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Miscellaneous FEM Formulation Topics

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This Chapter concludes Part II with miscellaneous topics in finite element formulation. This material is not intended to be covered in an introductory course. It provide sources for term projects and take-home exams as well as serving as a bridge to more advanced courses. The three topics are: natural strains and stresses, hierarchical shape functions, and curved bar elements.

§Q.1. *Natural Strains and Stresses

In the element formulations of Chapters 12–19, shape functions have been developed in natural coordinates. For example $\{\zeta_1, \zeta_2, \zeta_3\}$ in triangles and $\{\xi, \eta\}$ in quadrilaterals.

Strains and stresses are, however, expressed in terms of Cartesian coordinate components. This is no accident. It connects seemlessly with the elasticity formulations engineers are familiar with. Furthermore, the constitutive properties are often expressed with reference to Cartesian coordinates.

In advanced FEM formulations it is often convenient to proceed further, and express also strains and stresses in terms of natural coordinate components. For brevity these are called *natural strains* and *natural stresses*, natural strains natural stresses respectively. The advantages of doing so become apparent when one goes beyond the pure displacement formulations and assumes also strains and/or stress patterns. Since this book does not go that far, natural strains and stresses are introduced as objects deserving study on their own.

There is an exposition problem, however, related to "lack of uniqueness" in the definition of natural strains. First, the definitions vary according to element geometry: line, triangle, quadrilateral, tetrahedron, brick, and so on. This is to be expected since the natural coordinates vary with the geometry. Second, definitions are often author dependent because the topic is unsettled. The following subsections present natural strains and stresses only for two configurations in which there is reasonable agreement in the literature.

§Q.1.1. *Straight Line Elements

Suppose that a straight one-dimensional bar element is defined in terms of the local axis x and axial displacement u in the isoparametric form

$$x = x(\xi), \quad u = u(\xi) \tag{Q.1}$$

where ξ is a dimensionless natural coordinate that varies from -1 to $+1$. Derivatives with respect to ξ will be denoted by a subscript; for example $dx/d\xi = x_\xi$ and $u_\xi = du/d\xi$. The Cartesian strain is $e = du/dx = (du/d\xi)(d\xi/dx) = J^{-1}u_\xi$, where $J = dx/d\xi = x_\xi$ is the 1D Jacobian. The *natural strain* $e_{\xi\xi}$ is defined by

$$e_{\xi\xi} = \frac{dx}{d\xi}\frac{du}{d\xi} = J u_\xi \tag{Q.2}$$

Because ξ is dimensionless, $e_{\xi\xi}$ has dimension of length squared, that is, area. Obviously this does not lead to a simple physical interpretation, as is the case with Cartesian strains.

What is the relation between e and $e_{\xi\xi}$? This is easily obtained by performing some Jacobian manipulations:

$$e = \frac{du}{dx} = J^{-1} J \frac{du}{dx} = J^{-1}\frac{dx}{d\xi}\frac{du}{d\xi}\frac{d\xi}{dx} = J^{-1} e_{\xi\xi} J^{-1} = J^{-2} e_{\xi\xi} \tag{Q.3}$$

Since J has dimension of length, J^{-2} restores the expected non-dimensionality of e.

The *natural stress* $\sigma_{\xi\xi}$ may be defined in several ways. The most straightforward definition uses the invariance of strain energy density: $\sigma e = \sigma_{\xi\xi} e_{\xi\xi}$. Hence $\sigma_{\xi\xi} = (e/e_{\xi\xi})\sigma = J^{-2}\sigma$ and $\sigma_{\xi\xi} = J^2\sigma$. Note that $\sigma_{\xi\xi}$ has dimensions of force and not of force per unit area.

If $\sigma = E\, e$, the last step is to define a natural constitutive relation $\sigma_{\xi\xi} = E_{\xi\xi} e_{\xi\xi}$. Evidently $E_{\xi\xi} = \sigma_{\xi\xi}/e_{\xi\xi} = (J^2\sigma)/(J^{-2}e) = J^4 E$. This "natural modulus" has dimensions of force times area.

In the Assumed Natural Strain (ANS) formulation of finite element, one assumes directly a form for $e_{\xi\xi}$ as a *simplification* of the expression (Q.2). One common simplification is to take an average Jacobian J_0 instead of the variable Jacobian $J = x_\xi$; for example the value at the element midpoint $\xi = 0$. The implications are explored in Exercise Q.1.

§Q.1.2. *Plane Stress Quadrilaterals

The foregoing derivation for line elements looks like empty formalism. And indeed it does not help much. The power of the ANS method comes in two and three dimensions. In this section we restrict the exposition to plane quadrilaterals in plane stress referred to a Cartesian system $\{x, y\}$. Collect coordinates $\{x, y\}$ in a 2-vector $\vec{\mathbf{x}}$ and the inplane displacement field into a 2-vector $\vec{\mathbf{u}}$. Then the natural strains are defined as

$$e_{\xi\xi} = \frac{\partial\vec{\mathbf{x}}}{\partial\xi}\cdot\frac{\partial\vec{\mathbf{u}}}{\partial\xi}, \quad e_{\eta\eta} = \frac{\partial\vec{\mathbf{x}}}{\partial\eta}\cdot\frac{\partial\vec{\mathbf{u}}}{\partial\eta}, \quad e_{\xi\eta} = \frac{\partial\vec{\mathbf{x}}}{\partial\xi}\cdot\frac{\partial\vec{\mathbf{u}}}{\partial\eta}, \quad e_{\eta\xi} = \frac{\partial\vec{\mathbf{x}}}{\partial\eta}\cdot\frac{\partial\vec{\mathbf{u}}}{\partial\xi}, \quad \gamma_{\xi\eta} = e_{\xi\eta} + e_{\eta\xi}. \tag{Q.4}$$

where $\cdot$ denotes the vector dot product. Note that, unlike Cartesian strains, the shear strains $e_{\xi\eta}$ and $e_{\eta\xi}$ are generally different; consequently $\gamma_{\xi\eta} \neq 2e_{\xi\eta}$ and $\gamma_{\xi\eta} \neq 2e_{\eta\xi}$. To effect a transformation to Cartesian strains, it is convenient to go back to a tensor-like arrangement

$$\mathbf{e}_{\xi\eta} = \begin{bmatrix} e_{\xi\xi} & e_{\xi\eta} \\ e_{\eta\xi} & e_{\eta\eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial\xi} & \dfrac{\partial y}{\partial\xi} \\ \dfrac{\partial x}{\partial\eta} & \dfrac{\partial y}{\partial\eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u_x}{\partial\xi} & \dfrac{\partial u_x}{\partial\eta} \\ \dfrac{\partial u_y}{\partial\xi} & \dfrac{\partial u_y}{\partial\eta} \end{bmatrix} = \mathbf{J}\,\mathbf{G}, \tag{Q.5}$$

where $\mathbf{e}_{\xi\eta}$, $\mathbf{J}$ and $\mathbf{G}$ denote the indicated matrices ($\mathbf{J}$ is the Jacobian matrix introduced in Chapter 17). It is easily verified that

$$\mathbf{J}^{-1}\mathbf{e}_{\xi\eta}\mathbf{J}^{-T} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_y}{\partial x} \\ \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_y}{\partial y} \end{bmatrix} = \begin{bmatrix} e_{xx} & e_{yx} \\ e_{xy} & e_{yy} \end{bmatrix}. \tag{Q.6}$$

from which the Cartesian components may be extracted and rearranged as a 3-vector. Exercise Q.4 works out how to express the transformations as a matrix-vector product. The natural stresses are defined as the energy conjugates of the natural strains. The natural constitutive equation follows.

§Q.1.3. *Three Dimensional Bricks

The extension to three dimensions for brick elements is straightforward. We list here only the pertinent definitions and results. The natural coordinates are $\{\xi, \eta, \zeta\}$ and the Cartesian coordinates

$\{x, y, z\}$. Going directly to matrix-tensor form, the natural strains are defined as

$$\mathbf{e}_{\xi\eta\zeta} = \begin{bmatrix} e_{\xi\xi} & e_{\xi\eta} & e_{\xi\eta} \\ e_{\eta\xi} & e_{\eta\eta} & e_{\eta\zeta} \\ e_{\zeta\xi} & e_{\zeta\eta} & e_{\zeta\zeta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{bmatrix} \frac{\partial u_x}{\partial \xi} & \frac{\partial u_x}{\partial \eta} & \frac{\partial u_x}{\partial \zeta} \\ \frac{\partial u_y}{\partial \xi} & \frac{\partial u_y}{\partial \eta} & \frac{\partial u_y}{\partial \zeta} \\ \frac{\partial u_z}{\partial \xi} & \frac{\partial u_z}{\partial \eta} & \frac{\partial u_z}{\partial \zeta} \end{bmatrix} = \mathbf{J}\,\mathbf{G}, \tag{Q.7}$$

where $\mathbf{J}$ is the Jacobian matrix for bricks. To pass to Cartesian strains use

$$\mathbf{J}^{-1}\mathbf{e}_{\xi\eta\zeta}\mathbf{J}^{-T} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} e_{xx} & e_{yx} & e_{zx} \\ e_{xy} & e_{yy} & e_{zy} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix}. \tag{Q.8}$$

from which the 6-vector of Cartesian strains is easily extracted.

§Q.2. *Hierarchical Shape Functions

Conventional shape functions, as covered in Chapters 15–19, express total displacements from a reference state. Hierarchical shape functions express *differences* from a set of simpler shape functions. Elements derived with hierarchical shape functions are called *hierarchical elements*. This concept finds applications in the following areas: hierarchical shape functions hierarchical elements

(1) *Element Specialization* Hierarchical formulations simplify the derivation of special elements generated from a "parent" element by removing or transforming freedoms. In particular transition elements are easily produced.

(2) *Implementation of p-convergence*. This adaptive-discretization technique, which relies on systematic use of higher polynomial orders, relies on hierarchical elements implemented in a multilevel manner.

(3) *Advanced Element Formulations* Some advanced formulations rely on hierachical "splitting" of the element response. Although the split may not necessarily be done with shape functions, learning the topic helps.

In what follows we shall emphasize only the first application.

§Q.2.1. *The Six Node Triangle, Revisited

Consider (again) the six-node quadratic triangle, which is pictured in Figure Q.1. The isoparametric definition of this element is repeated here for convenience:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \\ N_5^{(e)} \\ N_6^{(e)} \end{bmatrix}. \tag{Q.9}$$

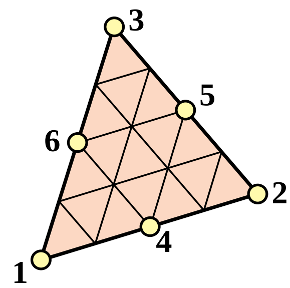

Figure Q.1. The six-node quadratic triangle (once more)

with the shape functions

$$\begin{aligned} N_1^{(e)} &= \zeta_1(2\zeta_1 - 1), & N_4^{(e)} &= 4\zeta_1\zeta_2 \\ N_2^{(e)} &= \zeta_2(2\zeta_2 - 1), & N_5^{(e)} &= 4\zeta_2\zeta_3 \\ N_3^{(e)} &= \zeta_3(2\zeta_3 - 1), & N_6^{(e)} &= 4\zeta_3\zeta_1 \end{aligned} \tag{Q.10}$$

For use below consider a generic scalar function, w, interpolated with the shape functions (Q.10):

$$w = [\, w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \,] \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_6 \end{bmatrix}. \tag{Q.11}$$

Symbol w may represent x, y, u_x or u_y in the isoparametric representation (Q.10), or other element-varying quantities such as thickness, temperature, etc. See Figure Q.2(a).

§Q.2.2. *Hierarchical Interpolation

To "go hierarchical" the key step is to express the values of w at the midnodes 4, 5 and 6 as *deviations from the linear interpolation*:

$$w_4 = \tfrac{1}{2}(w_1 + w_2) + \tilde{w}_4, \qquad w_5 = \tfrac{1}{2}(w_2 + w_3) + \tilde{w}_5, \qquad w_4 = \tfrac{1}{2}(w_3 + w_1) + \tilde{w}_6. \tag{Q.12}$$

These "deviations from linearity" $\tilde{w}_4$, $\tilde{w}_5$ and $\tilde{w}_6$ are called *hierarchical values*. They have a straightforward geometric interpretation if w is plotted normal to the plane of the triangle. For example, Figure Q.2(b) illustrates the meaning of the hierarchical value $\tilde{w}_4$ at midnode 4.

If we insert (Q.12) into (Q.11) we get the *hierarchical interpolation* formula

$$w = [\, w_1 \quad w_2 \quad w_3 \quad \tilde{w}_4 \quad \tilde{w}_5 \quad \tilde{w}_6 \,] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ 4\zeta_1\zeta_2 \\ 4\zeta_2\zeta_3 \\ 4\zeta_3\zeta_1 \end{bmatrix}. \tag{Q.13}$$

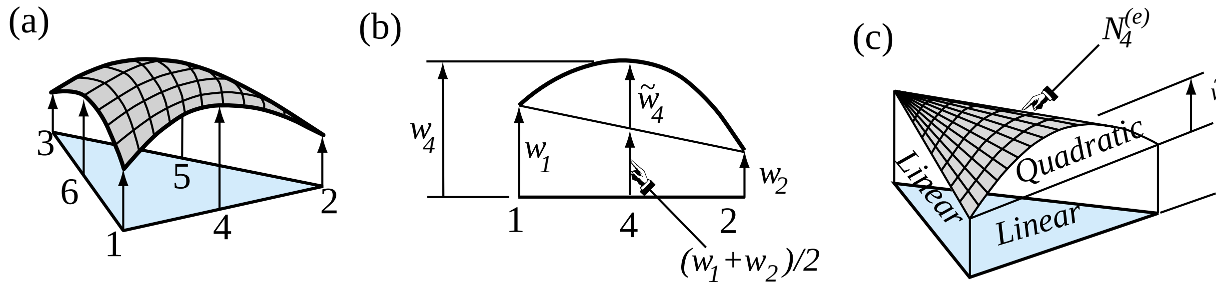

Figure Q.2. Interpretation of hierarchical midnode value $\tilde{w}_4$.

On comparing (Q.13) with the conventional interpolation two points become evident:

1. The shape functions for the three corner nodes (1, 2 and 3) have become the shape functions of the *linear triangle*.

2. The shape functions for the three midnodes (4, 5 and 6) stay the same.

These results are not surprising, for we have expressed the new (hierarchical) midnodes values as *corrections* from the expansion of the linear triangle. The midnode hierarchical shape functions have the same form, but are measured from the linear shape functions, as illustrated in Figure Q.2(b,c) for $\tilde{N}_4^{(e)} = 4\zeta_1\zeta_2$.

Insert now the hierarchical interpolation formula (Q.5) into rows 2 through 4 of (Q.1) by making $w \equiv x, y, u_x$ and u_y in turn. As for the first row, its last three entries vanish because the hierarchical deviation from a constant (the number 1) is zero. We thus arrive at the *hierarchical isoparametric representation* of the six noded triangle:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ x_1 & x_2 & x_3 & \tilde{x}_4 & \tilde{x}_5 & \tilde{x}_6 \\ y_1 & y_2 & y_3 & \tilde{y}_4 & \tilde{y}_5 & \tilde{y}_6 \\ u_{x1} & u_{x2} & u_{x3} & \tilde{u}_{x4} & \tilde{u}_{x5} & \tilde{u}_{x6} \\ u_{y1} & u_{y2} & u_{y3} & \tilde{u}_{y4} & \tilde{u}_{y5} & \tilde{u}_{y6} \end{bmatrix} \begin{bmatrix} \tilde{N}_1^{(e)} \\ \tilde{N}_2^{(e)} \\ \tilde{N}_3^{(e)} \\ \tilde{N}_4^{(e)} \\ \tilde{N}_5^{(e)} \\ \tilde{N}_6^{(e)} \end{bmatrix}, \qquad \text{(Q.14)}$$

where only the first three shape functions are different:

$$\begin{aligned} \tilde{N}_1^{(e)} &= \zeta_1, & \tilde{N}_4^{(e)} &= N_4^{(e)} = 4\zeta_1\zeta_2, \\ \tilde{N}_2^{(e)} &= \zeta_2, & \tilde{N}_5^{(e)} &= N_5^{(e)} = 4\zeta_2\zeta_3, \\ \tilde{N}_3^{(e)} &= \zeta_3, & \tilde{N}_6^{(e)} &= N_6^{(e)} = 4\zeta_3\zeta_1. \end{aligned} \qquad \text{(Q.15)}$$

The linear and quadratic parts of the element are now neatly separated:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \left[\begin{array}{c|c} \text{Linear} & \text{Quadratic} \end{array} \right] \begin{bmatrix} \text{Linear} \\ \cdots\cdots \\ \text{Quadratic} \end{bmatrix}. \tag{Q.16}$$

§Q.2.3. *Hierarchical Stiffness Matrix and Load Vector

The element stiffness equations for the conventional (non-hierarchical) shape function form are

$$\mathbf{K}^{(e)}\mathbf{u}^{(e)} = \mathbf{f}^{(e)}. \tag{Q.17}$$

We would like to transform these equations to the hierarchical form

$$\tilde{\mathbf{K}}^{(e)}\tilde{\mathbf{u}}^{(e)} = \tilde{\mathbf{f}}^{(e)}. \tag{Q.18}$$

The nodal displacement vector may be partitioned as

$$\tilde{\mathbf{u}}^{(e)} = \begin{bmatrix} \tilde{u}_x^{(e)} \\ \tilde{u}_y^{(e)} \end{bmatrix}, \tag{Q.19}$$

where $\tilde{\mathbf{u}}_x$ and $\tilde{\mathbf{u}}_y$ denote the 6×1 vectors

$$\tilde{\mathbf{u}}_x^{(e)} = \begin{bmatrix} \tilde{u}_{x1} \\ \tilde{u}_{x2} \\ \tilde{u}_{x3} \\ \tilde{u}_{x4} \\ \tilde{u}_{x5} \\ \tilde{u}_{x6} \end{bmatrix}, \qquad \tilde{\mathbf{u}}_y^{(e)} = \begin{bmatrix} \tilde{u}_{y1} \\ \tilde{u}_{y2} \\ \tilde{u}_{y3} \\ \tilde{u}_{y4} \\ \tilde{u}_{y5} \\ \tilde{u}_{y6} \end{bmatrix}. \tag{Q.20}$$

Let $\tilde{\mathbf{T}}_x$ and $\tilde{\mathbf{T}}_y$ denote the 6×6 transformation matrices that relate the non-hierarchical to the hierarchical displacement components along x and y, respectively:

$$\mathbf{u}_x^{(e)} = \tilde{\mathbf{T}}_x \tilde{\mathbf{u}}_x^{(e)}, \qquad \mathbf{u}_y^{(e)} = \tilde{\mathbf{T}}_y \tilde{\mathbf{u}}_y^{(e)}. \tag{Q.21}$$

To construct these transformation matrices, observe that

$$\begin{bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \\ u_{x4} \\ u_{x5} \\ u_{x6} \end{bmatrix} = \begin{bmatrix} \tilde{u}_{x1} \\ \tilde{u}_{x2} \\ \tilde{u}_{x3} \\ \frac{1}{2}(u_{x1} + u_{x2}) + \tilde{u}_{x4} \\ \frac{1}{2}(u_{x2} + u_{x3}) + \tilde{u}_{x5} \\ \frac{1}{2}(u_{x3} + u_{x1}) + \tilde{u}_{x6} \end{bmatrix} = \begin{bmatrix} \tilde{u}_{x1} \\ \tilde{u}_{x2} \\ \tilde{u}_{x3} \\ \frac{1}{2}(\tilde{u}_{x1} + \tilde{u}_{x2}) + \tilde{u}_{x4} \\ \frac{1}{2}(\tilde{u}_{x2} + \tilde{u}_{x3}) + \tilde{u}_{x5} \\ \frac{1}{2}(\tilde{u}_{x3} + \tilde{u}_{x1}) + \tilde{u}_{x6} \end{bmatrix}, \tag{Q.22}$$

with exactly the same relation holding for u_y. We can present (Q.22) in matrix form with

$$\tilde{\mathbf{T}}_x = \tilde{\mathbf{T}}_y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}. \tag{Q.23}$$

Combining the two matrix equations in one:

$$\mathbf{u}^{(e)} = \tilde{\mathbf{T}}\,\tilde{\mathbf{u}}^{(e)} \tag{Q.24}$$

where $\tilde{\mathbf{T}}$ is the 12×12 transformation matrix

$$\tilde{\mathbf{T}} = \begin{bmatrix} \tilde{\mathbf{T}}_x & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{T}}_y \end{bmatrix}. \tag{Q.25}$$

Inserting (Q.17) into (Q.9) and premultiplying by $\tilde{\mathbf{T}}^T$ we obtain (Q.10), in which

$$\begin{aligned} \tilde{\mathbf{K}}^{(e)} &= \tilde{\mathbf{T}}^T \mathbf{K}^{(e)} \tilde{\mathbf{T}} \\ \tilde{\mathbf{f}}^{(e)} &= \tilde{\mathbf{T}}^T \mathbf{f}^{(e)} \end{aligned}. \tag{Q.26}$$

In practice the entries of $\tilde{\mathbf{K}}^{(e)}$ and $\tilde{\mathbf{f}}^{(e)}$ are *not* computed by forming the conventional stiffness matrices and force vectors and applying the preceding transformation equations. They are formed directly instead. However, the transformation equations are useful for checking element derivations and computer programs.

§Q.2.4. *Equation Nesting and Node Dropping

To take full advantage of the "node dropping" feature described in §Q.6, it is convenient to order the hierarchical element equations so that all *corner degrees of freedom come first*. That is, we pass from the ordering

$$\begin{aligned} &[\,\tilde{u}_{x1} \quad \tilde{u}_{x2} \quad \tilde{u}_{x3} \quad \tilde{u}_{x4} \quad \tilde{u}_{x5} \quad \tilde{u}_{x6} \quad \tilde{u}_{y1} \quad \tilde{u}_{y2} \quad \tilde{u}_{y3} \quad \tilde{u}_{y4} \quad \tilde{u}_{y5} \quad \tilde{u}_{y6}\,] \\ \text{to} \quad & \\ &[\,\tilde{u}_{x1} \quad \tilde{u}_{y1} \quad \tilde{u}_{x2} \quad \tilde{u}_{y2} \quad \tilde{u}_{x3} \quad \tilde{u}_{y3} \quad \tilde{u}_{x4} \quad \tilde{u}_{y4} \quad \tilde{u}_{x5} \quad \tilde{u}_{y5} \quad \tilde{u}_{x6} \quad \tilde{u}_{y6}\,]. \end{aligned} \tag{Q.27}$$

With this rearrangement of nodal freedoms, the hierarchical stiffness equations of the six-node triangle take the *nested form*

$$\begin{bmatrix} \text{Linear} & \\ & \text{Quadratic} \end{bmatrix} \begin{bmatrix} \text{Linear} \\ \\ \text{Quadratic} \end{bmatrix} = \begin{bmatrix} \text{Linear} \\ \\ \text{Quadratic} \end{bmatrix}, \tag{Q.28}$$

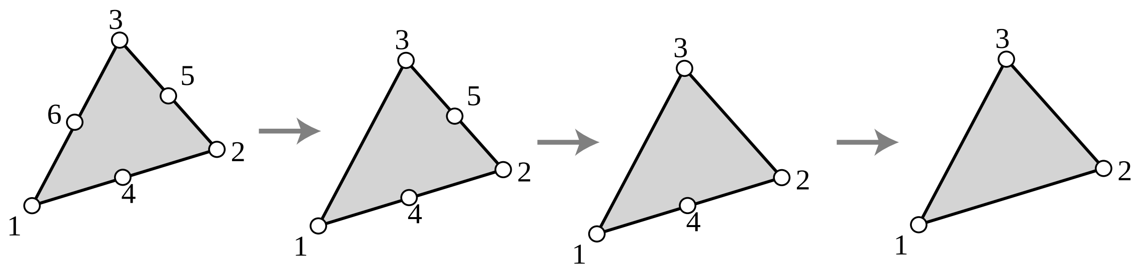

Figure Q.3. Transition triangular elements

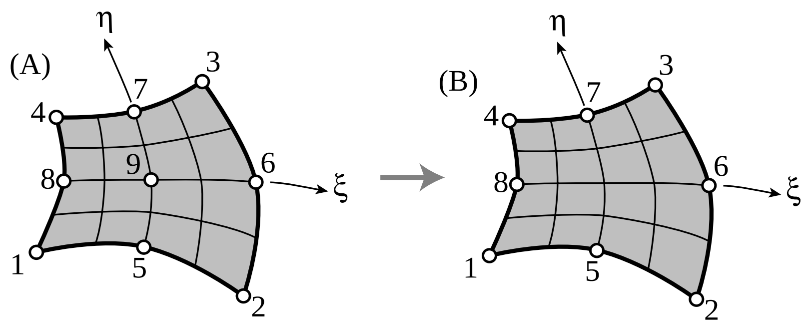

Figure Q.4. Conventional eight-node quadrilateral (A) and nine-node quadrilateral (B) with hierarchical internal node.

in which labels “Linear” and “Quadratic” refer to the association of the equations and degrees of freedom with the linear triangle and quadratic corrections thereto, respectively.

We would like to form the sequence of *transition triangles* illustrated in Figure Q.3 by simply “dropping” or “deleting” nodes in a simple and systematic way.

With the hierarchical stiffness equations in nested form, the task is simple. To get rid of node 6, we simply say that

$$\tilde{u}_{x6} = \tilde{u}_{y6} = 0, \tag{Q.29}$$

and treat this boundary condition as a zero-displacement constraint, dropping equations 11 and 12 from the nested-form of the element stiffness equations. We are left with 10 equations, and have effectively formed the five-node transition element ($n = 5$) shown in Figure Q.4.

Dropping node 5 through a similar procedure we get the four-node transition element ($n = 4$) shown in Figure Q.4. Finally, dropping node 4 we reduce the element to the three-node linear triangle.

§Q.2.5. *Internal Node Injection

The hierarchical representation can also be applied to *internal* nodes. The algebra is more elaborate because *all* external nodes participate in the definition of hierarchical value. The technique used for handling internal nodes is illustrated here on the two elements shown in Figure Q.4: a conventional eight-node quadrilateral (A), and a nine-node quadrilateral (B) with hierarchical internal node 9.

It should be emphasized that the midnodes of the two example elements are *not* hierarchical; in practice they would be, but such "hierarchical nesting" would obscure the presentation that follows.

The procedure followed in this example illustrates the *node injection* technique: from a simpler element we build a more complex one by inserting one or more nodes in a hierarchical manner. This is roughly the inverse of node dropping procedure used in §Q.6 to build transition elements.

The isoparametric representation of (A) is

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \\ N_5^{(e)} \\ N_6^{(e)} \\ N_7^{(e)} \\ N_8^{(e)} \end{bmatrix}, \tag{Q.30}$$

with the shape functions

$$\begin{aligned} N_1^{(e)} &= -\tfrac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta), & N_5^{(e)} &= \tfrac{1}{2}(1-\xi^2)(1-\eta) \\ N_2^{(e)} &= -\tfrac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta), & N_6^{(e)} &= \tfrac{1}{2}(1-\eta^2)(1+\xi) \\ N_3^{(e)} &= -\tfrac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta), & N_7^{(e)} &= \tfrac{1}{2}(1-\xi^2)(1+\eta) \\ N_4^{(e)} &= -\tfrac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta), & N_8^{(e)} &= \tfrac{1}{2}(1-\eta^2)(1-\xi) \end{aligned}. \tag{Q.31}$$

Express a generic quantity w over element (A) as

$$w^A = [\, w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad w_7 \quad w_8 \,] \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \\ N_5^{(e)} \\ N_6^{(e)} \\ N_7^{(e)} \\ N_8^{(e)} \end{bmatrix}. \tag{Q.32}$$

To construct (B) with 9 as a hierarchical node at the quadrilateral center ($\xi = \eta = 0$), we express the value at 9 as the sum of the interpolated center value in (A), plus a correction:

$$w_9 = w_9^A(0,0) + \tilde{w}_9 \tag{Q.33}$$

The center value is obtained by evaluating (Q.24) at $\xi = \eta = 0$, which gives

$$w_9^A(0,0) = -\tfrac{1}{4}(w_1 + w_2 + w_3 + w_4) + \tfrac{1}{2}(w_5 + w_6 + w_7 + w_8), \tag{Q.34}$$

so that the full form of (Q.25) is

$$w_9 = -\tfrac{1}{4}(w_1 + w_2 + w_3 + w_4) + \tfrac{1}{2}(w_5 + w_6 + w_7 + w_8) + \tilde{w}_9. \tag{Q.35}$$

The transformation between non-hierarchical and hierarchical values may be expressed in matrix form:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \\ \tilde{w}_5 \\ \tilde{w}_6 \\ \tilde{w}_7 \\ \tilde{w}_8 \\ \tilde{w}_9 \end{bmatrix}. \tag{Q.36}$$

The shape function associated with the internal node is the bubble function

$$N_9^{(e)} = (1 - \xi^2)(1 - \eta^2) \tag{Q.37}$$

With these results the generic interpolation formula for (B) becomes

$$w^B = [\, w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6 \quad w_7 \quad w_8 \quad \tilde{w}_9 \,] \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \\ N_5^{(e)} \\ N_6^{(e)} \\ N_7^{(e)} \\ N_8^{(e)} \\ N_9^{(e)} \end{bmatrix}, \tag{Q.38}$$

where the shape functions are (Q.36) and (Q.37). [No tildes on the N's are necessary, because none of these functions changes in going from (A) to (B).] Finally, on specializing this relation to x, y, etc, we obtain the isoparametric representation of element (B):

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & \tilde{x}_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & \tilde{y}_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & \tilde{u}_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & \tilde{u}_{y9} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \\ N_5^{(e)} \\ N_6^{(e)} \\ N_7^{(e)} \\ N_8^{(e)} \\ N_9^{(e)} \end{bmatrix}. \tag{Q.39}$$

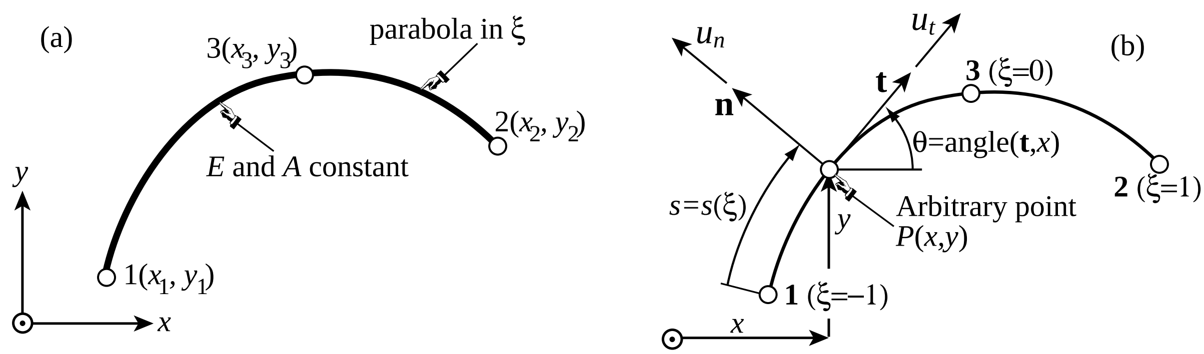

Figure Q.6. Figure (a) shows a 3-node curved bar element with constant modulus E and area A. This element has six degrees of freedom and can only transmit an axial force $N = EAe$. Figure Q.6(b) depicts geometric details covered in the text.

§Q.3. *A 3-Node Curved Bar Element

This section formulates a *curved* curved bar element 3-node bar isoparametric element in which node 3 is allowed to be off alignment from 1 and 2, as depicted in Figure Q.6(a). The derivation provides a first glimpse of techniques used in advanced FEM expositions to develop curved beam and shell elements.

§Q.3.1. *Element Description

The element moves in the $\{x, y\}$ plane. It has six degrees of freedom, namely, the displacements u_{xn} and u_{yn} at nodes $n = 1, 2, 3$. It has constant elastic modulus E and cross section area A, and can transmit only an axial force $N = E\,A\,e$, where e is the axial strain.

Additional geometric details are given in Figure Q.6(b). At an arbitrary point $P(x, y)$ of the element one defines the tangential and normal directions by the *unit* vectors **t** and **n**, respectively.[1] The positive sense of **t** corresponds to traversing along from 1 to 2. The normal vector **n** is at 90° CCW from **t**. The angle formed by **t** and x is θ, positive CCW. The arclength coordinate is s, where $ds^2 = dx^2 + dy^2$.

The Cartesian displacement vector at generic point P is $\mathbf{u} = [\,u_x \;\; u_y\,]^T$. The displacement components in the directions **t** and **n** are the tangential displacement $u_t = \mathbf{u}^T\mathbf{t} = \mathbf{t}^T\mathbf{u}$ and the normal displacement $u_n = \mathbf{u}^T\mathbf{n} = \mathbf{n}^T\mathbf{u}$. The axial strain is defined as

$$e = \frac{du_t}{ds} - \kappa u_n \tag{Q.40}$$

where κ is the inplane bar curvature, the expression of which is derived in §Q.3.3. The corrective term κu_n is known as the *hoop strain* in the theory of curved bars and rods.[2]

[1] "Unit vectors" means they have unit length: $\mathbf{t}^T\mathbf{t} = 1$ and $\mathbf{n}^T\mathbf{n} = 1$.

[2] If the bar is straight, $\kappa = 0$ and $e = du_t/ds$. If then x is taken along the bar axis, $s \equiv x$ and $e = du_x/dx$, which is the well known axial strain used in Chapter 12.

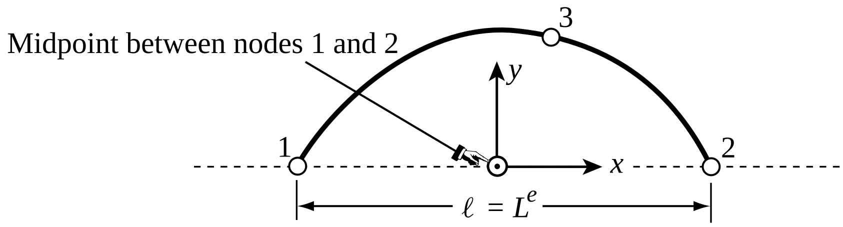

Figure Q.7. Choice of Cartesian local axes to simplify derivations.

The isoparametric definition of this element is[3]

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{Q.41}$$

Here

$$N_1 = -\tfrac{1}{2}\xi(1-\xi), \quad N_2 = \tfrac{1}{2}\xi(1+\xi), \quad N_3 = 1-\xi^2 \tag{Q.42}$$

are the shape functions defined in terms of an isoparametric coordinate ξ that takes the value -1, 1 and 0 at nodes 1, 2 and 3, respectively. Derivatives taken with respect to coordinate ξ will be denoted by a prime; thus $x' = dx/d\xi$, etc. The strain energy taken up by the element is

$$U^{(e)} = \tfrac{1}{2}\int_{\text{arclength}} E A\, e^2\, ds = \tfrac{1}{2}\int_{-1}^{1} E A\, e^2\, J\, d\xi = \tfrac{1}{2} E A \int_{-1}^{1} e^2\, J\, d\xi, \qquad J = ds/d\xi = s'. \tag{Q.43}$$

To simplify the analytical derivations that follow, we orient $\{x, y\}$ as shown in Figure Q.7, with origin at the midpoint of nodes 1 and 2. Take $x_3 = \alpha\ell$ and $y_3 = \beta\ell$ where α and β are dimensionless parameters that characterize the deviation of node 3 from the 1-2 midpoint, and ℓ is the 1-2 distance.[4]

The axial strain e, from the curved rod theory outlined in §Q.3.3, is

$$\begin{aligned} e &= \frac{du_n}{ds} - \kappa u_n = \frac{d(\mathbf{u}^T\mathbf{t})}{ds} - \kappa u_n = \left(\frac{d\mathbf{u}}{ds}\right)^T \mathbf{t} + \mathbf{u}^T \frac{d\mathbf{t}}{ds} - \kappa u_n \\ &= \left(\frac{d\mathbf{u}}{d\xi}\right)^T \frac{d\xi}{ds}\,\mathbf{t} + \mathbf{u}^T(\kappa\mathbf{n}) - \kappa u_n = J^{-1}\mathbf{t}^T\mathbf{u}' + \kappa\, u_n - \kappa\, u_n = J^{-1}\mathbf{t}^T\mathbf{u}'. \end{aligned} \tag{Q.44}$$

in which $J = s' = ds/d\xi$ is the curved-bar Jacobian, and the replacement of $d\mathbf{t}/ds$ by $\kappa\mathbf{n}$ comes from the first Frénet formula given in §Q.3.3. The values of $\mathbf{u}$ and $\mathbf{u}'$ can be calculated from the last two rows of the isoparametric definition:

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{Q.45}$$

[3] Supercript (e) is omitted for brevity.

[4] Axes $\{x, y\}$ in Figure Q.7 are actually the local axes called $\bar{x}$ and $\bar{y}$ in Chapters 2-3. The overbars are omitted to avoid cluttering; they are reintroduced in Figure Q.8. The role of the deviation parameters α and β is similar to the device used in Exercises 16.4 and 16.5 to study the effect of moving node 3 away from the midpoint.

Noting that $N_1' = dN_1/d\xi = \xi - \frac{1}{2}$, $N_2' = dN_2/d\xi = \xi + \frac{1}{2}$, $N_3' = dN_3/d\xi = -2\xi$, we get

$$\mathbf{u}' = \begin{bmatrix} u_x' \\ u_y' \end{bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix}. \tag{Q.46}$$

Collecting terms and replacing the expression of **t** given in §Q.3.3:

$$e = \mathbf{B}\,\mathbf{u}^{(e)}, \tag{Q.47}$$

$$\mathbf{B} = J^{-2}\,[\, x'N_1' \quad y'N_1' \quad x'N_2' \quad y'N_2' \quad x'N_3' \quad y'N_3' \,], \tag{Q.48}$$

The derivatives $x' = dx/d\xi$ and $y' = dy/d\xi$ are obtained from rows 2-3 of the isoparametric element definition:

$$x' = x_1N_1' + x_2N_2' + x_3N_3' = (\tfrac{1}{2} - 2\alpha\xi)\ell, \qquad y' = y_1N_1' + y_2N_2' + y_3N_3' = -2\beta\xi\ell, \tag{Q.49}$$

whence

$$J = \sqrt{(x')^2 + (y')^2} = \ell\sqrt{(\tfrac{1}{2} - 2\alpha\xi)^2 + 4\beta^2\xi^2}, \tag{Q.50}$$

Finally,

$$\mathbf{B} = \frac{\ell}{J^2}\,[\,(\tfrac{1}{2} - 2\alpha\xi)(\xi - \tfrac{1}{2}) \quad 2\beta\xi(\xi - \tfrac{1}{2}) \quad (\tfrac{1}{2} - 2\alpha\xi)(\xi + \tfrac{1}{2}) \quad -2\beta\xi(\xi + \tfrac{1}{2}) \quad -2(\tfrac{1}{2} - 2\alpha\xi)\xi \quad 2\beta\xi^2\,]. \tag{Q.51}$$

If the element is straight ($\beta = 0$), $s \equiv x$, $J^2 = \ell^2(\frac{1}{2} - 2\alpha\xi)^2$, $J = \ell(\frac{1}{2} - 2\alpha\xi)$, and **B** reduces to

$$\mathbf{B} = \frac{1}{\ell(\frac{1}{2} - 2\alpha\xi)}\,[\,\xi - \tfrac{1}{2} \quad 0 \quad \xi + \tfrac{1}{2} \quad 0 \quad -2\xi \quad 0\,]. \tag{Q.52}$$

The Jacobian coincides with the expression derived in Chapter 19 for the straight 3-bar element.

Item (b). Here is an implementation of the foregoing **B** in *Mathematica* 2.2, as a function that returns a 1×6 matrix:

```
B [xi_,alpha_,beta_,ell_]:={{(1/2-2*alpha*xi)*(xi-1/2),
        -2*beta*xi*(xi-1/2), (1/2-2*alpha*xi)*(xi+1/2),
        -2*beta*xi*(xi+1/2),-(1/2-2*alpha*xi)*(2*xi),
         4*beta*xi^2}}*ell/JJ[xi,alpha,beta,ell];
J [xi_,alpha_,beta_,ell_]:=ell*Sqrt[(1/2-2*alpha*xi)^2+4*beta^2*xi^2];
JJ[xi_,alpha_,beta_,ell_]:=ell^2*  ((1/2-2*alpha*xi)^2+4*beta^2*xi^2);
```

The function JJ above returns the squared Jacobian J^2; this separate definition is important for symbolic work, because it bypasses the hard-to-simplify square root. Function J, which returns the Jacobian J, is not needed here but it will be used in the stiffness matrix formation in Question 2.

Following is the verification that **B** predicts zero strains under the three two-dimensional rigid body modes (RBMs) for arbitrary α, β and ℓ. It is follow by a uniform-strain test on a straight bar: $\alpha = 0$, $\beta = 0$ but arbitrary ℓ. (The uniform strain test on a curved bar is far trickier and is not required in the test.)

```
ClearAll[alpha,beta,ell,xi];
rm1={1,0,1,0,1,0}; rm2={0,1,0,1,0,1};  (* translations along x,y *)
rm3={0,ell/2,0,-ell/2,beta*ell,-alpha*ell};  (* rotation about z *)
Print["Check zero strain for x-RBM=",Simplify[B[xi,alpha,beta,ell].rm1]];
Print["Check zero strain for y-RBM=",Simplify[B[xi,alpha,beta,ell].rm2]];
Print["Check zero strain for z-RBM=",Simplify[B[xi,alpha,beta,ell].rm3]];
ue={-1/2,0,1/2,0,0,0};
Print["Check unif strain for straight bar=",Simplify[B[xi,0,0,ell].ue]];
```

Running this gives

```
Check zero strain for x-RBM={0}
Check zero strain for y-RBM={0}
Check zero strain for z-RBM={0}
                                        1
Check unif strain for straight bar={---}
                                       ell
```

These checks can also be done by hand, but there is some error-prone algebra involved.

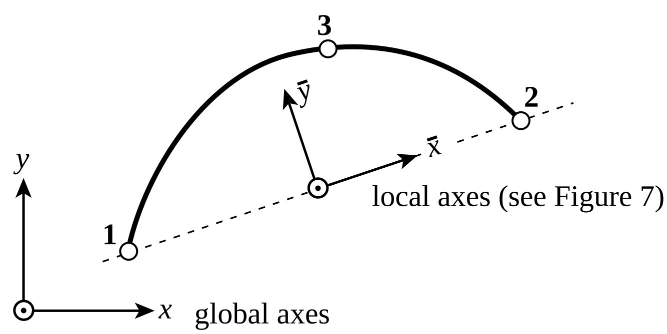

Figure Q.8. Global and local axes for the transformations used in forming the global stiffness matrix.

§Q.3.2. The Stiffness Matrix

Here is an implementation of the element stiffness matrix as module `Stiffness3NodePlaneCurvedBar` that returns a 6×6 matrix:

```
Stiffness3NodePlaneCurvedBar[ncoor_,mprop_,fprop_,opt_]:=
  Module[{x1,y1,x2,y2,x3,y3,x21,y21,xm,ym,alpha,beta,ell2,ell,
          Em,A,num,B,J,JJ,Kebar,T,Ke},
   B [xi_,alpha_,beta_,ell_]:={{(1/2-2*alpha*xi)*(xi-1/2),
       -2*beta*xi*(xi-1/2), (1/2-2*alpha*xi)*(xi+1/2),
       -2*beta*xi*(xi+1/2),-(1/2-2*alpha*xi)*(2*xi),
        4*beta*xi^2}}*ell/JJ[xi,alpha,beta,ell];
  J[xi_,alpha_,beta_,ell_]:=ell*Sqrt[(1/2-2*alpha*xi)^2+4*beta^2*xi^2];
  JJ[xi_,alpha_,beta_,ell_]:=ell^2*((1/2-2*alpha*xi)^2+4*beta^2*xi^2);
 {{x1,y1},{x2,y2},{x3,y3}}=ncoor; {Em}=mprop; A=fprop; num=opt;
  {x21,y21}={x2-x1,y2-y1}; {xm,ym}={x1+x2,y1+y2}/2;
  ell2=x21^2+y21^2; ell=PowerExpand[Sqrt[ell2]];
  alpha=Simplify[ ( (x3-xm)*x21+(y3-ym)*y21)/ell2];
```

```
beta= Simplify[ (-(x3-xm)*y21+(y3-ym)*x21)/ell2];
Print["alpha,beta in Stiffness3NodePlaneCurvedBar=",{alpha,beta}];
B1=B[-Sqrt[3]/3,alpha,beta,ell]; J1=J[-Sqrt[3]/3,alpha,beta,ell];
B2=B[ Sqrt[3]/3,alpha,beta,ell]; J2=J[ Sqrt[3]/3,alpha,beta,ell];
If [num, {B1,J1,B2,J2}=N[{B1,J1,B2,J2}]];
Kebar=Em*A*(J1*Transpose[B1].B1+J2*Transpose[B2].B2);
T={{x21, y21,0,0,0,0},{-y21,x21,0,0,0,0},{0,0, x21,y21,0,0},
   {0,0,-y21,x21,0,0},{0,0,0,0,x21,y21}, {0,0,0,0,-y21,x21}};
Ke=(Transpose[T].Kebar.T)/ell2; (* avoids taking Sqrt[ell2] *)
Return[Ke] ];
```

Arguments. The module has four arguments:

`ncoor`	Global node coordinates arranged as `{ { x1,y1 },{ x2,y2 },{ x3,y3 } }`.
`mprop`	Material properties supplied as `{ Em }`, which is the elastic modulus E.
`fprop`	Fabrication properties supplied as the cross-section area `A`.
`opt`	Processing option: contains logical flag `numer`; if `True` it forces floating-point computation.

Internal Functions. Functions `B`, `J` and `JJ` discussed in Question 1 are incorporated inside the body of `Stiffness3NodePlaneCurvedBar`, and their names made local to the module. This is just a precaution against name clashing when the module is incorporated in a larger FEM code.

Local Geometry Analysis. The node coordinates supplied to `Stiffness3NodePlaneCurvedBar` in `ncoor` are *global*, that is, referred to the global axes $\{x, y\}$. The local system $\{\bar{x}, \bar{y}\}$ is defined through the scheme depicted in Figure Q..8. To connect these two systems begin by computing $x_{21} = x_2 - x_1$, $y_{21} = y_2 - y_1$, $\ell^2 = x_{21}^2 + y_{21}^2$, $\ell = +\sqrt{\ell^2}$, $c = \cos\phi = x_{21}/\ell$ and $s = \sin\phi = y_{21}/\ell$. Local and global coordinates of arbitrary points $P(x, y) \equiv P(\bar{x}, \bar{y})$ are related by the transformations

$$
\begin{aligned}
\bar{x} &= (x - x_m)\,c + (y - y_m)\,s, & \bar{y} &= -(x - x_m)\,s + (y - y_m)\,c \\
x &= \bar{x}\,c - \bar{y}\,s + x_m, & y &= \bar{x}\,s + \bar{y}\,c + y_m,
\end{aligned}
\tag{Q.53}
$$

in which $x_m = \frac{1}{2}(x_1 + x_2)$ and $y_m = \frac{1}{2}(y_1 + y_2)$ are the global coordinates of the midpoint between 1 and 2, which is taken as origin of the local system as shown in Figure Q.8. Whence

$$
\begin{aligned}
\alpha = \bar{x}_3/\ell &= \frac{(x_3 - x_m)\,c + (y_3 - y_m)\,s}{\ell} = \frac{(x_3 - x_m)\,x_{21} + (y_3 - y_m)\,y_{21}}{\ell^2}, \\
\beta = \bar{y}_3/\ell &= \frac{-(x_3 - x_m)\,s + (y_3 - y_m)\,c}{\ell} = \frac{-(x_3 - x_m)\,y_{21} + (y_3 - y_m)\,x_{21}}{\ell^2}.
\end{aligned}
\tag{Q.54}
$$

The last expressions for α and β avoid square roots and are those implemented in the module. The local stiffness matrix $\bar{\mathbf{K}}^{(e)}$ is `Kebar= Em*A*(J1*Transpose[B1].B1+J2*Transpose[B2].B2`, where `B1`, `J1`, etc., are function evaluations at the Gauss points of the 2-point rule. From inspection, the global stiffness matrix is given by

$$
\mathbf{K}^{(e)} = \mathbf{T}^T\,\bar{\mathbf{K}}^{(e)}\,\mathbf{T}, \qquad \text{where} \quad \mathbf{T} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & c & s & 0 & 0 \\ 0 & 0 & -s & c & 0 & 0 \\ 0 & 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & 0 & -s & c \end{bmatrix}
\tag{Q.55}
$$

Since $c = x_{21}/\ell$ and $s = y_{21}/\ell$, taking square roots of $\ell^2 = x_{21}^2 + y_{21}^2$ can be avoided by slightly rearranging the foregoing transformation as follows:

$$\mathbf{K}^{(e)} = \frac{1}{\ell^2}\,\hat{\mathbf{T}}^T\,\bar{\mathbf{K}}^{(e)}\,\hat{\mathbf{T}}, \qquad \text{where} \quad \hat{\mathbf{T}} = \begin{bmatrix} x_{21} & y_{21} & 0 & 0 & 0 & 0 \\ -y_{21} & x_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{21} & y_{21} & 0 & 0 \\ 0 & 0 & -y_{21} & x_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{21} & y_{21} \\ 0 & 0 & 0 & 0 & -y_{21} & x_{21} \end{bmatrix}, \tag{Q.56}$$

This is the transformation implemented in `Stiffness3NodePlaneCurvedBar`, in which $\hat{\mathbf{T}}$ is called `T`.

Verification. Several tests on the element stiffness are performed now. The following statements form $\mathbf{K}^{(e)}$ for the straight bar with node 3 at the midpoint ($\alpha = \beta = 0$) while keeping E, A and ℓ symbolic:

```
ClearAll[Em,A,alpha,beta,ell,xi];
ncoor={{-ell/2,0},{ell/2,0},{0,0}};
Ke=Stiffness3NodePlaneCurvedBar[ncoor,{Em},A,False];  Ke=Simplify[Ke];
Print["Check Ke for straight bar with 3 at midpoint:"]
Print[Ke//MatrixForm];
```

The result is the stiffness matrix listed in Exercise 16.1. Next, a curved bar stiffness is formed numerically with $E = A = \ell = 1$, $\alpha = 1/5$, $\beta = 1/5$, and tested for rank and rigid body modes (RBMs):

```
Em=A=1; ell=1; alpha=1/5; beta=1/3;
ncoor={{-ell/2,0},{ell/2,0},{alpha,beta}*ell};
Ke=Stiffness3NodePlaneCurvedBar[ncoor, {Em},A,True];
Print["Ke for Em=A=ell=1, alpha=1/5 and beta=1/5=",Ke//MatrixForm];
rm1={1,0,1,0,1,0}; rm2={0,1,0,1,0,1};
rm3={0,ell/2,0,-ell/2,beta*ell,-alpha*ell};
Print["eigs of Ke=", Chop[Eigenvalues[N[Ke]]]];
Print["Ke.rm1=",Chop[Ke.rm1]]; Print["Ke.rm2=",Chop[Ke.rm2]];
Print["Ke.rm3=",Chop[Ke.rm3]];
```

The results are:

```
                                             1  1
alpha,beta in Stiffness3NodePlaneCurvedBar={-, -}
                                             5  3
Ke for Em=A=ell=1, alpha=1/5 and beta=1/5=
  1.1042       0.573266    0.137226   -0.0417379   -1.24142    -0.531528
  0.573266     0.31358    -0.0417379   0.141101    -0.531528   -0.454681
  0.137226    -0.0417379   0.816961   -1.1576      -0.954187    1.19933
 -0.0417379    0.141101   -1.1576      1.66183      1.19933    -1.80293
 -1.24142     -0.531528   -0.954187    1.19933      2.19561    -0.667806
 -0.531528    -0.454681    1.19933    -1.80293     -0.667806    2.25761
eigs of Ke={5.36231, 2.98748, 0, 0, 0, 0}
Ke.rm1={0, 0, 0, 0, 0, 0}
Ke.rm2={0, 0, 0, 0, 0, 0}
```

```
Ke.rm3={0, 0, 0, 0, 0, 0}
```

The eigenvalue distribution show a rank deficiency of 1 (4 zero eigenvalues, one more that $6-3=3$). This property is explained in §Q.3.4. The RBM checks work fine, as can be expected.

Finally, a local-to-global invariance test is performed by rotating this element by 30° about z and displacing it by 6 and -4 along x and y, respectively. The global coordinates are recomputed and the new $\mathbf{K}^{(e)}$ formed:

```
{{xbar1,ybar1},{xbar2,ybar2},{xbar3,ybar3}}=ncoor;
phi=Pi/6; c=Cos[phi]; s=Sin[phi]; xm=6; ym=-4;
x1=xbar1*c-ybar1*s+xm; y1=xbar1*s+ybar1*c+ym;
x2=xbar2*c-ybar2*s+xm; y2=xbar2*s+ybar2*c+ym;
x3=xbar3*c-ybar3*s+xm; y3=xbar3*s+ybar3*c+ym;
ncoor={{x1,y1},{x2,y2},{x3,y3}};
Ke=Stiffness3NodePlaneCurvedBar[ncoor, {Em},A,True]; Ke=N[Ke];
Print["Ke for 30-deg rotated & translated bar:",Ke//MatrixForm];
Print["eigs of Ke=", Chop[Eigenvalues[N[Ke]]]];
```

Running the test gives:

```
                                             1  1
alpha,beta in Stiffness3NodePlaneCurvedBar={-, -}
                                             5  3
Ke for 30-deg rotated & translated bar:
  0.41008      0.62898      0.17434      -0.0225469   -0.58442     -0.606433
  0.62898      1.0077       -0.0225469   0.103986     -0.606433    -1.11168
  0.17434      -0.0225469   2.03069      -0.944636    -2.20503     0.967183
  -0.0225469   0.103986     -0.944636    0.448104     0.967183     -0.552089
  -0.58442     -0.606433    -2.20503     0.967183     2.78945      -0.36075
  -0.606433    -1.11168     0.967183     -0.552089    -0.36075     1.66377
eigs of Ke={5.36231, 2.98748, 0, 0, 0, 0}
```

Note that $\mathbf{K}^{(e)}$ has changed completely. However, the eigenvalues are not changed because $\mathbf{T}$ is orthogonal. Furthermore α and β are also preserved because they are intrinsic geometric properties.

§Q.3.3. Geometric Properties of Plane Curves

The following geometric properties of plane curves are collected to help in the analytical derivations of §Q.3.1. They can be found in any book on differential geometry, differential geometry such as the well known textbook by Struik.[5]

Consider a smooth plane curve given in parametric form

$$x = x(\xi), \qquad y = y(\xi). \tag{Q.57}$$

The arclength s is also a function $s = s(\xi)$ of the coordinate ξ; cf. Figure Q.6.

First we need the Jacobian $J = s' = ds/d\xi$. The quickest way to get it is to differentiate both sides of the identity $ds^2 = dx^2 + dy^2$ with respect to ξ: $2ds\, s' = 2dx\, x' + 2dy\, y'$, whence

$$J = s' = x'\frac{dx}{ds} + y'\frac{dy}{ds} = x'\cos\theta + y'\sin\theta = \sqrt{(x')^2 + (y')^2}, \tag{Q.58}$$

[5] D. J. Struik, *Lectures on Classical Differential Geometry*, Addison-Wexley, New York, 2nd ed., 1961.

where θ is the angle formed by the tangent vector $\mathbf{t}$ (directed along increasing s) with the x axis; see Figure 1(b).

The tangent $\mathbf{t}$ and normal $\mathbf{n}$ at a point are *unit length* vectors defined by

$$\mathbf{t} = \begin{bmatrix} dx/ds \\ dy/ds \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = J^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{Q.59}$$

$$\mathbf{n} = \begin{bmatrix} -dy/ds \\ dx/ds \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = J^{-1} \begin{bmatrix} -y' \\ x' \end{bmatrix} \tag{Q.60}$$

Here $\mathbf{n}$ has been taken to be at $+90^\circ = +\pi/2$ radians, from $\mathbf{t}$. The curvature is given by

$$\kappa = \frac{x'y'' - x''y'}{J^3}. \tag{Q.61}$$

Expression (Q.61) is not needed for a bar element. It would be required, however, for a curved beam element as in Exercise Q.8.

The derivative of $\mathbf{t}$ with respect to the arclength s is given by the first Frénet formula

$$\frac{d\mathbf{t}}{ds} = \kappa\, \mathbf{n}, \tag{Q.62}$$

§Q.3.4. *Why Is the Stiffness Matrix Rank Deficient?

The rank deficiency is due to the presence of an *inextensional zero energy mode* or IZEM.[6] The IZEM is a bending-like motion of the element that is not a rigid body mode (RBM) but produces no axial stretching or contraction, hence the qualifier “inextensional.” The IZEM produces a rank deficiency of one, no matter how exact the integration rule is, for Gauss rules of 2 or more points. To get an idea of what the IZEM looks like, the bar element with $E = A = \ell = 1$, $\alpha = 1/5$ and $\beta = 1/5$ previously treated in §Q.3.2 is used. The eigenvector analysis of its stiffness $\mathbf{K}^{(e)}$ would not show the IZEM because the zero eigenvalue has multiplicity 4. But three of the eigenvectors of the associated invariant subspace are known: the RBMs. Using a “spectral inflation” technique the three RBMs are separated by raising their eigenvalues to nonzero values. This leaves one zero eigenvalue, which is the IZEM. This mode is now easily captured by an eigenvector analysis. In the following, `Ke`, `rm1`, `rm2`, `rm3` were generated by the *Mathematica* statements shown previously.

```
    Ke=Ke+Transpose[{rm1}].{rm1}+Transpose[{rm2}].{rm2}+
    Transpose[{rm3}].{rm3}; (* Spectrally separate RBMs to isolate 0-energy
mode *)
    Print["zero energy mode=",Chop[Eigenvectors[Ke][[6]] ]];
```

Running this gives the IZEM eigenvector:

```
    zero energy mode={0.531352,-0.165054,-0.70251,-0.194949,0.171158,0.360003}
```

These six numbers can be physically interpreted by using a graphic display. Here is a plotting module and the driving program:

[6] These are also called “kinematic mechanisms” in the FEM literature.

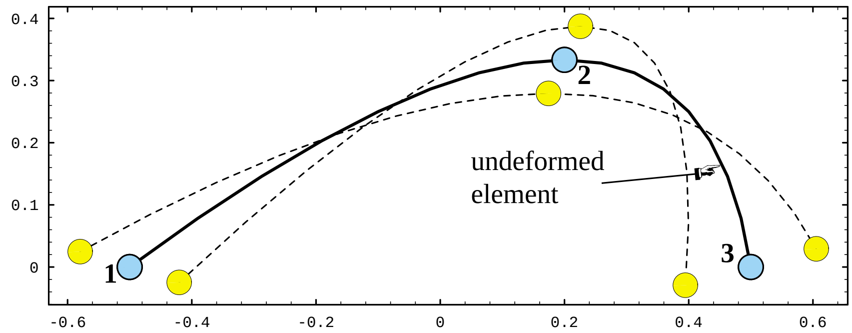

Figure Q.9. The inextensional zero-energy mode (IZEM) of a 3-node curved bar. Depicted for an element with $\ell = 1$, $\alpha = 1/5$ and $\beta = 1/3$.

```
PlotBar3Shape[ncoor_,ue_,amp_,nsub_,th_]:=Module[
  {x1,y1,x2,y2,x3,y3,x21,y21,ux1,uy1,ux2,uy2,ux3,uy3,
   k,xi,N1,N2,N3,m,a,atab,ttab,x,y,xold,yold,p={}},
   {{x1,y1},{x2,y2},{x3,y3}}=N[ncoor];
   {ux1,uy1,ux2,uy2,ux3,uy3}=N[ue];
   N1[xi_]:=-(1-xi)*xi/2; N2[xi_]:=(1+xi)*xi/2; N3[xi_]:=1-xi^2;
   {xold,yold}={x1+amp*ux1,y1+amp*uy1};  xi=-1;
   If [Length[amp]==0,atab={amp},atab=amp];
   If [Length[th ]==0,ttab={th}, ttab=th ];
   For [m=1,m<=Length[atab],m++, a=atab[[m]];
     AppendTo[p,Graphics[AbsoluteThickness[ttab[[m]] ]]];
     {xold,yold}={x1+a*ux1,y1+a*uy1}; xi=-1;
     For [k=1, k<=nsub, k++,   xi=N[xi+2/nsub];
       x=(x1+a*ux1)*N1[xi]+(x2+a*ux2)*N2[xi]+(x3+a*ux3)*N3[xi];
       y=(y1+a*uy1)*N1[xi]+(y2+a*uy2)*N2[xi]+(y3+a*uy3)*N3[xi];
       AppendTo[p,Graphics[Line[{{xold,yold},{x,y}}]]];
       xold=x; yold=y];
      AppendTo[p,Graphics[RGBColor[1,1,0]]];
      AppendTo[p,Graphics[Disk[{x1+a*ux1,y1+a*uy1},0.02]]];
      AppendTo[p,Graphics[Disk[{x2+a*ux2,y2+a*uy2},0.02]]];
      AppendTo[p,Graphics[Disk[{x3+a*ux3,y3+a*uy3},0.02]]];
      AppendTo[p,Graphics[RGBColor[0,0,0]]];
      AppendTo[p,Graphics[Circle[{x1+a*ux1,y1+a*uy1},0.02]]];
      AppendTo[p,Graphics[Circle[{x2+a*ux2,y2+a*uy2},0.02]]];
      AppendTo[p,Graphics[Circle[{x3+a*ux3,y3+a*uy3},0.02]]];
    ];
  Return[p];
  ];
ell=1; alpha=1/5; beta=1/3; ncoor={{-ell/2,0},{ell/2,0},{alpha,beta}*ell};
ue={0.531352, -0.165054, -0.70251, -0.194949, 0.171158, 0.360003};
p=PlotBar3Shape[ncoor,ue,{0,-0.15,.15},16,{2,1,1}];
Show[p,Frame->True,FrameTicks->Automatic,AspectRatio->Automatic];
```

The plot produced by this code, after some reformatting and labeling, is shown in Figure Q.9. The dashed curves therein depict the deformed element moving in the IZEM (being an eigenvector, it has arbitrary amplitude; two amplitudes of opposite signs are shown.) Note that the midline does not change length (in the linear approximation), and thus takes no axial strain energy. If bending energy is taking into account, however, the element becomes rank-sufficient.

Homework Exercises for Chapter Q

Miscellaneous FEM Formulation Topics

EXERCISE Q.1 [A:30] An assumed-strain 1D bar element is developed by making a simplifying assumption on $e_{\xi\xi}$: the jacobian J is taken constant and equal to an average value J_0. Discuss the implications of this assumption as regards compatibilty and completeness requirements. Explain how you would construct the strain-displacement matrix (this is not a trivial problem).

EXERCISE Q.2 [A:35] As above, but for a two-dimensional isoparametric element.

EXERCISE Q.3 [A:30] Develop a theory of natural strains and stresses for the 3-node curved bar element formulated in §Q.3.

EXERCISE Q.4 [A:20] Prove the transformation (Q.6). Then express it as a matrix vector product $\mathbf{e} = \mathbf{T}_e \mathbf{e}_{nat}$ where $\mathbf{e} = [\, e_{xx} \;\; e_{yy} \;\; 2e_{xy} \,]^T$ and $\mathbf{e}_{nat} = [\, e_{xi\xi} \;\; e_{\eta\eta} \;\; \gamma_{\xi\eta} \,]^T$.

EXERCISE Q.5 [A:25] As in the previous Exercise, but for the 3D case discussed in §Q.1.3.

EXERCISE Q.6 [A:25] Construct the hierarchical version of the 10-node triangular element of Exercise 17.1 proceeding in two stages:

First stage. Add the six node points 4 through 9 as cubic deviations from the linear shape functions of the three-node triangle. Compare those hierarchical shape functions for these nodes with those found in Exercise 17.1. The results for this stage should be a formula such as (Q.23) with 9 matrix columns plus a list of the shape functions for nodes 1–9. Verify that rigid body motions and constant strain states are preserved

Second stage. Inject the interior node point 10 as a hierarchical function. The results for this stage should be again a formula such as (Q.23) but with 10 matrix columns, and a 10×10 transformation matrix. Verify that rigid body motions and constant strain states are preserved.

EXERCISE Q.7 [A:25] Construct the hierarchical version of the 9-node biquadratic quadrilateral. Proceed in two stages as outlined in the previous exercise. Verify that rigid body motions and constant strain states are preserved.

EXERCISE Q.8 [A:35] Develop a 3-node, curved, plane bar-beam element with parabolic midline geometry defined by three nodes. Element has 8 degrees of freedom: three at the end nodes 1-2 (add the rotation θ_z to the freedoms of the curved bar element) and two at the midnode 3 (same as in the bar element). Assumed uniform flexural rigidity EI decoupled from the bar-axial constitutive relation. Investigate the rank of the numerically integrated stiffness matrix.