# STATISTICAL ORBIT DETERMINATION

ASEN 5070

Fall 2011

9/16/2011

LECTURE 10

Supplemental Reading – Ch 4

# Computational Algorithm for the Batch Processor

$$\hat{\chi}_k = \left(H^T W H + \bar{W}\right)^{-1}_{n\times n}\left(H^T W y + \bar{W}\,\bar{\chi}_k\right)_{n\times 1}$$

Look at $H^T W H$,

$$\begin{bmatrix} H_1^T & H_2^T & \dots & H_l^T \end{bmatrix}\begin{bmatrix} W_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W_l \end{bmatrix}\begin{bmatrix} H_1 \\ \vdots \\ H_l \end{bmatrix} = H_1^T W_1 H_1 + H_2^T W_2 H_2 + \cdots H_l^T W_l H_l$$

$$= \sum_{i=1}^{l} H_i^T W_i H_i$$

Likewise,

$$H^T W y = \sum_{i=1}^{l} H_i^T W_i y_i$$

and

$$\hat{\chi}_k = \left(\sum_{i=1}^{l} H_i^T W_i H_i + \bar{W}\right)^{-1}\left(\sum_{i=1}^{l} H_i^T W_i y_i + \bar{W}\,\bar{\chi}_k\right)$$

# Computational Algorithm for the Batch Processor

Hence, the computational algorithm involves forming the indicated summations, performing the Matrix inversion and solving for $\hat{\chi}_k$. We will see later that there are more computationally efficient ways to solve for $\hat{\chi}_k$ without inverting the normal matrix.

# The Batch (Least Squares) Algorithm

- Estimating $x_k$ begins with the following state propagation and observation-state relationships:

$$
\begin{aligned}
x(t) &= \Phi(t, t_k) x(t_k) \\
y &= H x_k + \varepsilon \\
H(t) &= \tilde{H}(t) \Phi(t, t_0)
\end{aligned}
\qquad (12)
$$

- The H matrix relates the state deviation vector at an epoch time to the observation deviation vector at another time.

# The Batch (Least Squares) Algorithm

- The normal equations are formed

$$\left(\sum_{i=0}^{m} H_i^T R_i^{-1} H_i + \bar{P}_0^{-1}\right)\hat{x}_0 = \sum_{i=0}^{m} H_i^T R_i^{-1} y_i + \bar{P}_0^{-1}\bar{x}_0 \qquad (13)$$

$$\Lambda \hat{x}_0 = N$$

- R is the observation error covariance matrix.

- $\bar{x}_0$ and $\bar{P}_0$ are the a priori state estimate and state error covariance, respectively.

# The Batch (Least Squares) Algorithm

- Eqn. (13) may now be solved for $\hat{x}_0$ .

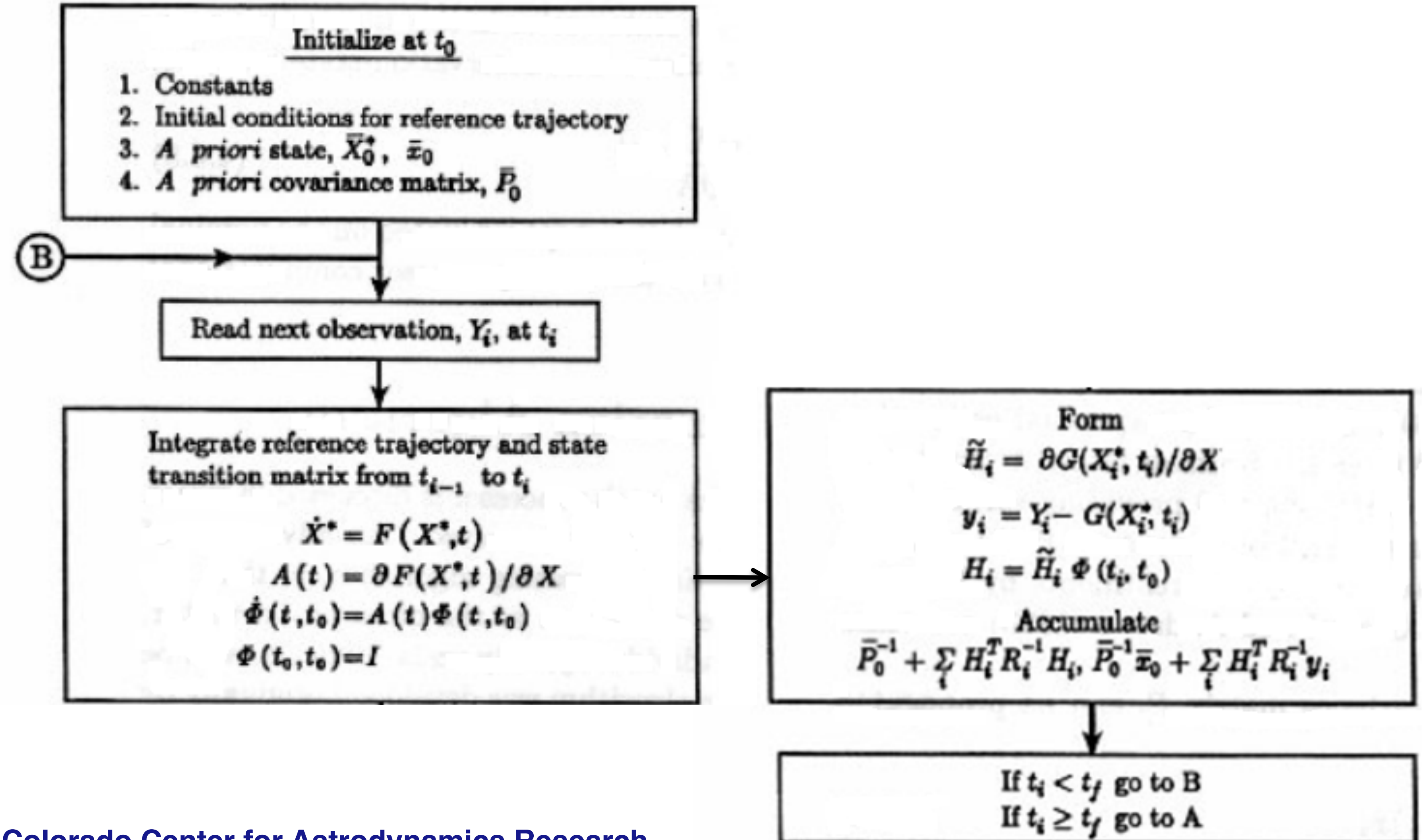
$$\left( \sum_{i=0}^{m} H_i^T R_i^{-1} H_i + \overline{P}_0^{-1} \right) \hat{x}_0 = \sum_{i=0}^{m} H_i^T R_i^{-1} y_i + \overline{P}_0^{-1} \overline{x}_0 \qquad (13)$$

$$\Lambda \hat{x}_0 = N$$

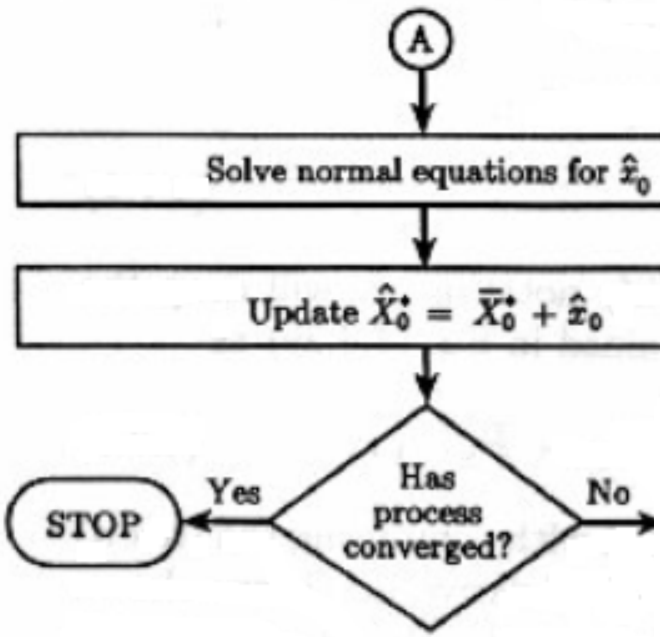
- The RMS of the observation residuals is given by

$$RMS = \sqrt{\frac{\sum_{i=0}^{m} (y_i - H_i \hat{x}_0)^2}{m}} \qquad (15)$$

# The Batch (Least Squares) Algorithm



# The Batch (Least Squares) Algorithm




Repeat initialization using $\hat{X}_0^*$ as the initial conditions for the reference trajectory.
Set $\bar{x}_0 = \bar{x}_0 - \hat{x}_0$ and use the original value of $\bar{P}_0$.
Reprocess all observations.

We want to maintain apriori Information; $\bar{x}_0$, *and* $\bar{P}_0$ hence,

$$X^*_{n+1} + \bar{x}_{n+1} = X^*_n + \bar{x}_n \quad (1)$$

$$BUT,\ X^*_{n+1} = X^*_n + \hat{x}_n \quad (2)$$

$$Subst\ (2) \rightarrow (1),\quad \bar{x}_{n+1} = \bar{x}_n - \hat{x}_n$$

# The Batch (Least Squares) Algorithm

- The batch (or least squares) processor processes all observation data at once and, then, determines the best estimate of the state deviation, thereby estimating $\hat{X}$ .

- The best estimate of x is chosen such that it minimizes the sum of the squares of the calculated observation errors.

# LEO Orbit Determination Example

- Instantaneous observation data is taken from three Earth fixed tracking stations over an approximate 5 hour time span (light time is ignored).

$$\rho = \left(x^2 + y^2 + z^2 + x_{s_i}^2 + y_{s_i}^2 + z_{s_i}^2 - 2(xx_{s_i} + yy_{s_i})\cos\theta + 2(xy_{s_i} - yx_{s_i})\sin\theta - 2zz_{s_i}\right)^{\frac{1}{2}}$$

$$\dot{\rho} = \frac{x\dot{x} + y\dot{y} + z\dot{z} - (\dot{x}x_{s_i} + \dot{y}y_{s_i})\cos\theta + \dot{\theta}(xx_{s_i} + yy_{s_i})\sin\theta + (\dot{x}y_{s_i} - \dot{y}x_{s_i})\sin\theta + \dot{\theta}(xy_{s_i} - yx_{s_i})\cos\theta - \dot{z}z_{s_i}}{\rho}$$

- where x, y, and z represent the spacecraft Earth Centered Inertial (ECI) coordinates and $x_s$, $y_s$, and $z_s$ are the tracking station Earth Centered, Earth Fixed (ECEF) coordinates.

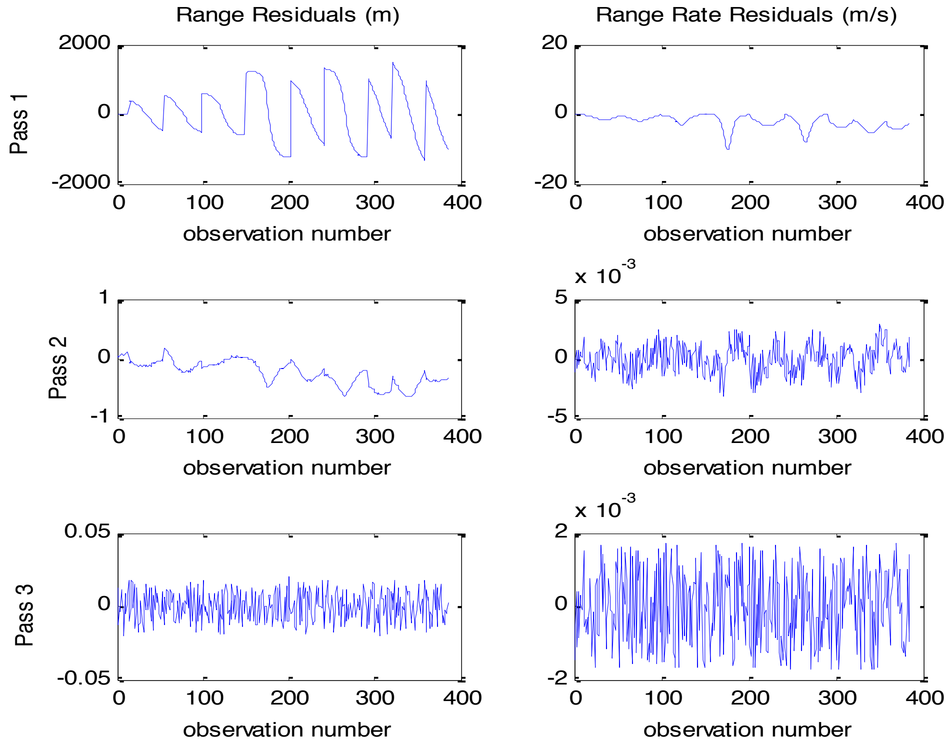
# LEO Orbit Determination Example

| State Deviation ( $\hat{x}_0$ ) | | |
|---|---|---|
| **Pass 1** | **Pass 2** | **Pass 3** |
| -0.036383127158274 | 0.326725527230386 | 2.16404427676782e-007 |
| -0.274523873228247 | -0.1480641161433 | -2.62437517889209e-007 |
| -0.18013801260102 | -0.0809938800450721 | 1.64403340270235e-007 |
| 0.0409346491303605 | -0.000317153824609643 | -4.38244130278385e-010 |
| 0.0327489396676593 | -3.92221769835543e-005 | 1.94288829502823e-010 |
| -0.0147524440475379 | 0.000338033953408322 | 2.49099138592487e-010 |
| -9463031.96726608 | -33306147.3255543 | -16.3113339128591 |
| -6.57367472046001e-007 | 2.98810508464489e-008 | 2.27735423982078e-015 |
| 0.147516774454061 | 0.0411458796251399 | -4.59840606133234e-007 |
| 1.86265905206096e-006 | -1.86970469476067e-006 | 8.73427771458123e-013 |
| 1.37829598816653e-006 | -1.38350949129108e-006 | 6.46310696714768e-013 |
| -2.54260084011498e-007 | 2.54404878085509e-007 | 2.24461403220845e-013 |
| -10.5635758182854 | 0.555182863038883 | 9.07302237892118e-008 |
| 9.98322113724009 | 0.0201549881875307 | -2.43030096543763e-007 |
| 5.79485500114413 | 0.18208202802521 | 2.37370032711892e-007 |
| -5.78184708032509 | 0.773198712686432 | 2.35005539856875e-008 |
| 2.34411863803746 | -0.323056978825581 | -8.52191767278272e-008 |
| 1.51222736642103 | 1.46363482699289 | 5.24955707897414e-007 |

**Batch State Deviation for Passes 1, 2, and 3**

# LEO Orbit Determination Example

## Batch (Least Squares) Residuals




| RMS Values | | | |
|---|---|---|---|
| | **Pass 1** | **Pass 2** | **Pass 3** |
| **Range (m)** | 732.748350225264 | 0.319570766726265 | 0.00974562719122707 |
| **Range Rate (m/s)** | 2.90016531897711 | 0.00119972978584721 | 0.000997930398398708 |

# Least Squares Example

Given the following system

$$y_1 = 2x_1 + x_2 + \varepsilon_1$$

$$y_2 = 4x_1 + 2x_2 + \varepsilon_2$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad H = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix},$$

a) Could we obtain a solution for $\hat{X}$ using the least squares equation? Why or why not? Will minimum norm yield a solution?

b) Assume we are given a priori information $\bar{X} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\bar{W} = I$. Can we now solve for $\hat{X}$ ?

# Three possibilities for m and n

$$
\begin{aligned}
\hat{\mathbf{x}} &= (H^T H)^{-1} H^T \mathbf{y}, && \text{if } m > n \\
\hat{\mathbf{x}} &= H^{-1} \mathbf{y}, && \text{if } m = n \\
\hat{\mathbf{x}} &= H^T (H H^T)^{-1} \mathbf{y}, && \text{if } m < n.
\end{aligned}
$$

# CH 4 Problem (19)

Given the observation-state relation

$$y(t) = \sum_{i=1}^{3} (t)^i x_i$$

and the observation sequence at $t = 1$, $y(1) = 2$, and at $t = 2$ , $y(2) = 1$.

Find the "best" estimate of $x_i$, $i = 1, 2, 3$.

# CH 4 Problem (19)

Find: $\hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_3, \quad X = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix}$

$$y(t_1) = \sum_{i=1}^{3} (1)^i \chi_i = \chi_1 + \chi_2 + \chi_3$$

$$y(t_2) = \sum_{i=1}^{3} (2)^i \chi_i = 2\chi_1 + 4\chi_2 + 8\chi_3$$

$$y = \begin{bmatrix} y(t_1) \\ y(t_2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have 2 observations and 3 unknowns; hence, we use the minimum norm solution given by Eq (4.3.13) **Note the rank of H is 2

# CH 4 Problem (19)

We have 2 observations and 3 unknowns; hence, we use the minimum norm solution given by Eq (4.3.13) . Note that the rank of H is 2

$$\hat{X} = H^T \left( HH^T \right)^{-1} y \quad (4.3.13)$$

# CH 4 Problem (19)

$$\hat{X} = H^T \left( HH^T \right)^{-1} y$$

Note that because $X \neq X(t)$, i.e., the state vector is not a function of time, $\Phi = I$ and $\tilde{H} = H$.

$$\hat{X} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1.5 & -.25 \\ -.25 & .05357 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\hat{\varepsilon} = y - H\hat{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \hat{\chi}_1 \\ \hat{\chi}_2 \\ \hat{\chi}_3 \end{bmatrix} = \begin{bmatrix} 1.86 \\ 0.964 \\ -0.821 \end{bmatrix}$$

Remember that we derived $\hat{\chi}$ with the constraint that $\hat{\varepsilon} = 0$

i.e. $y - H\chi = 0$

# CH 4 Problem (19)

- What if we had apriori information, e.g.,

$$\bar{X} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \bar{W} = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{bmatrix}$$

Can we now have a least squares solution?

# CH 4 Problem (19)

- Note that the rank of A+B is less than or equal to the rank of A plus the rank of B. Hence, in this example the rank of $\bar{W}$ and $H^T H$ need not be rank 3 but their sum must be rank 3.

Given

$$\ddot{x}_1 = ax_1 + b\dot{x}_2$$

$$\ddot{x}_2 = cx_1 + ex_2$$

with

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

and a, b, c, and e are constants.

a) Write as a first order system and derive the *A* matrix

Given

$$\ddot{x}_1 = ax_1 + b\dot{x}_2$$

$$\ddot{x}_2 = cx_1 + ex_2$$

with

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ u \\ v \end{bmatrix}$$

and a, b, c, and e are constants.

Write as a 1st order system

$$\dot{x} = u$$

$$x_2 = v$$

$$\dot{u} = ax_1 + bv$$

$$\dot{v} = cx_1 + ex_2$$

$$A = \frac{\partial F}{\partial X} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & b \\ c & e & 0 & 0 \end{bmatrix}$$

a) If $a = b = c = e = 0$ what is $\Phi(t, t_0)$ ?

Assume initial conditions, $x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}$ are given at $t_0 = 0$.

If the $H$ matrix is not of full rank, i.e. $\left(H^T H\right)^{-1}$ does not exist, can we make $\left(H^T W H\right)^{-1}$ exist by the proper choice of a weighting matrix, $W$ ?

1. T or F

2. Justify your answer

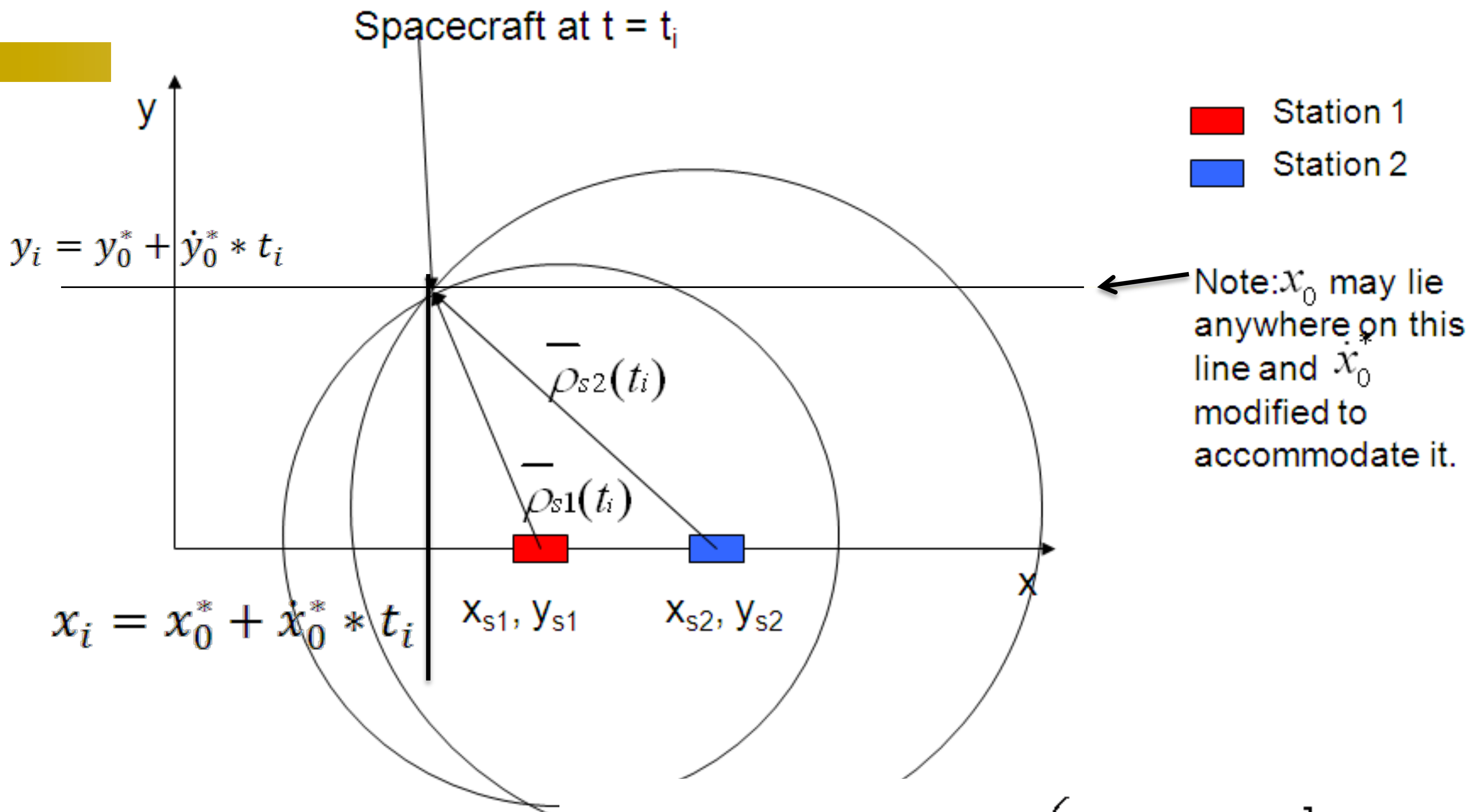
The rank of the product AB of two matrices is less than or equal to The rank of A **and** is less than or equal to the rank of B.  Hence the answer to (1) is false.

Given range observations in the 2-D flat earth problem, i.e.

$$\rho_i^2 = \left(x_0 + \dot{x}_0 t_i - x_s\right)^2 + \left(y_0 + \dot{y}_0 t_i - \frac{1}{2} g t_i^2 - y_s\right)^2$$

1. Assume all parameters except $x_0$ and $\dot{x}_0$ are known. We can solve for both $x_0$ and $\dot{x}_0$ from range measurements taken simultaneously from two well separated tracking stations. T or F?

2. Justify your answer in terms of the rank of $H$.




$$\rho^2 = \left(x_0 + \dot{x}_0 t_i - x_s\right)^2 + \left(y_0 + \dot{y}_0 t_i - \frac{1}{2} g t_i^2 - y_s\right)^2$$

$$\rho_i^2 = \left(x_0 + \dot{x}_0 t_i - x_s\right)^2 + \left(y_0 + \dot{y}_0 t_i - \frac{1}{2} g t_i^2 - y_s\right)^2$$

$$H = \begin{bmatrix} \dfrac{\partial \rho_{s1}}{\partial x_0} & \dfrac{\partial \rho_{s1}}{\partial \dot{x}_0} \\ \dfrac{\partial \rho_{s2}}{\partial x_0} & \dfrac{\partial \rho_{s2}}{\partial \dot{x}_0} \end{bmatrix} = \begin{bmatrix} \dfrac{x_0 + \dot{x}_0 t - x_{s1}}{\rho_{s1}} & \dfrac{(x_0 + \dot{x}_0 t - x_{s1})t}{\rho_{s1}} \\ \dfrac{x_0 + \dot{x}_0 t - x_{s2}}{\rho_{s2}} & \dfrac{(x_0 + \dot{x}_0 t - x_{s2})t}{\rho_{s2}} \end{bmatrix} = \begin{bmatrix} C & Ct \\ K & Kt \end{bmatrix}$$

Assume we had both range and range rate; can we now solve for $x_0$ and $\dot{x}_0$?

$$\rho^2 = \left(x_0 + \dot{x}_0 t_i - x_s\right)^2 + \left(y_0 + \dot{y}_0 t_i - \frac{1}{2} g t_i^2 - y_s\right)^2$$

$$\dot{\rho}^2 = \frac{1}{\rho}\left[\left(x_0 + \dot{x}_0 t_i - x_s\right)\left(\dot{x}_0 - \dot{x}_s\right) + \left(y_0 + \dot{y}_0 t_i - \frac{1}{2} g t_i^2 - y_s\right) * \left(\dot{y}_o - g t - \dot{y}_s\right)\right]$$

The differential equation

$$\ddot{x} + a\dot{x} + bx\dot{x} = 0$$

is (choose all correct answers)

1. $2^{nd}$ order and $2^{nd}$ degree
2. $2^{nd}$ order and $1^{st}$ degree
3. linear
4. nonlinear

See footnote on page 512 of text