Example of Kalman Filter Application
ASEN 5070
October 18, 2005
George H. Born

(Problem 43c of text)

Given:

$$\mathbf{X}(t_{i+1}) = \begin{bmatrix} x_1(t_{i+1}) \\ x_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix}$$

$$\mathbf{Y}(t_i) = \begin{bmatrix} y_1(t_i) \\ y_2(t_i) \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix} + \begin{bmatrix} \varepsilon_1(t_i) \\ \varepsilon_2(t_i) \end{bmatrix}.$$

Hence,

$$\Phi(t_{i+1}, t_i) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$\tilde{H}(t_i) = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}.$$

Note that both the state propagation and observation-state equations are linear. Assume the following a priori information is given:

$$\bar{\mathbf{X}}(t_0) = \begin{bmatrix} \bar{x}_1(t_0) \\ \bar{x}_2(t_0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ \bar{P}(t_0) = I$$

$$R(t_i) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ for all values of } i$$

$$\mathbf{Y}(t_1) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find: $\hat{\mathbf{X}}(t_1)$, $\hat{\mathbf{X}}(t_0)$, $P(t_1)$, and $P(t_0)$ using the sequential (Kalman) filter. Verify the value of $\hat{\mathbf{X}}(t_0)$ and $P(t_0)$ by using the batch processor algorithm.

Step #1 – Do a time update at t_1. Note that there is no observation at t_0. If this were the case we would first do a measurement update.

$$\bar{\mathbf{X}}(t_1) = \Phi(t_1, t_0)\bar{\mathbf{X}}(t_0)$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\bar{P}(t_1) = \Phi(t_1, t_0)\bar{P}_0\Phi^T(t_1, t_0)$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Step #2 – Measurement update at t_1

$$K(t_1) = \bar{P}(t_1)\tilde{H}^T(t_1)\left(\tilde{H}(t_1)\bar{P}(t_1)\tilde{H}^T(t_1) + R\right)^{-1}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\left(\begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^{-1}$$
$$= \begin{bmatrix} 1/8 & 11/24 \\ 1/4 & 1/4 \end{bmatrix}$$

$$\hat{\mathbf{X}}(t_1) = \bar{\mathbf{X}}(t_1) + K(t_1)\left(\mathbf{Y}(t_1) - \tilde{H}(t_1)\bar{\mathbf{X}}(t_1)\right)$$
$$= \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 1/8 & 11/24 \\ 1/4 & 1/4 \end{bmatrix}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 25/8 \\ 9/4 \end{bmatrix}$$

$$P(t_1) = \left[I - K(t_1)\tilde{H}(t_1)\right]\bar{P}(t_1)$$
$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/8 & 11/24 \\ 1/4 & 1/4 \end{bmatrix}\right)\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 19/24 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}.$$

We may now map $\hat{\mathbf{X}}(t_1)$ to t_0 to obtain $\hat{\mathbf{X}}(t_0)$, i.e.,

$$
\begin{aligned}
\hat{\mathbf{X}}(t_0) &= \Phi(t_0,t_1)\hat{\mathbf{X}}(t_1) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 25/8 \\ 9/4 \end{bmatrix} \\
&= \begin{bmatrix} 7/8 \\ 9/4 \end{bmatrix}.
\end{aligned}
$$

Note that $\Phi(t_0,t_1) = \Phi^{-1}(t_1,t_0)$.

$P(t_0)$ is given by

$$
\begin{aligned}
P(t_0) &= \Phi(t_0,t_1) P_1 \Phi^T(t_0,t_1) \\
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 19/24 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 19/24 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}.
\end{aligned}
$$

The batch processor may be used to compute $\hat{\mathbf{X}}(t_0)$,

$$\hat{\mathbf{X}}(t_0) = \left(H^T(t_0) R^{-1} H(t_0) + \bar{P}_0^{-1}\right)^{-1} \left(H^T(t_0) R^{-1} \mathbf{Y}(t_1) + \bar{P}_0^{-1} \bar{\mathbf{X}}(t_0)\right)$$

where

$$\begin{aligned} H(t_0) &= \tilde{H}(t_1)\Phi(t_1, t_0) \\ &= \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1 & 3/2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \hat{\mathbf{X}}(t_0) &= \left(\begin{bmatrix} 0 & 1 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1 & 3/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 0 & 1 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \\ &= \begin{bmatrix} 7/8 \\ 9/4 \end{bmatrix}. \end{aligned}$$

and

$$\begin{aligned} P(t_0) &= \left(\begin{bmatrix} 0 & 1 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1 & 3/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} 19/24 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}. \end{aligned}$$

Hence, the values of $\hat{\mathbf{X}}(t_0)$ and $P(t_0)$ from the batch processor agree with the results of the Kaman or sequential filter.