State Noise Compensation Algorithm

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We will develop a simple state noise compensation algorithm. This algorithm adds process noise to the acceleration equations of the system and results in an additional term being added to the time update of the estimation error covariance matrix for the Kalman filter.

Process-noise is added to the differential equation of motion given by Eq. (4.2.1), i.e.,

$$\dot{\boldsymbol{X}}(t) = F(\boldsymbol{X}, t) + B(t)\boldsymbol{u}(t) \tag{1}$$

where $\boldsymbol{u}$ is white Gaussian process noise with

$$\mathrm{E}\left[\boldsymbol{u}(t)\right] = 0,\ \mathrm{E}\left[\boldsymbol{u}(t_i)\ \boldsymbol{u}(t_j)^T\right] = Q(t_i)\delta_{ij}\,. \tag{2}$$

where δ_{ij} is the kronicker delta.

The matrix, B, maps the process noise into the state derivatives. Although it is not necessary, we will assume for simplicity that $Q(t_i)$ is constant.

Expanding Eq. (1) in a Taylor series about a reference trajectory, $\boldsymbol{X}^*(t)$, yields

$$\dot{\boldsymbol{X}}(t) = \dot{\boldsymbol{X}}^*(t) + \frac{\partial F\left(\boldsymbol{X}(t)\right)^*}{\partial \boldsymbol{X}(t)}\left(\boldsymbol{X}(t) - \boldsymbol{X}^*(t)\right) + B(t)\boldsymbol{u}(t) \tag{3}$$

define

$$\boldsymbol{x}(t) \equiv \boldsymbol{X}(t) - \boldsymbol{X}^*(t) \text{ and } A(t) \equiv \frac{\partial F\left(\boldsymbol{X}(t)\right)^*}{\partial \boldsymbol{X}(t)}$$

(4)

Thus the state deviation propagation equation including process noise for a linear system is given by (see Eq. (4.9.1))

$$\dot{\boldsymbol{x}}(t) = \mathrm{A}(t)\boldsymbol{x}(t) + \mathrm{B}(t)\boldsymbol{u}(t) \tag{5}$$

We will further assume that process noise is only being added to the acceleration components of the state; hence,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \ , \ \boldsymbol{u} = \begin{bmatrix} u_{\ddot{X}} \\ u_{\ddot{Y}} \\ u_{\ddot{Z}} \end{bmatrix} \ , \ \mathrm{B} = \begin{bmatrix} \mathrm{O}_{3x3} \\ \mathrm{I}_{3x3} \end{bmatrix}. \tag{6}$$

The time update for the estimation error covariance matrix is given by Eq. (4.9.50).

$$\overline{\mathrm{P}}_{k+1} = \Phi\left(t_{k+1}, t_k\right) \mathrm{P}_k \Phi^{\mathrm{T}}\left(t_{k+1}, t_k\right) + \Gamma\left(t_{k+1}, t_k\right) Q \Gamma^{\mathrm{T}}\left(t_{k+1}, t_k\right) \tag{7}$$

where $\Gamma\left(t_{k+1}, t_k\right)$ is given by Eq. (4.9.47)

$$\Gamma\left(t_{k+1} t_k\right) = \int_{t_k}^{t_{k+1}} \Phi\left(t_{k+1}, \tau\right) \mathrm{B}\left(\tau\right) d\tau \, . \tag{8}$$

In this case $\mathrm{B}\left(\tau\right)$ is a constant given by Eq. (6) and $\Phi\left(t_{k+1}, \tau\right)$ is given by

$$\Phi\left(t_{k+1}, \tau\right) = \begin{bmatrix} \dfrac{\partial \boldsymbol{r}\left(t_{k+1}\right)}{\partial \boldsymbol{r}\left(\tau\right)} & \dfrac{\partial \boldsymbol{r}\left(t_{k+1}\right)}{\partial \dot{\boldsymbol{r}}\left(\tau\right)} \\ \dfrac{\partial \dot{\boldsymbol{r}}\left(t_{k+1}\right)}{\partial \boldsymbol{r}\left(\tau\right)} & \dfrac{\partial \dot{\boldsymbol{r}}\left(t_{k+1}\right)}{\partial \dot{\boldsymbol{r}}\left(\tau\right)} \end{bmatrix}. \tag{9}$$

Each element of Φ is a 3x3 matrix, so for simplicity define

$$\Phi\left(t_{k+1}, \tau\right) \equiv \begin{bmatrix} \phi_1\left(\tau\right) & \phi_2\left(\tau\right) \\ \phi_3\left(\tau\right) & \phi_4\left(\tau\right) \end{bmatrix}. \tag{10}$$

Then

$$\Phi\left(t_{k+1}, \tau\right) \ \mathrm{B}\left(\tau\right) = \begin{bmatrix} \phi_1\left(\tau\right) & \phi_2\left(\tau\right) \\ \phi_3\left(\tau\right) & \phi_4\left(\tau\right) \end{bmatrix} \begin{bmatrix} \mathrm{O} \\ \mathrm{I} \end{bmatrix} = \begin{bmatrix} \phi_2\left(\tau\right) \\ \phi_4\left(\tau\right) \end{bmatrix} \tag{11}$$

and from Eq. (8)

$$\Gamma\left(t_{k+1}, t_k\right) = \int_{t_k}^{t_{k+1}} \begin{bmatrix} \phi_2\left(\tau\right) \\ \phi_4\left(\tau\right) \end{bmatrix} d\tau \, . \tag{12}$$

One could substitute values of ϕ_2 and ϕ_4 into Eq. (12) and carry out the integration as a quadrature. Alternatively one can approximate the values of ϕ_2 and ϕ_4 and carry out the integration analytically. We will do the latter.

From Eq. (9) we see that

$$\phi_2(\tau) = \frac{\partial \mathbf{r}(t_{k+1})}{\partial \dot{\mathbf{r}}(\tau)} = \begin{bmatrix} \dfrac{\partial X(t_{k+1})}{\partial \dot{X}(\tau)} & \dfrac{\partial X(t_{k+1})}{\partial \dot{Y}(\tau)} & \dfrac{\partial X(t_{k+1})}{\partial \dot{Z}(\tau)} \\ \dfrac{\partial Y(t_{k+1})}{\partial \dot{X}(\tau)} & \dfrac{\partial Y(t_{k+1})}{\partial \dot{Y}(\tau)} & \dfrac{\partial Y(t_{k+1})}{\partial \dot{Z}(\tau)} \\ \dfrac{\partial Z(t_{k+1})}{\partial \dot{X}(\tau)} & \dfrac{\partial Z(t_{k+1})}{\partial \dot{Y}(\tau)} & \dfrac{\partial Z(t_{k+1})}{\partial \dot{Z}(\tau)} \end{bmatrix}. \tag{13}$$

Because we are generally dealing with dense tracking data, the time between observations is usually 10 seconds or less. Therefore, we can assume that $\dot{X}(\tau)$ is constant over this interval and approximate $X(t_{k+1})$ by

$$X(t_{k+1}) = \dot{X}(\tau)[t_{k+1} - \tau] \tag{14}$$

and

$$\frac{\partial X(t_{k+1})}{\partial \dot{X}(\tau)} = t_{k+1} - \tau\ . \tag{15}$$

Here we have assumed that for $t_{k+1} - \tau$ small, $X(t_{k+1})$ will be negligibly affected by $\dot{Y}(\tau)$ and $\dot{Z}(\tau)$. The same arguments apply to $Y(t_{k+1})$ and $Z(t_{k+1})$, hence

$$\phi_2(\tau) \cong \begin{bmatrix} t_{k+1} - \tau & 0 & 0 \\ 0 & \tau_{k+1} - \tau & 0 \\ 0 & 0 & t_{k+1} - \tau \end{bmatrix}. \tag{16}$$

Also,

$$\phi_4(t_{k+1}, \tau) = \frac{\partial \dot{\mathbf{r}}(t_{k+1})}{\partial \dot{\mathbf{r}}(\tau)}\ . \tag{17}$$

Again the off diagonal terms are small and the diagonal terms are ≈ 1 since we assume that the velocity is constant over the internal. Therefore,

$$\phi_4(t_{k+1}, \tau) \approx \mathrm{I}\ . \tag{18}$$

Substituting Eqs. (16) and (18) into Eq. (12) yields

$$\Gamma\left(t_{k+1}, t_k\right) = \int_{t_k}^{t_{k+1}} \begin{bmatrix} t_{k+1}-\tau & 0 & 0 \\ 0 & t_{k+1}-\tau & 0 \\ 0 & 0 & t_{k+1}-\tau \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} d\tau \ . \tag{19}$$

Define

$$\Delta t \equiv \left(t_{k+1} - t_k\right)$$

then

$$\Gamma\left(t_{k+1}, t_k\right) = \begin{bmatrix} \frac{\Delta t^2}{2} & 0 & 0 \\ 0 & \frac{\Delta t^2}{2} & 0 \\ 0 & 0 & \frac{\Delta t^2}{2} \\ \Delta t & 0 & 0 \\ 0 & \Delta t & 0 \\ 0 & 0 & \Delta t \end{bmatrix} \tag{20}$$

or

$$\Gamma\left(t_{k+1}, t_k\right) = \Delta t \begin{bmatrix} \frac{\Delta t}{2} & \mathrm{I} \\ \mathrm{I} & \end{bmatrix} \tag{21}$$

where I is a 3x3 identity matrix.

Assuming that Q is given by

$$Q = \begin{bmatrix} \sigma_{\ddot{X}}{}^2 & 0 & 0 \\ 0 & \sigma_{\ddot{Y}}{}^2 & 0 \\ 0 & 0 & \sigma_{\ddot{Z}}{}^2 \end{bmatrix} \tag{22}$$

then the process noise contribution to the estimation error covariance matrix time update at t_{k+1} is given by

$$\begin{aligned} \Gamma\left(t_{k+1},t_k\right)\ Q\ \Gamma^{\mathrm{T}}\left(t_{k+1},t_k\right) &= \Delta t^2 \begin{bmatrix} \frac{\Delta t}{2} I \\ I \end{bmatrix} \begin{bmatrix} \sigma_{\ddot{X}}{}^2 & 0 & 0 \\ 0 & \sigma_{\ddot{Y}}{}^2 & 0 \\ 0 & 0 & \sigma_{\ddot{Z}}{}^2 \end{bmatrix} \begin{bmatrix} \frac{\Delta t}{2}\mathrm{I} & \mathrm{I} \end{bmatrix} \\ &= \Delta t^2 \begin{bmatrix} \frac{\Delta t^2}{4}\sigma_{\ddot{X}}{}^2 & 0 & 0 & \frac{\Delta t}{2}\sigma_{\ddot{X}}{}^2 & 0 & 0 \\ 0 & \frac{\Delta t^2}{4}\sigma_{\ddot{Y}}{}^2 & 0 & 0 & \frac{\Delta t}{2}\sigma_{\ddot{Y}}{}^2 & 0 \\ 0 & 0 & \frac{\Delta t^2}{4}\sigma_{\ddot{Z}}{}^2 & 0 & 0 & \frac{\Delta t}{2}\sigma_{\ddot{z}}{}^2 \\ \frac{\Delta t}{2}\sigma_{\ddot{X}}{}^2 & 0 & 0 & \sigma_{\ddot{X}}{}^2 & 0 & 0 \\ 0 & \frac{\Delta t}{2}\sigma_{\ddot{Y}}{}^2 & 0 & 0 & \sigma_{\ddot{Y}}{}^2 & 0 \\ 0 & 0 & \frac{\Delta t}{2}\sigma_{\ddot{Z}}{}^2 & 0 & 0 & \sigma_{\ddot{Z}}{}^2 \end{bmatrix}. \end{aligned} \tag{23}$$

One can check the units of this matrix by noting the units of the nonzero terms, i.e.,

$$T^2 \begin{bmatrix} T^2 \frac{L^2}{T^4} & T\frac{L^2}{T^4} \\ T\frac{L^2}{T^4} & \frac{L^2}{T^4} \end{bmatrix} = \begin{bmatrix} L^2 & \frac{L^2}{T} \\ \frac{L^2}{T} & \frac{L^2}{T^2} \end{bmatrix} \tag{24}$$

where L is length and T is time. Hence, the units are correct since the diagonal elements have units of position and velocity squared and off diagonal terms have units of position times velocity.

Eq. (23) represents the contribution to the estimation error covariance matrix from uncertainty in the accelerations acting on the system. The values chosen for $\sigma_{\ddot{X}}{}^2$, $\sigma_{\ddot{Y}}{}^2$ and $\sigma_{\ddot{Z}}{}^2$ should correspond to the magnitude of the uncertainty of the acceleration acting on the system. For example, if we were trying to compensate for atmospheric drag uncertainty this would primarily be along track. In the RIC frame one might assume that $\sigma_R{}^2 = \sigma_C{}^2 = 0$ and $\sigma_I{}^2$ corresponds to the uncertainty in the drag acceleration. A transformation to the X,Y,Z frame, which is generally ECI, would be made at each observation time so that (see Eq. (4.16.19))

$$Q_{ECI} = \gamma^T Q_{RIC} \gamma \tag{25}$$

where γ is the transformation matrix from ECI to RIC (see Eq. (4.16.22)). Note that Q_{ECI} is not diagonal nor is it constant and Eq. (23) must be reevaluated with the proper value of Q_{ECI} at each observation time.