Week 9

Random walk in trap environment

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9.1 Continuous time random walk

In this model the random walker wasits an exponential amount of time to perform a jump like a discrete time random walk. Consider $\{V_i : i \geq 1\}$ to be a collection of independent Exponential(λ) random variables. Let $\lambda = 1$. Define T_k to be the sum of the first k V_i's. Also, define N_t to be the number of T_k less than t. Hence,

- $\mathbf{P}(V_i \leq t) = 1 - e^{-t}$
- $T_k = \sum_{i=1}^{k} V_i$
- $N_t = \sum_{k=1}^{\infty} \mathbb{1}(T_k \leq t)$
- $\{N_t = k\} = \{T_k \leq T_{k+1}\}$

Theorem 9.1.1. *1. $N_t \sim Poisson(t)$*

2. $N_t - N_s$ is independent of N_r where $r \leq s \leq t$.

3. For $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n$

$$\{N_{t_{i+1}} - N_{t_i} : i = 0, \ldots, n-1\} \text{ are independent}$$

So, $N_{t_{i+1}} - N_{t_i} \sim Poisson(t_{i+1} - t_i)$

Definition 9.1.1. *Let $U_n : n \geq 0$ be a random walk on (Γ, μ). Define, a continuous time random walk on (Γ, μ) with rate 1 to be:*

$$Y_t = U_{N_t} \forall t \geq 0$$

.

Remark. The random variable Y_t is a random step function which is right continuous with left limits.

9.2 Random walk in trap environment

Continuous Time set-up

Consider the graph $\mathbb{Z}^d$ with natural weights. Let $\{X_t\}_{t\geq 0}$ be a continuous time random walk on $\mathbb{Z}^d$ starting at 0, with rate κ. Now, let us set up the traps, i.e., for each $y \in \mathbb{Z}^d$ let $N_y \sim$Poisson(ρ). This N_y denote the number of *traps* at y. Each trap$(Y^{j,y})$ perform a continuous time random walk$(\{Y_t^{j,y}\}_{t\geq 0})$ with rate ν; where $1 \leq j \leq N_y$. The random walk gets killed if it meets a trap. There are two ways of killing viz,

Hard The walk gets killed upon intersection with any $Y^{j,y}$.

Soft At each site x at time $t \geq 0$, define

$$\xi(t,x) := \sum_{y\in\mathbb{Z}^d, 1\leq j\leq N_y} \#\{Y^{j,y} \text{ at } x\}.$$

Now X_t gets killed at rate $\gamma\xi(t,x)$ where $\gamma \in \mathbb{R}$.

Remark. Hard killing infact corresponds to $\gamma = \infty$ case of soft killing.

The probability of survival is given by

$$Z_{\gamma,t} = \mathbf{E}^X[\exp(-\gamma \int_0^t \xi(s, X(s))ds)]$$

Discrete Time set-up

Let $\{X_t\}_{t\geq 0}$ be a random walk on $\mathbb{Z}^d$ with natural weights starting at 0. For each $y \in \mathbb{Z}^d$ let $N_y \sim$Poisson(ρ) denotes the number of traps at y. Each trap$(Y^{j,y})$ perform a lazy random walk$(\{Y_t^{j,y}\}_{t\geq 0})$ on $\mathbb{Z}^d$; where $1 \leq j \leq N_y$. The trap kills the random walk with probability q if it meets the random walk; $q \in (0,1)$. Let $\xi(n,x)$ denote the number of traps at location x, i.e.

$$\xi(n,x) = \sum_{y\in\mathbb{Z}^d, 1\leq j\leq N_y} \delta_x(Y_n^{j,y}).$$

Assume X_k has survived till $k \leq n$. Given X_n the probability that X_n will survive at time n is $(1-q)^{\xi(n,X_n)}$. Hence,

$$\begin{aligned}\sigma^X(n,\xi) &= \mathbf{P}(X \text{ has survived till time } n \text{ given } \{Y_m^{j,y}\}_{1\leq j\leq m, y\in\mathbb{Z}^d} \text{ where } m \leq n) \\ &= (1-q)^{\sum_{i=1}^n \xi(i,X_i)}.\end{aligned} \tag{9.1}$$

9.3 Pascal's Theorem

The average survival probability of a given trajectory X is given by $\sigma^X(n) = \mathbf{E}^\xi[(1-q)^{\sum_{i=1}^n \xi(i,X_i)}]$.

Theorem 9.3.1 (Pascal). *The survival probability is maximized by the trajectory $\underline{0}$ where $\underline{0}_k = 0$ for every $k \in \mathbb{N} \cup 0$, i.e,*

$$\sigma^X(n) \leq \sigma^{\underline{0}}(n).$$

Lemma 9.3.1. *$\sigma^X(n) = \exp(-\lambda \sum_{y\in\mathbb{Z}^d} W_X(n,y))$ where $W_X(n,y) = 1-\mathbf{E}^y[1-(1-q)^{\sum_{i=1}^n \delta(Y_i^y)}]$. The Y_i^y is a random variable with ditribution same as i.i.d. $Y_i^{j,y}$.*

Proof. Let $X : \mathbb{N}\cup 0 \to \mathbb{Z}^d$ with $X_0 = 0$ be the trajectory. Now,

$$\begin{aligned}
\sigma^X(n) &= \mathbf{E}^\xi[(1-q)^{\sum_{i=1}^n \xi(i,X_i)}] \\
&= \mathbf{E}^\xi[(1-q)^{\sum_{i=1}^n \sum_{y\in\mathbb{Z}^d} \sum_{1\le j\le N_y} \delta_{X_i}(Y_n^{j,y})}] \\
&= \prod_{y\in\mathbb{Z}^d} \mathbf{E}^\xi[\prod_{1\le j\le N_y} (1-q)^{\sum_{i=1}^n \delta_{X_i}(Y_n^{j,y})}] \\
&= \prod_{y\in\mathbb{Z}^d} \mathbf{E}^y\mathbf{E}^{N_y}[\prod_{1\le j\le N_y} (1-q)^{\sum_{i=1}^n \delta_{X_i}(Y_n^{j,y})}] \\
&= \prod_{y\in\mathbb{Z}^d} \sum_{k=0}^\infty \frac{e^{-\lambda}\lambda^k}{k!}\mathbf{E}^y[\prod_{1\le j\le k} (1-q)^{\sum_{i=1}^n \delta_{X_i}(Y_n^{j,y})}] \\
&= \prod_{y\in\mathbb{Z}^d} \sum_{k=0}^\infty \frac{e^{-\lambda}\lambda^k}{k!}(\prod_{1\le j\le k} (1-q)^{\sum_{i=1}^n \delta_{X_i}(Y_n^{j,y})})^k \\
&= \prod_{y\in\mathbb{Z}^d} e^{-\lambda(1-\mathbf{E}^y((1-q)^{\sum_{i=1}^n \delta_{X_i}(Y_n^{j,y})}))} \\
&= e^{-\lambda\sum_{y\in\mathbb{Z}^d} W_x(n,y)}.
\end{aligned}$$

□

Lemma 9.3.2. *$W_X(n,y) = 1-\mathbf{E}^y[1-(1-q)^{\sum_{i=1}^n \delta(Y_i^y)}] = \mathbf{P}_y^X(\tau \le n)$, where $\tau = min\{i \ge 0 | X_i = Y_i, Z_i = 1\}$.*

Lemma 9.3.3. *$\sum_{y\in\mathbb{Z}^d} \mathbf{P}_y^X(\tau \le n) \ge \sum_{y\in\mathbb{Z}^d} \mathbf{P}_y^0(\tau \le n)$.*

Proof. Let

$$\begin{aligned}
q &= \mathbf{P}(Z_n = 1) \\
&= \mathbf{P}_{X_n}(\bigcup_{y\in\mathbb{Z}^d} \{Z_n = 1, Y_n = y\}) \\
&= \sum_{y\in\mathbb{Z}^d} \mathbf{P}_{X_n}(Z_n = 1, Y_n = y) \\
&= \sum_{y\in\mathbb{Z}^d} \mathbf{P}_{X_n}(Z_n = 1, Y_n = X_n) \\
&= \sum_{y\in\mathbb{Z}^d} \left[\mathbf{P}^X(\tau = n) + \sum_{k=0}^{n-1} \mathbf{P}_y^X(\tau = k) p_{n-k}^y (X_n - X_k) q\right]
\end{aligned}$$

Lemma 9.3.4. *For a lazy symmetric random walk on $\mathbb{Z}^d$.*

$$p_n^Y(0) \ge p_n^Y(y), \forall y \in \mathbb{Z}^d$$

$$p_n^Y(0) \ge p_{n+1}^Y(0).$$

Therefore using the above lemma, we get:

$$q \leq \sum_{y\in\mathbb{Z}^d} \left[\mathbf{P}^X(\tau = n) + \sum_{k=0}^{n-1} \mathbf{P}_y^X(\tau = k)p_{n-k}^y(0)q\right].$$

Also, replacing $X = \underline{0}$ in $\sum_{y\in\mathbb{Z}^d} \left[\mathbf{P}^X(\tau = n) + \sum_{k=0}^{n-1} \mathbf{P}_y^X(\tau = k)p_{n-k}^y(X_n - X_k)q\right]$, we get:

$$q = \sum_{y\in\mathbb{Z}^d} \left[\mathbf{P}^0(\tau = n) + \sum_{k=0}^{n-1} \mathbf{P}_y^0(\tau = k)p_{n-k}^y(0)q\right]$$

.

Let

$$S_n^X = \sum_{y\in\mathbb{Z}^d} \mathbf{P}_y^X(\tau \leq n)$$
$$S_n^0 = \sum_{y\in\mathbb{Z}^d} \mathbf{P}_y^0(\tau \leq n)$$
$$S_n^X - S_{n-1}^X = \sum_{y\in\mathbb{Z}^d} \mathbf{P}_y^X(\tau = n)$$

We define $S_{-1}^X = S_{-1}^0 = 0$.

We have

$$q = \sum_{y\in\mathbb{Z}^d} \left[\mathbf{P}^X(\tau = n) + \sum_{k=0}^{n-1} \mathbf{P}_y^X(\tau = k)p_{n-k}^y(X_n - X_k)q\right]$$
$$\implies (S_n^X - S_n^0) \geq (1 - qp_i^Y(0))(S_{n-1}^X - S_{n-1}^0) + q\sum_{k=0}^{n-2}(S_k^X - S_k^0)(p_{n-k-1}^Y(0) - p_{n-k}^Y(0))$$

Now using induction, we get $S_n^X \geq S_n^0$.

$\square$

Remark. The continuous case has a similar proof and can be found here [**?**].