M

Converting IOMoDE to FOMoDE

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§M.1. Introduction

This Appendix summarizes transformations that find application in the method of Modified Differential Equations or MoDE. technically the most difficult operation is passing from an Infinite Order MoDE (IOMoDE) to a Finite Order MoDE (FOMoDE). There is no general method because involves the process involves identification of series. Case by case is the rule. Nonetheless there are some forms which recur in application problems and which may be processed in closed form. Such forms are collected here as a hub for references from other chapters.

§M.2. A Fourier First

The inhomogeneous, even-derivative, infinite-order ODE

$$f(x) = u(x) - \frac{1}{n^2} u''(x) + \frac{1}{n^4} u''''(x) + \ldots, \tag{M.1}$$

provides a simple example of infinite-to-finite reduction. Here as usual $(.)'$ is an abbreviation for $d(.)/dx$ while n is a nonzero constant. It is assumed that $u(x) \in C^\infty$ for any finite x whereas $f(x) \in C^2$. The reduction is immediate. By inspection (M.1) satisfies

$$u(x) = f(x) + \frac{1}{n^2} f''(x). \tag{M.2}$$

so the FOMoDE (M.2) is of zero order. This easy result is not relevant to modification methods, but allows the introduction of a historical curiosity.

Joseph Fourier used the transformation (M.1)→(M.2), in which n is an integer, on the way to finding a formula for the coefficients of what are now called Fourier series [270, pp. 187ff]. If $u(x)$ is given, the general solution of (M.2) is

$$f(x) = C \cos nx + D \sin nx + n \sin nx \int_0^x u(t) \cos nt \, dt - n \cos nx \int_0^x u(t) \sin nt \, dt. \tag{M.3}$$

From this Fourier's famous formula emerges after a few more gyrations. Euler had effortlessly found the same result 70 years earlier through term-by-term integration. But in taking the hard way Fourier became apparently the first person to use an ODE of infinite order.

Hardy [339, §2.10], a gentle spirit, beningly observes than Fourier's laborious derivations rely on divergent series *in passim* but that his final result turned out to be correct. Truesdell, a stickler for mathematical orthodoxy, is not so kind: "Fourier proved it [the fundamental theorem for expansion in trigonometric series] through a mass of divergent gobbledegook which every competent mathematician of his own day rejected." [753, p. 77]. [1]

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[1] Of course Fourier could retort that his name is well known in sciences and mathematics after two centuries. The ratio of Google hits for Fourier to those for Truesdell is 65:1 on June 4, 2012.

§M.3. Example 2: IOMoDE with Even Derivatives

As first nontrivial example consider the homogeneous, even-derivative, infinite-order ODE:

$$-\frac{\phi}{2a^2}u(x)+\frac{1}{2!}u''(x)+\frac{a^2\chi^2}{4!}u''''(x)+\frac{a^4\chi^4}{6!}u''''''(x)+\ldots=0,\quad a>0,\ \ \phi\neq 0,\ \ 0<\chi\leq 1. \tag{M.4}$$

Here ϕ and χ are dimensionless real parameters whereas a, which is a characteristic problem dimension, has dimension of length. The reduction to finite order can be obtained by a variant of Warming and Hyett's [783] derivative elimination procedure, Differentiate (M.4) $2(n-1)$ times ($n=1,2,\ldots$) with respect to x while discarding all odd derivatives. Truncate to the same level in χ, and set up a linear system in the even derivatives u'', u'''', ... The configuration of the elimination system is illustrated for $n=4$:

$$\begin{bmatrix} 1/2! & a^2\chi^2/4! & a^4\chi^4/6! & a^6\chi^6/8! \\ -\frac{1}{2}\phi a^{-2} & 1/2! & a^2\chi^2/4! & a^4\chi^4/6! \\ 0 & -\frac{1}{2}\phi a^{-2} & 1/2! & a^2\chi^2/4! \\ 0 & 0 & -\frac{1}{2}\phi a^{-2} & 1/2! \end{bmatrix}\begin{bmatrix} u'' \\ u'''' \\ u'''''' \\ u'''''''' \end{bmatrix}=\begin{bmatrix} \frac{1}{2}a^{-2}\phi\, u \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{M.5}$$

The coefficient matrix of this system is Toeplitz and Hessenberg but not Hermitian. This can be solved for u'' to yield a truncated FOMoDE. Solving (M.5) and expanding in Taylor series gives

$$u''=\frac{56\phi(360+60\lambda+\lambda^2)\,u}{20160+5040\lambda+252\lambda^2+\lambda^3}=\phi\left(1-\frac{1}{12}\lambda+\frac{1}{90}\lambda^2+\ldots\right)u. \tag{M.6}$$

where $\lambda=a^2\chi^2\phi$. Increasing n, the coefficients of the power series in λ are found to be generated by the recursion $c_1=1$, $c_{n+1}=-\frac{1}{2}n^2c_n/[(n+1)(2n+1)]$, $n\geq 1$, which produces the sequence $\{1,-1/12,1/90,-1/560,1/3150,-1/16632,\ldots\}$. The generating function [**?**] can be found by *Mathematica's* package `RSolve` by entering `<<DiscreteMath`RSolve`; g=GeneratingFunction[ a[n+1]==-n*n/(2*(n+1)*(2*n+1))*a[n],a[1]==1,a[n],n,λ]; Print[g]`. To verify the answer do `Print[Series[g,{ λ,0,8 }]`. The result is

$$\frac{4}{\lambda}\left(\text{arcsinh}\frac{\sqrt{\lambda}}{2}\right)^2=1-\frac{\lambda}{12}+\frac{\lambda^2}{90}-\frac{\lambda^3}{560}+\frac{\lambda^4}{3150}-\frac{\lambda^5}{16632}+\frac{\lambda^6}{84084}-\frac{\lambda^7}{411840}+\ldots \tag{M.7}$$

This yields the second-order FOMoDE

$$u''=\frac{4}{a^2\chi^2}\left(\text{arcsinh}\frac{\sqrt{\lambda}}{2}\right)^2u=\frac{4}{a^2\chi^2}\left(\text{arcsinh}\frac{\chi\sqrt{\phi}}{2}\right)^2u. \tag{M.8}$$

To give an example of matching suppose that the original ODE from which (M.4) comes is $u''=(w/a^2)u$, where $w>0$ is constant. For nodal exactness, $w=(4/\chi^2)\big(\text{arcsinh}(\frac{1}{2}\chi\sqrt{\phi}\big)^2$. If ϕ is the free parameter, solving for it gives

$$\phi=\frac{4}{\chi^2}\left(\sinh\frac{\chi\sqrt{w}}{2}\right)^2=\frac{2\big(\cosh(\chi\sqrt{w})-1\big)}{\chi^2}. \tag{M.9}$$

In this analysis no term of model equation is assumed to be small. The procedure for handling a forcing term $f(x)$ follows the same technique.

§M.3.1. *A More Advanced Derivation

The foregoing construction of (M.8) has a heuristic flavor: it relies on recognizing a series. A more direct derivation, which however requires more advanced mathematical tools, is presented here. The method relies on the following determinant theorem [501, p. 704]. Given the formal series expansion

$$\frac{1}{g(x)} = \frac{1}{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots} = A_0 - A_1 x + A_2 x^2 - A_3 x^3 + \ldots, \quad a_0 \neq 0, \tag{M.10}$$

then the Toeplitz determinants formed with the a_i coefficients satisfy

$$A_1 = a_0^{-1}\,|\,a_1\,|\,,\ \ A_2 = a_0^{-2}\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix},\ \ A_3 = a_0^{-3}\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix},\ \ A_4 = a_0^{-4}\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}, \ldots \tag{M.11}$$

with $A_0 = 1/a_0$. Now the determinants that appear in the foregoing FOMoDE derivation have the form

$$A_1 = |\,1/2!\,|\,, \quad A_2 = \begin{vmatrix} 1/2! & a^2\chi^2/4! \\ -\frac{1}{2}\phi a^{-2} & 1/2! \end{vmatrix}, \quad A_3 = \begin{vmatrix} 1/2! & a^2\chi^2/4! & a^4\chi^4/6! \\ -\frac{1}{2}\phi a^{-2} & 1/2! & a^2\chi^2/4! \\ 0 & -\frac{1}{2}\phi a^{-2} & 1/2! \end{vmatrix}, \tag{M.12}$$

Identifying to (M.10) and (M.11) one obtains by inspection

$$a^2\chi^2\, g(x) = \cosh(a\chi\sqrt{x}) - (1 + \tfrac{1}{2}\phi\chi^2) \tag{M.13}$$

The n^{th} approximation to the FOMoDE ($n > 1$) is $u'' = C_n u$, with $C_n = A_{n-1}/A_n$. If the series (M.10) has radius of convergence R, then $C_n \to 1/R$ as $n \to \infty$. But the radius of convergnce of $1/g(x)$ is the distance from $x = 0$ to the closest pole, or what is the the same, the closest zero of $g(x)$. This is obtained by solving $g(R) = 0$ or $\cosh(a\chi\sqrt{R}) = 1 + \frac{1}{2}\phi\chi^2$, whence $a\chi\sqrt{R} = \operatorname{arccosh}(1 + \frac{1}{2}\phi\chi^2)$, which for $\phi > 0$ is equivalent to $R = 4a^{-2}\chi^{-2}\big(\operatorname{arcsinh}(\frac{1}{2}\sqrt{\phi}\chi)\big)^2$. This leads to the same solution: $u'' = Ru = 4a^{-2}\chi^{-2}\big(\operatorname{arcsinh}(\frac{1}{2}\sqrt{\phi}\chi)\big)^2$ found before.

Note that this method bypasses determinant expansions and series identification, but it is restricted to Toeplitz matrices.

§M.4. Example 3: IOMoDE with Even and Odd Derivatives

As second example consider the homogeneous infinite-order ODE:

$$-\frac{\phi}{2a}u(x) + u'(x) + \frac{a}{2!}u''(x) + \frac{a^2\chi^2}{3!}u'''(x) + \frac{a^3\chi^3}{4!}u''''(x) + \ldots = 0, \quad a > 0, \ \ \phi \neq 0, \ \ 0 < \chi \leq 1. \tag{M.14}$$

Here ϕ and χ are dimensionless real parameters whereas a, which is a characteristic problem dimension, has dimension of length. Proceeding as above one forms the elimination system,

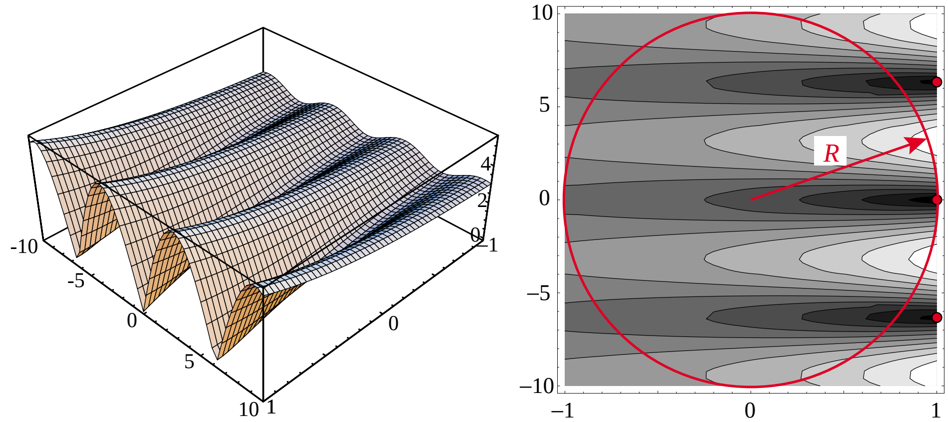

FIGURE M.1. Plot of the modulus $\sqrt{g_r^2 + g_i^2}$ of generating function (M.16) for $\{\phi = e-1, a = \chi = 1\}$, showing zeros on line $x = x_R = 1$ and convergence radius.

illustrated for $n = 4$:

$$\begin{bmatrix} 1 & a\chi/2! & a^2\chi^2/3! & a^3\chi^3/4! \\ -\frac{1}{2}\phi/a & 1 & a\chi/2! & a^2\chi^2/3! \\ 0 & -\frac{1}{2}\phi/a & 1 & a\chi/2! \\ 0 & 0 & -\frac{1}{2}\phi/a & 1 \end{bmatrix} \begin{bmatrix} u' \\ u'' \\ u''' \\ u'''' \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\phi\, u/a \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{M.15}$$

This is a Toeplitz matrix, so the quickest procedure is the determinant theorem. By inspection

$$a\chi\, g(x) = \exp(a\chi x) - (1 + \phi\chi). \tag{M.16}$$

The radius of convergence of the $1/g(z)$ series is obtained by solving $g(R) = 0$ for a complex $g(z)$, $z = x + yi$. The real and imaginary parts of $g(z)$ are $a\chi g_r = \exp(ax\chi)\cos(ay\chi) - (1 + \phi\chi)$, $a\chi g_i = \exp(ax\chi)\sin(ay\chi)$. Solving $g_r = g_i = 0$ gives the solutions $x_R = \log(1 + \phi\chi)/(a\chi)$, $y_R = \pi j/(a\chi)$, where j is an arbitrary integer. Thus all zeros lies on the $x = x_R$ line. This is pictured in Figure M.1, drawn for $\phi = e - 1$, $a = \chi = 1$ so $x_R = 1$. The zero closest to the origin corresponds to $y_R = 0$ whence the radius of convergence is $R = x_R = \log(1 + \phi\chi)/(a\chi)$, and the FOMoDE is

$$u' = \frac{\log(1 + \phi\chi)}{a\chi}\, u. \tag{M.17}$$

In this analysis no term of model equation is assumed to be small. The procedure for handling a forcing term $f(x)$ follows the same technique.

§M.4.1. Example 3: IOMoDE with All Derivatives

$$u'(t) + \frac{h}{2!}u''(t) + \frac{h^2}{3!}u'''(t) + \ldots = \lambda\, u(t). \tag{M.18}$$

To derive the FOMoDE, differentiate repeatedly with respect to t truncating after the same order in h, taking care to obtain a square coefficient matrix. For example, differentaiting 3 times gives

$$\begin{bmatrix} 1 & h/2! & h^2/3! & h^3/4! \\ -\lambda & 1 & h/2! & h^2/3! \\ 0 & -\lambda & 1 & h/2! \\ 0 & 0 & -\lambda & 1 \end{bmatrix} \begin{bmatrix} u' \\ u'' \\ u''' \\ u'''' \end{bmatrix} = \begin{bmatrix} \lambda\, u \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{M.19}$$

Solving produces

$$u' = \frac{4\lambda(6+6h\lambda+h^2\lambda^2)}{24+36h\lambda+14h^2\lambda^2+h^3\lambda^3} = \left(\lambda - \frac{h\lambda^2}{2} + \frac{h^2\lambda^3}{3} - \frac{h^3\lambda^4}{4} + \frac{29h^4\lambda^5}{144} - \ldots\right) u \tag{M.20}$$

The series on the right looks like that of $\log(1+h\lambda)$, which can be confirmed by increasing n. Therefore the FOMoDE has the explicit form

$$u' = \frac{\log(1+h\lambda)}{h} u \tag{M.21}$$

The explicit solution of this equation under the initial condition $u(0) = u_0$ is of course $u(t/h) = u_0(1+h\lambda)^{(t/h)}$. As $h \to 0$ this approaches $u_0 e^{\lambda t}$, which confirms the consistency of Forward Euler.