RA Assignment - 5

Q1 Given: 3 unique vectors $\vec{x_1}, \vec{x_2}, \vec{x_3}$

To Prove: Vector triple product is not associative in general

i.e. $\vec{x_1} \times (\vec{x_2} \times \vec{x_3}) = (\vec{x_1} \times \vec{x_2}) \times \vec{x_3}$

Proof: We can prove this using example as given in question

Let

$\vec{x_1} = \hat{i} + \hat{j}$ $\quad \vec{x_2} = \hat{j} + \hat{k}$ $\quad \vec{x_3} = \hat{k} + 2\hat{i}$

$\vec{x_1} \times \vec{x_2} = (\hat{i} + \hat{j}) \times (\hat{j} + \hat{k}) = \hat{k} - \hat{j} + \hat{i} = \hat{i} - \hat{j} + \hat{k}$

$(\vec{x_1} \times \vec{x_2}) \times \vec{x_3} = (\hat{i} - \hat{j} + \hat{k}) \times (2\hat{i} + \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -\hat{i} + \hat{j} - \hat{k} \quad - (1)$$

$$\vec{x_2} \times \vec{x_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{x_1} \times (\vec{x_2} \times \vec{x_3}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i} - 2\hat{j} - \hat{k} \quad - (2)$$

$(1) \neq (2)$

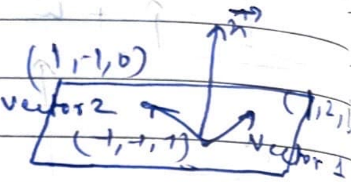
Hence vector triple product is not ~~same~~ associative in general

Hence, Proved.

Q2

Given: 3 points on Cartesian plane $(1,2,1), (-1,-1,-1), (1,-1,0)$

To find: Eqn of plane.



Soln: vector 1 on plane

$= \hat{i}+2\hat{j}+\hat{k} - [-\hat{i}-\hat{j}-\hat{k}]$

$\Rightarrow 2\hat{i}+3\hat{j}+2\hat{k}$

vector 2 on plane

$= \hat{i} - \cancel{\hat{j}} + \hat{k} - [-\hat{i} - \cancel{\hat{j}} - \hat{k}]$

$= 2\hat{i} + \hat{k}$

Perpendicular vector to the plane $(\vec{n})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \cancel{6\hat{i} + 2\hat{j} - 6\hat{k}} = 3\hat{i} + 2\hat{j} - 6\hat{k}$$

Any general pt. be (x,y,z)

Arbitrary vector on plane $= (x+1)\hat{i} + (y+1)\hat{j} + (z+1)\hat{k}$

Eqn of plane $= [(x+1)\hat{i} + (y+1)\hat{j} + (z+1)\hat{k}] \cdot [3\hat{i}+2\hat{j}-6\hat{k}] = 0$

$\Rightarrow \cancel{x+1-2-1=0}$

$\Rightarrow \cancel{\boxed{x-2=0}}$

$3x+3+2y+2-6z-6=0$

$$\boxed{3x+2y-6z=1}$$

vector form $\Rightarrow [\vec{r} - (-\hat{i}-\hat{j}-\hat{k})] \cdot \vec{n} = 0$

$\Rightarrow [\vec{r} + \hat{i}+\hat{j}+\hat{k}] \cdot [3\hat{i}+2\hat{j}-6\hat{k}] = 0$

Since we need to find it's intersection with $z=0$

2 pts having z co-ordinate $=0$ are.

$(1,-1,0)$ & $(-1,2,0)$
$\vec{a}$ $\vec{c}$

$\vec{c}-\vec{a} = -2\hat{i} + 3\hat{j} + 0\hat{k}$

$|\vec{c}-\vec{a}| = \sqrt{13}$

$$\vec{r} = \hat{i}-\hat{j}+0\hat{k} + \frac{\lambda}{\sqrt{13}}(-2\hat{i}+3\hat{j}+0\hat{k})$$

as eqn of line $\vec{r} = \vec{a} + \lambda \frac{(\vec{c}-\vec{a})}{|\vec{c}-\vec{a}|}$

λ = arbitrary constant.

Scalar form of eqn of line $\Rightarrow$ $\boxed{3x+2y=1}$

Q3 Given :- Vector $\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$, transform under rotation of co-ordinate system as:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

To find :- Conditions for direction cosines must very.

Soln :- x_1' = Multiplying matrixes on RHS

$x_1' = R_{11}x_1 + R_{12}x_2 + R_{13}x_3$

$x_2' = R_{21}x_1 + R_{22}x_2 + R_{23}x_3$

$x_3' = R_{31}x_1 + R_{32}x_2 + R_{33}x_3$

Since the magnitudes of (x_1', x_2', x_3') and (x_1, x_2, x_3) are same

$$\sqrt{(x_1')^2 + (x_2')^2 + (x_3')^2} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Squaring both sides.

$$x_1'^2 + x_2'^2 + x_3'^2 = x_1^2 + x_2^2 + x_3^2$$

$$x_1^2 + x_2^2 + x_3^2 = (R_{11}x_1 + R_{12}x_2 + R_{13}x_3)^2 + (R_{21}x_1 + R_{22}x_2 + R_{23}x_3)^2 + (R_{31}x_1 + R_{32}x_2 + R_{33}x_3)^2$$

$$x_1^2 + x_2^2 + x_3^2 = \left(\sum_{i=1}^{3} R_{i1}^2\right) x_1^2 + \left(\sum_{i=1}^{3} R_{i2}^2\right) x_2^2 + \left(\sum_{i=1}^{3} R_{i3}^2\right) x_3^2$$
$$+ (R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{32}) x_1 x_2$$
$$+ (R_{12}R_{13} + R_{22}R_{23} + R_{32}R_{33}) x_2 x_3$$
$$+ (R_{11}R_{13} + R_{21}R_{23} + R_{31}R_{33}) x_3 x_1$$

Comparing coefficients on both sides:

Condn's :-

$R_{11}^2 + R_{21}^2 + R_{31}^2 = 1$

$R_{12}^2 + R_{22}^2 + R_{32}^2 = 1$

$R_{13}^2 + R_{23}^2 + R_{33}^2 = 1$

$R_{11}R_{12} + R_{21}R_{22} + R_{31}R_{32} = 0$

$R_{12}R_{13} + R_{22}R_{23} + R_{32}R_{33} = 0$

$R_{11}R_{13} + R_{21}R_{23} + R_{31}R_{33} = 0$

Q4

Given: A force field $\vec{F} = x\sin y\,\hat{i} + y\hat{j}$ acts along the curve $\vec{r} = (1+t)\hat{i} + t^3\hat{j}$ for $t=1$ to $t=2$.

To find: work done by force field.

Soln:- Work done by force field along the curve.

$\Rightarrow W = \int \vec{F}\cdot d\vec{r}$

$\vec{r}(t) = (1+t)\hat{i} + t^3\hat{j}$; $\frac{d\vec{r}}{dt} = \hat{i} + 3t^2\hat{j}$

$\vec{F}(\vec{r}(t)) = (1+t)\sin t^3\hat{i} + t^3\hat{j}$

$W = \int_C \vec{F}\cdot d\vec{r} = \int_1^2 \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$

$= \int_1^2 ((1+t)\sin t^3\,\hat{i} + t^3\hat{j})\cdot(\hat{i} + 3t^2\hat{j})\,dt$

$= \int_1^2 ((1+t)\sin t^3 + 3t^5)\,dt$

$= \int_1^2 (1+t)\sin t^3\,dt + \int_1^2 3t^5\,dt$

$= \int_1^2 (1+t)\sin t^3\,dt + \frac{3}{6}t^6\Big|_1^2$

$= \int_1^2 (1+t)\sin t^3\,dt + \frac{1}{2}\times 63$

with using calculator.

$\Rightarrow 31.9412$

Q5 (1) Given: $\int xy^2 z \, ds$

~~Ref~~ (0) computing integration over line b/w $(0,2,1)$ and $(3,1,5)$.

$\vec{a}$ $\vec{b}$

$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$,

$\vec{r} = 0\hat{i} + 2\hat{j} + \hat{k} + \lambda(3\hat{i} - \hat{j} + 4\hat{k})$

$\left|\frac{d(\vec{r})}{d\lambda}\right| = \sqrt{26}$

we need to integrate $(0,2,1)$ to $(3,1,5)$

so $\lambda = 0$ to $\lambda = 1$

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$$\Rightarrow \int_0^1 (3\lambda)(2-\lambda)^2(1+4\lambda)\left|\frac{d\vec{r}'}{d\lambda}\right| d\lambda$$

$$\Rightarrow 3\sqrt{26}\int_0^1 (1+4\lambda)\lambda(\lambda-2)^2 \, d\lambda$$

$$\Rightarrow 3\sqrt{26}\int_0^1 (\lambda^2-4\lambda+4)(4\lambda^2+\lambda)\, d\lambda$$

$$\Rightarrow 3\sqrt{26}\int_0^1 (4\lambda^4 - 15\lambda^3 + 12\lambda^2 + 4\lambda)\, d\lambda$$

$$\Rightarrow 3\sqrt{26}\left[\frac{4}{5}\times 1 - \frac{15}{4}\times 1 + \frac{12}{3}\times 1 + \frac{4}{2}\times 1\right]$$

$$\Rightarrow 3\sqrt{26}\left[\frac{4}{5} - \frac{15}{4} + 6\right]$$

$$\Rightarrow 3\sqrt{26}\left[\frac{120-75+16}{20}\right]$$

$$\Rightarrow 3\sqrt{26}\left[\frac{61}{20}\right]$$

$$\Rightarrow \frac{\sqrt{26}\times 183}{20}$$

(b) Computing integration over line given by $\vec{r} = \hat{i}u + \hat{j}u^2 + \hat{k}u$.
for $u=1$ to $u=2$.

$$\left|\frac{d\vec{r}}{du}\right| = \sqrt{1^2+(2u)^2+1^2} = \sqrt{2+4u^2}$$

$$\Rightarrow \int_1^2 (u)(u^2)^2(u)\sqrt{2+4u^2}\,du$$

$$\Rightarrow \int_1^2 u^6\sqrt{2+4u^2}\,du$$

$\Rightarrow$ with using online calculator

Ans = 68.786