

Week 1

Finite length random walks on \mathbb{Z}

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1.1 Definitions

Random walks serve as very useful models in many applications. They are simple to state and understand, yet they lead to lots of intractable questions.

Notation. $\mathbb{N} = \{k \in \mathbb{Z} : k \geq 1\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

We now proceed to construct what is called a “simple random walk” on \mathbb{Z} of finite length $N \in \mathbb{N}$. The sample space Ω_N and the event space \mathcal{F}_N are described below.

$$\Omega_N := \{(\omega_1, \dots, \omega_N) : \omega_i \in \{-1, 1\} \forall 1 \leq i \leq N\}$$

$$\mathcal{F}_N := \{A : A \subseteq \Omega_N\}$$

The probability function $\mathbf{P}_N : \Omega_N \rightarrow [0, 1]$ is defined as

$$\mathbf{P}_N(A) := |A| 2^{-N}$$

We also define random variables X_k and S_k on Ω_N for $1 \leq k \leq N$ as

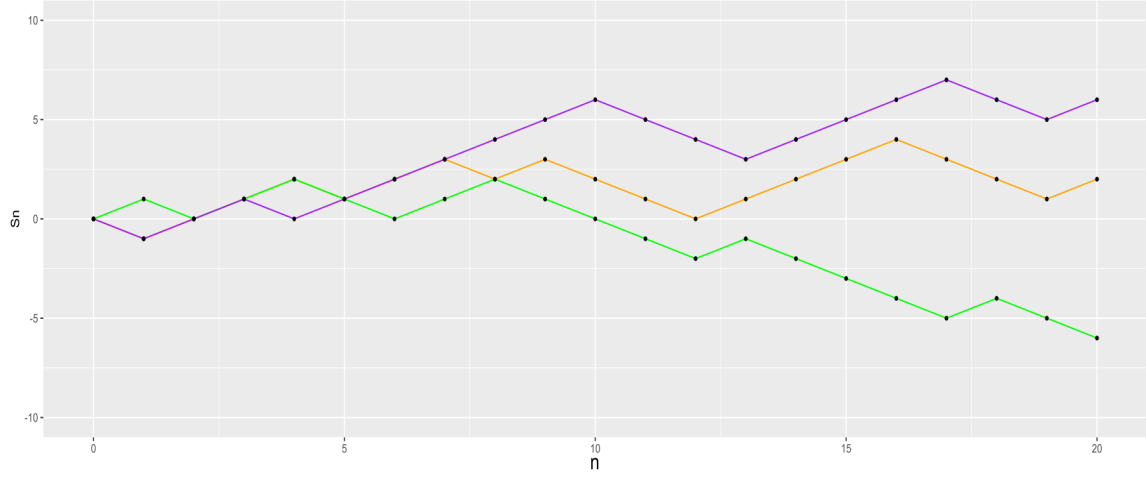
$$X_k : \Omega_N \rightarrow \{-1, 1\} ; X_k(\omega) := \omega_k$$

$$S_k : \Omega_N \rightarrow \mathbb{Z} ; S_k(\omega) := \sum_{i=1}^k X_i(\omega) ; S_0(\omega) := 0 \text{ for all } \omega \in \Omega_N$$

Definition 1.1.1. Fix $N \in \mathbb{N}$. The sequence of random variables $\{S_k\}_{k=1}^N$ on $(\Omega_N, \mathcal{F}_N, \mathbf{P}_N)$ is called a (symmetric) simple random walk on \mathbb{Z} , of finite length N , starting at 0.

[†] added illustrations

Figure 1.1: Three possible trajectories for $(S_n)_{n=0}^N$



In what follows, we suppress the subscript N while referring to the probability space $(\Omega_N, \mathcal{F}_N, \mathbf{P}_N)$, and we assume that $N \in \mathbb{N}$ is fixed.

Observations.

- (a) $\{X_k\}_{k=1}^N$ are iid, i.e. independent and identically distributed.

Proof.

$$\begin{aligned} \mathbf{P}(X_k = 1) &= \mathbf{P}(\{\omega \in \Omega : \omega_k = 1\}) = 2^{-N} |\{\omega \in \Omega : \omega_k = 1\}| \\ &= 2^{-N} 2^{N-1} \\ &= \frac{1}{2} \\ &= \mathbf{P}(X_k = -1) \end{aligned}$$

So $\{X_k\}_{k=1}^N$ are identically distributed. Independence is left as an exercise. \square

- (b) (Independent increments) For $1 \leq k_1 \leq k_2 \leq \dots \leq N$, $\{S_{k_i} - S_{k_{i-1}} : 1 \leq i \leq N\}$ are independent random variables.

Proof. Observe that, for $1 \leq k < l \leq N$, we have $S_l - S_k = \sum_{i=k+1}^l X_i$. Therefore, if $1 \leq a < b \leq c < d \leq N$, we see that $S_b - S_a$ and $S_d - S_c$ are functions of disjoint sets of independent random variables, and hence the claim is true. \square

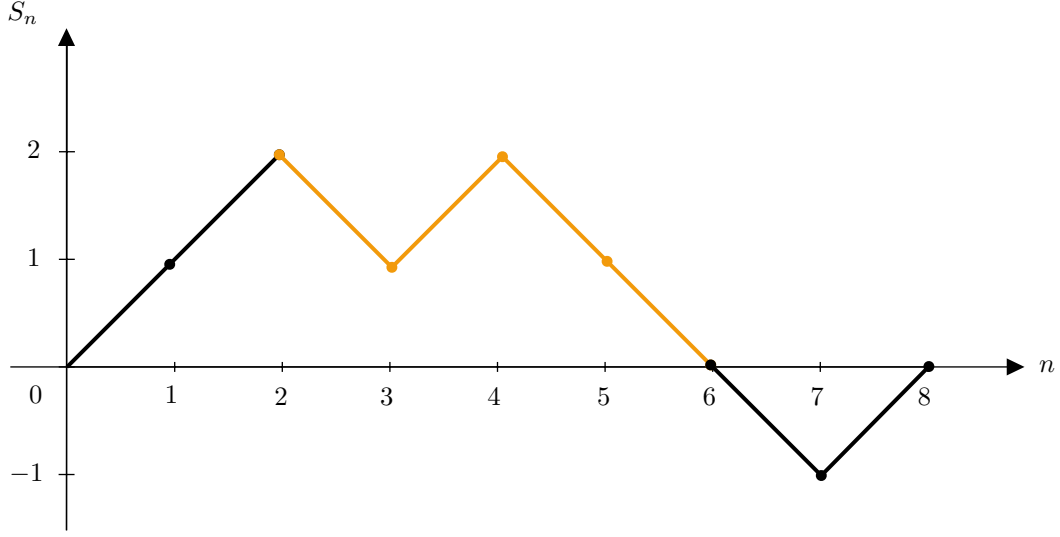


Figure 1.2: Independent (colored) increments in a simple random walk

- (c) (Stationary in increments) For $1 \leq k < m \leq N$, $\mathbf{P}(S_m - S_k = \alpha) = \mathbf{P}(S_{m-k} = \alpha)$ for every $\alpha \in \mathbb{Z}$.

Proof. We use the fact that $\{X_i\}_{i=1}^N$ are identically distributed in the following argument.

$$\mathbf{P}(S_m - S_k = \alpha) = \mathbf{P}\left(\sum_{i=k+1}^m X_i = \alpha\right) = \mathbf{P}\left(\sum_{i=1}^{m-k} X_i = \alpha\right) = \mathbf{P}(S_{m-k} = \alpha)$$

□

- (d) (Markov Property) For $\alpha_i \in \mathbb{Z}$, $1 \leq i \leq N$ and $0 \leq n \leq N$,

$$\mathbf{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}, \dots, S_1 = \alpha_1) = \mathbf{P}(S_n = \alpha_n \mid S_{n-1} = \alpha_{n-1}),$$

assuming (of course) that the conditional probabilities are well defined.

Proof. Left as an exercise.

□

- (e) (Conditional Law) For $1 \leq k < m \leq N$, $\mathbf{P}(S_m = b \mid S_k = a) = \mathbf{P}(S_{m-k} = b - a)$.

Proof. Left as an exercise.

□

- (f) (Moments) For $1 \leq k \leq N$, we have $\mathbf{E}[X_k] = \mathbf{E}[S_k] = 0$ and $\text{Var}[S_k] = k$.

Proof. By definition of expected value, $\mathbf{E}[X_k] = 1(1/2) - 1(1/2) = 0$. By linearity of expected values, $\mathbf{E}[S_k] = \sum_{i=1}^k \mathbf{E}[X_i] = 0$.

Since $\mathbf{E}[S_k] = 0$, $\text{Var}[S_k] = \mathbf{E}[(\sum_{i=1}^k X_i)^2] = \sum_{i=1}^k \mathbf{E}[X_k^2] = k$. As an exercise, show that $\mathbf{E}[(\sum_{i=1}^k X_i)^2] = \sum_{i=1}^k \mathbf{E}[X_k^2]$. \square

(g) (Distribution of S_n) For $x \in \{-n, -n+2, \dots, n-2, n\}$, we have

$$\mathbf{P}(S_n = x) = \mathbf{P}(S_n = -x) = \binom{n}{\frac{n+x}{2}} 2^{-n}$$

Proof. We only provide a sketch of the proof, which is left as an exercise. For $0 \leq j \leq N$, $\{S_n = 2j - n\} = \{S_n = j - (n - j)\}$. So there must be a total of j steps to the right and $n - j$ steps to the left. Therefore

$$\mathbf{P}(S_n = 2j - n) = 2^{-N} |\{\omega \in \Omega : \dots\}| = 2^{-n} \binom{n}{j}$$

\square

(h) (Mode) The mode of the above distribution is achieved in the middle, i.e. at $x = 0$ and at $x = 1, -1$ for S_{2n} and S_{2n-1} respectively.

Proof.

$$\mathbf{P}(S_{2n} = 0) = \mathbf{P}(S_{2n-1} = 1) = \binom{2n}{n} 2^{-2n}$$

\square

(i) (Stirling's formula) Using Stirling's approximation, for large n , we have

$$\binom{2n}{n} = \frac{2n!}{n!n!} \sim \frac{(2n)^{2n} e^{-2n} \sqrt{4\pi n}}{n^{2n} e^{-2n} \sqrt{2\pi n} \sqrt{2\pi n}} \sim \frac{2^{2n}}{\sqrt{\pi n}} \quad (*)$$

Therefore,

$$\mathbf{P}(S_{2n} = 0) = \binom{2n}{n} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}} \quad \text{as } n \rightarrow \infty$$

This approximation, although correct, has a caveat - we chose to keep N fixed, but as $n \rightarrow \infty$, we must also let $N \rightarrow \infty$, and this requires subtler arguments. A few consequences of this approximation are mentioned in the exercises.

1.2 Stopping times

Motivation for this section comes from the classic Gambler's ruin problem. We can interpret a simple random walk as a fair game between two players, where in round k , a player wins the amount X_k . Then S_n denotes the capital of one player over the other after n rounds.

We would like to answer the following question - "Is it possible to stop the game in a favorite moment, i.e., can clever stopping lead to a positive expected gain?". In other words, can we design a $T(\omega)$ for every $\omega \in \Omega$ such that $\mathbf{E}[S_T] > 0$? Of course, the decision to stop may only depend on the trajectory until that time: no "insider knowledge" about the future of the trajectory is permitted.

To formalize this setting, we make the following definition.

Definition 1.2.1. An event $A \subseteq \Omega$ is said to be observable by time n if it is a (possibly empty) union of basic / elementary events of the form

$$\{\omega \in \Omega : \omega_1 = o_1, \dots, \omega_n = o_n\}$$

where $o_i \in \{-1, 1\}$ for $1 \leq i \leq n$.

We also define $\mathcal{A}_0 = \{\phi, \Omega\}$ and set

$$\mathcal{A}_n := \{A \in \mathcal{F} : A \text{ is observable by time } n\}.$$

Immediately, we observe that

$$\{\phi, \Omega\} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \dots \subseteq \mathcal{A}_{N-1} \subseteq \mathcal{A}_N = \mathcal{F}$$

As an easy exercise, verify that each \mathcal{A}_n is closed with respect to taking complement, union and intersection. Such a sequence $\{\mathcal{A}_i\}_{i=0}^N$ is called a *filtration*.

Definition 1.2.2. A function $T : \Omega \rightarrow \{0, 1, \dots, N\} \cup \{\infty\}$ is called a *stopping time* if for each $0 \leq n \leq N$,

$$\{T = n\} = \{\omega \in \Omega : T(\omega) = n\} \in \mathcal{A}_n$$

Example. For $a \in \mathbb{Z}$, let $\sigma_a = \inf\{n : S_n = a, 0 \leq n \leq N\}$ denote the *first* hitting time of a . As an exercise, show that σ_a is a stopping time.

Example. For $a \in \mathbb{Z}$, let $L_a = \max\{n : S_n = a, 0 \leq n \leq N\}$ denote the *last* hitting time of a . As an exercise, show that L_a is NOT a stopping time.

Theorem 1.2.1. Let $T : \Omega \rightarrow \{0, 1, \dots, N\}$ be a stopping time. Then

$$\mathbf{E}[S_T] = 0$$

where $S_T : \Omega \rightarrow \mathbb{Z}$ maps $\omega \mapsto S_{T(\omega)}(\omega)$.

Proof.

$$\begin{aligned}
S_T &= \sum_{k=1}^N S_k \mathbb{1}\{T = k\} = \sum_{k=1}^N S_k (\mathbb{1}\{T \geq k\} - \mathbb{1}\{T \geq k+1\}) \\
&= \sum_{k=1}^N (S_k - S_{k-1}) \mathbb{1}\{T \geq k\} \\
&= \sum_{k=1}^N X_k \mathbb{1}\{T \geq k\}
\end{aligned}$$

where we take $\mathbb{1}\{T \geq N+1\} = 0$. Now, we can write $\mathbf{E}[S_T]$ as

$$\mathbf{E}[S_T] = \sum_{k=1}^N \mathbf{E}[X_k \mathbb{1}\{T \geq k\}] \quad (\dagger)$$

Observe that for $1 \leq k \leq N$, we have

$$X_k \mathbb{1}\{T \geq k\} = \begin{cases} 1, & \text{for } X_k = 1, T \geq k \\ -1, & \text{for } X_k = -1, T \geq k \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{E}[X_k \mathbb{1}\{T \geq k\}] = \mathbf{P}(X_k = 1, T \geq k) - \mathbf{P}(X_k = -1, T \geq k) \quad (\dagger\dagger)$$

Now,

$$\{T \geq k\} = \{T < k\}^c = \left(\bigcup_{l=0}^{k-1} \{T = l\} \right)^c \in \mathcal{A}_{k-1}$$

Using the fact that $\{T \geq k\} \in \mathcal{A}_{k-1}$, one can show that (details left as an exercise)

$$\mathbf{P}(X_k = 1, T \geq k) = \mathbf{P}(X_k = -1, T \geq k) = \frac{1}{2} \mathbf{P}(T \geq k)$$

Substituting the above values in (\dagger) and $(\dagger\dagger)$, we finally have

$$\mathbf{E}[S_T] = 0$$

□

As an exercise, compute $\text{Var}[S_T]$.

Definition 1.2.3. A bet sequence / game system is a sequence of random variables $V_k : \Omega \rightarrow \mathbb{R}$ such that

$$\{V_k = c\} \in \mathcal{A}_{k-1} \text{ for every } c \in \mathbb{R} \text{ and } 1 \leq k \leq N$$

Theorem 1.2.2. Let $\{V_k\}_{k=1}^N$ be a bet sequence. Then

$$\mathbf{E}[S_N^V] = 0 \quad \text{where} \quad S_N^V = \sum_{k=1}^N V_k X_k$$

In this setting, S_N^V is interpreted as the “total gain”.

Proof. Since Ω is finite, we may write

$$\text{Range}(V_k) = \{c_i^k : 1 \leq i \leq m_k\} \text{ where } c_i^k \in \mathbb{R}$$

$$V_k = \sum_{i=1}^{m_k} c_i^k \mathbb{1}\{V_k = c_i^k\}$$

Now, since $\mathbf{E}[X_k] = 0$, and since $X_k \perp \mathbb{1}\{V_k = c_i^k\}$, we get

$$\begin{aligned} \mathbf{E}[S_N^V] &= \sum_{k=1}^N \mathbf{E}[V_k X_k] = \sum_{k=1}^N \mathbf{E}\left[X_k \sum_{i=1}^{m_k} c_i^k \mathbb{1}\{V_k = c_i^k\}\right] \\ &= \sum_{k=1}^N \sum_{i=1}^{m_k} c_i^k \mathbf{E}[X_k \mathbb{1}\{V_k = c_i^k\}] \\ &= \sum_{k=1}^N \sum_{i=1}^{m_k} c_i^k \mathbf{E}[X_k] \mathbf{P}(V_k = c_i^k) \\ &= 0 \end{aligned}$$

□

1.3 Exercises

1. Show that $\{X_k\}_{k=1}^N$ are independent.
2. Show that $\{S_n\}_{n=0}^N$ satisfies the Markov property.
3. For $1 \leq k < m \leq N$, show that $\mathbf{P}(S_m = b \mid S_k = a) = \mathbf{P}(S_{m-k} = b - a)$.
4. Show that $\mathbf{E}[S_n^2] = \sum_{i=1}^n \mathbf{E}[X_i^2]$.
5. (a) Show that for any $a, b \in \mathbb{R}$,

$$\mathbf{P}(a \leq S_n < b) \leq (b - a) \mathbf{P}(S_n \in \{-1, 0, 1\}).$$

- (b) Using (a), conclude that

$$\mathbf{P}(a \leq S_n < b) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, we observe that the walk exits any finite interval as $n \rightarrow \infty$.

6. Verify that each \mathcal{A}_n , $0 \leq n \leq N$, is closed with respect to taking complement, union and intersection.
7. For $a \in \mathbb{Z}$, let $\sigma_a = \inf\{n : S_n = a, 0 \leq n \leq N\}$. Show that σ_a is a stopping time.
8. For $a \in \mathbb{Z}$, let $L_a = \max\{n : S_n = a, 0 \leq n \leq N\}$. Show that L_a is not a stopping time.
9. Let $T : \Omega \rightarrow \{0, 1, \dots, N\}$ be a stopping time. Compute $\text{Var}[S_T]$.
10. Show that X_k and $\mathbb{1}\{T \geq k\}$ are independent.