Semester Project

Deterministic Attitude Estimation of a Spacecraft with Sun and Magnetic Field Sensor Measurements

ASEN 5010

Mid-Project (a-e) Report Due Date: April 11$^{\text{th}}$ 2013
Final Project Report Due Date: April 25$^{\text{th}}$ 2013

You are to study the attitude estimation of a spacecraft that is equipped with sun and magnetic field direction sensors. The numerical software will simulate measurements in body $\mathcal{B}$ and Inertial frame $\mathcal{N}$ using the Optimal Linear Attitude Estimation (OLAE) technique (see separate write-up in the draft 3rd edition book section). In the simulation will be able to simulate noise sensor states, or errors in our perceived orbit location, etc., to determine the impact on the attitude estimation algorithm. The spacecraft is orbiting the Earth on a circular orbit, and no attitude control is present. Rather, the vehicle is doing a slow tumbling motion. The following steps outline the required project steps you must undertake. They are ordered in a sequence to first provide you with the required sub-routines for sensor modeling before combining all results in a full simulation.

The orbital position vector $\boldsymbol{r}$ is given in inertial frame components as (Eq. (9.147) in S&J):

$$
{}^{\mathcal{N}}\boldsymbol{r} = r \begin{pmatrix} \cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i \\ \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i \\ \sin\theta\sin i \end{pmatrix}
$$

Here $\Omega = 20$ degrees is the longitude of the ascending node, $i = 75$ degrees is the orbit inclination angle, and θ is the orbit position angle relative to the equator crossing. Because the orbit is circular, note that

$$
\theta(t) = \theta_0 + nt
$$

where the mean orbit rate $n = \sqrt{\mu/r^3}$ is the mean orbit rate, $r = 6878$ km is the constant orbit radius. The gravitation constant μ is 398600 km^3/s^2.

The spacecraft inertia tensor is given in body frame components as

$$
{}^{\mathcal{B}}[I] = \begin{bmatrix} 25 & 2.5 & 0.5 \\ 2.5 & 20 & 0 \\ 0.5 & 0 & 15 \end{bmatrix} \text{ kg m}^2
$$

There is no external torque acting on the spacecraft, and the initial attitude states in terms of 3-2-1 Euler angles are $\psi_0 = 5$ deg, $\theta_0 = 10$ deg and $\phi_0 = -5$ deg. The initial angular rate is ${}^{\mathcal{B}}\boldsymbol{\omega}_0 = (0.4, 0.3, 0.2)$ deg/s.

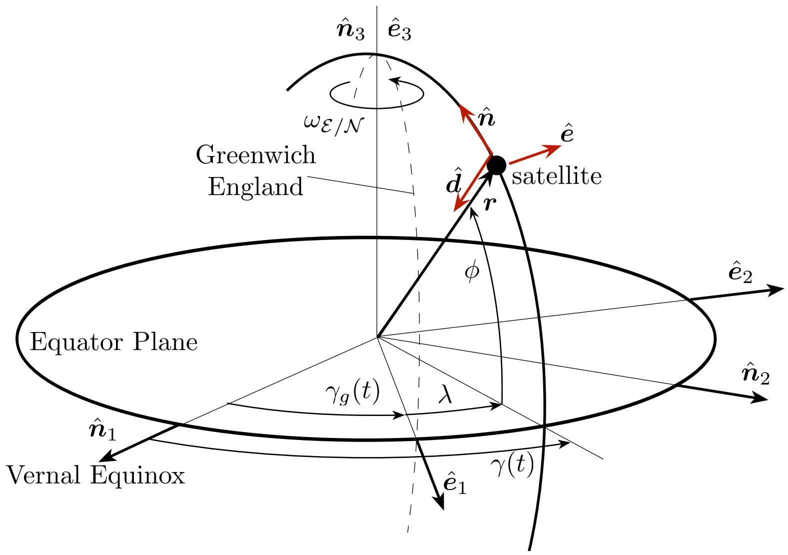

Figure 1: Coordinate Frames Illustration

a) **(5 points)** Write a sub-routine that accepts time t, and returns the inertial satellite position vector ${}^{\mathcal{N}}\boldsymbol{r}$. Plot the satellite position coordinates for 10 minutes.

b) **(10 points)** Assume the true inertial sun direction vector is ${}^{\mathcal{N}}\hat{\boldsymbol{s}} = (0, -1, 0)^T$. Write a sub-routine that has as inputs the true body orientation as a MRP set $\boldsymbol{\sigma}_{B/N}$, and returns a simulate sun direction vector measurement ${}^{\mathcal{B}}\hat{\boldsymbol{s}}$. Show the development of this math in your report.

c) **(10 points)** Consider the North-East-Down (NED) topographic frame $\mathcal{T} : \{\hat{\boldsymbol{n}}, \hat{\boldsymbol{e}}, \hat{\boldsymbol{d}}\}$ (see Figure 1). Develop the DCM $[TE]$ that maps from a Earth-fixed $\mathcal{E} : \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ to this NED frame $\mathcal{T}$. Show the development of this DCM in your report.

d) **(5 points)** The Earth-fixed frame $\mathcal{E}$ and the Earth-Centered-Inertial frame N differ by a 3-axis rotation by the angle γ. This angle γ is called the Greenwhich local sidereal time. Assume the Earth rotates at a constant $\omega_{E/N}$ =361 degrees per day, develop a sub-routine that returns the DCM $[EN]$ as a function of time. Set the initial $\gamma(t_0) = 20$ degrees. Show the development of this DCM in your report.

e) **(10 points)** A common magnetic field M model is the tilted-centered dipole model given by

$$\begin{bmatrix} M_{\text{North}} \\ M_{\text{East}} \\ M_{\text{Down}} \end{bmatrix} = -\left(\frac{r_{\text{eq}}}{r}\right)^3 \begin{bmatrix} -\cos\phi & \sin\phi\cos\lambda & \sin\phi\sin\lambda \\ 0 & \sin\lambda & -\cos\lambda \\ -2\sin\phi & -2\cos\phi\cos\lambda & -2\cos\phi\sin\lambda \end{bmatrix} \begin{bmatrix} 29900 \\ 1900 \\ -5530 \end{bmatrix} \text{ nT}$$

where nT is the unit of nano-Tesla, $r_{\text{eq}} = 6378$ km is the Earth's equatorial radius, r is the current orbit radius, λ is the satellite longitude relative to the Earth fixed frame, and ϕ is the satellite latitude. Note that this magnetic field direction $\boldsymbol{M}$ is given in NED frame components. Write a sub-routine which accepts an inertial satellite position vector ${}^{\mathcal{N}}\boldsymbol{r}$ and time t, and returns the magnetic field unit direction vector ${}^{\mathcal{N}}\hat{\boldsymbol{M}}$ in inertial frame vector components. In the report, outline the mathematics, and show the simulated results for $t = 0$ to 10 minutes.

f) **(10 points)** Write a numerical simulation that integrates both the true attitude $\boldsymbol{\sigma}_{B/N}$ and rates $\boldsymbol{\omega}_{B/N}$. In the code, use MRPs to express the orientation. Show the attitude and rate coordinates for a 10 minute simulation using the initial conditions on the first page.

g) **(10 points)** Integrate the sun and magnetic field sensor simulations into this dynamic simulation. Plot the expected sun and magnetic field direction measurements ${}^{\mathcal{B}}\hat{\boldsymbol{s}}$ and ${}^{\mathcal{B}}\hat{\boldsymbol{m}}$ for 10 minutes of simulation time, assuming no sensor corruptions or noise is present.

h) **(20 points)** Read up on the OLAE attitude estimation method. Outline in the report the algorithm, and implement this into your simulation. Assuming perfect sensor measurements, show that your estimated body orientation $\bar{\mathcal{B}}$ and true body orientation $\mathcal{B}$ are the same. Assume here that the weights on the measurements are the same.

i) **(20 points)** Apply corruptions to your sensor measurements, and numerically evaluate the impact on the estimate attitude accuracy (still using OLAE method). Discuss in the report the type of corruptions you are modeling, and justify why you chose the values of corruptions.

The final report must be written as an AIAA conference paper. Paper templates are available in MS Word and LaTeX on the CULearn site under the projects folder. Your final report must have an abstract, introduction, problem statement, sections explaining your development, numerical simulation, as well as a conclusion. Points will be deducted for poor presentation. The work you present must be your own. Be sure to properly reference other papers, figures or books that you use in support of this work. You need to submit your final report in electronic format (PDF preferred) to CULearn in response to the project report assignment.

- p. 134, After Section 3.9 Insert new section. With the adjusted section numbering, this should become section 3.10:

3.10 Deterministic Attitude Estimation

3.10.1 Attitude Estimation Overview

Thus far this chapter has shown how to describe a three-dimensional attitude of a rigid body or coordinate frame. This section illustrates select algorithms on how to determine the three-dimensional orientation from a series of attitude sensor readings. Attitude sensors often provide measurements in the form of unit direction vectors. For example, these measurements could be the sun-direction, a magnetic field heading vector, or a vector pointing to a known star. The process of combining various heading vector measurements, all taken at the same time, into a single three-dimensional attitude is called the deterministic attitude estimation problem.[34] In contrast, if attitude measurements at different times are combined with rate gyro information, this is called the dynamic attitude estimation problem, or an attitude filter solution. A common approach for the latter is to employ a Kalman filter. The detailed development of such a filter is beyond the scope of this book. Good references on aerospace-related Extended Kalman Filtering (EKF) are References 35 and 36. Popular quaternion-based Kalman filter solutions include the Additive Extended Kalman Filter (AEKF)[37] and the Multiplicative Extended Kalman Filter (MEKF).[35] Both the AEKF and MEKF necessitate restoring the norm constraint after the update. A recently introduced quaternion EKF called the norm-constraint Kalman Filter[38] avoids having to restore the quaternion constraint. Another recent attitude filter is the MRP-based extended Kalman filter (MRP-EKF) presented in Reference 39. By exploiting the original and shadow sets of MRPs and associated covariances, this filter is simpler to implement, faster to evaluate computationally, and retains the first-order accuracy of the quaternion-based filters. Appendix I provides a brief overview of this MRP-EKF method. The solutions of this section's deterministic attitude determinations would be the input to these EKFs.

Returning to the deterministic attitude estimation problem, assume a spacecraft is equipped with two attitude sensors. A fine sun sensor provides a unit direction vector $\hat{\boldsymbol{s}}$ pointing to the sun, while a magnetometer provides a magnetic field direction vector $\hat{\boldsymbol{m}}$ measurement. These attitude measurement vectors are taken in body frame components. Naturally, the accuracy of attitude sensors can vary greatly, often coupled with the sensor price. Let $\mathcal{B}$ be a body fixed frame, whose attitude relative to an inertial frame $\mathcal{N}$ is to be determined. The two measurements are then given by the unit direction vectors expressed in body frame components:

$$ {}^{\mathcal{B}}\hat{\boldsymbol{m}} \qquad {}^{\mathcal{B}}\hat{\boldsymbol{s}} $$

The attitude estimation problem requires knowledge of these measurement vectors in the inertial frame:

$$ {}^{\mathcal{N}}\hat{\boldsymbol{m}} \qquad {}^{\mathcal{N}}\hat{\boldsymbol{s}} $$

Thus, the spacecraft's orbit must be known to determine what physical quantities the attitude sensors would measure at this location. For the magnetometer, an onboard magnetic field model is needed to determine the inertial magnetic field ${}^{\mathcal{N}}\hat{\boldsymbol{m}}$ at the spacecraft's current position. To use a sun sensor, the Julian date must be known to determine the Earth's location about the sun to evaluate the inertial sun-direction vector ${}^{\mathcal{N}}\hat{\boldsymbol{s}}$.

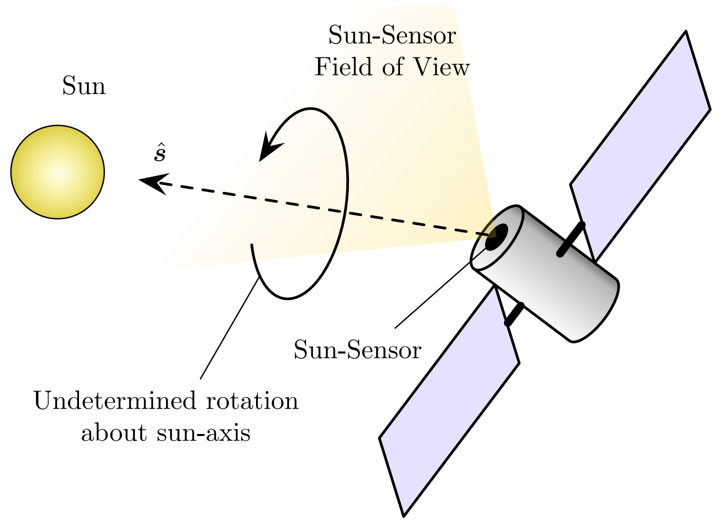

Figure 3.13.1: Illustration of Underdetermined Attitude Estimation

The goal of the deterministic attitude estimation process is to find the direction cosine matrix $[\bar{B}N]$ which will map these body frame observations into the known inertial quantities.

$$^{\mathcal{B}}\hat{\boldsymbol{m}} = [\bar{B}N]^{\mathcal{N}}\hat{\boldsymbol{m}} \tag{3.209.1a}$$

$$^{\mathcal{B}}\hat{\boldsymbol{s}} = [\bar{B}N]^{\mathcal{N}}\hat{\boldsymbol{s}} \tag{3.209.1b}$$

The over-bar notation is used to denote an estimate of the true orientation. Thus, $\bar{\mathcal{B}}$ is the estimated body frame, while $\mathcal{B}$ is the true body frame.

Note that each attitude observation is in the form of a unit direction vector. Thus, a single attitude observation will only provide information on two degrees of rotational freedom. While the observation vector is a three-dimensional vector, the unit length constraint implies that only two independent attitude degrees of freedom can be observed with this measurement. To visualize this, consider a spacecraft where a single sun sensor is used. In this case, the orientation of one body-fixed axis (the sun sensor axis) of the spacecraft is determined. However, the spacecraft orientation about this sun-axis is unknown. If the vehicle were to rotate about this axis, the identical sun-vector measurement would be taken as illustrated in Figure 3.13.1. As a consequence, the minimum number of attitude observation vectors required for a three-dimensional attitude estimate is 2. One observation leaves the problem underdetermined. However, adding the second vector results in an overdetermined estimation problem. Thus, with attitude observation vectors, it is not possible to have "just enough" information to have a uniquely determined three-dimensional attitude. Rather, one observation is too little, and two are already too much. The algorithms in this section account for this issue in different ways.

The following sections cover the TRIAD algorithm, Davenport's method, the QUaternion ESTimator (QUEST), as well as the Optimal Linear Attitude Estimator (OLAE). The TRIAD and QUEST algorithms have been very popular in the last few decades. Davenport's method is a classical result which many newer methods, including QUEST, seek to refine. OLAE is a recent solution that is shown to be simple to implement, and achieve equivalent accuracy to QUEST. Other methods of interest include FOAM,[40] Markley's SVD method,[41] the EULER-2 and EULER-n methods,[42] the ESOQ[43] and ESOQ-2[44] algorithms, and MRAD[45]. The reader is referred to the original publications to learn more about them.

3.10.2 TRIAD Method

The TRIAD method[46,47,48] provides a simple algorithm to determine the unknown attitude measure $[BN]$ using two, and only two vector observations $\hat{\boldsymbol{v}}_1$ and $\hat{\boldsymbol{v}}_2$. This algorithm establishes a third, intermediate coordinate frame called the triad frame $\mathcal{T} : \{\hat{\boldsymbol{t}}_1, \hat{\boldsymbol{t}}_2, \hat{\boldsymbol{t}}_3\}$. As before, the inertial vectors ${}^{\mathcal{N}}\hat{\boldsymbol{v}}_i$ must be know a priori.

Without loss of generality, let $\hat{\boldsymbol{v}}_1$ be the more accurate measurement of the two. The first $\mathcal{T}$-frame axis is then defined as

$$\hat{\boldsymbol{t}}_1 = \hat{\boldsymbol{v}}_1 \tag{3.209.2}$$

Note that all the information of $\hat{\boldsymbol{v}}_1$ is being used in this $\hat{\boldsymbol{t}}_1$ definition. Next, the second $\mathcal{T}$-frame axis is chosen to be orthogonal to both attitude vector measurements using

$$\hat{\boldsymbol{t}}_2 = \frac{\hat{\boldsymbol{v}}_1 \times \hat{\boldsymbol{v}}_2}{|\hat{\boldsymbol{v}}_1 \times \hat{\boldsymbol{v}}_2|} \tag{3.209.3}$$

Here only part of the second observation vector $\hat{\boldsymbol{v}}_2$ is being used. Thus, in essence, only 1.5 observation vectors are being used in the TRIAD method. This is why it is a good choice to let $\hat{\boldsymbol{v}}_1$ be the more accurate measurement to increase the quality of the attitude estimate. Finally, $\hat{\boldsymbol{t}}_3$ is found by completing the right-handed coordinate frame:

$$\hat{\boldsymbol{t}}_3 = \hat{\boldsymbol{t}}_1 \times \hat{\boldsymbol{t}}_2 \tag{3.209.4}$$

The corresponding $\mathcal{T}$-frame is illustrated in Figure 3.13.2

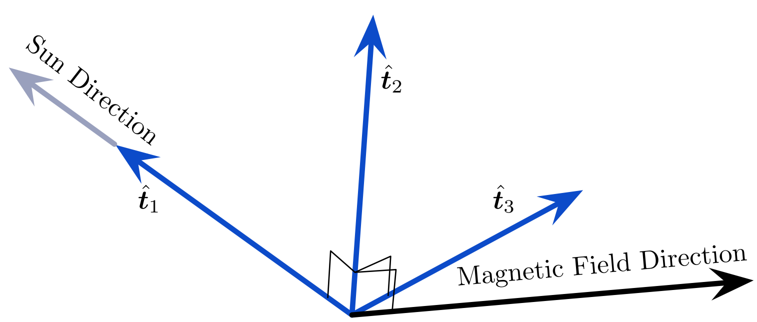

Figure 3.13.2: Triad Coordinate Frame Definition Illustration

Given the two attitude measurements ${}^{\mathcal{B}}\hat{\boldsymbol{v}}_1$ and ${}^{\mathcal{B}}\hat{\boldsymbol{v}}_2$, the $\mathcal{T}$-frame axes are computed in body frame $\mathcal{B}$ vector components using:

$${}^{\mathcal{B}}\hat{\boldsymbol{t}}_1 = {}^{\mathcal{B}}\hat{\boldsymbol{v}}_1 \tag{3.209.5a}$$

$${}^{\mathcal{B}}\hat{\boldsymbol{t}}_2 = \frac{({}^{\mathcal{B}}\hat{\boldsymbol{v}}_1) \times ({}^{\mathcal{B}}\hat{\boldsymbol{v}}_2)}{|({}^{\mathcal{B}}\hat{\boldsymbol{v}}_1) \times ({}^{\mathcal{B}}\hat{\boldsymbol{v}}_2)|} \tag{3.209.5b}$$

$${}^{\mathcal{B}}\hat{\boldsymbol{t}}_3 = ({}^{\mathcal{B}}\hat{\boldsymbol{t}}_1) \times ({}^{\mathcal{B}}\hat{\boldsymbol{t}}_2) \tag{3.209.5c}$$

Similarly, given the known inertial vector components of these two attitude observation vectors, the $\mathcal{T}$-frame axes are expressed in inertial frame $\mathcal{N}$ components using:

$${}^{\mathcal{N}}\hat{\boldsymbol{t}}_1 = {}^{\mathcal{N}}\hat{\boldsymbol{v}}_1 \tag{3.209.6a}$$

$${}^{\mathcal{N}}\hat{\boldsymbol{t}}_2 = \frac{({}^{\mathcal{N}}\hat{\boldsymbol{v}}_1) \times ({}^{\mathcal{N}}\hat{\boldsymbol{v}}_2)}{|({}^{\mathcal{N}}\hat{\boldsymbol{v}}_1) \times ({}^{\mathcal{N}}\hat{\boldsymbol{v}}_2)|} \tag{3.209.6b}$$

$${}^{\mathcal{N}}\hat{\boldsymbol{t}}_3 = ({}^{\mathcal{N}}\hat{\boldsymbol{t}}_1) \times ({}^{\mathcal{N}}\hat{\boldsymbol{t}}_2) \tag{3.209.6c}$$

Having defined the triad-frame axes in both body and inertial frame components, we can use these matrix descriptions of the $\mathcal{T}$-frame vectors to develop direction cosine matrices relating the estimated body and inertial frame orientation relative to the new triad frame:

$$[\bar{B}T] = \begin{bmatrix} {}^{\mathcal{B}}\hat{\boldsymbol{t}}_1 & {}^{\mathcal{B}}\hat{\boldsymbol{t}}_2 & {}^{\mathcal{B}}\hat{\boldsymbol{t}}_3 \end{bmatrix} \tag{3.209.7a}$$

$$[NT] = \begin{bmatrix} {}^{\mathcal{N}}\hat{\boldsymbol{t}}_1 & {}^{\mathcal{N}}\hat{\boldsymbol{t}}_2 & {}^{\mathcal{N}}\hat{\boldsymbol{t}}_3 \end{bmatrix} \tag{3.209.7b}$$

At this point it is straight forward to assemble the desired attitude matrix $[\bar{B}N]$ using

$$[\bar{B}N] = [\bar{B}T][NT]^T \tag{3.209.8}$$

The DCM attitude measure can be readily converted to other attitude descriptions if needed. While the TRIAD method is simple to implement and evaluate, it does suffer from the issue that not all information of the two vector observations is being used. Further, the classical TRIAD algorithm can only be used with two observations. Thus, even if more than two sensors are available, this extra attitude sensing capability cannot be exploited with this algorithm. The next methods discussed in this section seek to address these issues by employing a best least-squares fit across a general number of observations.

Example 3.12.1

To test the TRIAD algorithm, assume the true spacecraft body frame is $\mathcal{B}$, while the estimated body frame orientation is given through $\bar{\mathcal{B}}$. The true attitude of $\mathcal{B}$ relative to the inertial frame $\mathcal{N}$ is described by a 30°, 20° and -10° 3-2-1 Euler axis rotation. The resulting $[BN]$ matrix is

$$[BN] = \begin{bmatrix} 0.8138 & 0.4698 & -0.3420 \\ -0.5438 & 0.8232 & -0.1632 \\ 0.2049 & 0.3188 & 0.9254 \end{bmatrix}$$

By knowing the location of the spacecraft, and the physical environment (such as having an onboard Earth magnetic field model), the inertial vector components of the two observations are known a priori. For this example, we assume they have the following values:

$${}^{\mathcal{N}}\hat{\boldsymbol{v}}_1 = {}^{\mathcal{N}}\!\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad {}^{\mathcal{N}}\hat{\boldsymbol{v}}_2 = {}^{\mathcal{N}}\!\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Using Eq. (3.209.1), the true unit vectors each attitude sensor should provide are

$${}^{\mathcal{B}}\hat{\boldsymbol{v}}_{1,\text{true}} = {}^{\mathcal{B}}\!\begin{pmatrix} 0.8138 \\ -0.5438 \\ 0.2049 \end{pmatrix} \qquad {}^{\mathcal{B}}\hat{\boldsymbol{v}}_{2,\text{true}} = {}^{\mathcal{B}}\!\begin{pmatrix} -0.3420 \\ -0.1632 \\ 0.9250 \end{pmatrix}$$

However, no attitude measurement is devoid of sensor errors. To simulate imperfect sensor readings, the following two corrupted sensor vectors are used to test the TRIAD algorithm:

$${}^{\mathcal{B}}\hat{\boldsymbol{v}}_1 = {}^{\mathcal{B}}\!\begin{pmatrix} 0.8190 \\ -0.5212 \\ 0.2242 \end{pmatrix} \qquad {}^{\mathcal{B}}\hat{\boldsymbol{v}}_2 = {}^{\mathcal{B}}\!\begin{pmatrix} -0.3138 \\ -0.1584 \\ 0.9362 \end{pmatrix}$$

Without any indication which sensor is more accurate, the first sensor is selected as the more accurate reading. The resulting body-frame triad vectors are

$$
{}^{\mathcal{B}}\hat{\boldsymbol{t}}_1 = {}^{\mathcal{B}}\!\begin{pmatrix} 0.8190 \\ -0.5282 \\ 0.2242 \end{pmatrix} \qquad {}^{\mathcal{B}}\hat{\boldsymbol{t}}_2 = {}^{\mathcal{B}}\!\begin{pmatrix} -0.4593 \\ -0.8377 \\ -0.2956 \end{pmatrix} \qquad {}^{\mathcal{B}}\hat{\boldsymbol{t}}_3 = {}^{\mathcal{B}}\!\begin{pmatrix} 0.3439 \\ 0.1392 \\ -0.9286 \end{pmatrix}
$$

while the inertial frame components of $\hat{\boldsymbol{t}}_i$ are:

$$
{}^{\mathcal{N}}\hat{\boldsymbol{t}}_1 = {}^{\mathcal{N}}\!\begin{pmatrix} 1.0000 \\ 0.0000 \\ 0.0000 \end{pmatrix} \qquad {}^{\mathcal{N}}\hat{\boldsymbol{t}}_2 = {}^{\mathcal{N}}\!\begin{pmatrix} 0.0000 \\ -1.0000 \\ 0.0000 \end{pmatrix} \qquad {}^{\mathcal{N}}\hat{\boldsymbol{t}}_3 = {}^{\mathcal{N}}\!\begin{pmatrix} 0.0000 \\ 0.0000 \\ -1.0000 \end{pmatrix}
$$

Evaluating the direction cosine matrices $[\bar{B}T]$ and $[NT]$ yields

$$
[\bar{B}T] = \begin{bmatrix} 0.8190 & -0.4593 & 0.3439 \\ -0.5282 & -0.8377 & 0.1392 \\ 0.2242 & -0.2956 & -0.9286 \end{bmatrix} \qquad [NT] = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 \end{bmatrix}
$$

Finally, the estimated attitude description $[\bar{B}N]$ is obtained using

$$
[\bar{B}N] = [\bar{B}T] \cdot [NT]^T = \begin{bmatrix} 0.8190 & 0.4593 & -0.3439 \\ -0.5282 & 0.8377 & -0.1392 \\ 0.2242 & 0.2956 & 0.9286 \end{bmatrix}
$$

The attitude estimation error is found by evaluating the relative attitude of the estimate $\bar{\mathcal{B}}$ relative to the true attitude of $\mathcal{B}$.

$$
[\bar{B}B] = [\bar{B}N] \cdot [BN]^T = \begin{bmatrix} 0.9999 & -0.0112 & -0.0041 \\ 0.0114 & 0.9995 & 0.0301 \\ 0.0037 & -0.0301 & 0.9995 \end{bmatrix}
$$

While this $[\bar{B}B]$ attitude estimation error matrix provides a complete three-dimensional error measure, it is not very convenient. A scalar attitude estimation error value is obtain by computing the principle rotation angle of $[\bar{B}B]$. For the presented scenario, the principle attitude error is $\Phi = 1.8548^{\text{o}}$.

3.10.3 Davenport's q-Method

In 1965, Wahba presented the deterministic attitude estimation as a solution to the following non-negative cost function:[49]

$$
J\left([\bar{B}N]\right) = \frac{1}{2}\sum_{k=1}^{N} w_k \left|{}^{\mathcal{B}}\hat{\boldsymbol{v}}_k - [\bar{B}N]\, {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right|^2 \geq 0 \tag{3.209.9}
$$

Her formulation allows for a general set of N vector observations to be processed, where each sensor reading has an associated weight w_k. Note that only the relative magnitude of w_k is important. Thus, having equal weights of 10 or 1 will yield the same result. The optimal attitude estimate $[\bar{B}N]$ will provide the best least-squares answer satisfying Wahba's cost function.

Solving this cost function directly in terms of $[\bar{B}N]$ can be challenging because the direction cosine matrix is a heavily over-parameterized attitude description. Davenport was the first to show that Wahba's problem can be rewritten as a quadratic cost function in terms of the quaternions (Euler Parameters), hence the name Davenport's q-method.[50] To illustrate this result, the cost function in Eq. (3.209.9) is rewritten as

$$J = \frac{1}{2}\sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\hat{\boldsymbol{v}}_k - [\bar{B}N]\ {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right)^T \left({}^{\mathcal{B}}\hat{\boldsymbol{v}}_k - [\bar{B}N]\ {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right) \tag{3.209.10}$$

Carrying out the matrix algebra, and making use of $\hat{\boldsymbol{v}}_k^T\hat{\boldsymbol{v}}_k = 1$, the cost function is simplified to

$$J = \sum_{k=1}^{N} w_k \left(1 - \left({}^{\mathcal{B}}\hat{\boldsymbol{v}}_k\right)^T [\bar{B}N]\ {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right) \tag{3.209.11}$$

Note that minimizing the cost function J is equivalent to maximizing the auxiliary function g given by

$$g([\bar{B}N]) = \sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\hat{\boldsymbol{v}}_k\right)^T [\bar{B}N]\,{}^{\mathcal{N}}\hat{\boldsymbol{v}}_k \tag{3.209.12}$$

After substituting the direction cosine matrix definition in terms of the Euler parameters in Eq. (3.93) into Eq. (3.209.12) and simplifying, the auxiliary cost function g is expressed in the elegant Euler parameter quadratic form:

$$g(\bar{\boldsymbol{\beta}}) = \bar{\boldsymbol{\beta}}^T[K]\bar{\boldsymbol{\beta}} \tag{3.209.13}$$

Here $\bar{\boldsymbol{\beta}}$ is the estimated quaternion set relating the body frame to the inertial frame. Thus, to solve Wahba's problem, the optimal set $\bar{\boldsymbol{\beta}} = (\bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)$ must be found such that $g(\bar{\boldsymbol{\beta}})$ is maximized. The 4×4 matrix $[K]$ is defined as

$$[K] = \begin{bmatrix} \sigma & Z^T \\ Z & S - \sigma I_{3\times 3} \end{bmatrix} \tag{3.209.14}$$

where

$$[B] = \sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\hat{\boldsymbol{v}}_k\right)\left({}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right)^T \tag{3.209.15a}$$

$$[S] = [B] + [B]^T \tag{3.209.15b}$$

$$\sigma = \operatorname{tr}([B]) \tag{3.209.15c}$$

$$[Z] = \begin{bmatrix} B_{23} - B_{32} & B_{31} - B_{13} & B_{12} - B_{21} \end{bmatrix}^T \tag{3.209.15d}$$

To maximize g subject to the Euler parameter constraint function $\bar{\boldsymbol{\beta}}^T\bar{\boldsymbol{\beta}} = 1$, the augmented function g' is used

$$g'(\bar{\boldsymbol{\beta}}) = \bar{\boldsymbol{\beta}}^T[K]\bar{\boldsymbol{\beta}} - \lambda(\bar{\boldsymbol{\beta}}^T\bar{\boldsymbol{\beta}} - 1) \tag{3.209.16}$$

where λ is a Lagrange multiplier. The maximum constrained solution of g' is found by setting its derivative with respect to $\bar{\boldsymbol{\beta}}$ to zero.

$$\frac{\mathrm{d}g'}{\mathrm{d}\bar{\boldsymbol{\beta}}} = 2[K]\bar{\boldsymbol{\beta}} - 2\lambda\bar{\boldsymbol{\beta}} = 0 \tag{3.209.17}$$

This leads to the following elegant necessary condition for an optimal solution:

$$[K]\bar{\boldsymbol{\beta}} = \lambda\bar{\boldsymbol{\beta}} \tag{3.209.18}$$

Note that the optimization problem has been rewritten as an eigenvector-eigenvalue problem. Because $[K]$ is a 4×4 matrix, this equation will yield 4 possible eigen-solutions. The question is, which of these solutions is the optimal attitude estimate. Substituting Eq. (3.209.18) into Eq. (3.209.13), the auxiliary cost function g is simplified to

$$g(\bar{\boldsymbol{\beta}}) = \bar{\boldsymbol{\beta}}^T[K]\bar{\boldsymbol{\beta}} = \bar{\boldsymbol{\beta}}^T\lambda\bar{\boldsymbol{\beta}} = \lambda\bar{\boldsymbol{\beta}}^T\bar{\boldsymbol{\beta}} = \lambda \tag{3.209.19}$$

Thus, Davenport's solution to Wahba's problem, finding the maximum value of g, is reduced to finding the largest eigenvalue λ of the $[K]$ matrix. After finding this optimal eigenvalue $\lambda_{\max}$, the associated unit-length eigenvector is the desired optimal attitude estimate $\bar{\boldsymbol{\beta}}$.

In contrast to the TRIAD method, this algorithm has the benefit that an arbitrary number of vector observations can be processed at once. Further, the sensor accuracy information of each observation can be included through the weight w_k. If the sensor provides a more accurate reading, the relative weight with respect to the less accuracy sensor readings should be larger. A drawback to this method is that solving a 4×4 eigenvector-eigenvalue problem is computationally challenging. The following QUEST method will seek to have the same advantages as Davenport's q-method, but with a significantly reduced computation overhead.

Example 3.12.2

To illustrate how to numerically employ Davenport's method, the two sensor measurements of Example 3.12.1 are reused. The weights are set to be the same with $w_1 = w_2 = 1$ as both sensors are assumed to provide equivalent accuracy. The first step is to evaluate the components of the 4×4 $[K]$ matrix:

$$[B] = w_1({}^{\mathcal{B}}\hat{\boldsymbol{v}}_1)({}^{\mathcal{N}}\hat{\boldsymbol{v}}_1)^T + w_1({}^{\mathcal{B}}\hat{\boldsymbol{v}}_2)({}^{\mathcal{N}}\hat{\boldsymbol{v}}_2)^T = \begin{bmatrix} 0.8190 & 0.0000 & -0.3138 \\ -0.5282 & 0.0000 & -0.1584 \\ 0.2242 & 0.0000 & 0.9362 \end{bmatrix}$$

$$[S] = [B] + [B]^T = \begin{bmatrix} 1.638 & -0.5282 & -0.0896 \\ -0.5282 & 0.0000 & -0.1584 \\ -0.0896 & -0.1584 & 1.8724 \end{bmatrix}$$

$$\sigma = B_{11} + B_{22} + B_{33} = 1.7552$$

$$Z = \begin{bmatrix} B_{23} - B_{32} \\ B_{31} - B_{13} \\ B_{12} - B_{21} \end{bmatrix} = \begin{bmatrix} -0.1584 \\ 0.5380 \\ 0.5282 \end{bmatrix}$$

These results are next assembled into the $[K]$ matrix to yield:

$$[K] = \begin{bmatrix} \sigma & Z^T \\ Z & S - \sigma I_{3\times3} \end{bmatrix} = \begin{bmatrix} 1.7552 & -0.1584 & 0.538 & 0.5282 \\ -0.1584 & -0.1172 & -0.5282 & -0.0896 \\ 0.5380 & -0.5282 & -1.7552 & -0.1584 \\ 0.5282 & -0.0896 & -0.1584 & 0.1172 \end{bmatrix}$$

Using a numerical solver such as Matlab or Mathematica, the four possible eigenvalues λ_i and associated eigenvectors $\bar{\boldsymbol{\beta}}_i$ (normalized to unit-length) are found to be

$$\lambda_1 = 1.9997 \quad \lambda_2 = -1.9997 \quad \lambda_3 = 0.0365 \quad \lambda_4 = -0.0365$$

$$\bar{\boldsymbol{\beta}}_1 = \begin{bmatrix} 0.9481 \\ -0.1172 \\ 0.1414 \\ 0.2597 \end{bmatrix} \quad \bar{\boldsymbol{\beta}}_2 = \begin{bmatrix} 0.1414 \\ -0.2597 \\ -0.9481 \\ -0.1172 \end{bmatrix} \quad \bar{\boldsymbol{\beta}}_3 = \begin{bmatrix} -0.1008 \\ 0.5704 \\ -0.2665 \\ 0.7704 \end{bmatrix} \quad \bar{\boldsymbol{\beta}}_4 = \begin{bmatrix} -0.2665 \\ -0.7704 \\ 0.1008 \\ 0.5704 \end{bmatrix}$$

The optimal attitude estimate satisfying Wahba's problem is the unit-length eigenvector associated with the largest eigenvalue, in this numerical example it is $\bar{\boldsymbol{\beta}}_1$. Using Eq. (3.93) to map these Euler parameters to the equivalent direction cosine matrix yields

$$[\bar{B}N(\bar{\boldsymbol{\beta}}_1)] = \begin{bmatrix} 0.8252 & 0.4593 & -0.3289 \\ -0.5255 & 0.8377 & -0.1488 \\ 0.2072 & 0.2956 & 0.9326 \end{bmatrix}$$

Comparing this orientation to the true orientation yields the following estimation error attitude matrix.

$$[\bar{B}B] = [\bar{B}N] \cdot [BN]^T = \begin{bmatrix} 0.9998 & -0.0170 & 0.0111 \\ 0.0168 & 0.9996 & 0.0217 \\ -0.0114 & -0.0215 & 0.9997 \end{bmatrix}$$

The corresponding principal error angle is $\Phi = 1.699^{\circ}$. Davenport's method typically provides attitude estimates which are more accurate than the TRIAD method. However, note that this is not guaranteed. It is possible to construct particular errors where the TRIAD method (by not using all the vector observation information) actually yields a better answer. While such results are possible, they are statistically unlikely. In particular, if more than 2 observations vectors are being processed, as can occur when evaluating star heading vectors in a star tracker, this method will yield more robust results. However, solving the 4×4 eigenvalue-eigenvector problem is a large computation overhead with this method.

3.10.4 QUEST

While Davenport's method transforms the solution search for Wahba's problem into an elegant quadratic form of Euler parameters, its numerical implementation is challenging due to the evaluation of the largest eigenvalue. In 1981 Malcom Shuster published his seminal work on the Quaternion Estimator, or QUEST algorithm.[47] This deterministic attitude estimator has since become a very popular solution, and been employed on numerous missions. QUEST also solves Wahba's problem to find the attitude estimate, and builds on Davenport's earlier solution. The main advantage of QUEST is the speed at which the numerical computation can be carried out. Since 1981, other algorithms by Danielle Mortari have shown even faster speeds to solve Wahba's problem such as ESOQ[43] and ESOQ-2[44]. However, QUEST remains to this day a popular algorithm.

When using Davenport's method, the critical solution component is the evaluation of the maximum eigenvalue $\lambda_{\max}$ of the $[K]$ matrix. However, note that in Example 3.12.2 the maximum eigenvalue is 1.9996, while the sum of the weights is 2. It is easy for the reader to try different weights. In each case the sum of the weights is very close to the maximum eigenvalue. Let us investigate why. For the optimal attitude, the associated cost function is $g = \lambda_{\max}$. Substituting this insight into the Wahba's original cost function in Eq. (3.209.11) leads to

$$J = \sum_{k=1}^{N} w_k - g = \sum_{k=1}^{N} w_k - \lambda_{\max} \tag{3.209.20}$$

If the attitude sensor noise and corruption levels are reasonable low, then the optimal attitude should yield a near-zero cost function J. Thus, applying $J \approx 0$ to Eq. (3.209.20) leads to the important insight:

$$\lambda_{\max} \approx \sum_{k=1}^{N} w_k \tag{3.209.21}$$

While this estimate of $\lambda_{\max}$ can be used to quickly determine an attitude, with a few numerical iterations the maximum eigenvalue estimate can be determined to machine precision.

The eigenvalues of $[K]$ must satisfy the characteristic equation

$$f(s) = \det([K] - s[I_{3\times3}]) = 0 \tag{3.209.22}$$

which is a 4^{th} order polynomial. To search for the 4 possible eigenvalues is to search for the 4 roots of $f(s)$. With the insight of Eq. (3.209.21), a good estimate to the maximum eigenvalue is already available. Newton's root solving method allows for very rapid convergence of the true optimal eigenvalue using the following iterative procedure of maximum eigenvalue estimates λ_i:

$$\begin{aligned}
\lambda_0 &= \sum_{k=1}^{N} w_k \\
\lambda_1 &= \lambda_0 - \frac{f(\lambda_0)}{f'(\lambda_0)} \\
&\vdots \\
\lambda_{\max} = \lambda_i &= \lambda_{i-1} - \frac{f(\lambda_{i-1})}{f'(\lambda_{i-1})}
\end{aligned}$$

Having a quick-to-evaluate maximum eigenvalue $\lambda_{\max}$, Eq. (3.209.18) can now be solved directly for the desired attitude estimate by using the Classical Rodrigues Parameter (CRP) set $\bar{\boldsymbol{q}}_i = \bar{\beta}_i/\bar{\beta}_0$. Dividing Eq. (3.209.18) by $\bar{\beta}_0$ yields

$$\begin{bmatrix} \sigma & Z^T \\ Z & S - \sigma I_{3\times3} \end{bmatrix} \begin{pmatrix} 1 \\ \bar{\boldsymbol{q}} \end{pmatrix} = \lambda_{\max} \begin{pmatrix} 1 \\ \bar{\boldsymbol{q}} \end{pmatrix} \tag{3.209.23}$$

The second line of this matrix equation is solved directly for the optimal CRP set using

$$\bar{\boldsymbol{q}} = \Big((\lambda_{\text{max}} + \sigma)[I_{3\times3}] - [S]\Big)^{-1}[Z] \tag{3.209.24}$$

The CRP attitude measure can easily be translated to any desired attitude description.

One issue with QUEST becomes readily apparent when the solution is written in terms of CRPs, a singular attitude description. If the matrix inverse in Eq. (3.209.24) is near-singluar, then the CRPs are approaching infinity. This indicates that the estimated body frame is nearly 180° apart from the inertial frame. To avoid this kinematic singularity, the concept of successive rotations is employed. To determine the attitude of a rigid body, an infinity of body-fixed frames can be chosen from. If the selected body-frame $\mathcal{B}$ results in singular evaluations of Eq. (3.209.24), then a second body frame, rotated 90° from the original body frame $\mathcal{B}$, can be chosen instead. Using this new body frame with a 90° offset, the associate attitude estimate will be nonsingular. This CRP attitude from the QUEST algorithm can then be translated into a nonsingular description, and adjusted for the 90° offset through a successive rotation.

Example 3.12.3

The same attitude vector observations used for the TRIAD and Davenport's method are used to illustrate how to evaluate an attitude using QUEST. Using the sum of the sensor weights as the initial eigenvalue guess, Newton's method is employed to numerically iterate for the maximum eigenvalue λ_{max}.

Iteration	λ_i	$f(\lambda_i)$
1	2.000000000000000	$5.34 \cdot 10^{-3}$
2	1.999666272285886	$2.23 \cdot 10^{-6}$
3	1.999666132970526	$3.88 \cdot 10^{-13}$
True Value	1.999666132970501	

In this example the third iteration already yields a near machine precision value of λ_{max}. Substituting the estimated maximum eigenvalue into Eq. (3.209.24) yields the optimal CRP attitude estimate:

$$\bar{\boldsymbol{q}} = \begin{pmatrix} -0.1236 \\ 0.1491 \\ 0.2739 \end{pmatrix}$$

The principal rotation angle difference with Wahba's solution obtained via Davenport's method is only $\Phi = 3.99 \cdot 10^{-13}$ degrees.

3.10.5 OLAE Method

Novel deterministic attitude estimation methods continue to be developed. A recently published solution is the Optimal Linear Attitude Estimator (OLAE).[51,52,53] To rewrite the attitude estimation into a perfectly linear form, recall that each vector observation must satisfy

$${}^{\mathcal{B}}\hat{\boldsymbol{v}}_i = [\bar{B}N]^{\mathcal{N}}\hat{\boldsymbol{v}}_i \tag{3.209.25}$$

The desired attitude estimate $[\bar{B}N]$ is decomposed using the elegant Cayley Transform in Eq. 3.133.

$$[\bar{B}N] = ([I_{3\times3}] + [\tilde{\bar{\boldsymbol{q}}}])^{-1}([I_{3\times3}] - [\tilde{\bar{\boldsymbol{q}}}]) \tag{3.209.26}$$

where $\bar{\boldsymbol{q}}$ is the desired attitude estimate in terms of CRPs, and $[\tilde{\bar{\boldsymbol{q}}}]$ is the matrix-tilde operator defined in Eq. (3.23). Substituting Eq. (3.209.26) into (3.209.25), and multiplying through by $([I_{3\times3}] + [\tilde{\bar{\boldsymbol{q}}}])$, yields

$$([I_{3\times3}] + [\tilde{\bar{\boldsymbol{q}}}])\ {}^{\mathcal{B}}\hat{\boldsymbol{v}}_i = ([I_{3\times3}] - [\tilde{\bar{\boldsymbol{q}}}])\ {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i \tag{3.209.27}$$

Rearranging this equation results in

$${}^{\mathcal{B}}\hat{\boldsymbol{v}}_i - {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i = -[\tilde{\bar{\boldsymbol{q}}}]({}^{\mathcal{B}}\hat{\boldsymbol{v}}_i + {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i) \tag{3.209.28}$$

Defining the summation and difference matrices $\boldsymbol{s}_i$ and $\boldsymbol{d}_i$ as

$$\boldsymbol{s}_i = {}^{\mathcal{B}}\hat{\boldsymbol{v}}_i + {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i \tag{3.209.29a}$$

$$\boldsymbol{d}_i = {}^{\mathcal{B}}\hat{\boldsymbol{v}}_i - {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i \tag{3.209.29b}$$

the observation vector relationship to the desired $\bar{\boldsymbol{q}}$ is written in the rigorously linear form

$$\boldsymbol{d}_i = [\tilde{\boldsymbol{s}}_i]\bar{\boldsymbol{q}} \tag{3.209.30}$$

Studying this elegant linear relationship, it is apparent again that a single observation cannot yield a three-dimensional attitude. A single copy of Eq. (3.209.30) cannot be solved for $\bar{\boldsymbol{q}}$ because $[\tilde{\boldsymbol{s}}_i]$ is not full-rank. However, if two or more measurements are available, then these can be all solved simultaneously using a classical least-square solution.

Assume each of the N measurements has an associated weight w_k. The attitude estimation can now be written as a weighted least-squares problem, and solved in the following manner. The N summation and difference matrices are stacked to create a $3N \times 1$ matrix $\boldsymbol{d}$, a $3N \times 3$ matrix $[S]$, and the weights are arranged to yield a $3N \times 3N$ matrix $[W]$ through:

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_1 \\ \cdots \\ \boldsymbol{d}_N \end{bmatrix} \qquad [S] = \begin{bmatrix} \tilde{\boldsymbol{s}}_1 \\ \cdots \\ \tilde{\boldsymbol{s}}_N \end{bmatrix} \qquad [W] = \begin{bmatrix} w_1 I_{3\times3} & 0_{3\times3} & \ddots \\ 0_{3\times3} & \ddots & 0_{3\times3} \\ \ddots & 0_{3\times3} & w_N I_{3\times3} \end{bmatrix} \tag{3.209.31}$$

The optimal CRP attitude description is now easily evaluated using a weighted least-squares solution:

$$\bar{\boldsymbol{q}} = \left([S]^T[W][S]\right)^{-1}[S]^T[W]\boldsymbol{d} \tag{3.209.32}$$

The inclusion of the weights has the advantage that relative sensor accuracy information (consider sun pointing and magnetometer sensor errors) can be considered in the optimal attitude evaluation. Since both the $[S]$ and $\boldsymbol{d}$ matrices contain measurement errors, this least-squares solution (as with most of the Wahba methods) is not a minimal variance estimate. However, its direct linear algebraic solution makes it a very attractive solution. With modern computer graphic card's ability to solve linear algebra problems very fast, the linear nature of this method provides increasing computational benefits as the number of observations is increased.

Because this method yields a CRP attitude measure, it must address the same 180° singularity issues as the QUEST method. The solution is also to use the method of sequential rotations, and alternate between two body frames which are 90° apart.

Example 3.12.4

Let us use the two attitude observations of Example 3.12.1 to illustrate how OLAE is used. The difference matrix $\boldsymbol{d}$ is

$$\boldsymbol{d} = \begin{bmatrix} -0.181 & -0.5282 & 0.2242, & -0.3138 & -0.1584 & -0.06380 \end{bmatrix}^T$$

and the summation matrix $[S]$ is

$$[S] = \begin{bmatrix} 0 & -0.2242 & -0.5282 \\ 0.2242 & 0 & -1.8190 \\ 0.5282 & 1.8190 & 0 \\ 0 & -1.9362 & -0.1584 \\ 1.9362 & 0 & 0.3138 \\ 0.1584 & -0.3138 & 0 \end{bmatrix}$$

Assuming unit weights, the optimal CRP solution is

$$\bar{\boldsymbol{q}} = \begin{bmatrix} -0.1236 \\ 0.1488 \\ 0.2742 \end{bmatrix}$$

The principal rotation angle difference with truth attitude is $\Phi = 1.6903°$.
