Week 5

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Part I

Killed process and Green's function

5.1 Introduction

(Γ, μ) is a weighted graph which is H1(Locally finite) and H2(Connected).$\{X_n\}$ is a simple random walk on it.

Transition density: $p_n^x(y) = \frac{\mathbb{P}^x(X_n)=y}{\mu_y}$

$p_0(x,y) = \frac{\mathbf{1}_x(y)}{\mu_y}$

The transition density satisfies the following:

- $p_{n+m}(x,y) = \sum_{z\in\mathbb{V}} p_n(x,z)p_m(z,y)\mu_z$ [Chapman-Kolmogorov Equation]
- $p_n(x,y) = p_n(y,x)$ [Symmetry]
- $P(p_n^x(y)) = \sum_{z\in\mathbb{V}} p(y,z)p_n^x(z)\mu_z = \sum_{z\in\mathbb{V}} p(y,z)p_n(x,z)\mu_z = p_{n+1}^x(y)$ [*details left as Exercise*]
- $p_t^x(y) = P(t;x,y) = \dfrac{e^{-(x-y)^2}/2t}{\sqrt{2\pi t}}$

  $\Leftrightarrow \frac{\delta}{\delta t}p_t^x = \Delta p_t^x = \frac{\delta^2}{\delta y^2}p_t^x$
- $\Delta p_n^x(y) = (P-I)p_n^x y = p_{n+1}^x(y) - p_n^x(y)$
- $\|p_n^x\|_2^2 = \langle p_n^x, p_n^x\rangle = p_{2n}(x,x) = \frac{\mathbb{P}^x(X_2 n=x)}{\mu_x} \leq \frac{1}{\mu_x}$

Dirichlet form/Energy form

$\varepsilon(f,g) = \frac{1}{2}\sum_{x\in\mathbb{V}}\sum_{y\in\mathbb{V}}$

Domain of $\varepsilon : D(\varepsilon) = \{f : \mathbb{V} \to \mathbb{R} | \varepsilon(f,f) < \infty\}$

$$\begin{aligned}\varepsilon(f,g) &= -\langle \Delta f, g\rangle \\ &= -\langle (P-I)f, g\rangle \\ &= -\langle Pf, g\rangle + \langle f, g\rangle\end{aligned}$$

where the first equality comes from Discrete Gauss-Green theorem.

$$\varepsilon \leftrightarrow \Delta \leftrightarrow P \leftrightarrow \{X_n\}_{n\geq 1}$$

on $\mathbb{R}^n$

$$\varepsilon(f,g) = \int_{\mathbb{R}^n} \nabla f(x)\nabla g(x)dx$$

it can be shown that if $f \in D(\varepsilon)$, $-\langle \Delta f, g\rangle_n$

$$\varepsilon \leftrightarrow \Delta \leftrightarrow \{P_t\}_{t\geq 0} \leftrightarrow \{X_t\}_{t\geq 0}$$

$$\begin{aligned}\varepsilon(p_n^x, p_m^y) &= -\langle \Delta p_n^x, p_m^y\rangle \\ &= -\langle p_{n+1}^x - p_n^x, p_m^y\rangle \\ &= -\langle p_{n+1}, p_m^y\rangle + \langle p_n^x, p_m^y\rangle \\ &= -p_{n+m+1}(x,y) + p_{n+m}(x,y)\end{aligned}$$

where the first equality comes from Discrete Gauss-Green theorem. *As an Exercise* check that $p_n^x(.)$ and $p_m^y(.)$ satisfies the hypothesis of Discrete Gauss- Green Theorem.

$x \in \mathbb{V}$, $I_x(z) = \begin{cases} 1, z = x \\ 0, otherwise\end{cases}$

$$
\begin{aligned}
\varepsilon(I_x, I_y) &= -\langle \Delta I_x, I_y \rangle \\
&= -\sum_{z\in\mathbb{V}} I_y(x)\Delta I_x(z)\mu_z \\
&= -\Delta I_x(y)\mu_y \\
&= \mu_y \frac{\sum_{z\in\mathbb{V}}(I_x(z) - I_x(y)\mu_{zy}}{\mu_y} \\
&= \begin{cases} -\mu_{xy}, if y \neq x \\ \mu_x - \mu_{xx}, if y = x \end{cases}
\end{aligned}
$$

5.2 Killed Process

Gambler's ruin

N: Total capital of 2 players

X_k : Capital of Player 1 in k^{th} step

$$
\mathbb{P}^x(X_{T_{\{0,N\}}} = 0) = h(X) \leftrightarrow h(x) = \begin{cases} \frac{1}{2}h(x-1) + \frac{1}{2}h(x+1), 0 < x < N \\ 1, x = 0 \\ 1, x = N \end{cases}
$$

$$
h = Ph \Leftrightarrow \Delta h = 0
$$

Let the graph $\Gamma = (\mathbb{V}, E)$ be H1 and H2 with weights μ. $A \subset \mathbb{V}$.

$\tau_A = \tau_{A^c} = inf\{n \geq 1 | X_n \in A^c\}$

We define the kill density, i.e. the transition density of the random walk until it exits A by:

$$
p_n^A(x, y) = \frac{\mathbb{P}^x(X_n = y, n < \tau_A}{\mu_y}
$$

- if $y \notin A$, then $p_n^A(x, y) = 0 \; \forall n \geq 1$
- $I_A f(x) = I_A(x) f(x)$
- $n \geq 1$, $P_n^A f(x) = \sum_{z\in\mathbb{V}} p_n^A(x, z) f(z)\mu_z = F^x[f(X_n); n < \tau_A]$
- $\Delta^A := P^A - I^A$

Lemma 5.2.1. *(a)* $p_n^A(x, y) = 0 \; \forall x, y \notin A, n \geq 1$

(b) $p_{n+1}^A(x, y) = \sum_{z\in\mathbb{V}} p_n^A(x, z) p^A(z, y)\mu_z$

(c) $\Delta p_n^{A,x} = p_{n+1}^{A,x} - p_n^{A,x}$

[$p_n^{A,x} = p_n^A(x, y)$*]*

(d) $p_n^A(x, y) = p_n^A(y, x) \; \forall x, y \in \mathbb{V}$

(e) $P_n^A f(x) = (P^A)^n f(x) \; \forall n \geq 1$

(f) $P^A f(x) = I_A P I_A f(x)$

Proof. Left as an Exercise. □

5.3 Green's function

Let $A \subset \mathbb{V}$. We define Green's function of $\{X_n\}_{n\geq 0}$ as:

$$g_A(x,y) = \sum_{n=0}^{\infty} p_n^A(x,y)$$

$x, y \in \mathbb{V}$.

Notation.

- *if $A = \mathbb{V}$ then $g_A = g$*
- *$x \in \mathbb{V}$ fixed, then $g_A^x(y) = g_A(x, y\ \forall y \in \mathbb{V}$*

Observations.

- *$g_A(x,y) = g_A(y,x)\ \forall\ x, y \in \mathbb{V}$.*
- *Define Local time at y before exiting A i.e. time spent by the walk at y before exiting A by $L_{\tau_A}^y = \sum_{n=0}^{\infty} \mathbf{1}_{X_n=y}$.*

$$\begin{aligned} g_A(x,y) &= \sum_{n=0}^{\infty} p_n^A(x,y) \\ &= \frac{\sum_{n=0}^{\infty} E^x[\mathbf{1}_{X_n=y}; n < \tau_A]}{\mu_y} \\ &= \frac{E^x[\sum_{n=0}^{\infty}(\mathbf{1}_{X_n=y}\mathbf{1}_{n<\tau_A})]}{\mu_y} \\ &= \frac{E^x[\sum_{n=0}^{\tau_A-1}(\mathbf{1}_{X_n=y})]}{\mu_y} \\ &= \frac{E^x[L_{\tau_A}^y]}{\mu_y}. \end{aligned}$$

- *if $A = \mathbb{V}$ and $\mathbb{V}$ is recurrent then $g(x,.) = \infty$*

Theorem 5.3.1. *$A \subset \mathbb{V}$. Suppose either (Γ, μ) is transient or $A \neq V$.Then*

1. *$g_A(x,y) = \mathbb{P}(\tau_y < \tau_A) g_A(y,y)$*
2. *$g_A(y,y) = \frac{1}{\mu_y \mathbb{P}(\tau_a \leq \tau_y^+)}$*

Lemma 5.3.1. *Let $x, y \in A$. Then,*

1. *$\mathbf{P} g_A^x(y) = g_A(x,y) - \frac{\mathbf{1}_x(y)}{\mu_x}$*
2. *$\Delta g_A^x(y) = \begin{cases} -\frac{1}{\mu_x} \text{ if } y{=}x \\ 0 \text{ otherwise} \end{cases}$*

Proof. 1.

$$\begin{aligned}
Pg_A^x &= \sum_{z\in\mathbb{V}} p(y,z)g_A^x(z)\mu_z \\
&= \sum_{z\in\mathbb{V}} p(y,z)\mu_z(\sum_{n=0}^{\infty} p_n^A(xz) \\
&= \sum_{n=0}^{\infty}\sum_{z\in\mathbb{V}} p(y,z)\mu_z p_n^A(x,z) \\
&= \sum_{n=0}^{\infty}\sum_{z\in A} p(y,z)\mu_z p_n^A(x,z) \\
&= \sum_{n=0}^{\infty}\sum_{z\in A} p_1^A(y,z)p_n^A(x,z)\mu_z \\
&= \sum_{n=0}^{\infty} p_{n+1}^A(x,y) \\
&= g_A(x,y) - p_0^A(x,y) \\
\Rightarrow Pg_A^x(y) &= g_A(x,y) - \frac{\mathbf{1}_x(y)}{\mu_x}
\end{aligned}$$

2. follows from definition of $D = P - I$ $\square$

Proof of Theorem.
Notations: Given $f : \mathbb{V} \to \mathbb{R}$, $E^X f(X_n) = \sum_{y\in\mathbb{V}} \mathbb{P}^x(X_n = y)f(y)$.
let ξ be a random variable. $h_n(\xi) = E^\xi f(X_n)$
1.

$$\begin{aligned}
g_A(x,y)\mu_y &= E^x(L_{\tau_A}^y \\
&= E^x(\mathbf{1}_{\tau_y<\tau_A} \times L_{\tau_A}^y \\
&= E^x(\mathbf{1}_{\tau_y<\tau_A}\mathbb{E}^y(L_{\tau_A}^y) \\
\Rightarrow g_A(x,y) &= g_A(y,y)\mathbb{P}^x(\tau_y < \tau_A)\square
\end{aligned}$$

2. $p = \mathbb{P}(\tau_y^+ < \tau_A)$
if (Γ,μ) is transient then $p < 1$ and if recurrent and $A \neq \mathbb{V}$ then $p < 1$.[$\exists z \in A^c$ such that $\mathbb{P}^y(\tau_A < \tau_y^+) \geq \mathbb{P}^y(\tau_z < \tau_y^+) > 0$]
$\therefore p < 1$

$$\begin{aligned}
\mathbb{P}^y(L_{\tau_A}^y = k) &= p^k(1-p) \\
\Rightarrow \mu_y g_A(y,y) &= E^y(L_{\tau_A}^y) \\
&= \sum_{k=0}^{\infty} p^k(1-p) \\
&= \frac{1}{1-p} \\
&= \frac{1}{\mathbb{P}(\tau_A \leq \tau_y^+)} \\
\Rightarrow g_A(y,y) &= \frac{1}{\mu_y\mathbb{P}(\tau_A \leq \tau_y^+)}\square
\end{aligned}$$

Combining 1 and 2, we get

$$g_A(x,y) = \frac{\mathbb{P}^x(\tau_y < \tau_A)}{\mu_y \mathbb{P}(\tau_A \leq \tau_y^+)}.$$

Part II

Martingales

$\{Z_n\}$ is a Martingale
$E[Z_n|Z_i, Z_{i-1}, ..., Z_1] = Z_i$ where $1 \leq i \leq n]$
$E[Z_n] = E[Z_1]$

5.4 Stopping time and Stopped process

Definition 5.4.1. *Let $(\Omega, A, \mathbb{P})$ be a probability space on which $\{Z_n\}_{n\geq 1}$ is defined.*
$\mathscr{A}_k$= events determined by $Z_1, Z_2, ..., Z_k$.
*$T : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ is called a **stopping time** for $\{Z_n\}_{n\geq 1}$ if $\{T = k\} \in \mathscr{A}_k$, i.e. $\mathbf{1}_{T=k}$ = "function" of $Z_1, Z_2, ..., Z_k$.*

Definition 5.4.2. *for any stopping time T, we define the **stopped process**:*
$$Z_n^T(w) = Z_{n\wedge T(w)}(w) = \begin{cases} Z_n & \text{if } n < T \\ Z_T & \text{if } n \geq T \end{cases}$$

Theorem 5.4.1. *Given a sequence of random variables $\{Z_n\}_{n\geq 1}$ and $T : \Omega \longrightarrow \mathbb{N} \cup \{\infty\}$, a stopping time of $\{Z_n\}_{n\geq 1}$. Then $\{Z_n^T\}_{n\geq 1}$ is a martingale iff $\{Z_n\}_{n\geq 1}$ is a martingale*

Idea of the proof: $\mathbb{E}(Z_n^T|Z_{n-1}^T, ..., Z_1^T) = \mathbb{E}(Z_{n-1}^T)$
Take $Z_1 = z_1, ..., Z_{n-1} = z_{n-1} \rightarrow$ determine if T has happened by time n-1 or not
$\rightarrow$ if $T \geq n$, $Z_n^T = Z_n$
if $T < n$, $Z_n^T = z_{n-1}$ $\square$
Let $\{X_i\}, X, Y, Z$ be discrete random variables.

$$\mathbb{E}[Y|X = x_1] = \sum_{k\in Range(Y)} k\mathbb{P}(Y = k|X = x_1) \tag{5.1}$$

$$\mathbb{E}[Y|X_1 = x_1, ..., X_n = x_n] = \sum_{k\in Range(Y)} k\mathbb{P}(Y = k|X_1 = x_1, ..., X_n = x_n) \tag{5.2}$$

where $\mathbb{E}[Y|X_1 = x_1, ..., X_n = x_n] \equiv f(x_1, x_2, ..., x_n)$
$f : \prod_{i=i}^{n} Range(X_i) \rightarrow \mathbb{R}$

$$\mathbb{E}[Y|X_1, ..., X_n](\omega) = \sum_{x\in Range(X_i)} k\mathbb{E}(Y = k|X_1 = x_1, ..., X_n = x_n)\mathbf{1}_{(X_1=x_1,...,X_n=x_n)}(\omega) \tag{5.3}$$

where $\mathbb{E}[Y|X_1, ..., X_n] \equiv \mathbb{E}[Y|\mathscr{A}_n]$, i.e. events observable by time n.

5.5 Tower Property

Let $\mathscr{A}_n \subset \mathscr{A}_m$, $n \leq m$ then $\mathbb{E}[E[Y|\mathscr{A}_m]|\mathscr{A}_n] = \mathbb{E}[Y|\mathscr{A}_n]$

5.6 Markov property and Strong Markov Property

Property for $\{X_n\}$ random walk on (Γ, y).
$\Omega = \mathbb{V}^{\mathbb{Z}_+}$.
$X_n : \Omega \rightarrow \mathbb{V}$.
$X_n(\omega) = \omega(n)$.
$\mathscr{A}_n$= events determined by $X_1, ..., X_n$.
$\mathbb{P}^x(X_0 = x_0, X_1 = x_1, ..., X_n = x_n) = \mathbf{1}_x(x_0) \prod_{i=0}^{n} \mathscr{P}(x_{i-1}, x_i)$
$\mathscr{P}(x, y) = \frac{\mu_{xy}}{\mu_y}$
$\xi \rightarrow$ random variable that is determinable by $\mathscr{A}_n$ i.e. $\xi = g(X_1, X_2, ..., X_n$ for some g.

$\forall k \geq 1$, $\theta_k : \Omega \to \mathbb{V}^{\mathbb{Z}_+}$, $\theta_k(\omega) = (\omega(k), \omega(k+1), ...)$
Let $\eta : \Omega \to \mathbb{R}$ be any random variable.
$\mathbb{E}[\xi"\eta \text{ after time } n"|\mathscr{A}_n] = \mathbb{E}[\xi\mathbb{E}^{X_n}[\eta \text{ after time } n"]]$
Markov Property:

$$\mathbb{E}[(\xi) \times (\eta.\theta_n)|\mathscr{A}_n] = \mathbb{E}[\xi\mathbb{E}^{X_n}[\eta]] \tag{5.4}$$

Strong Markov Property:
T is a stopping time of $\{X_n\}_{n\geq 1}$.
$\mathscr{A}_n$ $\equiv$events determined by time T.
if ξ is determinable by time T, then

$$\mathbb{E}[(\xi) \times (\eta.\theta_T)|\mathscr{A}_T] = \mathbb{E}[\xi\mathbb{E}^{X_T}[\eta]] \tag{5.5}$$