Week 5

Killed process and Green's function

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5.1 Introduction

(Γ, μ) is a weighted graph which is H1(Locally finite) and H2(Connected).$\{X_n\}$ is a simple random walk on it.

Transition density: $p_n^x(y) = \frac{\mathbf{P}^x(X_n)=y}{\mu_y}$

$p_0(x,y) = \frac{\mathbf{1}_x(y)}{\mu_y}$

The transition density satisfies the following:

- $p_{n+m}(x,y) = \sum_{z\in\mathbb{V}} p_n(x,z)p_m(z,y)\mu_z$ [Chapman-Kolmogorov Equation]
- $p_n(x,y) = p_n(y,x)$ [Symmetry]
- $P(p_n^x(y)) = \sum_{z\in\mathbb{V}} p(y,z)p_n^x(z)\mu_z = \sum_{z\in\mathbb{V}} p(y,z)p_n(x,z)\mu_z = p_{n+1}^x(y)$ [*details left as Exercise*]
- $p_t^x(y) = P(t;x,y) = \frac{e^{-(x-y)^2}/2t}{\sqrt{2\pi t}}$

  $\Leftrightarrow \frac{\delta}{\delta t}p_t^x = \Delta p_t^x = \frac{\delta^2}{\delta y^2}p_t^x$
- $\Delta p_n^x(y) = (P-I)p_n^x y = p_{n+1}^x(y) - p_n^x(y)$
- $\|p_n^x\|_2^2 = \langle p_n^x, p_n^x\rangle = p_{2n}(x,x) = \frac{\mathbf{P}^x(X_2 n=x)}{\mu_x} \leq \frac{1}{\mu_x}$

Dirichlet form/Energy form

$\varepsilon(f,g) = \frac{1}{2}\sum_{x\in\mathbb{V}}\sum_{y\in\mathbb{V}}$

Domain of $\varepsilon : D(\varepsilon) = \{f : \mathbb{V} \to \mathbb{R} | \varepsilon(f,f) < \infty\}$

$$\begin{aligned}\varepsilon(f,g) &= -\langle \Delta f, g\rangle \\ &= -\langle (P-I)f, g\rangle \\ &= -\langle Pf, g\rangle + \langle f, g\rangle\end{aligned}$$

where the first equality comes from Discrete Gauss-Green theorem.

$$\varepsilon \leftrightarrow \Delta \leftrightarrow P \leftrightarrow \{X_n\}_{n\geq 1}$$

on $\mathbb{R}^n$

$$\varepsilon(f,g) = \int_{\mathbb{R}^n} \nabla f(x) \nabla g(x) dx$$

it can be shown that if $f \in D(\varepsilon)$, $-\langle \Delta f, g \rangle_n$

$$\varepsilon \leftrightarrow \Delta \leftrightarrow \{P_t\}_{t\geq 0} \leftrightarrow \{X_t\}_{t\geq 0}$$

$$\begin{aligned} \varepsilon(p_n^x, p_m^y) &= -\langle \Delta p_n^x, p_m^y \rangle \\ &= -\langle p_{n+1}^x - p_n^x, p_m^y \rangle \\ &= -\langle p_{n+1}, p_m^y \rangle + \langle p_n^x, p_m^y \rangle \\ &= -p_{n+m+1}(x,y) + p_{n+m}(x,y) \end{aligned}$$

where the first equality comes from Discrete Gauss-Green theorem. *As an Exercise* check that $p_n^x(.)$ and $p_m^y(.)$ satisfies the hypothesis of Discrete Gauss- Green Theorem.

$x \in \mathbb{V}$, $I_x(z) = \begin{cases} 1, z = x \\ 0, otherwise \end{cases}$

$$\begin{aligned} \varepsilon(I_x, I_y) &= -\langle \Delta I_x, I_y \rangle \\ &= -\sum_{z\in\mathbb{V}} I_y(x) \Delta I_x(z) \mu_z \\ &= -\Delta I_x(y) \mu_y \\ &= \mu_y \frac{\sum_{z\in\mathbb{V}} (I_x(z) - I_x(y)\mu_{zy}}{\mu_y} \\ &= \begin{cases} -\mu_{xy}, if y \neq x \\ \mu_x - \mu_{xx}, if y = x \end{cases} \end{aligned}$$

5.2 Killed Process

<u>Gambler's ruin</u>

N: Total capital of 2 players

X_k : Capital of Player 1 in k^{th} step

$$\mathbf{P}^x(X_{T_{\{0,N\}}} = 0) = h(X) \leftrightarrow h(x) = \begin{cases} \frac{1}{2}h(x-1) + \frac{1}{2}h(x+1), 0 < x < N \\ 1, x = 0 \\ 1, x = N \end{cases}$$

$$h = Ph \Leftrightarrow \Delta h = 0$$

Let the graph $\Gamma = (\mathbb{V}, E)$ be H1 and H2 with weights μ. $A \subset \mathbb{V}$.

$\tau_A = \tau_{A^c} = inf\{n \geq 1 | X_n \in A^c\}$

We define the kill density, i.e. the transition density of the random walk until it exits A by:

$$p_n^A(x,y) = \frac{\mathbf{P}^x(X_n = y, n < \tau_A}{\mu_y}$$

- if $y \notin A$, then $p_n^A(x,y) = 0 \; \forall n \geq 1$
- $I_A f(x) = I_A(x) f(x)$
- $n \geq 1$, $P_n^A f(x) = \sum_{z \in \mathbb{V}} p_n^A(x,z) f(z) \mu_z = F^x[f(X_n); n < \tau_A]$
- $\Delta^A := P^A - I^A$

Lemma 5.2.1. *(a)* $p_n^A(x,y) = 0 \; \forall x, y \notin A, n \geq 1$

(b) $p_{n+1}^A(x,y) = \sum_{z \in \mathbb{V}} p_n^A(x,z) p^A(z,y) \mu_z$

(c) $\Delta p_n^{A,x} = p_{n+1}^{A,x} - p_n^{A,x}$
[$p_n^{A,x} = p_n^A(x,y)$*]*

(d) $p_n^A(x,y) = p_n^A(y,x) \; \forall x, y \in \mathbb{V}$

(e) $P_n^A f(x) = (P^A)^n f(x) \; \forall n \geq 1$

(f) $P^A f(x) = I_A P I_A f(x)$

Proof. Left as an Exercise. □

5.3 Green's function

Let $A \subset \mathbb{V}$. We define Green's function of $\{X_n\}_{n \geq 0}$ as:

$$g_A(x,y) = \sum_{n=0}^{\infty} p_n^A(x,y)$$

$x, y \in \mathbb{V}$.

Notation. • *if* $A = \mathbb{V}$ *then* $g_A = g$

- $x \in \mathbb{V}$ *fixed, then* $g_A^x(y) = g_A(x,y \; \forall y \in \mathbb{V}$

Observations. • $g_A(x,y) = g_A(y,x) \; \forall \; x, y \in \mathbb{V}$.

- *Define Local time at* y *before exiting* A *i.e. time spent by the walk at* y *before exiting* A *by* $L^y_{\tau_A} = \sum_{n=0}^{\infty} \mathbf{1}_{X_n=y}$.

$$\begin{aligned} g_A(x,y) &= \sum_{n=0}^{\infty} p_n^A(x,y) \\ &= \frac{\sum_{n=0}^{\infty} E^x[\mathbf{1}_{X_n=y}; n<\tau_A]}{\mu_y} \\ &= \frac{E^x[\sum_{n=0}^{\infty}(\mathbf{1}_{X_n=y}\mathbf{1}_{n<\tau_A})]}{\mu_y} \\ &= \frac{E^x[\sum_{n=0}^{\tau_A-1}(\mathbf{1}_{X_n=y})]}{\mu_y} \\ &= \frac{E^x[L^y_{\tau_A}]}{\mu_y}. \end{aligned}$$

- *if* $A=\mathbb{V}$ *and* $\mathbb{V}$ *is recurrent then* $g(x,.)=\infty$

Theorem 5.3.1. $A \subset \mathbb{V}$. *Suppose either* (Γ,μ) *is transient or* $A \neq V$. *Then*

1. $g_A(x,y) = \mathbb{P}(\tau_y < \tau_A) g_A(y,y)$
2. $g_A(y,y) = \frac{1}{\mu_y \mathbb{P}(\tau_a \leq \tau_y^+)}$

Lemma 5.3.1. *Let* $x,y \in A$. *Then,*

1. $\mathbf{P} g_A^x(y) = g_A(x,y) - \frac{\mathbf{1}_x(y)}{\mu_x}$
2. $\Delta g_A^x(y) = \begin{cases} -\frac{1}{\mu_x} \text{ if } y{=}x \\ 0 \text{ otherwise} \end{cases}$

Proof. 1.

$$\begin{aligned}
Pg_A^x &= \sum_{z\in\mathbb{V}} p(y,z) g_A^x(z)\mu_z \\
&= \sum_{z\in\mathbb{V}} p(y,z)\mu_z(\sum_{n=0}^{\infty} p_n^A(xz) \\
&= \sum_{n=0}^{\infty}\sum_{z\in\mathbb{V}} p(y,z)\mu_z p_n^A(x,z) \\
&= \sum_{n=0}^{\infty}\sum_{z\in A} p(y,z)\mu_z p_n^A(x,z) \\
&= \sum_{n=0}^{\infty}\sum_{z\in A} p_1^A(y,z) p_n^A(x,z)\mu_z \\
&= \sum_{n=0}^{\infty} p_{n+1}^A(x,y) \\
&= g_A(x,y) - p_0^A(x,y) \\
\Rightarrow Pg_A^x(y) &= g_A(x,y) - \frac{\mathbf{1}_x(y)}{\mu_x}
\end{aligned}$$

2. follows from definition of $D = P - I$ $\square$

Proof of Theorem.
Notations: Given $f : \mathbb{V} \to \mathbb{R}$, $E^X f(X_n) = \sum_{y\in\mathbb{V}} \mathbf{P}^x(X_n = y) f(y)$.
let ξ be a random variable. $h_n(\xi) = E^\xi f(X_n)$
1.

$$\begin{aligned}
g_A(x,y)\mu_y &= E^x(L_{\tau_A}^y \\
&= E^x(\mathbf{1}_{\tau_y<\tau_A} \times L_{\tau_A}^y \\
&= E^x(\mathbf{1}_{\tau_y<\tau_A}\mathbf{E}^y(L_{\tau_A}^y) \\
\Rightarrow g_A(x,y) &= g_A(y,y)\mathbf{P}^x(\tau_y<\tau_A)\square
\end{aligned}$$

2. $p = \mathbf{P}(\tau_y^+ < \tau_A)$
if (Γ,μ) is transient then $p < 1$ and if recurrent and $A \neq \mathbb{V}$ then $p < 1$.[$\exists z \in A^c$ such that $\mathbf{P}^y(\tau_A < \tau_y^+) \geq \mathbf{P}^y(\tau_z < \tau_y^+) > 0$]
$\therefore p < 1$

$$
\begin{aligned}
\mathbf{P}^y(L^y_{\tau_A} = k) &= p^k(1-p) \\
\Rightarrow \mu_y g_A(y,y) &= E^y(L^y_{\tau_A}) \\
&= \sum_{k=0}^{\infty} p^k(1-p) \\
&= \frac{1}{1-p} \\
&= \frac{1}{\mathbf{P}(\tau_A \leq \tau_y^+)} \\
\Rightarrow g_A(y,y) &= \frac{1}{\mu_y \mathbf{P}(\tau_A \leq \tau_y^+)} \square
\end{aligned}
$$

Combining 1 and 2, we get

$$
g_A(x,y) = \frac{\mathbf{P}^x(\tau_y < \tau_A)}{\mu_y \mathbf{P}(\tau_A \leq \tau_y^+)}.
$$