ORBIT DETERMINATION

ASEN 5070

Fall 2011

LECTURE 9

9/12/2011

Supplemental Reading:

Sections 4.1 – 4.4

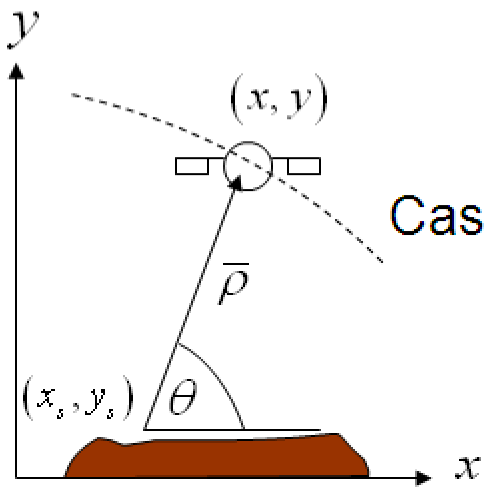
Concept Test

True or False

1) If the relationship between the observations and the state is linear we do not have to iterate the Newton-Raphson equation. ____________

2) Given observations of range rate $\dot{\rho}_j$, j=1----5, all elements of the following state vectors can be solved for (indicate T or F for case A and B)

$$\dot{\rho}_j = \frac{1}{\rho_j}\left[\left(x_0 - x_s + \dot{x}_0 t_j\right)\dot{x}_0 + \left(y_0 - y_s + \dot{y}_0 t_j - \frac{g t_j^{2}}{2}\right)\left(\dot{y}_0 - g t_j\right)\right]$$



Case A: $X = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ y_0 \\ \dot{y}_0 \\ y_s \end{bmatrix}$ T or F __.

Case B: $X = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ y_s \\ g \end{bmatrix}$ T or F __.

Concept Test

3) The state vector in case 2.B could be solved for uniquely with one observation each of $\rho, \dot{\rho}, \theta, \dot{\theta}$ at one instant in time. _______

4) For problem 2.B we may use any initial guess for the state but it may take many iterations to converge. _______

5) Given:

$$\ddot{\theta} + \omega^2 \theta = 0 \qquad X_0 = \begin{bmatrix} \theta_0 \\ \dot{\theta}_0 \end{bmatrix} \qquad Y(t) = \begin{bmatrix} \theta_0(t) \\ \dot{\theta}_0(t) \end{bmatrix} + \varepsilon(t)$$

Since the observation-state and state propagation equations are linear we do not have to use a state deviation vector. ________

Concept Test

6) The differential equation in each column of $\dot{\Phi} = A\Phi$ is independent of the equations in other columns. __________

7) The least squares solution minimizes the sum of the residuals. ______

8) For the equation $y = Hx + \varepsilon$ we always have more unknowns than equations. ________

9) If the determinant of a symmetric matrix is negative (answer T or F)

a) It is not positive definite _______
b) It's inverse does not exist. _______
c) Some eigenvalues are imaginary. ______

10) The derivative of a scalar WRT a vector is a scalar. _____

Concept Test

11) The rank of $H = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 6 & 12 \end{bmatrix}$ is 2. ________

Variable
$\tilde{H}$
ϵ
ρ
$\boldsymbol{y}$
θ
$\boldsymbol{Y}$
$\hat{\boldsymbol{X}}_k$
$G(\boldsymbol{X}, t)$
$\dot{\rho}$
$\boldsymbol{x}$
$\Phi(t, t_0)$
$\dot{\theta}$
$\boldsymbol{X}_k$
$\boldsymbol{X}^*$

Alternate Description of Φ using Taylor Series Expansion

Assume that we can write the solution for $X^*(t)$ based on initial conditions X_0^*

$$X^*(t) = F\left(X_0^*, t\right)$$

Expand the true solution about $X^*(t) = F\left(X_0^*, t\right)$ and retain 1st order terms

$$X(t) = X^*(t) + \frac{\partial X^*(t)}{\partial X_0}\left(X(t_0) - X^*(t_0)\right) + \ldots$$

$$X(t) - X^*(t) = \frac{\partial X^*(t)}{\partial X_0}\left(X(t_0) - X^*(t_0)\right)$$

Define $\chi(t) = X(t) - X^*(t)$, then

$$\chi(t) = \frac{\partial X^*(t)}{\partial X_0}\chi(t_0) = \Phi(t, t_0)\chi_0$$

Alternate Description of Φ using Taylor Series Expansion

For example, assume that
$$X(t) = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \\ \alpha \end{bmatrix}$$
then
$$\Phi(t,t_0) = \frac{\partial X(t)}{\partial X(t_0)} = \frac{\partial X(t)}{\partial X_0}$$

α is a constant

and

$$\Phi(t,t_0) = \begin{bmatrix} \partial x(t)/\partial x_0 & \partial x(t)/\partial y_0 & \partial x(t)/\partial \dot{x}_0 & \partial x(t)/\partial \dot{y}_0 & \partial x(t)/\partial \alpha \\ \partial y(t)/\partial x_0 & \partial y(t)/\partial y_0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial \alpha/\partial x_0 & \partial \alpha/\partial y_0 & \cdots & \cdots & \partial \alpha/\partial \alpha \end{bmatrix}$$

*Note that the last row will be
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

State Transition Matrix Differential Equation for a General State Vector

Let β represent a $j \times 1$ vector of force model parameters and γ a $k \times 1$ vector of measurement model parameters. β and γ are constants.

$$X = \begin{bmatrix} \bar{r} \\ \bar{v} \\ \bar{\beta} \\ \bar{\gamma} \end{bmatrix} \qquad \dot{X} = F(X,t) = \begin{bmatrix} \dot{\bar{r}} \\ \dot{\bar{v}} \\ \dot{\bar{\beta}} = 0 \\ \dot{\bar{\gamma}} = 0 \end{bmatrix}$$

$$A = \frac{\partial F(X,t)}{\partial X} = \begin{bmatrix} \left[\partial \dot{\bar{r}} / \partial \bar{r}\right]_{3\times 3} = 0 & \left[\partial \dot{\bar{r}} / \partial \bar{v}\right]_{3\times 3} = I & \left[\partial \dot{\bar{r}} / \partial \bar{\beta}\right]_{3\times j} = 0 & \left[\partial \dot{\bar{r}} / \partial \bar{\gamma}\right]_{3\times k} = 0 \\ \left[\partial \dot{\bar{v}} / \partial \bar{r}\right]_{3\times 3} & \left[\partial \dot{\bar{v}} / \partial \bar{v}\right]_{3\times 3} & \left[\partial \dot{\bar{v}} / \partial \bar{\beta}\right]_{3\times j} & \left[\partial \dot{\bar{v}} / \partial \bar{\gamma}\right]_{3\times k} = 0 \\ \left[\partial \dot{\bar{\beta}} / \partial \bar{r}\right]_{j\times 3} = 0 & \left[\partial \dot{\bar{\beta}} / \partial \bar{v}\right]_{j\times 3} = 0 & \left[\partial \dot{\bar{\beta}} / \partial \bar{\beta}\right]_{j\times j} = 0 & \left[\partial \dot{\bar{\beta}} / \partial \bar{\gamma}\right]_{j\times k} = 0 \\ \left[\partial \dot{\bar{\gamma}} / \partial \bar{r}\right]_{k\times 3} = 0 & \left[\partial \dot{\bar{\gamma}} / \partial \bar{v}\right]_{k\times 3} = 0 & \left[\partial \dot{\bar{\gamma}} / \partial \bar{\beta}\right]_{k\times j} = 0 & \left[\partial \dot{\bar{\gamma}} / \partial \bar{\gamma}\right]_{k\times k} = 0 \end{bmatrix}_{(6+j+k)\times(6+j+k)}$$

State Transition Matrix Differential Equation for a General State Vector

Hence, $A(t)$ may be written as:

$$A = \begin{bmatrix} [0]_{3\times 3} & [\mathrm{I}]_{3\times 3} & [0]_{3\times j} & [0]_{3\times k} \\ \left[\partial \ddot{\bar{r}} / \partial \bar{r}\right]_{3\times 3} & \left[\partial \ddot{\bar{r}} / \partial \bar{v}\right]_{3\times 3} & \left[\partial \ddot{\bar{r}} / \partial \bar{\beta}\right]_{3\times j} & [0]_{3\times k} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mathrm{I} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{(6+j+k)\times(6+j+k)}$$

State Transition Matrix Differential Equation for a General State Vector

Then $\dot{\Phi} = A\Phi$ yields

$$\dot{\Phi} = \begin{bmatrix} 0 & I & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & 0 \\ \phi_{21} & \phi_{22} & \phi_{23} & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}_{(6+j+k)\times(6+j+k)}$$

$$= \begin{bmatrix} [\phi_{21}]_{3x3} & [\phi_{22}]_{3x3} & [\phi_{23}]_{3xk} & [0]_{3xk} \\ [A_{21}\phi_{11} + A_{22}\phi_{21}]_{3x3} & [A_{21}\phi_{12} + A_{22}\phi_{22}]_{3x3} & [A_{21}\phi_{13} + A_{22}\phi_{23} + A_{23}]_{3xj} & [0]_{3xk} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

State Transition Matrix Differential Equation for a General State Vector

Hence, we need only to integrate the $6\times(6+j)$ matrix of differential equations,

$$\dot{\Phi}_1 = \begin{bmatrix} [\phi_{21} & \phi_{22} & \phi_{23} \\ [A_{21}\phi_{11} + A_{22}\phi_{21}] & [A_{21}\phi_{12} + A_{22}\phi_{22}] & [A_{21}\phi_{13} + A_{22}\phi_{23} + A_{23}] \end{bmatrix}_{6\times(6+j)}$$

within I.C.

$$\Phi_1(t_0, t_0) = \Big[[\mathrm{I}]_{6\times 6} \quad [0]_{6\times j} \Big]$$

The remaining elements of Φ simply are the elements of an identity matrix.

Sympletic Property of Φ

Under certain conditions on *A(t)* the state transition matrix may be inverted analytically (Battin, 1987). Under these conditions Φ is referred to as being sympletic.

If the matrix *A(t)* can be partitioned in the form

$$A(t) = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \qquad (4.2.12)$$

where the submatrices have the properties that

$$A_1^T = -A_4\,,\quad A_2^T = A_2\,,\ \text{and}\ \ A_3^T = A_3 \qquad (4.2.13)$$

Sympletic Property of Φ

Then $\Phi(t, t_k)$ can be similarly partitioned as

$$\Phi(t, t_k) = \left[\begin{array}{c:c} \Phi_1 & \Phi_2 \\ \hdashline \Phi_3 & \Phi_4 \end{array}\right]$$

and $\Phi^{-1}(t, t_k)$ may be written as

$$\Phi^{-1}(t, t_k) = \left[\begin{array}{c:c} \Phi_4^T & -\Phi_2^T \\ \hdashline -\Phi_3^T & \Phi_1^T \end{array}\right] \qquad (4.2.14)$$

If $\ddot{\bar{r}} = \nabla U$ then Eq. (4.2.13) is true

Sympletic Property of Φ

In this case (consider a 2-D case for simplicity)

$$\ddot{x} = \frac{\partial U}{\partial x} \qquad X = \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} \longrightarrow \qquad \dot{x} = u \qquad \dot{y} = v$$

$$\ddot{y} = \frac{\partial U}{\partial y} \qquad\qquad \dot{u} = \frac{\partial U}{\partial x} \qquad \dot{v} = \frac{\partial U}{\partial y}$$

$$A = \frac{\partial F}{\partial X} = \left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hdashline {\partial^2 U}/{\partial x^2} & {\partial^2 U}/{\partial x \partial y} & 0 & 0 \\ {\partial^2 U}/{\partial y \partial x} & {\partial^2 U}/{\partial y^2} & 0 & 0 \end{array}\right]$$

and $A_1^T = -A_4$,

$A_2^T = A_2$, $A_3^T = A_3$

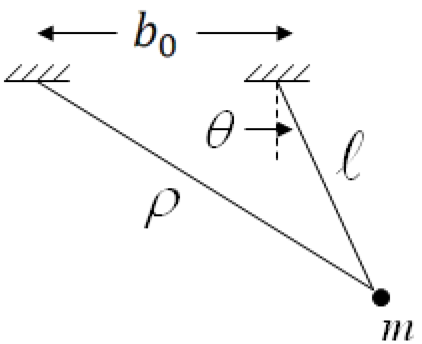
Because $\dfrac{\partial^2 U}{\partial x \partial y} = \dfrac{\partial^2 U}{\partial y \partial x}$,

Φ is symplectic

Example, Problem 14, Chapter 4 of Text

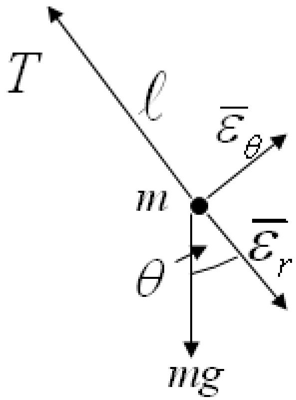
14a) Generate the $A(t)$ and $\tilde{H}$ matrix for the pendulum problem. Assume that we wish to estimate θ, $\dot{\theta}$, and b_0 at some epoch time.

Derive the equations of motion from the free body diagram.



$$\bar{r} = \ell\bar{\varepsilon}_r \qquad \dot{\bar{r}} = \ell\dot{\bar{\varepsilon}}_r = \ell\dot{\theta}\bar{\varepsilon}_\theta$$

$$\ddot{\bar{r}} = \ell\ddot{\theta}\bar{\varepsilon}_\theta - \ell\dot{\theta}^2\bar{\varepsilon}_r$$



$$m\ddot{\bar{r}} = \sum \bar{F}$$

$$m\ddot{\bar{r}} = mg\cos\theta\bar{\varepsilon}_r - T\bar{\varepsilon}_r - mg\sin\theta\bar{\varepsilon}_\theta$$

$$m\left(\ell\ddot{\theta}\bar{\varepsilon}_\theta - \ell\dot{\theta}^2\bar{\varepsilon}_r\right) = \left(mg\cos\theta - T\right)\bar{\varepsilon}_r - mg\sin\theta\bar{\varepsilon}_\theta$$

Example, Problem 14, Chapter 4 of Text

In component form: $\bar{\varepsilon}_r \quad -m\ell\dot{\theta}^2 = mg\cos\theta - T \quad (1)$

$\bar{\varepsilon}_\theta \quad m\ell\ddot{\theta} = -mg\sin\theta \quad (2)$

Eq. (2) gives us $\theta(t)$, $\dot{\theta}(t)$ and Eq. (1) gives the tension in the cord,

$$T = mg\cos\theta + m\ell\dot{\theta}^2$$

Hence, we need to solve $\ddot{\theta} = \frac{-g}{\ell}\sin\theta$, $\omega \equiv \sqrt{\frac{g}{\ell}}$

$$= -\omega^2 \sin\theta$$

writing Eqs. in 1st order form:

$$\dot{\theta} = \alpha$$
$$\dot{\alpha} = -\omega^2 \sin\theta$$
$$\dot{b}_0 = 0$$

Then, $X = \begin{bmatrix} \theta \\ \alpha \\ b_0 \end{bmatrix}, \dot{X} = \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \dot{b}_0 \end{bmatrix} = \begin{bmatrix} \alpha \\ -\omega^2 sin\theta \\ 0 \end{bmatrix}$

Example, Problem 14, Chapter 4 of Text

$$A(t) = \frac{\partial \dot{X}(t)}{\partial X(t)} = \begin{bmatrix} \partial\alpha/\partial\theta & \partial\alpha/\partial\alpha & \partial\alpha/\partial b_0 \\ \partial\dot{\alpha}/\partial\theta & \partial\dot{\alpha}/\partial\alpha & \partial\dot{\alpha}/\partial b_0 \\ \partial\dot{b}_0/\partial\theta & \partial\dot{b}_0/\partial\alpha & \partial\dot{b}_0/\partial b_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\omega^2 cos\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{\Phi}(t,t_o) = A(t)\Phi(t,t_o) = \begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos\theta & 0 \end{bmatrix}^* \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

Can we use LaPlace Transforms to solve for $\Phi(t,t_o)$?

Where $[\ ]^*$ indicates that $A(t)$ is evaluated on a reference solution for $\theta(t)$.

Example, Problem 14, Chapter 4 of Text

Choose initial conditions $\theta_0^*, \dot{\theta}_0^*$ and generate the reference trajectory while simultaneously integrating $\dot{\Phi} = A(t)\Phi$ i.e.

$$\dot{\theta} = \alpha$$

$$\dot{\alpha} = -\omega^2 \sin\theta$$

$$\dot{\phi}_{11} = \phi_{21}$$

$$\dot{\phi}_{12} = \phi_{22}$$

$$\dot{\phi}_{21} = -\omega^2 \cos(\theta)\phi_{11}$$

$$\dot{\phi}_{22} = -\omega^2 \cos(\theta)\phi_{12}$$

Note that we do not need to integrate equations for b_0 since it is a constant

Example, Problem 14, Chapter 4 of Text

IC: $\theta^*(t_0)$, $\dot{\theta}^*(t_0)$, $\Phi(t_0,t_0) = \mathrm{I}$

i.e., $\phi_{11}(t_0,t_0) = \phi_{22}(t_0,t_0) = 1$, $\phi_{12}(t_0,t_0) = \phi_{21}(t_0,t_0) = 0$

To do this in Matlab we would use the Reshape command. Which would write a matrix as a vector and vice versa. (see hints under handouts on web – "Matlab help for solving problem 4.10"). The vector derivatives are:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \dot{\phi}_{11} \\ \dot{\phi}_{12} \\ \dot{\phi}_{21} \\ \dot{\phi}_{22} \end{bmatrix}$$

Example, Problem 14, Chapter 4 of Text

Compute $\tilde{H}$

From the law of cosines

$$\rho^2 = b_0^2 + l^2 - 2b_0 l\cos(90^\circ + \theta)$$

$$= b_0^2 + l^2 + 2b_0 l\sin\theta$$

$$= \frac{1}{\rho}\left[b_0 l\cos\theta \quad 0 \quad b_0 + l\sin\theta\right]$$

$$\text{Hence,} \quad \tilde{H} = \frac{\partial\rho}{\partial x} = \left[\frac{\partial\rho}{\partial\theta} \quad \frac{\partial\rho}{\partial\alpha} \quad \frac{\partial\rho}{\partial b_0}\right]$$

$\tilde{H}$ is evaluated on the reference solution

Example, Problem 14, Chapter 4 of Text

The observations are related to a reference state deviation vector by,

$$y(t_1) = \tilde{H}(t_1)\Phi(t_1, t_0)x_0 + \varepsilon_1$$

$$y(t_2) = \tilde{H}(t_2)\Phi(t_2, t_0)x_0 + \varepsilon_2$$

$$y(t_m) = \tilde{H}(t_m)\Phi(t_m, t_0)x_0 + \varepsilon_m$$

Defining

$$y = \begin{bmatrix} y(t_1) \\ \vdots \\ y(t_m) \end{bmatrix}, \quad H = \begin{bmatrix} \tilde{H}(t_1)\Phi(t_1, t_0) \\ \vdots \\ \tilde{H}(t_m)\Phi(t_m, t_0) \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{bmatrix}$$

Then $y = Hx_0 + \varepsilon$ and $\hat{x}_0 = \left(H^T H\right)^{-1} H^T y$

$$= \left(\sum_{i=1}^{m} H_i^T H_i\right)^{-1} \sum_{i=1}^{m} H_i^T y_i$$

Example, Problem 14, Chapter 4 of Text

Here

$$x_0 = (X_0 - X_0^*) = \begin{bmatrix} \theta_0 - \theta_0^* \\ \alpha_0 - \alpha_0^* \\ b_0 - b_0^* \end{bmatrix} \text{ at reference time, } t_0$$

and $y(t_i) = \rho(t_i)$observed - $\rho(t_i)$computed

14 b) Assume small oscillations, i.e., $\sin\theta = \theta$, $\cos\theta = 1$. Then the equations of motion become

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\ddot{\theta} + \omega^2\theta = 0$$

Define

$$\omega \equiv \sqrt{g/l}$$

Example, Problem 14, Chapter 4 of Text

This is the equation for a harmonic oscillator which has the solution

$$\theta(t) = A\cos\omega t + B\sin\omega t$$

The constants are evaluated by noting that $t = t_0 = 0,\ \theta = \theta_0,\ \dot{\theta} = \dot{\theta}_0$

Hence,

$$\theta(t) = \theta_0 \cos\omega t + \frac{\dot{\theta}_0}{\omega}\sin\omega t$$

$$\dot{\theta}(t) = \alpha(t) = -\theta_0\omega\sin\omega t + \dot{\theta}_0\cos\omega t$$

Example, Problem 14, Chapter 4 of Text

We may now write the state transition matrix directly by differentiating the solution for $\theta(t)$ and $\alpha(t)$, i.e.,

$$\Phi(t, t_0 = 0) = \frac{\partial X(t)}{\partial X(t_0)}$$

$$= \begin{bmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t & 0 \\ -\omega \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example, Problem 14, Chapter 4 of Text

Alternatively write $\ddot{\theta} = -\omega^2\theta$ as a 1st order system

$$\dot{\theta} = \alpha$$
$$\dot{\alpha} = -\omega^2\theta$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \end{bmatrix}$$

$$(SI - A)^{-1} = \begin{bmatrix} s & -1 \\ \omega^2 & s \end{bmatrix}^{-1} = \frac{1}{s^2 + \omega^2} \begin{bmatrix} s & 1 \\ -\omega^2 & s \end{bmatrix}$$

Taking the inverse Laplace Transform gives us

$$= \begin{bmatrix} \cos\omega t & \frac{1}{\omega}\sin\omega t & 0 \\ -\omega\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14c) The assumption that θ is small restricts this solution to small values of θ. However, if we linearize $\theta(t)$ about a reference solution, we do not require that $\theta(t)$ be small, only that the deviation of $\theta(t)$ from the reference, $\theta^*(t)$, be small.

Least Squares with apriori Information

If an apriori value is available for x_k (call it $\bar{x}_k$) and an associated symmetric weighting matrix $\bar{w}_k$, the weighted least squares estimate of $\hat{x}_k$ can be obtained.

Least Squares with apriori Information

Given

$$y = Hx_k + \varepsilon$$

$$\bar{x}_k = x_k + \eta_k$$

Where η_k is the error in $\bar{x}_k$ and its influence on $\hat{x}_k$ is reflected in the weighting matrix $\bar{W}_k$

and $[y]_{m\times 1}$, $[\bar{x}_k]_{n\times 1}$

Choose $\hat{x}_k$ to minimize the performance index

$$J(x_k) = \frac{1}{2}\varepsilon^T w \varepsilon + \frac{1}{2}\eta_k \bar{w}_k \eta_k^T$$

Least Squares with apriori Information

Writing $J(x_k)$ explicitly in terms of x_k

$$J(x_k) = \frac{1}{2}(y - Hx_k)^T w(y - Hx_k) + \frac{1}{2}(\bar{x}_k - x_k)^T \bar{w}_k(\bar{x}_k - x_k)$$

(4.3.24)

$$\frac{\partial J(x_k)}{\partial x_k} = 0$$

Results in (See Eq B.7.4)

$$\frac{\partial J(x_k)}{\partial x_k} = -(y - Hx_k)^T wH - (\bar{x}_k - x_k)^T w_k = 0$$

$$= -y^T wH + x_k{}^T H^T wH - \bar{x}_k{}^T w_k + x_k{}^T w_k = 0$$

Least Squares with apriori Information

Solving for x_k yields $\hat{x}_k$

$$\mathbf{x}_k^T \left(H^T wH + \mathbf{w}_k \right) = y^T wH + \bar{\mathbf{x}}_k^T \mathbf{w}_k$$

$$\hat{\mathbf{x}}_k^T = \left(y^T wH + \bar{\mathbf{x}}_k^T \mathbf{w}_k \right) \left(H^T wH + \mathbf{w}_k \right)^{-1}$$

$$\hat{x}_k = \left(H^T wH + w_k \right)^{-1} \left(H^T wy + w_k \bar{x}_k \right) \quad (4.3.25)$$

Least Squares with apriori Information

Note that $\left(H^T w H + w_k\right)^{-1}$ is symmetric

also

$$\frac{\partial^2 J\left(x_k\right)}{\partial x_k^2} = H^T w H + w_k$$

which will be positive definite if *H* and/or w_k is full rank. Hence, $\hat{x}_k$ minimizes $J\left(x_k\right)$.