Attitude Determination

ASEN 5010

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Notation in Notes

- Here is a quick guide to the notational differences in the notes by Dr. Hall and the notation used in the text book and during lessons:

Book vs Notes	Comment
${}^{\mathcal{B}}\hat{\boldsymbol{s}} \equiv \hat{\boldsymbol{s}}_b$	Vector component label
$[BI] \equiv \boldsymbol{R}^{bi}$	Rotation matrix notation
$\boldsymbol{\beta} \equiv \bar{\boldsymbol{q}}$ $(\beta_1, \beta_2, \beta_3, \beta_0) \equiv (q_1, q_2, q_3, q_4)$	Variable used for Euler Parameter or quaternion vector

Introduction

- Attitude determination is broken up into two areas
  - Static attitude determination: All measurements are taken at the same time. Using this snap shot in time concept, the problem becomes up of optimally solving the geometry of the measurements
  - Dynamic attitude determination: Here measurements are taken over time. This is a much harder problem, in that attitude measurements are taken over time, along with some gyro (rotation rate) measurements, which then need to be optimally blended together (Kalman filter).

Basic Concept

- Consider the 2D attitude problem. How many direction measurements (unit direction vectors) does it take to determine your heading?

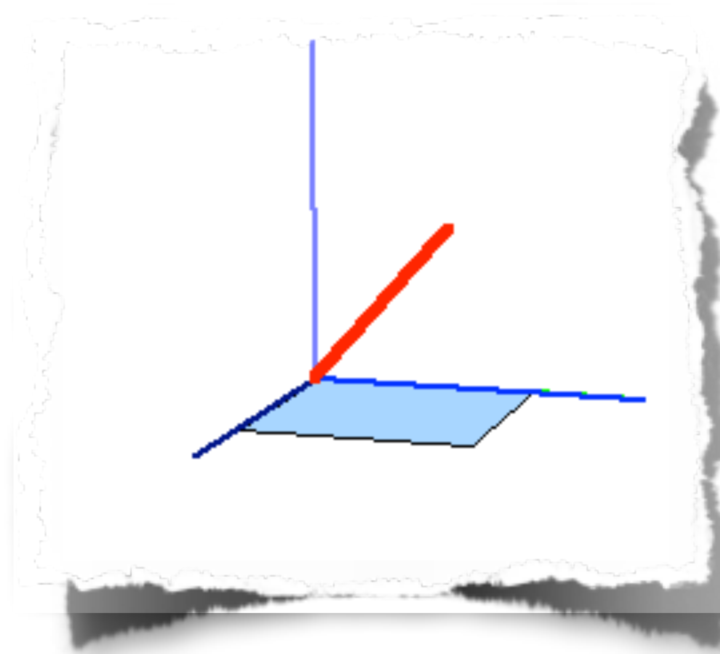
Answer: You only need one direction measurement for the 2D case.

Explanation: Headings in a 2D environment is a 1D measure. The unit direction vector (with the unit length constraint) provides all the required information.

- Next, let us consider the three dimensional orientation measurement. How many observation vectors (unit direction vectors) are required here?

Answer: You will need a minimum of two observation vectors.

Explanation: With only one measurement, you cannot sense rotations about this axis. Measuring a second direction will fix the complete three dimension orientation in space.



- To determine attitude, we assume you already know the inertial direction to certain objects (sun, Earth, magnetic field direction, stars, moon, etc.)
- Assume the sun direction is given by $\hat{\boldsymbol{s}}$ and the local magnetic field direction is given by $\hat{\boldsymbol{m}}$.
- If the vehicle has sensors on board that measure these directions, then these unit vectors are measured with components taken in the vehicle fixed body frame $\mathcal{B}$.

Measured: ${}^{\mathcal{B}}\hat{\boldsymbol{m}}$ ${}^{\mathcal{B}}\hat{\boldsymbol{s}}$

Given: ${}^{\mathcal{I}}\hat{\boldsymbol{m}}$ ${}^{\mathcal{I}}\hat{\boldsymbol{s}}$

Mapping:

$$ {}^{\mathcal{B}}\hat{\boldsymbol{m}} = [BI]\ {}^{\mathcal{I}}\hat{\boldsymbol{m}} $$

$$ {}^{\mathcal{B}}\hat{\boldsymbol{s}} = [BI]\ {}^{\mathcal{I}}\hat{\boldsymbol{s}} $$

Challenge: How do we find $[BI]$?

Under or Over?

- Note that each observation vector (unit direction vector) contains two independent degrees of freedom.
- The 3D attitude problem is a three-degree of freedom problem.
- Thus, by measuring two observation directions, the attitude determination problem is always an over-determined problem!

Deterministic Attitude Estimation

Vector Triad Method

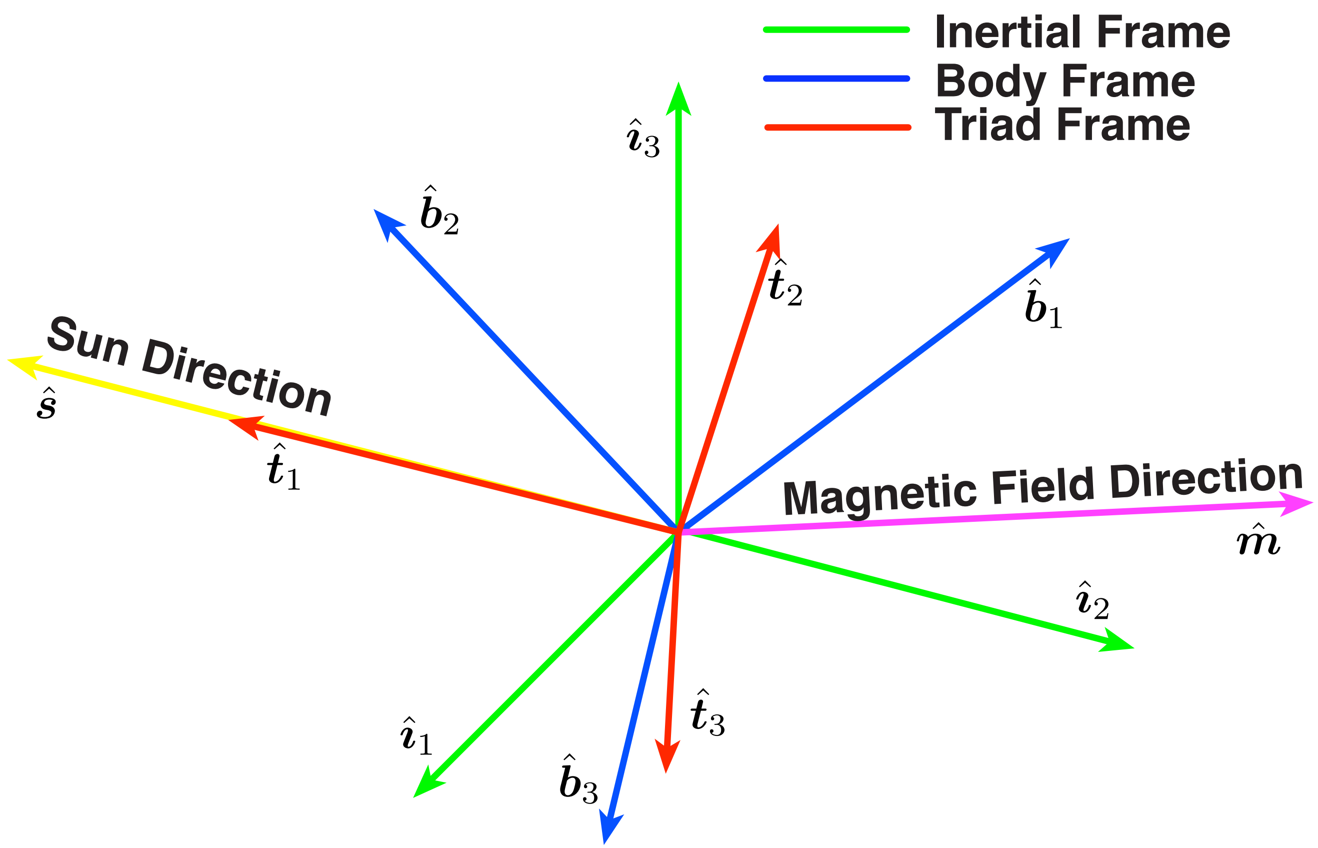
- To determine the desired [*BI*] matrix, we first introduce the triad coordinate frame *T*.

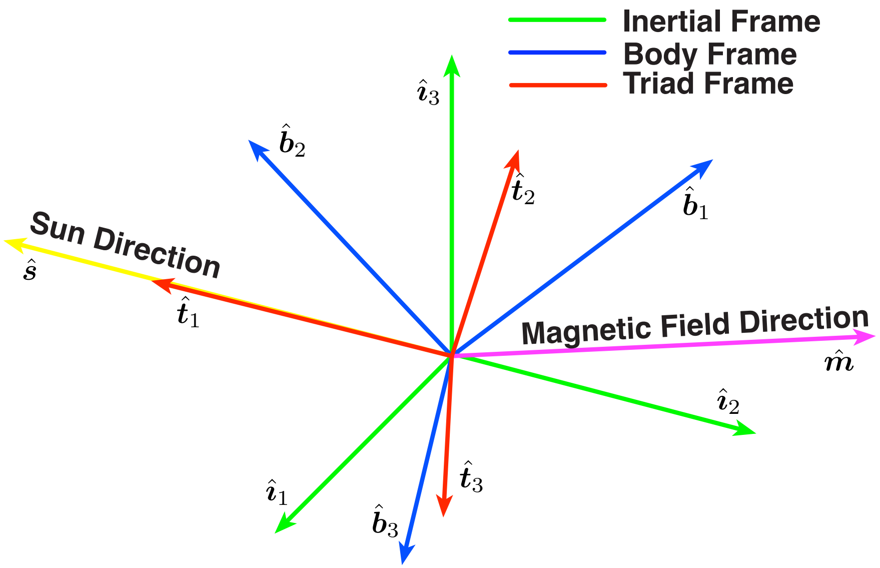
Assume: $\hat{\boldsymbol{t}}_1 = \hat{\boldsymbol{s}}$

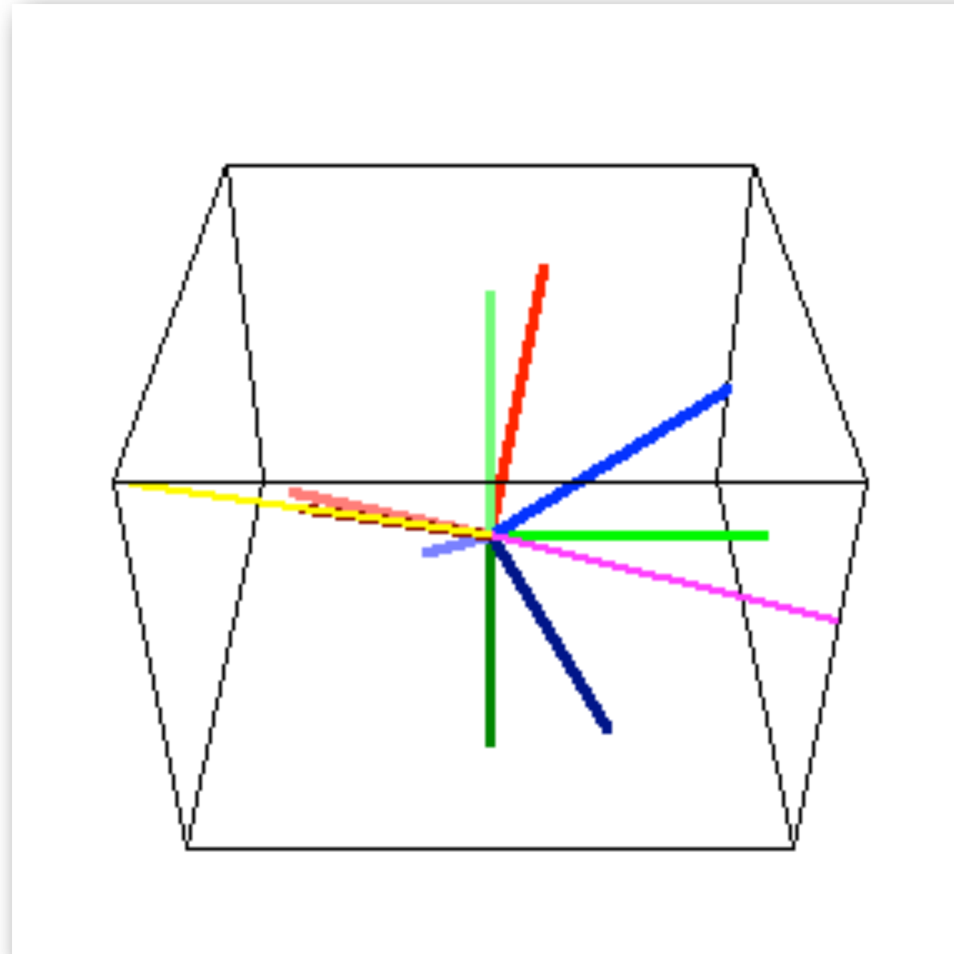
Then define:

$$\hat{\boldsymbol{t}}_2 = \frac{\hat{\boldsymbol{s}} \times \hat{\boldsymbol{m}}}{|\hat{\boldsymbol{s}} \times \hat{\boldsymbol{m}}|}$$

$$\hat{\boldsymbol{t}}_3 = \hat{\boldsymbol{t}}_1 \times \hat{\boldsymbol{t}}_2$$







3D Illustration of Triad Coordinate Frame

- We can compute the *T* frame direction axes using both *B* and I frame components using

$$
\begin{aligned}
{}^{\mathcal{B}}\hat{\boldsymbol{t}}_1 &= {}^{\mathcal{B}}\hat{\boldsymbol{s}} \\
{}^{\mathcal{B}}\hat{\boldsymbol{t}}_2 &= \frac{({}^{\mathcal{B}}\hat{\boldsymbol{s}}) \times ({}^{\mathcal{B}}\hat{\boldsymbol{m}})}{|({}^{\mathcal{B}}\hat{\boldsymbol{s}}) \times ({}^{\mathcal{B}}\hat{\boldsymbol{m}})|} \\
{}^{\mathcal{B}}\hat{\boldsymbol{t}}_3 &= ({}^{\mathcal{B}}\hat{\boldsymbol{t}}_1) \times ({}^{\mathcal{B}}\hat{\boldsymbol{t}}_2)
\end{aligned}
$$

Body Frame Triad Vectors

$$
\begin{aligned}
{}^{\mathcal{I}}\hat{\boldsymbol{t}}_1 &= {}^{\mathcal{I}}\hat{\boldsymbol{s}} \\
{}^{\mathcal{I}}\hat{\boldsymbol{t}}_2 &= \frac{({}^{\mathcal{I}}\hat{\boldsymbol{s}}) \times ({}^{\mathcal{I}}\hat{\boldsymbol{m}})}{|({}^{\mathcal{I}}\hat{\boldsymbol{s}}) \times ({}^{\mathcal{I}}\hat{\boldsymbol{m}})|} \\
{}^{\mathcal{I}}\hat{\boldsymbol{t}}_3 &= ({}^{\mathcal{I}}\hat{\boldsymbol{t}}_1) \times ({}^{\mathcal{I}}\hat{\boldsymbol{t}}_2)
\end{aligned}
$$

Inertial Frame Triad Vectors

- In the absence of measurement errors, both sets of Triad frame representations should be the same.

- We can write the various rotation matrices as

$$
[BT] = \begin{bmatrix} {}^{\mathcal{B}}\hat{\boldsymbol{t}}_1 & {}^{\mathcal{B}}\hat{\boldsymbol{t}}_2 & {}^{\mathcal{B}}\hat{\boldsymbol{t}}_3 \end{bmatrix} \qquad [IT] = \begin{bmatrix} {}^{\mathcal{I}}\hat{\boldsymbol{t}}_1 & {}^{\mathcal{I}}\hat{\boldsymbol{t}}_2 & {}^{\mathcal{I}}\hat{\boldsymbol{t}}_3 \end{bmatrix}
$$

- Finally, we can compute the desired [*BI*] matrix using

$$[BI] = [BT][IT]^T$$

- From the rotation matrix, we can now extract any desired set of attitude coordinates!

- Note that with this method we do not use the full magnetic field direction vector $\hat{\boldsymbol{m}}$. If this measurement were more accurate, then we could modify this method to define $\hat{\boldsymbol{t}}_1 = \hat{\boldsymbol{m}}$ instead.

Example 4.2 *Suppose a spacecraft has two attitude sensors that provide the following measurements of the two vectors* $\hat{\mathbf{v}}_1$ *and* $\hat{\mathbf{v}}_2$*:*

$$\mathbf{v}_{1b} = [0.8273 \;\; 0.5541 \;\; -0.0920]^{\mathsf{T}} \tag{4.29}$$
$$\mathbf{v}_{2b} = [-0.8285 \;\; 0.5522 \;\; -0.0955]^{\mathsf{T}} \tag{4.30}$$

These vectors have known inertial frame components of

$$\mathbf{v}_{1i} = [-0.1517 \;\; -0.9669 \;\; 0.2050]^{\mathsf{T}} \tag{4.31}$$
$$\mathbf{v}_{2i} = [-0.8393 \;\; 0.4494 \;\; -0.3044]^{\mathsf{T}} \tag{4.32}$$

Applying the Triad algorithm, we construct the components of the vectors $\hat{\mathbf{t}}_j, j = 1, 2, 3$ *in both the body and inertial frames:*

$$\mathbf{t}_{1b} = [0.8273 \;\; 0.5541 \;\; -0.0920]^{\mathsf{T}} \tag{4.33}$$
$$\mathbf{t}_{2b} = [-0.0023 \;\; 0.1671 \;\; 0.9859]^{\mathsf{T}} \tag{4.34}$$
$$\mathbf{t}_{3b} = [0.5617 \;\; -0.8155 \;\; 0.1395]^{\mathsf{T}} \tag{4.35}$$

and

$$\mathbf{t}_{1i} = [-0.1517 \ \ -0.9669 \ \ 0.2050]^{\mathsf{T}} \tag{4.36}$$

$$\mathbf{t}_{2i} = [0.2177 \ \ -0.2350 \ \ -0.9473]^{\mathsf{T}} \tag{4.37}$$

$$\mathbf{t}_{3i} = [0.9641 \ \ -0.0991 \ \ 0.2462]^{\mathsf{T}} \tag{4.38}$$

Using these results with Eq. (4.28), we obtain the approximate rotation matrix

$$\mathbf{R}^{bi} = \begin{bmatrix} 0.4156 & -0.8551 & 0.3100 \\ -0.8339 & -0.4943 & -0.2455 \\ 0.3631 & -0.1566 & -0.9185 \end{bmatrix} \tag{4.39}$$

Applying this rotation matrix to $\mathbf{v}_{1i}$ *gives* $\mathbf{v}_{1b}$ *exactly, because we used this condition in the formulation; however, applying it to* $\mathbf{v}_{2i}$ *does not give* $\mathbf{v}_{2b}$ *exactly. If we know* a priori *that sensor 2 is more accurate than sensor 1, then we can use* $\hat{\mathbf{v}}_2$ *as the exact measurement, hopefully leading to a more accurate estimate of* $\mathbf{R}^{bi}$.

Statistical Attitude Determination

Wahba's Problem

- Assume we have N>1 observation measurements (i.e. measured directions to sun, magnetic field, stars, etc.), and we know the corresponding inertial vector directions. Then we can write attitude determination problem as

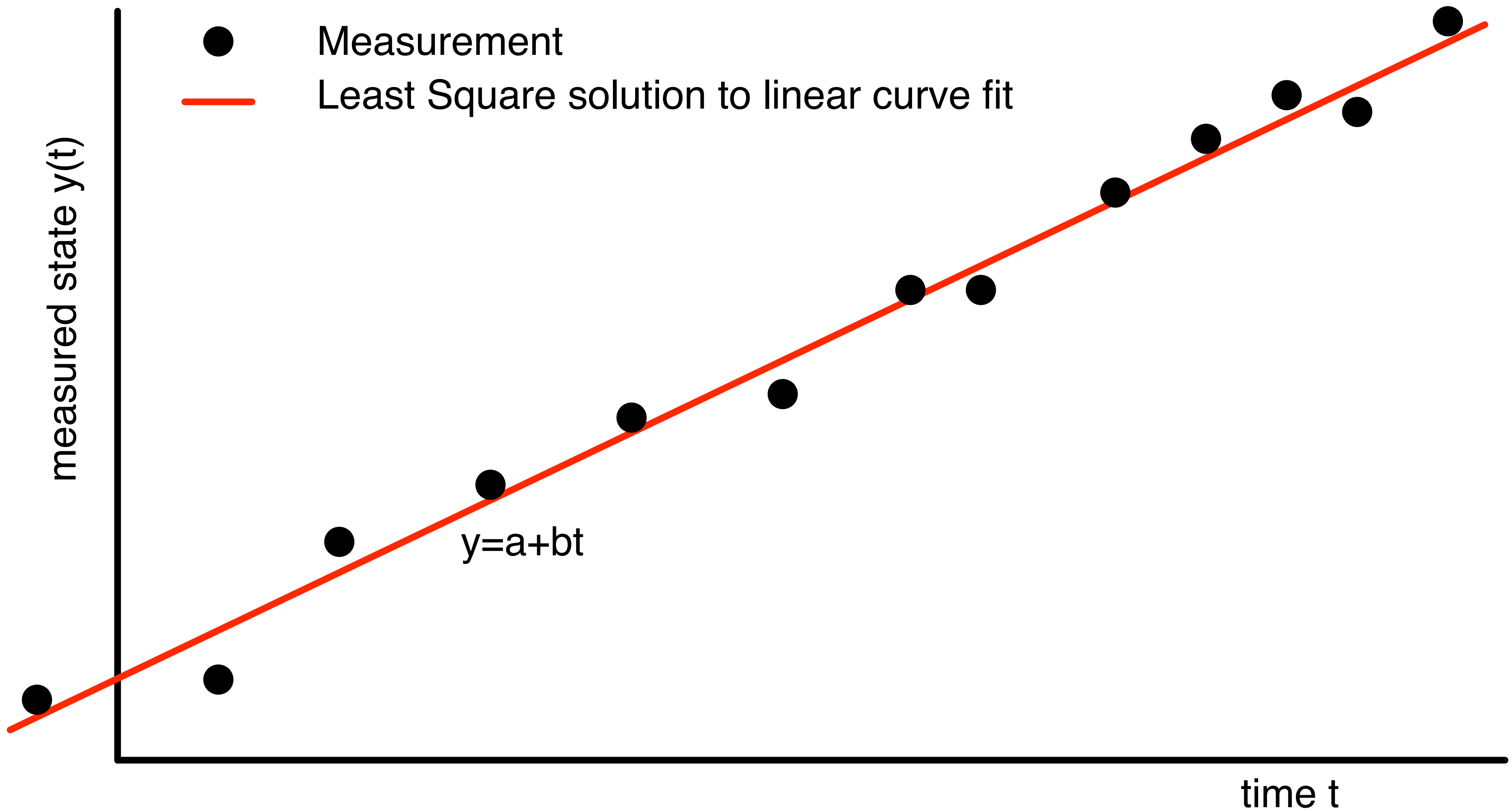
$$^{\mathcal{B}}\boldsymbol{v}_k = [BI]\ ^{\mathcal{I}}\boldsymbol{v}_k \qquad \text{for } k = 1, \cdots N$$

with the goal to find the rotation matrix [*BI*] such that the following loss function is minimized:

- If all measurements are perfect, then J = 0.

$$J\left([BI]\right) = \frac{1}{2}\sum_{k=1}^{N} w_k \left| ^{\mathcal{B}}\boldsymbol{v}_k - [BI]\ ^{\mathcal{I}}\boldsymbol{v}_k \right|^2$$

- Think of the cost function *J* as the error of the common least squares curve fitting problem:



Devenport's q-Method

- Let the 4-D Euler parameter (quaternion) vector be defined as

$$\bar{\boldsymbol{q}} = (\boldsymbol{q}, q_4)^T = (\beta_1, \beta_2, \beta_3, \beta_0)^T$$

- The cost function can be rewritten

$$J = \frac{1}{2}\sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\boldsymbol{v}_k - [BI]\ {}^{\mathcal{I}}\boldsymbol{v}_k\right)^T \left({}^{\mathcal{B}}\boldsymbol{v}_k - [BI]\ {}^{\mathcal{I}}\boldsymbol{v}_k\right)$$

$$J = \frac{1}{2}\sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\boldsymbol{v}_k^T\ {}^{\mathcal{B}}\boldsymbol{v}_k + {}^{\mathcal{I}}\boldsymbol{v}_k^T\ {}^{\mathcal{I}}\boldsymbol{v}_k - 2({}^{\mathcal{B}}\boldsymbol{v}_k)^T[BI]\ {}^{\mathcal{I}}\boldsymbol{v}_k\right)$$

$$J = \sum_{k=1}^{N} w_k \left(1 - \left({}^{\mathcal{B}}\boldsymbol{v}_k\right)^T [BI]\ {}^{\mathcal{I}}\boldsymbol{v}_k\right)$$

- Minimizing *J* is equivalent to maximizing the gain function *g*:

$$g = \sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\boldsymbol{v}_k\right)^T [BI]\,{}^{\mathcal{I}}\boldsymbol{v}_k$$

- The rotation matrix can be written in terms of Euler parameters as

$$[BI] = (q_4^2 - \boldsymbol{q}^T\boldsymbol{q})[I_{3\times 3}] + 2\boldsymbol{q}\boldsymbol{q}^T - 2q_4[\tilde{\boldsymbol{q}}]$$

- This allows us to rewrite the gain function *g* using the 4x4 matrix [*K*]

$$\boxed{g(\bar{\boldsymbol{q}}) = \bar{\boldsymbol{q}}^T[K]\bar{\boldsymbol{q}}} \qquad [K] = \begin{bmatrix} S - \sigma I_{3\times 3} & Z \\ Z^T & \sigma \end{bmatrix}$$

$$[B] = \sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\boldsymbol{v}_k\right)\left({}^{\mathcal{I}}\boldsymbol{v}_k\right)^T \qquad [S] = [B] + [B]^T$$

$$\sigma = \mathrm{tr}\left([B]\right)$$

$$[Z] = \begin{bmatrix} B_{23} - B_{32} & B_{31} - B_{13} & B_{12} - B_{21} \end{bmatrix}^T$$

- However, since the Euler parameter vector must abide by the unit length constraint, we cannot solve this gain function directly. Instead, we use Lagrange multipliers to yield a new gain function *g'*

$$g'(\bar{\boldsymbol{q}}) = \bar{\boldsymbol{q}}^T [K] \bar{\boldsymbol{q}} - \lambda \left(\bar{\boldsymbol{q}}^T \bar{\boldsymbol{q}} - 1 \right)$$

- We differentiate *g'* and set it equal to zero to find the extrema point of this function.

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\boldsymbol{q}}} \left(g'(\bar{\boldsymbol{q}}) \right) = 2[K]\bar{\boldsymbol{q}} - 2\lambda\bar{\boldsymbol{q}} = 0 \qquad \Rightarrow \qquad \boxed{[K]\bar{\boldsymbol{q}} = \lambda\bar{\boldsymbol{q}}}$$

- Clearly the desired Euler parameter vector is the eigenvector of the [*K*] matrix.

- To maximize the gain function, we need to choose the largest eigenvalue of the [*K*] matrix.

$$g(\bar{\boldsymbol{q}}) = \bar{\boldsymbol{q}}^T [K] \bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}^T \lambda \bar{\boldsymbol{q}} = \lambda \bar{\boldsymbol{q}}^T \bar{\boldsymbol{q}} = \lambda$$

- In summary, to use the q-Method, we must
  - Compute the 4x4 matrix [K]
  - Find the eigenvalue and eigenvector of the [K] matrix
  - Choose the largest eigenvalue and associated eigenvector.
  - This eigenvector is the Euler parameter vector
- Note that solving this eigenvalue, eigenvector problem is numerically rather intensive for real-time applications.

$$\bar{\boldsymbol{q}} = (\boldsymbol{q}, q_4)^T = (\beta_1, \beta_2, \beta_3, \beta_0)^T$$

Example 4.3 *We use a two-sensor satellite to demonstrate the q-method. First, we use the Triad algorithm to generate an attitude estimate which we compare with the known attitude as well as with the q-method result.*

Given two vectors known in the inertial frame:

$$\mathbf{v}_{1i} = \begin{bmatrix} 0.2673 \\ 0.5345 \\ 0.8018 \end{bmatrix} \quad \mathbf{v}_{2i} = \begin{bmatrix} -0.3124 \\ 0.9370 \\ 0.1562 \end{bmatrix} \tag{4.76}$$

The "known" attitude is defined by a 3-1-3 Euler angle sequence with a 30° *rotation for each angle. The true attitude is represented by the rotation matrix between the inertial and body frames:*

$$\mathbf{R}^{bi}_{exact} = \begin{bmatrix} 0.5335 & 0.8080 & 0.2500 \\ -0.8080 & 0.3995 & 0.4330 \\ 0.2500 & -0.4330 & 0.8660 \end{bmatrix} \tag{4.77}$$

If the sensors measured the two vectors without error, then in the body frame the vectors would be:

$$\mathbf{v}_{1b_{exact}} = \begin{bmatrix} 0.7749 \\ 0.3448 \\ 0.5297 \end{bmatrix} \quad \mathbf{v}_{2b_{exact}} = \begin{bmatrix} 0.6296 \\ 0.6944 \\ -0.3486 \end{bmatrix} \tag{4.78}$$

Sensor measurements are not perfect however, and to model this uncertainty we introduce some error into the body-frame sensor measurements. A uniformly distributed random error is added to the sensor measurements, with a maximum value of $\pm 5°$. *The two "measured" vectors are:*

$$\mathbf{v}_{1b} = \begin{bmatrix} 0.7814 \\ 0.3751 \\ 0.4987 \end{bmatrix} \quad \mathbf{v}_{2b} = \begin{bmatrix} 0.6163 \\ 0.7075 \\ -0.3459 \end{bmatrix} \tag{4.79}$$

Using the Triad algorithm and assuming $\mathbf{v}_1$ *is the "exact" vector, the satellite attitude is estimated by :*

$$\mathbf{R}^{bi}_{triad} = \begin{bmatrix} 0.5662 & 0.7803 & 0.2657 \\ -0.7881 & 0.4180 & 0.4518 \\ 0.2415 & -0.4652 & 0.8516 \end{bmatrix} \tag{4.80}$$

A useful approach to measuring the value of the attitude estimate makes use of the orthonormal nature of the rotation matrix; i.e., $\mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{1}$*. Since the Triad algorithm's estimate is not perfect, we compare the following to the identity matrix:*

$$\mathbf{R}_{triad}^{bi^{\mathsf{T}}}\mathbf{R}_{exact}^{bi} = \begin{bmatrix} 0.9992 & 0.03806 & 0.0094 \\ -0.0378 & 0.9989 & -0.0268 \\ -0.0104 & 0.02645 & 0.9996 \end{bmatrix} \tag{4.81}$$

This new matrix is the rotation matrix from the exact attitude to the attitude estimated by the Triad algorithm. The principal Euler angle of this matrix, and therefore the attitude error of the estimate, is $\Phi = 2.72°$*. For later comparison, the loss function for this rotation matrix is* $J = 7.3609 \times 10^{-4}$*.*

Using the q-method with the same inertial and measured vectors produces the $\mathbf{K}$ *matrix:*

$$\mathbf{K} = \begin{bmatrix} -1.1929 & 0.8744 & 0.9641 & 0.4688 \\ 0.8744 & 0.5013 & 0.3536 & -0.4815 \\ 0.9641 & 0.3536 & -0.5340 & 1.1159 \\ 0.4688 & -0.4815 & 1.1159 & 1.2256 \end{bmatrix} \tag{4.82}$$

Each measurement is equally weighted in the loss function. The largest eigenvalue and corresponding eigenvector of this matrix are:

$$\lambda_{max} = 1.9996 \tag{4.83}$$

$$\bar{\mathbf{q}} = \begin{bmatrix} 0.2643 \\ -0.0051 \\ 0.4706 \\ 0.8418 \end{bmatrix} \tag{4.84}$$

The corresponding rotation matrix is:

$$\mathbf{R}_q^{bi} = \begin{bmatrix} 0.5570 & 0.7896 & 0.2575 \\ -0.7951 & 0.4173 & 0.4402 \\ 0.2401 & -0.4499 & 0.8602 \end{bmatrix} \tag{4.85}$$

We determine the accuracy of this solution by computing the Euler angle of $\mathbf{R}_q^{bi^\mathsf{T}} \mathbf{R}_{exact}^{bi}$. *For the q-method estimate of attitude, the attitude error and loss function values are:*

$$\Phi = 1.763^\circ \tag{4.86}$$

$$J = 3.6808 \times 10^{-4} \tag{4.87}$$

QUEST

- Recall the cost function *J* and the gain function *g*

$$J = \sum_{k=1}^{N} w_k \left(1 - \left({}^{\mathcal{B}}\boldsymbol{v}_k\right)^T [BI] \, {}^{\mathcal{I}}\boldsymbol{v}_k\right)$$
$$g = \sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\boldsymbol{v}_k\right)^T [BI] \, {}^{\mathcal{I}}\boldsymbol{v}_k$$

- Further, we found that the optimal *g* will be

$$g(\bar{\boldsymbol{q}}) = \lambda_{\text{opt}}$$

- This can now be rewritten in the useful form

- Finally, the optimality condition can be written as

$$\lambda_{\text{opt}} = \sum_{k=1}^{N} w_k - J$$

- Note that *J* should be small for an optimal solution. This assumes that the measurement noise is reasonable small and Gaussian. The QUEST method then makes the elegant assumption that

$$\lambda_{\text{opt}} \approx \sum_{k=1}^{N} w_k$$

- This allows us to avoid the numerically intensive eigenvalue problem!

- However, we still need to find a solution for the eigenvector.

- Let use introduce the classical Rodrigues parameter vector $\boldsymbol{p}$

$$\boldsymbol{p} = \hat{\boldsymbol{e}} \tan\frac{\Phi}{2} = \frac{1}{\beta_0}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} = \frac{1}{q_4}\boldsymbol{q}$$

- Note that

$$\frac{\bar{\boldsymbol{q}}}{q_4} = \begin{pmatrix}\boldsymbol{p}\\ 1\end{pmatrix}$$

- The eigenvector problem is now re-written as

$$[K]\frac{\bar{\boldsymbol{q}}}{q_4} = \lambda_{\text{opt}}\frac{\bar{\boldsymbol{q}}}{q_4}$$

$$\begin{bmatrix} S - \sigma I_{3\times 3} & Z\\ Z^T & \sigma\end{bmatrix}\begin{pmatrix}\boldsymbol{p}\\ 1\end{pmatrix} = \lambda_{\text{opt}}\begin{pmatrix}\boldsymbol{p}\\ 1\end{pmatrix}$$

$$\left([S] - \sigma[I_{3\times 3}]\right)\boldsymbol{p} + [Z] = \lambda_{\text{opt}}\boldsymbol{p}$$

- Finally, the classical Rodrigues parameter vector is found

$$\boldsymbol{p} = \Big((\lambda_{\text{opt}} + \sigma)[I_{3\times 3}] - [S]\Big)^{-1}[Z]$$

- Note that we still have to take an inverse of a 3x3 matrix here. However, this is numerically a very fast process.

- To solve for the corresponding 4-D Euler parameter vector, we use

$$\bar{\boldsymbol{q}} = \frac{1}{\sqrt{1 + \boldsymbol{p}^T\boldsymbol{p}}}\begin{bmatrix}\boldsymbol{p} \\ 1\end{bmatrix}$$

Example 4.4 *We repeat Example 4.3 using the QUEST method. Recall that the vector measurements are equally weighted, so we use a weighting vector of:*

$$\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.98}$$

Using $\lambda_{opt} \approx \sum w_k = 2$, *the QUEST method produces an attitude estimate of*

$$\mathbf{R}^{bi}_{QUEST} = \begin{bmatrix} 0.5571 & 0.7895 & 0.2575 \\ -0.7950 & 0.4175 & 0.4400 \\ 0.2399 & -0.4499 & 0.8603 \end{bmatrix} \tag{4.99}$$

For the QUEST estimate of attitude, the attitude error and loss function values are:

$$\Phi = 1.773^\circ \tag{4.100}$$

$$J = 3.6810 \times 10^{-4} \tag{4.101}$$

The QUEST method produces a rotation matrix which has a slightly larger loss function value, but without solving the entire eigenproblem. The actual attitude error of the estimate is comparable to that obtained using the q-method.