Research Notes for Kim Lab

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The non-uniform error modeling paper that I'm working on is based on the Topological Quantum Memory paper by Dennis et al. On page 20 of the paper, they give the derivation of $\text{Prob}_{\text{fail}}$ by adding up the probabilities of self-avoiding polygons (SAPs) with a certain number of horizontal and vertical links. Certainly, a homologically non-trivial path must contain at least L horizontal links and they give the upper bound on the probability of failure as

$$\text{Prob}_{\text{fail}} \leq \sum_{V} \sum_{H \geq L} \text{Prob}_{\text{SAP}}(H, V).$$

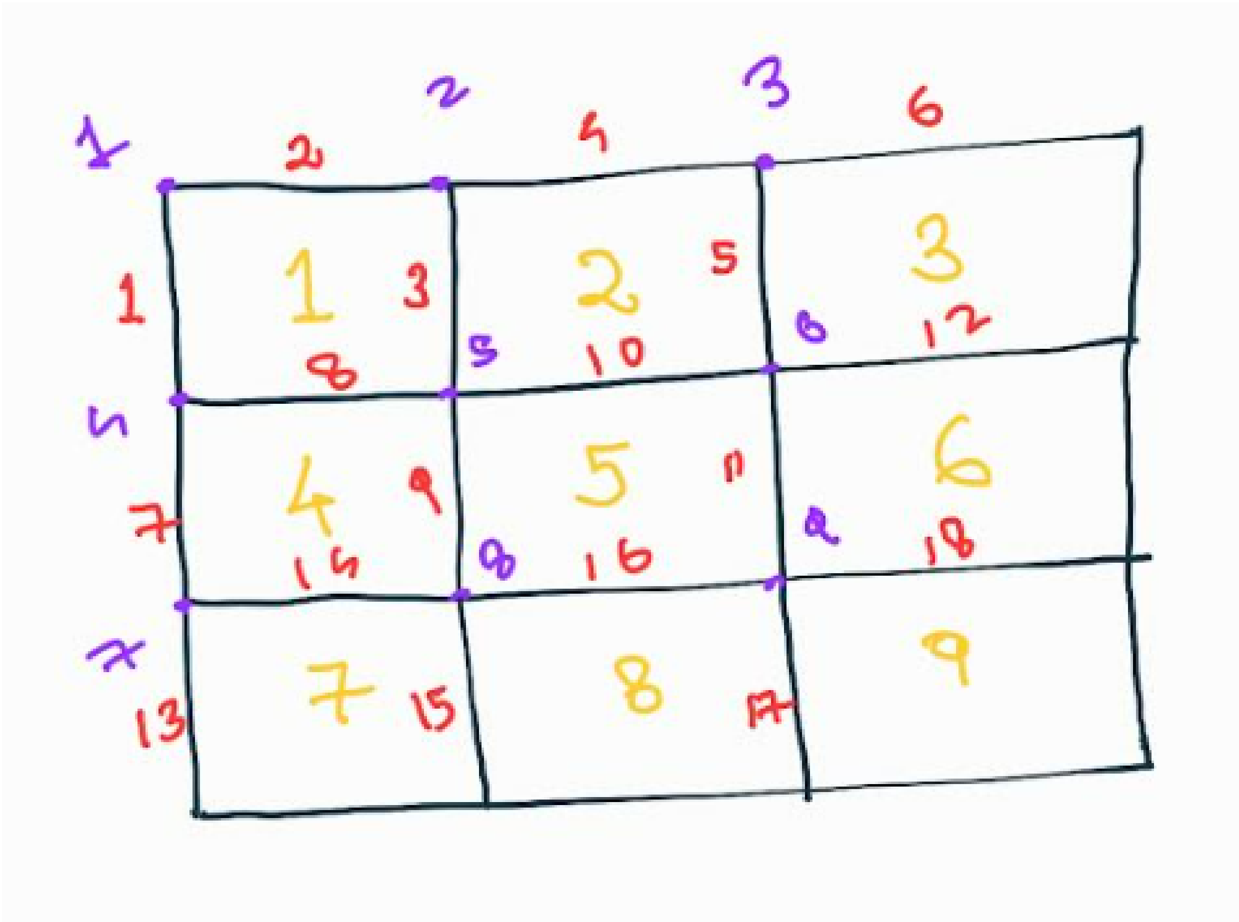
That's an upper bound but I don't think it's a good upper bound. Moreover, their estimation of SAPs seems sub-optimal to me. For one, the estimate $\text{Prob}_{\text{fail}} \leq \sum_{V} \sum_{H \geq L} \text{Prob}_{\text{SAP}}(H, V)$ fails to neglect homologically trivial paths which also have a length of at least L. Second, they don't seem to be using the much neater and more powerful tools from algebraic topology, considering that after all, they're working on a lattice torus over $\mathbb{Z}/2$ with T layers.

For starters, let's say $T = 1$ and $q = 0$. Let's just consider one horizontal lattice torus slice. A MathOverflow user pointed out to me that there is a neat way to find the number (and in turn probability, if edge qubit has failure rate p) of homologically non-trivial cycles based on the definition of the first homology group $H_1(\mathbb{Z}/2)$. Consider an $n \times n$ lattice torus, and let C_2 be the cells (the plaquettes), let C_1 be the 1-cells (the edges), and C_0 be the 0-cells (the vertices).

We can denote our chain complex like this:

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$$

δ_2 is a linear map that maps each plaquette to its boundary edges. δ_1 is a linear map that maps each edge to its boundary vertices. C_2 is itself, of course, the vector space whose basis elements are the n^2 plaquettes, and C_1 is the vector space whose basis elements are the $2n^2$ edges. C_0 is the vector space whose basis elements are the n^2 vertices.



Consider a lattice torus with the above labellings. The matrix of δ_2 looks like this:

$$\begin{bmatrix} 1 & 0 & 1 & \dots \\ 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ \vdots & & & \end{bmatrix}$$

Each column of this $2.n^2 \times n^2$ matrix has 4 non-zero 1s and any two columns of this matrix share at most 1 non-zero entry.

Question 1. The first homology group is defined as $H_1 = \ker \delta_1 / \mathrm{im}\ \delta_2$. Thus, the essential (non-trivial) cycles lie in $\ker \delta_1 - \mathrm{im}\ \delta_2$. Using the methods from *matrix analysis*, is it possible to find a good estimation of the number of elements in $\mathrm{im}(\delta)$ that have a specific weight $\ell \geq n$? Denis T. from MO suggested its approximately ℓ^{n^2-1} upto a constant, but I still don't understand how. It's apparently using the operator norm or Lipchitz constant of approximate SVD.

We could certainly put a crude upper bound on weight ℓ elements in im δ_2. Suppose we're looking for the number of elements in im δ_2 with a specific weight ℓ, then for any of the $\binom{2n^2}{\ell}$ choices there are less than 2^ℓ options. But this is quite bad.

The matrix of δ_1 looks like this:

$$\begin{bmatrix} 1 & 1 & 0 & 0\dots \\ 0 & 1 & 1 & 1\dots \\ 0 & 0 & 0 & 1\dots \\ 1 & 0 & 0 & 0\dots \\ 0 & 0 & 1 & 0\dots \\ 0 & 0 & 0 & 0\dots \\ 0 & 0 & 0 & 0\dots \\ 0 & 0 & 0 & 0\dots \\ \vdots & & & \end{bmatrix}$$

Each column of this $n^2 \times 2.n^2$ matrix has 2 non-zero 1 entries and any two columns share at most 1 non-zero entry. We need to estimate the total number of 1-cycles of a certain length $\ell \geq n$ in the kernel of this matrix and subtract the trivial cycles from this number to get the estimate of essential cycles. **Question 2.** Can matrix analysis be used for this? I'm thinking about a crude upper bound but that may not be any better than the SAP calculation...

One point is that calculating $\mathrm{im}(\delta_2)$ is quite similar to calculating **cluster size distribution** n_s in percolation theory, which is a hard problem with no known closed form expression. I don't know if this is relevant here (?).

The number of homologically non-trivial paths of length ℓ is basically the number of 1-cycles in the kernel of δ_1 (with length ℓ, with the constraint that we neglect disjoint 1-cycles or self-intersecting 1-cycles) minus the number of 1-cycles in the image of δ_2 (with length ℓ, again subject to the constraint that we neglect disjoint 1-cycles or self-intersecting 1-cycles). This should be a standard problem at the intersection of linear algebra and combinatorics but I don't see a straightforward way to attack this. I tried a bare hands calculation but the algebraic expression gets unmanageable very quickly. The user Denis T. on MO gave me the estimate $\ell^{n^2+1} - \ell^{n^2-1}$, but I don't quite understand how. I'm thinking about it this week. :-)

This was the first part of the problem. If this method succeeds, then I suspect we can improve the accuracy threshold from $p < .0373$ as given in eqn. (67) to a value closer to $p_c = .1094 \pm .0002$ as given in eqn (41) which they found using the random-bond Ising model.

The second and more important part of the problem is this: Suppose we have a probability distribution of the edge qubit errors. That is the edge qubits are labelled $p_k = p_{ij}$ and these p_{ij}s satisfy a distribution whose mean is μ, variance is σ and range is (say) $[P_1, P_2]$ where $P_1, P_2 \in [0, 1]$. Then what is the maximum probability of non-trivial cycles of length $\ell \geq n$ occurring? At a minimum, it would help if we could put a reasonable upper bound on this probability for all $\ell \geq n$. If this upper bound gets arbitrarily small as system size increases, then quantum information can be reliably stored. (Indeed, in the real world, all the qubits in a quantum chip won't have the same error rate. However, the experimentalists do usually know the distribution of error rates of the edge qubits.)

Question 3. Can this be solved as a matrix analysis and statistics problem? I think so but I perhaps need to talk to statisticians about this and also try to search for the relevant literature. (I asked Greg about this and am waiting for his response.) Surely, it's possible to put a crude upper bound again, but I should try to optimize my result/theorem, whatever comes up. In the simplest case, it seems like a much simpler problem: Given a set of n numbers: $x_1, x_2, \dots, x_n$ satisfying a certain distribution with mean μ, variance σ and range $[N_1, N_2]$ what is the maximum sum of products of any subset of size $k \leq n$ numbers, where $k \geq c$? Here, however, there are quite a few additional constraints including the fact that the probability of a non-trivial geometric 1-cycles must come from the kernel of δ_1 modulo the 1-cycles coming from the image of δ_2.