*

$$\forall\ f,g \in L^2(V,\mu),\ \langle \mathcal{P}f,\ g\rangle = \langle f,\ \mathcal{P}g\rangle$$

Proof.

$$\begin{aligned}
\langle \mathcal{P}f,\ g\rangle &= \sum_{x\in V} \mathcal{P}f(x)g(x)\mu_x \\
&= \sum_{x\in V}(\sum_{z\in V} f(z)p(x,z)\mu_z)g(x)\mu_x \\
&\overset{Ex}{=} \sum_{z\in V} f(z)\mu_z \sum_{x\in V} p(z,x)g(x)\mu_x \\
&= \sum_{z\in V} f(z)\mathcal{P}g(z)\mu_z \\
&= \langle f,\ \mathcal{P}g\rangle
\end{aligned}$$

□

4.3 Dirichlet forms

Definition 4.3.1. *We define the quadratic form on $L^2(V,\mu)$, $\mathcal{E}$ as*

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x\in V}\sum_{y\in V}(f(x)-f(y))(g(x)-g(y))\mu_{xy}$$

whenever the series converges absolutely.

Theorem 4.3.1 (Discrete Green's Theorem)**.** *$\forall\ f,g \in C(V)$,*

$$\begin{aligned}
&\sum_{x\in V}\sum_{y\in V} |f(x)-f(y)|\,|g(x)|\,\mu_x < \infty \\
&\implies \mathcal{E}(f,g) = -\langle \Delta f,\ g\rangle
\end{aligned}$$

We present an application of (4.3.1)

Lemma 4.3.1. *Let $(\Gamma,\ \mu)$ be a weighted graph such that $\mu(V) < \infty$. Then, $(\Gamma,\ \mu)$ is* ***recurrent.***

Proof. Fix $Z \in V$ Define $\Phi : V \to \mathbb{R}$ where $\Phi(x) \coloneqq \mathbf{P}^x(\mathcal{T}_z = \infty)$

1. Firstly observe that $\Phi(z) = \mathbf{P}^z(\mathcal{T}_z = \infty) = 0$

2. $\forall\ n \geq 1,\ x \neq z$
   $\Phi_n(x) \coloneqq \mathbf{P}^x(\mathcal{T}_z = n) = \sum_{u\in V}\mathcal{P}(x,u)\Phi_{n-1}(x)$

   This holds true from a simple logical argument. Starting from x, hitting z in n steps is equivalent to jumping from x to some vertex u and hitting z in $n-1$ steps.

3. $1 - \Phi(x) = \mathbf{P}^x(\mathcal{T}_z < \infty) = \sum_{n=0}^{\infty} \mathbf{P}^x(\mathcal{T}_z = n) = \sum_{n=1}^{\infty} \Phi_n(x)$
4. $\Phi \equiv 0$

$$\begin{aligned}
\sum_{n=1}^{k} \Phi_n(x) &= \sum_{n=1}^{k} \sum_{u \in V} \mathcal{P}(x,u)\Phi_{n-1}(u) \\
\implies \sum_{n=1}^{k} \Phi_n(x) &= \sum_{u \in V} \mathcal{P}(x,u) \sum_{n=1}^{k} \Phi_{n-1}(u) \\
\implies \sum_{n=1}^{\infty} \Phi_n(x) &= \sum_{u \in V} \mathcal{P}(x,u) \sum_{n=1}^{\infty} \Phi_{n-1}(u) \\
\implies 1 - \Phi(x) &= \sum_{u \in V} \mathcal{P}(x,u)(1 - \Phi(u)) \\
\implies 1 - \Phi(x) &= \sum_{u \in V} p(x,u)(1 - \Phi(u))\mu_u \\
\implies 1 - \Phi &= \mathcal{P}(1 - \Phi) \\
\implies \Delta(1 - \Phi) &= 0
\end{aligned}$$

Then, by theorem (4.3.1),

$$\begin{aligned}
&\mathcal{E}(1-\Phi, 1-\Phi) = \langle \Delta(1-\Phi),\ 1-\Phi \rangle = 0 \\
\implies &\frac{1}{2} \sum_{x \in V} \sum_{y \in V} (\Phi(x) - \Phi(y))^2 \mu_{xy} = 0 \\
\implies &\Phi(x) = \Phi(y) \qquad \forall\, x, y \in V \\
\implies &\Phi \text{ is constant} \\
\implies &\Phi \equiv 0
\end{aligned}$$

The last equality holds as $\Phi(z) = 0$.

Since, $\Phi \equiv 0$ for arbitrary z, $(\Gamma,\ \mu)$ is recurrent.

□

To proof theorem (4.3.1), we start with some prerequisites.

Definition 4.3.2.

$$\begin{aligned}
\mathcal{H}^2(V) &= \{f : f \in C(v),\ \mathcal{E}(f,f) < \infty\} \\
\|f\|_{\mathcal{H}^2} &= \sqrt{\mathcal{E}(f,f) + f^2(\rho)} \qquad \textit{for some fixed } \rho \in V
\end{aligned}$$

Proposition 4.3.1. *Let $(\Gamma,\ \mu)$ be a graph satisfying properties, H1 and H2.*

1. $|f(x) - f(y)| \leq \frac{1}{\sqrt{\mu_{xy}}} \sqrt{\mathcal{E}(f,f)} \qquad \forall\, x \sim y$
2. $\mathcal{E}(f,f) = 0 \iff f$ *is constant*

3. $f \in L^2 \implies \mathcal{E}(f,f) \leq 2\|f\|_2^2$

Proof. 1. If $\mathcal{E}(f,f) = \infty$, then we are done

Let $\mathcal{E}(f,f)$ be finite

$$\begin{aligned}\mathcal{E}(f,f) &= \frac{1}{2}\sum_{x\in V}\sum_{y\in V}(f(x)-f(y))^2\mu_{xy} \\ &\geq (f(x)-f(y))^2\mu_{xy} \qquad \forall\, x,y \in V\end{aligned}$$

2. The forward direction is left as an exercise. The reverse direction follows from the definition.

3.

$$\begin{aligned}\mathcal{E}(f,f) &\leq \frac{1}{2}\sum_{x\in V}\sum_{y\in V}(f(x)-f(y))^2\mu_{xy} \\ &\leq \sum_{x\in V}\sum_{y\in V}(|f(x)|^2+|f(y)|^2)\mu_{xy} \\ &\overset{Ex}{=} \sum_{x\in V}|f(x)|^2\mu_x + \sum_{y\in V}|f(y)|^2\mu_y \\ &= 2\|f\|_2^2\end{aligned}$$

The second last equality is left as an exercise.

□

Proposition 4.3.2. *Let $f \in \mathcal{H}^2(V)$. Then,*

$$\|\Delta f\|_2^2 \leq 2\mathcal{E}(f,f)$$

Proof.

$$\begin{aligned}\|\Delta f\|_2^2 &= \sum_{x\in V}(\Delta f(x))^2\mu_x \\ &= \sum_{x\in V}\left[\frac{1}{\mu_x}\sum_{y\in V}(f(y)-f(x))^2\mu_{xy}\right]^2\mu_x \\ &= \sum_{x\in V}\frac{1}{\mu_x}\left[\sum_{y\in V}(f(x)-f(y))^2\mu_{xy}\right]^2 \\ &\overset{Ex}{\leq} \sum_{x\in V}\frac{1}{\mu_x}\left[\sum_{y\in V}(f(x)-f(y))^2\mu_{xy}\right]\left[\sum_{y\in V}\mu_{xy}\right] \\ &= 2\mathcal{E}(f,f)\end{aligned}$$

The second last inequality is an exercise and can be shown using Cauchy-Schwarz inequality. □

Proof of Discrete Green's Theorem (4.3.1).

$$\begin{aligned}
\langle \Delta f,\ g \rangle &= \sum_{x \in V} \Delta f(x) g(x) \mu_x \\
&= \sum_{x \in V} \frac{1}{\mu_x} \sum_{y \in V} (f(y) - f(x)) \mu_{xy} g(x) \mu_x \\
&= - \sum_{x \in V} \sum_{y \in V} (f(x) - f(y)) \mu_{xy} g(x)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(f,g) &= \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y))(g(x) - g(y)) \mu_{xy} \\
&= \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y)) g(x) \mu_{xy} - \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y)) g(y) \mu_{xy} \\
&= -\frac{1}{2} \langle \Delta f,\ g \rangle - \frac{1}{2} \sum_{y \in V} \sum_{x \in V} (f(x) - f(y)) g(y) \mu_{xy} \\
&\overset{*}{=} -\frac{1}{2} \langle \Delta f,\ g \rangle - \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(y) - f(x)) g(x) \mu_{yx} \\
&= -\frac{1}{2} \langle \Delta f,\ g \rangle + \frac{1}{2} \sum_{x \in V} \sum_{y \in V} (f(x) - f(y)) g(x) \mu_{xy} \\
&= -\frac{1}{2} \langle \Delta f,\ g \rangle - \frac{1}{2} \langle \Delta f,\ g \rangle \\
&= - \langle \Delta f,\ g \rangle
\end{aligned}$$

where * is obtained by flipping the labels of x and y. $\square$

Example.
Let $V = \mathbb{N}$ and μ be the usual weights.
Define $f, g : \mathbb{N} \to \mathbb{R}$ such that

$$\begin{aligned}
f(n) &\coloneqq \sum_{i=1}^{n} \frac{(-1)^i}{i} \\
g(n) &\coloneqq 1
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{E}(f,f) &= \frac{1}{2} \left[\sum_{k \geq 1} (f(k+1) - f(k))^2 + \sum_{k \geq 1} (f(k-1) - f(k))^2 \right] \\
&\leq \sum_{k \geq 2} \frac{1}{k^2} < \infty
\end{aligned}$$

$$\mathcal{E}(g,g) = 0$$
$$\mathcal{E}(f,g) = 0$$

$$\begin{aligned}
\Delta f(n) &= \frac{1}{2}[f(n+1) + f(n-1) - 2f(n)] \\
&= \frac{(-1)^{n+1}}{2}\frac{2n+1}{n(n+1)} \\
\implies \langle \Delta f,\ g\rangle &= \frac{3}{4} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{2}\frac{2n+1}{n(n+1)}2 \\
&= \frac{3}{4} - \frac{1}{2} \neq 0
\end{aligned}$$

which contradicts the Discrete Green's Theorem (4.3.1)