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The Origins of the Finite Element Method

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§O.1. Introduction

This Appendix summarizes the history of structural finite elements since 1950 to date. It functions as a hub for chapter-dispersed historical references.

For exposition convenience, structural "finitelementology" may be divided into four generations that span 10 to 15 years each. There are no sharp intergenerational breaks, but noticeable change of emphasis. The following summary does not cover the conjoint evolution of Matrix Structural Analysis into the Direct Stiffness Method from 1934 through 1970. This was the subject of a separate essay [238], which is also given in Appendix H.

§O.1.1. Who Invented Finite Elements?

Not just one individual, as this historical sketch will make clear. But if the question is tweaked to: who created the FEM in everyday use? there is no question in the writer's mind: M. J. (Jon) Turner at Boeing over the period 1950–1962. He generalized and perfected the Direct Stiffness Method, and forcefully got Boeing to commit resources to it while other aerospace companies were mired in the Force Method. During 1952–53 he oversaw the development of the first continuum based finite elements. In addition to Turner, major contributors to current practice include: B. M. Irons, inventor of isoparametric models, shape functions, the patch test and frontal solvers; R. J. Melosh, who recognized the Rayleigh-Ritz link and systematized the variational derivation of stiffness elements; and E. L. Wilson, who developed the first open source (and widely imitated and distributed) FEM software.

All of these pioneers were in the aerospace industry at least during part of their careers. That is not coincidence. FEM is the confluence of three ingredients, one of which is digital computation. And only large industrial companies (as well as some government agencies) were able to afford mainframe computers during the 1950s.

Who were the popularizers? Four academicians: J. H. Argyris, R. W. Clough, H. C. Martin, and O. C. Zienkiewicz are largely responsible for the "technology transfer" from the aerospace industry to a wider range of engineering applications during the 1950s and 1960s. The first three learned the method from Turner directly or indirectly. As a consultant to Boeing in the early 1950s, Argyris, a Force Method expert then at Imperial College, received reports from Turner's group, and weaved the material into his influencial 1954 serial [22]. To Argyris goes the credit of being the first in constructing a displacement-assumed continuum element [22, p. 62].

Clough and Martin, then junior professors at U.C. Berkeley and U. Washington, respectively, spent "faculty internship" summers at Turner's group during 1952 and 1953. The result of this seminal collaboration was a celebrated paper [758], widely considered the start of the present FEM. Clough baptized the method in 1960 [136] and went on to form at Berkeley the first research group to propel the idea into Civil Engineering applications. Olek Zienkiewicz, originally an expert in finite difference methods who learned the trade from Southwell, was convinced in 1964 by Clough to try FEM. He went on to write the first textbook on the subject [821] and to organize another important Civil Engineering research group in the University of Wales at Swansea.

§O.1.2. G1: The Pioneers

The 1956 paper by Turner, Clough, Martin and Topp [758], henceforth abbreviated to TCMT, is recognized as the start of the current FEM, as used in the overwhelming majority of commercial

codes. Along with Argyris' serial [22] they prototype the first generation, which spans 1950 through 1962. A panoramic picture of this period is available in two textbooks [572,596]. Przemieniecki's text is still reprinted by Dover. The survey by Gallagher [288] was influential at the time but is now difficult to access outside libraries.

The pioneers were structural engineers, schooled in classical mechanics. They followed a century of tradition in regarding structural elements as a device to transmit forces. This "element as force transducer" was the standard view in pre-computer structural analysis. It explains the use of flux assumptions to derive stiffness equations in TCMT. Element developers worked in, or interacted closely with, the aircraft industry. (As noted above, only large aerospace companies were then able to afford mainframe computers.) Accordingly they focused on thin structures built up with bars, ribs, spars, stiffeners and panels. Although the Classical Force Method dominated stress analysis during the 1950s [238], stiffness methods were kept alive by use in dynamics and vibration. It is not coincidence that Turner was a world-class expert in aeroelasticity.

§O.1.3. G2: The Golden Age

The next period spans the golden age of FEM: 1962–1972. This is the "variational generation." Melosh showed [485] that conforming displacement models are a form of Rayleigh-Ritz based on the minimum potential energy principle. This influential paper marks the confluence of three lines of research: Argyris' dual formulation of energy methods [22], the Direct Stiffness Method (DSM) of Turner [759,761], and early ideas of interelement compatibility as basis for error bounding and convergence [274,484]. G1 workers thought of finite elements as idealizations of structural components. From 1962 onward a two-step interpretation emerges: discrete elements approximate continuum models, which in turn approximate real structures.

By the early 1960s FEM begins to expand into Civil Engineering through Clough's Boeing-Berkeley connection [144,145] and had been baptized [136,138]. Reading Fraeijs de Veubeke's famous article [275] side by side with TCMT [758] one can sense the ongoing change in perspective opened up by the variational framework. The first book devoted to FEM appears in 1967 [821]. Applications to nonstructural problems had started in 1965 [820], and were treated in some depth by Martin and Carey [470].

From 1962 onwards the displacement formulation dominates. This was given a big boost by the invention of the isoparametric formulation and related tools (numerical integration, fitted natural coordinates, shape functions, patch test) by Irons and coworkers [394,397]. Low order displacement models often exhibit disappointing performance. Thus there was a frenzy to develop higher order elements. Other variational formulations, notably hybrids [573,578], mixed [353,723] and equilibrium models [275] emerged. G2 can be viewed as closed by the monograph of Strang and Fix [698], the first book to focus on the mathematical foundations.

§O.1.4. G3: Consolidation

The post-Vietnam economic doldrums are mirrored during this post-1972 period. Gone is the youthful exuberance of the golden age. This is consolidation time. Substantial effort is put into improving the stock of G2 displacement elements by tools initially labeled "variational crimes" [697], but later justified. Textbooks by Hughes [385] and Bathe [54] reflect the technology of this period. Hybrid and mixed formulations record steady progress [39]. Assumed strain formulations

appear [455]. A booming activity in error estimation and mesh adaptivity is fostered by better understanding of the mathematical foundations [714].

Commercial FEM codes gradually gain importance. They provide a reality check on what works in the real world and what doesn't. By the mid-1980s there was gathering evidence that complex and high order elements were commercial flops. Exotic gadgetry interweaved amidst millions of lines of code easily breaks down in new releases. Complexity is particularly dangerous in nonlinear and dynamic analyses conducted by novice users. A trend back toward simplicity starts [457,461].

§O.1.5. G4: Back to Basics

The fourth generation begins by the early 1980s. More approaches come on the scene, notably the Free Formulation [82,86], orthogonal hourglass control [264], Assumed Natural Strain methods [57,691], stress hybrid models in natural coordinates [576,599], as well as variants and derivatives of those approaches: ANDES [225,492], EAS [671,672] and others. Although technically diverse the G4 approaches share two common objectives:

(i)   Elements must fit into DSM-based programs since that includes the vast majority of production codes, commercial or otherwise.

(ii)   Elements are kept simple but should provide answers of engineering accuracy with relatively coarse meshes. These were collectively labeled "high performance elements" in 1989 [219].

Two more recent trends can be noted: increased abstraction on the mathematical side,[1] and canned recipes for running commercial software on the physical side.

"Things are always at their best in the beginning," said Pascal. Indeed. By now FEM looks like an aggregate of largely disconnected methods and recipes. The blame should not be placed on the method itself, but on the community split noted in the book Preface.

§O.1.6. Precursors

As used today, FEM represents the confluence of three ingredients: Matrix Structural Analysis (MSA), variational approximation theory, and the digital computer. These came together in the early 1950. The reader should not think, however, that they simultaneouly appeared on the table through some alchemy. MSA came on the scene in the mid 1930s when desk calculators became popular, as narrated in Appendix H. And variational approximation schemes akin to those of modern FEM were proposed before digital computers. Three examples:

●   The historical sketch of [470] says that "Archimedes used finite elements in determining the volume of solids." The alleged linkage is tenuous. Indeed he calculated areas, lengths and volumes of geometrical objects by dividing them into simpler ones and adding their contributions, passing to the limit as necessary. Where does "variational approximation" come in? Well, one may argue that the volume (area, length) measure of an object is a scalar functional of its geometry. Transmute "measure" into "energy" and "simpler objects" into "elements" and you capture one of the FEM tenets: the energy of the system is the sum of element energies. But for Archimedes to reach modern FEM "long is the way, and hard," since physical energy calculations require derivatives and Calculus would not be invented for 20 centuries.

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[1] "If you go too far up, abstraction-wise, you run out of oxygen." (Joel Spolsky).

- In his studies leading to the creation of variational calculus, Euler divided the interval of definition of a one-dimensional functional into finite intervals and assumed a linear variation over each, defined by end values [434, p. 53]. Passing to the limit he obtained what is now called the Euler-Lagrange differential equation of variational calculus. Thus Euler deserves credit for being the first to use a piecewise linear function with discontinuous derivatives at nodes to produce, out of the hat, an ODE with second derivatives. He did not use those functions, however, to obtain an approximate value of the functional.[2]

- In the early 1940s Courant wrote an expository article [153] advocating the variational treatment of partial differential equations. The Appendix of this article contains the first FEM-style calculations on a triangular net for determining the torsional stiffness of a hollow shaft. He used piecewise linear interpolation over each triangle as Rayleigh-Ritz trial functions, and called his idea "generalized finite differences."

- A direct variational approach similar to Courant's was continued by Synge and Prager in the context of functional analysis [592] and exposed in Synge's book [712] as the "hypercircle" method.[3]

- The seminal paper by Turner et al [758] cites two immediate DSM precursors, both dated 1953, by Levy [443] and Schuerch [657]. (Only the former is available as a journal article; both have "delta wings" in the title.) From [758], p. 806: "In a recent paper Levy has presented a method of analysis for highly redundant structures that is particularly suited to the use of high-speed digital computing machines. ... The stiffness matrix for the entire structure is computed by simple summation of of the stiffness matrices of the elements of the structure."

Precursors prior to 1950 had no influence on the rapid developments of Generation 1 outlined in §O.7.2. Two crucial pieces were missing. First, and most important, was the programmable digital computer. Without computers FEM would be a curiosity, worth perhaps a footnote in an arcane book. Also missing was a *driving application* that could get the long-term attention of scientists and engineers as well as industrial resources to fund R&D work. Aerospace structural mechanics provided the driver because the necessary implementation apparatus of MSA was available since the late 1930s [282].

Matrix procedures had to be moved from desk calculators and punched-tape accounting machines to digital computers, which affluent aerospace companies were able to afford amidst Cold War paranoia. Can you imagine defense funds pouring into hypercircles or Courant's triangles? Once all pieces were in place, synergy transformed the method into a *product*, and FEM took off.

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[2] That would have preceded the invention of direct variational methods (Rayleigh-Ritz) for over one century, while representing also the first FEM-style calculation. A near miss indeed.

[3] Curiously this book does not mention, even in passing, the use of digital computers that had already been commercially available for several years. The few numerical examples, all in 2D, are done by hand via relaxation methods.