RA Assignment (2022101070)

Q1 Given: A and B are bounded sets of R.

To Show:→ $A \cup B$ is a bounded set.

$$\sup(A \cup B) = \sup(\sup(A), \sup(B))$$

Proof:→ A and B are bounded sets

A has greatest lower bound (inf)

A has lowest upper bound (sup)

B has greatest lower bound (inf)

B has lowest upper bound (sup)

Let $\sup A = a_1$ $\quad$ $\sup B = b_1$

$\inf A = a_2$ $\quad$ $\inf B = b_2$

Let $\max(a_1, b_1) = l$ $\quad$ $\min(a_2, b_2) = m$

$\sup(A \cup B) = \max(a_1, b_1) = l$

$\inf(A \cup B) = \min(a_2, b_2) = m$

Since $(A \cup B)$ has both greatest lower bound and lower upper bound.

$A \cup B$ is a bounded set.

Since $\max(a_1, b_1) = \sup(a_1, b_1)$

as sup of 2 elements is similar to max.

So $\sup(A \cup B) = \sup(\sup A, \sup B)$.

Hence, Proved.

Q2) Given: A Non empty set $S \subset R$

To Prove:→ Non empty set finite $S \subset R$ contains its supremum.

Proof:→ We will prove this using mathematical induction.

For $n = 1$

The supremum is the element itself.

Assume that the finite set S has a supremum for 'n' elements.

Let the supremum be x

For $(n+1)$ elements set the supremum will be x or the new element that is added.

So the supremum is contained in the set itself.

Hence, Proved.

Q3 (a) <u>Given:</u> A set S, $a > 0$, A set aS.

<u>To Prove:</u> $\inf(aS) = a \inf(S)$.
$\sup(aS) = a \sup(S)$

<u>Proof:</u> Let $x \in S$

$\inf(S) \leq x \leq \sup(S)$

Multiply by 'a'

$a \inf(S) \leq ax \leq a \sup(S)$

$a \inf(S) \leq aS \leq a \sup(S)$ — (1)

From (1), we can conclude that:

$\inf(aS) = a \inf(S)$

$\sup(aS) = a \sup(S)$.

(b) <u>Given:</u> A set S, $b<0$, A set bS.

<u>To Prove:</u> $\sup(bS) = b\inf(S)$

$\inf(bS) = b\sup(S)$.

<u>Proof:</u> Let $x \in S$

$\inf(S) \leq x \leq \sup(S)$

Multiply both sides by b

$b\inf(S) \geq bx \geq b\sup(S)$ — (1)

Sign changes as $b<0$

~~Sup~~ From (1), we can conclude that :

$\inf(bS) = b\sup(S)$

$\sup(bS) = b\inf(S)$.

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Q4 Given: x, y be 2 elements of group G.

To Prove: $(xy)^{-1} = y^{-1}x^{-1}$

$(x^{-1})^{-1} = x$

Proof:- Using the axiom of group that inverse of the element of group exists.

we can conclude that:-

$x.x^{-1} =$ Identity — (1)

$y.y^{-1} =$ Identity — (2)

$(xy).(xy)^{-1} =$ Identity — (3)

Pre multiply (3) with x^{-1}

$x^{-1}xy.(xy)^{-1} = x^{-1}$ (Identity)

Using (1)

$y.(xy)^{-1} = x^{-1}$ — (4)

Pre multiply (4) with y^{-1}

$y^{-1}y(xy)^{-1} = y^{-1}x^{-1}$

Using (2)

$(xy)^{-1} = y^{-1}x^{-1}$

$x.x^{-1} =$ Identity — (5)

$x^{-1}.(x^{-1})^{-1} =$ Identity (6)

Pre multiply (6) with x

$xx^{-1}.(x^{-1})^{-1} = x$ (Identity)

$(x^{-1})^{-1} = x$

Hence, Proved.

Q5 (a) <u>Given</u>:→ A ring with identity whose all non zero elements are invertible and is commutative w.r.t to multiplication.

<u>To Prove</u>:→ Ring is a field.

Proof:- A ring becomes a field when it is commutative under multiplication and all elements of the ring are invertible s.t.

$x \cdot x^{-1} = 1$

Ring Axioms:-

1. $\forall a, b \in R \quad a+b \in R$
2. $\forall a, b, c \in R \quad (a+b)+c = a+(b+c)$
3. $a+b = b+a \quad \forall a, b \in R$
4. Identity (0) s.t. $a+0 = 0+a = a$
5. For each $a \in R$, $\exists (-a)$ [Additive Inverse] s.t. $a+(-a) = 0$
6. $a \cdot b \in R \quad \forall a, b \in R$
7. $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in R$
8. $a \cdot (b+c) = a \cdot b + a \cdot c$

Field Axioms:-

1. $x+y = y+x \quad \forall x, y \in F$
2. $x \cdot y = y \cdot x \quad \forall x, y \in F$
3. $x \cdot (y+z) = x \cdot y + x \cdot z \quad \forall x, y, z \in F$
4. $\exists\ 0 \in F$ s.t. $x+0 = x \quad \forall x \in F$
5. $x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \forall x, y, z \in F$
6. For each $x \in F$ $\exists -x \in F$ s.t. $x+(-x) = 0$
7. $\exists\ 1 \in F$ with $1 \neq 0$ satisfying $1 \cdot x = x \quad \forall x \in F$
8. For each $x \neq 0$, $\exists\ x^{-1}$ s.t. $x \cdot x^{-1} = 1$
9. $\forall a, b \in F \quad a+b \in F$
10. $\forall a, b \in F \quad a \cdot b \in F$

Since the remaining 2 axioms of ring to become a field is clearly stated

Axioms:→ (that are needed for ring to become field)

① Ring should be commutative under (·) which is clearly given

② The elements should be invertible of ring then it become field which is also clearly given.

∴ Ring ~~becomes~~ is a field.

Hence, Proved.

Q3 (b)

Given :→ A = Set of all $N \times N$ sized matrices with elements as real Numbers ($\mathbb{R}$).

To Prove :→ The set A is not a field.

Proof :→ Set A has matrices that have elements whose determinant is ~~not~~ equal to 0.

Let $x \in A$ → For eg matrices where all elements are equal.

$|x| = 0$ for some x.

~~Then~~ Since the determinant is 0 this matrix inverse does not exist.

∴ This violates the axiom of existence of unique inverse $\forall x$ $x \neq 0$ $x \cdot x^{-1} =$ Identity

Hence, the set is not a field.

Hence, Proved.

It has other matrices where $|x| = 0$ for $x \in A$ where x is a matrix in which all the elements are same. Hence x is a singular matrix.

Q5 (2) Given :→ Set A of even integers.

To Prove :→ set A is ~~field~~ ring.

Proof :→ Let 3 even integers be '2k', '2l', '2m'
$\forall (k, l, m \in N)$

(1) Sum $= 2k + 2l = 2(k+l)$ $\forall k, l \in N$,
Sum is also even. (So set is closed).
(A1) satisfies.

(2) Sum $= (2k + 2l) + 2m = 2l + (2k + 2m)$
Sum is also associative
(A2) satisfied.

(3) Sum $= (2k + 2l) = (2l + 2k)$
Sum is also commutative
(A3) satisfied.

(4) $0 \in A$ as it is even.
$2k + 0 = 0 + 2k = 2k$
Additive identity exists.
(A4) satisfied.

(5) $2k \in A$, then $-(2k) \in A$ $\forall k \in N$
as $-(2k)$ is also an even integer.
$\therefore 2k + (-2k) = 0$
(A5) satisfied.

(6) Multiplication = $2k \cdot 2l = 4kl$ which is also ...

$4kl \in A$ $\forall (k, l \in A)$

So set is closed under Multiplication.

(A6) satisfied.

(7) Product = $(2k \cdot 2l)\, 2m = 2k\, (2l \cdot 2m)$

LHS $\Rightarrow$ $(4kl)\, 2m = 8klm$

RHS $\Rightarrow$ $2k\,(4lm) = 8klm$

LHS = RHS

Hence both are equal.

Multiplication is associative.

(A7) satisfied

(8) $2k(2l + 2m) = 4kl + 4km$

LHS $\Rightarrow$ $2k(2)[l + m]$

$\Rightarrow 4k(l+m)$

$\Rightarrow 4kl + 4km$

LHS = RHS

Multiplication is distributive over Addition.

(A8) Satisfied.

Since it satisfies all axioms the set A is ring.

Hence, Proved.

2 is singular as its inverse does not exist.

Q6 Given :→ $I_m = \{0, 1, 2 \ldots m-1\}$, $m > 0$

Two numbers $a, b \in I_m$

Sum = $a+b = (a+b) \% m$

Product = $a \cdot b = (ab) \% m$

Remainder after division by m of ordinary sum and product.

To Prove :→ I_m with these operations is a ring.

Proof :→ ① Axiom 1 : Remainder < Divisor.

So remainder is always less than m

Therefore sum is closed in set I_m

~~If sum > divisor, then we can subtract (divisor) till new remainder > 0~~

Axiom 2: (Associative under addn)

To Prove:

$$(a+b)+c = a+(b+c)$$

$$((a+b) \bmod n + c) \bmod n = (a + (b+c) \bmod n) \bmod n$$

LHS ord

Proof: Case I: $0 \le a+b < m$

$(a+b) \bmod (m) = a+b$

as $a+b < m$

$[a+b+c] \bmod (m)$

Case II: $m \le a+b \le 2m-2$

$(a+b) \bmod (m) = a+b-m$

as $m \le a+b \le 2m-2$

$(a+b-m+c) \bmod m$

$(a+b+c \cancel{-m+m}) \bmod n$

$[a+b+c] \bmod (m)$

we can add m as mod(m) does not changes the soln

Thus in all cases LHS: $[a+b+c] \bmod (m)$

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RHS: $a+(b+c) = [a+(b+c) \bmod (m)] \bmod m$

Case I: $0 \le b+c < m$

$(b+c) \bmod (m) = b+c$

as $0 \le b+c \le m$

$[a+b+c] \bmod (m)$

Case II: $m \le b+c \le 2m-2$

$(b+c) \bmod (m) = b+c-m$

as $m \le b+c \le 2m-2$

$[a+b+c-m] \bmod (m)$

$[a+b+c-\cancel{m}+\cancel{m}] \bmod (m)$

$[a+b+c] \bmod (m)$

we can add m

as $m \bmod (m) \equiv 0$

Thus for RHS every case $= [a+b+c] \bmod (m)$

LHS = RHS for every case

Hence axiom 2 is true.

③ <u>Axiom 3:</u> → $a+b = (a+b) \bmod (m)$ — ①

$b+a = (b+a) \bmod (m)$ — ②

As ordinary sum is commutative

① = ②

④ <u>Axiom 4:</u> → $a+0 = (a+0) \bmod (m) = (a) \bmod (m) = a$

As ordinary addn $(a+0) = a$

Axiom 5:- $a+b=0$

then b is additive inverse of a

If $a+b=0$, then $(a+b) \bmod (m) = 0$

This means ord. sum $(a+b)$ is a multiple of m

This also implies that ord. $(a+b) = m$ as max value of ord. $(a+b) = 2m-2$. ~~So b or additive inverse~~

So b or additive inverse of $a = m-a$ for $a \neq 0$.

for $a=0$, additive inverse is 0.

Axiom 6:- Remainder < Divisor. If Remainder > Divisor ~~Rx / great~~ we can just subtract k(Divisor), $k \in N$ till new remainder > 0.

Axiom 7:- $ab = (ab) \bmod (m)$

In our question $c = c \bmod m$ as $0 \leq c \leq m-1$

$(ab).c = ((ab) \bmod m \times c) \bmod m$

~~$= ((a \bmod m \cdot b \bmod m) \bmod m \times c) \bmod m$~~

using property $= ((ab) \bmod m \cdot (\bmod m)) \bmod m$

$= (abc) \bmod m$ — ①

$(ab) \bmod m = (a \bmod m \; b \bmod m) \bmod m$. (Proof at the end)

$a(b.c) \Rightarrow (a \times (bc) \bmod m) \bmod m$

$\Rightarrow (a \bmod m \times (bc) \bmod m) \bmod m$

$\Rightarrow (abc) \bmod m$ — ②

① = ② Hence, Proved.

Multiplication is associative.

(8) Axiom 8: To Prove: $x(y+z) = xy + xz$

LHS: $(x(y+z) \bmod m) \bmod m$

Case I: $0 \le y+z < m$

$(y+z) \bmod m = y+z$

as $0 \le y+z < m$

$\Rightarrow (xy + xz) \bmod m$

$\Rightarrow (xy + xz) \bmod m$

Case II: $m \le y+z \le 2m-2$

$(y+z) \bmod m = y+z-m$

as $m \le y+z \le 2m-2$

$(x(y+z-m)) \bmod m$

$(xy + xz - xm) \bmod m$

$(xy + xz \cancel{- xm + xm}) \bmod (m)$

we can add xm as

$(xm) \bmod m = 0$

$\Rightarrow (xy + xz) \bmod (m)$

RHS: $[(xy) \bmod m + (xz) \bmod m] \bmod m$

$a.b + a.c$

$\Rightarrow [xy - km + xz - lm] \bmod m$

s.t. $k, l \in N$ & $a.b - kn$ & $a.c - lm \in [0, m]$

$\Rightarrow [xy + xz - (k+l)m + (k+l)m] \bmod (m)$

as $m(k+l) \bmod m = 0$

$\Rightarrow [xy + xz] \bmod (m)$

Hence, Proved.

Since it satisfies all axioms of ring.

I_m is a ring.

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To Prove:

$(ab) \bmod m = (a \bmod m \; b \bmod m) \bmod m.$

Proof: let $a = k_1 m + r_1$

$b = k_2 m + r_2$

LHS: $ab = k_1 k_2 m^2 + (k_1 + k_2) m + r_1 r_2$

$(ab) \bmod m = (r_1 r_2) \bmod m$

RHS: $a \bmod m = r_1$

$b \bmod m = r_2$

$\Rightarrow (r_1 r_2) \bmod m$

LHS = RHS

Hence, Proved.

Q7

Given:- Sequence $\{a_n\}_{n\geq 0}^{\infty}$ where $a_n = \frac{1}{n}$

To Prove:- $\lim_{n\to\infty} a_n = 0$

Proof:- Defination of limit of a sequence is

$\forall \varepsilon > 0$, $\exists\, n_0$ s.t. $|a_n - L| < \varepsilon$ $\forall\, n > n_0$

where $L = \lim_{n\to\infty} a_n$

Using Archimedian property we can say that

$\exists$ some $n_0 \in N$ $\forall$ $n_0 > \frac{1}{\varepsilon}$ and $\forall\, n > n_0$

$$\frac{1}{n_0} < \varepsilon \quad -(1)$$

$\forall\, \varepsilon > 0$ & $n_0 \in N$ $\forall$ $n \geq n_0$

$$\frac{1}{n_0} \geq \frac{1}{n} \quad -(2)$$

$\therefore$ Using (1) & (2)

$$\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$$

Thus we can say that:

$$\frac{1}{n} < \varepsilon$$

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

Comparing with $|a_n - L| < \varepsilon$

$$L = 0$$

Hence Proved.

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Given :- $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are 2 convergent sequences i.e. their limits exist.

To Prove :- $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

Proof :- Let $\lim_{n\to\infty} a_n = L_1$,

$\lim_{n\to\infty} b_n = L_2$

Using defination of limit :-

$\forall \varepsilon > 0$, $\exists\, n_0$ $|a_n - L_1| < \varepsilon$ ① $\forall$ $n > n_0$

$\forall \varepsilon > 0$, $\exists\, n_0'$ $|b_n - L_2| < \varepsilon$ ② $\forall$ $n > n_0'$

① + ②

$(a_n - L_1) + |b_n - L_2| < 2\varepsilon$

let 2ε be ε'

$|a_n - L_1| + |b_n - L_2| < \varepsilon'$

Using triangular's inequality

$|a_n + b_n - (L_1 + L_2)| < \varepsilon'$

Comparing it with defination of limit :-

$\lim_{n\to\infty}(a_n + b_n) = L_1 + L_2$

Hence, Proved.

09

Given: Given $\{x_n\}$ of a sequence of $\mathbb{R}$

To Prove: $x \in \mathbb{R}$ is a limit pt. of $\{x_n\}$ iff a subsequence $\{x_{k_n}\}$ of x_n s.t. $\lim x_n = x$

Proof: Since it is an iff statement we proceed in 2 steps

① **To Prove:** If a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ converges to x then it is a limit pt.

Proof: Def^n of convergent seq.

$\forall \varepsilon > 0 \ \exists \ n_0 \in \mathbb{N}$ s.t. $n > n_0 \ |x_n - x| < \varepsilon$

Since x_{k_n} is a subsequence ~~the~~ it is strictly increasing in nature.

Def^n. of limit pt. of seq.

$\forall \varepsilon > 0 \quad \forall n \in \mathbb{N}, \quad \exists k > n$ s.t. $|x_k - x| < \varepsilon$

So, we have to show that for any (m, ε) we can find $p > m$ s.t. $|x_p - x| < \varepsilon$

For a fixed ε, we are guaranteed the existence of n_0 s.t. $|x_n - x| < \varepsilon \ \forall n > n_0$

Since $\{k_1, k_2 \ldots k_n\}$ is increasing strictly

$\exists k_s \in \{k_1 \ldots k_n\}$ s.t. $k_s > n_0$

i.e. $|x_{k_s} - x| < \varepsilon$ for our chosen ε

Now, consider any $s > \max\{m, n_0\}$

$x_{k_s} = x_p$

$p > m$, $p > n_0$ and $|x_p - x| < \varepsilon$ by defn.

Hence, Proved.

(2) To Prove: If x is a limit pt, $\exists$ a convergent subsequence $\{x_{k_n}\}$ of $\{x_n\}$ which converges to x.

Proof: Since we need to prove existence of a subsequence, we can just construct a single subsequence for any sequence $\{x_n\}$

By proof by mathematical induction. For the first element we choose an index k_1 in $\{x_n\}$ s.t. $|x_{k_1} - x| < 1$

If $k_1 \ldots k_n$ are selected already, we choose an index k_{n+1} of $\{x_n\}$ s.t. $k_{n+1} > k_n$.

This selection is guaranteed to exist by defn of limit pt.

$$|x_{k_{n+1}} - x| < \frac{1}{n+1}$$

~~As we ch~~ i.e. $\forall \varepsilon > 0$, $\exists k_n$ s.t.

$\forall k_{n+i}$ $i > 1$

$$|x_{k_{n+i}} - x| < \varepsilon$$

(For any ε, we choose $n = \frac{1}{\varepsilon} + 1$, $\frac{1}{n} < \varepsilon$)

Thus we have constructed a subsequence which converges to x.

Q10

Given: A bounded seq $\{x_n\}$ of $\mathbb{R}$.

To Prove: $\{x_n\}$ converges iff. $\lim\inf x_n = \lim\sup x = \lim x_n$
$= x$.

Proof: ① **Given:** $\lim x_n$ converges to x

To Prove: $\lim\sup x_n = \lim\inf x_n = x = \lim x_n$

Proof: From def^n of convergence

$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N}$ s.t. $|x_n - x| < \varepsilon \ \forall n \geq n_0$

$-\varepsilon < (x_n - x) < \varepsilon$

$x - \varepsilon < x_n < x + \varepsilon$

$x - \varepsilon$ and $x + \varepsilon$ have become arbitrarily lower and upper bounds for term x_n.

$\Rightarrow x - \varepsilon \leq \inf_{k \geq n}^{\infty} x_k$ $\qquad$ $x + \varepsilon \geq \sup_{k \geq n}^{\infty} x_k$

$\Rightarrow x - \varepsilon \leq \inf_{k \geq n}^{\infty} x_k < \sup_{k \geq n}^{\infty} x_k \leq x + \varepsilon$

$\Rightarrow \left| x - \inf_{k \geq n}^{\infty} x_k \right| < \varepsilon$ $\qquad$ $\left| x - \sup_{k \geq n}^{\infty} x_k \right| < \varepsilon$

This is valid $\forall \varepsilon > 0$ since steps follow from defn of convergence.

$\therefore$ $\inf_{k \geq n}^{\infty} x_k$ converges to x & $\sup_{k \geq n}^{\infty} x_k$ converges

$\lim(\inf) = \lim(\sup) = x$

Hence, proved.

(2) Given: $\limsup x_n = x = \liminf x_n$

To Prove: $\lim x_n = x$ converges.

Proof: Let a k-tail of a sequence represent the seq. $\{x_k, x_{k+1} \ldots \infty\}$ i.e. it consists of all the terms of the sequence from the term k onwards.

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LEMMA: $\forall k$ $\quad x_k - \liminf \leq \bigvee_{l \geq k}^{\infty} x_k - \bigwedge_{l \geq k}^{\infty} x_k$ &

$$\limsup - x_k \leq \bigvee_{l \geq k}^{\infty} x_l - \bigwedge_{l \geq k}^{\infty} x_l$$

Proof: $\limsup x_n = \bigwedge_{n \geq 1}^{\infty} \left[\bigvee_{k \geq n}^{\infty} x_k \right]$, $\liminf x_n = \bigvee_{n \geq 1}^{\infty} \left[\bigwedge_{k \geq n}^{\infty} x_k \right]$

$\forall n \in \mathbb{N}$ $\quad \bigwedge_{k \geq n}^{\infty} x_k \leq x_n \leq \bigvee_{k \geq n}^{\infty} x_k$ — (1)

$\liminf x_n \geq \bigwedge_{k \geq n}^{\infty} x_k$ — (2) $\qquad \limsup x_n \leq \bigvee_{k \geq n}^{\infty} x_k$ — (3)

Using (1) & (2) & (1) & (3) we prove lemma.

However we are given that:

$\limsup x_n = \liminf x_n = x$

Hence, $x_k - x \leq \bigvee_{k \geq n}^{\infty} x_k - \bigwedge_{k \geq n}^{\infty} x_k$ & $x - x_k \leq \bigvee_{k \geq n}^{\infty} x_k - \bigwedge_{k \geq n}^{\infty} x_k$

$\Rightarrow |x_k - x| \leq \bigvee_{k \geq n}^{\infty} x_k - \bigwedge_{k \geq n}^{\infty} x_k$, However

$\lim_{n \to \infty} \left[\bigvee_{k \geq n}^{\infty} x_k \right] = x$ & $\lim_{n \to \infty} \left[\bigwedge_{k \geq n}^{\infty} x_k \right] = x$

by defn of convergence.

So. $|x_k - x| \to 0$ as $n \to \infty$

$\lim x_n = x$

Hence, Proved.