classmate

Date

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Real Analysis (H2)

Assignment 2

Q1) For a Vector to be a conservative field, it should satisfy 3 properties :-

(1) $\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$ (2) $\frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$ (3) $\frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$

$\frac{\partial F_y}{\partial x} = 2x$ - (1)

(1) = (2)

$\frac{\partial F_x}{\partial y} = 2x$ - (2)

$\frac{\partial F_z}{\partial x} = 2z$ - (3) $\frac{\partial F_x}{\partial z} = 2z$ - (4)

(3) = (4)

$\frac{\partial F_y}{\partial z} = 2y$ - (5) $\frac{\partial F_z}{\partial y} = 2y$ - (6)

(5) = (6)

From above we can say that $\vec{V}$ is a conservative field.

$\vec{V} = \nabla \phi$, where $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$\vec{V} = \frac{\partial}{\partial x}(\phi)\hat{i} + \frac{\partial}{\partial y}$

$\phi = x^2y + y^2z + z^2x + c$

where c is a constant.

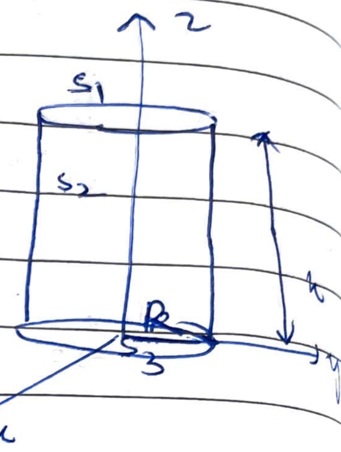
$\vec{\nabla}\phi = \vec{V}$ can be seen by taking partial derivatives of $\vec{V}$.

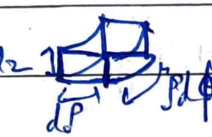
Q2 Given: A right circular cylinder with axis along z axis

To Prove: Divergence thm for this cylinder.

Proof: Divergence thm:

$$\oiint_S \vec{F} \cdot \vec{ds} = \iiint_V (\vec{\nabla} \cdot \vec{F}) \, \vec{dV}$$





we know that $dV = \rho \, d\phi \, d\rho \, dz$

Let $\vec{F} = F_\phi \hat{e}_\phi + F_\rho \hat{e}_\rho + F_z \hat{e}_z = \vec{F}(\phi, \rho, z)$

$$RHS = \iiint (\vec{\nabla} \cdot \vec{F}) \, \vec{dV}$$

$$= \iiint \left(\frac{\partial F_z}{\partial z} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho}\right)\rho\, d\phi\, dz\, d\rho$$

$$= \int_0^R\int_0^{2\pi}\int_0^H \frac{\partial F_z}{\partial z}\, dz\,(\rho\, d\phi\, d\rho) + \int_0^R\int_0^H\int_0^{2\pi}\frac{\partial F_\phi}{\partial\phi}\,d\phi\,(dz\,d\rho)$$

$$+ \int_0^H\int_0^{2\pi}\int_0^R \frac{\partial(\rho F_\rho)}{\partial\rho}\, d\rho\,(d\phi\,dz)$$

we know that $\int_{x_1}^{x_2}\frac{d(f(x))}{dx}\,dx = f(x_2) - f(x_1)$

$$\Rightarrow \cancel{\int_0^R\int_0^{2\pi}\left[\vec F(\theta,\rho,H) - \vec F(\theta,\rho,0)\right] + \iint}$$

$$\Rightarrow \int_0^R\int_0^{2\pi}\left(\vec F_{(\theta,\rho,H)} - \vec F_{(\theta,\rho,0)}\right)\rho\,d\phi\,d\rho + \int_0^R\int_0^H \overbrace{\cancel{\left(\vec F_{(2\pi,\rho,z)} - \vec F_{(0,\rho,z)}\right)}}^{0}(dz\,d\rho)$$

$$+ \int_0^H\int_0^{2\pi}\left(R\,F_{(\theta,R,z)} - 0\right)d\phi\,dz$$

$2\pi \equiv 0$ as we complete circle and come back to same pt

$$\Rightarrow \int_0^R\int_0^{2\pi}\left[\vec F_{(\theta,\rho,H)} - \vec F_{(\theta,\rho,0)}\right]\rho\,d\phi\,d\rho + \int_0^H\int_0^{2\pi} F_{(\theta,R,z)}\,R\,d\phi\,dz \quad (1)$$

LHS: $\oiint_S \vec{F} \cdot \vec{ds}$

$= \iint_{S_1} \vec{F} \cdot \hat{n}_1 \, dA_1 + \iint_{S_2} \vec{F} \cdot \hat{n}_2 \, dA_2 + \iint_{S_3} \vec{F} \cdot \hat{n}_3 \, dA_3$

$\hat{n}_1 = \hat{e_z}$ $\quad \hat{n}_2 = \hat{e_\rho}$ $\quad \hat{n}_3 = -\hat{e_z}$

For A_1 and A_3 z is fixed $\therefore$ $dz = 0$ and ~~dA_1~~
$dA_1 = dA_3 = \rho \, d\phi \, d\rho$

For A_2, $d\rho = 0$ as R is fixed. (as R is projection vector) $dA_2 = \rho \, d\phi \, dz = R \, d\phi \, dz$.

$$= \int_0^R \int_0^{2\pi} [\vec{F}_{(\theta, \rho, H)}] \, \rho \, d\phi \, d\rho + \int_0^H \int_0^{2\pi} [F_{(\theta, R, z)}] \, (R \, d\phi \, dz)$$

$$+ \int_0^R \int_0^{2\pi} -[\vec{F}_{(\theta, \rho, 0)}] \, (\rho \, d\phi \, d\rho)$$

$$= \int_0^R \int_0^{2\pi} [\vec{F}_{(\theta, \rho, H)} - \vec{F}_{(\theta, \rho, 0)}] \rho \, d\phi \, d\rho + \int_0^H \int_0^{2\pi} F_{(\theta, R, z)} \, R \, d\phi \, dz \quad -(2)$$

From (1) & (2)

(1) = (2)

LHS = RHS

Hence, Proved.

Q3 Given: A vector $\vec{A}$.

To Prove: $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2 \vec{A}$.

Proof: Let $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

LHS: $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$= \left(\frac{\partial(A_z)}{\partial y} - \frac{\partial(A_y)}{\partial z}\right)\hat{i} - \hat{j}\left(\frac{\partial(A_z)}{\partial x} - \frac{\partial(A_x)}{\partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial(A_y)}{\partial x} - \frac{\partial(A_x)}{\partial y}\right)$$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial(A_z)}{\partial y} - \frac{\partial(A_y)}{\partial z} & \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{vmatrix}$$

$$\Rightarrow \hat{i}\left(\frac{\partial^2 A_y}{\partial y\,\partial x} - \frac{\partial^2 A_x}{\partial y^2} - \left(\frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_z}{\partial z\,\partial x}\right)\right]$$

$$-\hat{j}\left(\frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_x}{\partial x\,\partial y} - \left[\frac{\partial^2 A_z}{\partial z\,\partial y} - \frac{\partial^2 A_y}{\partial z^2}\right]\right)$$

$$+\hat{k}\left(\frac{\partial^2 A_x}{\partial x\,\partial z} - \frac{\partial^2 A_z}{\partial x^2} - \left[\frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_y}{\partial y\,\partial z}\right]\right)$$

~~RHS:~~ $\cancel{\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$

$$\cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\,\partial y} + \frac{\partial^2 A_z}{\partial x\,\partial z} + \right.}$$

$$\Rightarrow \left[\left(\frac{\partial^2 V_y}{\partial y\,\partial x} + \frac{\partial^2 V_z}{\partial z\,\partial x}\right) - \left(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right)\right]\hat{i}$$

$$+\left[\left(\frac{\partial^2 V_x}{\partial x\,\partial y} + \frac{\partial^2 V_z}{\partial z\,\partial y}\right) - \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial z^2}\right)\right]\hat{j}$$

$$+\left[\left(\frac{\partial^2 V_x}{\partial x\,\partial z} + \frac{\partial^2 V_y}{\partial y\,\partial z}\right) - \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2}\right)\right]\hat{k}$$

$$= \left(\left(\frac{d^2 V_x}{dx^2} + \frac{d^2 V_y}{dy\,dx} + \frac{d^2 V_z}{dz\,dx}\right) - \left(\frac{d^2 V_x}{dx^2} + \frac{d^2 V_y}{dy^2} + \frac{d^2 V_z}{dz^2}\right)\right] \hat{i}$$

$$+ \left(\left[\frac{d^2 V_x}{dx\,dy} + \frac{d^2 V_y}{dy^2} + \frac{d^2 V_z}{dz\,dy}\right] - \left(\frac{dV_y{}^2}{dx^2} + \frac{dV_y{}^2}{dy^2} + \frac{dV_y{}^2}{dz^2}\right)\right] \hat{j}$$

$$+ \left(\left[\frac{d^2 V_x}{dx\,dz} + \frac{d^2 V_y}{dy\,dz} + \frac{d^2 V_z}{dz^2}\right] - \left(\frac{d^2 V_z}{dx^2} + \frac{d^2 V_z}{dy^2} + \frac{d^2 V_z}{dz^2}\right)\right] \hat{k}$$

$$= \left(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}\right) \cdot \left(\frac{dV_x}{dx} + \frac{dV_y}{dy} + \frac{dV_z}{dz}\right)$$

$$- \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$\Rightarrow \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Hence, Proved.

Q4 Given: A complex number z.

To Prove: $\lim_{z \to \infty} \frac{2z^3-1}{z^2+1} = \infty$

Proof: Let $f(z) = \frac{2z^3-1}{z^2+1}$

If $\lim_{z \to \infty} f(z) = \infty$

then $\lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$

$f(\frac{1}{z}) = \frac{\frac{2}{z^3} - 1}{\frac{1}{z^2} + 1}$

$= \frac{2-z^3}{z+z^3}$

we need to prove that $\lim_{z \to 0} \frac{z+z^3}{2-z^3} = 0$

Let $z = x+iy$

$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{z+z^3}{2-z^3}$

we do this by approaching above eqn

1. $x=0$ and $y \to 0$ (i.e. by imaginary axis)
2. $x \to 0$ and $y=0$ (i.e. by real axis)

$(y^3 - 3x^2y)$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x+iy+(x+iy)^3}{2-(x+iy)^3}$$

(1) $$\lim_{y\to 0} \frac{+iy+(iy)^3}{2-(iy)^3}$$

$$\lim_{y\to 0} \frac{iy-iy^3}{2+iy^3} = \frac{\to 0}{\to 2}$$

$= 0$

(2) $$\lim_{\substack{x\to 0\\ y=0}} \frac{x+iy+(x+iy)^3}{2-(x+iy)^3}$$

$$\lim_{x\to 0} \frac{x+x^3}{2-x^3} \Rightarrow \frac{\to 0}{\to 2}$$

$= 0$

Since $\lim_{z \to 0} \dfrac{1}{f(1/z)} = 0$

$\therefore \lim_{z \to \infty} f(z) = \infty$

Hence, Proved.

Q5 Given: $f(z) = \sqrt{r}\, e^{i\theta/2}$

To show: Derivative is $f'(z) = \dfrac{1}{2f(z)}$

Shown: $f(z) = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\theta/2\right)$

Using Cauchy Reiman condition in polar form.

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$u = \sqrt{r}\cos\theta/2 \qquad v = \sqrt{r}\sin\theta/2$$

$$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\theta/2 \qquad r\frac{\partial u}{\partial r} = \frac{\sqrt{r}}{2}\cos(\theta/2) \quad -1$$

$$\frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2}\cos\theta/2 \quad -2$$

$$1 = 2$$

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}\sin\theta/2}{2} \quad -3$$

$$\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\theta/2 \qquad -\frac{r\,\partial v}{\partial r} = -\frac{\sqrt{r}}{2}\sin\theta/2 \quad 4$$

$$3 = 4$$

Hence Cauchy Reiman condition satisfy. Derivative exists.

Let $z = re^{i\theta}$

$f(z) = \sqrt{r}\, e^{i\theta/2}$

$\ln(f(z)) = \frac{\ln r}{2} + \frac{i\theta}{2}$

$= \frac{1}{2} \ln(re^{i\theta})$

$\ln f(z) = \frac{1}{2} \ln z$

$f(z) = z^{1/2}$

$f'(z) = \frac{1}{2 z^{1/2}}$

$= \frac{1}{2 f(z)}$

Hence, showed.

Q6

Given: $f(z) = e^{-\theta}(\cos\ln r + i\sin\ln r)$ $r>0$

To Shown: $f'(z) = \dfrac{i\,f(z)}{z}$

Show: Using Cauchy Reimann conditions in polar form

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$u = e^{-\theta}\cos\ln r \qquad v = e^{-\theta}\sin\ln r$$

$$\frac{\partial u}{\partial r} = -e^{-\theta}\sin\ln r\,\frac{1}{r} \qquad \frac{\partial v}{\partial \theta} = -e^{-\theta}\sin\ln r \quad -(1)$$

$$r\frac{\partial u}{\partial r} = -e^{-\theta}\sin\ln r \quad -(2)$$

(1) = (2)

$$\frac{\partial u}{\partial \theta} = -e^{-\theta}\cos\ln r \quad -(3) \qquad \frac{\partial v}{\partial r} = e^{-\theta}\cos(\ln r)\frac{1}{r}$$

$$-r\frac{\partial v}{\partial r} = -e^{-\theta}\cos(\ln r) \quad -(4)$$

(3) = (4)

Hence, Cauchy Reimann conditions satisfy & Hence, Derivative exists.

$$f(z) = e^{-\theta} e^{i \ln r}$$

$$f(z) = e^{i \ln r - \theta}$$

Take log both sides

$$\log f(z) = i \ln r - \theta$$

Differentiate with respect to z.

$$\frac{1}{f(z)} f'(z) = i \frac{1}{r} \frac{dr}{dz} - \frac{d\theta}{dz} \quad -①$$

we know that $z = re^{i\theta}$

Differentiate w.r.t. to z both sides

$$1 = e^{i\theta} \frac{dr}{dz} + e^{i\theta} ri \frac{d\theta}{dz}$$

$$e^{-i\theta} = \frac{dr}{dz} + i \frac{d\theta}{dz} r$$

Multiply (i) both sides & Divide by r

$$\frac{i e^{-i\theta}}{r} = \frac{i}{r} \frac{dr}{dz} - \frac{d\theta}{dz}$$

Substituting in ①

$$\frac{i}{re^{i\theta}} = \frac{f'(z)}{f(z)}$$

$$\frac{i f(z)}{z} = f'(z)$$

Hence, showed.