Research Notes for Kim Lab

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Quantum Erasure Channel

Delfosse-Zemor discuss the *peeling decoder* for the *quantum erasure channel*. In the quantum erasure channel each qubit is lost or erased independently with probability p. Such a loss can be detected and the missing qubit is replaced by a totally mixed state

$$\frac{I}{2} = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$$

.

The new qubit can be interpreted as the original state which suffers from a Pauli error I, X, Y and Z chosen uniformly at random.

Delfosse-Zemor

Just like when dealing with Pauli noise, one can then measure the stabilizer generators $X(v)$ and $Z(f)$ and try to recover the error from its syndrome. The main difference with the Pauli channels is the knowledge of the erasure pattern ε. Since operators of S act trivially on the code space, the goal of the decoder is to identify the coset PS of the error, knowing the syndrome ε and the syndrome σ of P.
The optimal strategy, called maximum likelihood decoding (MLD), is to maximize this conditional probability.

To illustrate how the knowledge of the erasure ε simplifies the decoding problem, assume that we found an error $\tilde{P}$ whose syndrome matches σ. Both errors P and $\tilde{P}$ have the same syndome, hence P and $\tilde{P}$ differ in a logical operator $L \subset \varepsilon$, trivial or not.

Due to the fact that errors $Q \subset \varepsilon$ are uniformly distributed (?), $\mathbb{P}(QS|\varepsilon, \sigma)$ is proportional to the number $|(QS) \cap \varepsilon|$ of Pauli errors of that coset that are included in ϵ. *This number depends only on the number $|S \cap \epsilon|$ of stabilizers having support inside ε, which shows that all cosets are equiprobable. Therefore, MLD consists of returning an error coset $\tilde{P}S$ such that $\tilde{P} \subset S$ and the syndrome of $\tilde{P}$ is equal to a given σ*

The idea of the peeling decoder for surface codes

Measurement of $X(v)$ operators can detect Z errors. The syndrome of the Z error P is a subset of the $\sigma(P) \subset V$ of vertices v such that $X(v)$ anticommutes with this error. Equivalently, it is the set of vertices, surrounded by an odd number of qubits supporting an error Z.

Graphically, we can call $\partial(A)$ the set of vertices that a subset $A \subset E$ encounters an odd number of times and call it the boundary of A. The syndrome of the Z-error pattern supported on A is exactly $\partial(A)$. Given $\varepsilon \subset E$ and $\sigma \subset V$, we are looking for the subset of edges $A \subset \varepsilon$ such that $\partial(A) = \sigma$.

The idea of the peeling decoder for surface codes

The idea of the peeling decoder is to select a spanning forest F_ε inside ε. Equipped with a spanning forest F_ε that contains all the syndrome vertices we can find the required subset $A \subseteq \varepsilon$ s.t. $\partial(A) = \sigma$ very efficiently in a recursive manner.
Pick a leaf, that is an edge $e = \{u, v\}$ connected to the forest through only one of its endpoints, say v. The vertex u is called pendant vertex. Assume first that $u \in \sigma$, then we add the edge e to A and flip the vertex v. By flipping, we mean that v is added to the set σ if $v \notin \sigma$ and it is removed from σ in the case $v \in \sigma$.
Then, e is removed from the forest F_ε. In the case when $u \notin \sigma$, this edge is simply removed from F_ϵ and A is kept unchanged.

Delfosse-Hickerson

Delfosse-Hickerson say that the erasure channel is where a qubit is subjected to a Z error with $\frac{1}{2}$ probability.

The union-find decoder is based on the philosophy that erasure errors are easier to decode than Pauli errors.

Union-find decoder

The decoder is divided into two stages. The first stage is syndrome validation. The goal of the first stage is to take a syndrome generated by both Pauli error and an erasure ε and from this generate a modified erasure ε'.

To perform syndrome validation, we identify invalid clusters of erasures and iteratively grow them until the updated state is correctable by the peeling decoder.

Union-find decoder

Algorithm 1: Union-Find decoder – Naive version

input : The set of erased positions $\varepsilon \subset E$ and the syndrome $\sigma \subset V$ of an error E_Z.

output: An estimation $\mathcal{C}$ of E_Z up to a stabilizer.

```
Create the list of all odd clusters C_1, ..., C_m,
  and initialize the modified erasure ε' = ε.
while there exists an odd cluster do
    for all odd cluster C_i do
        Grow C_i by increasing its radius by one
          half-edge.
        If C_i meets another cluster, fuse and
          update parity.
        If C_i is even, remove it from the odd
          cluster list.
Add full edges that are in the grown clusters to
  ε'.
Apply the peeling decoder to the erasure to
  find C.
```

Union-find decoder

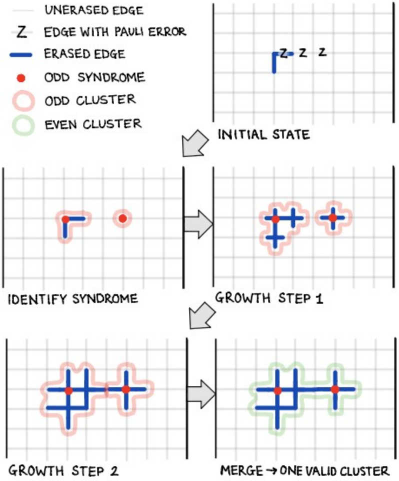


Figure 3: Schematic representation of syndrome validation.

Union-find as percolation

1. The main barrier to interpreting the union-find decoder as a percolation-like problem seems to be that we do know really know much much the clusters can grow when joining and merging (?)
2. The erased qubit edges don't affect the syndrome σ. But why exactly?

Stace-Barrett-Doherty

Their paper Thresholds for topological codes in the presence of loss analytically shows that the maximum tolerable qubit loss rate is 0.5 which is the same as the bond percolation threshold on a square lattice. But this is not exactly our problem even in the union-find decoder setting, because for given qubit erasure rate p we don't know how large the modified erasure cluster ε' can be after the merge and join process.

Question: Is the definition of "loss" in Stace-Barrett-Doherty the same as the definition of "erasure" in Delfosse-Zemor?

Stace-Barrett-Doherty

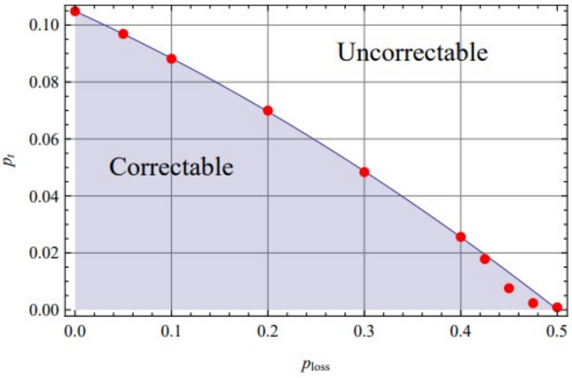
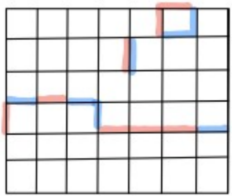


FIG. 3: Correctability phase diagram. The shaded region is correctable in the limit $L \to \infty$. The threshold, p_t, is calculated by fitting the universal scaling law $p_{fail} = f[(p_{com} - p_t)L^{1/\nu_0}]$. The curve is a quadratic fit to the points for which $p_{loss} \leq 0.4$ (where universal scaling is unaffected by the finite lattice size). It extrapolates through $(p_{loss}, p_t) = (0.5, 0)$.

Bounding average logical error rate - Isaac's approach



: E (Error chain)

: E_{min} (minimum error chain)

P_ℓ : Error probability for link ℓ.

(assume $P_\ell < \frac{1}{2}$)

Consider a cycle C.

We define E_{min} to be an error chain s.t.

1. $\partial E_{min} = \partial E$
2. $\sum_{\ell \in E_{min}} \log \frac{1-P_\ell}{P_\ell}$ is minimized.
3. $C = E + E_{min}$

Let H and V be the sets of horizontal and vertical links in C. Let H_e and V_e be the sets of horizontal and vertical links in E. Let H_m and V_m be the sets of horizontal and vertical links in E_{min}.

Bounding average logical error rate - Isaac's approach

Lemma 1. $\left(\sum_{\ell \in H_m} \log \frac{1-p_\ell}{p_\ell}\right) + \left(\sum_{\ell \in H_e} \log \frac{1-p_\ell}{p_\ell}\right) \geq \left(\sum_{\ell \in H} \log \frac{1-p_\ell}{p_\ell}\right)$

Lemma 2. $\left(\sum_{\ell \in V_m} \log \frac{1-p_\ell}{p_\ell}\right) + \left(\sum_{\ell \in V_e} \log \frac{1-p_\ell}{p_\ell}\right) \geq \left(\sum_{\ell \in V} \log \frac{1-p_\ell}{p_\ell}\right)$

proof. Since $p_\ell < \frac{1}{2}$, $\log \frac{1-p_\ell}{p_\ell} > 0$ and $H_m \cup H_e \supset H$, $V_m \cup V_e \supset V$, the claims follow. $\blacksquare$

Proposition 1. $\left(\prod_{\ell \in H_m} \frac{p_\ell}{1-p_\ell}\right)\left(\prod_{\ell \in V_m} \frac{p_\ell}{1-p_\ell}\right) \cdot \left(\prod_{\ell \in H_e} \frac{p_\ell}{1-p_\ell}\right)\left(\prod_{\ell \in V_e} \frac{p_\ell}{1-p_\ell}\right)$

$$\leq \left(\prod_{\ell \in H} \frac{p_\ell}{1-p_\ell}\right)\left(\prod_{\ell \in V} \frac{p_\ell}{1-p_\ell}\right)$$

Bounding average logical error rate - Isaac's approach

In total, there are at most $2^{|C|}$ choices of assigning E to C.

So for a cycle C, probability that $E + E_{min} = C$ is bounded from above by

$$\prod_{\ell \in C} 2\sqrt{(1-p_\ell)p_\ell}$$

Thus, the logical error probability is bounded by

$$\sum_{C:\text{ noncontractive cycle}} \prod_{\ell \in C} 2\sqrt{(1-p_\ell)p_\ell}.$$

Bounding average logical error rate - Isaac's approach

Upon averaging over p_ℓ, this upper bound is

$$\sum_{C:\text{ non contractible cycle}} \mathbb{E}_{\{p_\ell\}}\left[\prod_{\ell \in C} 2\sqrt{(1-p_\ell)p_\ell}\right] = \sum_{C} \prod_{\ell \in C} 2\,\mathbb{E}_{p_\ell}\left[\sqrt{(1-p_\ell)p_\ell}\right]$$

$$= \sum_{C} \prod_{\ell \in C} (2\tilde{p}),$$

where $\tilde{p} = \mathbb{E}_{p_\ell}\left[\sqrt{(1-p_\ell)p_\ell}\right]$. So we can define a threshold for $\tilde{p}$.