Q) Given :- $f(z) = \sqrt[3]{r}\, e^{i\theta/3}$

To Prove: $f'(z)$ exists

and $f'(z) = \dfrac{1}{3(f(z))^2}$

Proof :- $f(z) = r^{1/3}\left(\cos\frac{\theta}{3} + i\sin\theta/3\right)$

For $f'(z)$ to exist it should satisfy Cauchy Riemann eqn's in polar form which is!

$f(z) = u(r,\theta) + i\, v(r,\theta)$

$r\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial \theta}$ & $\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$

① $\dfrac{\partial u}{\partial r} = \dfrac{1}{3}\dfrac{1}{r^{2/3}}\cos\theta/3$ ② $\dfrac{\partial v}{\partial \theta} = \dfrac{1}{3}r^{1/3}\cos\theta/3$

$r\dfrac{\partial u}{\partial r} = \dfrac{\partial v}{\partial \theta}$ $\Rightarrow$ $\dfrac{r\cos\theta/3}{3r^{2/3}} = \dfrac{1}{3}r^{1/3}\cos\theta/3$

1st eqn satisfies

③ $\dfrac{\partial u}{\partial \theta} = \dfrac{-r^{1/3}}{3}\sin\theta/3$ ④ $\dfrac{\partial v}{\partial r} = \dfrac{1}{3}r^{-2/3}\sin\theta/3$

$-r\dfrac{\partial v}{\partial r} = -\dfrac{r^{1/3}}{3}\sin\theta/3$

$-r\dfrac{\partial v}{\partial r} = \dfrac{\partial u}{\partial \theta}$ 2nd CR eqn satisfies.

As $f(z)$ satisfies CR conditions.
$f'(z)$ exists.

Now to find $f'(z)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\cancel{z+}\Delta z \cancel{- z}}$$

where $f(z) = r^{1/3} e^{i\theta/3}$ & $z = re^{i\theta}$

Let $f(z) = u + iv$ $\qquad z =$

$$f'(z) = \lim_{\substack{\delta r \to 0 \\ \delta\theta \to 0}} \frac{\cancel{u} + \Delta u + i(\cancel{v} + \Delta v) - \cancel{(u + iv)}}{\delta[r(\cos\theta + i\sin\theta)]}$$

$$f'(z) = \lim_{\substack{\delta r \to 0 \\ \delta\theta \to 0}} \frac{\delta u + i\delta v}{\delta r(\cos\theta + i\sin\theta) + r(-\sin\theta + i\cos\theta)\delta\theta}$$

Let $\delta\theta = 0$ Take derivative along path $\delta r \to 0$

$$f'(z) = \lim_{\delta r \to 0} \frac{\delta u + i\delta v}{\delta r(\cos\theta + i\sin\theta) + 0}$$

$$f'(z) = \frac{1}{e^{i\theta}} \lim_{\delta r \to 0} \frac{\delta u}{\delta r} + i\frac{\delta v}{\delta r}$$

$u = r^{1/3}\cos\theta/3$

$v = r^{1/3}\sin\theta/3$

$$f'(z) = \frac{1}{e^{i\theta}}\left[\frac{1}{3r^{2/3}}\cos\theta/3 + i\frac{1}{3r^{2/3}}\sin\theta/3\right]$$

$$= \frac{1}{e^{i\theta}} \times \frac{1}{3r^{2/3}} e^{i\theta/3}$$

$$\Rightarrow \frac{1}{3r^{2/3} e^{i2\theta/3}}$$

$$f'(z) = \frac{1}{3\left(r^{1/3} e^{i\theta/3}\right)^2}$$

$$= \frac{1}{3(f(z))^2}$$

Hence, Proved.

Q2 Given: $\oint_C \frac{z^2}{z-4} dz$ where C is circle $|z|=1$ in anticlockwise direction.

To Calculate :-
$$\oint_C \frac{z^2}{z-4} dz$$

Solution :- $\oint_C \frac{z^2}{z-4} dz$ can be compared by

$$\oint_C \frac{f(z)}{z-z_0} dz$$ where $f(z) = z^2$, $z_0 = 4$

If we can say that $f(z)$ is analytic function, we can use Cauchy Integral theorem with contour C.

$$f(z) = z^2 = (x+iy)^2 = x^2-y^2+2xyi$$

$f(z)$ should satisfy Cauchy Riemann Conditions and all partial derivatives should be continuous.

$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = -2y$ $\frac{\partial v}{\partial x} = 2y$ $\frac{\partial v}{\partial y} = 2x$

where $u = x^2-y^2$ $v = 2xy$

We ~~can~~ see that they satisfy Cauchy Reimann Condition.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ —(1) $\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$ —(2)

We can also say that all 4 partial derivatives are continuous.

$\therefore$ $f(z) = z^2$ is a analytic function.

$\therefore$ Cauchy integral theorem can be applied on this problem and we can say that

$$\oint_C \frac{z^2}{z-4}\, dz \quad \text{as } z_0 = 4 \text{ is an outside point of } |z| = 1$$

[$|z| = 1$ is a circle with centre $(0,0)$ and radius. And ~~p~~ putting $(4,0)$ we can say that z_0 is the outside the curve C]

$\therefore$ By Cauchy integral Thm

$$\frac{1}{2\pi i}\oint \frac{z^2}{z-4}\, dz = 0$$

$$\therefore \oint \frac{z^2}{z-4}\, dz = 0$$

Hence, ~~Proved~~. solved.

Q3 Given: C a ~~r~~rectangle whose sides are along lines $x = \pm 5$ and $y = \pm 6$.

To evaluate :-
$$\oint_C \frac{z}{2z+1}\, dz$$

Evaluation:
$$\oint \frac{z}{2z+1}\, dz = \frac{1}{2}\oint \frac{z}{z+\frac{1}{2}}\, dz$$

$$\Rightarrow \frac{1}{2}\oint \frac{z}{z-\left(-\frac{1}{2}\right)}\, dz$$

Let $I = \oint_C \frac{z}{z-\left(-\frac{1}{2}\right)} dz$

We need to find $I/2$.

We can compare I with $\oint_C \frac{f(z)}{z-z_0} dz$

where $f(z) = z$ & $z_0 = -1/2$

If we can prove $f(z)$ to be analytic function, ~~and~~
we can use Cauchy Integral Theorem with contour C.

$f(z) = z = x+iy$ $\quad u = x$ & $v = y$

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 1$

$f(z)$ should satisfy Cauchy Riemann Condition and all partial derivatives should be continous.

CR condition i.e. :

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied.

Also all partial derivatives are continuous.

$\therefore$ $f(z) = z$ is a analytic function.

$\therefore$ Cauchy Integral Theorem could be used in this function.

As z_0 lies inside the contour C and $f(z)$ is analytic

By using Cauchy's integral formula

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$\Rightarrow \oint \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)$

$\Rightarrow \oint \frac{z}{z-(-\frac{1}{2})} = 2\pi \times i \times f(-\frac{1}{2})$

$I = -\pi i$

$\frac{I}{2} = \frac{-\pi i}{2}$

Therefore $\oint_c \frac{z}{2z+1}\, dz = \frac{-\pi i}{2}$

3) Given: $|z| < \infty$

To Prove:- $z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}\, z^n}{(n-2)!}$

Proof:- Let $f(z) = e^{3z}$

Let us check if $f(z)$ is analytic function or not.

If $f(z)$ is analytic we can use Taylor Series to expand it.

Let $z = x+iy$.

$f(z) = e^{3x+3iy} = e^{3x} \cdot e^{i3y} = e^{3x}(\cos 3y + i\sin 3y)$

$f(z) = e^{3x}\cos 3y + i\, e^{3x}\sin 3y$ Here $u = e^{3x}\cos 3y$

$v = e^{3x}\sin 3y$

$\frac{\partial u}{\partial x} = 3e^{3x}\cos 3y$; $\frac{\partial u}{\partial y} = -3e^{3x}\sin 3y$; $\frac{\partial v}{\partial x} = 3e^{3x}\sin 3y$; $\frac{\partial v}{\partial y} = 3e^{3x}\cos 3y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

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So Cauchy Riemann Condition Satisfies.

And product of 2 continuous functions is a continuous function, (e^{3x}, $\sin 3x$, $\cos 3x$) are all continuous.

$\therefore$ CR conditions satisfied and partial derivatives are continuous everywhere.

$\therefore$ $f(z) = e^{3z}$ is a analytic function.

$\therefore$ we can write $f(z)$ as Σ infinite summation using ~~Taylor series~~. Maclaurian series.

$$f(z) = e^{3z} = \sum_{n=0}^{\infty} a_n z^n$$

where $a_n = \dfrac{f^n(0)}{\lfloor n}$

$$f^n(0) = e^{3\times 0} \times 3^n = 3^n$$

$$f(z) = e^{3z} = \sum_{n=0}^{\infty} \frac{3^n z^n}{\lfloor n}$$

multiplying z^2 both sides

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n . z^{n+2}}{\lfloor n}$$

$\cancel{z^2 e^{3z} = \sum 3^{n-2} . z}$

$n \to t-2$

$0 = t-2$ Lower limit

$t = 2$

$\infty = t-2$

$t = \infty$ (Upper limit)

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$$z^2 e^{3z} = \sum_{t=2}^{\infty} \frac{3^{t-2} \times z^t}{(t-2)!}$$

$$\therefore \quad z^2 e^{3z} = \sum_{t=2}^{\infty} \frac{3^{t-2} \cdot z^t}{\lfloor t-2}$$

Hence, proved.