Week 3

Random Walks on Graphs

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3.1 Introduction

- A random walk on a graph is basically a reversible Markov chain on the graph.
- many results of random walks will hold true for general markov chains but we will not go into it
- we will study some of the geometric properties of the Graph which translate to different properties of the Random walks

$\Gamma = (V, E)$
$V \equiv$ Vertex set = finite or countably infinite set.
$E \equiv$ Edge set $= E \subset \mathcal{P}(V) = \{\{x, y\} : |x, y \in V, x \neq y\}$.
(No self loops, No multiple edges)

1. $x \in V; y \in V$ is a neighbour of x in $\{x, y\} \in E$ $(x \sim y)$
2. A path $\gamma \in \Gamma$ is any sequence $\{x_i\}_{i=0}^n$ such that $x_{i-1} \sim x_i$ in Γ for some $n \geq 1, x_i \in V, 1 \leq i \leq n$
   •γ is a loop if $x_0 = x_n$
   •γ is self avoiding if $x_i \neq x_j \ \forall \ i \neq j$.
3. "chemical metric" $d : V \times V \longrightarrow [0, \infty) \bigcup \{\infty\}$
   $d(x, x) = 0$,
   $$d(x, y) = \begin{cases} \text{length of smallest path from x to y} \\ \infty \text{ if no path exixts} \end{cases}$$
4. Γ is connected if $d(x, y) < \infty, \forall x, y \in V$ **(H1 property)**
5. Γ is locally finite if $\forall x \in V$,
   $N(x) = \{y \in V | y \sim x\} \Rightarrow |N(x)| < \infty$ **(H2 property)**
6. we say Γ has a bounded geometry if $\sup_{x \in V} |N(x)| < \infty$ **(H3 property)**

Definition 3.1.1. *$\forall \ x, y \in V$, we assume that thre is a weight μ_{xy} such that:*

1. $\mu_{xy} = \mu_{yx}$

2. $\mu_{xy} \geq 0$

3. if $x \neq y$ *then,* $\mu_{xy} > 0 \Leftrightarrow x \sim y$

we will call (Γ, μ) *a weighted graph.*
Using property 3 above, $E = \{\{x, y\} | x, y \in V, \mu_{xy} > 0, x \neq y\}$

Definition 3.1.2. (Γ, μ) *has bounded weights if* $\exists\ C_1, C_2 > 0$ *such that* $C_1 < \mu_{xy} \leq C_2\ \forall\ x, y \in V, x \neq y$. *This is called the* ***(H4 Property)***.

Definition 3.1.3. (Γ, μ) *has controlled weights if* $\exists\ c > 0$ *such that* $\frac{\mu_{xy}}{\mu_x} \geq c^{-1}\ \forall\ x, y \in V, x \neq y$. *This is called the* ***(H5 Property)***.

Define for $x \in V$: $\mu_x = \sum_{y \sim x} \mu_{xy}$

Definition 3.1.4. *Natural weights:*

$$\mu_{xy} = \begin{cases} 1 \text{ if } x \sim y \\ 0 \text{ otherwise} \end{cases}$$

Lemma 3.1.1. *Suppose* (Γ, μ) *is a weighted graph then,*

1. (H3), (H5) holds.

2. $\forall x \in V, n > 0\ , B(x, n) = \{y \in V | d(x, y) \leq n\}$ *(balls are not exponentially large)*

3. $\forall x \in V, n \geq 0, \mu(B(x, n)) = \sum_{y \in B(x,n)} \mu_y \leq 2\mu_x(c_2)^n$ *(Balls have bounded weights)*

Proof. 1. Take $x \in V$.

$$\begin{aligned} N(x) &= c \sum_{y \in V} \frac{1}{c} 1_{\{x \sim y\}} \\ &\leq c \sum_{y \in V} \frac{\mu_{xy}}{\mu_x} 1_{\{x \sim y\}} \\ &= c \frac{1}{\mu_x} \sum_{y \in V} \mu_{xy} = c \end{aligned}$$

2. $S(x, n) = \{y \in V | d(x, y) = n\}$

$$|S(x, n)| \leq c|S(x, n-1)|\ \ \forall\ n \geq 1$$

Arguing inductively,

$$\begin{aligned}|B(x,n)| &= \sum_{k=0}^{n} |S(x,k)| \\ &\le \sum_{k=0}^{n} c^k \\ &= \frac{c^{n+1}-1}{c-1} \le 2c^n\end{aligned}$$

3. $n = 1$.

$$\begin{aligned}\mu(B(x,1)) &= \mu_x + \sum_{y\sim x} \mu_y \\ &\le c\sum_{y\sim x} \mu_{xy} + \mu_x \\ &= c\mu_x + \mu_x\end{aligned}$$

Second step follows from the H5 assumption.
We also note
$$\mu(B(x,2)) = \sum_{y\in B(x,2)} \mu_y = \mu(B(x,1)) + \sum_{y\sim x}\sum_{z\sim y} \mu_z$$

Therefore

$$\begin{aligned}\mu(B(x,2)) &\le \mu_x + c\mu_x + \sum_{y\sim x} c \sum_{z\sim y} \mu_{zy} \\ &= \mu_x + c\mu_x c \sum_{y\sim x} \mu_y \\ &\le \mu_x + c\mu_x + c^2\mu_x\end{aligned}$$

$\square$

Example. $V = \mathbb{Z}^d$. Take $x, y \in V, |x-y| = \sum_{i=1}^{d} |x_i - y_i|$
$E = \{(x,y) \mid |x-y| = 1\}$. $\mu_{xy} = 1$ whenever $(x,y) \in E$. $N(x) = 2d \ \ \forall x \in V$
$|B(x,n)| \sim n^d \le 2c^n \ \ \forall c \ge 2$.
Example. Rooted Binary Tree- Let the root be $B_0 = \{\rho\}$.
$\forall\ n \ge 1, B_n = \{0,1\}^n$

$$V = \cup_{n=1}^{\infty} B_n \cup \{\rho\}$$

For $x \in B_n, n \ge 2, x = (x_1, \dots, x_n), x_i \in \{0,1\}$.
Let the parent of x be- $\alpha(x) = (x_1, \dots, x_{n-1})$
For $n = 1, x \in B_1, \alpha(x) = \rho$

$$E = \{(x, \alpha(x)) \mid\ x \in V, x \notin B_0\}$$
$$|N(\rho)| = 2, |N(x)| = 3 \ \ \forall\ x \notin B_0$$

Canopy Tree

$$\bar{V} = \{x \in V | x = (x_1, \ldots, x_n) \text{ and } x_i = 0 \ \ \forall\, 1 \leq i \leq n \text{ forsome } n \geq 1\} \cup \{\rho\}$$

$f(x)$ is the element in $\bar{V}$ closest to x.
V_{canopy} is a subset of V such that-

$$V_{canopy} = \{x \in V | \ d(x, f(x)) \leq d(\rho, f(x))\}$$

Observe that in the canopy tree, there is only one self-avoiding path to infinity, but the size of the balls $B(\rho, n)$ still grows exponentially. It shows that one does not need too many paths to infinity for the size of your graph to grow exponentially. Denoted by $\mathbb{T}^2_{canopy}$

3.2 Random Walks on Weighted Graphs

(This section will be done as a discrete time reversible Markov Chain)
Formally, X_n jumps from $x \sim y_i$ with probability proportional to μ_{xy_i}. It stays at x with probability proportional to μ_{xx}.

Our graph is denoted by $\Gamma = (V, E)$. We assume there are no isolated edges that is $\{\mu_x \neq 0 \ \ \forall x \in V\}$. Also assume H(1) and H(2).

$$\Omega = \{f : \mathbb{N} \cup \{0\} \to V\} \equiv V^{\mathbb{N} \cup \{0\}}$$

$\forall\, n \geq 0$, $X_n : \Omega \to V$ where $X_n(\omega) = \omega(n)$
Let $\mathcal{A}_n \equiv$ observable events upto time n (all events that can be derived from $X_1, \ldots, X_n$). This will be a filtration.

$$\mathcal{F} \equiv \cup_{n \geq 1} \ \mathcal{A}_n$$

Set $\mathcal{P}(x, y) = \frac{\mu_{xy}}{\mu_x} \ \ \forall x, y \in V$.
$\forall x \ \in V$, there exists a unique $\mathcal{P}^x(.)$ on $(\Omega, \mathcal{F})$.
(Existence can be shown using Kolmogorov consistency theorem).
$\forall\, n \geq 1$

$$\mathbb{P}^x(X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = 1_{\{x\}}(x_0) \prod_{i=1}^{n} P(x_n, x_{n-1})$$

$$\begin{aligned}
\mathbb{P}^x(X_1 = y) &= \mathbb{P}^x(X_1 = y, \cup_{z \in V} X_0 = z) \\
&= \sum_{z \in V} \mathbb{P}^x(X_1 = y, X_0 = z) \\
&= \sum_{z \in V} \mathcal{P}(y, z) 1_{\{x\}}(z) \\
&= \mathcal{P}(y, x)
\end{aligned}$$

One-step transition probability-

$$\mathbb{P}(X_n = y | X_{n-1} = z) = \frac{\mathbb{P}(X_n = y, X_{n-1} = z)}{\mathbb{P}(X_{n-1} = z)} = \mathcal{P}(y, z)$$

The last equality is left as an exercise.
Reversibility-

$$\mu_x \mathcal{P}(x, y) = \mu_x \frac{\mu_{xy}}{\mu_x} = \mu yx = \mu_y \mathcal{P}(y, x)$$

$(X_n, \mathcal{P})$ markov chain is symmetric with repsect to $\{\mu_x\}_{x \in V}$

Lemma 3.2.1. *Let $x_0, \ldots, x_n \in V$*

$$\mu_{x_0} \mathbb{P}^{x_0}(X_n = x_n, \ldots, X_0 = x_0) = \mu_{x_n} \mathbb{P}^{x_n}(X_n = x_0, \ldots, X_0 = x_n)$$

The above shows the reversibility of the markov chain wrt μ.

Proof.

$$\begin{aligned}
\mu_{x_0} \mathbb{P}^{x_0}(X_n = x_n, \ldots, X_0 = x_0) &= \mu_{x_0} \prod_{i=1}^{n} \mathcal{P}(x_i, x_{i-1}) \\
&= \mu_{x_0} \prod_{i=1}^{n} \frac{\mu_{x_i, x_{i-1}}}{\mu_{x_{i-1}}} \\
&= \mu_{x_n} \prod_{i=1}^{n} \frac{\mu_{x_{n-i}, x_{n-i+1}}}{\mu_{x_{n-i+1}}} \\
&= \mu_{x_n} \mathbb{P}^{x_n}(X_n = x_0, \ldots, X_0 = x_n)
\end{aligned}$$

□

Remark. If $\mu(V) = \sum_{x \in V} \mu_x = 1$ and $\mu(A) = \sum_{x \in A}$, then μ is the reversible distribution for $\{X_n\}_{n \geq 0}$ that is

$$\mu_x \mathcal{P}(x, y) = \mu_y \mathcal{P}(y, x)$$

Hence $\{\mu_x\}_{x \in V}$ is the stationary distribution.

Definition 3.2.1. *$A \subseteq V$. The hitting time of A be given by*

$$T_A = \min\{n \geq 0 | X_n \in A\}$$

By convention, $T_A = \infty$ iff X_n does not visit A.

Definition 3.2.2. *The return time of A is defined as -*

$$T_A^+ = \min\{n \geq 1 | X_n \in A\}$$

Note that $X_0 \notin A \implies T_A^+ = T_A$

Definition 3.2.3. *The exit time of A is-*

$$\tau_A = T_{A^c}$$

Theorem 3.2.1. *Let Γ be H(1) and H(2) and $|V| = \infty$. Then TFAE-*

1. $\exists\ x \in V$ *such that* $\mathbb{P}^x(\tau_x^+ < \infty) < 1$

2. $\forall\ x \in V$, $\mathbb{P}^x(\tau_x^+ < \infty) < 1$

3. $\forall\ x \in V$, $\sum_{n=0}^{\infty} \mathbb{P}^x(X_n = x) < \infty$

4. $\forall\ x, y \in V$, $\mathbb{P}^x(\tau_y < \infty) < 1$

5. $\mathbb{P}^x(\sum_{n\geq 0} 1_{\{X_n = x\}} < \infty) = 1 \quad \forall x, y \in V$

If the above is satisfied, the Markov Chain is transient.

Theorem 3.2.2. *Let Γ be H(1) and H(2) and $|V| = \infty$. Then TFAE-*

1. $\exists\ x \in V$ *such that* $\mathbb{P}^x(\tau_x^+ < \infty) = 1$

2. $\forall\ x \in V$, $\mathbb{P}^x(\tau_x^+ < \infty) = 1$

3. $\forall\ x \in V$, $\sum_{n=0}^{\infty} \mathbb{P}^x(X_n = x) = \infty$

4. $\forall\ x, y \in V$, $\mathbb{P}^x(\tau_y < \infty) = 1$

5. $\mathbb{P}^x(\sum_{n\geq 0} 1_{\{X_n = x\}} = \infty) = 1 \quad \forall x, y \in V$

If the above is satisfied, the Markov Chain is recurrent.

Definition 3.2.4. *If $\{X_n\}_{n\geq 0}$ random walk on (Γ, μ) satisfies*

1. *any statement of theorem 1.6, the graph (Γ, μ) is transient.*

2. *any statement of theorem 1.7, the graph (Γ, μ) is recurrent.*

3.3 Exercises

1. Show that $H_3, H_4 \Rightarrow H_5$

2. When is (Γ, μ) transient or recurrent?
   Partial answer- When $|V| < \infty$, (Γ, μ) is recurrent.

3. **Kesten Problem-** G is a finitely generated group with generating set A. Look at the Cayley graph of G. Which groups provide transient graphs?