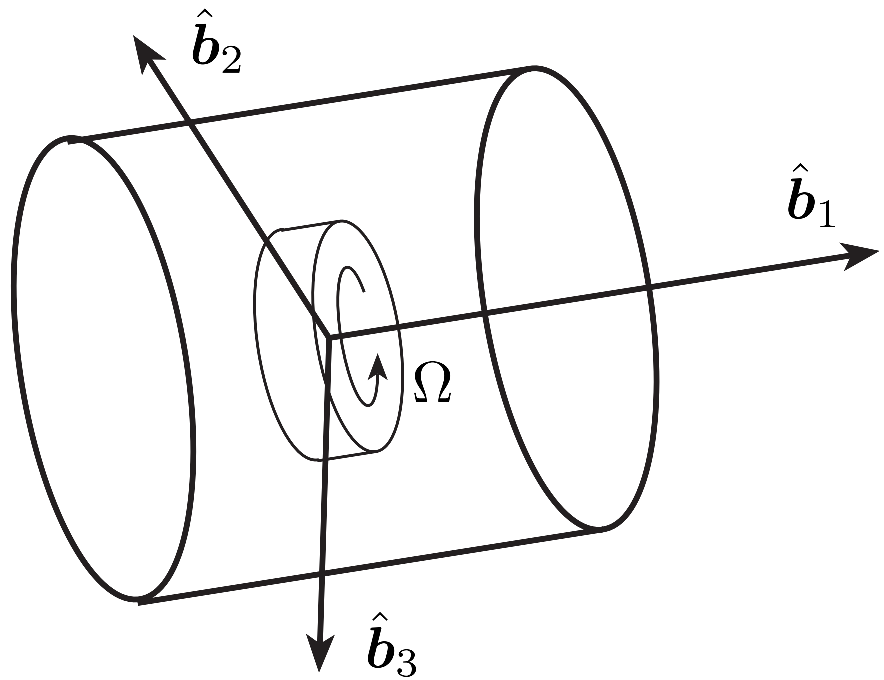
Dual Spin Spacecraft

Elegant attitude stabilization method...

Equations of Motion

- Assume a rigid spacecraft has an internal fly wheel, whose *constant* spin axis is aligned with the first body axis.



Note: The spacecraft inertia magnitude about each body axis is still free to be chosen.

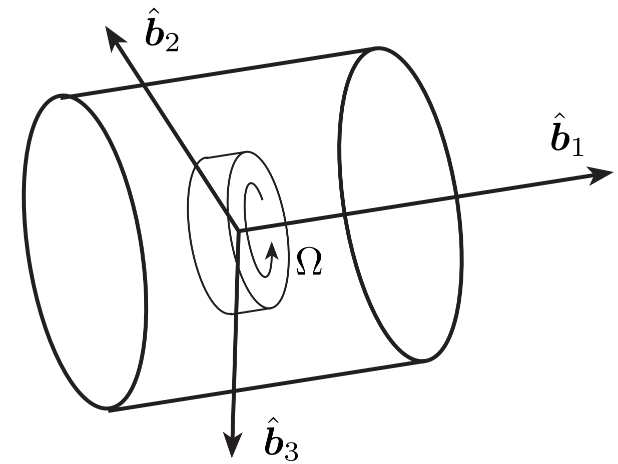
Total inertia matrix:

$$[I] = [I_s] + [I_W] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Total ang. momentum:

$$\boldsymbol{H} = [I]\boldsymbol{\omega} + \underbrace{I_W \Omega}_{h} \hat{\boldsymbol{b}}_1$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$



Differentiate to use Euler's equation:

$$\begin{aligned} \dot{\boldsymbol{H}} &= \frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{H}) + \boldsymbol{\omega} \times \boldsymbol{H} \\ &= [I]\dot{\boldsymbol{\omega}} + \dot{h}\hat{\boldsymbol{b}}_1 + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{\omega} \times (h\hat{\boldsymbol{b}}_1) = \boldsymbol{L} \end{aligned}$$

Differential equations of motion with no external torque:

$$\boxed{[I]\dot{\boldsymbol{\omega}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \dot{h}\hat{\boldsymbol{b}}_1 - h\omega_3\hat{\boldsymbol{b}}_2 + h\omega_2\hat{\boldsymbol{b}}_3}$$

with $\dot{h} = I_w \dot{\Omega}$

- Using Euler's equation, we find the spacecraft equations of motion with a constant speed fly wheel to be:

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = - \begin{pmatrix} (I_3 - I_2)\omega_2\omega_3 \\ (I_1 - I_3)\omega_1\omega_3 \\ (I_2 - I_1)\omega_1\omega_2 \end{pmatrix} + I_W \begin{pmatrix} -\dot{\Omega} \\ -\Omega\omega_3 \\ \Omega\omega_2 \end{pmatrix}$$

- This vector equation can also be written as three scalar equations:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_3}\omega_2\omega_3 - \frac{I_W}{I_1}\dot{\Omega}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_1\omega_3 - \frac{I_W}{I_2}\omega_3\Omega$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{I_W}{I_3}\omega_2\Omega$$

Linear Stability Analysis

- To determine the stability of this dual-spin spacecraft with constant wheel rate, we assume that the $\boldsymbol{\omega}_e$ angular rate vector is an equilibrium rotation rate.
- Next, we study small variations in angular rates about this equilibrium position.

$$\boldsymbol{\omega} = \boldsymbol{\omega}_e + \delta\boldsymbol{\omega}$$

- For the equilibrium motion, note that

$$\dot{\omega}_{e_1} = \frac{I_2 - I_3}{I_1}\omega_{e_2}\omega_{e_3} = 0$$
$$\dot{\omega}_{e_2} = \frac{I_3 - I_1}{I_2}\omega_{e_1}\omega_{e_3} - \frac{I_{W_s}}{I_2}\omega_{e_3}\Omega = 0$$
$$\dot{\omega}_{e_3} = \frac{I_1 - I_2}{I_3}\omega_{e_1}\omega_{e_2} + \frac{I_{W_s}}{I_3}\omega_{e_2}\Omega = 0$$

- Substituting $\boldsymbol{\omega} = \boldsymbol{\omega}_e + \delta\boldsymbol{\omega}$ into the rigid body equations of motion yields:

$$(\dot{\omega}_{e_1} + \delta\dot{\omega}_1) = \frac{I_2 - I_3}{I_1}(\omega_{e_2} + \delta\omega_2)(\omega_{e_3} + \delta\omega_3) = 0$$
$$(\dot{\omega}_{e_2} + \delta\dot{\omega}_2) = \frac{I_3 - I_1}{I_2}(\omega_{e_1} + \delta\omega_1)(\omega_{e_3} + \delta\omega_3) - \frac{I_{W_s}}{I_2}(\omega_{e_3} + \delta\omega_3)\Omega = 0$$
$$(\dot{\omega}_{e_3} + \delta\dot{\omega}_3) = \frac{I_1 - I_2}{I_3}(\omega_{e_1} + \delta\omega_1)(\omega_{e_2} + \delta\omega_2) + \frac{I_{W_s}}{I_3}(\omega_{e_2} + \delta\omega_2)\Omega = 0$$

- Next, assume that the space craft is spinning nominally about its first body axis

$$\boldsymbol{\omega}_e = \omega_{e_1}\hat{\boldsymbol{b}}_1 = \begin{pmatrix} \omega_{e_1} \\ 0 \\ 0 \end{pmatrix}$$

$$\omega_{e_2} = \omega_{e_3} = 0$$

- Dropping the higher order terms, and assuming that the equilibrium spin condition of interest is $\boldsymbol{\omega}_e = \omega_{e_1}\hat{\boldsymbol{b}}_1$, we find the following departure motion differential equations of motion.

$$\delta\dot{\omega}_1 = 0$$

$$\delta\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_{e_1}\delta\omega_3 - \frac{I_{W_s}}{I_2}\delta\omega_3\Omega$$

$$\delta\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_{e_1}\delta\omega_2 + \frac{I_{W_s}}{I_3}\delta\omega_2\Omega$$

- Note that $\delta\omega_1$ is constant and does not appear in the other two equations (decoupled from them).

$$\delta\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\omega_{e_1} - \frac{I_{W_s}}{I_2}\Omega\right)\delta\omega_3$$

$$\delta\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\omega_{e_1} + \frac{I_{W_s}}{I_3}\Omega\right)\delta\omega_2$$

- Next, we take the derivative of $\delta\dot{\omega}_2$:

$$\delta\ddot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\omega_{e_1} - \frac{I_{W_s}}{I_2}\Omega\right)\delta\dot{\omega}_3$$

- Substituting in the $\delta\dot{\omega}_3$ result from the previous page, we find the following decoupled body rate departure dynamics about the 2nd body axis:

$$\delta\ddot{\omega}_2 + \underbrace{\left(\frac{I_1 - I_3}{I_2}\omega_{e_1} + \frac{I_{W_s}}{I_2}\Omega\right)\left(\frac{I_1 - I_2}{I_3}\omega_{e_1} + \frac{I_{W_s}}{I_3}\Omega\right)}_{k}\delta\omega_2 = 0$$

Compare this to: $\delta\ddot{\omega}_2 + k\delta\omega_2 = 0$

Stability requires that $k > 0$

- The parameter k can be written as:

$$k = \frac{\omega_{e_1}^2}{I_2 I_3}\left(I_1 - I_3 + I_{W_s}\hat{\Omega}\right)\left(I_1 - I_2 + I_{W_s}\hat{\Omega}\right)$$

where $$\hat{\Omega} = \frac{\Omega}{\omega_{e_1}}$$

Zero RW Spin Rate

- First, let's verify the classical rigid body spin stability analysis if the RW spin rate is zero.

- For stability, we require *k*>0:

$$k = \frac{\omega_{e_1}^2}{I_2 I_3} \left(I_1 - I_3\right) \left(I_1 - I_2\right) > 0$$

True if: $I_1 > I_3$ $I_1 > I_2$ $\Rightarrow$ Max. Inertia Case

$I_1 < I_3$ $I_1 < I_2$ $\Rightarrow$ Min. Inertia Case

Non-Zero RW Spin

- Next, let's look at the stability requirement if the RW spin rate is nonzero:

$$k = \frac{\omega_{e_1}^2}{I_2 I_3}\left(I_1 - I_3 + I_{W_s}\hat{\Omega}\right)\left(I_1 - I_2 + I_{W_s}\hat{\Omega}\right) > 0$$

True if:

$$I_1 > I_3 - I_{W_s}\hat{\Omega} \qquad I_1 > I_2 - I_{W_s}\hat{\Omega}$$

$$I_1 < I_3 - I_{W_s}\hat{\Omega} \qquad I_1 < I_2 - I_{W_s}\hat{\Omega}$$

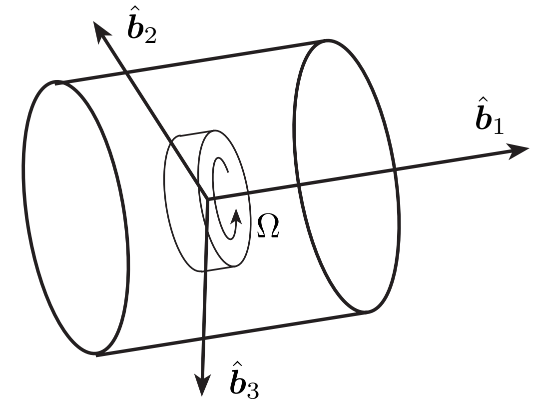
Note: The spacecraft spin can be made stable, regardless if I_1 is a major, intermediate or minor inertia!

Note: Careless use of the RW mode can also cause the spacecraft spin too become unstable.

Example

- Consider a spacecraft to have the following inertias:

$$
\begin{aligned}
I_1 &= 350 \text{ kgm}^2 \\
I_2 &= 300 \text{ kgm}^2 \\
I_3 &= 400 \text{ kgm}^2 \\
I_{W_s} &= 10 \text{ kgm}^2
\end{aligned}
$$



- Without the fly-wheel, note that spinning abut the first body axis would be unstable.

- The spacecraft spin about $\hat{\boldsymbol{b}}_1$ is 60 RPM.

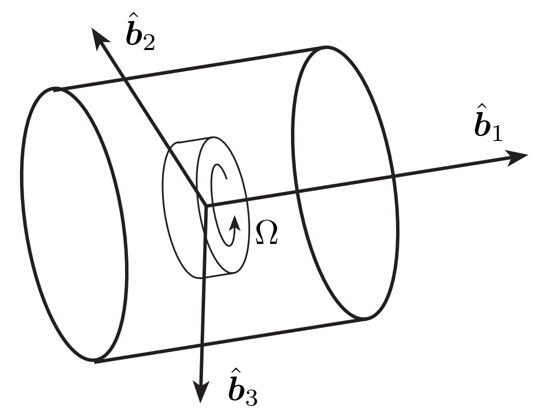
- How fast does the wheel have to spin to make this spacecraft a stable dual-spin system?

- The stability conditions are:

Set 1: $I_1 > I_3 - I_{W_s}\hat{\Omega}$ $\quad$ $I_1 > I_2 - I_{W_s}\hat{\Omega}$

Set 2: $I_1 < I_3 - I_{W_s}\hat{\Omega}$ $\quad$ $I_1 < I_2 - I_{W_s}\hat{\Omega}$

Since $I_1 > I_2$, the second condition of set 1 is satisfied if $\hat{\Omega} > -5$

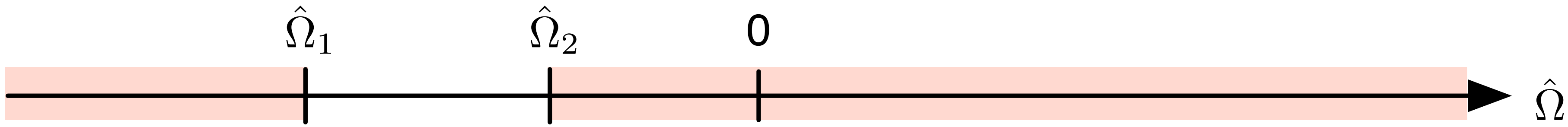


The first condition of set 1 then requires that:

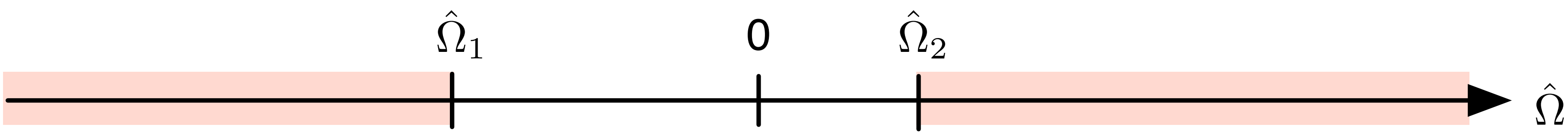
$$
\begin{aligned}
I_1 &> I_3 - I_{W_s}\hat{\Omega} \\
I_{W_s}\hat{\Omega} &> I_3 - I_1 \\
\hat{\Omega} &> \frac{I_3 - I_1}{I_{W_s}} \\
\hat{\Omega} &> 5
\end{aligned}
$$

$$
\Rightarrow \quad
\begin{aligned}
\Omega &= \hat{\Omega}\omega_{e_1} \\
&= 300 \text{ RPM}
\end{aligned}
$$

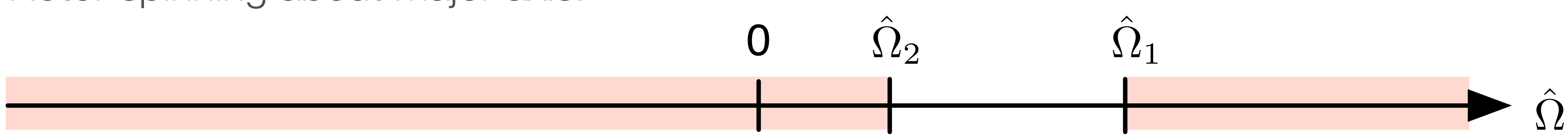
Rotor spinning about major axis:



Rotor spinning about major axis:

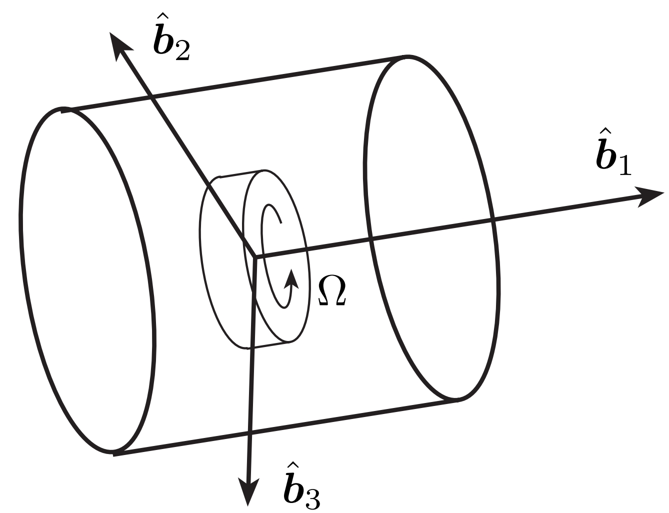


Rotor spinning about major axis:



Spin-Up Study

- Next we investigate a classical spin-up maneuver with a dual-spin spacecraft. Assume the wheel is initial at rest relative to the spacecraft.
- The spacecraft is assumed to have a pure spin about a principal axis which his not aligned with the reaction wheel spin axis.
- Then the RW is spun up until it has the same amount of angular momentum as the spacecraft had initially.



- What will happen to the spacecraft attitude during this spin-up maneuver?

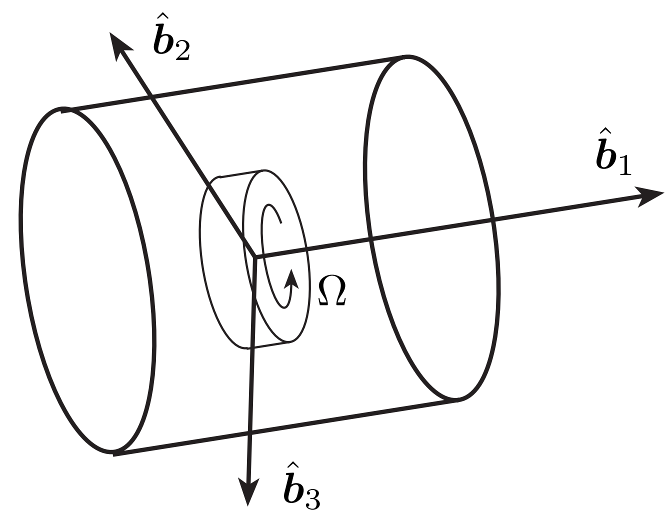
- Because no external torques are present, the angular momentum magnitude is constant and given by:

$$H = |\boldsymbol{H}(t_0)|$$

- The wheel angular momentum is increased at a constant rate through:

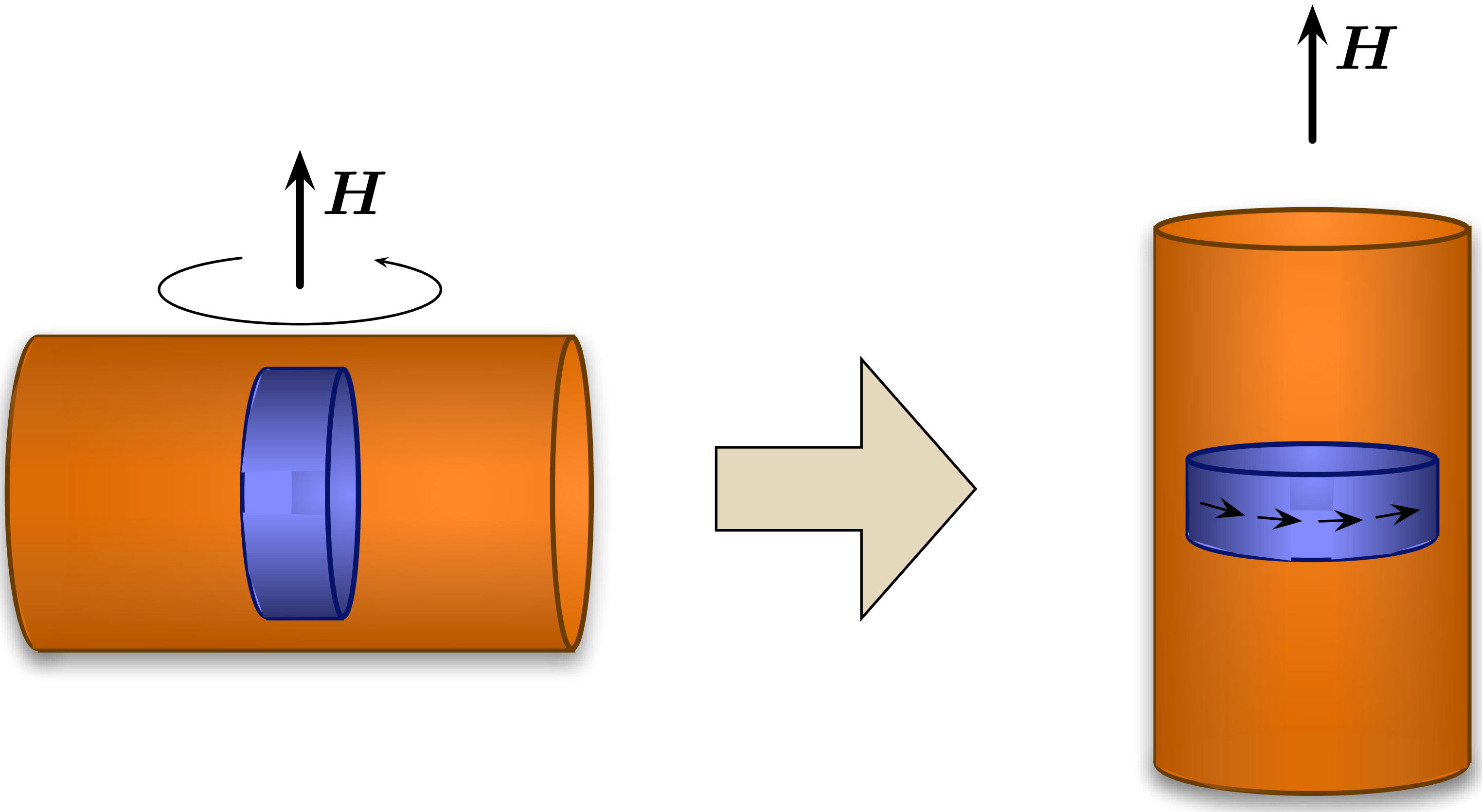
$$\dot{h} = I_W \dot{\Omega} = C = \text{constant}$$

$$h(t) = Ct$$



- The total maneuver time is

$$T_{\max} = \frac{H}{C} = \frac{H}{I_W \dot{\Omega}}$$

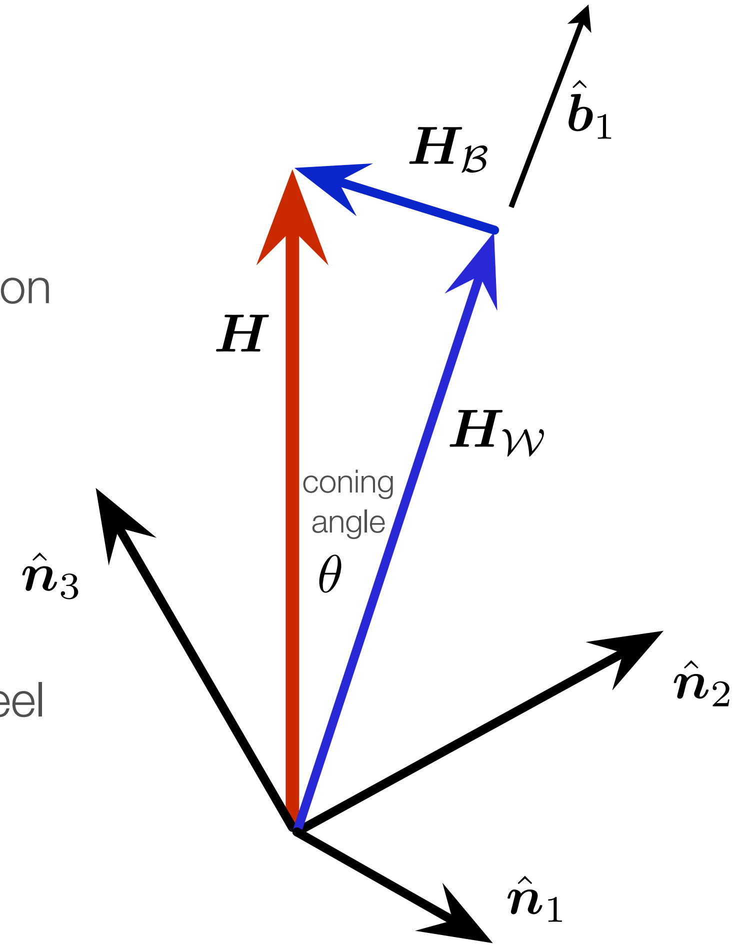


Question: As the RW assumes the same amount of angular momentum as the initially spinning spacecraft possessed, won't the angular momentum conservation cause the craft to realign RW spin axis along the momentum vector with the spacecraft at rest?

The answer is, not necessarily...

We are only controlling a single-degree of freedom, which is influencing the three-dimension motion of the spacecraft.

If the wheel angular momentum has the same magnitude as the initial system angular momentum ***H***, this does not mean that the wheel angular momentum $\boldsymbol{H}_W$ is aligned with the ***H*** vector.



Let's study this through an numerical example...

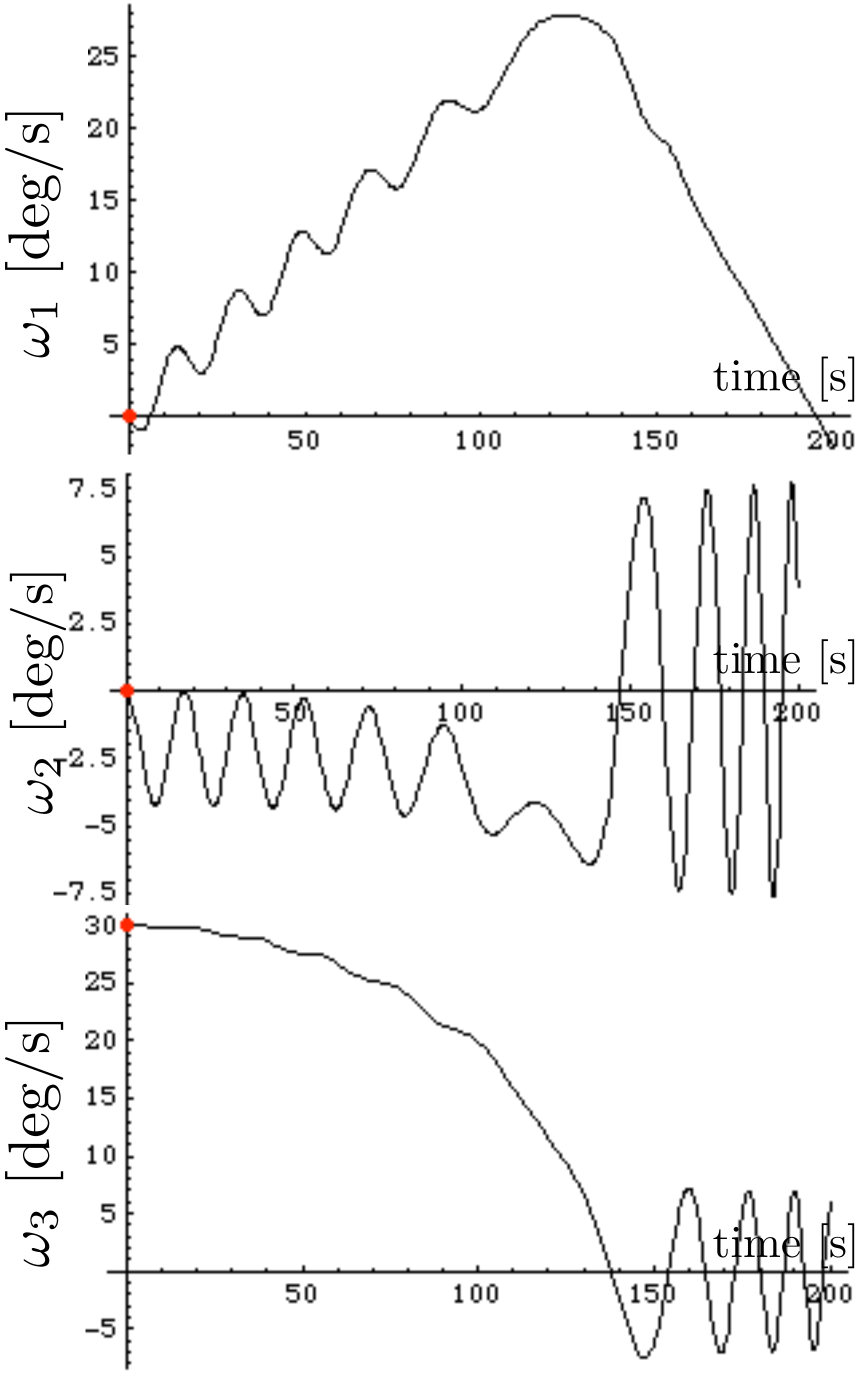
Example: Numerical Simulations of Spin up Maneuvers

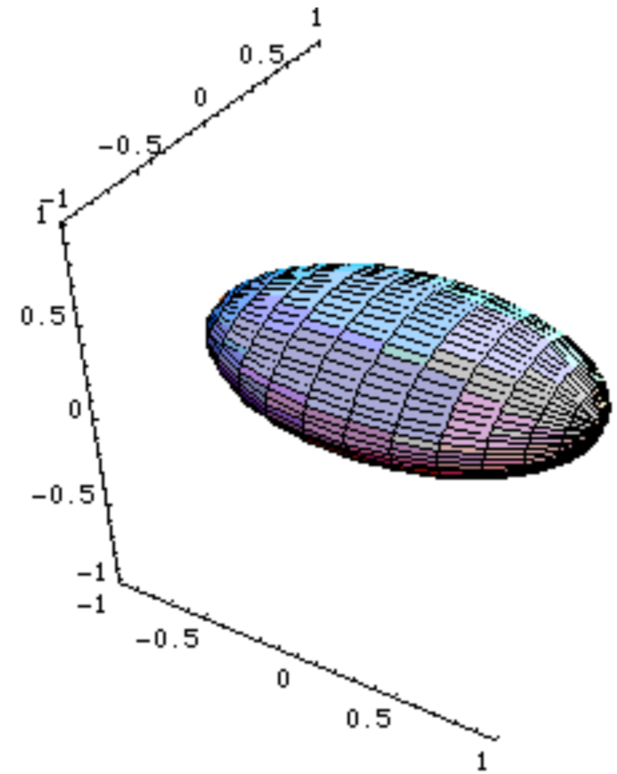
Spacecraft is initially in a flat spin about the major inertia principal axis.

The RW is spun up at different rates and the final coning angle θ is investigated.

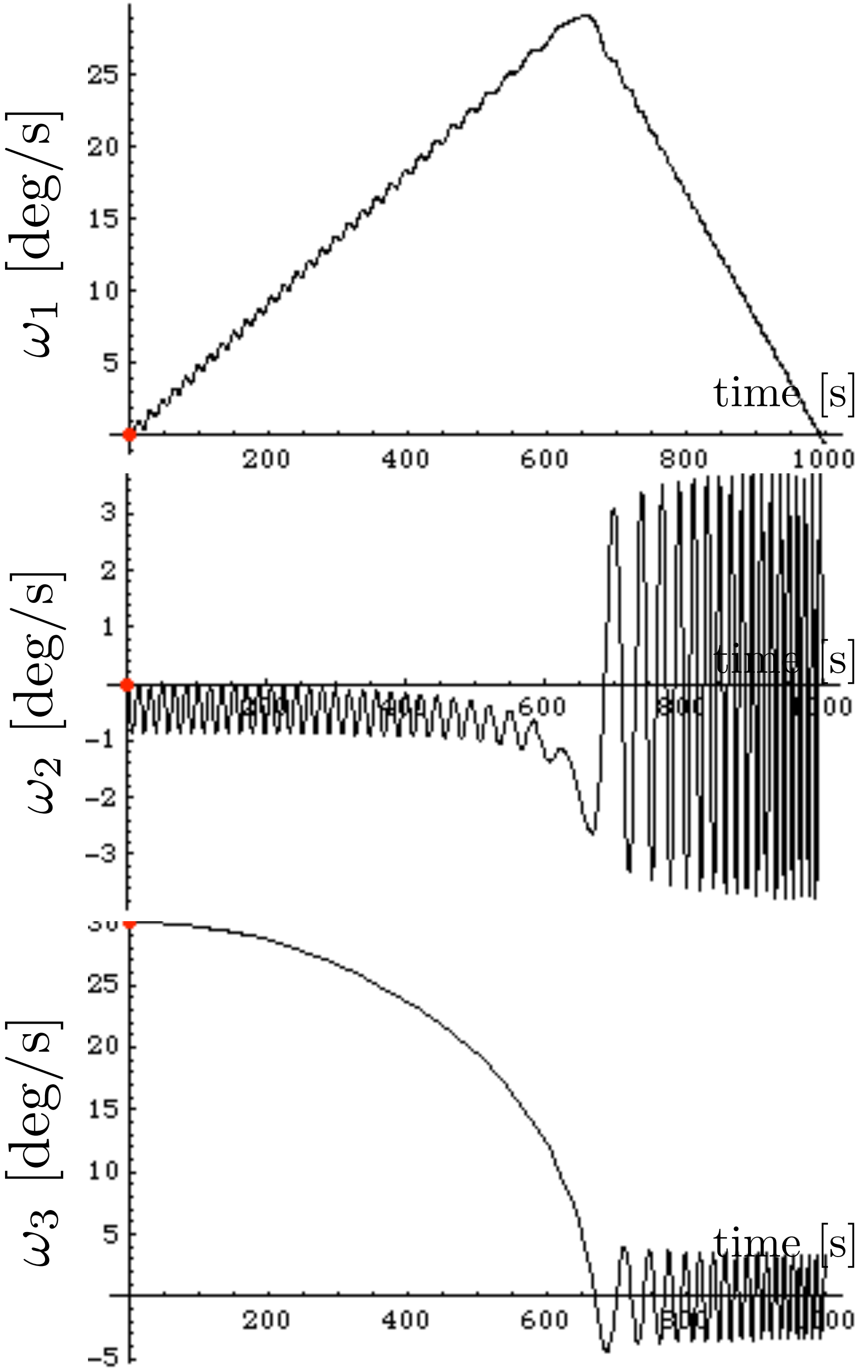
Simulation Parameters

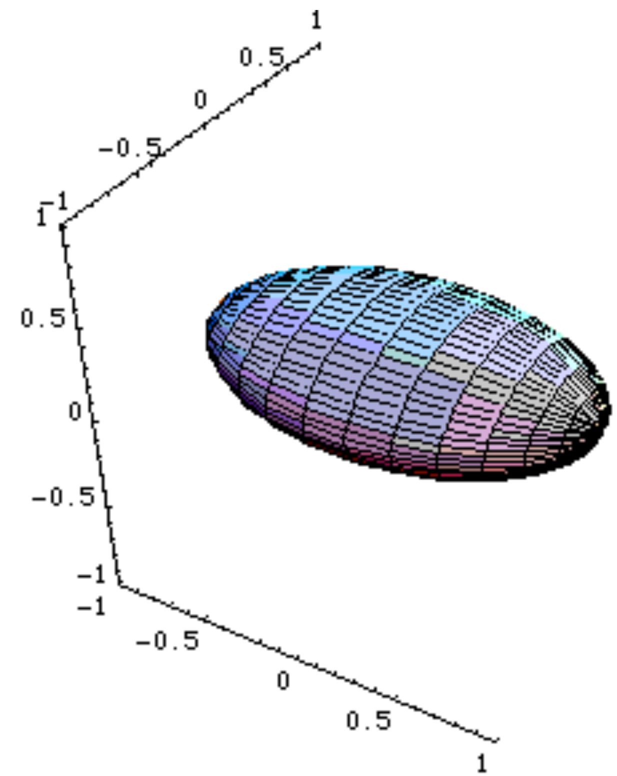
I_1	9.47 kgm^2
I_2	21.90 kgm^2
I_3	27.57 kgm^2
$\omega_3(t_0)$	30 °/s
$\omega_1(t_0)$, $\omega_2(t_0)$	0 °/s
$\Omega(t_0)$	0 °/s
I_W	1.89 kgm^2



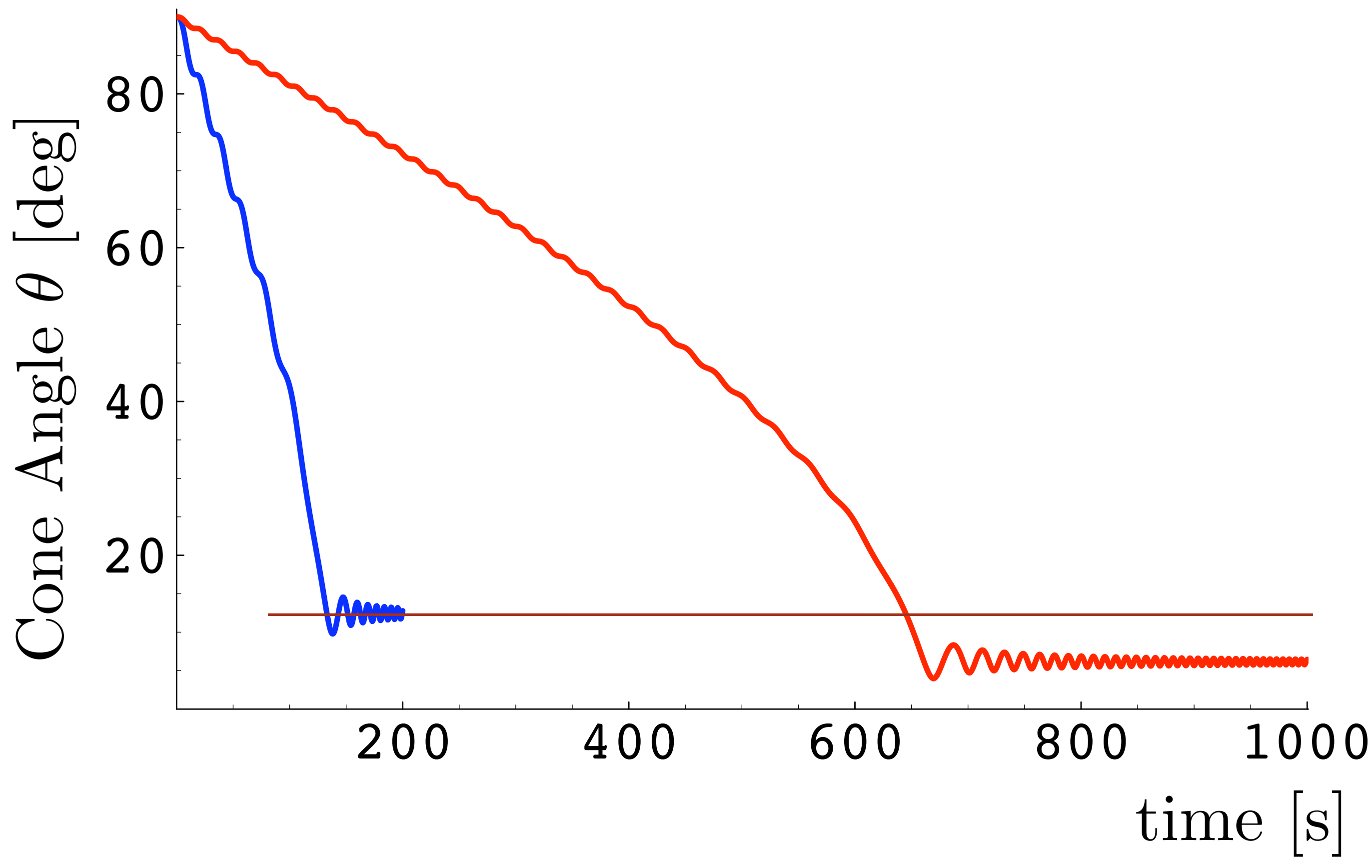


Maneuver Time: 200 seconds





Maneuver Time: 1000 seconds



Cone Angle θ [deg]
80
60
40
20
200
400
600
800
1000
time [s]

- To study the spin-up dynamics, we can use the momentum sphere –energy ellipsoid method.*

Spacecraft Kinetic Energy:

$$E^* = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

The control objective is to drive this positive definite measure of the spacecraft motion to zero!

Momentum Sphere:

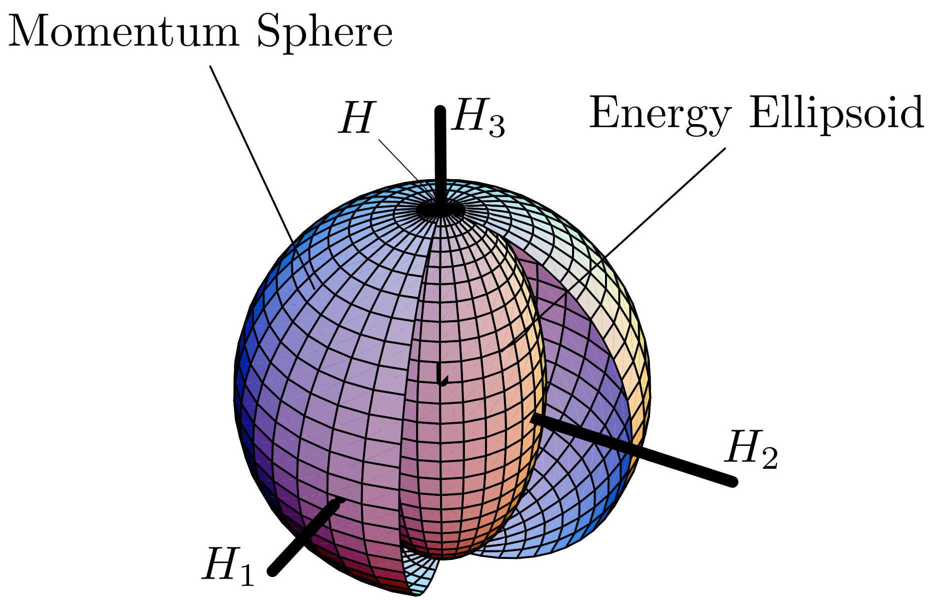
$$H^2 = H_1^2 + H_2^2 + H_3^2$$

Energy Ellipsoid:

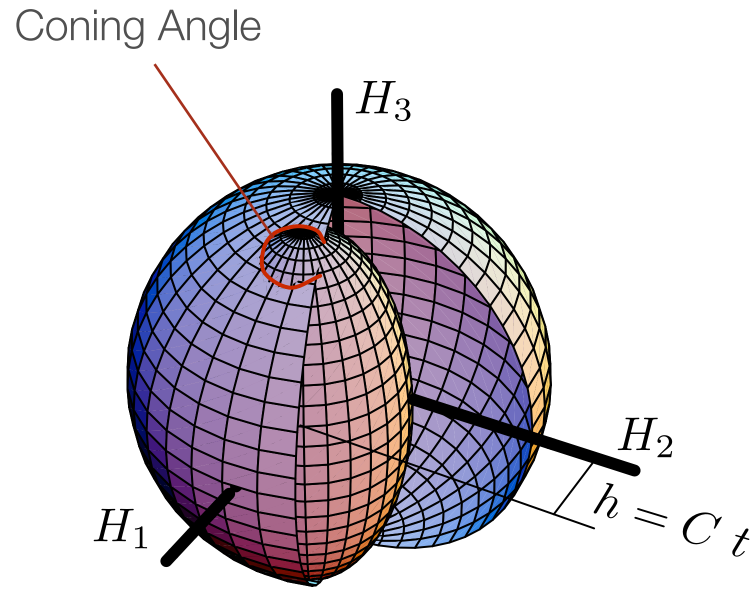
$$1 = \frac{(H_1 - h)^2}{2I_1E^*} + \frac{H_2^2}{2I_21E^*} + \frac{H_3^2}{2I_3E^*}$$

Note how the energy ellipsoid size will vary as the spacecraft kinetic energy is reduced, and how the ellipsoid center will shift along the first body axis.

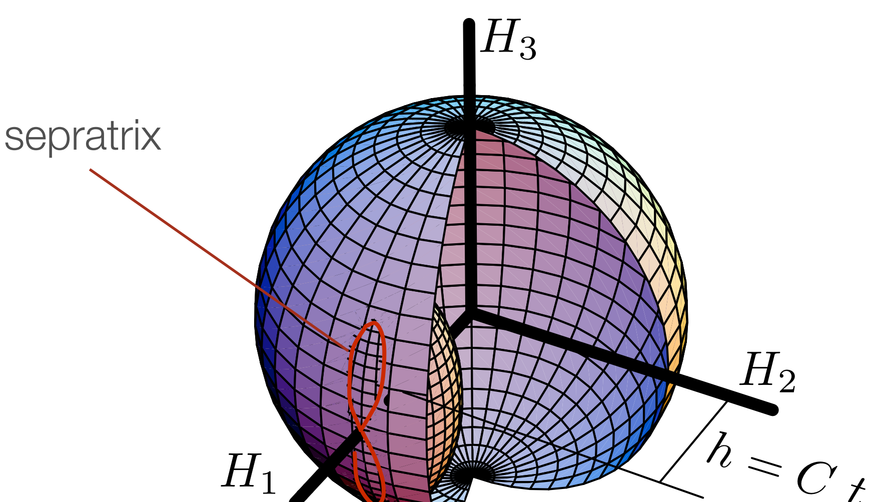
*Barba, P., and Auburn, J., “Satellite Attitude Acquisition by Momentum Transfer,” Paper #AAS-75-053, Presented at the AAS/AIAA Astrodynamics Conference, Nasau, Bahamas, July 1975.



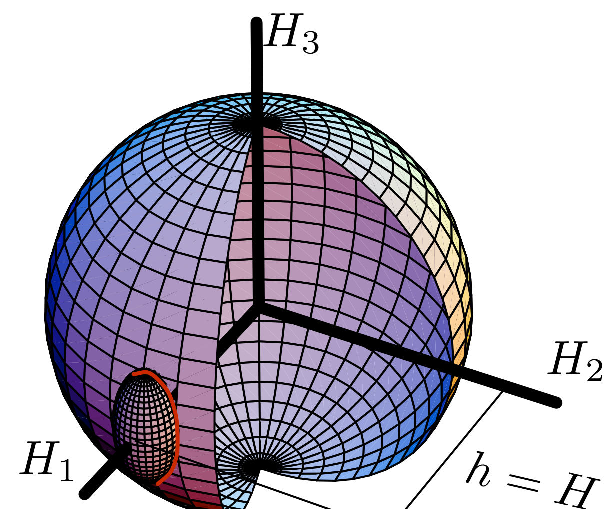
Spacecraft with pure spin about
3rd axis, RW at rest



The RW is spun up and reduces
the spacecraft kinetic energy.



A critical energy condition is
reached with the sepratrix.



The final stage has the spacecraft with a
non-zero energy state, and some coning.

Gravity Gradients

Nature’s free stabilization method...

Gravity Gradient Satellite

- For a rigid body in space, the "lower" parts of the body will be heavier than the "upper" parts!
- Techsat 21 satellites were originally planned to be G^2 satellites
- This tidal force will produce a torque onto the body, and cause the CM to move.

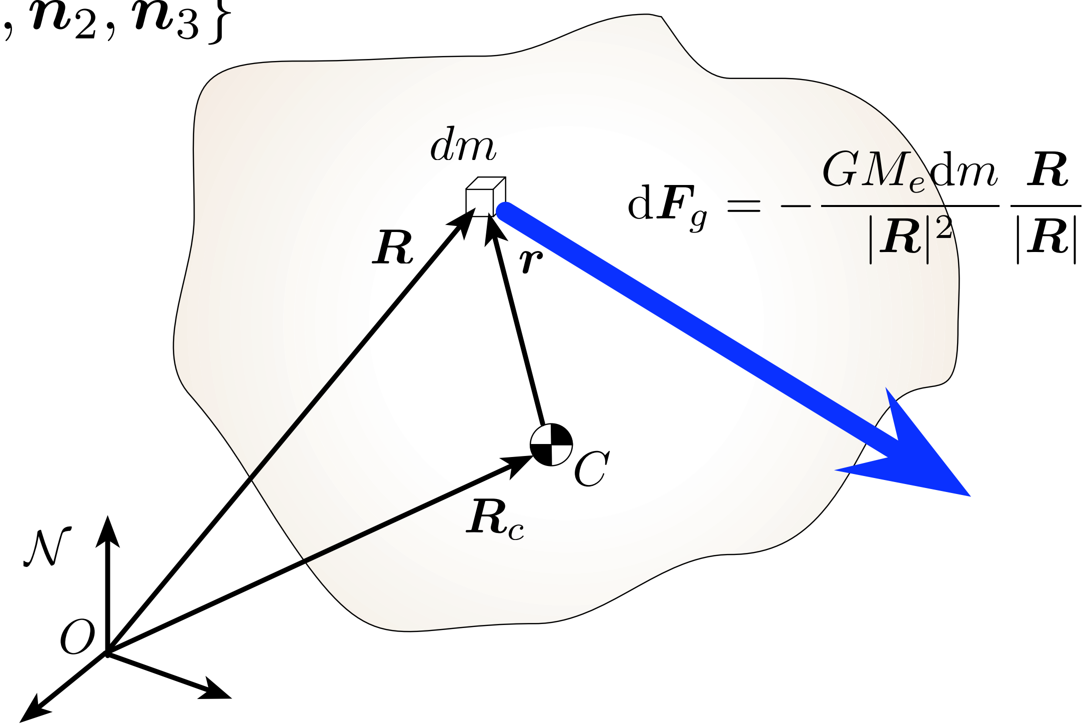
Gravity Gradient Torque

Inertial Frame: $\mathcal{N} : \{O, \hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$

Inertial Vectors: $\boldsymbol{R},\ \boldsymbol{R}_c$

Relative Vectors: $\boldsymbol{r}$

Note: $\boldsymbol{R} = \boldsymbol{R}_c + \boldsymbol{r}$



The Gravity gradient torque acting on the spacecraft is:

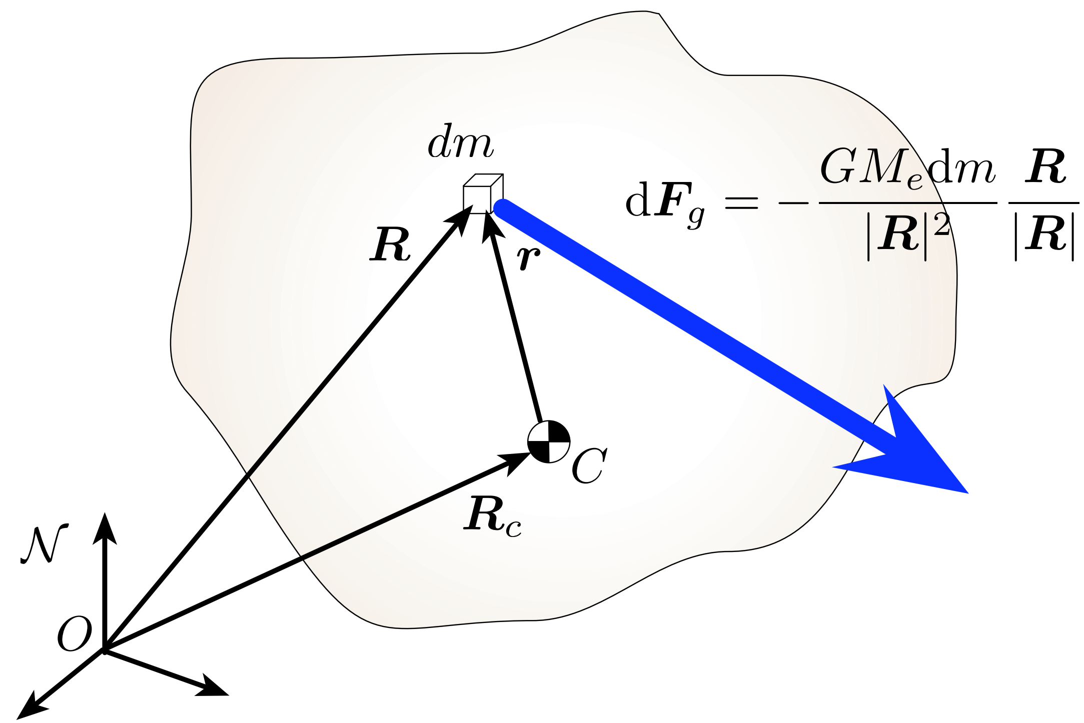
$$\boldsymbol{L}_G = \int_{\mathcal{B}} \boldsymbol{r} \times \mathrm{d}\boldsymbol{F}_G$$

The gravity force acting on the mass element is:

$$\mathrm{d}\boldsymbol{F}_G = -\frac{GM_e}{|\boldsymbol{R}|^3}\boldsymbol{R}\mathrm{d}m$$

The gravity gradient torque is then written as:

$$\boldsymbol{L}_G = -\int_{\mathcal{B}} \boldsymbol{r} \times \frac{GM_e}{|\boldsymbol{R}|^3}\left(\boldsymbol{R}_c + \boldsymbol{r}\right)\mathrm{d}m$$



- Taking all the constants outside of the integral term, we find:

$$\boldsymbol{L}_G = GM_e\boldsymbol{R}_c \times \int_{\mathcal{B}} \frac{\boldsymbol{r}}{|\boldsymbol{R}|^3}\mathrm{d}m$$

- Note that the inertial position vector ***R*** contains both ***R**$_c$* and ***r***. We can simplify the denominator using:

$$\begin{aligned} |\boldsymbol{R}|^{-3} = |\boldsymbol{R}_c + \boldsymbol{r}|^{-3} &= \left(R_c^2 + 2\boldsymbol{R}_c \cdot \boldsymbol{r} + r^2\right)^{-3/2} \\ &= \frac{1}{R_c^3}\left(1 + \frac{2\boldsymbol{R}_c \cdot \boldsymbol{r}}{R_c^2} + \left(\frac{r}{R_c}\right)^2\right)^{-3/2} \\ &\approx \frac{1}{R_c^3}\left(1 - \frac{3\boldsymbol{R}_c \cdot \boldsymbol{r}}{R_c^2} + \cdots\right) \end{aligned}$$

- Using this approximation for the $1/R^3$ term, the gravity gradient torque can be approximated as

$$\boldsymbol{L}_G = \frac{GM_e}{R_c^3}\boldsymbol{R}_c \times \int_{\mathcal{B}} \boldsymbol{r}\left(1 - \frac{3\boldsymbol{R}_c \cdot \boldsymbol{r}}{R_c^2}\right)\mathrm{d}m$$

- Using the center of mass definition

$$\int_{\mathcal{B}} \boldsymbol{r}\mathrm{d}m = 0$$

the gravity gradient torque is reduced to

$$\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\boldsymbol{R}_c \times \int_{\mathcal{B}} -\boldsymbol{r}\left(\boldsymbol{r} \cdot \boldsymbol{R}_c\right)\mathrm{d}m$$

- Using the vector identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$$

we can rewrite the following term

$$-\,(\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} = -\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$$

$$-(\boldsymbol{r} \cdot \boldsymbol{R}_c)\boldsymbol{r} = -\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{R}_c) - (\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{R}_c$$

- The torque vector is now written as

$$\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\boldsymbol{R}_c \times \int_{\mathcal{B}} -\,(\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{R}_c) + (\boldsymbol{r} \cdot \boldsymbol{r})\,\boldsymbol{R}_c)\,\mathrm{d}m$$

- Using the tilde matrix definition, we can reduce this expression to

$$\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\boldsymbol{R}_c \times \left(\int_{\mathcal{B}} -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m\right)\boldsymbol{R}_c - \frac{3GM_e}{R_c^5}\left(\int_{\mathcal{B}} r^2 \mathrm{d}m\right)\boldsymbol{R}_c \times \boldsymbol{R}_c$$

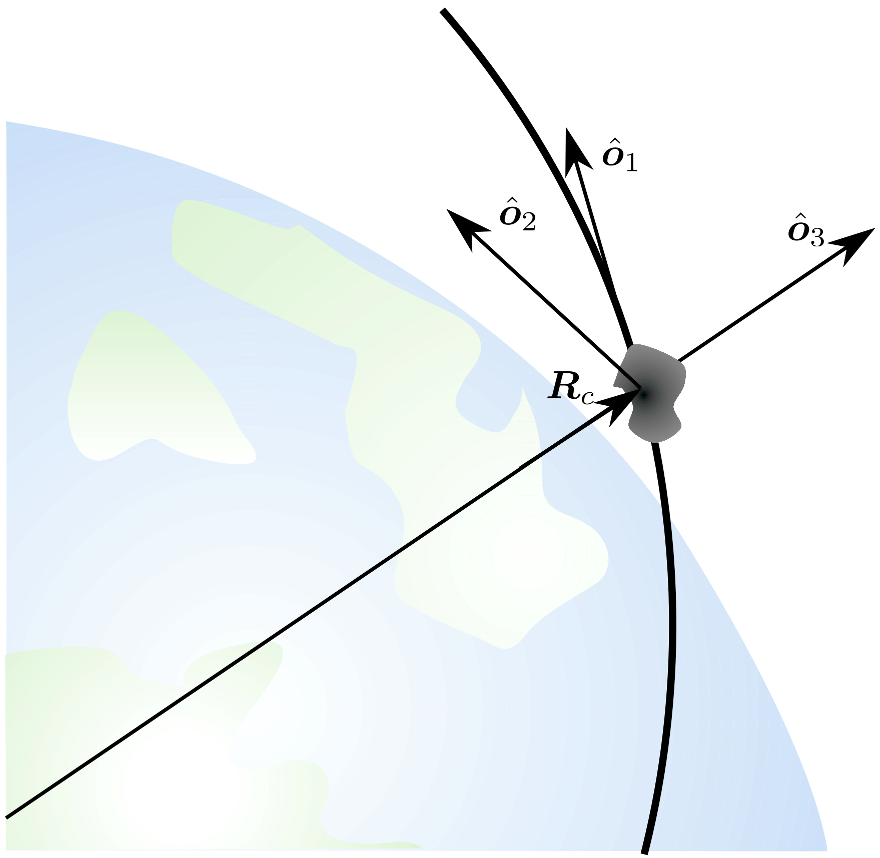
- Using the matrix definition

$$^{\mathcal{B}}[I_c] = \int_B -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m = \int_B {}^{\mathcal{B}}\begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} \mathrm{d}m$$

the gravity torque on a rigid body is **finally** written as

$$\boxed{\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\boldsymbol{R}_c \times [I]\boldsymbol{R}_c}$$

- The previous expression was still a general vector/matrix expression where the specific vector coordinate frame was not specified.
- Let us introduce the orbit frame *O* which tracks the center of mass of the rigid body as it rotates about the Earth.

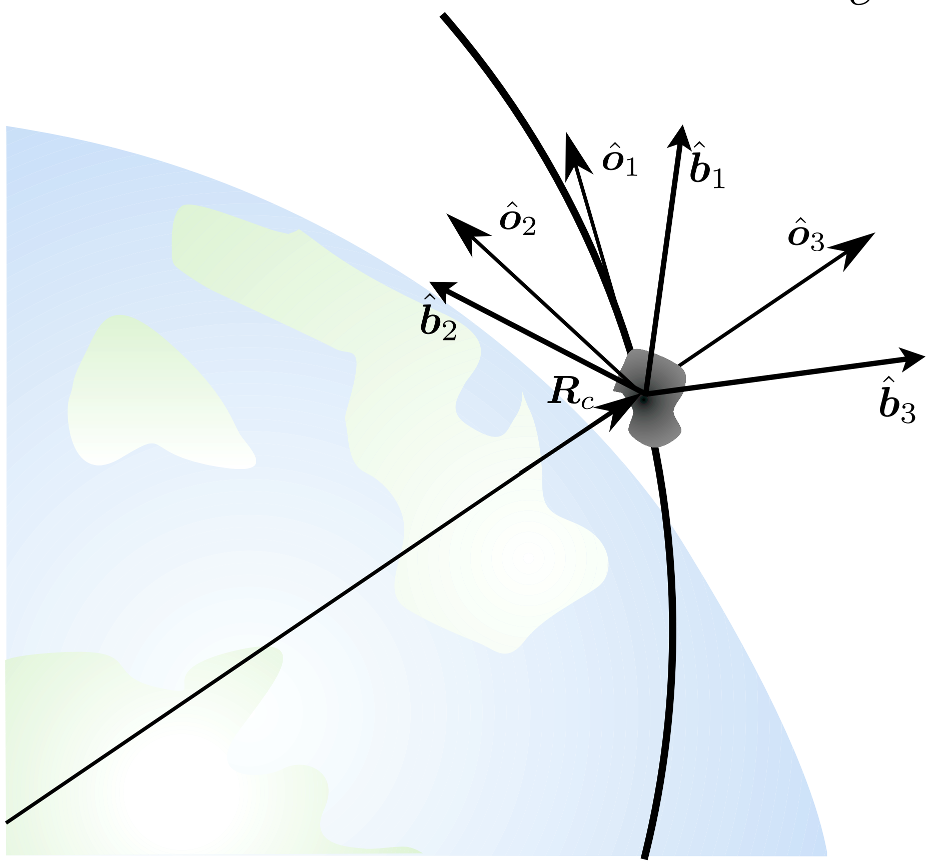


$$O : \{\hat{\boldsymbol{o}}_1, \hat{\boldsymbol{o}}_2, \hat{\boldsymbol{o}}_3\}$$

$$\boldsymbol{R}_c = R_c \hat{\boldsymbol{o}}_3 = {}^{\mathcal{O}}\begin{pmatrix} 0 \\ 0 \\ R_c \end{pmatrix}$$

- Since the rigid body dynamics are written in the body frame B, we can write the center of mass position vector in the body frame using:

$$\boldsymbol{R}_c = {}^{\mathcal{B}}\begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix} = [BO]\ {}^{\mathcal{O}}\begin{pmatrix} 0 \\ 0 \\ R_c \end{pmatrix}$$



- Assuming that the inertia matrix [*I*] is taken with respect to a principal coordinate system (i.e. [*I*] is diagonal), the gravity torque vector can now be written as

$$\boldsymbol{L}_G = \frac{3GM_e}{R_c^5} \begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix} \times \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix}$$

$$\boxed{\boldsymbol{L}_G = \begin{pmatrix} L_{G_1} \\ L_{G_2} \\ L_{G_3} \end{pmatrix} = \frac{3GM_e}{R_c^5} \begin{pmatrix} R_{c_2} R_{c_3} \left(I_{33} - I_{22} \right) \\ R_{c_1} R_{c_3} \left(I_{11} - I_{33} \right) \\ R_{c_1} R_{c_2} \left(I_{22} - I_{11} \right) \end{pmatrix}}$$

This torque (or components thereof) can be zero if:

$$I_{ii} = I_{jj} \qquad\qquad \boldsymbol{R}_c = R_c \hat{\boldsymbol{b}}_i$$

Center of Mass Motion

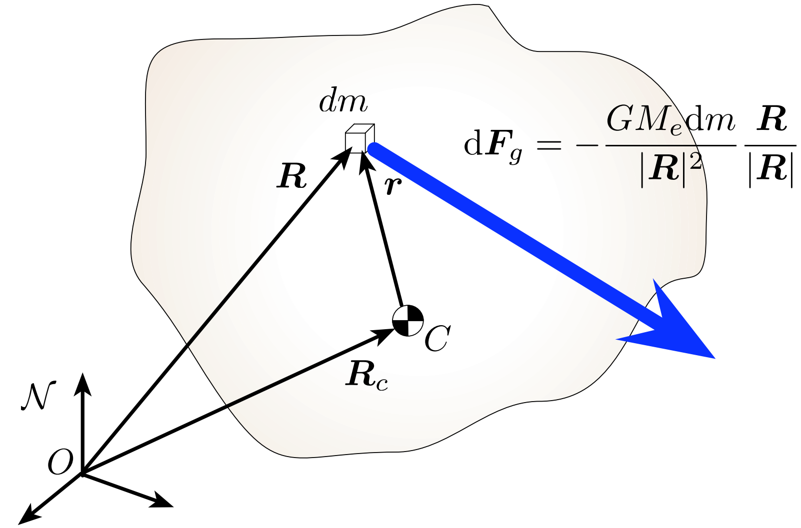
- Next, let's study how the center of mass of a rigid body will move while in orbit.
- From astrodynamics, we have seen that the orbit of a point mass *m* has the differential equations of motion:

$$\ddot{\boldsymbol{R}}_c = -\frac{GM_e}{R_c^3}\boldsymbol{R}_c \qquad \text{or} \qquad m\ddot{\boldsymbol{R}}_c = -\frac{GM_e}{R_c^3}m\boldsymbol{R}_c$$

Question: *How will the center of mass of a rigid body move? The gravitation force on the various mass elements will contribute to accelerate the CM.*

- The total gravity force is computed by integrating all gravity forces over the entire body:

$$\boldsymbol{F}_G = \int_{\mathcal{B}} \mathrm{d}\boldsymbol{F}_G = -GM_e \int_{\mathcal{B}} \frac{\boldsymbol{R}}{|\boldsymbol{R}|^3}\mathrm{d}m$$



Using the simplifying assumption:

$$\frac{1}{|\boldsymbol{R}|^3} \approx \frac{1}{R_c^3}\left[1 - \frac{3}{2}\left(2\frac{\boldsymbol{R}_c\cdot\boldsymbol{r}}{R_c^2} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}}{R_c^2}\right) + \frac{15}{2}\frac{(\boldsymbol{R}_c\cdot\boldsymbol{r})^2}{R_c^4}\right]$$

we find the gravity force expression:

$$\boldsymbol{F}_G = -\frac{GM_e}{R_c^3}\left[\cancel{\int_{\mathcal{B}} \boldsymbol{r}\mathrm{d}m} - \frac{3}{R_c^2}\int_{\mathcal{B}}(\boldsymbol{r}\cdot\boldsymbol{R}_c)\boldsymbol{r}\mathrm{d}m - \cancel{\frac{3}{R_c^2}\int_{\mathcal{B}}(\boldsymbol{R}_c\cdot\boldsymbol{r})\boldsymbol{R}_c\mathrm{d}m}\right.$$

$$\left. + \boldsymbol{R}_c\int_{\mathcal{B}}\mathrm{d}m - \frac{3}{2R_c^2}\int_{\mathcal{B}}\boldsymbol{R}_c(\boldsymbol{r}\cdot\boldsymbol{r})\mathrm{d}m + \frac{15}{2R_c^4}\int_{\mathcal{B}}(\boldsymbol{R}_c\cdot\boldsymbol{r})^2\boldsymbol{R}_c\mathrm{d}m\right]$$

- Using the center of mass definition and the vector identity

$$-(\boldsymbol{r}\cdot\boldsymbol{R}_c)\boldsymbol{r} = -\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{R}_c) - (\boldsymbol{r}\cdot\boldsymbol{r})\boldsymbol{R}_c$$

leads to:

$$\boldsymbol{F}_G = -\frac{GM_e}{R_c^3}\Bigg[m\boldsymbol{R}_c - \frac{3}{R_c^2}\int_{\mathcal{B}}\Big(\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{R}_c) + r^2\boldsymbol{R}_c\Big)\mathrm{d}m - \frac{3}{2R_c^2}\int_{\mathcal{B}} r^2\boldsymbol{R}_c\mathrm{d}m + \frac{15}{2R_c^4}\int_{\mathcal{B}}\boldsymbol{R}_c\cdot\Big(\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{R}_c) + r^2\boldsymbol{R}_c\Big)\boldsymbol{R}_c\mathrm{d}m\Bigg]$$

Next, note the identities:

$$\int_{\mathcal{B}} r^2\mathrm{d}m = \frac{1}{2}\mathrm{tr}([I]) \qquad \hat{\boldsymbol{\imath}}_r = \boldsymbol{R}_c/R_c$$

- Using the inertia matrix definition

$$[I_c] = \int_B -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m$$

The gravity force is finally written as

$$\boxed{\boldsymbol{F}_G = -\frac{\mu m}{R_c^3}\left[1 + \frac{3}{mR_c^2}\left([I] + \frac{1}{2}\Big(\mathrm{tr}([I]) - 5(\hat{\boldsymbol{\imath}}_r^T[I]\hat{\boldsymbol{\imath}}_r)\Big)[I_{3\times 3}]\right)\right]\boldsymbol{R}_c}$$

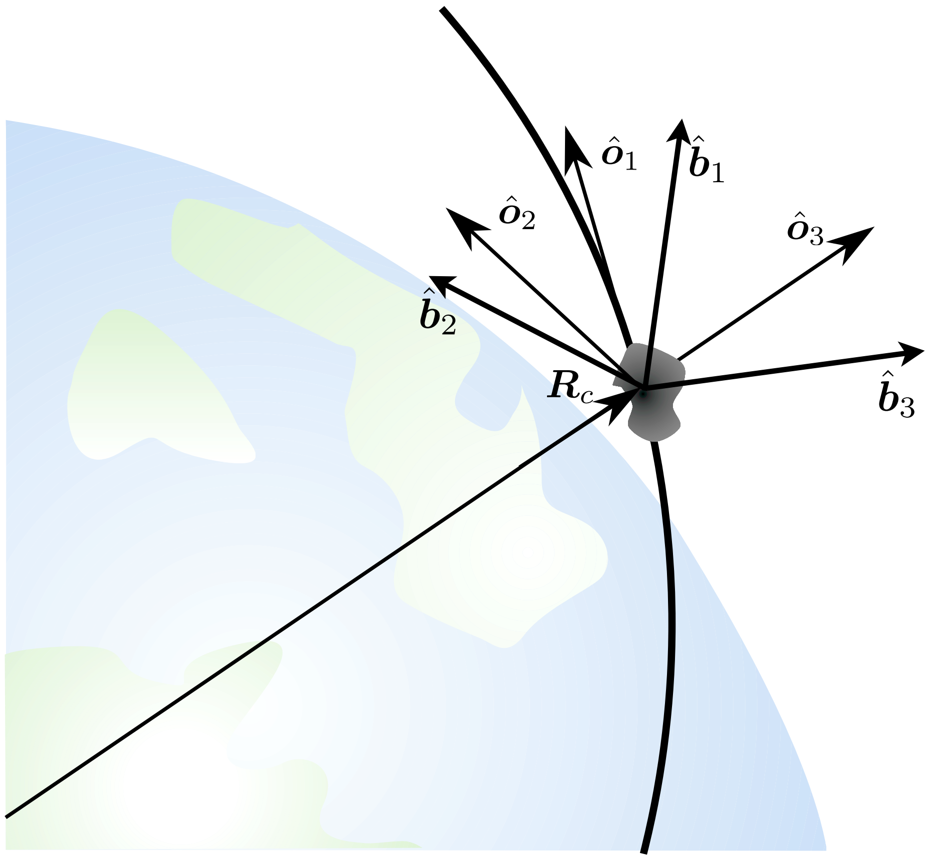
Compare to typical orbit force expression of point mass:

$$m\ddot{\boldsymbol{R}}_c = -\frac{GM_e}{R_c^3}m\boldsymbol{R}_c$$

The rigid body coupling to the center of mass motion is often ignored, because it is many, many orders of magnitude smaller than the orbital acceleration.

Relative Equilibrium State

We seek a rigid body attitude/orientation where the craft will remain stationary as seen by the rotating orbit frame.



Equations of motion:

$$[I]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{L}_G$$

Angular velocities:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} + \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}}$$

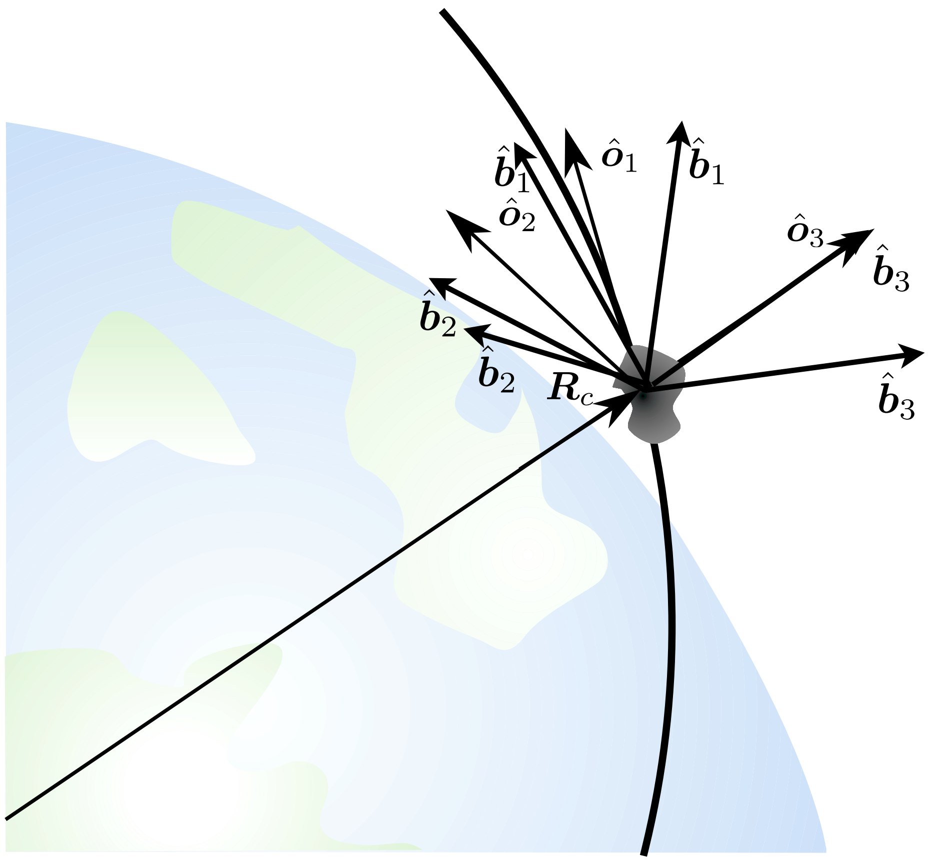
$$\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = n\hat{\boldsymbol{o}}_2$$

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \boldsymbol{0}$$

Relative Equilibria Condition

Equations of motion:

$$[I]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{L}_G$$



Gravity Gradient Torque:

$$\boldsymbol{L}_G = \boldsymbol{0}$$

$$\Downarrow$$

$$\hat{\boldsymbol{b}}_3 = \hat{\boldsymbol{o}}_3$$

must be principal axis of body

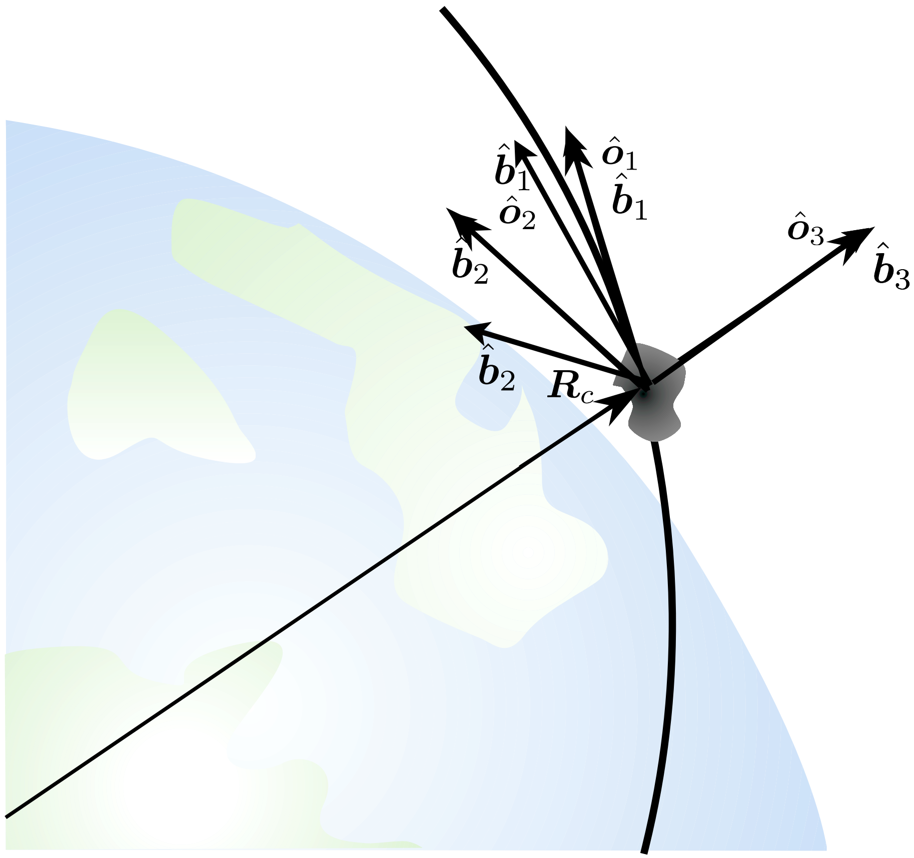
$$\Downarrow$$

$$[I] = {}^{\mathcal{O}}\begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$

This leads to this block diagonal form

Equations of motion:

$$[I]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\omega}_{\mathcal{B}/\mathcal{N}}][I]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{L}_G$$



Angular velocity condition:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \mathbf{0} \qquad \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{O}} = \mathbf{0}$$

$$\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = n\hat{\boldsymbol{o}}_2 \qquad \dot{\boldsymbol{\omega}}_{\mathcal{O}/\mathcal{N}} = \mathbf{0}$$

$$[\tilde{\omega}_{\mathcal{B}/\mathcal{N}}][I]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \mathbf{0}$$

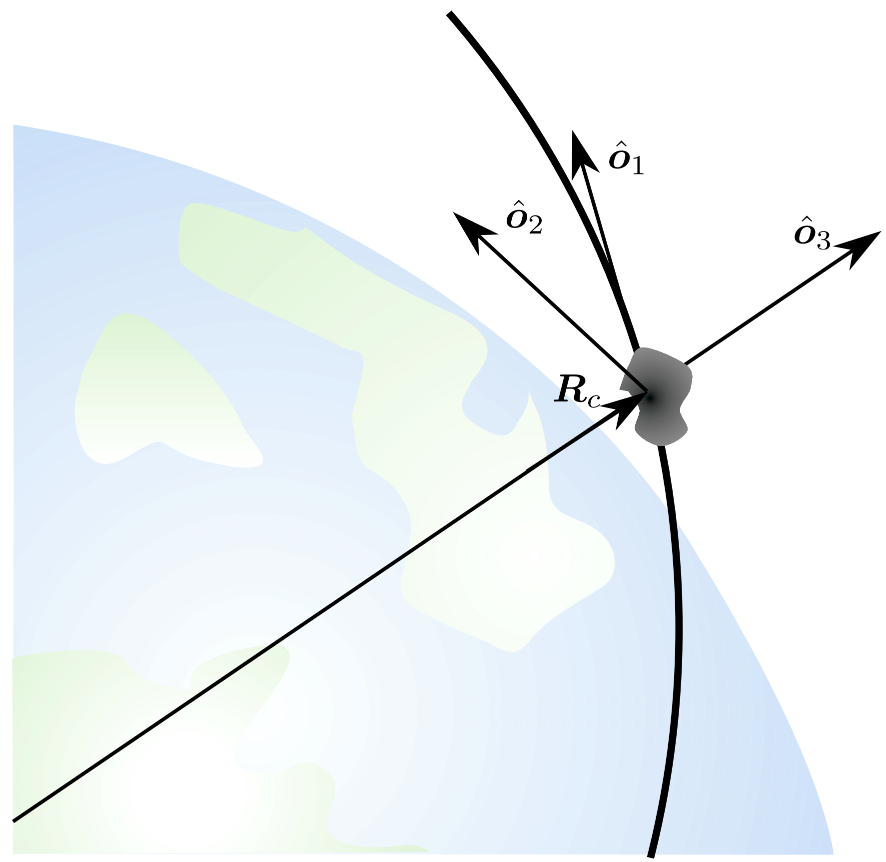
$${}^{\mathcal{O}}\begin{pmatrix} 0 \\ 0 \\ -I_{12}n^2 \end{pmatrix} = \mathbf{0}$$

$$I_{12} = 0$$

As a result, we find that relative equilibria must have all principal axes aligned with the orbit frame.

Gravity Gradient Motion

- Next, we study how the gravity torque vector will rotate the craft.
- The gravity torque is the only external force acting on the single-rigid body spacecraft.
- We use the “airplane” and “ship” like orbit frame *O*.
- Note, roll is about $\hat{o}_1$, pitch is about $\hat{o}_2$, yaw is about $\hat{o}_3$.



- The inertial angular velocity of the orbital frame O is:

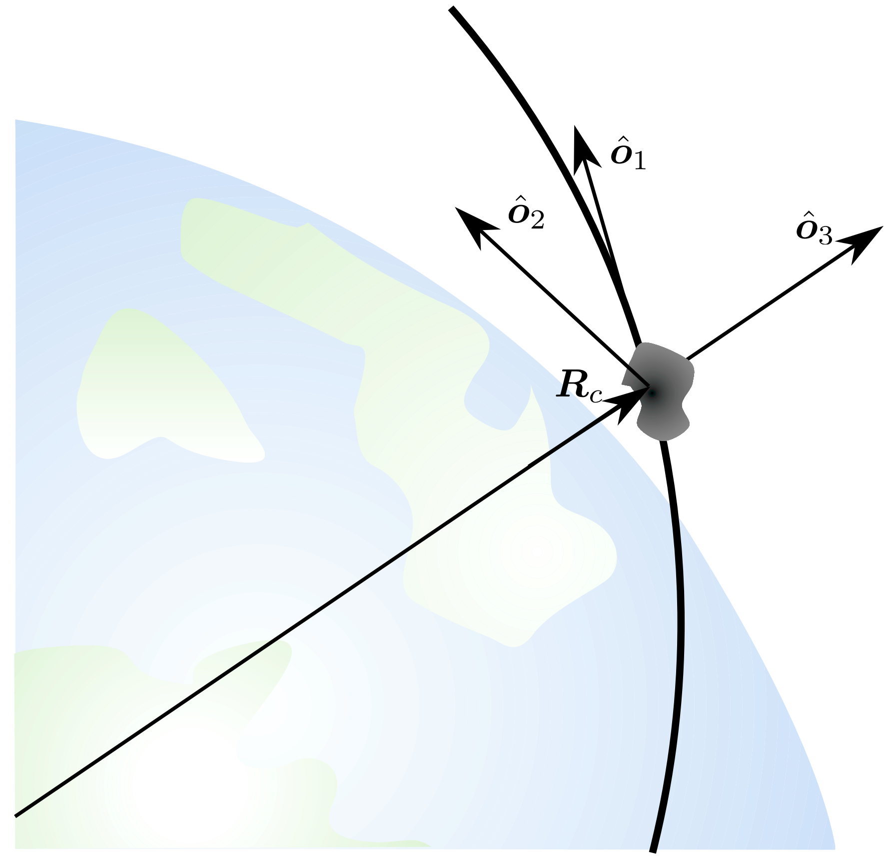
$$\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = \Omega \hat{\boldsymbol{o}}_2$$

- For a circular orbit, the constant orbit rate magnitude is given through Kepler's equation:

$$\Omega^2 = \frac{GM_e}{R_c^3}$$

- The spacecraft frame B is assumed to be a principal coordinate system. It's angular rate relative to the orbit frame O is:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}}$$



- We would like to study the yaw, pitch and roll motion (3-2-1 Euler angles) of the rigid body, if the gravitational torque is acting on it.

- From rigid body kinematics, we can relative the yaw, pitch and roll rates to body angular velocities through:

$$ {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} $$

- The spacecraft angular velocity relative to the *inertial* frame *N* is:

$$ \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} + \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} $$

- Further, also from rigid body kinematics, we can express the [*BO*] rotation matrix using the yaw, pitch and roll angles of the spacecraft with respect to the orbit frame *O* as:

$$[BO] = \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta \\ c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta \end{bmatrix}$$

- To be able to add up the angular rate vectors in *B* frame components, we find

$$\begin{aligned} {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} &= [BO]\,{}^{\mathcal{O}}\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = [BO](\Omega \hat{\boldsymbol{o}}_2) \\ &= \Omega \begin{pmatrix} c\theta s\psi \\ s\phi s\theta s\psi + c\phi c\psi \\ c\phi s\theta s\psi - s\phi c\psi \end{pmatrix} \end{aligned}$$

- Now we are able to compute the inertial body angular velocity vector:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} + \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}}$$

$${}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = {}^{\mathcal{B}}\begin{pmatrix} \dot{\phi} - s\theta\dot{\psi} + \Omega c\theta s\psi \\ s\phi c\theta\dot{\psi} + c\phi\dot{\theta} + \Omega(s\phi s\theta s\psi + c\phi c\psi) \\ c\phi c\theta\dot{\psi} - s\phi\dot{\theta} + \Omega(c\phi s\theta s\psi - s\phi c\psi) \end{pmatrix}$$

- This expression is valid for any large rotation of the spacecraft with respect to the orbit frame.

- Next, we would like to look at small rotation about the *O* frame. Here the yaw, pitch and roll angles are all treated as small angles.

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega} = {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \approx {}^{\mathcal{B}}\begin{pmatrix} \dot{\phi} + \Omega\psi \\ \dot{\theta} + \Omega \\ \dot{\psi} - \Omega\phi \end{pmatrix}$$

- The inertial angular acceleration is approximated as:

$$\dot{\boldsymbol{\omega}} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega}) + \boldsymbol{\omega} \times \boldsymbol{\omega} \approx {}^{\mathcal{B}}\begin{pmatrix} \ddot{\phi} + \Omega\dot{\psi} \\ \ddot{\theta} \\ \ddot{\psi} - \Omega\dot{\phi} \end{pmatrix}$$

- These two equations can later be used in the rigid body equations of motion:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{L}_c$$

- We still need to simplify the external gravity gradient torque expression for the case where yaw, pitch and roll are small angles

$$\boldsymbol{R}_c = {}^{\mathcal{B}}\begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix} = [BO]\, {}^{\mathcal{O}}\begin{pmatrix} 0 \\ 0 \\ R_c \end{pmatrix}$$

$${}^{\mathcal{B}}\begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix} = {}^{\mathcal{B}}\begin{pmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{pmatrix} R_c$$

Substituting this $\boldsymbol{R}_c$ expression into the gravity gradient torque definition, we find:

$${}^{\mathcal{B}}\boldsymbol{L}_G = \frac{3}{2}\Omega^2\, {}^{\mathcal{B}}\begin{pmatrix} (I_{33} - I_{22})\cos^2\theta\sin 2\phi \\ -(I_{11} - I_{33})\cos\phi\sin 2\theta \\ -(I_{22} - I_{11})\sin\phi\sin 2\theta \end{pmatrix}$$

- The nonlinear gravity torque vector is repeated here as:

$$
{}^{\mathcal{B}}\boldsymbol{L}_G = \frac{3}{2}\Omega^2 \; {}^{\mathcal{B}}\!\begin{pmatrix} (I_{33} - I_{22})\cos^2\theta \sin 2\phi \\ -(I_{11} - I_{33})\cos\phi \sin 2\theta \\ -(I_{22} - I_{11})\sin\phi \sin 2\theta \end{pmatrix}
$$

- Note that the body frame torque components **do not depend on the yaw angle**.

- Linearizing this torque for small attitude angles, we find:

$$
{}^{\mathcal{B}}\boldsymbol{L}_G \approx 3\Omega^2 \; {}^{\mathcal{B}}\!\begin{pmatrix} (I_{33} - I_{22})\,\phi \\ -(I_{11} - I_{33})\,\theta \\ 0 \end{pmatrix}
$$

Note: the linearized torque will never have a yaw component.

- Now we are able to write the equations of motion of a rigid spacecraft in a circular orbit subject to an inverse square gravity field. We use

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{L}_c$$

and substitute in the previous linearized results to find

$$\begin{aligned}
I_{11}\left(\ddot{\phi} + \Omega\dot{\psi}\right) &= -\left(I_{33} - I_{22}\right)\left(\dot{\theta} + \Omega\right)\left(\dot{\psi} - \Omega\phi\right) + 3\Omega^2\left(I_{33} - I_{22}\right)\phi \\
I_{22}\ddot{\theta} &= -\left(I_{11} - I_{33}\right)\left(\dot{\psi} - \Omega\phi\right)\left(\dot{\phi} + \Omega\psi\right) - 3\Omega^2\left(I_{11} - I_{33}\right)\theta \\
I_{33}\left(\ddot{\psi} - \Omega\dot{\phi}\right) &= -\left(I_{22} - I_{11}\right)\left(\dot{\phi} + \Omega\psi\right)\left(\dot{\theta} + \Omega\right)
\end{aligned}$$

Note: These expression contain products of angles! Since we are assuming small angles here, these equations can be further simplified.

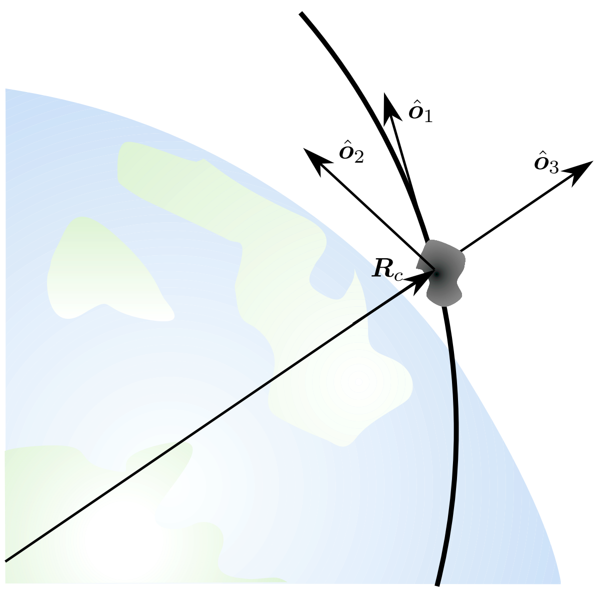
- The pitch equations can be decoupled from the yaw and roll equations!

$$\ddot{\theta} + 3\Omega^2 \left(\frac{I_{11} - I_{33}}{I_{22}} \right) \theta = 0$$

- Compare these equations to the spring-mass system

$$\ddot{x} + \frac{k}{m} x = 0$$

- This spring system is stable if the spring stiffness $k \geq 0$. Thus, the decoupled pitch motion is stable if



$$3\Omega^2 \left(\frac{I_{11} - I_{33}}{I_{22}} \right) \geq 0 \quad \Rightarrow \quad \boxed{I_{11} \geq I_{33}}$$

- The yaw and roll motion of the spacecraft are coupled through the gravity gradient torque:

$$\begin{pmatrix} \ddot{\phi} \\ \ddot{\psi} \end{pmatrix} + \begin{bmatrix} 0 & \Omega\left(1-k_Y\right) \\ \Omega\left(k_R-1\right) & 0 \end{bmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} + \begin{bmatrix} 4\Omega^2 k_Y & 0 \\ 0 & \Omega^2 k_R \end{bmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$$

where we introduce the inertia ratios:

$$k_R = \frac{I_{22} - I_{11}}{I_{33}} \qquad k_Y = \frac{I_{22} - I_{33}}{I_{11}}$$

- To prove stability of this coupled linear time-invariant system, we need to examine the characteristic equation.

$$\lambda^4 + \lambda^2\Omega^2\left(1 + 3k_Y + k_Y k_R\right) + 4\Omega^4 k_Y k_R = 0$$

The system is stable if **NO** roots have positive real components!

- Let's rewrite the characteristic equation into the convenient form:

$$\lambda^4 + b_1\lambda^2 + b_0 = 0$$

- We can solve this as a quadratic equations for λ^2:

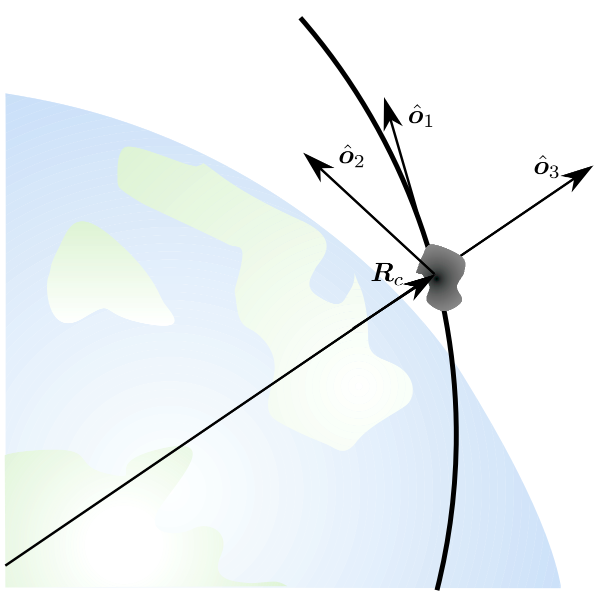
$$\lambda^2 = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_0}}{2}$$

- Next, we need to check what conditions apply to guarantee that no root of this characteristic equation has positive real components.

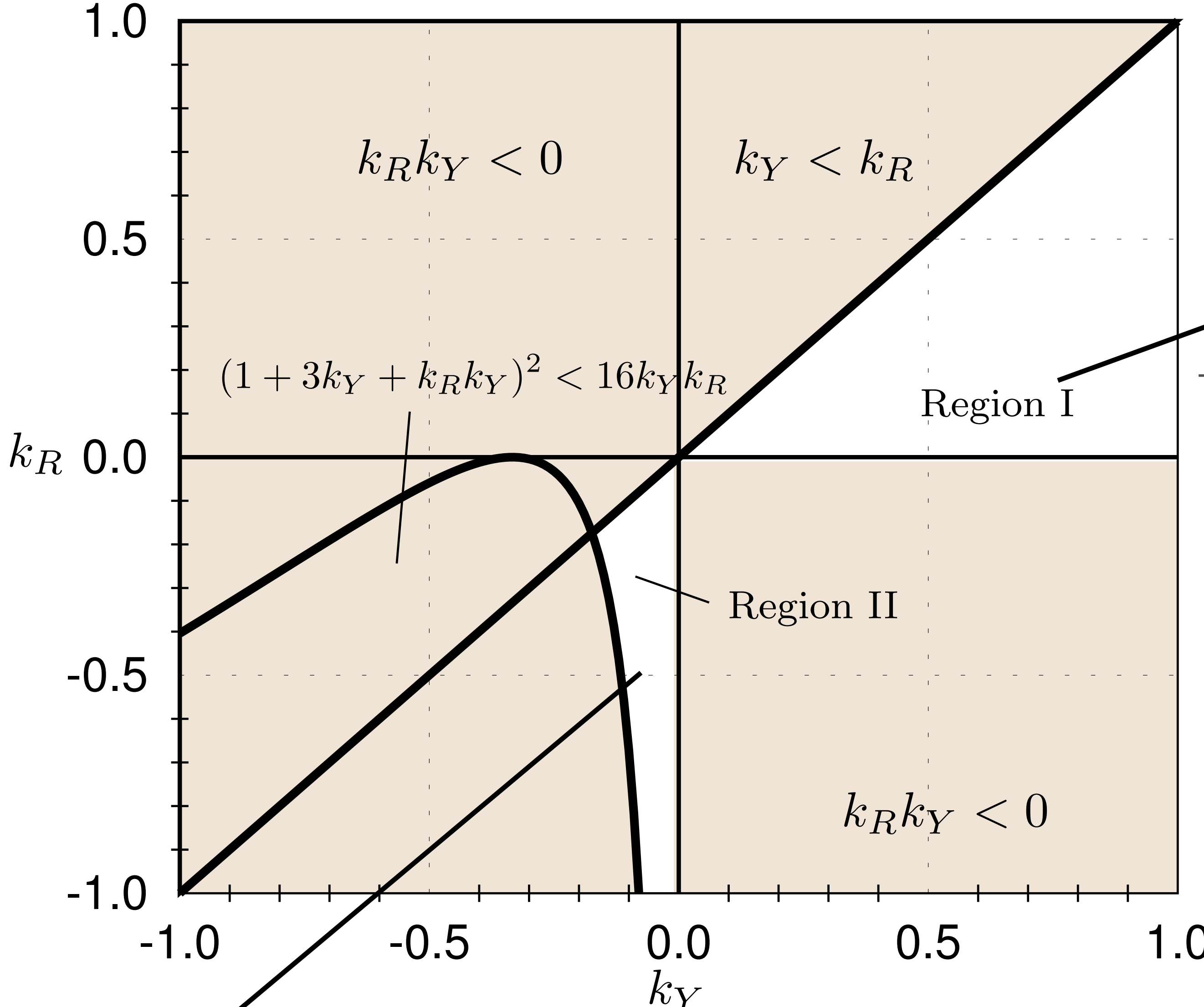
- The yaw-roll motion stability conditions can be summarized as:

$$\begin{aligned} k_R k_Y &> 0 \\ 1 + 3k_Y + k_Y k_R &> 0 \\ (1 + 3k_Y + k_Y k_R)^2 &> 16 k_Y k_R \\ k_Y &> k_R \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} b_0 &> 0 \\ b_1 &> 0 \\ b_1^2 - 4b_0 &> 0 \\ I_{11} &> I_{33} \end{aligned}$$

Note: the first condition states that the only solutions will be in the first and third quadrant. This is equivalent to



$$\begin{aligned} I_{22} &> I_{11}, I_{33} \\ I_{22} &< I_{11}, I_{33} \end{aligned}$$



$I_2 \geq I_1 \geq I_3$

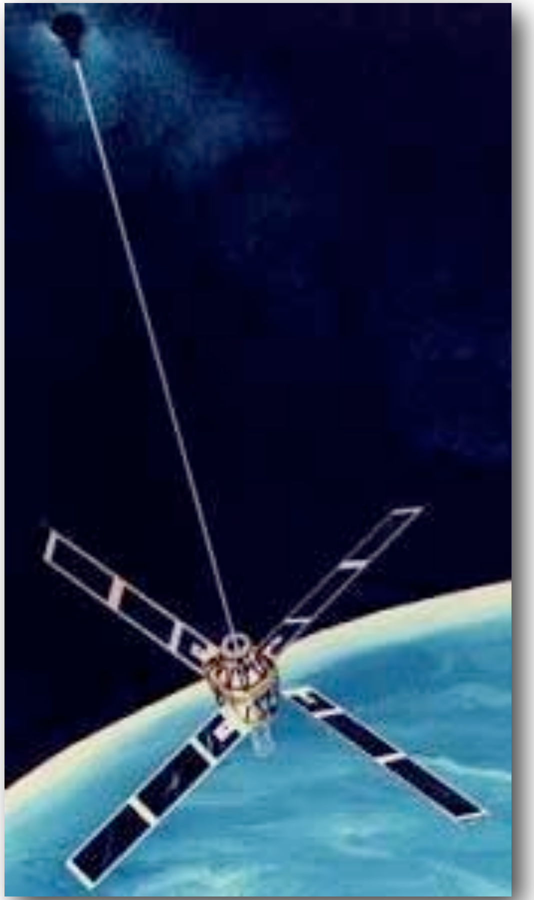
Typically used.

$I_1 \geq I_3 \geq I_2$

Only marginally stable.
Not typically used.

Polar Bear G^2 Mission

- Mission: Polar Bear (P87-1)
- Launched: Nov. 1986
- Goal: measure near-Earth plasma properties
- Attitude: gravity stabilized spacecraft
- Mass: 125 kg



Polar Bear G^2 Mission

- February 1987: After completing its first period of fully sunlit orbit, the attitude degraded significantly!
- May 1987: spacecraft inverted its attitude.
- Several attempts were undertaken to re-invert.
- Third attempt proved successful when the momentum wheel was allowed to despin for an orbit, before returning to max spin rate.