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LA Assignment - 4

Q1 Given: An $n \times n$ matrix A over field F. let $\bar{\alpha}_1, \bar{\alpha}_2, \ldots \bar{\alpha}_n \in F^{n}$ be the row ~~matrix~~ vectors of A s.t. they are a linearly independant set of vectors.

To Prove: A is invertible.

Proof: $\bar{\alpha}_1, \bar{\alpha}_2$ are linearly independant,
$\therefore \sum_{i=1}^{n} c_i \bar{\alpha}_i \neq 0$, where $c_1, c_2 \ldots c_n \in F$ & $\forall c_i \neq 0$.

$\therefore$ by row oper^ns we can't modify any row to be a zero-row ①
Also, suppose $\alpha_i = \bar{0}$, for some $i = j$. Then $\exists$ $(0, 0, \ldots c_j, 0, 0) \in F^n$ s.t. $\sum c_i \alpha_i = 0$ when $c_j \neq 0$.
This implies that $\forall i$ $\alpha_i \neq 0$ — ②

From ①, ②,
A does not have any zero rows initially and we also can't do row operations to modify it into a zero row.
Let $\beta_1, \beta_2 \ldots \beta_n$ be row vectors in row-echelon form of A.
Here, $\beta_i \neq 0$ $\forall i$
So we can do a set of elementary row operations on I to get A. i.e.
$$A = e_n e_{n-1} \ldots e_2 e_1 (A)$$
Corresponding to each we have elementary matrices that provides these row operations.
$$\text{So, } A = E_n E_{n-1} \ldots E_2 E_1 I$$
Multiplying by $E_1^{-1} E_2^{-1} \ldots E_n^{-1}$ on both sides,
$$E_1^{-1} E_2^{-1} \ldots E_{n-1}^{-1} E_n^{-1} A = I$$
$$\therefore A^{-1} = E_1^{-1} E_2^{-1} \ldots E_n^{-1}$$
Hence, proved.

Q2 Given: Two finite dimensional subspaces W_1, W_2 with dimensions $\dim W_1$, $\dim W_2$

To Prove :- (i) $W_1 + W_2$ is also finite dimensional.
(ii) $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

Proof: Intersection of two subspaces is also a subspace.
So, $W_1 \cap W_2$ is also a subspace.

Let basis of $W_1 = \{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_{\dim W_1}\}$ and
basis of $W_2 = \{\beta_1, \beta_2 \dots \beta_{\dim W_2}\}$

Now basis of $W_1 \cap W_2 = \{a_1, a_2 \dots a_n\}$ $n = \dim(W_1 \cap W_2)$.
Here, $a_1, a_2, a_3 \dots a_n \in$ (Basis of W_1), (basis of W_2)
so, $a_1, a_2, a_3 \dots a_n$ is linearly independent.
($W_1 \cap W_2$ is a subspace of W_1, W_2). So, it is finitely dimensional.

So, Basis of $W_1 = \{\alpha_1, \alpha_2 \dots \alpha_{\dim W_1 - n}, a_1, a_2 \dots a_n\}$

Basis of $W_2 = \{\beta_1, \beta_2 \dots \beta_{\dim W_2 - n}, a_1, a_2 \dots a_n\}$.
we need to show that,
$W_1 + W_2$ spanned by (basis of W_1) $\cup$ (basis of W_2) i.e.
$\beta = \{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_{\dim W_1 - n}, a_1, a_2 \dots a_n,$
$\beta_1, \beta_2, \dots \beta_{\dim W_2 - n}\}$ is linearly independant.

Let $$\sum_{i=1}^{\dim W_1 - n} b_i \alpha_i + \sum_{j=1}^{\dim W_2 - n} c_j \beta_j + \sum_{k=1}^{n} d_k a_k = 0$$

where $b_i, c_j, d_k \in F$.

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So, $\sum_i b_i \alpha_i + \sum_k d_k a_k = - \sum_j c_j \beta_j$.

Let $\sum_j c_j \beta_j = \sum_l e_l \beta_l = w_1 \cap w_2$ (as LHS has a_i)

$- \sum_j c_j \beta_j = \sum_k \delta_k a_k \quad (d_k \in F \ \forall k)$

So, $\sum_k \delta_k a_k + \sum_j c_j \beta_j = 0 \quad - (1)$

Here $\{a_1, a_2 \dots a_n, \beta_1, \beta_2 \dots \beta_{\dim w_2 - n}\}$ is basis of w_2.

(1) $\Rightarrow$ they are linearly dependant which is a contradiction.
i.e. B is linearly independant.

Thus B = basis of $(w_1 + w_2)$.

So, $\dim(w_1 + w_2) = \dim(w_1) + \dim(w_2) - n$
which is finite.

$LHS = \dim w_1 + \dim w_2$

$RHS = \dim(w_1 \cap w_2) + \dim(w_1 + w_2)$
$= \cancel{n} + \dim(w_1) + \dim(w_2) - \cancel{n}$
$= \dim(w_1) + \dim(w_2)$

$LHS = RHS$

Hence, proved.

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Q3 Given: An non invertible matrix P over F, an n-dimensional vector space V over F and β, the ordered basis of V.

To Prove :- (i) $[\alpha]_\beta = P[\alpha]_{\beta'}$

(ii) $[\alpha]_{\beta'} = P^{-1}[\alpha]_\beta$ for every vector α in V

Proof: Let $\beta = \{\bar{\alpha}_1, \bar{\alpha}_2 \dots \bar{\alpha}_n\}$ be the ordered basis of V, $\beta' = \{\bar{\alpha}_1', \bar{\alpha}_2' \dots \bar{\alpha}_n'\}$.

Let us define $\bar{\alpha}_j' = \sum_{i=1}^{n} P_{ij} \bar{\alpha}_i \quad j \in \{1, 2, 3 \dots n\}$

Here $P_{ij} \in F \mid \forall i, \forall j \in \{1, 2, 3 \dots n\}$.

Consider $A = P^{-1}$

Then $\sum_j A_{jk} \bar{\alpha}_j' = \sum_j A_{jk} \sum_i P_{ij} \vec{\alpha}_i$

$= \sum_j \sum_i P_{ij} A_{jk} \bar{\alpha}_i$

$= \sum_i \sum_j (P_{ij} A_{jk}) \bar{\alpha}_i = \alpha_i$

So, the subspace spanned by β' contains β and also span k

In $[\alpha]_\beta = P[\alpha]_{\beta'}$

$[\alpha]_\beta = 0$ iff $[\alpha]_{\beta'} = 0$. So, β' is linearly independent set.

β' is a linearly independent set, span V.

$\therefore \beta'$ is an ordered basis of V.

Acc. to theorem, for an n-dimensional vector space over field F, which contains β and β' as two ordered basis of V, $\exists$ unique invertible matrix P s.t.

$[\alpha]_\beta = P[\alpha]_{\beta'}$ & $[\alpha]_{\beta'} = P^{-1}[\alpha]_\beta$.

Hence, Proved.