Research Notes for Kim Lab

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The non-uniform error modeling paper that I'm working on is based on the Topological Quantum Memory paper by Dennis et al. On page 20 of the paper, they give the derivation of $\text{Prob}_{\text{fail}}$ by adding up the probabilities of self-avoiding polygons (SAPs) with a certain number of horizontal and vertical links. Certainly, a homologically non-trivial path must contain at least L horizontal links and they give the upper bound on the probability of failure as

$$\text{Prob}_{\text{fail}} \leq \sum_{V} \sum_{H \geq L} \text{Prob}_{\text{SAP}}(H, V).$$

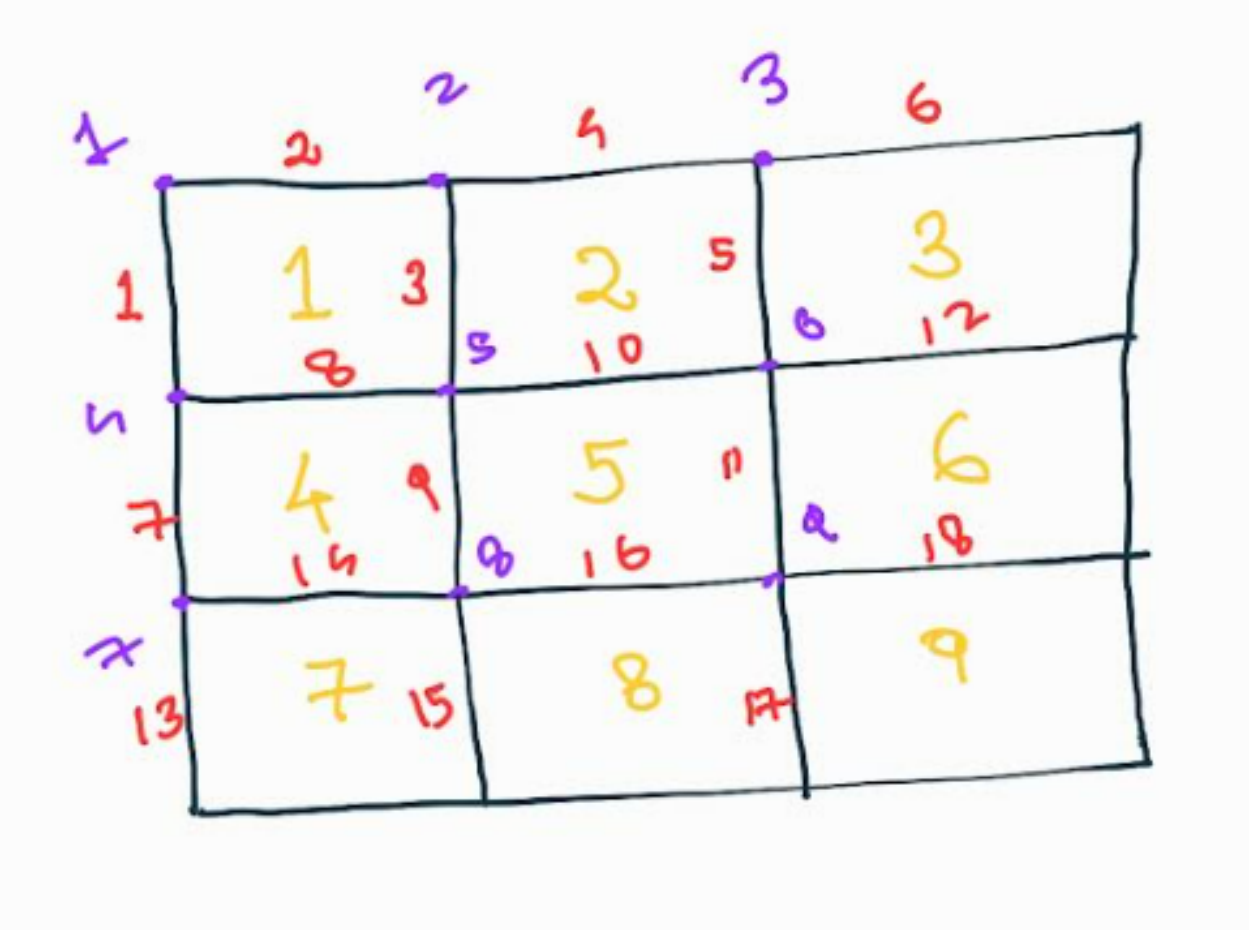
That's an upper bound but I don't think it's a good upper bound. Moreover, their estimation of SAPs seems sub-optimal to me. For one, the estimate $\text{Prob}_{\text{fail}} \leq \sum_V \sum_{H \geq L} \text{Prob}_{\text{SAP}}(H, V)$ fails to neglect homologically trivial paths which also have a length of at least L. Second, they don't seem to be using the much neater and more powerful tools from algebraic topology, considering that after all, they're working on a lattice torus over $\mathbb{Z}/2$ with T layers.

For starters, let's say $T = 1$ and $q = 0$. Let's just consider one horizontal lattice torus slice. A MathOverflow user pointed out to me that there is a neat way to find the number (and in turn probability, if edge qubit has failure rate p) of homologically non-trivial cycles based on the definition of the first homology group $H_1(\mathbb{Z}/2)$. Consider an $n \times n$ lattice torus, and let C_2 be the cells (the plaquettes), let C_1 be the 1-cells (the edges), and C_0 be the 0-cells (the vertices).

We can denote our chain complex like this:

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$$

δ_2 is a linear map that maps each plaquette to its boundary edges. δ_1 is a linear map that maps each edge to its boundary vertices. C_2 is itself, of course, the vector space whose basis elements are the n^2 plaquettes, and C_1 is the vector space whose basis elements are the $2n^2$ edges. C_0 is the vector space whose basis elements are the n^2 vertices.



Consider a lattice torus with the above labellings. The matrix of δ_2 looks like this:

$$\begin{bmatrix} 1 & 0 & 1 & \dots \\ 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ \vdots & & & \end{bmatrix}$$

Each column of this $2.n^2 \times n^2$ matrix has 4 non-zero 1s and any two columns of this matrix share at most 1 non-zero entry.

Question 1. The first homology group is defined as $H_1 = \ker \delta_1 / \text{im } \delta_2$. Thus, the essential (non-trivial) cycles lie in $\ker \delta_1 - \text{im } \delta_2$. Using the methods from *matrix analysis*, is it possible to find a good estimation of the number of elements in $\text{im}(\delta)$ that have a specific weight $\ell \geq n$? Denis T. from MO suggested its approximately ℓ^{n^2-1} upto a constant, but I still don't understand how. It's apparently using the operator norm or Lipchitz constant of approximate SVD.

We could certainly put a crude upper bound on weight ℓ elements in im δ_2. Suppose we're looking for the number of elements in im δ_2 with a specific weight ℓ, then for any of the $\binom{2n^2}{\ell}$ choices there are less than 2^ℓ options. But this is quite bad.

The matrix of δ_1 looks like this:

$$\begin{bmatrix} 1 & 1 & 0 & 0\ldots \\ 0 & 1 & 1 & 1\ldots \\ 0 & 0 & 0 & 1\ldots \\ 1 & 0 & 0 & 0\ldots \\ 0 & 0 & 1 & 0\ldots \\ 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0\ldots \\ \vdots & & & \end{bmatrix}$$

Each column of this $n^2 \times 2.n^2$ matrix has 2 non-zero 1 entries and any two columns share at most 1 non-zero entry. We need to estimate the total number of 1-cycles of a certain length $\ell \geq n$ in the kernel of this matrix and subtract the trivial cycles from this number to get the estimate of essential cycles. **Question 2.** Can matrix analysis be used for this? I'm thinking about a crude upper bound but that may not be any better than the SAP calculation...

One point is that calculating $\text{im}(\delta_2)$ is quite similar to calculating **cluster size distribution** n_s in percolation theory, which is a hard problem with no known closed form expression. I don't know if this is relevant here (?).

The number of homologically non-trivial paths of length ℓ is basically the number of 1-cycles in the kernel of δ_1 (with length ℓ, with the constraint that we neglect disjoint 1-cycles or self-intersecting 1-cycles) minus the number of 1-cycles in the image of δ_2 (with length ℓ, again subject to the constraint that we neglect disjoint 1-cycles or self-intersecting 1-cycles). This should be a standard problem at the intersection of linear algebra and combinatorics but I don't see a straightforward way to attack this. I tried a bare hands calculation but the algebraic expression gets unmanageable very quickly. The user Denis T. on MO gave me the estimate $\ell^{n^2+1} - \ell^{n^2-1}$, but I don't quite understand how. I'm thinking about it this week. :-)

This was the first part of the problem. If this method succeeds, then I suspect we can improve the accuracy threshold from $p < .0373$ as given in eqn. (67) to a value closer to $p_c = .1094 \pm .0002$ as given in eqn (41) which they found using the random-bond Ising model.

The second and more important part of the problem is this: Suppose we have a probability distribution of the edge qubit errors. That is the edge qubits are labelled $p_k = p_{ij}$ and these p_{ij}s satisfy a distribution whose mean is μ, variance is σ and range is (say) $[P_1, P_2]$ where $P_1, P_2 \in [0, 1]$. Then what is the maximum probability of non-trivial cycles of length $\ell \geq n$ occurring? At a minimum, it would help if we could put a reasonable upper bound on this probability for all $\ell \geq n$. If this upper bound gets arbitrarily small as system size increases, then quantum information can be reliably stored. (Indeed, in the real world, all the qubits in a quantum chip won't have the same error rate. However, the experimentalists do usually know the distribution of error rates of the edge qubits.)

Question 3. Can this be solved as a matrix analysis and statistics problem? I think so but I perhaps need to talk to statisticians about this and also try to search for the relevant literature. (I asked Greg about this and am waiting for his response.) Surely, it's possible to put a crude upper bound again, but I should try to optimize my result/theorem, whatever comes up. In the simplest case, it seems like a much simpler problem: Given a set of n numbers: $x_1, x_2, \ldots, x_n$ satisfying a certain distribution with mean μ, variance σ and range $[N_1, N_2]$ what is the maximum sum of products of any subset of size $k \leq n$ numbers, where $k \geq c$? Here, however, there are quite a few

additional constraints including the fact that the probability of a non-trivial geometric 1-cycles must come from the kernel of δ_1 modulo the 1-cycles coming from the image of δ_2.

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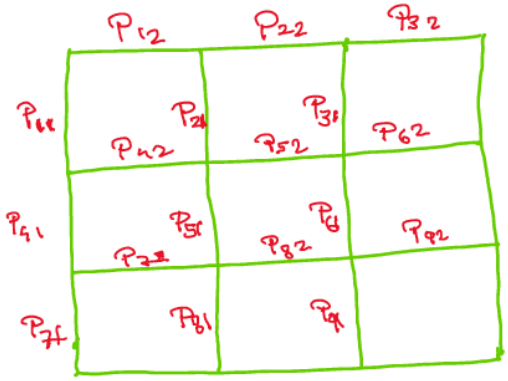
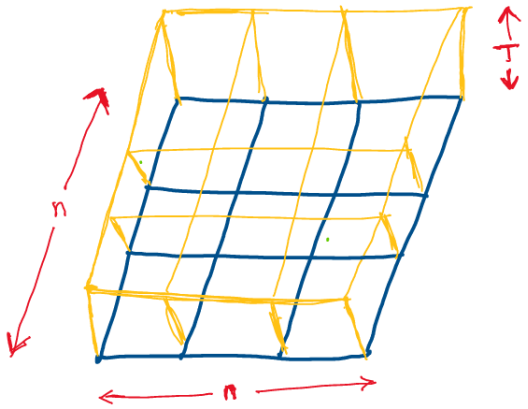
We're trying to find an accurate mathematical model for non-uniform errors on the toric code. Each edge qubit has some error probability p_{ij} and these probabilities come from a probability distribution with some mean μ, variance σ and some other general parameters.

Approach 1: Writing an upper bound on $\mathbf{P_{fail}}$ as a formal sum



In the topological quantum memory paper they consider T layers, with the vertical links denoting syndrome measurements, with a syndrome error probability of q.



Suppose a particular cycle in the spacetime lattice contains H horizontal links and V vertical links. Let H_m, V_m be the number of those links contained in $E_{\min}$ and H_e, V_e be the number of those links contained in E. Then the following inequality must be satisfied:

$$\left(\frac{p}{1-p}\right)^{H_e}\left(\frac{q}{1-q}\right)^{V_e} \leq \left[\left(\frac{p}{1-p}\right)^{H}\left(\frac{q}{1-q}\right)^{V}\right]^{\frac{1}{2}} \tag{1}$$

Question: Can we write this inequality in terms of general p_{ij}, q?

The probability $\text{Prob}(H, V)$ that a particular connected path with (H, V) horizontal and vertical links is contained in $E + E_{\min}$, once the error locations are specified, is:

$$p^{H_e}(1-p)^{H-H_e}q^{V_e}(1-q)^{V-V_e} \tag{2}$$

With these error locations, the cycle can be in $E+E_{\min}$ only if equation (1) is satisfied. Combining these observations, they obtain that:

$$\text{Prob}(H,V) \leq 2^{H+V}(\overline{p}^H\overline{q}^V)^{\frac{1}{2}} \tag{3}$$

where $\overline{p} = p(1-p)$ and $\overline{q} = q(1-q)$.

Then they bound the probability that $E + E_{\min}$ contains any closed cycle with (H, V) links by counting such self-avoiding polygons (SAPs).

$$\text{Prob}_{\text{SAP}}(H,V) \leq L^2T \cdot n_{\text{SAP}}(H,V) \cdot 2^{H+V}(\overline{p}^H\overline{q}^V)^{\frac{1}{2}} \tag{4}$$

The encoded quantum information cannot be reliably stored if $E + E_{\min}$ contains homologically non-trivial paths. At a minimum, the homologically nontrivial (self-avoiding) path must contain at least L horizontal links. The failure probability is bound as

$$\text{Prob}_{\text{fail}} \leq \sum_V \sum_{H \geq L} \text{Prob}_{\text{SAP}}(H,V) \leq L^2T \sum_V \sum_{H \geq L} n_{\text{SAP}}(H,V) \cdot (4\overline{p})^{H/2}(4\overline{q})^{V/2}. \tag{5}$$

We note that bound is fairly sub-optimal as it does not even neglect the homologically trivial cycles.

Question: Maybe we can write this as a formal sum involving p_{ij}s in some complicated way. However, I doubt we can extract any useful information or any useful upper bound from that expression, mainly since we don't know which SAP *transits* through which edges!

Nevertheless, I believe that the intuition that if the average error rate is kept below a certain threshold, we can say that the probability of homologically non-trivial cycles vanishes as $L \to \infty$. The crucial question is how to formalize this intuition. Ideas? One possible simplification is that the edges in each self-avoiding polygon has roughly the same error distribution as the global distribution of edge errors. This is crude but it may be a start.

As a side comment, the path-wise approach may be hopeless from a global statistical point of view. If we after all want a statistical picture, it may help to take inspiration from the approaches that statistical mechanics people use to study phase transitions in non-uniform systems – for instance, the theory of inhomogeneous percolation or non-uniform analogues of the Ising model.

Approach 2: Taking inspiration from Percolation Theory and the Edwards-Anderson Model

There is a notion of homological percolation on a torus, with an argument very similar to that of percolation in a square lattice. The bond percolation threshold is conjectured to be $\frac{1}{2}$ by a self-duality argument.

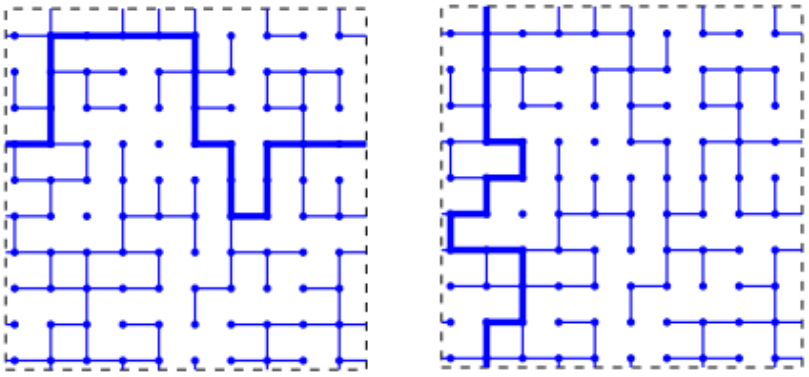


FIGURE 1. Two giant cycles for a random system of 1-dimensional plaquettes (bonds) on a 2-dimensional torus, shown in a square with opposite sides identified.

It can also be conjectured that each homology class in $H_1(\mathbb{T}^2)$ is represented with equal probability of $\sim 25\%$ beyond the percolation threshold $p_c = \frac{1}{2}$.

I don't have a citation for this but Greg gave me a heuristic reasoning: *For simplicity, say that you do tile some shape of torus by regular hexagons (or you use the truncated square tiling), so that 1-cycles are composed disjoint loops rather than any more complicated type of subgraph. I don't care what shape of torus either, as long as the tiling is rescaled to send the number of qubits to infinity while keeping the torus a fixed shape. Then above the phase transition, some of the components of the 1-cycle are very long twisty noodles.*

The energy penalty depends only on the total length of these twisty noodles and not on the number of them, nor which homology class they represent. It's true that the three non-trivial $\mathbb{Z}/2$ *homology classes have different minimum lengths from each other, but the twisty noodles are far from geodesics anyway. (In the scaling limit, they probably have Hausdorff dimension greater than 1.) So for instance, if it is a* 3×4 *rectangular torus with a diagonal length of 5, then 200 noodles in the shortest direction seem about as likely as 150 noodles in the middle direction or 120 noodles in the longest direction. Note also that they will be hidden by a background of short simple cycles.*

The claim that all the four homology classes occur with equal probability beyond the computational fault tolerance threshold is also mentioned in Stace, Barrett, Doherty - 2009:

The case $p_{loss} = 0$ and $p_{com} > 0$ has been well studied [9, 15, 16]. Briefly, physical bit-flip errors lead to logical bit-flip ($\bar{X}_i$) errors but not logical phase errors, and vice-versa. An *error chain*, E, is a set of lattice edges (i.e. physical qubits) where a bit-flip error has occurred. The plaquette operator eigenvalues change to -1 only at the *boundary*, ∂E, of the chain. Measuring the plaquette operators therefore yields information about the endpoints of connected sub-chains of E. If E crosses $\bar{Z}_i$ an odd number of times, then the logical qubit suffers a $\bar{X}_i$ error. These errors may be corrected if the endpoints, ∂E, can be matched by a correction chain, E', such that the closed chain $C = E + E'$ crosses $\bar{Z}_i$ an even number of times, i.e. C is homologially trivial. The error rate below which the correction chain E' may be successfully constructed is closely related to the phase boundary of the random-bond Ising model (RBIM) [15, 17]. If $p_{com} < p_{c0} = 0.104$ [18], then in the limit $L \to \infty$, the most probable chain, $C_{max} = E + E'_{max}$, is almost surely homologically trivial and recovery succeeds. If $p_{com} > p_{c0}$, then in the limit $L \to \infty$, the chain is homologically trival only 25% of the time, and recovery fails.

Indeed, this is not exactly a percolation problem. Surprisingly, Stace et al have studied qubit loss errors as a percolation problem but not computational errors on qubits as a percolation problem – maybe it's not possible to exactly frame computational X/Z errors on qubits as percolation after all. However, it does look very similar.

One obvious factor is that in the case of the toric code we're concerned with the homology classes of $C = E + E_{\min}$ rather than just the error chain E. In a homogeneous percolation model, the edges of the $n \times n$ lattice torus would each have some error rate p. Then beyond $p = p_c = \frac{1}{2}$ we can say that the error chain E has an equal probability of belonging to one of the four homology classes of $H_1(\mathbb{T}^2)$. However, that is not really the question here, is it? The recovery chain $E_{\min}$ generally comes from a minimum weight perfect matching decoder or something like that. The recovery chain isn't generated by edge errors, although it's generally chosen to be the most likely path sharing the endpoints of E, once the error chain E has already been generated. Though, even this observation is not enough to explain the difference between the values $\frac{1}{2}$ and .1031.

However, if at least one thing is clear from here, it is that (in my opinion) we should be more concerned about the statistics of the cluster size distribution of the large connected components rather than focusing on individual paths.

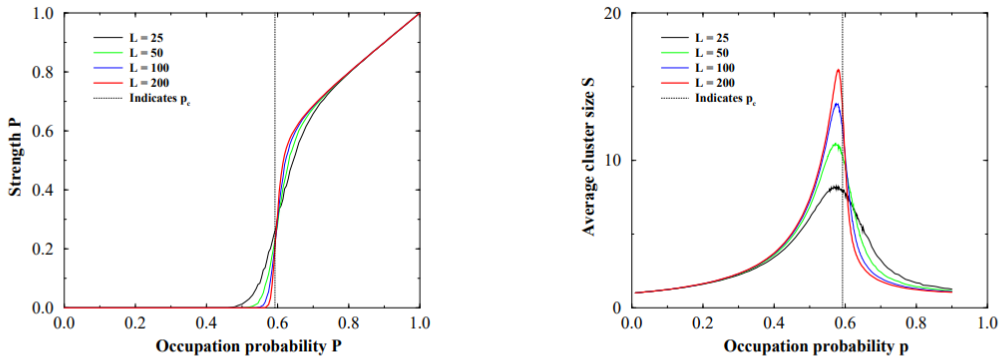


Figure 1.13: (a) The strength $P(p)$ measured in a $2d$ square lattice for finite system sizes as a function of occupation probability p. When p is far away from p_c, $L \gg \xi$ and the strength is determined by $\xi^{-\frac{\beta}{\nu}}$. However, with p closer to p_c, the finite-size effect is seen and $P(p)$ decays with system size as $L^{-\frac{\beta}{\nu}}$. (b) The average cluster size $S(p)$ measured in a $2d$ square lattice for finite system sizes L as a function of occupation probability p. When p is far away from p_c, $L \gg \xi$ and the average cluster size is determined by the correlation length $\xi^{\frac{\gamma}{\nu}}$. However, with p closer to p_c, the finite-size effect is seen and $S(p)$ increases with system size as $L^{\frac{\gamma}{\nu}}$.

The cluster size distribution looks like this in a homogeneous percolation model on a square $n \times n$ lattice. We might investigate analogues of this in inhomogeneous percolation models. Two relevant studies are:

1. *Inhomogeneous percolation models for spreading phenomena in random graphs* by Luca Dall'Asta. They study a situation similar to ours, where the edge probabilities really come from a distribution with a certain mean μ, variance σ and other general distribution parameters. Their approach is based on generating functions. The issue is that they consider random graphs rather than lattices. I don't know if their ideas can be used for our problem but it should be interesting.

2. *Inhomogeneous bond percolation on square, triangular and hexagonal lattices* by Grimmett and Manolescu. However, they do not deal with the more general scenario where the edge probabilities come from a distribution. Rather, they deal with a specific inhomogeneous lattice model like this, which has translational symmetry but not rotational symmetry.

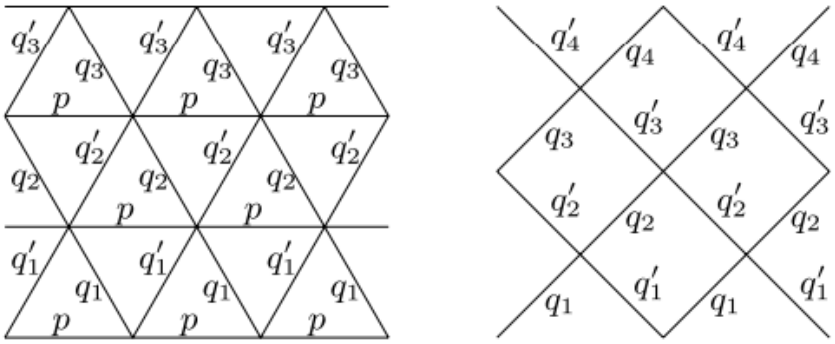


FIG. 2. *Left: The triangular lattice with the highly inhomogeneous product measure* $\mathbb{P}^{\triangle}_{p,\mathbf{q},\mathbf{q}'}$. *The probability for each edge to be open is described in the picture: all horizontal edges have probability p of being open, while the other edges have probability* q_n *(right edges of upward pointing triangles) or* q'_n *(left edges of upward pointing triangles) of being open, with n being their height. Right: The square lattice with a highly inhomogeneous product measure* $\mathbb{P}^{\square}_{\mathbf{q},\mathbf{q}'}$, *rotated by* $\pi/4$. *Edges inclined at angle* $\pi/4$ *have probability* q_n *of being open, while edges inclined at angle* $3\pi/4$ *have probability* q'_n *of being open, with n being their height.*

There's now a more relevant motivation that comes from the Ising model and the Edwards-Anderson model.

Preskill et al had published a separate paper explaining their RBIM model for the fault tolerance threshold of $.1031 \pm .0001$. I think the main physics idea in the paper is this:

$$\sum_{E} \text{prob}(E) \cdot \sum_{D \text{ nontrivial}} \text{prob}[(E+D)|E] = 0. \tag{33}$$

Eq. (33) says that error chains that differ from the actual error chain by a homologically nontrivial cycle have probability zero. Therefore, the outcome of the measurement of the check operators is sure to point to the correct homology class, in the limit of an arbitrarily large code block.

This criterion is identical to the criterion for long-range order in the two-dimensional RBIM, along the Nishimori line. The error chain E can be identified with the

chain of antiferromagnetic bonds of a sample, bounded by Ising vortices that are pinned down by the measurement of the local check operators. The ensemble of all the chains $\{E'\}$ with a specified boundary can be interpreted as a thermal ensemble. If the temperature T and the error rate p obey Nishimori's relation, then the chain E' and the chain E have the same bond concentration. At low temperature along the Nishimori line, the cycle $D = E + E'$ contains no large connected loops for typical samples and typical thermal fluctuations—the spin system is magnetically ordered and error recovery succeeds with high probability. But at higher temperature, the quenched chain E and the thermal chain E' fluctuate more vigorously. At the Nishimori point, D contains loops that "condense," disordering the spins and compromising the effectiveness of error correction. Thus, the critical concentration p_c at the Nishimori point of the two-dimensional RBIM coincides with the accuracy threshold for quantum memory using toric codes (where p_c is the largest acceptable probability for either an X error or a Z error).

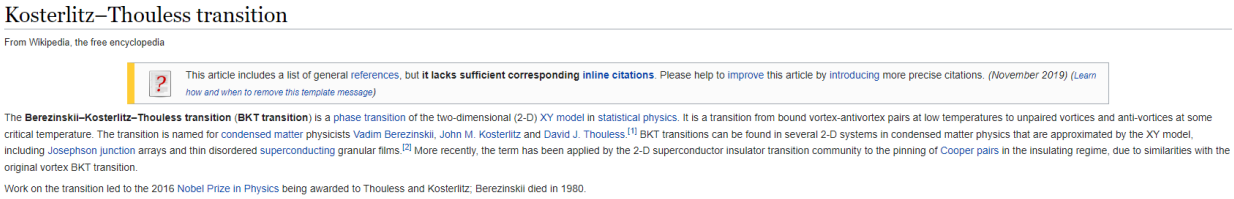
The Edwards-Anderson model is similar to the Ising model, with the spin Hamiltonian given as

$$H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j \tag{6}$$

The J_{ij} terms represent spin-spin interactions and the distribution of values of J_{ij} is taken to be a Gaussian with mean J_0 and a variance J^2. It might be helpful to study the Nishimori line value for this model. (Literature search required.)

We've mostly been talking about models for cases where the syndrome measurements are perfect and $q \neq 0$. However, when syndrome measurements are imperfect and we have the general lattice torus with T layers picture, fault-tolerance is said to be similar to the Kosterlitz–Thouless transition model is statistical physics (this was pointed out to me by Greg). We can think of the error syndromes

as point particles that are created in pairs, and the idea is to draw them together and annihilate them before any point particle wanders around a non-contractible loop in the torus. Whether this has an inhomogeneous version that could be useful to us, I do not know.



Kosterlitz–Thouless transition

From Wikipedia, the free encyclopedia

This article includes a list of general references, but **it lacks sufficient corresponding inline citations**. Please help to improve this article by introducing more precise citations. *(November 2019)* *(Learn how and when to remove this template message)*

The **Berezinskii–Kosterlitz–Thouless transition** (**BKT transition**) is a phase transition of the two-dimensional (2-D) XY model in statistical physics. It is a transition from bound vortex-antivortex pairs at low temperatures to unpaired vortices and anti-vortices at some critical temperature. The transition is named for condensed matter physicists Vadim Berezinskii, John M. Kosterlitz and David J. Thouless.[1] BKT transitions can be found in several 2-D systems in condensed matter physics that are approximated by the XY model, including Josephson junction arrays and thin disordered superconducting granular films.[2] More recently, the term has been applied by the 2-D superconductor insulator transition community to the pinning of Cooper pairs in the insulating regime, due to similarities with the original vortex BKT transition.

Work on the transition led to the 2016 Nobel Prize in Physics being awarded to Thouless and Kosterlitz; Berezinskii died in 1980.

Well, at least one thing is evident: We're looking to study the phase transition threshold in *some* statistical physics model that corresponds to inhomogeneous computational errors on a lattice torus with T layers and imperfect syndrome measurement. The fully general case might be an open problem given the Kosterlitz-Thouless Model was awarded the 2016 Nobel Prize.

Some further literature search for "toric code" + "percolation":

1. Quantum erasure-correcting codes and percolation on regular tilings of the hyperbolic plane by Nicolas Delfosse and Gilles Zémor.

2. Self-Correcting Quantum Memories Beyond the Percolation Threshold by Hastings, Watson and Melko.

It might be worth communicating with Nicholas Delfosse and Matthew Hastings at some point.

Approach 3: Simplicial complexes and matrix analysis

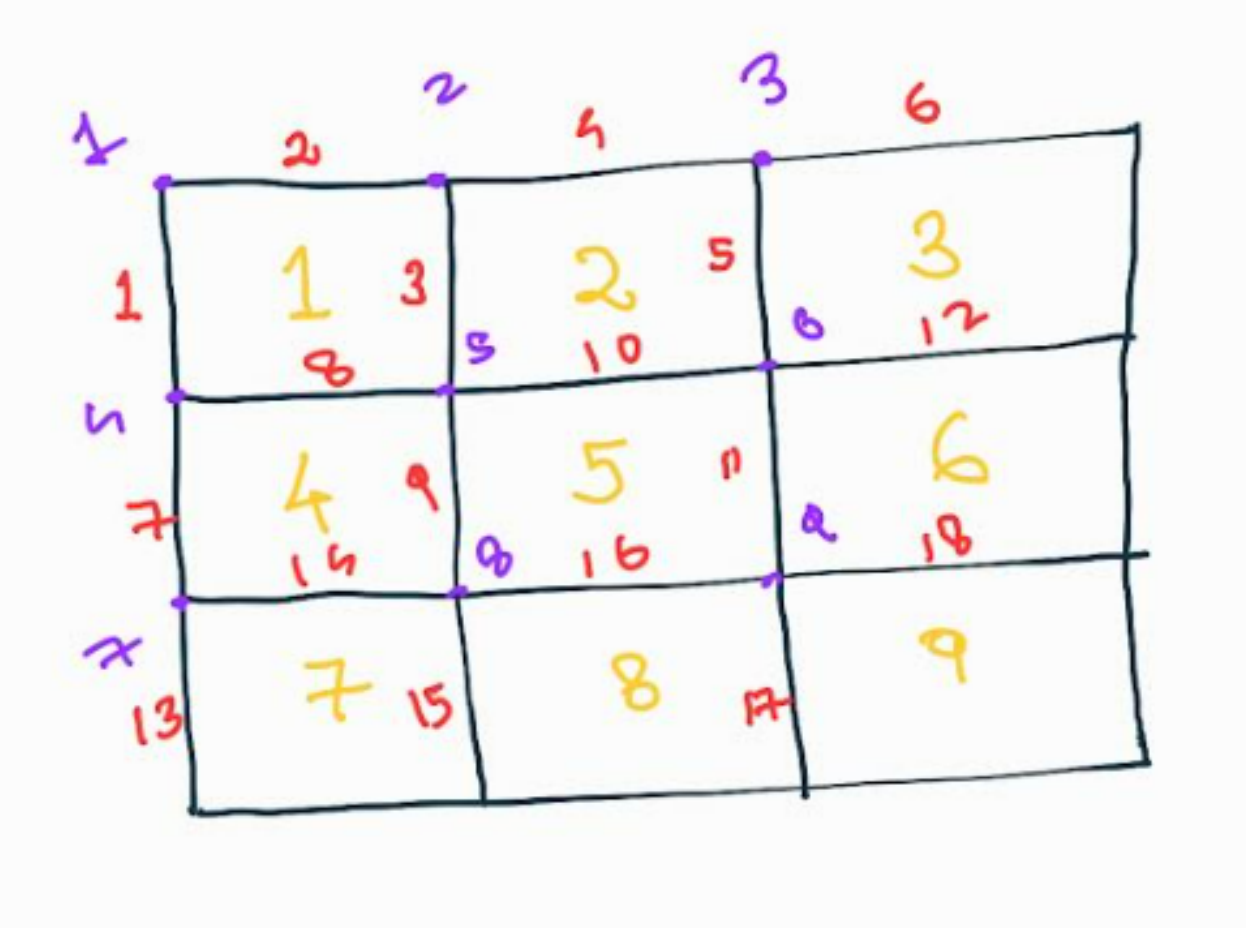
We discussed this last week. This approach looks hopeless so far, unless there is a new big idea.

For starters, let's say $T = 1$ and $q = 0$. Let's just consider one horizontal lattice torus slice. A MathOverflow user pointed out to me that there is a neat way to find the number (and in turn probability, if edge qubit has failure rate p) of homologically non-trivial cycles based on the definition of the first homology group $H_1(\mathbb{Z}/2)$. Consider an $n \times n$ lattice torus, and let C_2 be the cells (the plaquettes), let C_1 be the 1-cells (the edges), and C_0 be the 0-cells (the vertices).

We can denote our chain complex like this:

$$C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0$$

δ_2 is a linear map that maps each plaquette to its boundary edges. δ_1 is a linear map that maps each edge to its boundary vertices. C_2 is itself, of course, the vector space whose basis elements are the n^2 plaquettes, and C_1 is the vector space whose basis elements are the $2n^2$ edges. C_0 is the vector space whose basis elements are the n^2 vertices.



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Each column of this $2.n^2 \times n^2$ matrix has 4 non-zero 1s and any two columns of this matrix share at most 1 non-zero entry.

Question: The first homology group is defined as $H_1 = \ker \delta_1 / \text{im } \delta_2$. Thus, the essential (non-trivial) cycles lie in $\ker \delta_1 - \text{im } \delta_2$. Using the methods from *matrix analysis*, is it possible to find a good estimation of the number of elements in $\text{im}(\delta)$ that have a specific weight $\ell \geq n$? Denis T. from MO suggested its approximately ℓ^{n^2-1} upto a constant, but I don't understand how and neither does Greg. It's probably wrong and as per Greg such simple crisp arguments in statistical physics are usually considered a swindle by the mathematical physics community.