ASEN5070

Statistical Orbit determination I

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Professor George H. Born

ECNT 316

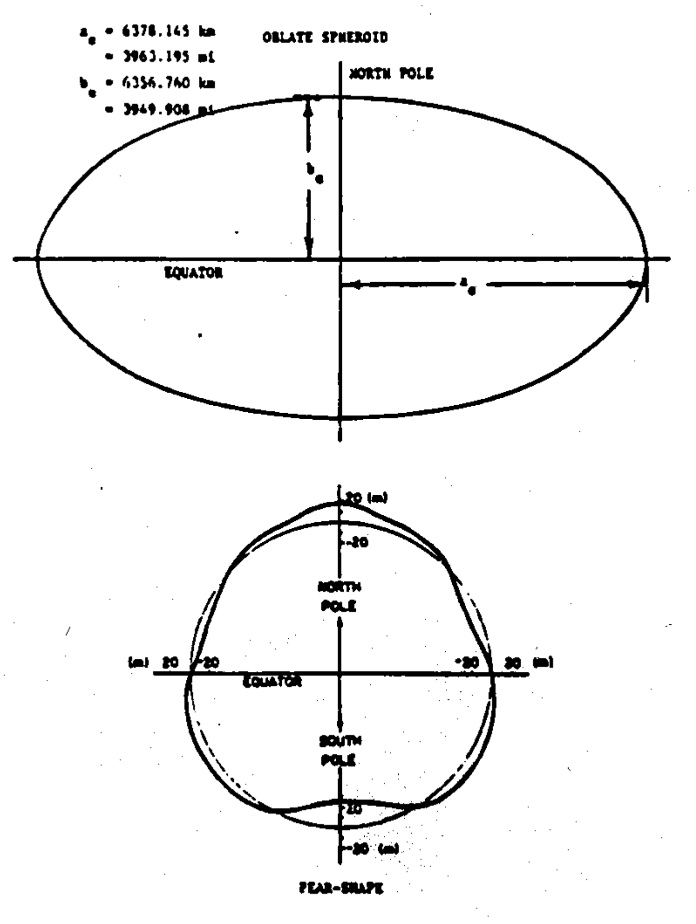
georgeb@colorado.edu

http://ccar.colorado.edu/ASEN5070/

Lecture 3&4

Colorado Center for Astrodynamics Research

Shape of Earth: J_2, J_3



- U.S. Vanguard satellite launched in 1958, used to determine J_2 and J_3
- J_2 represents most of the oblateness; J_3 represents a pear shape
- $J_2 = 1.08264 \times 10^{-3}$
- $J_3 = -2.5324 \times 10^{-6}$

Vanguard

- Determination of Earth gravity coefficients resulted from Vanguard-I (NRL project)
- First network of tracking stations, known as Minitrack, was deployed to support objectives: "determine atmospheric density and the shape of the Earth"
- To achieve objectives, all basic elements of orbit determination were involved and a state of the art IBM 704 computer was used to determine the orbit

J_2 and Orbit Design

- As altitude increases, J_2 perturbation diminishes (from a great distance the Earth is equivalent to a point mass)
- Use J_2 perturbation in orbit design, e.g., solar synchronous satellite
  - If $d\Omega/dt = +360°/365.25$ days, the line of nodes will keep a fixed (in an average sense) orientation with respect to the Earth-Sun direction
  - Must be retrograde; for 600 km altitude, i=98°

Perturbations from Spherical Harmonics

- Mean Ω, ω, M exhibit secular variation (caused by even degree J_n)
- Mean a, e, i are constant
- Odd degree J_n cause long period perturbations (period of argument of perigee motion)
- All harmonic coefficients cause short period perturbations (period is 1, ½, 1/3, etc multiple of the orbital period)
- $m \neq 0$ harmonic coefficients cause m-daily perturbations (i.e., 1, ½, 1/3, etc multiple of one day)
- Special category: resonant perturbations (e.g., geosynchronous, GPS, …)

Secular Variations

- Secular variations of Ω (positive J_2)
  - $0° < i < 90°$: $d\Omega/dt < 0$
  - $i = 90°$: $d\Omega/dt = 0$
  - $90° < i < 180°$: $d\Omega/dt > 0$
- Secular variations of ω (positive J_2)
  - $i=63.4°$ or $116.6°$, $d\omega/dt = 0$ (critical i)
  - See Table 2.3.3 for more details
- Secular variations produced by all even-degree zonal harmonics

Influence of J_2 on Satellite Motion

$$\dot{\Omega}_s = -\frac{3}{2} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \cos i$$

$$\dot{\omega}_s = \frac{3}{4} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 (5\cos^2 i - 1)$$

$$\dot{M}_s = \bar{n} + \frac{3}{4} J_2 \frac{n}{(1-e^2)^{3/2}} \left(\frac{a_e}{a}\right)^2 (3\cos^2 i - 1) \,.$$

$$a(t) = \bar{a} + 3\bar{n}\,\bar{a} J_2 \left(\frac{a_e}{\bar{a}}\right)^2 \sin^2 \bar{i} \frac{\cos(2\omega + 2M)}{2\dot{\omega}_s + 2\dot{M}_s}$$

- Oblateness produces linear (secular) changes in Ω, ω, M
- Periodic variations in all elements; e.g., semimajor axis exhibits a twice per orbital revolution variation
- Approximate equations for variations in semimajor axis shown at left

Example

- GPS known as PRN 05 (p. 70)
- Observed elements and rates:
  - a=26560.5 km, e=0.0015, i=54.5°
  - $d\Omega/dt$ = -0.04109°/day
- Contributions from analytical rates:
  - J2: -0.03927°/day
  - Moon: -0.00097°/day
  - Sun: -0.00045°/day
  - Total: -0.04069°/day (difference with observed is 0.0004°/day, or 1%)

J_2 and Orbit Design

- As altitude increases, J_2 perturbation diminishes (from a great distance the Earth is equivalent to a point mass)
- Use J_2 perturbation in orbit design, e.g., solar synchronous satellite
  - If $d\Omega/dt = +360°/365.25$ days, the line of nodes will keep a fixed (in an average sense) orientation with respect to the Earth-Sun direction
  - Must be retrograde; for 600 km altitude, i=98°

Atmospheric Resistance

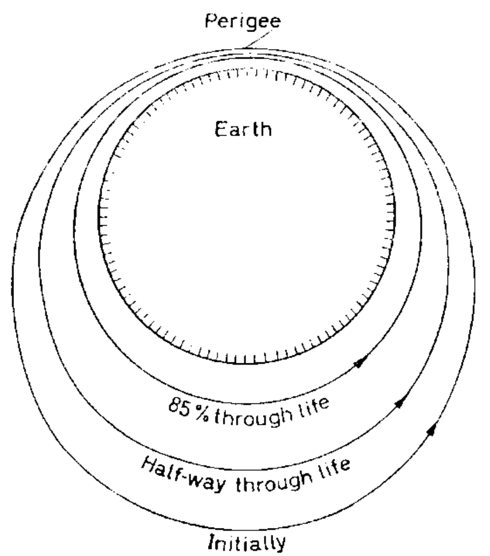


Figure 1.3. Contraction of a satellite orbit under the action of air drag

From D. King-Hele, 1964, *Theory of Satellite Orbits in an Atmosphere*

- Atmospheric drag is the dominant nongravitational force at low altitudes if the celestial body has an atmosphere
- Depending on nature of the satellite, lift force may exist
- Drag removes energy from the orbit and results in $da/dt < 0$, $de/dt < 0$
- Orbital lifetime of satellite strongly influenced by drag
- Drag Paradox – If drag increases velocity, why does specific energy decrease?

Other Forces

Other Forces

- The Earth (and all planets) are not rigid bodies
  – Gravitationally induced deformation (tides), both in fluid parts of the planet and solid
  – Section 2.3.6 provides more detail
  – Earth ΔJ_2 from luni-solar tides is ~ 10^{-8}
- Relativity (small effect on motion of perigee)
- Nongravitational forces
  – Atmospheric drag (dependent on C_D A/m)
    - Responsible for orbit decay, da/dt < 0
  – Solar radiation pressure, SRP (dependent on C_R A/m)
  – Earth radiation pressure
  – Other (including thermal radiation)
  – Unknown or not well understood forces

Coordinate Systems and Time: I

- The transformation between ECI and ECF is required in the equations of motion
- ECI is represented by ICRF (International Celestial Reference Frame, usually close to J2000)
- ECF is represented by ITRF (International Terrestrial Reference Frame), e.g., ITRF-2000 which gives coordinates of international space geodetic global sites

Coordinate Systems and Time: II

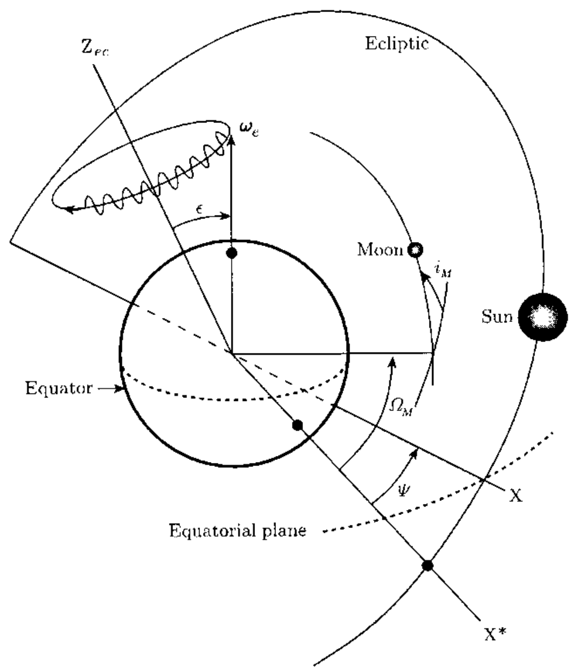


Figure 2.4.1: Precession and nutation. The Sun and Moon interact with the oblateness of the Earth to produce a westward motion of the vernal equinox along the ecliptic (Ψ) and oscillations of the obliquity of the ecliptic (ϵ).

- Equinox location is function of time
  - Sun and Moon interact with Earth J2 to produce
    - Precession of equinox (ψ)
    - Nutation (ε)
- Newtonian time (independent variable of equations of motion) is represented by atomic time scales (dependent on Cesium Clock)

Precession/Nutation

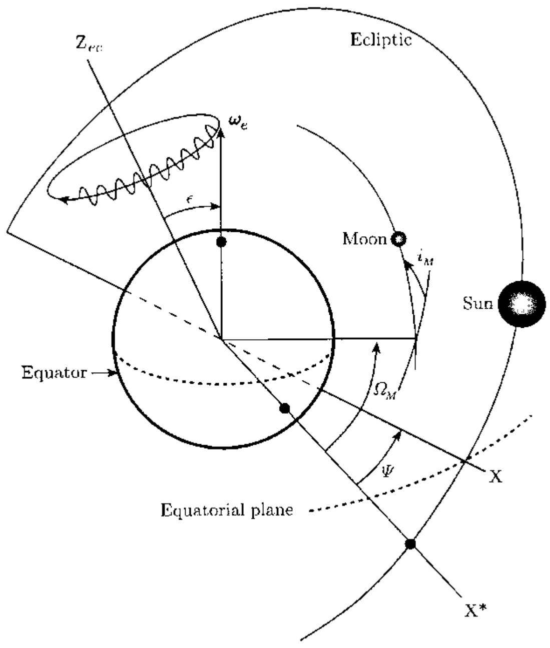


Figure 2.4.1: Precession and nutation. The Sun and Moon interact with the oblateness of the Earth to produce a westward motion of the vernal equinox along the ecliptic (Ψ) and oscillations of the obliquity of the ecliptic (ϵ).

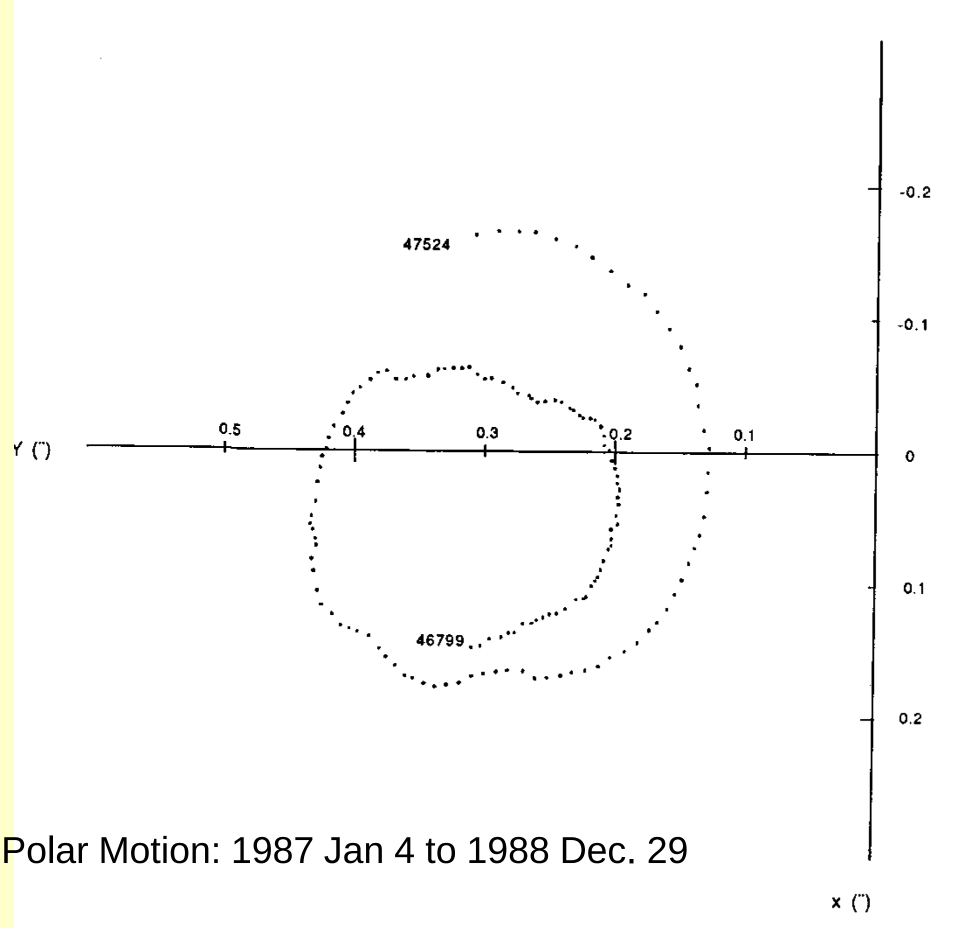
- Precession

$$\dot{\Psi} = -\frac{3}{2}\left(\frac{\mu J_2 a_e^2}{C\omega_e}\right)\frac{M_p}{a_p^3}\cos\bar{\epsilon}$$

- Nutation (main term):

$$\Delta\epsilon = \frac{3}{2}\left(\frac{\mu J_2 a_e^2}{C\omega_e}\right)\frac{M_M}{a_M^3}\cos i_M \sin i_M \cos\epsilon \frac{\cos(\Omega_M - \Psi)}{(\dot{\Omega}_M - \dot{\Psi})}$$

Earth Rotation



Polar Motion: 1987 Jan 4 to 1988 Dec. 29

- The angular velocity vector $\boldsymbol{\omega}_E$ is not contstant in direction or magnitude
  - Direction: polar motion
    - Chandler period: 430 days
    - Solar period: 365 days
  - Magnitude: related to length of day (LOD)
    - LOD dependent on atmospheric winds
- Components of $\boldsymbol{\omega}_E$ depend on observations; difficult to predict over long periods

Earth Rotation and Time

- Sidereal rate of rotation: ~2π/86164 rad/day
- Variations exist in magnitude of $\boldsymbol{\omega}_E$, from upper atmospheric winds, tides, etc.
- UT1 is used to represent such variations
  - UTC is kept within 0.9 sec of UT1 (leap second)
- Polar motion and UT1 observed quantities
- Different time scales: GPS-Time, TAI, UTC, TDT
- Time is independent variable in satellite equations of motion; relates observations to equations of motion (TDT is usually taken to represent independent variable in equations of motion)

Transformation Between ECI and ECF

- Transformation between ECI and ECF
- P is the precession matrix (~50 arcsec/yr)
- N is the nutation matrix (main term is 9 arcsec with 18.6 yr period)
- S' is sidereal rotation (depends on changes in angular velocity magnitude; UT1)
- W is polar motion
- Caution: small effects may be important in particular application

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ECF} = T_{XYZ}^{xyz} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{ECI}$$

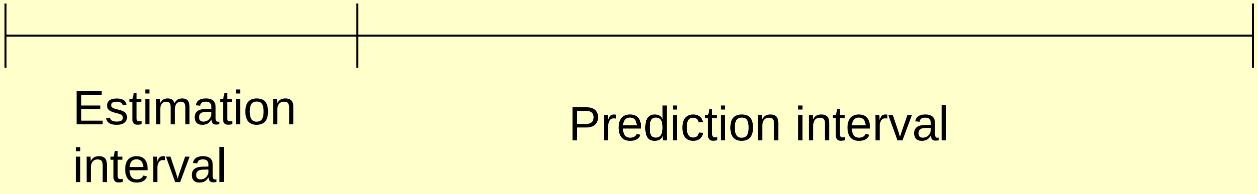
$$T_{XYZ}^{xyz} = W S' N P$$

Orbit Accuracy: I

- Orbit accuracy
  - Accuracy of method used to solve the equations of motion (*solution technique accuracy*)
  - Accuracy of the force model used in the equations of motion (*force model accuracy*)
    - Potential errors
      - Mismodeled forces (arise from poorly understood nature of the forces)
      - Force model parameters

Orbit Accuracy: II

- Orbit accuracy
  - Accuracy of determined orbit
  - Accuracy of predicted orbit (how the error in the determined orbit propagates with time)



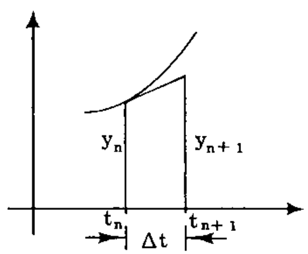
- Requirements
  - What are the orbit accuracy requirements for a particular mission?
    - Requirements may allow some forces to be ignored
  - Are the requirements based on the determined orbit accuracy or the predicted orbit?

Numerical Integration

$$\frac{dy}{dt} = f(y, t)$$

Euler's Method

$$y_{n+1} = y_n + \Delta t\, f(y_n, t_n)$$



Modified Euler as predictor-corrector

$$y_{n+1}^{(P)} = y_n + \Delta t\, f(y_n, t_n)$$

$$y_{n+1}^{(c)} = y_n + \Delta t \left[\frac{f(y_n, t_n) + f(y_{n+1}^{(P)}, t_{n+1})}{2} \right]$$

- Initial value problem:
  - Given initial conditions
  - Integration accuracy controlled by order of method and step size
  - Second order ordinary D.E. may be integrated as system of first order, or method may integrate second order directly
- Single step methods:
  - Euler (shown)
  - Runge-Kutta
    - 8th order requires 9 function evaluations/step
- Multi-step methods:
  - Require table of starting points (e.g., from RK), number depends on order of integrator

Runge-Kutta Method

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A method of numerically integrating ordinary differential equations by using a trial step at the midpoint of an interval to cancel out lower-order error terms. The second-order formula is

$$\begin{aligned} k_1 &= h\, f(x_n, y_n) \\ k_2 &= h\, f\left(x_n + \tfrac{1}{2}h,\, y_n + \tfrac{1}{2}k_1\right) \\ y_{n+1} &= y_n + k_2 + O(h^3), \end{aligned}$$

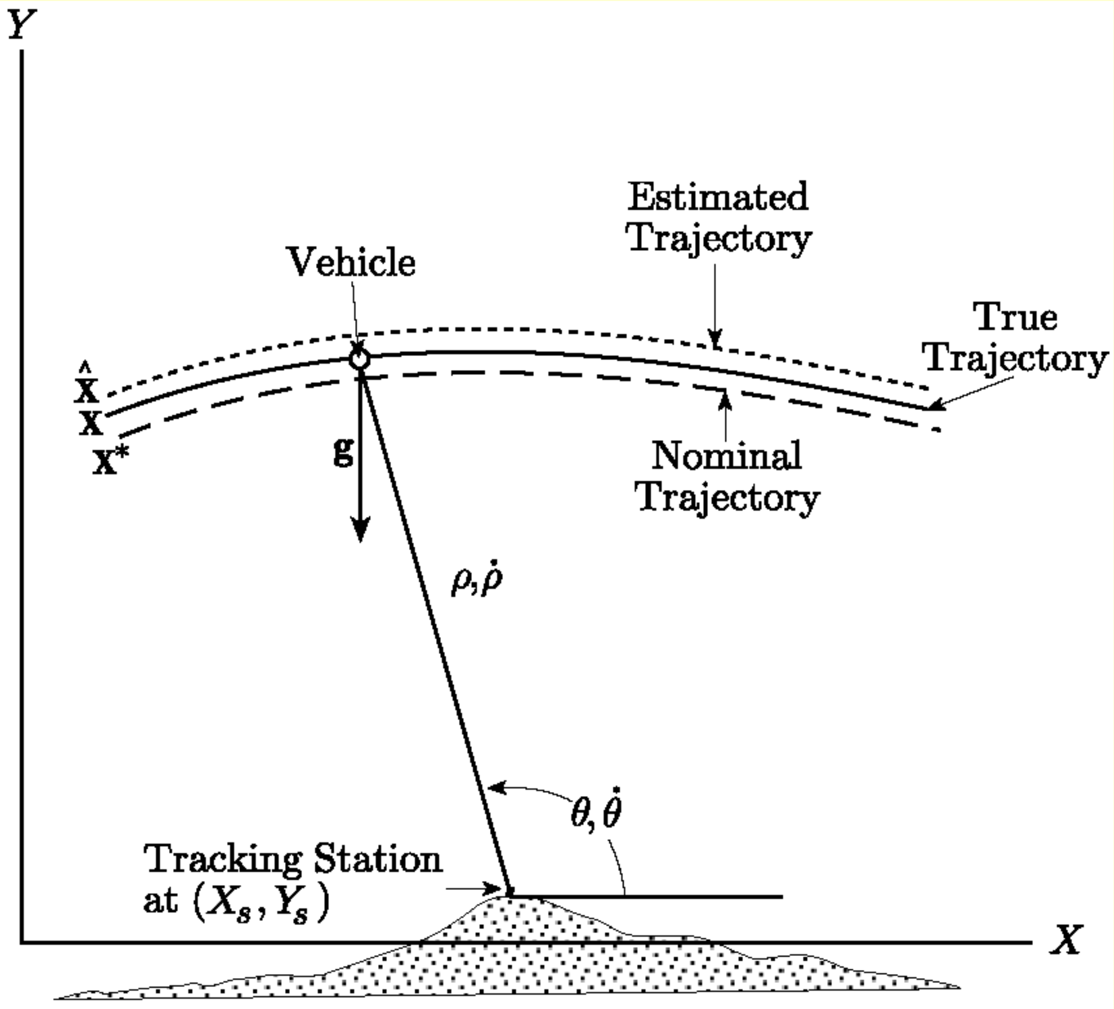
sometimes known as RK2, and the fourth-order formula is

$$\begin{aligned} k_1 &= h\, f(x_n, y_n) \\ k_2 &= h\, f\left(x_n + \tfrac{1}{2}h,\, y_n + \tfrac{1}{2}k_1\right) \\ k_3 &= h\, f\left(x_n + \tfrac{1}{2}h,\, y_n + \tfrac{1}{2}k_2\right) \\ k_4 &= h\, f(x_n + h,\, y_n + k_3) \\ y_{n+1} &= y_n + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4 + O(h^5) \end{aligned}$$

(Press *et al.* 1992), sometimes known as RK4. This method is reasonably simple and robust and is a good general candidate for numerical solution of differential equations when combined with an intelligent adaptive step-size routine.

A Simple Example: Uniform Gravity Model

Figure 1.2.1: Uniform gravity field trajectory.



1.2.1 Formulation of the Problem

For the uniform gravity field model, the differential equations of motion can be expressed as

$$\begin{aligned} \ddot{X}(t) &= 0 \\ \ddot{Y}(t) &= -g, \end{aligned} \tag{1.2.1}$$

where g is the gravitational acceleration and is assumed to be constant. Integration of Eq. (1.2.1) leads to

$$\begin{aligned} X(t) &= X_0 + \dot{X}_0 t \\ Y(t) &= Y_0 + \dot{Y}_0 t - g\frac{t^2}{2}, \\ \dot{X}(t) &= \dot{X}_0 \\ \dot{Y}(t) &= \dot{Y}_0 - gt \end{aligned} \tag{1.2.2}$$

where t represents the time, and the reference time, t_0, is chosen to be zero. The subscript 0 indicates values of the quantities at t_0.

As an example of the orbit determination procedure, consider the situation in which the state is observed at some time epoch, t_j. Then if X_j, Y_j, $\dot{X}_j$ and $\dot{Y}_j$ are given at t_j, Eq. (1.2.2) can be used to form four equations in terms of four unknowns. This system of equations can be used to determine the unknown components of the initial state. For example, from Eq. (1.2.2) it follows that

$$\begin{bmatrix} X_j \\ Y_j + gt_j^2/2 \\ \dot{X}_j \\ \dot{Y}_j + gt_j \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_j & 0 \\ 0 & 1 & 0 & t_j \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ \dot{X}_0 \\ \dot{Y}_0 \end{bmatrix}$$

Then, the initial state can be determined as follows:

$$\begin{bmatrix} X_0 \\ Y_0 \\ \dot{X}_0 \\ \dot{Y}_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_j & 0 \\ 0 & 1 & 0 & t_j \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} X_j \\ Y_j + gt_j^2/2 \\ \dot{X}_j \\ \dot{Y}_j + gt_j \end{bmatrix} . \qquad (1.2.5)$$

Unfortunately, in an actual trajectory or orbit determination process, the individual components of the state generally cannot be observed directly. Rather, the observations consist of nonlinear functions of the state; for example, range, elevation, range rate, and so on. In this case, the nonlinear observation-state relationship is

$$
\begin{aligned}
\rho &= \sqrt{(X - X_s)^2 + (Y - Y_s)^2} \\
\tan\theta &= (Y - Y_s)/(X - X_s) \\
\dot{\rho} &= \frac{1}{\rho}[(X - X_s)(\dot{X} - \dot{X}_s) + (Y - Y_s)(\dot{Y} - \dot{Y}_s)] \qquad (1.2.6) \\
\dot{\theta} &= \frac{1}{\rho^2}[(X - X_s)(\dot{Y} - \dot{Y}_s) - (\dot{X} - \dot{X}_s)(Y - Y_s)],
\end{aligned}
$$

$$
\begin{aligned}
\rho_j &= \sqrt{(X_0 - X_s + \dot{X}_0 t_j)^2 + (Y_0 - Y_s + \dot{Y}_0 t_j - g t_j^2/2)^2} \\
\theta_j &= \tan^{-1}[(Y_0 - Y_s + \dot{Y}_0 t_j - g\frac{t_j^2}{2})/(X_0 - X_s + \dot{X}_0 t_j)] \\
\dot{\rho}_j &= \frac{1}{\rho_j}[(X_0 - X_s + \dot{X}_0 t_j)(\dot{X}_0 - \dot{X}_s) \\
&\quad + (Y_0 - Y_s + \dot{Y}_0 t_j - g\frac{t_j^2}{2})(\dot{Y}_0 - g t_j - \dot{Y}_s)]
\end{aligned}
\qquad (1.2.7)
$$

$$
\begin{aligned}
\dot{\theta}_j &= \frac{1}{\rho_j^2}[(X_0 - X_s + \dot{X}_0 t_j)(\dot{Y}_0 - g t_j - \dot{Y}_s) \\
&\quad -(\dot{X}_0 - \dot{X}_s)(Y_0 - Y_s + \dot{Y}_0 t_j - g\frac{t_j^2}{2})].
\end{aligned}
$$

$$\boldsymbol{Y}^T = \left[\rho_j, \theta_j, \dot{\rho}_j, \dot{\theta}_j\right] \qquad G(\boldsymbol{X}_0, t) = RHS\ Eq.\ 1.2.7$$

$$J(\boldsymbol{X}_0) \equiv \boldsymbol{Y} - G(\boldsymbol{X}_0, t) = 0 \qquad \text{(eqn 1.2.8)}$$

Solve using Newton-Raphson iteration method

Newton-Raphson Method

- Re-write function as Taylor Series Expansion
- Ignore H.O.T
- Iterate until convergence

Newton-Raphson Method

$$J(\mathbf{X}_0^{n+1}) \cong J(\mathbf{X}_0^n) + \left[\frac{\partial J}{\partial \mathbf{X}_0^n}\right] [\mathbf{X}_0^{n+1} - \mathbf{X}_0^n],$$

Set equal to zero and solve for $\mathbf{X}_0^{n+1}$

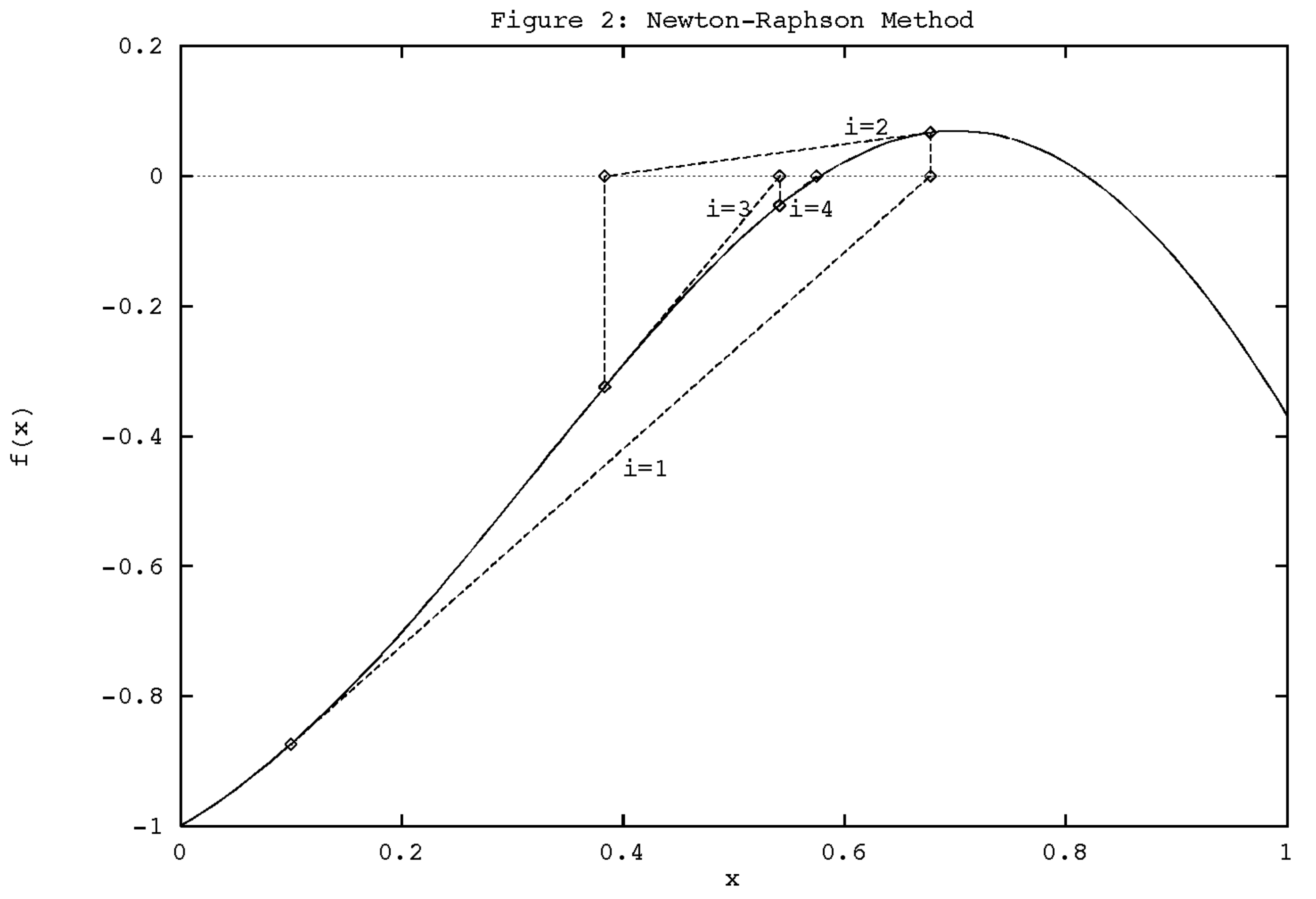
$$\mathbf{X}_0^{n+1} = \mathbf{X}_0^n - \left[\frac{\partial J}{\partial \mathbf{X}_0^n}\right]^{-1} J(\mathbf{X}_0^n), \qquad (1.2.9)$$

where

$$\frac{\partial J}{\partial \mathbf{X}_0^n} = -\frac{\partial G(\mathbf{X}_0^n, t)}{\partial \mathbf{X}_0^n}.$$

The process can be repeated until $||\mathbf{X}_0^{n+1} - \mathbf{X}_0^n|| \leq \epsilon$, where ϵ is a small positive number.

Newton-Raphson Method



Reality

- Due to this imperfect knowledge, cannot determine planar motion trajectory with only four observations
- In real-life operational situation # of observations are much larger than # of unknowns
- To obtain the 'best' estimate the problem is generally linearized by expanding the equations of motion and observation-state relationship about a reference trajectory

Linearization Procedure (1.2.4)

- Consider flat Earth example
  - Assume errors in position, velocity, and *g*
  - State vector is: $X^T = \left[X, Y, \dot{X}, \dot{Y}, g\right]$
  - Replace components of Eq. (1.2.2) with:

$$
\begin{gathered}
X \rightarrow X^* + \delta X \\
Y \rightarrow Y^* + \delta Y \\
\ldots \\
g \rightarrow g^* + \delta g
\end{gathered}
$$

  - Where ( )* are assumed or specified values

Linearization Procedure (1.2.4)

- Subtracting the reference trajectory, designated by ( )* values, from result of substitution
- Equations for the **state deviations** obtained:

$$
\begin{aligned}
\delta X &= \delta X_0 + \delta \dot{X}_0 t \\
\delta Y &= \delta Y_0 + \delta \dot{Y}_0 t - \delta g \frac{t^2}{2} \\
\delta \dot{X} &= \delta \dot{X}_0 \\
\delta \dot{Y} &= \delta \dot{Y}_0 - \delta g t \\
\delta g &= \delta g .
\end{aligned}
\qquad (1.2.11)
$$

Linearization Procedure (1.2.4)

Equation (1.2.11) in matrix form:

$$\begin{bmatrix} \delta X \\ \delta Y \\ \delta \dot{X} \\ \delta \dot{Y} \\ \delta g \end{bmatrix} = \begin{bmatrix} 1 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & t & -t^2/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta X_0 \\ \delta Y_0 \\ \delta \dot{X}_0 \\ \delta \dot{Y}_0 \\ \delta g \end{bmatrix} . \tag{1.2.12}$$

Equation (1.2.12) can be used to predict the deviation of the vehicle from the reference trajectory, $\mathbf{X}^*(t)$, at any time $t > 0$. Note that the deviations from the true state, $\delta\mathbf{X}(t) \equiv \mathbf{x}(t)$, will be caused by deviations in the original state or deviations in the gravitational acceleration, δg. However, the quantities $\delta X_0, \delta Y_0, \ldots,$ are not known, and it is the problem of the orbit determination procedure to estimate their values.

Linearization Procedure (1.2.4)

As mentioned earlier, the observations ρ and θ, which are taken as the satellite follows the true trajectory, are nonlinear functions of X, Y, X_s, and Y_s. Furthermore, they contain random and systematic errors represented by ϵ_ρ and ϵ_θ; hence,

$$\begin{aligned}\rho &= \sqrt{(X-X_s)^2+(Y-Y_s)^2}+\epsilon_\rho \\ \theta &= \tan^{-1}\left(\frac{Y-Y_s}{X-X_s}\right)+\epsilon_\theta.\end{aligned} \tag{1.2.13}$$

Our objective is to linearize the observations with respect to the reference trajectory. This can be accomplished by expanding Eq. (1.2.13) in a Taylor series about the reference or nominal trajectory as follows:

$$\begin{aligned}\rho &\cong \rho^* + \left[\frac{\partial\rho}{\partial X}\right]^*(X-X^*)+\left[\frac{\partial\rho}{\partial Y}\right]^*(Y-Y^*)+\epsilon_\rho \\ \theta &\cong \theta^* + \left[\frac{\partial\theta}{\partial X}\right]^*(X-X^*)+\left[\frac{\partial\theta}{\partial Y}\right]^*(Y-Y^*)+\epsilon_\theta.\end{aligned} \tag{1.2.14}$$

Note that the partials with respect to $\dot{X}$, $\dot{Y}$, and g are zero since they do not appear explicitly in Eq. (1.2.13).

Linearization Procedure (1.2.4)

Define:

$$\begin{aligned} \delta\rho &= \rho - \rho^* \\ \delta\theta &= \theta - \theta^* \\ \delta X &= X - X^* \\ \delta Y &= Y - Y^*, \end{aligned} \qquad (1.2.15)$$

can write Eq. (1.2.14) as

$$\begin{aligned} \delta\rho &= \left[\frac{\partial\rho}{\partial X}\right]^* \delta X + \left[\frac{\partial\rho}{\partial Y}\right]^* \delta Y + \epsilon_\rho \\ \delta\theta &= \left[\frac{\partial\theta}{\partial X}\right]^* \delta X + \left[\frac{\partial\theta}{\partial Y}\right]^* \delta Y + \epsilon_\theta. \end{aligned} \qquad (1.2.16)$$

The symbol [ ]* indicates that the value in brackets is evaluated along the nominal trajectory. In Eq. (1.2.16), terms of order higher than the first in the state deviation values have been neglected assuming that these deviations are small. This requires that the reference trajectory and the true trajectory be close at all times in the

$$
\begin{aligned}
\mathbf{y}^T &\equiv [\delta\rho\ \delta\theta] \\
\widetilde{H} &\equiv \begin{bmatrix} \left[\frac{\partial\rho}{\partial X}\right]^* & \left[\frac{\partial\rho}{\partial Y}\right]^* & 0 & 0 & 0 \\ \left[\frac{\partial\theta}{\partial X}\right]^* & \left[\frac{\partial\theta}{\partial Y}\right]^* & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{x}^T &\equiv [\delta X\ \delta Y\ \delta\dot{X}\ \delta\dot{Y}\ \delta g] \\
\boldsymbol{\epsilon}^T &\equiv [\epsilon_\rho\ \epsilon_\theta],
\end{aligned}
$$

Linearization Procedure (1.2.4)

Equation (1.2.16) can then be expressed as the matrix equation:

$$\mathbf{y} = \widetilde{H}\mathbf{x} + \boldsymbol{\epsilon}. \qquad (1.2.18)$$

y = *observation deviation vector*

x = *state deviation vector*

$\widetilde{H}$ = mapping matrix relating observation deviation vector to the state deviation vector

ϵ = random vector representing noise or error in the observations

State Transition Matrix (1.2.5)

$$\begin{bmatrix} \delta X \\ \delta Y \\ \delta \dot{X} \\ \delta \dot{Y} \\ \delta g \end{bmatrix} = \begin{bmatrix} 1 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & t & -t^2/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta X_0 \\ \delta Y_0 \\ \delta \dot{X}_0 \\ \delta \dot{Y}_0 \\ \delta g \end{bmatrix} \quad (1.2.12)$$

The matrix multiplying the initial state vector in Eq. (1.2.12) is referred to as the *state transition matrix*, $\Phi(t, t_0)$. For the state deviation vector, $\mathbf{x}$, defined in Eq. (1.2.17), this matrix is given by

$$\Phi(t, t_0) = \begin{bmatrix} 1 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & t & -t^2/2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.2.19)$$

State Transition Matrix (1.2.5)

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0$$
$$\mathbf{y}(t) = \widetilde{H}(t)\mathbf{x}(t) + \boldsymbol{\epsilon}(t). \qquad (1.2.20)$$

y(t) can be written in terms of x_0 as:

$$\mathbf{y}(t) = H(t)\mathbf{x}_0 + \boldsymbol{\epsilon}(t), \qquad (1.2.21)$$

where

$$H(t) = \widetilde{H}(t)\Phi(t, t_0). \qquad (1.2.22)$$

State Transition Matrix (1.2.5)

The problem that remains now is to determine the best estimate of $\mathbf{x}_0$ given the linearized system represented by Eqs. (1.2.20) and (1.2.21). Our problem can now be summarized as follows. Given an arbitrary epoch, t_k, and the state propagation equation and observation-state relationship

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t, t_k)\mathbf{x}_k \\ \mathbf{y} &= H\mathbf{x}_k + \boldsymbol{\epsilon}, \end{aligned} \tag{1.2.23}$$

find the "best" estimate of $\mathbf{x}_k$. In Eq. (1.2.23), $\mathbf{y}$ is an $m \times 1$ vector, $\mathbf{x}$ is an $n \times 1$ vector, $\boldsymbol{\epsilon}$ is an $m \times 1$ vector, and H is an $m \times n$ mapping matrix, where n is the number of state variables and m is the total number of observations. If m is sufficiently large, the essential condition $m \geq n$ is satisfied.

State Transition Matrix (1.2.5)

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t, t_k)\mathbf{x}_k \\ \mathbf{y} &= H\mathbf{x}_k + \boldsymbol{\epsilon}, \end{aligned} \qquad (1.2.23)$$

In contrast to the algebraic solutions described earlier in this chapter, the system represented by the second of Eq. (1.2.23) is always underdetermined. That is, there are m-knowns (e.g., the observation deviations, $\mathbf{y}$) and $m + n$ unknowns (e.g., the m-observation errors, $\boldsymbol{\epsilon}$, and the n-unknown components of the state deviation vector, $\mathbf{x}_k$). In Chapter 4, several approaches for resolving this problem are discussed. The most straightforward is based on the method of least squares as proposed by Gauss (1809).

Method of Least Squares

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t, t_k)\mathbf{x}_k \\ \mathbf{y} &= H\mathbf{x}_k + \epsilon, \end{aligned} \qquad (1.2.23)$$

In this approach, the best estimate for the unknown state vector, $\mathbf{x}_k$, is selected as the value $\hat{\mathbf{x}}_k$, which minimizes the sum of the squares of the calculated values of the observations errors. That is, if $\mathbf{x}_k^0$ is any value of $\mathbf{x}_k$, then $\epsilon^0 = \mathbf{y} - H\mathbf{x}_k^0$ will be the m-calculated values of the observation residuals corresponding to the value $\mathbf{x}_k^0$. Then, the best estimate of $\mathbf{x}_k$ will be the value that leads to a minimal value of the performance index, $J(\mathbf{x}_k^0)$, where

$$J(\mathbf{x}_k^0) = 1/2(\epsilon^{0T}\epsilon^0) = 1/2(\mathbf{y} - H\mathbf{x}_k^0)^T(\mathbf{y} - H\mathbf{x}_k^0) \quad (1.2.24)$$

Method of Least Squares

For a minimum of this quantity, it is necessary and sufficient that:

$$\left.\frac{\partial J}{\partial \mathbf{x}_k^0}\right|_{\hat{\mathbf{x}}_k} = 0; \quad \delta\mathbf{x}_k^{0^T} \left.\frac{\partial^2 J}{\partial \mathbf{x}_k^0 \partial \mathbf{x}_k^0}\right|_{\hat{\mathbf{x}}_x} \delta\mathbf{x}_k^0 > 0. \quad (1.2.25)$$

From the first of the conditions given by Eq 1.2.25) it follows that:

$$(\mathbf{y} - H\hat{\mathbf{x}}_k)^T H = 0$$

Method of Least Squares

If rearranged:

$$H^T H \hat{\mathbf{x}}_k = H^T \mathbf{y}. \qquad (1.2.26)$$

If the n x n matrix $(H^T H)$ has an inverse the solution can be expressed as:

$$\hat{\mathbf{x}}_k = (H^T H)^{-1} H^T \mathbf{y}.$$

1.6 EXERCISES

1. Write a computer program that computes $\rho(t_i)$ for a uniform gravity field using Eq. (1.2.10). A set of initial conditions, X_0, $\dot{X}_0$, Y_0, $\dot{Y}_0$, g, X_s, Y_s, and observations, ρ, follow. With the exception of the station coordinates, the initial conditions have been perturbed so that they will not produce exactly the observations given. Use the Newton iteration scheme of Eq. (1.2.9) to recover the exact initial conditions for these quantities; that is, the values used to generate the observations. Assume that X_s and Y_s are known exactly. Hence, they are not solved for.

Unitless Initial Conditions

$$
\begin{aligned}
X_0 &= 1.5 \\
Y_0 &= 10.0 \\
\dot{X}_0 &= 2.2 \\
\dot{Y}_0 &= 0.5 \\
g &= 0.3 \\
X_s &= Y_s = 1.0
\end{aligned}
$$

Time	Range Observation, ρ
0	7.0
1	8.00390597
2	8.94427191
3	9.801147892
4	10.630145813

Answer

$$
\begin{aligned}
X_0 &= 1.0 \\
Y_0 &= 8.0 \\
\dot{X}_0 &= 2.0 \\
\dot{Y}_0 &= 1.0 \\
g &= 0.5
\end{aligned}
$$

2. In addition to the five state variables of Exercise 1, could X_s and Y_s be solved for given seven independent observations of ρ? Why or why not? Hint: Notice the relationship between X_0 and X_s and between Y_0 and Y_s in Eq. (1.2.7) for ρ_j. See also Section 4.12 on observability.

$$
\begin{aligned}
\rho_j &= \sqrt{(X_0 - X_s + \dot{X}_0 t_j)^2 + (Y_0 - Y_s + \dot{Y}_0 t_j - g t_j^2/2)^2} \\
\theta_j &= \tan^{-1}[(Y_0 - Y_s + \dot{Y}_0 t_j - g\frac{t_j^2}{2})/(X_0 - X_s + \dot{X}_0 t_j)] \\
\dot{\rho}_j &= \frac{1}{\rho_j}[(X_0 - X_s + \dot{X}_0 t_j)(\dot{X}_0 - \dot{X}_s) \\
&\quad +(Y_0 - Y_s + \dot{Y}_0 t_j - g\frac{t_j^2}{2})(\dot{Y}_0 - g t_j - \dot{Y}_s)] \qquad (1.2.7)
\end{aligned}
$$