Assignment - 4

Q1 (a) We will form general term

$$Sum = \sum a_r = \sum\left(\frac{1}{2^r} + \frac{1}{3^r}\right)$$

Using Ratio test :

$$l = \lim_{x \to \infty} \left(\frac{\frac{1}{2^{r+1}} + \frac{1}{3^{r+1}}}{\frac{1}{2^r} + \frac{1}{3^r}} \right)$$

$$l = \lim_{x \to \infty} \frac{3^{r+1} + 2^{r+1} / 6 \times \cancel{3^r \cdot 2^r}}{\frac{3^r + 2^r}{3^r \cdot \cancel{2^r}}}$$

$$l = \lim_{x \to \infty} \frac{\frac{3^r}{2} + \frac{2^r}{3}}{3^r + 2^r}$$

Divide N and D by 3^r

$$l = \lim_{x \to \infty} \left(\frac{\frac{1}{2} + \frac{1}{3} \times \left(\frac{2}{3}\right)^r}{1 + \left(\frac{2}{3}\right)^r} \right)$$

as $r \to \infty$ $\left(\frac{2}{3}\right)^r \to 0$

$l = \frac{1}{2}$

as $l < 1$, the ~~that~~ $\sum a_n$ converges.

Q1 (a) $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2} \dots \infty$

Using root test in $\sum \frac{1}{2^k}+\frac{1}{3^k}$

$$\alpha = \lim_{k\to\infty}\left(\frac{1}{2^k}+\frac{1}{3^k}\right)^{1/k}$$

$$\alpha = \lim_{k\to\infty}\left(\frac{3^k+2^k}{6^k}\right)^{1/k}$$

$$\alpha = \lim_{k\to\infty}\frac{(3^k+2^k)^{1/k}}{6}$$

$$\alpha = \frac{1}{6}\lim_{k\to\infty}\left[3^k\left(1+\left(\frac{2}{3}\right)^k\right)\right]^{1/k}$$

$$\alpha = \frac{3}{6}\lim_{k\to\infty}\left[1+\left(\frac{2}{3}\right)^k\right]^{1/k}$$

as $k\to\infty$ $\left(\frac{2}{3}\right)^k \to 0$ and $1+\left(\frac{2}{3}\right)^k \to 1$

so series $(\to 1)^{\to 0} \Rightarrow 1$

$$\alpha = \frac{1}{2}$$

as $\alpha < 1$ the series converges.

(b) This sum general term is :

$$\text{sum} = \sum_{r=0}^{\infty} a_r = \sum_{r=0}^{\infty} \frac{1}{2^r}$$

Using Root test :

$$\alpha = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

$$\alpha = \lim_{r \to \infty} \left(\frac{1}{2^r}\right)^{\frac{1}{r}}$$

$$\alpha = \lim_{r \to \infty} \frac{1}{2^{r/r}}$$

$$\alpha = \frac{1}{2}$$

as $\alpha < 1$, $\therefore \sum a_n$ converges .

The series $\sum \frac{1}{2^n}$

Using ratio test:

$$\alpha = \frac{1}{2^{n+1}\left(\frac{1}{2}\right)^n} = \frac{1}{2}$$

Since $\alpha < 1$, the series converges.

Hence, the series converges.

Q2

Given: A metric space (X, d)

To Prove:- ① X and ϕ are closed sets.

② Arbitrary intersection of closed sets are closed

③ Finite union of closed sets is closed.

Proof:- (i) A set A is closed set if

$A^c = X / \{A\}$ is an open set.

$X^c = \phi$ for the metric space (X, d)

$\phi^c = X$ for the metric space (X, d)

~~Since ϕ has no element so open ball around it is not defined.~~

~~∴ ϕ is open set. This implies X is a closed set~~

Since ϕ has no element, $\nexists$ no x in ϕ in which open ball exists. So ϕ is open set.

∴ X is a closed set.

Also $\forall x \in X$, $\exists B(x, r) \subseteq X$, Hence X is open set

∴ ϕ is a closed set.

To

(2) Given :- $A_1, A_2, A_3 \ldots$ be closed sets.

To Prove :- $A_1 \cap A_2 \cap A_3 \ldots$ be closed sets.

Proof :- $A_1, A_2, A_3 \ldots$ are closed sets.

$A_1^c, A_2^c, A_3^c \ldots$ are open sets.

Since $A_1^c, A_2^c, A_3^c \ldots$ are open sets the following ~~sets have~~ sets ~~sets~~ $x \in A$, $\exists$ some $r>0$ s.t. $B(x,r) \subseteq A$.

$B_1(x_1, r_1) \subseteq A_1^c$

$B_2(x_2, r_2) \subseteq A_2^c$

$B_3(x_3, r_3) \subseteq A_3^c$

$\vdots$

Union all of these sets.

$\underbrace{B_1(x_1, r_1) \cup B_2(x_2, r_2) \ldots}_{\text{we can write this as } B(x,r)} \subseteq A_1^c \cup A_2^c \cup A_3^c \ldots$

$B(x,r) \subseteq (A_1 \cap A_2 \cap A_3 \ldots)^c$ → [DeMorgan's theorem]

Since $B(x,r) \subseteq (A_1 \cap A_2 \cap A_3 \ldots)^c$ so set $A_1 \cap A_2 \cap A_3 \ldots$ is closed.

Hence, Proved.

(3) Given: $A_1, A_2, A_3 \ldots A_n$ are closed sets.

To Prove: $A_1 \cup A_2 \cup A_3 \ldots \cup A_n$ are closed sets.

Finite unions of closed sets are closed.

Proof:- ~~To A~~ $\bigcup_{i=1}^{n} A_i$ ~~are~~ is a closed set.

The above is a closed set when $\left(\bigcup_{i=1}^{n} A_i\right)^c$ is an ~~of~~ open ~~closed~~ set.

We need to prove that $\bigcap_{i=1}^{n} A_i^c$ is an open set (1)

[DeMorgan's Law]

A_i is an closed ~~open~~ set A_i^c is an open set.

$B(x, r_i) \subseteq A_i^c$

$\forall\ r_i > 0$

consider the minimum of $\{r_1, r_2, r_3 \ldots r_n\} = r$

so $B(x, r) \subseteq A_1^c \cap A_2^c \cap A_3^c \ldots A_n^c$

Hence, we proved the (1) statement.

Hence, proved that finite union of closed sets is closed.

Q3 Given: A f^n, $f : (X, d) \to (Y, \rho)$ b/w 2 metric spaces:

To Prove: Following statements are equivalent.

(1) f is continuous on X.

(2) $f^{-1}(O)$ is an open subset of X whenever O is an open subset of Y.

(3) If $\lim x_n = x$ holds in X then $f(x_n) = f(x)$ holds in Y.

(4) $f(\overline{A}) \subseteq \overline{f(A)}$ holds for every subset A of X.

(5) $f^{-1}(C)$ is closed subset of X whenever C is closed subset of Y.

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Proof: (i) $\to$ (ii) O be open subset of Y.

$a \in f^{-1}(O)$, This implies $f(a) \in O$ and O is open.

$\exists\ r>0$, s.t. $B(f(a), r) \subseteq O$.
{ Open ball definition

Now using defination of continuity that $f:(X,d) \to (Y,\rho)$ b/w 2 metric space is continuous at pt. $a \in X$ if $\forall\ \varepsilon>0\ \exists\ \delta>0$ s.t. $\rho[f(x), f(a)] < \varepsilon$ whenever $d(a,x)<\delta$.

$\Rightarrow\ \delta>0$ s.t. $d(x,a)<\delta$

$\Rightarrow \rho(f(x), f(a)) < r$.

The above shows that $B(a,\delta) \subseteq f^{-1}(O)$.

Therefore a is an interior pt. of $f^{-1}(O)$. Hence $f^{-1}(O)$ is open.

(ii) $\to$ (iii) Let $\lim x_n = x$ in X and $r>0$.

Let $V \equiv B(f(x), r)$. By our assm $f^{-1}(V)$ is an open subset of X, since x belongs to it, $\exists\ \delta>0$, s.t. $B(x,\delta) \subseteq f^{-1}(V)$, $\exists\ k$ s.t. $x_n \in B(x,\delta)\ \forall\ n>k$.

$f(x_n) \in V\ \forall\ n>k$, which shows that $\lim f(x_n) = f(x)$

(iii) $\to$ (iv) A be a subset of X.

$y \in f(\bar{A})$, $\exists\ x \in \bar{A}$ s.t. $y = f(x)$.

Since $x \in \bar{A}$, $\exists$ a sequence $\{x_n\}$ of A s.t. $\lim x_n = x$.

But, $\{f(x_n)\}$ is a sequence of $f(A)$ & by assm

$\lim f(x_n) = f(x) = y$.

By theorem A is a subset of metric space (X,d).
Then a pt. $x \in X$ belongs to $\bar{A}$ iff $\exists$ seq. of A s.t. $\lim x_n = x$

From above theorem we can conclude that:

$y \in f(A)$

that is $f(\overline{A}) \subseteq \overline{f(A)}$

(iv) → (v) Let C be a closed subset of Y.

$C = \overline{C}$ holds in Y as it is a closed set.

Applying assumption above to set $A = f^{-1}(C)$,

we get $f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{C} = C$.

$\therefore \overline{A} \subseteq f^{-1}(C) = A$.

Since, $A \subseteq \overline{A}$ is always True, it follows $A = \overline{A}$

This shows that $A = f^{-1}(C)$ is a closed subset of X.

(v) → (i) $a \in X$ & $\varepsilon > 0$. consider closed set:

$$C = [B(f(a), \varepsilon)]^c = \{y \in Y : \rho(f(a), y) \geq \varepsilon\}$$

Set $f^{-1}(C)$ is closed subset of X.

$a \notin f^{-1}(C)$, $\exists\ \delta > 0$

s.t. $B(a, \delta) \subseteq [f^{-1}(C)]^c$.

if $d(x, a) < \delta$; then $\rho(f(x), f(a)) < \varepsilon$ holds

so f is continuous at a.

Since a is arbitrary, f is continuous at X.