Q1 **Given:-** A set of all positive increasing sequences of positive integers.

To Prove:- The set is uncountable.

Proof:- We can proof this by mapping the sequence to the set of infinite length binary strings. AND we know that the set of infinite length binary strings is uncountable by following proof.

Consider the following binary strings. Let this be countable.

$S_1 = 10101010 \ldots$

$S_2 = 01010101 \ldots$

$S_3 = 100100100 \ldots$

$S_4 = 001001001 \ldots$

$\vdots$

Since the above set is countable we can say that the set has each and every possible string and it is mapped to set of natural numbers.

Now consider a new string s'.

$s' = a_1 a_2 a_3 a_4 \ldots a_n$

Let be $a_1 \Rightarrow$ complement of 1st character of S_1

Similarly $a_n \Rightarrow$ Complement of nth character of S_n

Therefore, the s' string constructed is different from all the above strings.

So, we have constructed a new string which is not in above set.

So this contradicts our ~~stat~~ assumption
Hence, set of binary strings of infinite length is not countable.

Now positive integers is set of Natural No's.

~~If~~ $N = \{1, 2, 3, 4 \ldots\}$
~~If 1 is a part of the sequence put 1 (in string) if 1 is not part of the sequence~~
$i \in N$
If i^{th} number is a part of sequence put 1, at the ith position of the sequence If it is not the part then put 0 at that position.

Following ~~Hence the~~ above procedure we can produce binary strings of infinite length and since that set is uncountable the set of positive ~~ad~~ increasing sequence of positive integers is also uncountable.

Hence, proved.

Q2 To Prove:- Collection of all finite subsets of N is countable.

Proof:- Firstly we consider a mapping of N to the set of all prime numbers, so we replace the nth Natural number with the nth prime number.
Let S be this set and S_F be the set of all finite subsets of S (which is equivalent to set of all finite subsets of N).

Let $f: S_F \to N$ be defined as $\left(\prod_{i=1}^{n} P_i\right)$ for any set $F \in S_F$ with n elements.
i.e. f computes the product of the elements comprising any finite subset of primes.

By fundamental theorem of arithmetic, each $m \in N$ has a unique factorization i.e. $f(F_1) = f(F_2)$
$\therefore F_1 = F_2$

Hence, f is a one-one function from S_F to N.

Since, a one-one function exists from S_F to a subset of N, S_F is countable set.

Hence, proved.

Q3 Given:→ $(a,b) \sim (c,d)$ s.t. $ad = bc$.

To Prove:→ $\sim$ is an equivalence relation.

Proof:→ R is an equivalence relation if it satisfies following 3 properties:→

① Reflexive:→ $(a,b) \sim (a,b)$

$a \times b = b a$

so Relation is Reflexive.

② Symmetric:→ $(a,b) \sim (c,d)$ — ①

$(c,d) \sim (a,b)$ — ②

$ad = bc$ $\qquad$ $ad = bc$

So relation is symmetric.

③ Transitive:→ $(a,b) \sim (c,d)$ — ①

$ad = bc$ — ③

$(c,d) \sim (e,f)$ — ②

$cf = de$ — ④

Multiply ③ and ④

$a\cancel{d}\cancel{c}f = b\cancel{c}\cancel{d}e$

$af = be$

$\therefore (a,b) \sim (e,f)$.

So relation is transitive.

$\therefore$ Relation is equivalence.

Hence, Proved.

Q4 Given:→ Relation m is a multiple of n.

To Prove:→ The relation is a partial order relation.

Proof:→ For relation to be Partial order it should satisfy 3 properties:→

① Reflexive:→ m is a multiple of m.

② Antisymmetric:→ m is a multiple of n

$\therefore m = k_1 n$ — ① where $k_1 \in N$

n is a multiple of m

$n = k_2 m$ — ② $k_2 \in N$

From ① & ②

$m = k_1 k_2 m$

This is possible $k_1 k_2 = 1$ which possible when $k_1 = k_2 = 1$ as $k_1, k_2 \in N$.

So, $m = n$

③ Transitive:→ m is a multiple of n

n is a multiple of a.

$m = k_1 n$ $k_1, k_2 \in N$

$n = k_2 a$

$m = k_1 k_2 a$ So m is multiple of a.

$\therefore$ It satisfies all 3 above properties. It is partial order relation.

Q5 Given :- $A, B \in P(S)$ $\quad ARB \Rightarrow |A| \leq |B|$

To Prove :- R is a partial order relation.

Proof :- R is partial order when it follows 3 properties :-

① Reflexive :- $|A| \leq |A|$ True.

② Antisymmetric :- $|A| \leq |B|$ AND $|B| \leq |A|$

$\therefore |A| = |B|$

$\therefore$ Relation is Antisymmetric.

③ Transitive :- $A, B, C \in P(S)$

$|A| \leq |B|$ — ①

$|B| \leq |C|$ — ②

From ① & ②

① × ②

$\Downarrow$

$|A|\,|B| \leq |B|\,|C|$ Since |B| are +ve

$\therefore |A| \leq |C|$

$\therefore$ Relation is transitive

Since relation is reflexive, Antisymmetric and transitive, the relation is partially ordered

Hence Proved.

Q6 1. Since $x \sim y$ if $x^2 = y^2$

For $x = 1$ $\quad y^2 = 1$

$\boxed{y = \pm 1}$

For $x = \sqrt{2}$ $\quad y^2 = 2$

$y = \pm\sqrt{2}$

For $x = 0$ $\quad y^2 = 0$

$y = 0$

2. Given :- xRy s.t. $x^2 = y^2$

To Prove :- R is equivalence relation.

Proof :- An equivalence relation follows following 3 properties :-

① Reflexive :- $x^2 = x^2$ $\quad xRx$

$\therefore$ This relation is reflexive

② Symmetric :- $x^2 = y^2$ $\quad xRy$

$y^2 = x^2$ $\quad yRx$

$\therefore$ This relation is symmetric.

③ Transitive :- $x^2 = y^2$ - ① $\quad xRy$

$y^2 = z^2$ - ② $\quad yRz$

from ① & ②

$x^2 = z^2$ $\quad \therefore xRz$

$\therefore$ This relation is transitive.

Since this ~~function~~ relation satisfies all 3 properties.

The relation is an equivalence relation

Hence, Proved.

Q7 <u>Given</u>:→ A set B which is countable.

<u>To Prove</u>:→ $X \subseteq B$ s.t. X is ~~finite~~ at most countable.

<u>Proof</u>:→ Since B is a countable set it has one + one correspondance to set of Natural numbers.

Let set $X = \{a_1, a_2, a_3 \ldots\}$

Since it is a subset of B which has one one correspondance with N. So X satisfies well ordering principle.

∴ Elements of X is ordered.

Remove the first element of X say a_1 map it with 1.

Repeat this process.

$a_1 \to 1$

$a_2 \to 2$

$a_3 \to 3$

So we can map every element of X with N with one - one correspondence.

∴ It is a countable set.

Hence, Proved.

Q8

Given:- A_i is a countable set for $i \geq$.

To Prove:- $A = \bigcup_{i \geq 1}^{\infty} A_i$ is a countable set.

Proof:- Let the set $A_n = \{a_1^n, a_2^n, a_3^n \ldots\}$

This ~~set~~ indexing is possible because each A_i is countable and A_i's are getting disjointed.

Now for any element a_k^n, consider $f: (n,k) = p_1^n p_2^k$ for any primes p_1 and p_2.

Since primes p_1 and p_2 are distinct, expression $p_1^n p_2^k$ gives distinct results for different (n,k). i.e.

$f(n_1, k_1) = f(n_2, k_2)$

$\therefore n_1 = n_2 \qquad k_1 = k_2$

Hence, it is a one-one function from A to a subset of natural numbers (set of combinations of powers of prime p_1 and p_2). Hence, the set is countable.