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Q1

Given: A is a $m \times n$ matrix

B is a $q \times r$ matrix

with product BC defined.

$\Rightarrow$ C is a $r \times p$ matrix.

$\Rightarrow$ (BC) is a $q \times p$ matrix.

$\rightarrow$ A(BC) is also defined.

To Prove: AB, (AB) C are defined.

& (AB) C = A (BC)

Proof: For $A_{m \times n} (BC)_{q \times p}$ to be defined,

$n = q$

~~Since~~ $\Rightarrow$ A is $m \times n$ matrix.

Since B is a $q \times r$ matrix

and $n = q$.

Product (AB) is defined of $m \times r$ matrix.

Since C is $r \times p$ matrix.

$\Rightarrow$ (AB) C is also defined.

and is a $m \times p$ matrix.

For (AB) C = A(BC)

$[(AB)C]_{ij} = [A(BC)]_{ij}$

$\forall\ 1 \le i \le m$ & $\forall\ 1 \le j \le p$.

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$$[(AB)C]_{ij} = \sum_{k} (AB)_{ik} C_{kj}.$$

$$= \sum_{k} \left(\sum_{l} A_{il} B_{lk} \right) C_{kj}.$$

$$= \sum_{k} \sum_{l} A_{il} B_{lk} C_{kj} \quad (C_{kj} \text{ is independent of } l)$$

$$= \sum_{k} A_{il} \left(\sum_{k} B_{lk} C_{kj} \right)$$

(since A_{il} is independent of k)

$$= \sum_{l} A_{il} (BC)_{lj}.$$

$$= [A(BC)]_{ij}$$

Hence, proved.

Q2 Given: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

To Prove: A is invertible iff $ad - bc \neq 0$

Proof: PART-1

let's us assume $ad-bc \neq 0$

$\therefore \exists$ a matrix $B = \begin{bmatrix} d(ad-bc)^{-1} & -b(ad-bc)^{-1} \\ -c(ad-bc)^{-1} & a(ad-bc)^{-1} \end{bmatrix}$

over field F.

Product AB:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d(ad-bc)^{-1} & -b(ad-bc)^{-1} \\ -c(ad-bc)^{-1} & a(ad-bc)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} ad(ad-bc)^{-1} - bc(ad-bc)^{-1} & -ab(ad-bc)^{-1} + ab(ad-bc)^{-1} \\ cd(ad-bc)^{-1} - cd(ad-bc)^{-1} & -bc(ad-bc)^{-1} + ad(ad-bc)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarily we can prove that

$BA = I$

$\therefore \exists$ a matrix B over F s.t. $AB = BA = I$

$\therefore$ A is invertible.

Thus, $ad-bc \neq 0 \Rightarrow A$ is invertible —(1)

PART 2:

Assumption A is invertible.

Lemma: If A is invertible, both a & c cannot be zero.

CASE 1: $a \neq 0$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equivalent to $\begin{bmatrix} c & bca^{-1} \\ c & d \end{bmatrix}$

[Multiply row 1 by ca^{-1}]

$\begin{bmatrix} c & bca^{-1} \\ 0 & d \end{bmatrix}$ ← This is equivalent $R_2 - R_1$

For A to be invertible it has to be row equivalent to I.

$\therefore$ $d - bca^{-1} \neq 0$

$ad - bc \neq a \cdot 0$

$ad - bc \neq 0$

Case 2: $c \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\substack{\text{equivalent} \\ \text{multiply by } ac^{-1} \\ R_2}]{\text{row}} \begin{bmatrix} a & b \\ a & ac^{-1}d \end{bmatrix}$$

$$\begin{bmatrix} 0 & b - ac^{-1}d \\ a & ac^{-1}d \end{bmatrix} \xleftarrow[R_1 - R_2]{\text{Row equivalent}}$$

For A to be invertible, A has to be row equivalent to I.

i.e. $b - ac^{-1}d \neq 0$

$bc - ad \neq c \cdot 0$

$bc - ad \neq 0$. $\Rightarrow$ $ad - bc \neq 0$.

Thus A is invertible, iff $ad - bc \neq 0$.

Hence, proved.

Q3 Given: A is $m \times n$ matrix

B is $n \times m$ matrix

$m > n$.

To Prove: $(AB)_{m \times m}$ is not invertible.

Proof: Since $(AB)_{m \times m}$ is a square matrix, let us assume that $(AB)^{-1}_{m \times m}$ exists.

Let $(AB)^{-1}_{m \times m} = C_{m \times m}$

$\therefore \quad (AB)C = I$

I = identity matrix $m \times m$.

Since we know $A(BC) = (AB)C$
when BC & $A(BC)$ is defined.
↳ This can be checked easily.

$A(BC) = (AB)C = I$
BC is inverse of A.
But A cannot have an inverse
as it is not a square matrix.
Hence this is a contradiction.
Hence, (AB) does not have a inverse.

Hence, proved.

Q4 Thm: For a $n \times n$ matrix A, foll. statements are equivalent.

(i) A is invertible.

(ii) A is row equivalent to $n \times n$ identity matrix

(iii) A is a product of elementary matrices

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(i) → (ii)

Given: A is invertible.

To Prove: A is row equi. to $I_{n\times n}$.

Proof: ∴ ∃. a matrix A^{-1} s.t.

$$A^{-1}.A = A.A^{-1} = I$$

∴ ∃ a sequence of elementary row opn's.
$e_1, e_2, e_3 \ldots e_k$ s.t.

$$e_k(e_{k-1}(e_{k-2} \ldots (e_3(e_2(e_1(A)))\ldots))) = I$$

Thus A is row equivalent to I.
Hence, proved.

(ii) → (iii)

Given: $A_{n\times n}$ is row equivalent to $I_{n\times n}$,

To Prove: A is product of elementary matrices.

Proof: Thus, ∃ a sequence of row transformations.
$e_1, e_2, e_3 \ldots e_k$ s.t.

$$A = e_k(e_{k-1}(e_{k-2} \ldots (e_3(e_2(e_1 A)))\ldots))$$

Now for every row opn e, ∃ an elementary matrix E s.t. $e(I) = EI$.

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let E_i be elementary matrix corresponding to operation e_i $\forall$ $1 \leq i \leq k$.

Thus, $A = E_k E_{k-1} \cdots E_2 E_1 \cdot I$.

$A = E_k E_{k-1} \cdots E_2 E_1$

$\therefore$ A is product of elementary matrices.

Hence, Proved.

(iii) → (i)

Given: A is product of elementary matrices.

To Prove: A is invertible.

Proof: $A = E_1 . E_2 \cdots E_k$

Since we know that E_k^{-1} also exists and are elementary matrices.

$E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1} A = I$.

Since product of inverse are also inverse invertible $\because$ ($= I$ here)

$\Rightarrow$ A is invertible too.

Hence, Proved.