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LA - Assignment - 5

Q1 Given: Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose V is finite - dimensional.

To Prove:- rank(T) + Nullity(T) = dim(V).

Proof:- If V is finite - dimensional (given), the rank of T is the dimension of the range of T and the nullity of T is the dimension of the null space of T.

Let basis of V be $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$.
$\Rightarrow \dim(V) = n$

Let $\{\alpha_1, \alpha_2 \ldots, \alpha_k\}$ be the basis for the null space of T, say K.
$\Rightarrow \dim(K) = K$.
nullity(T) = dimension of null space = $\dim(K) = K$.
$\{T_{\alpha_{k+1}}, T_{\alpha_{k+2}}, \ldots T_{\alpha_n}\}$ should be a basis for the range of T. To prove it is a basis for the range of T we need to show that,

① The vectors $T_{\alpha_{k+1}}, \ldots, T_{\alpha_n}$ span the range of T.

The vectors $T_{\alpha_1}, T_{\alpha_2} \ldots T_{\alpha_n}$ span the range of T (as domain of T is $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$). As $(T_{\alpha_1}, T_{\alpha_2} \ldots T_{\alpha_n}) = 0$ (They belong to the Null space).

$\Rightarrow T_{\alpha_{k+1}}, T_{\alpha_{k+2}} \ldots T_{\alpha_n}$ span the range of T.

② Show that these vectors are independent :-

Proof: Let there be scalars m_i s.t.

$$\sum_{i=k+1}^{n} m_i (T\alpha_i) = 0 \quad \Rightarrow T\left(\sum_{i=k+1}^{n} m_i \alpha_i\right) = 0 \quad -①$$

From ① we can say that,
$\exists$ a vector $\alpha = \sum_{i=k+1}^{n} m_i \alpha_i$ — ② in the null space of T.

Since $\alpha_1, \alpha_2 \ldots \alpha_k$ for a basis for the null space of T,
$\exists$ scalars $n_1, n_2 \ldots n_k$ s.t.

$$\alpha = \sum_{j=1}^{k} n_j \alpha_j \quad -③$$

From ② and ③ we can say that,

$$\alpha = \sum_{i=k+1}^{n} m_i \alpha_i = \sum_{j=1}^{k} n_j \alpha_j$$

$$\Rightarrow \sum_{j=1}^{k} n_j \alpha_j - \sum_{i=k+1}^{n} m_i \alpha_i = 0$$

Since $\alpha_1, \alpha_2 \ldots \alpha_n$ are linearly independent vectors
$(c_1\alpha_1 + c_2\alpha_2 \ldots + c_n\alpha_n = 0$ iff $c_1 = c_2 = c_3 \ldots c_n = 0)$

$\Rightarrow n_1 = n_2 = \ldots n_k = m_{k+1} = m_{k+2} \ldots = m_n = 0$

$\Rightarrow \{T\alpha_{k+1}, \ldots, T\alpha_n\}$ are linearly independent vectors.

$\Rightarrow \{T\alpha_{k+1}, \ldots T\alpha_n\}$ is the basis for the range of T.

$\Rightarrow$ dim (range of T) = n-k

$\Rightarrow$ rank (T) = dim. (range of T) = n-k

$\Rightarrow$ nullity (T) + rank (T) = $k + n-k = n = \dim(V)$

$\Rightarrow$ nullity (T) + rank (T) = dim. V.

Hence, Proved.

Q2 Given: V and W are vector spaces over field F.
T and U be linear transformations from V to W.
Function $(T+U)$ defined as $\rightarrow$
$(T+U)(\alpha) = T\alpha + U\alpha$
$c \in F$, Function (cT) defined as $\rightarrow$
$(cT)(\alpha) = c(T\alpha)$.

To Prove :- ① Function $(T+U)$ is a linear transformation from V to W.
② Function (cT) is a linear transformation from V to W.
③ The set of all linear transformation from V to W, together with the addition and scalar multiplication, defined above, is a vector space over field F.

Proof: ① A function f is a linear transformation iff. for $\alpha, \beta \in V$, $c \in F$ $f(c\alpha + \beta) = c f(\alpha) + f(\beta)$.

$\Rightarrow (T+U)(c\alpha + \beta) = T(c\alpha + \beta) + U(c\alpha + \beta)$ (Fn defn)
$= cT\alpha + T\beta + cU\alpha + U\beta$ (linear transformation).
$= c(T\alpha + U\alpha) + (T\beta + U\beta)$
$= c(T+U)(\alpha) + (T+U)(\beta)$

$\Rightarrow (T+U)$ is a linear transformation from V to W.

(2) Similar to 1^{st} part,

$$(cT)(m\alpha+\beta) = c(T(m\alpha+\beta))$$ [Function defination]

$$= c(mT\alpha + T\beta)$$ [Linear transformation]

$$= cm(T\alpha) + c(T\beta)$$

$$= m(c(T\alpha)) + c(T\beta)$$

$$= m((cT)\alpha) + (cT)\beta$$

$\Rightarrow$ cT is a linear transformation from V to W.

(3) To prove the set of all linear transformations from V to W, together with the given addition and scalar multiplication is a vector space over field F, we need to show 8 properties of vector space:

(1) Associativity:

$$(T+U)\alpha + V(\alpha) = T\alpha + U\alpha + V\alpha = T\alpha + (U\alpha + V\alpha)$$

$$= T\alpha + (U+V)(\alpha)$$

$\Rightarrow$ $(T+U)+V = T + (U+V)$

Hence, associativity exists.

(2) Commutativity:

$$(T+U)(\alpha) = T\alpha + U\alpha = U\alpha + T\alpha = (U+T)(\alpha)$$ (Commutative addn in a field F)

$\Rightarrow$ $T+U = U+T$

Hence, commutativity exists.

(3) Identity in addition:

$0(\alpha) = 0 \quad \forall\ \alpha \in V$ in $0: V \to W$.

$\Rightarrow$ $(T+0)(\alpha) = T\alpha + 0(\alpha) = T\alpha = T(\alpha)$.

$\Rightarrow$ $T+0 = T$ (Defn of additive identity in a Field F)

Hence, identity exists.

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④ Inverse in addition:

$T: V \to W.$

we define $(-T)(\alpha) = -T\alpha \quad \forall\ \alpha \in V$

$(T+(-T))(\alpha) = T\alpha + (-T\alpha) = 0$ [Defn of additive inverse in a field F]

$\Rightarrow \quad T+(-T) = 0$

Hence, inverse exists.

⑤ Identity in multiplication:

$T: V \to W$

$(1.T)(\alpha) = T(\alpha)$

$\Rightarrow 1.T = T \quad \forall\ \alpha \in V$ [Defn of multiplicative identity in a field F]

Hence, identity exists.

⑥ Associativity in multiplication:

$(cd)\,T(\alpha) = c(d\,T(\alpha))$ (Associativity of multiplication in a field F)

$(cd)T = c(dT)$

Hence, associativity in multiplication exists.

[Distributive of multiplication over addition in a field F]

⑦ $(c+d)\,T(\alpha) = cT(\alpha) + dT(\alpha)$

$\Rightarrow (c+d)\,T = cT + dT$

Hence, distributive law exists.

⑧ $c(T+U)(\alpha) = c(T(\alpha)+U(\alpha)) = cT(\alpha) + cU(\alpha)$

$\Rightarrow c(T+U) = cT + cU$

Hence, distributive law exists.

As all 8 properties are satisfied, the vector space over this addition and multiplication is a vector space.

Hence, Proved.

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Q3 Given: V is a n-dimensional vector space over the field F. W is a m-dimensional vector space over field F.

To prove: The space L(V,W) is finite dimensions and has dimensions mn.

Proof: Let $\beta = \{\alpha_1, \dots \alpha_n\}$ and $\beta' = \{\alpha_1', \dots \alpha_m'\}$ be the ordered basis of V & W respectively.

For each pair of integers (p, q) in F s.t. $1 \le p \le m$ & $1 \le q \le n$, then the linear transformation E^{pq} from V to W is defined by

$$E^{pq}(\alpha_i^{m}) = \begin{cases} 0 & \text{if } i \neq q \\ \beta_p & \text{if } i = q \end{cases}$$

$$= \delta_{iq} \beta_p$$

Using the thm that there is a unique linear transformation from V to W satisfying these conditions T from V to W:

$T\alpha_j = \beta_j$

where $V = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ & $W = \{\beta_1, \beta_2 \dots \beta_n\}$

Thus we can say that mn transformation of E^{pq} from the prev. theorem are unique. Let T be the linear transformations from V into W.

For each j s.t. $1 \le j \le n$,

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Let $A_{ij}, \ldots A_{mj}$ be the co-ordinates of the vector $T\alpha_j$ in the ordered basis β' i.e.

$$\therefore T\alpha_j = \sum_{x=1}^{m} A_{xj} \beta_x$$

Let U be a linear transformation s.t.

$$U\alpha_j = \sum_{x=1}^{m} \sum_{y=1}^{n} A_{xy} E(x,y)(\alpha_j)$$

$$= \sum_x \sum_y A_{xy} \delta_{jy} \beta_x$$

$$= \sum_{x=1}^{m} A_{xj} \beta_x$$

$$\therefore U\alpha_j = T\alpha_j$$

$$U = T.$$

$U = \sum_x \sum_y A_{xy} E(x,y)$ is the zero transformation

$$\therefore U\alpha_j = 0 \quad \forall j$$

$$\therefore \sum_{x=1}^{m} A_{xj} \beta_x = 0$$

β_x is the basis of finite dimensional vector space W.

$\therefore \beta_x$ cannot be all 0 at same time.

$\therefore A_{xj} = 0 \quad \forall\ x, j.$

$\therefore L(V,W)$ is finite dimensional

$\therefore \beta = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ & $\beta' = \{\alpha_1', \alpha_2' \ldots \alpha_n'\}$

$\dim(L(V,W)) = mn$

Hence, Proved.