ASEN5070

Statistical Orbit determination I

Fall 2011, 8/22, 24/2011

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ECNT 316

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http://ccar.colorado.edu/ASEN5070/

Lectures 1&2

Course Outline

- Instructor
    - Professor George Born
        - Office Hour: Wed 2-3 PM
- Course Assistants

 Jason Leonard<**Jason.Leonard@colorado.edu**>

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Office Hours:

Monday/Wednesday 3-4 PM

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    - Office: ECNT 418
    - Office Hours: TBD


HUNDLEY

Course Website

- Course website: ccar.colorado.edu/asen5070
  - Homework, project, and reference materials

Course Grade

- Homework = 20%

  ~11 assignments

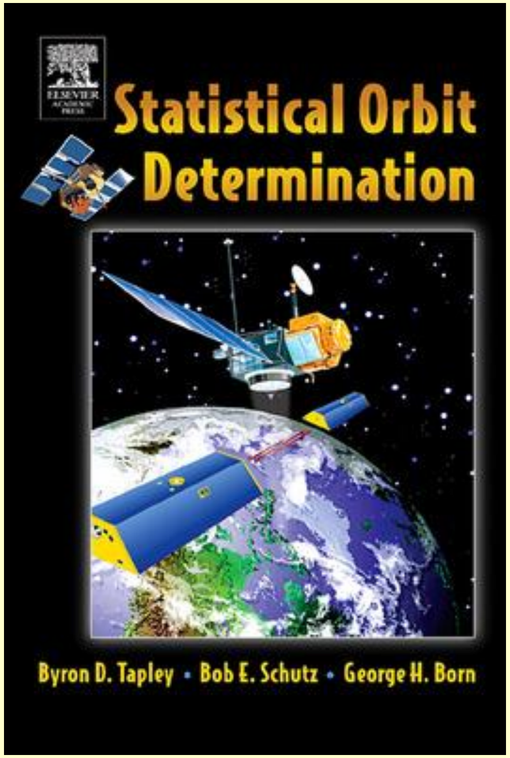
- Quizzes/Exams = 50%

  - 2 mid-terms
  - 1 take-home final

- Course Project = 30 %

Course Textbook

Tapley, B.D., B.E. Schutz, and G.H. Born, *Statistical Orbit Determination*, Elsevier Academic Press, New York, 2004.



Course Topics

- Introduction
  - -Overview, Background, Notation, References
  - -Review of Matrix Theory (App. B in Text)
  - -Uniform Gravity Field Problem (1.2)
- The Orbit Determination (OD) Problem
  - -The Observation – State Relationship
  - -Linearization of the OD Process (1.2.4, 4.2)
  - -Transformation to a Common Epoch – The State Transition Matrix (1.2.5, 4.2, 4.2.3)

Course Topics

• Solution Methods
  - Least Squares (4.3)
  - Weighted Least Squares (4.3.3)
  - Minimum Norm (4.3.1)
  - Least Squares with A priori (4.3.3, 4.4.2)

• Review of Probability and Statistics (App. A in Text)
  - Density/Distribution Functions
  - Moment Generating Functions
  - Bivariate Density Functions
  - Properties of Covariance and Correlation

Course Topics

- Review of Probability and Statistics (App. A in Text)
  - Central Limit Theorem
  - Bayes Theorem
  - Stochastic Processes
  - Statistical Interpretation of Least Squares

- Computational Algorithms
  - Cholesky (5.2)
  - Square Root Free Cholesky (5.2.2)
  - Givens Algorithm (Orthogonal Transformations 5.3, 5.4

Course Topics

- The Sequential Estimation Algorithm (4.7)
  - -The Extended Sequential Estimation Algorithm
  - -Numerical Problems with the Kalman Filter Algorithm
  - -Square Root Filter Algorithms
  - -Potter Algorithm
  - -State Noise Compensation Algorithms
  - -Information Filters
  - -Smoothing Algorithms
  - -Gauss-Markoff Theorem
- The Probability Ellipsoid (4.16)
- Combining Estimates (4.17)

Homework # 1

- Problem 1:

1. Given the following Earth orbiting spacecraft position and velocity vectors in Cartesian coordinates, solve for the Keplerian elements (a, e, i, Ω, ω, M). See cart2kep.htm.

$$\bar{R} = -2436.45\,\hat{i} - 2436.45\,\hat{j} + 6891.037\,\hat{k} \ km$$

$$V = \dot{\bar{R}} = 5.088611\,\hat{i} - 5.088611\,\hat{j} + 0.0\hat{k} \ km/s$$

assume $\mu = 398600.5\ km^3/s^2$

M is mean anomaly

- Problem 2:

2. Convert the Keplerian elements from problem 1 back to position and velocity. See kep2cart.htm.

Homework # 1

- Problem 3:

3. Given the gravity potential function $U = \frac{\mu}{R}$ solve for the two-body acceleration due to gravity, i.e. $\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$.

$$R = \sqrt{x^2 + y^2 + z^2}$$

- Problem 4:

4. Develop the necessary code to integrate the equations of motion using the position and velocity from part 1 as the initial conditions. Compute the future position and velocity at 20 second intervals for two full orbits.

Homework # 1

- Problem 5:

5. Plot the magnitude of the position, velocity, and acceleration as a function of time for two full orbits.

- Problem 6:

6. Compute the specific kinetic energy and specific potential energy as a function of time and plot the change in total specific energy to show that it remains constant over the two orbits. (i.e. plot $dTE = TE(t) - TE(t0)$)

Homework # 1

•Problem 7:

1. Write a computer program that computes $\rho(t_i)$ for a uniform gravity field using Eq. (1.2.10). A set of initial conditions, X_0, $\dot{X}_0$, Y_0, $\dot{Y}_0$, g, X_s, Y_s, and observations, ρ, follow. With the exception of the station coordinates, the initial conditions have been perturbed so that they will not produce exactly the observations given. Use the Newton iteration scheme of Eq. (1.2.9) to recover the exact initial conditions for these quantities; that is, the values used to generate the observations. Assume that X_s and Y_s are known exactly. Hence, they are not solved for.

Unitless Initial Conditions

$$
\begin{aligned}
X_0 &= 1.5\\
Y_0 &= 10.0\\
\dot{X}_0 &= 2.2\\
\dot{Y}_0 &= 0.5\\
g &= 0.3\\
X_s &= Y_s = 1.0
\end{aligned}
$$

Time	Range Observation, ρ
0	7.0
1	8.00390597
2	8.94427191
3	9.801147892
4	10.630145813

Problem of Two Bodies

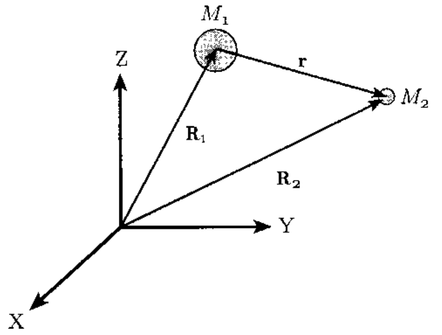


Figure 2.2.1: The problem of two bodies: M_1 and M_2 are spheres of constant, uniform density. The position vectors of the spheres refer to their respective geometrical center, which also coincide with the center of mass of the sphere.

$$M_1 \ddot{\mathbf{R}}_1 = \frac{G M_1 M_2 \mathbf{r}}{r^3}$$

$$M_2 \ddot{\mathbf{R}}_2 = -\frac{G M_1 M_2 \mathbf{r}}{r^3}$$

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3}$$

$\mu = G(M_1 + M_2)$

XYZ is nonrotating, with zero acceleration; an inertial reference frame

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Integrals of Motion

- Center of mass of two bodies moves in straight line with constant velocity
- Angular momentum per unit mass (**h**) is constant, **h** = **r** x **V** = **constant**, where **V** is velocity of M_2 with respect to M_1, **V**= d**r**/dt
  - Consequence: motion is planar
- Energy per unit mass (scalar) is constant

$$\xi = \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{2} - \frac{\mu}{r} = \text{constant}$$

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Orbit Plane in Space

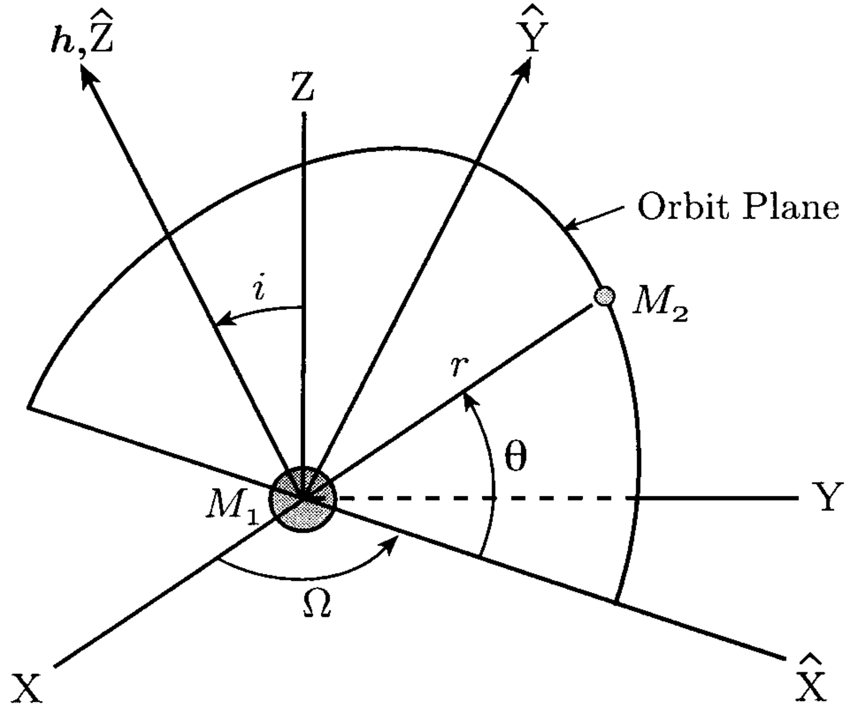

Figure 2.2.2: The orbit plane in space. Since the angular momentum is constant, the motion of M_2 with repect to M_1 takes place in a plane, known as the orbit plane. This plane is perpendicular to the angular momentum vector and the orientation is defined by the angles i and Ω.

Equations of Motion in the Orbit Plane

$$\mathbf{u}_r \text{ component}: \quad \ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

$$\mathbf{u}_\theta \text{ component}: \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0\,.$$

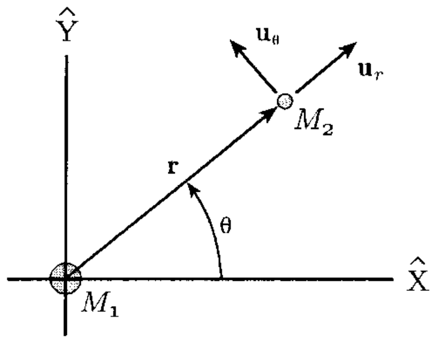

Figure 2.2.3: The planar motion described with polar coordinates.

The $\mathbf{u}_\theta$ component yields:

$$\frac{d}{dt}(r^2\dot{\theta}) = 0\,.$$

which is simply h = constant

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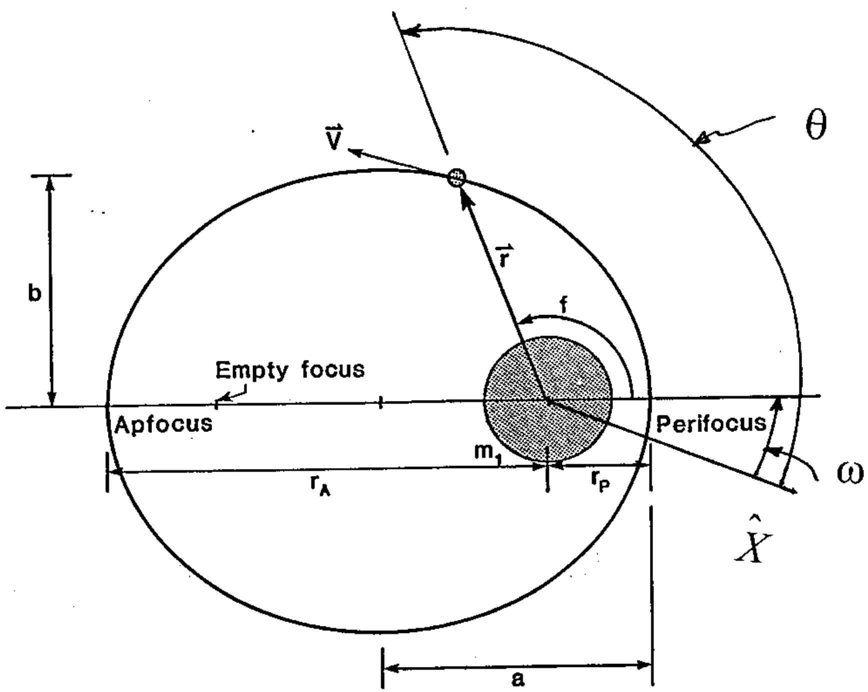
Solution of $\mathbf{u}_r$ Equations of Motion



- The solution of the $\mathbf{u}_r$ equation is (as function of θ instead of t):

$$r = \frac{h^2/\mu}{1 + e\cos(\theta - \omega)} \; .$$

where e and ω are constants of integration.

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The Conic Equation

- Constants of integration: e and ω
  - $e = (1 + 2 \xi h^2/\mu^2)^{1/2}$
  - ω corresponds to θ where r is minima
- Let $f = \theta - \omega$, then

$$r = p/(1 + e \cos f)$$

which is "conic equation" from analytical geometry (e is conic "eccentricity", p is "semi-latus rectum" or "semi-parameter", and f is the "true anomaly")

- Conclude that motion of M_2 with respect to M_1 is a "conic section"
  - Circle (e=0), ellipse (0<e<1), parabola (e=1), hyperbola (e>1)

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Types of Orbital Motion

Table 2.2.1: Classes of Orbital Motion

Orbit Type	Eccentricity	Energy	Orbital Speed
Circle	$e = 0$	$\xi = -\frac{\mu}{2a}$	$V = \sqrt{\mu/r}$
Ellipse	$e < 1$	$\xi < 0$	$\sqrt{\mu/r} < V < \sqrt{2\mu/r}$
Parabola	$e = 1$	$\xi = 0$	$V = \sqrt{2\mu/r}$
Hyperbola	$e > 1$	$\xi > 0$	$V > \sqrt{2\mu/r}$

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The Orbit and Time

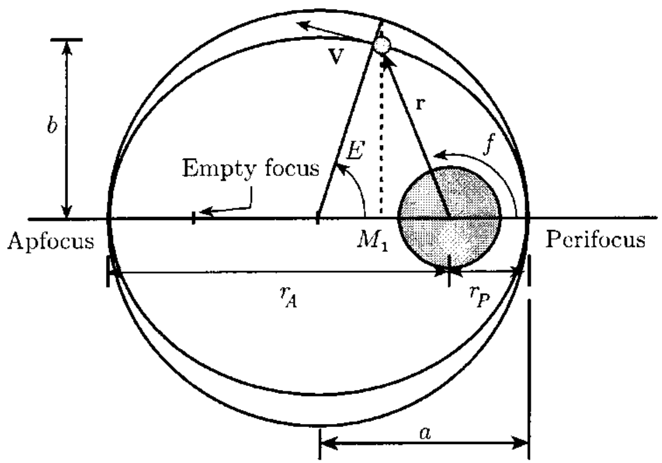

Figure 2.2.4: Characteristics of the elliptic orbit. The circle has radius equal to a and is used to illustrate the eccentric anomaly, E. The dashed line is perpendicular to the major axis of the ellipse.

- If angle f is known, r can be determined from conic equation
- Time is preferred independent variable instead of f
- Introduce E, "eccentric anomaly" related to time t by Kepler's Equation:

$$E - e \sin E = M = n\,(t - t_p)$$

where M is "mean anomaly"

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Orbit in Space

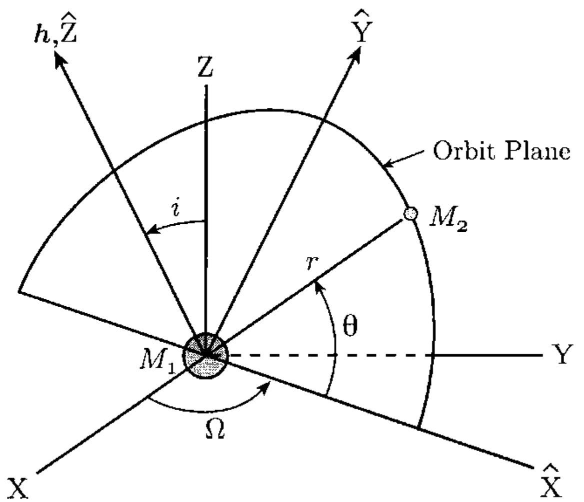

Figure 2.2.2: The orbit plane in space. Since the angular momentum is constant, the motion of M_2 with repect to M_1 takes place in a plane, known as the orbit plane. This plane is perpendicular to the angular momentum vector and the orientation is defined by the angles i and Ω.

- **h = constant**
- Components of h:
  - h_X, h_Y, h_Z
- Inclination, i (angle between Z-axis and h), 0 ≤ I ≤ 180°
- Line of nodes is line of intersection between orbit plane and (X,Y) plane
  - Ascending node (AN) is point where M_2 crosses (X,Y) plane from –Z to +Z
  - Ω is angle from X-axis to AN

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Six Orbit Elements

- The six orbit elements (or Kepler elements) are constant in the problem of two bodies (two gravitationally attracting spheres, or point masses)
  - Define shape of the orbit
    - a: semimajor axis (or p: semilatus rectum)
    - e: eccentricity
  - Define the orientation of the orbit in space
    - i: inclination
    - Ω: angle defining location of ascending node (AN)
    - ω: angle from AN to perifocus; argument of perifocus
  - Reference time:
    - t_p: time of perfocus (or mean anomaly at specified time)

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Orbit Determination

To satisfy fundamental navigation functions:

- Verification of initial orbit
- Monitor evolution of orbit
  - Establish mission lifetime
  - Conduct mission objectives
- Calculate orbit modification maneuvers
- Navigate science data records
- Calculate de-orbit maneuvers

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Orbit Determination and Orbit Prediction

- With known μ and 6 orbit elements (OE), the position/velocity vectors at t_0 can be computed: the orbit has been determined (but how μ and 6 OE were determined is part of the *orbit determination process*); conversion between OE and XYZ described in Section 2.2.4
- Refer to the position/velocity vectors as the *state vector*
- Predict state vector at any other time (forward or backward) by solving Kepler's Equation. Prediction is usually given in terms of an ephemeris (a table of state vectors, or perhaps an ephemeris of positions only, or *position ephemeris*)

  t_1 X_1 Y_1 Z_1 U_1 V_1 W_1 (XYZ are position; UVW are velocity)
  t_2 X_2 Y_2 Z_2 U_2 V_2 W_2
  etc

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Application to Earth Satellites

- Satellite mass is << Earth mass

$$\mu = G\,(M_1 + M_2)$$

Let M_1 represent Earth mass and

M_2 represent satellite mass

$\mu = GM_{Earth}$ since satellite mass << Earth

- Must M_1 be $\gg M_2$?

- Periapsis -> Perigee, Apoapsis -> Apogee

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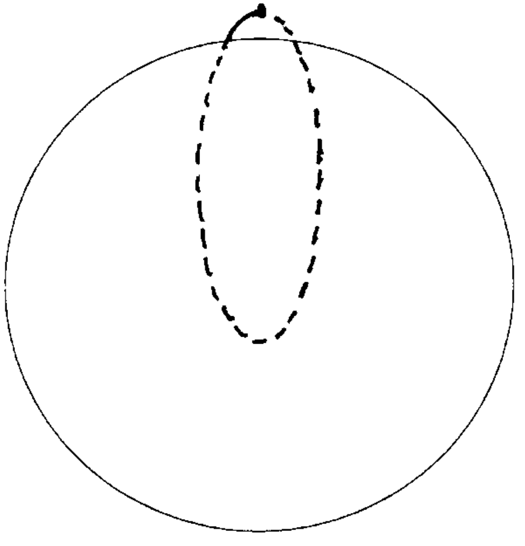
Everyday Orbital Motion



- Ballistic motion is orbital motion
- Solid Earth prevents a body in ballistic motion from reaching perigee
- A body dropped from rest, at the equator, is shown
  - Perigee: 11.2 km
  - Eccentricity: 0.9965

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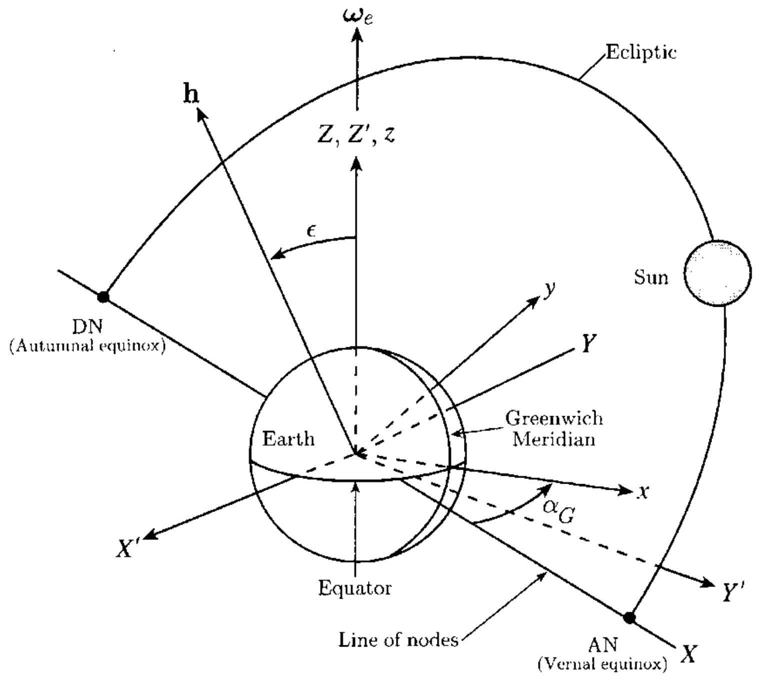
Sun-Earth Two Body Motion: I



- Describe motion of the Sun with respect to the Earth
- Choosing smaller of the masses to be M_1
- Orbit plane is "ecliptic"
- Earth spherical model is rotating
- Inclination is known as "obliquity of the ecliptic", ε, ~23.5°

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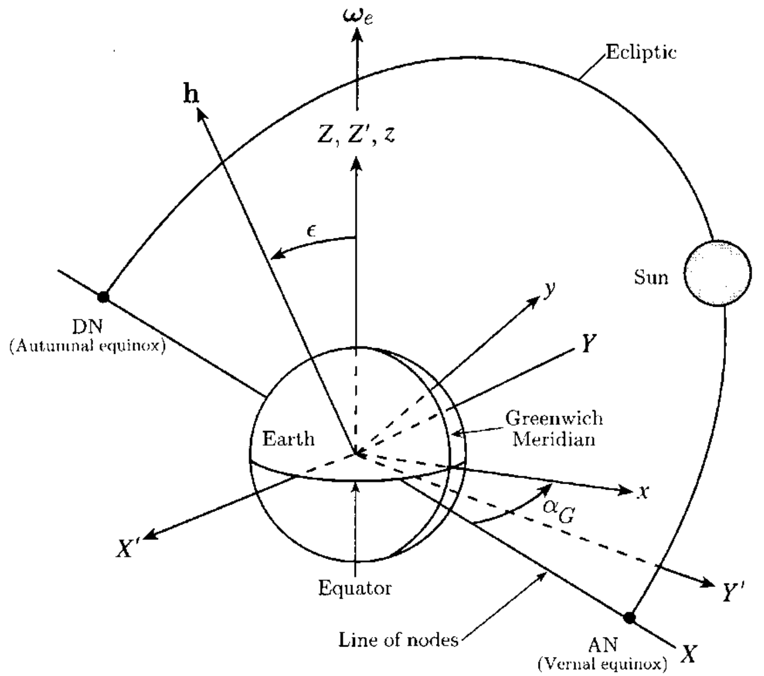
Sun-Earth Two Body Motion: II



- Ascending node is the point where the Sun crosses the equator moving from the southern hemisphere to the northern hemisphere: vernal equinox (~ March 21)
- The descending node is autumnal equinox (~Sept 21)

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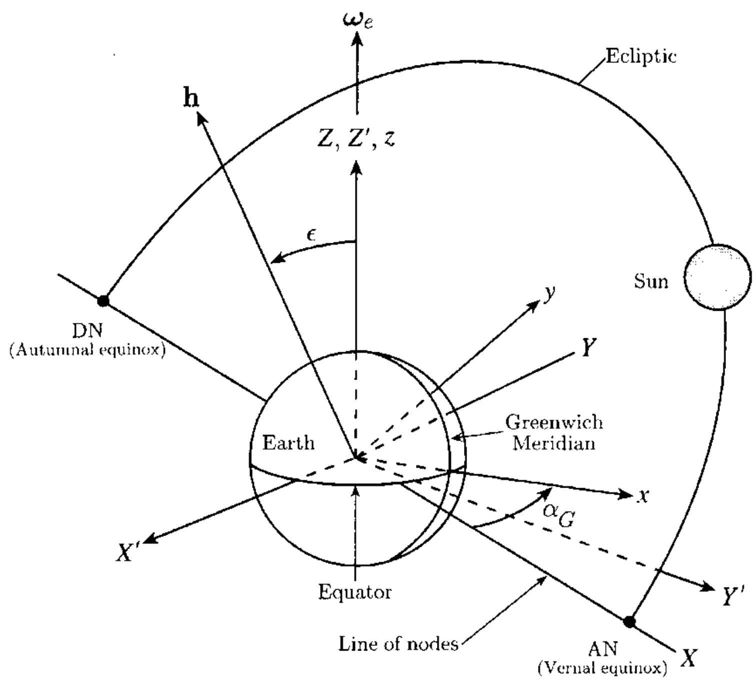
Basic Coordinate Systems: I



- Choose X-axis (coinciding with vernal equinox) as inertial direction; Z-axis coincident with Earth angular velocity vector ($\boldsymbol{\omega}_e$), period of rotation = 86164 sec, "sidereal" period
- GMST=$\alpha_G = \omega_e(t - t_0) + \alpha_{G0}$

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Basic Coordinate Systems: II

- (XYZ) represents a nonrotating coordinate system with X directed to the vernal equinox, and origin coinciding with Earth center (geometric center of the spherical Earth, or more precisely, the Earth center of mass)
- In reality, the location of the equinoxes change with time (use the equinox of a particular date as reference, e.g., January 1, 2000, 12:00 or more specifically, mean equator and vernal equinox of J2000)
- (xyz) is an Earth-fixed frame (ECF) and rotates with it, with x coincident with the intersection of the Greenwich meridian and the equator

$$
\begin{aligned}
x &= X\cos\alpha_G + Y\sin\alpha_G \\
y &= -X\sin\alpha_G + Y\cos\alpha_G \\
z &= Z\,.
\end{aligned}
$$

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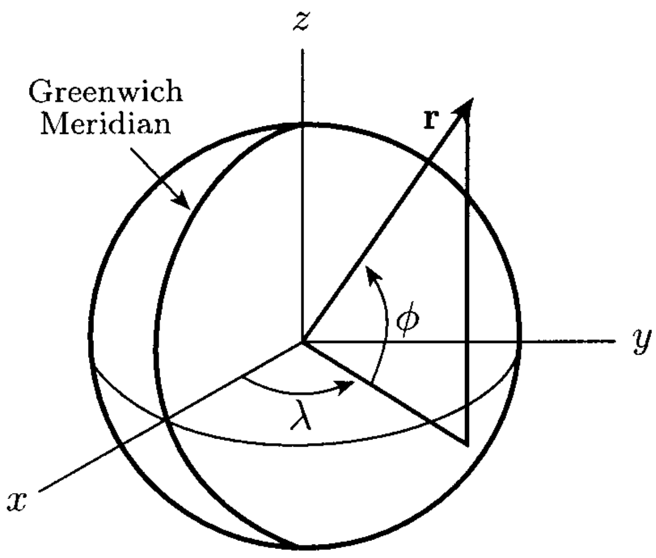
Basic Coordinate Systems: III



- Define xyz reference frame (Earth centered, Earth fixed; ECEF or ECF), fixed in the solid (and rigid) Earth and rotates with it
- Longitude λ measured from Greenwich Meridian

  $0 \leq \lambda < 360°$ E; or measure λ East (+) or West (-)
- Latitude (geocentric latitude) measured from equator (φ is North (+) or South (-))
  - At the poles, φ = + 90° N or φ = -90° S

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Perturbed Motion

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3} + \mathbf{f},$$

- The 2-body problem provides us with a foundation of orbital motion
- In reality, other forces exist which arise from gravitational and nongravitational sources
- In the general equation of satellite motion, **f** is the perturbing force (causes the actual motion to deviate from exact 2-body)

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Perturbation: Planetary Mass Distribution

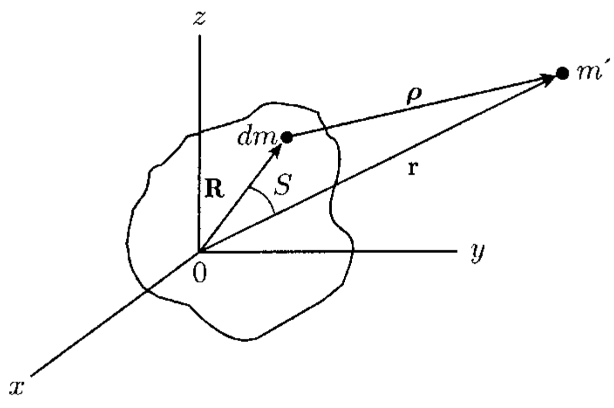

Figure 2.3.4: Definition of position vectors and differential mass for a body with arbitrary mass distribution.

$$U = \int_M \frac{G\, dm}{\rho}$$

- Sphere of constant mass density is not an accurate representation for planets
- Define gravitational potential, U, such that the gravitational force is

$$\mathbf{F} = \nabla U$$

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Gravitational Potential

$$
\begin{aligned}
U &= \frac{\mu}{r} + U' \\
U' &= -\frac{\mu^*}{r}\sum_{\ell=1}^{\infty}\left(\frac{a_e}{r}\right)^{\ell} P_\ell(\sin\phi) J_\ell \qquad (2.3.12)\\
&\quad + \frac{\mu^*}{r}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\ell}\left(\frac{a_e}{r}\right)^{\ell} P_{\ell m}(\sin\phi)[C_{\ell m}\cos m\lambda + S_{\ell m}\sin m\lambda]
\end{aligned}
$$

$$
J_\ell = -\left(\frac{1}{M^* a_e^\ell}\right)\int_M R^\ell P_\ell(\sin\phi')\,dm
$$

- The commonly used expression for the gravitational potential is given in terms of mass distribution coefficients J_n, C_{nm}, S_{nm}
- n is degree, m is order
- Coordinates of external mass are given in spherical coordinates: r, geocentric latitude φ, longitude λ

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Gravity Coefficients

- The gravity coefficients (J_n, C_{nm}, S_{nm}) are also known as Stokes Coefficients and Spherical Harmonic Coefficients
- J_n:
  - Gravitational potential represented in zones of latitude; referred to as *zonal coefficients*
- C_{nm}, S_{nm}:
  - If n=m, referred to as *sectoral coefficients*
  - If n≠m, referred to as *tesseral coefficients*

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Degree 2 Coefficients

$$J_2 = \frac{2C - B - A}{2Ma_e^2}$$

$$C_{2,1} = \frac{I_{xz}}{Ma_e^2} \qquad S_{2,1} = \frac{I_{yz}}{Ma_e^2}$$

$$C_{2,2} = \frac{B - A}{4Ma_e^2} \qquad S_{2,2} = \frac{I_{xy}}{2Ma_e^2}$$

- Degree 2 (n=2) spherical harmonic coefficients are related to moments and products of inertia of the planet
- In equations at left:
  - A, B are equatorial moments of inertia (x, y axes), C is polar (z-axis) moment of inertia
  - Products of inertia denoted as I
  - M is mass of planet, a_e is mean radius

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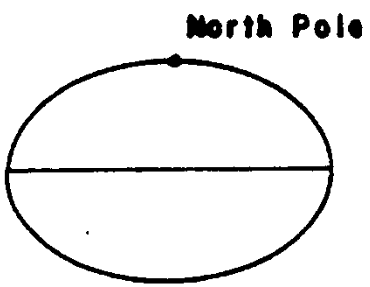
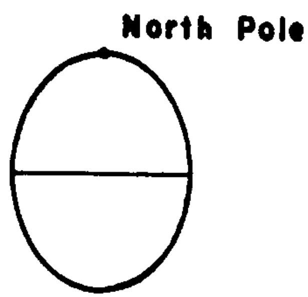
Earth J_2 (Degree 2 Zonal Harmonic)

Oblate spheroid



Prolate spheroid



- J_2 represents a dominant characteristic of the shape of the planet
  - Positive J_2: oblate spheroid
  - Negative J_2: prolate spheroid
- Scientific controversy in 1735: was Earth oblate or prolate?

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Resolution of Controversy

- In 1735, one view of the shape of the Earth was based on work of Newton, who had argued for the oblate shape (centrifugal forces)
- Another view was based on measurements of the length of 1° of latitude in France, supported a prolate spheroid
- French Academy of Sciences funded two expeditions to make measurements of 1° of latitude near the Arctic Circle (northern Scandinavia) and near the equator (now Ecuador)
- It took ~10 years for the equator team to complete, so the first results were from Scandinavia, and equator verified it: the Earth was an oblate spheroid, J_2 is +

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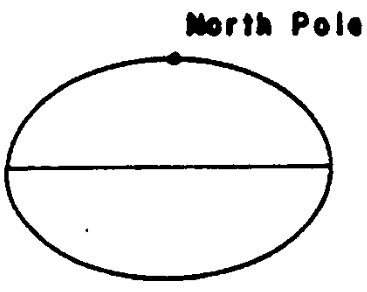
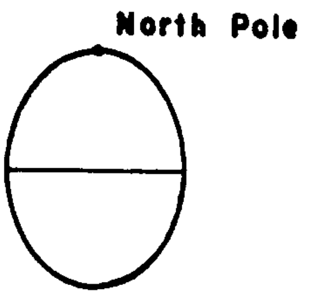
Length of A degree of Latitude for the Earth

Oblate spheroid



Prolate spheroid



Length of a degree of latitude at the equator = 110.574 KM
Length of a degree of latitude at the pole = 111.694 KM

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Forces on a particle, P, on the Earth's surface

$\bar{F}_c$ = centripetal force
$\bar{F}_G$ = gravitational force
$\bar{F}_g$ = resultant force
$\bar{r}$ = radius vector of
$\bar{\omega}$= angular velocity of Earth

$$\bar{F}_c = -\bar{\omega} \times (\bar{\omega} \times \bar{r})$$

Centripetal force exerts a force component directed toward the equator in both hemispheres. The Earth, or any rotating planet, reaches equilibrium when the resultant force is perpendicular to the tangent plane.

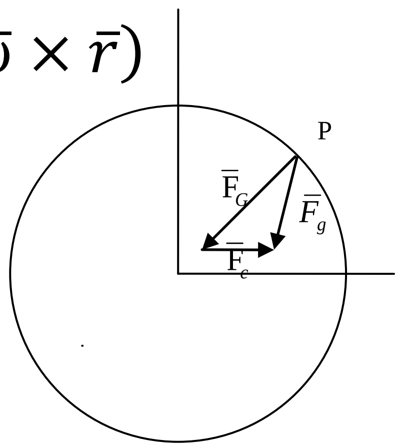

Figure 5. Forces on P for a spherical Earth

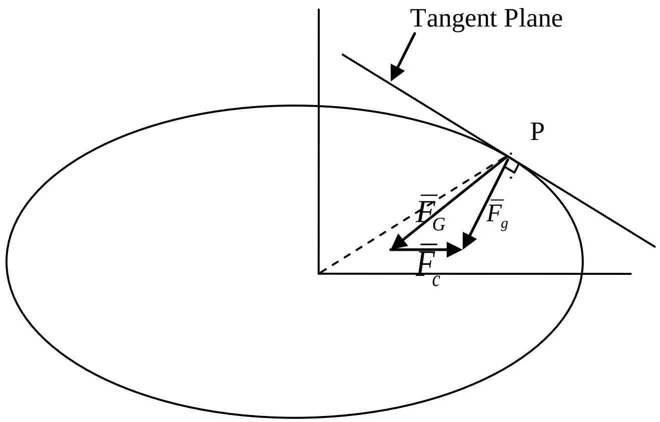

Figure 6. Forces on P for an oblate Earth

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