Real Analysis Project

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Ques Part.

Question :→ State mathematical induction and prove the n^{th} term of Fibonacci sequence using mathematical induction.

Fibonacci Sequence $\{F_n\}$

$F_0 = 0 \qquad F_1 = 1$

$F_n = F_{n-1} + F_{n-2} \quad \forall\ n \geq 2$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad \forall\ n \geq 0$$

Bonus:→ Solve the n^{th} term of Fibonacci sequence using generating functions.

Answer:→ Mathematical induction theorem states that if for every ~~positive~~ non negative integer n there is a corresponding statement P_n, then all of P_n are true if following two conditions are also satisfied:

① P_0 is true

② whenever k is a positive integer such that P_k is true, then P_{k+1} is true also.

classmate
Date
Page

Given:→ Fibonacci sequence (F_n) where
$F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ $\forall\ n \geq 2$, $n \in N$.

To Prove:→ $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ $\forall\ n \geq 0$
$n \in Z^+$

Proof:→ we will prove this using theorem of mathematical induction.

Prove for $n=0$:→
$$F_0 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right]$$
$$= \frac{1}{\sqrt{5}}\left[1 - 1\right] = 0$$

Prove for $n=1$:→
$$F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$$
$$\Rightarrow \frac{1}{\sqrt{5}} \times \frac{2\sqrt{5}}{2} = 1$$

Let F_{n-1} and F_{n-2} be true: (Assm)

$$F_{n-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$
$$F_{n-2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right]$$

According to defination: $F_n = F_{n-1} + F_{n-2}$

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right]$$

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \left[\frac{1+\sqrt{5}}{2} + 1 \right] - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \left[\frac{1-\sqrt{5}}{2} + 1 \right] \right]$$

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \left[\frac{3+\sqrt{5}}{2} \right] - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \left[\frac{3-\sqrt{5}}{2} \right] \right]$$

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \left[\frac{6+2\sqrt{5}}{4} \right] - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \left[\frac{6-2\sqrt{5}}{4} \right] \right]$$

$\cancel{F_n}$ $(1+\sqrt{5})^2 = 1+5+2\sqrt{5} = 6+2\sqrt{5}$

$(1-\sqrt{5})^2 = 1+5-2\sqrt{5} = 6-2\sqrt{5}$

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2} \left(\frac{1-\sqrt{5}}{2}\right)^2 \right]$$

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n} \right]$$

This implies that we have obtained the expression for F_n using the assumption that F_{n-1} and F_{n-2} are true

This formula works for $n=0$ and $n=1$ and if it works for $n=\cancel{k}-2$ and $n=\cancel{k}-1$ then it also works for $n=n$. Therefore the formula works for all $n \in Z^+$ by principle of mathematical induction.

Hence, Proved.

Date
Page

Bonus Part:-

$$F_r = F_{r-1} + F_{r-2} \quad r \geq 2$$

Take $\sum_{r=2}^{\infty}$ and multiply with z^r on both sides.

$$\Rightarrow \sum_{r=2}^{\infty} F_r z^r = \sum_{r=2}^{\infty} F_{r-1} z^r + \sum_{r\geq 2}^{\infty} F_{r-2} z^r$$

$$\Rightarrow \sum_{r\geq 0}^{\infty} F_r z^r - F_1 z - F_0 = z\left[\sum_{r=0}^{\infty} F_r z^r - F_0\right] + z^2 \sum_{r\geq 0}^{\infty} F_r z^r$$

Let $\sum_{r=0}^{\infty} F_r z^r = A(z)$

$$\Rightarrow A(z) - z - 0 = z[A(z) - 0] + z^2 A(z)$$

$$\Rightarrow A(z) = \frac{z}{1-z-z^2}$$

$A(z)$ is the generating function of the numeric function $\{F_0, F_1, F_2 \ldots F_n\}$.

Let $A(z) = \frac{z}{1-z-z^2} = \frac{z}{(1-\alpha z)(1-\beta z)}$

where $1/\alpha$ and $1/\beta$ are roots of quadratic polynomial $1-z-z^2=0$

$$A(z) = \frac{z}{(1-\alpha z)(1-\beta z)} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$$

On equating coeffecients both sides we get
$A = \frac{1}{\alpha-\beta}$ and $B = \frac{-1}{\alpha-\beta}$

$$1-z-z^2 = (1-\alpha z)(1-\beta z) = 1-(\alpha+\beta)z + \alpha\beta z^2$$

$\alpha+\beta = 1$ $\quad$ $\alpha\beta = -1$ $\quad$ $\alpha-\beta = \sqrt{5}$ $\quad$ when $\alpha > \beta$

$$A(z) = \frac{1}{\sqrt{5}}\left[\frac{1}{1-\alpha z} - \frac{1}{1-\beta z}\right]$$

$$A(z) = \frac{1}{\sqrt{5}}\left[\sum_{r=0}^{\infty} \alpha^r z^r - \sum_{r=0}^{\infty} \beta^r z^r\right]$$

$$A(z) = \sum_{r=0}^{\infty} F_r z^r = \left[\sum_{r=0}^{\infty} \frac{1}{\sqrt{5}}(\alpha^r - \beta^r) z^r\right]$$

$$F_r = \frac{1}{\sqrt{5}}\left[\alpha^r - \beta^r\right]$$

$\alpha + \beta = 1$ $\qquad\qquad$ $\alpha - \beta = \sqrt{5}$

$\alpha = \frac{1+\sqrt{5}}{2}$ $\qquad\qquad$ $\beta = \frac{1-\sqrt{5}}{2}$

$$F_r = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^r - \left(\frac{1-\sqrt{5}}{2}\right)^r\right], \quad n \geq 0$$

Hence, Proved.

Date
Page

Applications.

(1) Use of real analysis in chemical kinetics of biochemical rxns. (chemistry)

Real analysis also deal with the study of derivatives or differentiability. Derivatives have many real life applications.

Derivatives are mainly defined as rate of change of one particular quantity with respect to time.

One particular case of derivatives is the use of derivatives in the chemical kinetics of a reaction.

Rate of a reaction is defined as rate of change of reactant's concentration with time in a broad manner.

$$R \rightarrow P$$

$$\text{rate} = -\frac{d[R]}{dt}$$

This rate is of utmost importance. With the help of rate of rxn we can determine how fast the reaction will proceed. If the rate of rxn is very high this means that the reaction will get completed very early and vice versa.

So in biochemical reactions with the help of rate we can determine the time reactant requires to get converted into products. In this way we can estimate the time required by a medicine to starts its work.

$R \rightarrow P$

k = Rate Constant

n = Order of a reaction

$$\text{rate} = \frac{-d[R]}{dt} = k[R]^n$$

$$\frac{dR}{[R]^n} = -k\,dt$$

Integrating both sides and taking limits.

$$\int_{R_0}^{R_t} \frac{dR}{R^n} = -k\int_0^t dt$$

Solving and putting limits we get

$$\frac{1}{R_t^{\,n-1}} - \frac{1}{R_0^{\,n-1}} = k(n-1)t$$

By knowing the percentage of reactant to decompose for medicine (biochemical rxn) to work we can get a estimated time for the ~~reaction~~ ~~to~~ medicine or any other substance to show it's work.

Hence, derivatives and real analysis have a very importance in biochemical industry.

PS: This is a very basic model. working of a medicine is a very complex analysis. This is a very basic part which capable of understanding by me.

(2) Application in motion under gravity. [Finite time to stop the ball].
[Physics]

- For eg we throw a ball upwards with a speed (v)
Ignoring air resistance. Coeffecient of restitution is e. $(0<e<1)$
Using real analysis we can ~~speed~~ determine that it takes finite time to stop after throwing the ball.

Time taken by the ball to return to ground again $= \frac{2v}{g}$

where g is

Ground

Speed after 1^{st} bounce $= ev$
Time taken to return after 1^{st} bounce $= \frac{2ev}{g}$

Speed after 2^{nd} bounce $= (ev)e = e^2v$
Time taken to return after 2^{nd} bounce $= \frac{2e^2v}{g}$.

Similarly, total time $= \frac{2v}{g} + \frac{2ev}{g} + \frac{2e^2v}{g} \ldots \infty$

$\Rightarrow \frac{2v}{g}\left[1+e+e^2 \ldots \infty\right]$

$\Rightarrow \frac{2v}{g}\left[\sum_{r\geq 0}^{\infty} e^r\right]$

Using ratio test on $\sum_{r\geq 0}^{\infty} e^r$

$\alpha = \lim_{n\to\infty} \sup \left(\frac{e^{r+1}}{e^r}\right)$

$\alpha = \lim_{n\to\infty} \sup (e)$

$\lim_{n\to\infty} \sup (a_n) = $ ~~[illegible]~~ $\bigwedge_{n\geq 1}^{\infty} \left[\bigvee_{k=n}^{\infty} a_k \right]$

$\lim (\sup)(e) = e$

as it is a constant and infimum of supremum is equal to e.

as $\alpha = e < 1$ the series $\sum_{k\geq 0}^{\infty} e^k$ converges.

So the series $\frac{2v}{g} \sum_{k\geq 0}^{\infty} e^k$ also converges.

Thus in this way we can prove that it takes finite amount of time for a ball thrown upwards to stop.

BONUS:→ Using formula $1+e+e^2 \ldots \infty = \frac{1}{1-e}$

when $|e|<1$

Total time taken $= \frac{2v}{g} \left[\frac{1}{1-e} \right]$

Also we can compute the total distance travelled by the ball.

Distance travelled between throwing and 1st bounce =

$$\frac{v^2}{2g}$$

Distance travelled b/w 1st bounce and 2nd bounce =

$$\frac{e^2v^2}{2g}.$$

Therefore total distance $\Rightarrow \frac{v^2}{2g} + \frac{e^2v^2}{2g} + \frac{e^4v^2}{2g} \dots \infty$

$$\Rightarrow \frac{v^2}{2g}\left[\sum_{n=0}^{\infty} e^{2n}\right]$$

Again using root test we can prove above sequence is convergent.

Total distance $\Rightarrow \frac{v^2}{2g} \times \frac{1}{[1-e^2]}$