H

An Outline of MSA History

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A Historical Outline of Matrix Structural Analysis: A Play in Three Acts

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Abstract

The evolution of Matrix Structural Analysis (MSA) from 1930 through 1970 is outlined. Hightlighted are major contributions by Collar and Duncan, Argyris, and Turner, which shaped this evolution. To enliven the narrative the outline is configured as a three-act play. Act I describes the pre-WWII formative period. Act II spans a period of confusion during which matrix methods assumed bewildering complexity in response to conflicting demands and restrictions. Act III outlines the cleanup and consolidation driven by the appearance of the Direct Stiffness Method, through which MSA completed morphing into the present implementation of the Finite Element Method.

Keywords: matrix structural analysis; finite elements; history; displacement method; force method; direct stiffness method; duality

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§H.1. INTRODUCTION

Who first wrote down a stiffness or flexibility matrix?

The question was posed in a 1995 paper [1]. The educated guess was "somebody working in the aircraft industry of Britain or Germany, in the late 1920s or early 1930s." Since then the writer has examined reports and publications of that time. These trace the origins of Matrix Structural Analysis to the aeroelasticity group of the National Physics Laboratory (NPL) at Teddington, a town that has now become a suburb of greater London.

The present paper is an expansion of the historical vignettes in Section 4 of [1]. It outlines the major steps in the evolution of MSA by highlighting the fundamental contributions of four individuals: Collar, Duncan, Argyris and Turner. These contributions are lumped into three milestones:

Creation. Beginning in 1930 Collar and Duncan formulated discrete aeroelasticity in matrix form. The first two journal papers on the topic appeared in 1934-35 [2,3] and the first book, couthored with Frazer, in 1938 [4]. The representation and terminology for discrete dynamical systems is essentially that used today.

Unification. In a series of journal articles appearing in 1954 and 1955 [5] Argyris presented a formal unification of Force and Displacement Methods using dual energy theorems. Although practical applications of the duality proved ephemeral, this work systematized the concept of assembly of structural system equations from elemental components.

FEMinization. In 1959 Turner proposed [6] the Direct Stiffness Method (DSM) as an efficient and general computer implementation of the then embryonic, and as yet unnamed, Finite Element

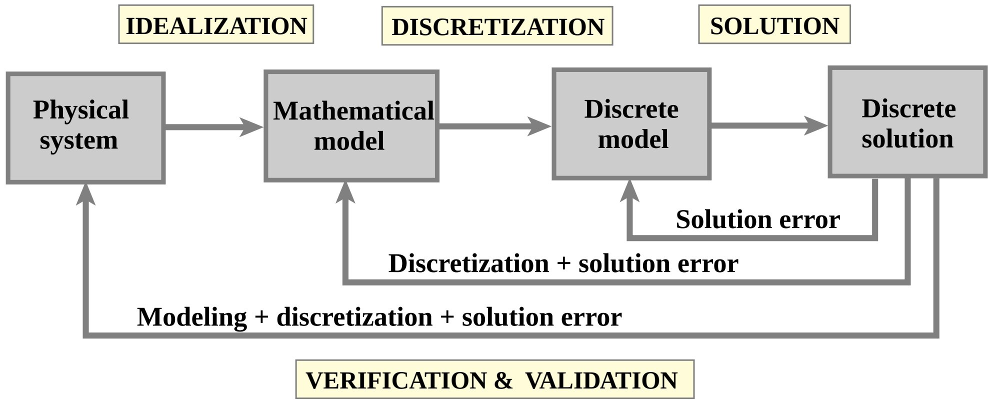

Figure 1. Flowchart of model-based simulation (MBS) by computer.

Method. This technique, fully explained in a follow-up article [7], naturally encompassed structural and continuum models, as well as nonlinear, stability and dynamic simulations. By 1970 DSM had brought about the demise of the Classical Force Method (CFM), and become the dominant implementation in production-level FEM programs.

These milestones help dividing MSA history into three periods. To enliven and focus the exposition these will be organized as three acts of a play, properly supplemented with a “matrix overture” prologue, two interludes and a closing epilogue. Here is the program:

Prologue - *Victorian Artifacts*: 1858–1930.

Act I - *Gestation and Birth*: 1930–1938.

Interlude I - *WWII Blackout*: 1938–1947.

Act II - *The Matrix Forest*: 1947–1956.

Interlude II - *Questions*: 1956–1959.

Act III - *Answers*: 1959–1970.

Epilogue - *Revisiting the Past*: 1970-date.

Act I, as well as most of the Prologue, takes place in the U.K. The following events feature a more international cast.

§H.2. Background and Terminology

Before departing for the theater, this Section offers some general background and explains historical terminology. Readers familiar with the subject should skip to Section 3.

The overall schematics of model-based simulation (MBS) by computer is flowcharted in Figure 1. For mechanical systems such as structures the Finite Element Method (FEM) is the most widely used discretization and solution technique. Historically the ancestor of FEM is MSA, as illustrated in Figure 2. The morphing of the MSA from the pre-computer era — as described for example in the first book [4] — into the first programmable computers took place, in wobbly gyrations, during the transition period herein called Act II. Following a confusing interlude, the young FEM begin to settle, during the early 1960s, into the configuration shown on the right of Figure 2. Its basic components have not changed since 1970.

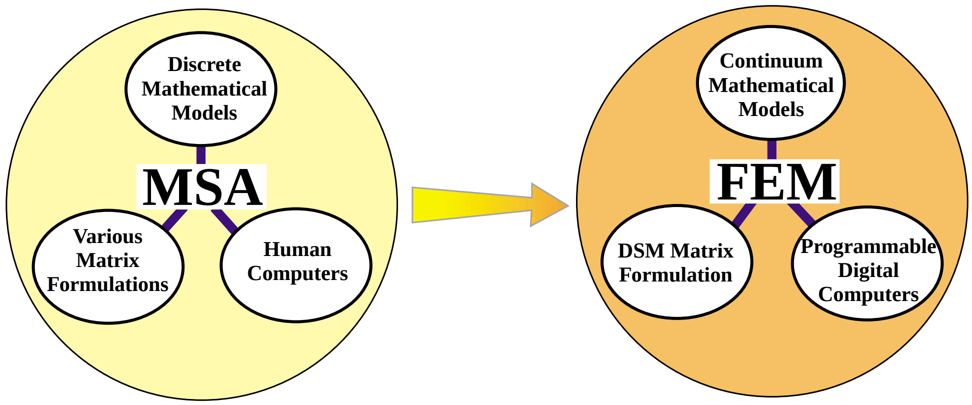

Figure 2. Morphing of the pre-computer MSA (before 1950) into the present FEM. On the left "human computer" means computations under direct human control, possibly with the help of analog devices (slide rule) or digital devices (desk calculator). The FEM configuration shown on the right settled by the mid 1960s.

MSA and FEM stand on three legs: mathematical models, matrix formulation of the discrete equations, and computing tools to do the numerical work. Of the three legs the latter is the one that has undergone the most dramatic changes. The "human computers" of the 1930s and 1940s morphed by stages into programmable computers of analog and digital type. The matrix formulation moved like a pendulum. It began as a simple displacement method in Act I, reached bewildering complexity in Act II and went back to conceptual simplicity in Act III.

Unidimensional structural models have changed little: a 1930 beam is still the same beam. The most noticeable advance is that pre-1955 MSA, following classical Lagrangian mechanics, tended to use spatially discrete energy forms from the start. The use of space-continuum forms as basis for multidimensional element derivation was pioneered by Argyris [5], successfully applied to triangular geometries by Turner, Clough, Martin and Topp [8], and finalized by Melosh [9] and Irons [10,11] with the precise statement of compatibility and completeness requirements for FEM.

Matrix formulations for MSA and FEM have been traditionally classified by the choice of *primary unknows*. These are those solved for by the human or digital computer to determine the system state. In the Displacement Method (DM) these are physical or generalized displacements. In the Classical Force Method (CFM) these are amplitudes of redundant force (or stress) patterns. (The qualifier "classical" is important because there are other versions of the Force Method, which select for example stress function values or Lagrange multipliers as unknowns.) There are additional methods that involve combinations of displacements, forces and/or deformations as primary unknowns, but these have no practical importance in the pre-1970 period covered here.

Appropriate mathematical names for the DM are *range-space method* or *primal method*. This means that the primary unknowns are the same type as the primary variables of the governing functional. Appropriate names for the CFM are *null-space method*, *adjoint method*, or *dual method*. This means that the primary unknowns are of the same type of the adjoint variables of the governing functional, which in structural mechanics are forces. These names are not used in the historical outline, but are useful in placing more recent developments, as well as nonstructural FEM applications, within a general framework.

The terms Stiffness Method and Flexibility Method are more diffuse names for the Displacement and

Force Methods, respectively. Generally speaking these apply when stiffness and flexibility matrices, respectively, are important part of the modeling and solution process.

§H.3. Prolog - Victorian Artifacts: 1858-1930

Matrices — or “determinants” as they were initially called — were invented in 1858 by Cayley at Cambridge, although Gibbs (the co-inventor, along with Heaviside, of vector calculus) claimed priority for the German mathematician Grassmann. Matrix algebra and matrix calculus were developed primarily in the U.K. and Germany. Its original use was to provide a compact language to support investigations in mathematical topics such as the theory of invariants and the solution of algebraic and differential equations. For a history of these early developments the monograph by Muir [12] is unsurpassed. Several comprehensive treatises in matrix algebra appeared in the late 1920s and early 1930s [13–15].

Compared to vector and tensor calculus, matrices had relatively few applications in science and technology before 1930. Heisenberg’s 1925 matrix version of quantum mechanics was a notable exception, although technically it involved infinite matrices. The situation began to change with the advent of electronic desk calculators, because matrix notation provided a convenient way to organize complex calculation sequences. Aeroelasticity was a natural application because the stability analysis is naturally posed in terms of determinants of matrices that depend on a speed parameter.

The non-matrix formulation of Discrete Structural Mechanics can be traced back to the 1860s. By the early 1900s the essential developments were complete. A readable historical account is given by Timoshenko [16]. Interestingly enough, the term “matrix” never appears in this book.

§H.4. Act I - Gestation and Birth: 1930-1938

In the decade of World War I aircraft technology begin moving toward monoplanes. Biplanes disappeared by 1930. This evolution meant lower drag and faster speeds but also increased disposition to flutter. In the 1920s aeroelastic research began in an international scale. Pertinent developments at the National Physical Laboratory (NPL) are well chronicled in a 1978 historical review article by Collar [17], from which the following summary is extracted.

§H.4.1. The Source Papers

The aeroelastic work at the Aerodynamics Division of NPL was initiated in 1925 by R. A. Frazer. He was joined in the following year by W. J. Duncan. Two years later, in August 1928, they published a monograph on flutter [18], which came to be known as “The Flutter Bible” because of its completeness. It laid out the principles on which flutter investigations have been based since. In January 1930 A. R. Collar joined Frazer and Duncan to provide more help with theoretical investigations. Aeroelastic equations were tedious and error prone to work out in long hand. Here are Collar’s own words [17, page 17] on the motivation for introducing matrices:

> “Frazer had studied matrices as a branch of applied mathematics under Grace at Cambridge; and he recognized that the statement of, for example, a ternary flutter problem in terms of matrices was neat and compendious. He was, however, more concerned with formal manipulation and transformation to other coordinates than with numerical results. On the other hand, Duncan and I were in search of numerical results for the vibration characteristics of airscrew blades; and we recognized that we could only advance by breaking the blade into, say, 10 segments and treating it as having 10 degrees

> of freedom. This approach also was more conveniently formulated in matrix terms, and readily expressed numerically. Then we found that if we put an approximate mode into one side of the equation, we calculated a better approximation on the other; and the matrix iteration procedure was born. We published our method in two papers in *Phil. Mag.* [2,3]; the first, dealing with conservative systems, in 1934 and the second, treating damped systems, in 1935. By the time this had appeared, Duncan had gone to his Chair at Hull."

The aforementioned papers appear to be the earliest *journal publications* of MSA. These are amazing documents: clean and to the point. They do not feel outdated. Familiar names appear: mass, flexibility, stiffness, and dynamical matrices. The matrix symbols used are $[m]$, $[f]$, $[c]$ and $[D] = [c]^{-1}[m] = [f][m]$, respectively, instead of the $\mathbf{M}$, $\mathbf{F}$, $\mathbf{K}$ and $\mathbf{D}$ in common use today. A general inertia matrix is called $[a]$. As befit the focus on dynamics, the displacement method is used. Point-mass displacement degrees of freedom are collected in a vector $\{x\}$ and corresponding forces in vector $\{P\}$. These are called $[q]$ and $[Q]$, respectively, when translated to generalized coordinates.

The notation was changed in the book [4] discussed below. In particular matrices are identified in [4] by capital letters without surrounding brackets, in more agreement with the modern style; for example mass, damping and stiffness are usually denoted by A, B and C, respectively.

§H.4.2. The MSA Source Book

Several papers on matrices followed, but apparently the traditional publication vehicles were not viewed as suitable for description of the new methods. At that stage Collar notes [17, page 18] that

> "Southwell [Sir Richard Southwell, the "father" of relaxation methods] suggested that the authors of the various papers should be asked to incorporate them into a book, and this was agreed. The result was the appearance in November 1938 of "Elementary Matrices" published by Cambridge University Press [4]; it was the first book to treat matrices as a branch of applied mathematics. It has been reprinted many times, and translated into several languages, and even now after nearly 40 years, stills sells in hundreds of copies a year — mostly paperback. The interesting thing is that the authors did not regard it as particularly good; it was the book we were instructed to write, rather than the one we would have liked to write."

The writer has copies of the 1938 and 1963 printings. No changes other than minor fixes are apparent. Unlike the source papers [2,3] the book feels dated. The first 245 pages are spent on linear algebra and ODE-solution methods that are now standard part of engineering and science curricula. The numerical methods, oriented to desk calculators, are obsolete. That leaves the modeling and application examples, which are not coherently interweaved. No wonder that the authors were not happy about the book. They had followed Southwell's "merge" suggestion too literally. Despite these flaws its direct and indirect influence during the next two decades was significant. Being first excuses imperfections.

The book focuses on dynamics of a complete airplane and integrated components such as wings, rudders or ailerons. The concept of *structural element* is primitive: take a shaft or a cantilever and divide it into segments. The assembled mass, stiffness or flexibility is given directly. The source of damping is usually aerodynamic. There is no static stress analysis; pre-WWII aircraft were overdesigned for strength and typically failed by aerodynamic or propulsion effects.

Readers are reminded that in aeroelastic analysis stiffness matrices are generally unsymmetric, being the sum of a a symmetric elastic stiffness and an unsymmetric aerodynamic stiffness. This clean decomposition does not hold for flexibility matrices because the inverse of a sum is not the sum of inverses. The treatment of [4] includes the now called load-dependent stiffness terms, which represent another first.

On reading the survey articles by Collar [17,19] one cannot help being impressed by the lack of pretension. With Duncan he had created a tool for future generations of engineers to expand and improve upon. Yet he appears almost apologetic: "I will complete the matrix story as briefly as possible" [17, page 17]. The NPL team members shared a common interest: to troubleshoot problems by understanding the physics, and viewed numerical methods simply as helpers.

§H.5. Interlude I - WWII Blackout: 1938-1947

Interlude I is a "silent period" taken to extend from the book [4] to the first journal publication on the matrix Force Method for aircraft [20]. Aeroelastic research continued. New demands posed by high strength materials, higher speeds, combat maneuvers, and structural damage survival increased interest in stress analysis. For the beam-like skeletal configurations of the time, the traditional flexibility-based methods such as CFM were appropriate. Flexibilities were often measured experimentally by static load tests, and fitted into the calculations. Punched-card computers and relay-calculators were increasingly used, and analog devices relied upon to solve ODEs in guidance and ballistics. Precise accounts of MSA work in aerospace are difficult to trace because of publication restrictions. The blackout was followed by a 2-3 year hiatus until those restrictions were gradually lifted, R&D groups restaffed, and journal pipelines refilled.

§H.6. Act II - The Matrix Forest: 1947-1956

As Act II starts MSA work is still mainly confined to the aerospace community. But the focus has shifted from dynamics to statics, and especially stress, buckling, fracture and fatigue analysis. Turbines, supersonic flight and rocket propulsion brought forth thermomechanical effects. The Comet disasters forced attention on stress concentration and crack propagation effects due to cyclic cabin pressurization. Failsafe design gained importance. In response to these multiple demands aircraft companies staffed specialized groups: stress, aerodynamics, aeroelasticity, propulsion, avionics, and so on. A multilevel management structure with well defined territories emerged.

The transition illustrated in Figure 2 starts, driven by two of the legs supporting MSA: new computing resources and new mathematical models. The matrix formulation merely reacts.

§H.6.1. Computers Become Machines

The first electronic commercial computer: Univac I, manufactured by a division of Remington-Rand, appeared during summer 1951. The six initial machines were delivered to US government agencies [21]. It was joined in 1952 by the Univac 1103, a scientific-computation oriented machine built by ERA, a R-R acquisition. This was the first computer with a drum memory. T. J. Watson Sr., founder of IBM, had been once quoted as saying that six electronic computers would satisfy the needs of the planet. Turning around from that prediction, IBM launched the competing 701 model in 1953.

Big aircraft companies began purchasing or leasing these expensive wonders by 1954. But this did not mean immediate access for everybody. The behemoths had to be programmed in machine or assembly code by specialists, who soon formed computer centers allocating and prioritizing cycles. By 1956 structural engineers were still likely to be using their slides rules, Marchants and punched card equipment. Only after the 1957 appearance of the first high level language (Fortran I, offered on the IBM 704) were engineers and scientists able (and allowed) to write their own programs.

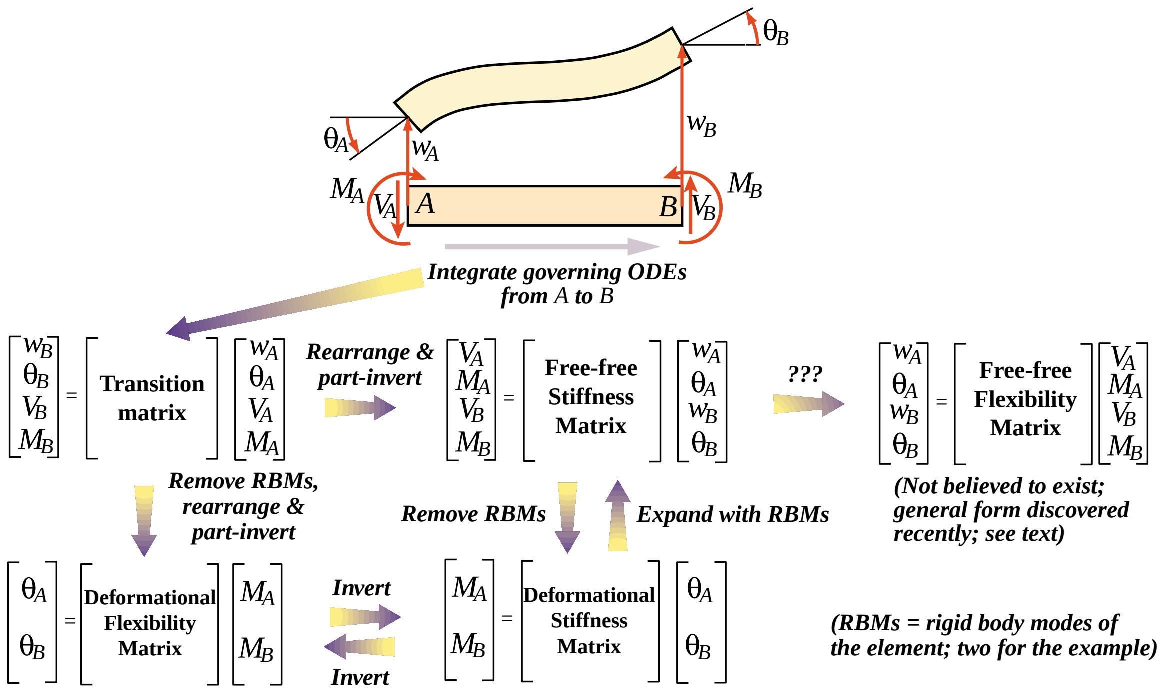

Figure 3. Transition, flexibility and stiffness matrices for unidimensional linear structural elements, such as the plane beam depicted here, can be obtained by integrating the governing differential equations, analytically or numerically, over the member to relate end forces and displacements. Clever things were done with this "method of lines" approach, such as including intermediate supports or elastic foundations.

§H.6.2. The Matrix CFM Takes Center Stage

In static analysis the non-matrix version of the Classical Force Method (CFM) had enjoyed a distinguished reputation since the source contributions by Maxwell, Mohr and Castigliano. The method provides directly the internal forces, which are of paramount interest in stress-driven design. It offers considerable scope of ingenuity to experienced structural engineers through clever selection of redundant force systems. It was routinely taught to Aerospace, Civil and Mechanical Engineering students.

Success in hand-computation dynamics depends on "a few good modes." Likewise, the success of CFM depends crucially on the selection of good redundant force patterns. The structures of pre-1950 aircraft were a fairly regular lattice of ribs, spars and panels, forming beam-like configurations. If the panels are ignored, the selection of appropriate redundants was well understood. Panels were modeled conservatively as inplane shear-force carriers, circumventing the difficulties of two-dimensional elasticity. With some adjustments and experimental validations, sweptback wings of high aspect ratio were eventually fitted into these models.

A matrix framework was found convenient to organize the calculations. The first journal article on the matrix CFM, which focused on sweptback wing analysis, is by Levy [20], followed by publications of Rand [22], Langefors [23], Wehle and Lansing [24] and Denke[25]. The development culminates in the article series of Argyris [5] discussed in Section 6.5.

§H.6.3. The Delta Wing Challenge

The Displacement Method (DM) continued to be used for vibration and aeroelastic analysis, although as noted above this was often done by groups separated from stress and buckling analysis. A new modeling challenge entered in the early 1950s: delta wing structures. This rekindled interest in stiffness methods.

The traditional approach to obtain flexibility and stiffness matrices of unidimensional structural members such as bars and shafts is illustrated in Figure 3. The governing differential equations are integrated, analytically or numerically, from one end to the other. The end quantities, grouping forces and displacements, are thereby connected by a transition matrix. Using simple algebraic manipulations three more matrices shown in Figure 3 can be obtained: deformational flexibility, deformational stiffness and free-free stiffness. This well known technique has the virtue of reducing the number of unknowns since the integration process can absorb structural details that are handled in the present FEM with multiple elements.

Notably absent from the scheme of Figure 3 is the free-free flexibility. This was not believed to exist since it is the inverse of the free-free stiffness, which is singular. A general closed-form expression for this matrix as a Moore-Penrose generalized stiffness inverse was not found until recently [26,27].

Modeling delta wing configurations required two-dimensional panel elements of arbitrary geometry, of which the triangular shape, illustrated in Figure 4, is the simplest and most versatile. Efforts to follow the ODE-integration approach lead to failure. (One particularly bizarre proposal, for solving exactly the wrong problem, is mentioned for fun in the label of Figure 4.) This motivated efforts to construct the stiffness matrix of the panel directly. The first attempt in this direction is by Levy [28]; this was only partly successful but was able to illuminate the advantages of the stiffness approach.

The article series by Argyris [5] contains the derivation of the 8×8 free-free stiffness of a flat rectangular panel using bilinear displacement interpolation in Cartesian coordinates. But that geometry was obviously inadequate to model delta wings. The landmark contribution of Turner, Clough, Martin and Topp [8] finally succeeded in directly deriving the stiffness of a triangular panel. Clough [29] observes that this paper represents the delayed publication of 1952-53 work at Boeing. It is recognized as one of the two sources of present FEM implementations, the second being the DSM discussed later. Because of the larger number of unknowns compared to CFM, competitive use of the DM in stress analysis had necessarily to wait until computers become sufficiently powerful to handle hundreds of simultaneous equations.

§H.6.4. Reduction Fosters Complexity

For efficient digital computation on present computers, data organization (in terms of fast access as well as exploitation of sparseness, vectorization and parallelism) is of primary concern whereas raw problem size, up to certain computer-dependent bounds, is secondary. But for hand calculations minimal problem size is a key aspect. Most humans cannot comfortably solve by hand linear systems of more than 5 or 6 unknowns by direct elimination methods, and 5–10 times that through problem-oriented "relaxation" methods. The first-generation digital computers improved speed and reliability, but were memory strapped. For example the Univac I had 1000 45-bit words and the IBM 701, 2048 36-bit words. Clearly solving a full system of 100 equations was still a major challenge.

It should come as no surprise that problem reduction techniques were paramount throughout this period, and exerted noticeable influence until the early 1970s. In static analysis reduction was

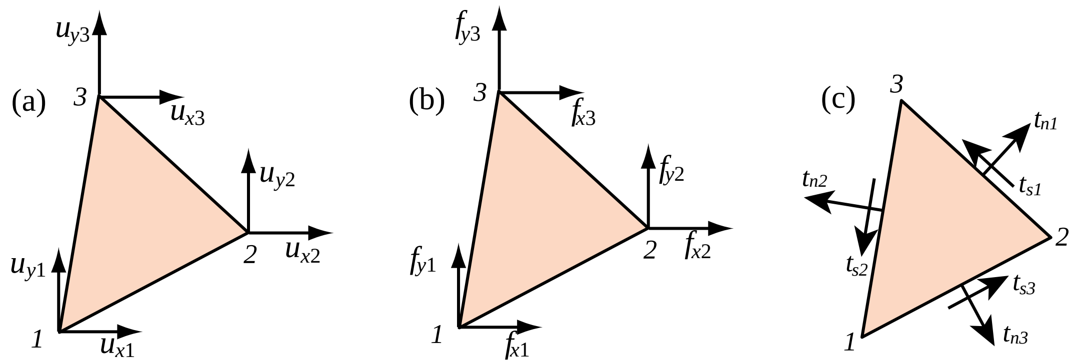

Figure 4. Modeling delta wing configurations required panel elements of arbitrary geometry such as the triangles depicted here. The traditional ODE-based approach of Figure 3 was tried by some researchers who (seriously) proposed finding the corner displacements in (a) produced by the concentrated corner forces in (b) on a supported triangle from the elasticity equations solved by numerical integration! Bad news: those displacements are infinite. Interior fields assumptions were inevitable, but problems persisted. A linear inplane displacement field is naturally specified by corner displacements, whereas a constant membrane force field is naturally defined by edge tractions (c). Those quantities "live" on different places. The puzzle was first solved in [8] by lumping edge tractions to node forces on the way to the free-free stiffness matrix.

achieved by elaborated *functional groupings* of static and kinematic variables. Most schemes of the time can be understood in terms of the following classification:

$$
\begin{array}{ll}
\text{generalized forces} & \left\{\begin{array}{ll} \text{primary} & \left\{\begin{array}{l}\text{applied forces } \mathbf{f}_a \\ \text{redundant forces } \mathbf{y}\end{array}\right. \\ \text{secondary} & \left\{\begin{array}{l}\text{condensable forces } \mathbf{f}_c = \mathbf{0} \\ \text{support reactions } \mathbf{f}_s\end{array}\right. \end{array}\right. \\
\text{generalized displacements} & \left\{\begin{array}{ll} \text{primary} & \left\{\begin{array}{l}\text{applied displacements } \mathbf{u}_a \\ \text{redundant displacements } \mathbf{z}\end{array}\right. \\ \text{secondary} & \left\{\begin{array}{l}\text{condensable displacements } \mathbf{u}_c \\ \text{support conditions } \mathbf{u}_s = \mathbf{0}\end{array}\right. \end{array}\right.
\end{array}
\tag{H.1}
$$

Here *applied forces* are those acting with nonzero values, that is, the ones visibly drawn as arrows by an engineer or instructor. In reduction-oriented thinking zero forces on unloaded degrees of freedom are classified as *condensable* because they can be removed through static condensation techniques. Similarly, nonzero *applied displacements* were clearly differentiated from zero-displacements arising from support conditions because the latter can be thrown out while the former must be retained. Redundant displacements, which are the counterpart of redundant forces, have been given many names, among them "kinematically indeterminate displacements" and "kinematic deficiencies."

Matrix formulation evolved so that the unknowns were the force redundants $\mathbf{y}$ in the CFM and the displacement redundants $\mathbf{z}$ in the DM. Partitioning matrices in accordance to (H.1) fostered exuberant growth culminating in the *matrix forest* that characterizes works of this period.

To a present day FEM programmer familiar with the DSM, the complexity of the matrix forest would strike as madness. The DSM master equations can be assembled without functional labels. Boundary conditions are applied on the fly by the solver. But the computing limitations of the time must be kept in mind to see the method in the madness.

§H.6.5. Two Paths Through the Forest

A series of articles published by J. H. Argyris in four issues of *Aircraft Engrg.* during 1954 and 1955 collectively represents the second major milestone in MSA. In 1960 the articles were collected in a book, entitled "Energy Theorems and Structural Analysis" [5]. Part I, sub-entitled General Theory, reprints the four articles, whereas Part II, which covers additional material on thermal analysis and torsion, is co-authored by Argyris and Kelsey. Both authors are listed as affiliated with the Aerospace Department of the Imperial College at London.

The dual objectives of the work, stated in the Preface, are "to generalize, extend and unify the fundamental energy principles of elastic structures" and "to describe in detail practical methods of analysis of complex structures — in particular for aeronautical applications." The first objective succeeds well, and represents a key contribution toward the development of continuum-based models. Part I carefully merges classical contributions in energy and work methods with matrix methods of discrete structural systems. The coverage is methodical, with numerous illustrative examples. The exposition of the Force Method for wing structures reaches a level of detail unequaled for the time.

The Displacement Method is then introduced by duality — called "analogy" in this work:

> "The analogy between the developments for the flexibilities and stiffnesses ... shows clearly that parallel to the analysis of structures with forces as unknowns there must be a corresponding theory with deformations as unknowns."

This section credits Ostenfeld [30] with being the first to draw attention to the parallel development. The duality is exhibited in a striking Form in Table II, in which both methods are presented side by side with simply an exchange of symbols and appropriate rewording. The steps are based on the following decomposition of internal deformation states $\mathbf{g}$ and force patterns $\mathbf{p}$:

$$\mathbf{p} = \mathbf{B}_0\,\mathbf{f}_a + \mathbf{B}_1\,\mathbf{y}, \qquad \mathbf{g} = \mathbf{A}_0\,\mathbf{u}_a + \mathbf{A}_1\,\mathbf{z}, \tag{H.2}$$

Here the $\mathbf{B}_i$ and $\mathbf{A}_i$ denote system equilibrium and compatibility matrices, respectively. The vector symbols on the right reflect a particular choice of the force-displacement decomposition (H.1), with kinematic deficiencies taken to be the condensable displacements: $\mathbf{z} \equiv \mathbf{u}_c$.

This unification exerted significant influence over the next decade, particularly on the European community. An excellent textbook exposition is that of Pestel and Leckie [31]. This book covers both paths, following Argyris' framework, in Chapters 9 and 10, using 83 pages and about 200 equations. These chapters are highly recommended to understand the organization of numeric and symbolic hand computations in vogue at that time, but it is out of print. Still in print (by Dover) is the book by Przemieniecki [32], which describes the DM and CFM paths in two Chapters: 6 and 8. The DM coverage is strongly influenced, however, by the DSM; thus duality is only superficially used.

§H.6.6. Dubious Duality

A key application of the duality in [5] was to introduce the DM by analogy to the then better known CFM. Although done with good intentions this approach did not anticipate the direct development of continuum-based finite elements through stiffness methods. These can be formulated from the total potential energy principle via shape functions, a technique not fully developed until the mid 1960s.

The side by side presentation of Table II of [5] tried to show that CFM and DM were going through exactly the same sequence of steps. Some engineers, eventually able to write Fortran programs, concluded that the methods had similar capabilities and selecting one or the other was a matter of taste. (Most structures groups, upholding tradition, opted for the CFM.) But the few engineers who tried implementing both noticed a big difference. And that was before the DSM, which has no dual counterpart under the decomposition (H.2), appeared.

The paradox is explained in Section 4 of [1]. It is also noted there that (H.2) is not a particularly useful state decomposition. A better choice is studied in Section 2 of that paper; that one permits all known methods of Classical MSA, including the DSM, to be derived for skeletal structures as well as for a subset of continuum models.

§H.7. Interlude II - Questions: 1956-1959

Interlude I was a silent period dominated by the war blackout. Interlude II is more vocal: a time of questions. An array of methods, models, tools and applications is now on the table, and growing. Solid-state computers, Fortran, ICBMs, the first satellites. So many options. Stiffness or flexibility? Forces or displacements? Do transition matrix methods have a future? Is the CFM-DM duality a precursor to general-purpose programs that will simulate everything? Will engineers be allowed to write those programs?

As convenient milestone this outline takes 1959, the year of the first DSM paper, as the beginning of Act III. Arguments and counter-arguments raised by the foregoing questions will linger, however, for two more decades into diminishing circles of the aerospace community.

§H.8. Act III - Answers: 1959-1970

The curtain of Act III lifts in Aachen, Germany. On 6 November 1959, M. J. Turner, head of the Structural Dynamics Unit at Boeing and an expert in aeroelasticity, presented the first paper on the Direct Stiffness Method to an AGARD Structures and Materials Panel meeting [6]. (AGARD is NATO's Advisory Group for Aeronautical Research and Development, which had sponsored workshops and lectureships since 1952. Bound proceedings or reports are called AGARDographs.)

§H.8.1. A Path Outside the Forest

No written record of [6] seem to exist. Nonetheless it must have produced a strong impression since published contributions to the next (1962) panel meeting kept referring to it. By 1960 the method had been applied to nonlinear problems [33] using incremental techniques. In July 1962 Turner, Martin and Weikel presented an expanded version of the 1959 paper, which appeared in an AGARDograph volume published by Pergamon in 1964 [7]. Characteristic of Turner's style, the Introduction goes directly to the point:

> "In a paper presented at the 1959 meeting of the AGARD Structures and Material Panel in Aachen, the essential features of a system for numerical analysis of structures, termed the direct-stiffness method, were described. The characteristic feature of this particular version of the displacement method is the assembly procedure, whereby the stiffness matrix for a composite structure is generated by direct addition of matrices associated with the elements of the structure."

The DSM is explained in six text lines and three equations:

> "For an individual element e the generalized nodal force increments $\{\Delta X^e\}$ required to maintain a set of nodal displacement increments $\{\Delta u\}$ are given by a matrix equation

$$\{\Delta X^e\} = K^e \{\Delta u\} \tag{H.3}$$

in which K^e denotes the stiffness matrix of the individual element. Resultant nodal force increments acting on the complete structure are

$$\{\Delta X\} = \sum \{\Delta X^e\} = K\,\{\Delta u\} \tag{H.4}$$

wherein K, the stiffness of the complete structure, is given by the summation

$$K = \sum K^e \tag{H.5}$$

which provides the basis for the matrix assembly procedure noted earlier."

Knowledgeable readers will note a notational glitch. For (H.3)-(H.5) to be correct matrix equations, K^e must be an element stiffness fully expanded to global (in that paper: "basic reference") coordinates, a step that is computationally unnecessary. A more suggestive notation used in present DSM expositions is $K = \sum (L^e)^T K^e L^e$, in which L^e are Boolean localization matrices. Note also the use of Δ in front of u and X and their identification as "increments." This simplifies the extension to nonlinear analysis, as outlined in the next paragraph:

"For the solution of linear problems involving small deflections of a structure at constant uniform temperature which is initially stress-free in the absence of external loads, the matrices K^e are defined in terms of initial geometry and elastic properties of the materials comprising the elements; they remain unchanged throughout the analysis. Problems involving nonuniform heating of redundant structures and/or large deflections are solved in a sequence of linearized steps. Stiffness matrices are revised at the beginning of each step to account for charges in internal loads, temperatures and geometric configurations."

Next are given some computer implementation details, including the first ever mention of user-defined elements:

"Stiffness matrices are generally derived in local reference systems associated with the elements (as prescribed by a set of subroutines) and then transformed to the basic reference system. It is essential that the basic program be able to acommodate arbitrary additions to the collection of subroutines as new elements are encountered. Associated with these are a set of subroutines for generation of stress matrices S^e relating matrices of stress components σ^e in the local reference system of nodal displacements:

$$\{\sigma^e\} = S^e\,\{\bar{u}\} \tag{H.6}$$

The vector $\{\bar{u}\}$ denotes the resultant displacements relative to a local reference system which is attached to the element. ... Provision should also be made for the introduction of numerical stiffness matrices directly into the program. This permits the utilization and evaluation of new element representations which have not yet been programmed. It also provides a convenient mechanism for introducing local structural modifications into the analysis."

The assembly rule (H.3)-(H.5) is insensitive to element type. It work the same way for a 2-node bar, or a 64-node hexahedron. To do dynamics and vibration one adds mass and damping terms. To do buckling one adds a geometric stiffness and solves the stability eigenproblem, a technique first explained in [33]. To do nonlinear analysis one modifies the stiffness in each incremental step. To apply multipoint constraints the paper [7] advocates a master-slave reduction method.

Some computational aspects are missing from this paper, notably the treatment of simple displacement boundary conditions, and the use of sparse matrix assembly and solution techniques. The latter were first addressed in Wilson's thesis work [34,35].

§H.8.2. The Fire Spreads

DSM is a paragon of elegance and simplicity. The writer is able to teach the essentials of the method in three lectures to graduate and undergraduate students alike. Through this path the old MSA and the young FEM achieved smooth confluence. The matrix formulation returned to the crispness of the source papers [2,3]. A widely referenced MSA correlation study by Gallagher [36] helped dissemination. Computers of the early 1960s were finally able to solve hundreds of equations. In an ideal world, structural engineers should have quickly razed the forest and embraced DSM.

It did not happen that way. The world of aerospace structures split. DSM advanced first by word of mouth. Among the aerospace companies, only Boeing and Bell (influenced by Turner and Gallagher, respectively) had made major investments in DSM by 1965. Among academia the Civil Engineering Department at Berkeley become a DSM evangelist through Clough, who made his students — including the writer — use DSM in their thesis work. These codes were freely disseminated into the non-aerospace world since 1963. Martin established similar traditions at Washington University, and Zienkiewicz, influenced by Clough, at Swansea. The first textbook on FEM [37], which appeared in 1967, makes no mention of force methods. By then the application to non-structural field problems (thermal, fluids, electromagnetics, ...) had begun, and again the DSM scaled well into the brave new world.

§H.8.3. The Final Test

Legacy CFM codes continued, however, to be used at many aerospace companies. The split reminds one of Einstein's answer when he was asked about the reaction of the old-guard school to the new physics: "we did not convince them; we outlived them." Structural engineers hired in the 1940s and 1950s were often in managerial positions in the 1960s. They were set in their ways. How can duality fail? All that is needed are algorithms for having the computer select good redundants automatically. Substantial effort was spent in those "structural cutters" during the 1960s [32,38].

That tenacity was eventually put to a severe test. The 1965 NASA request-for-proposal to build the NASTRAN finite element system called for the simultaneous development of Displacement and Force versions [39]. Each version was supposed to have identical modeling and solution capabilities, including dynamics and buckling. Two separate contracts, to MSC and Martin, were awarded accordingly. Eventually the development of the Force version was cancelled in 1969. The following year may be taken as closing the transition depicted in Figure 2, and as marking the end of the Force Method as a serious contender for general-purpose FEM programs.

§H.9. Epilogue - Revisiting the Past: 1970-date

Has MSA, now under the wider umbrella of FEM, attained a final form? This seems the case for general-purpose FEM programs, which by now are truly "1960 heritage" codes.

Resurrection of the CFM for special uses, such as optimization, was the subject of a speculative technical note by the writer [40]. This was motivated by efforts of numerical analysts to develop sparse null-space methods [41–45]. That research appears to have been abandoned by 1990. Section 2 of [26] elaborates on why, barring unexpected breakthroughs, a resurrection of the CFM is unlikely.

A more modest revival involves the use of non-CFM *flexibility methods* for multilevel analysis. The structure is partitioned into subdomains or substructures, each of which is processed by DSM; but the subdomains are connected by Lagrange multipliers that physically represent node forces. A key driving application is massively parallel processing in which subdomains are mapped on distributed-memory processors and the force-based interface subproblem solved iteratively by FETI methods [46]. Another set of applications include inverse problems such as system identification and damage detection. Pertinent references and a historical sketch may be found in a recent article [47] that presents a hybrid variational formulation for this combined approach.

The true duality for structural mechanics is now known to involve displacements and stress functions, rather than displacements and forces. This was discovered by Fraeijs de Veubeke in the 1970s [48]. Although extendible beyond structures, the potential of this idea remains largely unexplored.

§H.10. Concluding Remarks

The patient reader who has reached this final section may have noticed that this essay is a critical overview of MSA history, rather than a recital of events. It reflects personal interpretations and opinions. There is no attempt at completeness. Only what are regarded as major milestones are covered in some detail. Furthermore there is only spotty coverage of the history of FEM itself as well as its computer implementation; this is the topic of an article under preparation for Applied Mechanics Reviews.

This outline can be hopefully instructive in two respects. First, matrix methods now in disfavor may come back in response to new circumstances. An example is the resurgence of flexibility methods in massively parallel processing. A general awareness of the older literature helps. Second, the sweeping victory of DSM over the befuddling complexity of the "matrix forest" period illustrates the virtue of Occam's proscription against multiplying entities: when in doubt chose simplicity. This dictum is relevant to the present confused state of computational mechanics.

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