ASEN 5007 Introduction to FEM Fall 2013 (CAETE Offering) Take Home Final Exam

Important: Work individually. No consultation with others is permitted

On campus students: Take home posted Tu April 30, 2013 by noon. Due on or before Tu May 6 by 6pm, at ECAE 187. If I am not there leave in instructor's mailbox at ECAE 196 or ECOT 613 (note that the latter is locked after 5pm)
If you leave in mailbox, place in **clearly marked envelope**. **Do not** *leave under door or in hallway as it could be accidentally discarded. Attach pages 1–4 of exam as cover with your name on it.*
Remote CAETE students may email the exam

This test contains five Questions, all pertaining to the same problem. The first four are analytical; Q1 is fairly short. The algebra in Q2-Q4 may be helped by a CAS. The last one is computer oriented. Any language can be used to solve it; however, a *Mathematica* Notebook called `PlaneThermalFEM.nb` can be downloaded to help. The Notebook will be posted, along with this document, on April 30, 2013 by noon.

Reference to formulas in the Notes where appropriate is encouraged instead of explicitly listing them. Technical background for the first four Questions is given in Addendum A to this exam.

QUESTION 1. 15 pts = 3 + 4 + 4 + 4

After reading Addendum A, answer the following conceptual questions. *Please be brief*: no more than one short paragraph for each.

(a) What is the variational index m for the heat conduction problem? (Explain)

(b) Does bilinear interpolation over 4-node quadrilaterals provide the correct temperature continuity between elements? If so, why?

(c) Why is a 2 x 2 Gauss integration rule used for $\mathbf{K}^e$ of the 4-node quad? Wouldn't 1 x 1 be enough?

(d) If a 6-node triangle element were to be developed for this problem, how many Gauss points would be needed so that $\mathbf{K}^e$ is rank sufficient?

Note: the notion of element rank sufficiency (and the equivalent of "rigid body modes") for (c) and (d) is briefly covered on page Q3-7.

QUESTION 2. 20 pts = 10 + 6 + 4

(a) Show that the 4×4 element stiffness matrix of the 4-node bilinear iso-P quadrilateral for the heat conduction problem described in Addendum A and pictured in Figure A.2(b), has the form

$$\mathbf{K}^e = \int_{\Omega^e} kh\, \mathbf{B}^T \mathbf{B}\, d\Omega^e \tag{1}$$

Here k is the coefficient of thermal conduction, h is the element thickness and the 2×4 matrix $\mathbf{B}$ is defined in equation (A.8) of Addendum A. Both k and h can be assumed to be constant over the element and therefore taken out of the integral.

(b) Check that for a *rectangular element* of sizes $a \times b$ along the x and y axes, respectively, this formula, integrated either exactly or by a 2×2 Gauss rule, gives

$$\mathbf{K}^e = \tfrac{1}{6} kh \begin{bmatrix} \frac{2a}{b} + \frac{2b}{a} & \frac{a}{b} - \frac{2b}{a} & -\frac{a}{b} - \frac{b}{a} & \frac{-2a}{b} + \frac{b}{a} \\ \frac{a}{b} - \frac{2b}{a} & \frac{2a}{b} + \frac{2b}{a} & \frac{-2a}{b} + \frac{b}{a} & -\frac{a}{b} - \frac{b}{a} \\ -\frac{a}{b} - \frac{b}{a} & \frac{-2a}{b} + \frac{b}{a} & \frac{2a}{b} + \frac{2b}{a} & \frac{a}{b} - \frac{2b}{a} \\ \frac{-2a}{b} + \frac{b}{a} & -\frac{a}{b} - \frac{b}{a} & \frac{a}{b} - \frac{2b}{a} & \frac{2a}{b} + \frac{2b}{a} \end{bmatrix} \tag{2}$$

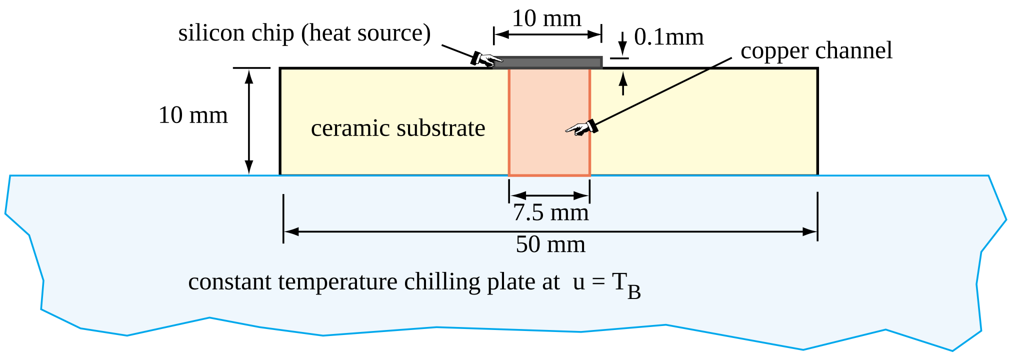

Figure Q3.1. The heat conduction problem for Question 4.

(c) Assuming that k, h, a and b are all positive, how many zero eigenvalues does the matrix (2) have? Is that the correct number? If so, why?

Material relevant to (a) is given in Addendum A (formulation) and §16.5.2 of Notes (shape functions of 4-node quad). Verification of (b,c) by computer algebra is fine. Check also the posted Notebook.

QUESTION 3. 15 pts = 10+5

Assume the source term of equation (A.1) varies bilinearly over the quadrilateral so that it is interpolated from the node values as

$$s = N_1 s_1 + N_2 s_2 + N_3 s_3 + N_4 s_4 = \mathbf{N}\,\mathbf{s} \tag{3}$$

where $N_i(\xi, \eta)$ are the shape functions of the bilinear quadrilateral, and $\mathbf{N}$ the 1×4 shape function matrix. Assume that all prescribed fluxes $\bar{q}_n$ over element sides are zero.

(a) With an argument similar to that of §14.4 but using the W^e of (A.6), show that the consistent element nodal force vector $\mathbf{f}^e$ can be expressed as

$$\mathbf{f}^e = \int_{\Omega^e} h\,\mathbf{N}^T\,s\,d\Omega^e = \int_{\Omega^e} h\,\mathbf{N}^T\mathbf{N}\,\mathbf{s}\,d\Omega^e \tag{4}$$

(b) What $\mathbf{f}^e$ would you get for an $a \times b$ rectangular element if s is constant over it? Hint: check Cell 4 of posted Notebook.

QUESTION 4. 15 pts

Suppose side 1–2 of the individual quadrilateral of Figure A.2(b) is subject to a uniform prescribed thermal flux $\bar{q}_n$, constant along the side, in which $\bar{q}_n$ is positive (negative) if heat flows out of (into) the element. Assume that the source term s is identically zero. Using the W^e of (A.6) show that the consistent element node forces are

$$\mathbf{f}^e = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = -\tfrac{1}{2}\bar{q}_n h \ell_{12} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{5}$$

where ℓ_{12} is the length of side 1–2. Verification by computer algebra is fine. Hint: check the module in Cell 5 of posted Notebook, in which the four sides of a quadrilateral are processed for specified fluxes.

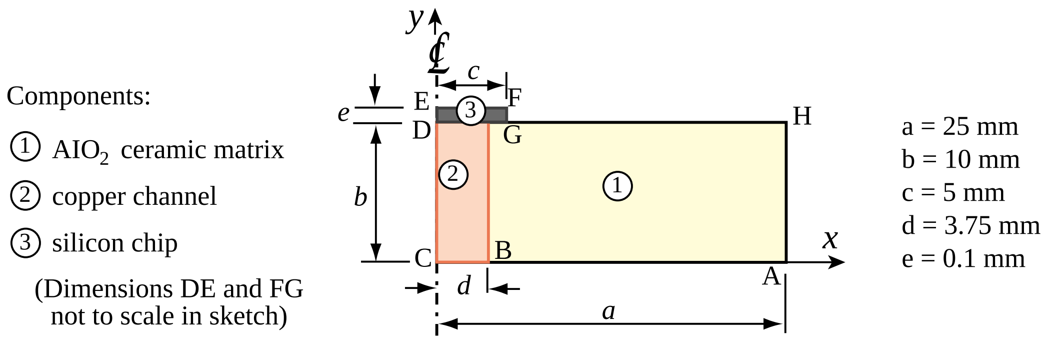

Figure Q3.2. The computational domain ABCDEFGH for the problem of Figure Q3.1.

QUESTION 5. 35 pts

This question deals with the FEM analysis of the 2D thermal problem depicted in Figure Q3.1. We want to find the temperature distribution on an electronic package consisting of a heat-producing silicon chip sitting on a ceramic substrate placed over a chilling plate. In the center of the substrate a copper channel is placed to enhance cooling. The following assumptions are made to simplify the model.

- Time dependency is neglected. Consequently the problem is steady state heat conduction and is modeled by the Poisson's equation presented in Addendum A.
- Temperature gradients in the z direction are negligible; hence the problem is modeled in 2D, with a nominal unit thickness in the z direction.
- The materials are in perfect contact. Any effects due to bonding or imperfect interfaces are ignored.
- The bottom plate is at constant temperature T_{bot}.
- The vertical boundaries are thermally isolated.
- Fluxes along the upper boundary are specified. Note that this is *not* an appropriate boundary condition if this is an air interface because heat will be actually dissipated by *convection*, which depends on the differences of temperature of the chip and ceramic matrix boundary from the air temperature. The boundary condition term for convection, however, is not provided in the functional given in Addendum A. The addition of such term is the subject of a Bonus Question.
- Radiation dissipation is neglected.
- The heat source in the chip material is taken to be constant over its volume. The heat sources in the other two materials are zero.

Table Q3.1 Geometric and physical properties for the electronic package heat conduction problem

Properties	Units	Values
Lengths	mm	In Figure Q3.2: $a = 25$, $b = 10$, $c = 5$, $d = 3.75$, $e = 0.1$
Specified temperature	$^\circ$C	Chilling plate boundary AB and BC: $T_{bot} = -20$.
Conductivity	mW/mm/$^\circ$C	Silicon: $k = 150$, copper: $k = 400$, ceramic: $k = 40$.
Heat body source	mW/mm^3	Silicon: $s = 16000$, copper: $s = 0$, ceramic: $s = 0$.
Heat flux, top	mW/mm^2	Over EF and FG: $\bar{q}_n = 25$; over GH: $\bar{q}_n = 10$.
Heat flux, lateral	mW/mm^2	Over CD, DE and HA: $\bar{q}_n = 0$.

The problem has one line of symmetry. Using that feature the computational domain can be reduced to that shown in Figure Q3.2. The necessary geometric and physical properties are provided in Table Q3.1.

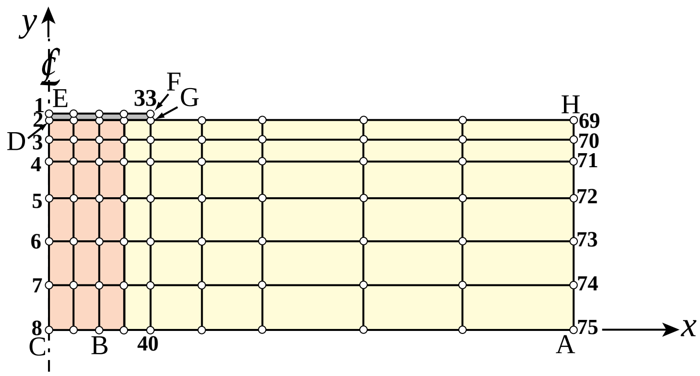

Figure Q3.3. Recommended FE mesh for Question 5. There is only one degree of freedom at each node n: the temperature u_n.

All units are in the SI system. The boundary conditions are as follows.

- Over ABC the temperature is prescribed: $u = T_{bot}$, the chilling plate temperature.
- Over EF and FG the heat flux is specified to the value listed in Table Q3.1.
- Over GH, the heat flux is also specified to the value listed in Table Q3.1.
- Over CDE and HA, the heat flux is zero. Note that CDE is the symmetry line.

Cell 13 of Notebook `PlaneThermalFEM.nb` contains a driver program for this problem using a very coarse mesh with 15 nodes and 8 elements. This mesh is depicted in a plot that appears under that cell when it is executed. To answer Question 5 you must use a *finer* idealization, such as the graded mesh depicted in Figure Q3.3. This has 75 nodes and 58 elements. You may use a finer mesh if so desired, but that shown in Figure Q3.3 is sufficient.

As answer to this Question, provide:

- The computed temperatures $\mathbf{u}$ at the nodes, such as those shown in Figure A.5 for a demo problem. Mark or highlight where the *highest* temperature occurs and its value.
- The nodal forces $\mathbf{f}$ recovered from $\mathbf{f} = \mathbf{K}\,\mathbf{u}$, such as those shown in Figure A.5 for a demo problem.
- A plot of the mesh showing element and node numbers, similar to that shown in Figure A.6(a)
- A contour plot of the temperature distribution, similar to those shown in Figure A.6(b,c). If you produce both a polygon plot and a band plot, pick one.

Bonus Question. Up to 6 pts, total may exceed 100. (Requires good knowledge of variational calculus, else don't even try it. Purely analytical: no computer work required.)

Change the boundary conditions over EFGH (the air interface) to be of convection type. Over this boundary portion, mathematically called Γ_3,

$$q_n = \chi\,(u - T_{air}) \quad \text{on} \quad \Gamma_3, \tag{6}$$

where u is the (unknown) boundary temperature of the electronic package, T_{air} the given air (environment, ambient) temperature, and χ a convection coefficient. In applied math, this boundary condition is said to be *mixed* or of Robbins type.

(a) Add an appropriate surface convective term (a surface integral over Γ_3) to (A.3)–(A.4) to account for this term.

(b) Show that the Euler-Lagrange equations of this expanded functional yield (6) as natural BC.

Addendum A: Technical Background for Exam Questions

The exam questions pertain to the development and implementation of a finite element model to numerically solve the two-dimensional Poisson equation over a plane domain Ω (see Figure A.1):

$$k\nabla^2 u = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = s \quad \text{in } \Omega, \tag{A.1}$$

Here ∇^2 is the Laplacian, $u = u(x, y)$ is a *scalar* unknown function, k a known constitutive coefficient that may be function of x and y, and $s = s(x, y)$ is a given source function. The actual body has a thickness h in the z direction; often this thickness is uniform and taken equal to unity.

The Poisson partial differential equation (A.1) models many important problems in mathematical physics. Some of them are: steady-state heat conduction (the topic of this exam), St-Venant torsion of arbitrary cross sections, potential fluid flow, linear acoustics, hydrostatics and electrostatics. The physical meaning of u (as well as of that of k and s) depends on the application. In the *heat conduction* problem considered here, u is the temperature, k the coefficient of thermal conductivity, $q_n = -k(\partial u/\partial n)$ the heat flux † along a direction n and s the internal heat source density (heat produced per unit volume in the material).

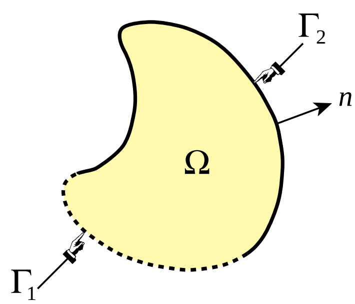

Figure A.1. The problem domain for the 2D Poisson equation.

To finish the problem specification, the Poisson equation (A.1) has to be complemented by boundary conditions (BC) on the domain boundary Γ. The classical BC are of two types. Over a portion Γ_1 of the domain boundary Γ (distinguished by a dashed line in Figure A.1) the value of u is prescribed to be $\bar{u}$; for example $u = 0$ or $u = 100$. Over the complementary portion Γ_2 the flux $q_n = -k(\partial u/\partial n)$, where n is the *exterior* normal to Γ, is prescribed to be $\bar{q}_n$. See sketch in Figure A.1. Mathematically:

$$u = \bar{u} \quad \text{on } \Gamma_1, \qquad q_n = -k\frac{\partial u}{\partial n} = \bar{q}_n \quad \text{on } \Gamma_2. \tag{A.2}$$

The heat flux is considered positive if heat flows away from Ω. In the mathematical literature the conditions on Γ_1 and Γ_2 are referred to as Dirichlet and Neumann boundary conditions, respectively. A third type of BC called *mixed*, which is appropriate for convective heat transfer interfaces (e.g., air-cooled electronics) is the topic of the Bonus Question.

Variational Formulation

A variational form equivalent to (A.1)–(A.2) is $\delta\Pi = 0$, where δ denotes variation with respect to the unknown function u, and Π is the total energy functional

$$\Pi(u) = U(u) - W(u). \tag{A.3}$$

U and W represent internal energy and and external potential, respectively:

$$U(u) = \tfrac{1}{2}\int_\Omega kh\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] d\Omega, \qquad W(u) = \int_\Omega s\,h\,u\,d\Omega - \oint_{\Gamma_2} \bar{q}_n h\,u\,d\Gamma, \tag{A.4}$$

† This is Fourier's law of heat conduction.

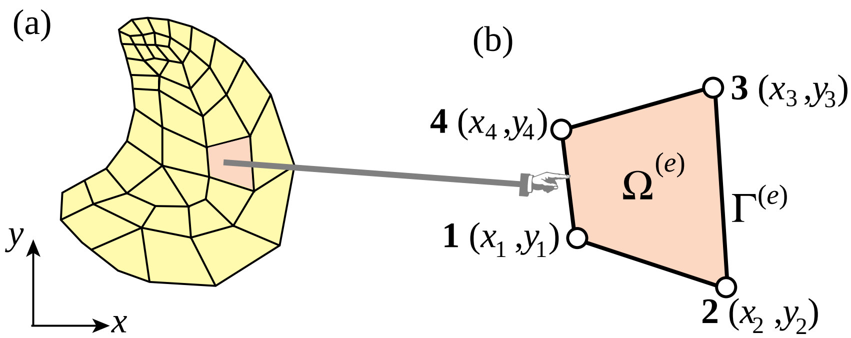

Figure A.2. (a) FEM discretization of the problem domain of Figure A.1 with quadrilateral elements; (b) an individual quadrilateral with 1,2,3,4 as local nodes.

in which $u = \bar{u}$ on Γ_1 is satisfied *a priori*, and h is the thickness in the z direction. With respect to this variational principle the Dirichlet BC $u = \bar{u}$ on Γ_1 is essential, whereas the Neumann BC $\bar{q}_n = -k\partial u/\partial n$ on Γ_2 is natural.

Finite Element Discretization of Heat Conduction Functional

The domain Ω is discretized with 4-node bilinear-quadrilateral finite elements as sketched in Figure A.2(a). Functionals U and W specialize to the element level in the usual way:

$$U^e = \tfrac{1}{2}\int_{\Omega^e} kh \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] d\Omega^e = \tfrac{1}{2}\int_{\Omega^e} kh \left(g_x^2 + g_y^2\right) d\Omega^e = \tfrac{1}{2}\int_{\Omega^e} kh\, \mathbf{g}^T \mathbf{g}\, d\Omega^e \tag{A.5}$$

$$W^e = \int_{\Omega^e} s\, h\, u\, d\Omega^e - \oint_{\Gamma^e} \bar{q}_n h\, u\, d\Gamma^e. \tag{A.6}$$

in which $g_x = \partial u/\partial x$ and $g_y = \partial u/\partial y$ are the temperature gradients and $\mathbf{g}^T = [\, g_x \;\; g_y \,]$.

Over a generic quadrilateral 1-2-3-4, depicted in Figure A.2(b), the temperature u is approximated by the bilinear iso-P interpolation:

$$u^e = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 = [\, N_1 \quad N_2 \quad N_3 \quad N_4 \,] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \mathbf{N}\mathbf{u}^e, \tag{A.7}$$

where u_1, u_2, u_3 and u_4 are the node temperatures, and N_1, N_2, N_3 and N_4 are the usual iso-P shape functions for the 4-node quadrilateral given in Chapter 16; e.g., $N_1(\xi, \eta) = (1-\xi)(1-\eta)/4$. Because the functional (A.3) of the thermal conduction problem has variational index one, the interpolation (A.7) can be shown to satisfy the requirements of continuity and completeness.

The temperature gradients $g_x = \partial u/\partial x$ and $g_y = \partial u/\partial y$ are mathematically analogous to strains in mechanical problems. We call $\mathbf{g}^T = [\, g_x \;\; g_y \,]$ the thermal gradient vector and the matrix $\mathbf{B}$ that relates $\mathbf{g}$ to the element node values

$$\mathbf{g} = \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = \begin{bmatrix} \partial N_1/\partial x & \partial N_2/\partial x & \partial N_3/\partial x & \partial N_4/\partial x \\ \partial N_1/\partial y & \partial N_2/\partial y & \partial N_3/\partial y & \partial N_4/\partial y \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \mathbf{B}\mathbf{u}^e, \tag{A.8}$$

the thermal-gradient-to-temperature matrix. To answer Questions 2–4, (A.5) and (A.6) must be maneuvered into the standard quadratic form

$$\Pi^e = U^e - W^e = \tfrac{1}{2}(\mathbf{u}^e)^T \mathbf{K}^e \mathbf{u}^e - (\mathbf{f}^e)^T \mathbf{u}^e \tag{A.9}$$

whence the first variation $\delta\Pi^e = (\delta\mathbf{u}^e)^T(\mathbf{K}^e\mathbf{u}^e - \mathbf{f}^e) = 0$ yields the element equations

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e. \tag{A.10}$$

Here $\mathbf{K}^e$ is the 4×4 element stiffness matrix, which derives from the element internal energy U^e, and $\mathbf{f}^e$ is the element node force vector, which derives from the element external potential W^e.† A 2×2 Gauss integration rule is recommended to evaluate $\mathbf{K}^e$.

Rank Sufficiency of $\mathbf{K}^e$

In the heat conduction problem the rank of $\mathbf{K}^e$ is still given by formula (19.6) with the following adjustments:

(1) The number of DOFs per node is 1

(2) In 2D, the dimension of the constitutive matrix is 2 (it connects two temperature gradients to two fluxes)

(3) The number of "rigid body modes" is 1

The last one requires some clarification. A "rigid body mode" in this context is a a vector $\mathbf{u}^e_R$ of *equal temperatures at each node*. If the 4 node temperatures are equal, the whole element is at constant temperature. Postmultiplying $\mathbf{K}^e\mathbf{u}^e_R = \mathbf{0}$ because a constant temperature state produces no heat fluxes: the gradients are zero. This is analogous to the definition of RBM in structures: postmultiplying $\mathbf{K}^e$ by a RBM evaluated at the nodes gives zero forces.

Addendum B: Using the PlaneThermalFEM Notebook

The *Mathematica* Notebook and `PlaneThermalFEM.nb` (compatible with versions 4.2 and higher) is posted on the course Web site. This Notebook may be used to solve the thermal conduction problem by the finite element method using 4-node iso-P quadrilaterals, as requested in Question 5. "Read me" instructions are provided in its top cell.

The very simple thermal problem shown in Figure A.3(a), discretized with just two quadrilateral elements as depicted in Figure A.3(b), will be used to illustrate the preparation of a driver program. This driver program for this problem actually appears in Cell 12B of the posted Notebook and hence it may be quickly used for verification.

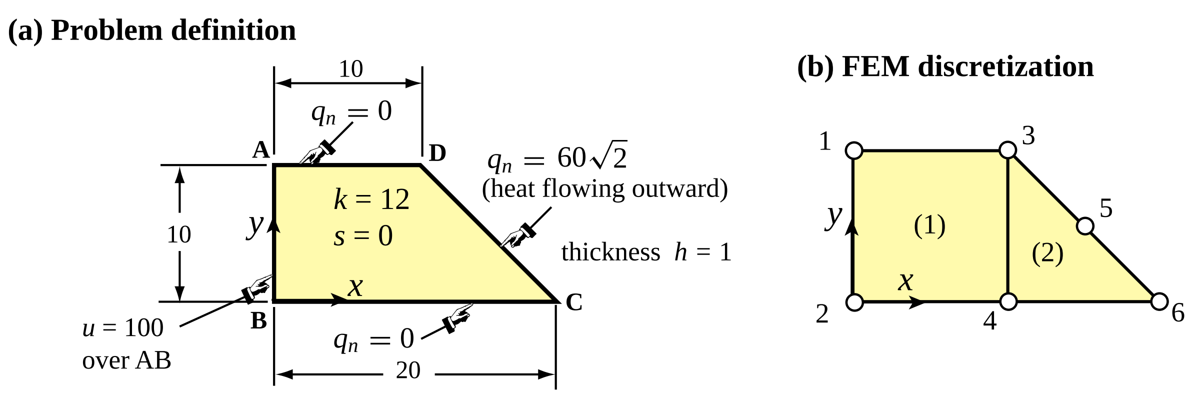

Figure A.3. A simple "dam" thermal conduction benchmark case: (a) problem definition, (b) FEM discretization with 5 nodes and 2 quadrilateral finite elements. With the properties and BCs shown in (a), the exact temperature distribution depends only on x and varies linearly from $u = +100$ on AB ($x = 0$) to $u = -100$ at node 6 ($x = 20$). This exact solution must be reproduced by any FEM mesh as long as the elements satisfy completeness.

The input cell that defines and solves this sample problem is shown in Figure A4. The input is self-explanatory except possibly for `ElemTypes` `ElementForces`, `FreedomTags` and `FreedomValues`.

`ElemTypes` should be set to `"Quad4"` for all elements. For processing, this is only a placeholder since this is the only element available in the Notebook. But it is required by the plotting programs.

`ElemForces` is a list of length equal to the number of elements. Each entry of this list, for a specific element, is of the form `{s,{q12,q23,q34,q41}}`. Here `s` is the heat source, assumed constant over the element. Entry `q12` is the

† The names "stiffness matrix" and "force vector," which are standard in the FE literature for all applications, exploit the mathematical analogy with structural mechanics. In the heat conduction problem $\mathbf{K}^e$ is physically a heat reactance matrix whereas $\mathbf{f}^e$ has dimensions of heat flux times volume.

```
(*  Define FE model *)

NodeCoordinates= N[{{0,10},{0,0},{10,10},{10,0},{15,5},{20,0}}];
ElemNodes={{1,2,4,3},{3,4,6,5}};
PrintPoissonNodeCoordinates[NodeCoordinates,
      "Node Coordinate Data",{8,4}];
numele=Length[ElemNodes]; numnod=Length[NodeCoordinates];
ElemTypes=Table["Quad4",{numele}];
ElemMaterial=Table[12,{numele}]; ElemFabrication=Table[1,{numele}];
ElemForces=Table[{0,{0,0,0,0}},{numele}];
ElemForces[[2]]={0,{0,0,N[60*Sqrt[2]],N[60*Sqrt[2]]}};
PrintPoissonElementNodesMatFab[ElemNodes,ElemMaterial,
       ElemFabrication,"Element Data",{9,4}];
PrintPoissonElementForces[ElemNodes,ElemForces,
      "Element Forces",{6,3}];
FreedomValues=FreedomTags=Table[0,{numnod}];
FreedomValues[[1]]=FreedomValues[[2]]=100; (* T @ 1,2*)
FreedomTags[[1]]=FreedomTags[[2]]=1; (* prescribed T *)
PrintPoissonFreedomActivity[FreedomTags,FreedomValues,
       "DOF Activity Data",{6,3}];
elepar={9,1.5,1,12,{0.15,1,1}};
nodpar={3.5,1.5,-8,5,12,{0.7,0.2,0.9}}; typspec={};
Plot2DMesh[NodeCoordinates,ElemTypes,ElemNodes,{},typspec,
  nodpar,elepar,{False,True,True,True,True},Automatic,
  "Plot of FEM Mesh"];
ProcessOptions={True};

(*  Solve problem and print results *)

{u,f}=LinearSolutionOfPoissonModel[NodeCoordinates,
      ElemTypes,ElemNodes,ElemMaterial,ElemFabrication,
      ElemForces,FreedomTags,FreedomValues,ProcessOptions];
PrintPoissonNodeTempForces[u,f,"Computed Solution",{6,4}];

(* Contour plot temperature distribution: 2 plotters tested *)

umax=Max[Abs[u]]; Nsub=8;
ContourPlotNodeFuncOver2DMesh[NodeCoordinates,ElemNodes,
  u,umax,Nsub,1/2,"Computed Temp Dist: Polygon Plotter"];
ContourBandPlotNodeFuncOver2DMesh[NodeCoordinates,
 ElemNodes,u,{-umax,umax,umax/10},{True,False,False,False,
 False,False},{},1/2,"Computed Temp Dist: Band Plotter"];
```

Figure A4. Driver script for benchmark problem of Figure A.3. Available in Cell 12B of posted Notebook.

specified flux $\bar{q}_n$ over side 1–2 of the element (1 and 2 being the local node numbers); `q23` is the specified flux over side 2–3 and so on. If the flux over an element side is unknown, a zero is entered. If the `q` list is completely omitted so only `s` appears, `q12=q23=q34=q41=0` is assumed.

Lists `FreedomTags` and `FreedomValues` are used to mark freedom BCs. Because there is only one DOF per node, these lists have a flat, array-like, node-by-node configuration. The tag `FreedomTags[[n]]=1` indicates that the temperature u at node `n` is prescribed, in which case its value is found in `FreedomValues[[n]]`. Else the tag is zero and `FreedomValues[[n]]` is ignored.

Executing the above cell, upon being sure that Cells 1-11 of the Notebook have been initialized, should give the answers shown in Figure A.5 under the title “Computed Solution.” Observe that the computed temperatures at nodes 3, 4, 5 and 6 of the two-element mesh are 0, 0, -50 and -100, respectively, in accordance with the exact solution. The computed temperatures at 1 and 2 are 100, as expected since those are prescribed values.

Graphic output is shown in Figure A6, which merges output from three different cells. Figure A.6(a) is the mesh plot produced by the call to module `Plot2DEMesh`. This is a recently written mesh plotter that replaced the old `Plot2DElementsAndNodes`. Arguments `typspec`, `elepar` and `nodpar`, as well as the string of logical flags, are used to control the configuration and appearance of the plot. Since these are still undocumented, the values shown for

```
Node Coordinate Data
node        x-coor        y-coor
   1        0.0000       10.0000
   2        0.0000        0.0000
   3       10.0000       10.0000
   4       10.0000        0.0000
   5       15.0000        5.0000
   6       20.0000        0.0000

Element Data
elem       nodelist  conductivity    thickness
   1  {1, 2, 4, 3}       12.0000       1.0000
   2  {3, 4, 6, 5}       12.0000       1.0000

Element Forces
elem       nodelist  source s  flux q12  flux q23  flux q34  flux q41
   1  {1, 2, 4, 3}      0.000     0.000     0.000     0.000     0.000
   2  {3, 4, 6, 5}      0.000     0.000     0.000    84.853    84.853

DOF Activity Data
node  DOF-tag  DOF-value
   1        1    100.000
   2        1    100.000
   3        0      0.000
   4        0      0.000
   5        0      0.000
   6        0      0.000

Computed Solution
node  temperature  thermal-force
   1     100.0000       600.0000
   2     100.0000       600.0000
   3       0.0000      -300.0000
   4       0.0000         0.0000
   5     -50.0000      -600.0000
   6    -100.0000      -300.0000
```

Required in Exam

Figure A5. Output from driver script of Figure A.4. Upper four tables echoprint input data, whereas the lower table gives the computed solution.

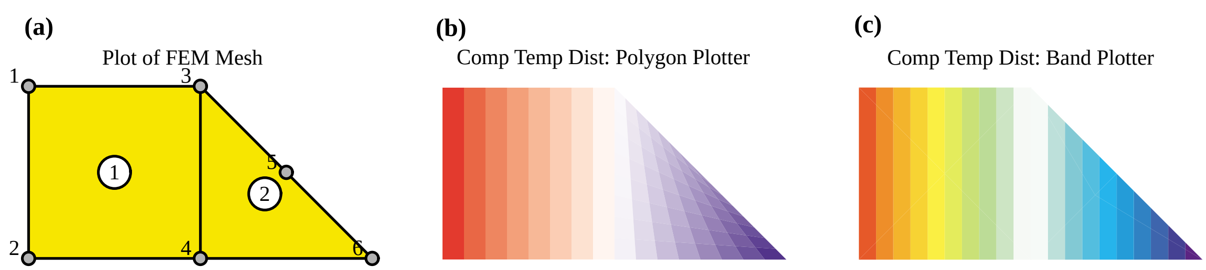

Figure A.6. Plot output from driver script of Figure A.4.

`elepar` and `nodpar` in the script of Figure A4 represent reasonable choices.

The call to the "polygon potter" `ContourPlotNodeFuncOver2DMesh` produces the contour plot of temperatures shown in Figure A.6(b). Here bright red corresponds to the highest temperature (+100 at nodes 1 and 2), bright blue corresponds to the lowest (−100 at node 6) and white to zero temperature. The value-to-color mapping is obtained by linear interpolation over the red-to-blue RGB table, which is implemented in *Mathematica* by function `RGBColor`. This mapping cannot display as many intermediate colors as the band plotter described below.

A second temperature contour plot, is shown in Figure A6(c). This one is produced by a "band plotter" which is invoked as `ContourBandPlotNodeFunctionOver2DMesh`, Here each color band is associated with certain tempearture values that range from red at the highest: +100, through white at zero, through blue at the lowest: −100. But unlike the polygonal plotter `ContourPlotNodeFuncOver2DMesh` here intermediate values are plotted using a richer value-to-hue mapping that may range over all colors of the rainbow.