16

The Isoparametric Representation

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§16.1. Introduction

The procedure used in Chapter 15to formulate the stiffness equations of the linear triangle can be formally extended to quadrilateral elements as well as higher order triangles. But one quickly encounters technical difficulties:

1. The construction of shape functions that satisfy consistency requirements for higher order elements with curved boundaries becomes increasingly complicated.
2. Integrals that appear in the expressions of the element stiffness matrix and consistent nodal force vector can no longer be evaluated in simple closed form.

These two obstacles can be overcome through the concepts of *isoparametric elements* and *numerical quadrature*, respectively. The combination of these two ideas transformed the field of finite element methods in the late 1960s. Together they support a good portion of what is presently used in production finite element programs.

In the present Chapter the concept of isoparametric representation is introduced for two dimensional elements. This representation is illustrated on specific elements. In the next Chapter these techniques, combined with numerical integration, are applied to quadrilateral elements.

§16.2. Isoparametric Representation

§16.2.1. Motivation

The linear triangle presented in Chapter 115*is* an isoparametric element although was not originally derived as such. The two key equations are (15.10), which defines the triangle geometry, and (15.16), which defines the primary variable, in this case the displacement field. These equations are reproduced here for convenience:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}, \tag{16.1}$$

$$\begin{aligned} u_x &= u_{x1} N_1^e + u_{x2} N_2^e + u_{x3} N_3^e = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3, \\ u_y &= u_{y1} N_1^e + u_{y2} N_2^e + u_{y3} N_3^e = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3. \end{aligned} \tag{16.2}$$

The interpretation of these equations is as follows. The triangular coordinates define the element geometry via (16.1). The displacement expansion (16.2) is defined by the shape functions, which are in turn expressed in terms of the triangular coordinates. For the linear triangle, shape functions and triangular coordinates coalesce.

These relations are diagrammed in Figure 16.1. Evidently geometry and displacements are not treated equally. If we proceed to higher order triangular elements while keeping straight sides, only the displacement expansion is refined whereas the geometry definition remains the same.

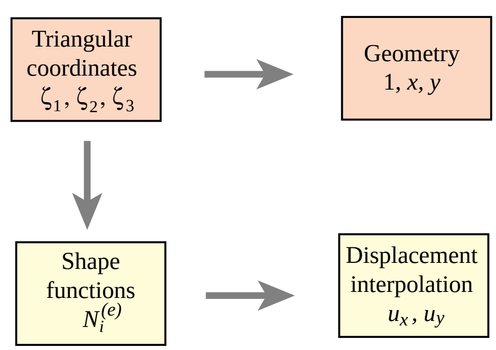

FIGURE 16.1. Superparametric representation of triangular element.

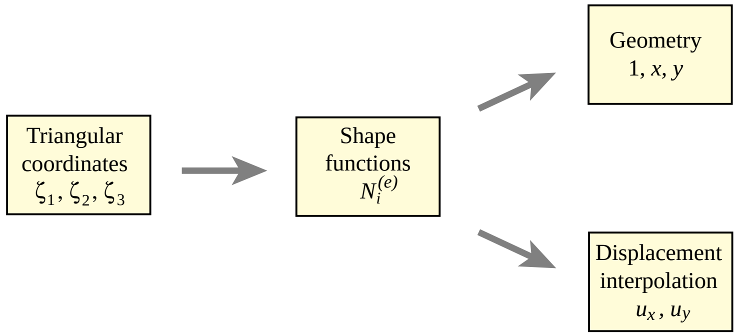

FIGURE 16.2. Isoparametric representation of triangular elements.

Elements built according to the foregoing prescription are called *superparametric*, a term that emphasizes that unequal treatment.

§16.2.2. Equalizing Geometry and Displacements

On first inspection (16.2) and (16.1) do not look alike. Their inherent similarity can be displayed, however, if the second one is rewritten and adjoined to (16.1) to look as follows:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{y3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{y3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}. \tag{16.3}$$

This form emphasizes that geometry and displacements are given by the *same* parametric representation, as shown in Figure 16.2.

The key idea is to use the shape functions to represent *both the element geometry and the problem unknowns*, which in structural mechanics are displacements. Hence the name *isoparametric element* ("iso" means equal), often abbreviated to *iso-P element*. This property may be generalized to arbitrary elements by replacing the term "triangular coordinates" by the more general one "natural coordinates." This generalization is illustrated in Figure 16.3.

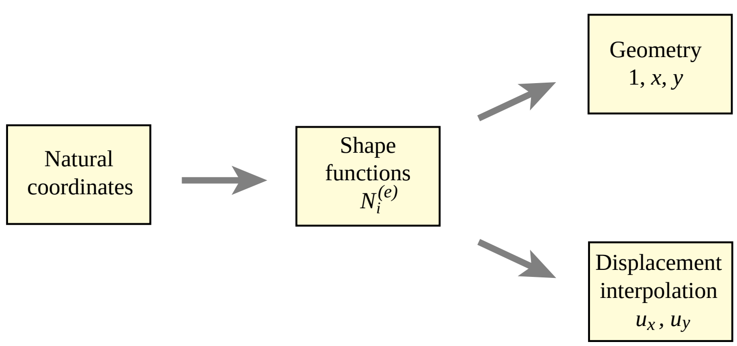

FIGURE 16.3. Isoparametric representation of arbitrary two-dimensional elements: triangles or quadrilaterals. For 3D elements, expand the geometry list to $\{1, x, y, z\}$ and the displacements to $\{u_x, u_y, u_z\}$.

Under this generalization, natural coordinates (triangular coordinates for triangles, quadrilateral coordinates for quadrilaterals) appear as *parameters* that define the shape functions. The shape functions connect the geometry with the displacements.

Remark 16.1. The terms *isoparametric* and *superparametric* were introduced by Irons and coworkers at Swansea in 1966. See **Notes and Bibliography** at the end of this Chapter. There are also *subparametric* elements whose geometry is more refined than the displacement expansion.

§16.3. General Isoparametric Formulation

The generalization of (16.3) to an arbitrary two-dimensional element with n nodes is straightforward. Two set of relations, one for the element geometry and the other for the element displacements, are required. Both sets exhibit the same interpolation in terms of the shape functions.

Geometric relations:

$$\boxed{1 = \sum_{i=1}^{n} N_i^e, \quad x = \sum_{i=1}^{n} x_i N_i^e, \quad y = \sum_{i=1}^{n} y_i N_i^e.} \tag{16.4}$$

Displacement interpolation:

$$\boxed{u_x = \sum_{i=1}^{n} u_{xi} N_i^e, \quad u_y = \sum_{i=1}^{n} u_{yi} N_i^e.} \tag{16.5}$$

These two sets of equations may be combined in matrix form as

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}. \tag{16.6}$$

The first three scalar equations in (16.6) express the geometry definition, and the last two the displacement expansion. Note that additional rows may be added to this matrix expression if more variables are interpolated by the same shape functions. For example, suppose that the thickness h and a temperature field T are both interpolated from the n node values:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \\ h \\ T \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \\ h_1 & h_2 & \dots & h_n \\ T_1 & T_2 & \dots & T_n \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}. \tag{16.7}$$

Note that the column of shape functions does not change.

To illustrate the use of the isoparametric concept, we take a look at specific 2D isoparametric elements that are commonly used in structural and non-structural applications. These are separated into triangles and quadrilaterals because different natural coordinates are used.

§16.4. Triangular Elements

§16.4.1. The Linear Triangle

The three-noded linear triangle, studied in Chapter 15 and pictured in Figure 16.4, may be presented as an isoparametric element:

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}. \tag{16.8}$$

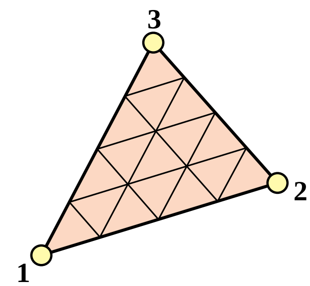

FIGURE 16.4. The 3-node linear triangle.

The shape functions are simply the triangular coordinates:

$$N_1^e = \zeta_1, \qquad N_2^e = \zeta_2, \qquad N_3^e = \zeta_3. \tag{16.9}$$

The linear triangle is the only triangular element that is both superparametric and isoparametric.

§16.4.2. The Quadratic Triangle

The six node triangle shown in Figure 16.5 is the next complete-polynomial member of the isoparametric triangle family. The isoparametric definition is

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ N_5^e \\ N_6^e \end{bmatrix} \tag{16.10}$$

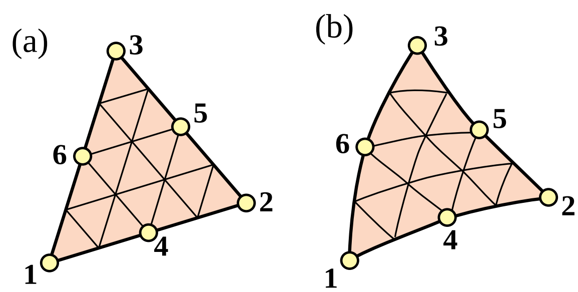

FIGURE 16.5. The 6-node quadratic triangle: (a) the superparametric version, with straight sides and midside nodes at midpoints; (b) the isoparametric version.

The shape functions are

$$\begin{aligned} &N_1^e = \zeta_1(2\zeta_1 - 1), \quad N_2^e = \zeta_2(2\zeta_2 - 1), \quad N_3^e = \zeta_3(2\zeta_3 - 1), \\ &N_4^e = 4\zeta_1\zeta_2, \quad N_5^e = 4\zeta_2\zeta_3, \quad N_6^e = 4\zeta_3\zeta_1. \end{aligned} \tag{16.11}$$

The element may have parabolically curved sides defined by the location of the midnodes 4, 5 and 6. The triangular coordinates for a curved triangle are no longer straight lines, but form a curvilinear system as can be observed in Figure 16.5(b).

§16.4.3. *The Cubic Triangle

The cubic triangle has ten nodes. This shape functions of this element are the subject of an Exercise in Chapter 18. The implementation is studied in Chapter 24.

§16.5. Quadrilateral Elements

§16.5.1. Quadrilateral Coordinates and Iso-P Mappings

Before presenting examples of quadrilateral elements, we must introduce the appropriate *natural coordinate system* for that geometry. The natural coordinates for a triangular element are the triangular coordinates ζ_1, ζ_2 and ζ_3. The natural coordinates for a quadrilateral element are ξ and η, which are illustrated in Figure 16.6 for both straight sided and curved side quadrilaterals. These are called *quadrilateral coordinates*.

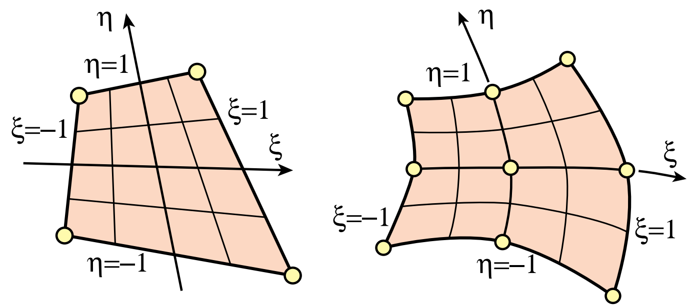

FIGURE 16.6. Quadrilateral coordinates.

These coordinates vary from -1 on one side to $+1$ at the other, taking the value zero over the quadrilateral medians. This particular variation range (instead of taking, say, 0 to 1) was chosen by Irons and coworkers to facilitate use of the standard Gauss integration formulas. Those formulas are discussed in the next Chapter.

Remark 16.2. In some FEM derivations it is convenient to visualize the quadrilateral coordinates plotted as Cartesian coordinates in the $\{\xi, \eta\}$ plane. This is called the *reference plane*. All quadrilateral elements in the reference plane become a square of side 2, called the *reference element*, which extends over $\xi \in [-1, 1]$, $\eta \in [-1, 1]$. The transformation between $\{\xi, \eta\}$ and $\{x, y\}$ dictated by the second and third equations of (16.4), is called the *isoparametric mapping*. A similar version exists for triangles. An important application of this mapping is discussed in §16.6; see Figure 16.9 there.

§16.5.2. The Bilinear Quadrilateral

The four-node quadrilateral shown in Figure 16.7 is the simplest member of the quadrilateral family. It is defined by

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \end{bmatrix}. \tag{16.12}$$

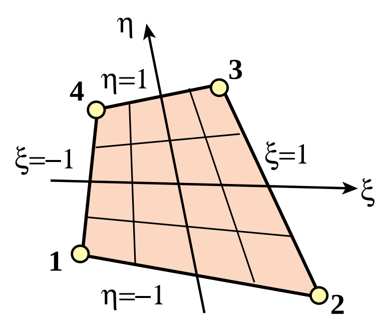

FIGURE 16.7. The 4-node bilinear quadrilateral.

The shape functions are

$$\begin{aligned} N_1^e &= \tfrac{1}{4}(1-\xi)(1-\eta), & N_2^e &= \tfrac{1}{4}(1+\xi)(1-\eta), \\ N_3^e &= \tfrac{1}{4}(1+\xi)(1+\eta), & N_4^e &= \tfrac{1}{4}(1-\xi)(1+\eta). \end{aligned} \tag{16.13}$$

These functions vary *linearly* on quadrilateral coordinate lines $\xi = const$ and $\eta = const$, but are not linear polynomials as in the case of the three-node triangle.

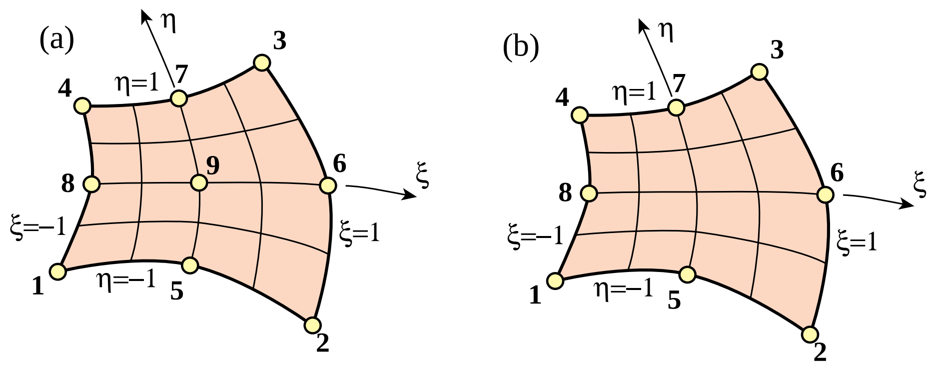

FIGURE 16.8. Two widely used higher order quadrilaterals: (a) the nine-node biquadratic quadrilateral; (b) the eight-node "serendipity" quadrilateral.

§16.5.3. The Biquadratic Quadrilateral

The nine-node quadrilateral shown in Figure 16.8(a) is the next *complete* member of the quadrilateral family. It has eight external nodes and one internal node. It is defined by

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_9^e \end{bmatrix}. \tag{16.14}$$

This element is often referred to as the *Lagrangian quadrilateral* in the FEM literature, a term explained in the **Notes and Bibliography**. Its shape functions are

$$\begin{aligned} N_1^e &= \tfrac{1}{4}(1-\xi)(1-\eta)\xi\eta, & N_5^e &= -\tfrac{1}{2}(1-\xi^2)(1-\eta)\eta, \\ N_2^e &= -\tfrac{1}{4}(1+\xi)(1-\eta)\xi\eta, & N_6^e &= \tfrac{1}{2}(1+\xi)(1-\eta^2)\xi, \qquad N_9^e = (1-\xi^2)(1-\eta^2) \\ &\ldots & &\ldots \end{aligned} \tag{16.15}$$

These functions vary *quadratically* along the coordinate lines $\xi = const$ and $\eta = const$. The shape function associated with the internal node 9 is called a *bubble function* because of its geometric shape, which is pictured in §18.4.2.

Figure 16.8(a) depicts a widely used eight-node variant called the "serendipity" quadrilateral. (A name that originated from circumstances surrounding the element discovery.) The internal node is eliminated by kinematic constraints as worked out in an Exercise of Chapter 18.

§16.6. Completeness Properties of Iso-P Elements

Some general conclusions as regards the range of applications of isoparametric elements can be obtained from a *completeness analysis*. More specifically, whether the general prescription (16.6) that combines (16.4) and (16.5) satisfies the *completeness* criterion of finite element trial expansions. This is one of the conditions for convergence to the analytical solution. The requirement is treated generally in Chapter 19, and is stated here in recipe form.

§16.6.1. *Completeness Analysis

The plane stress problem has variational index $m = 1$. A set of shape functions is complete for this problem if they can represent exactly any *linear* displacement motions such as

$$u_x = \alpha_0 + \alpha_1 x + \alpha_2 y, \qquad u_y = \beta_0 + \beta_1 x + \beta_2 y. \tag{16.16}$$

To carry out the check, evaluate (16.16) at the nodes

$$u_{xi} = \alpha_0 + \alpha_1 x_i + \alpha_2 y_i \qquad u_{yi} = \beta_0 + \beta_1 x_i + \beta_2 y_i, \qquad i = 1, \ldots n. \tag{16.17}$$

Insert this into the displacement expansion (16.5) to see whether the linear displacement field (16.16) is recovered. Here are the computations for the displacement component u_x:

$$u_x = \sum_{i=1}^{n} (\alpha_0 + \alpha_1 x_i + \alpha_2 y_i)\, N_i^e = \alpha_0 \sum_i N_i^e + \alpha_1 \sum_i x_i N_i^e + \alpha_2 \sum_i y_i N_i^e = \alpha_0 + \alpha_1 x + \alpha_2 y. \tag{16.18}$$

For the last step we have used the geometry definition relations (16.4), reproduced here for convenience:

$$\boxed{1 = \sum_{i=1}^{n} N_i^e, \quad x = \sum_{i=1}^{n} x_i N_i^e, \quad y = \sum_{i=1}^{n} y_i N_i^e.} \tag{16.19}$$

A similar calculation may be made for u_y. It appears that the isoparametric displacement expansion represents (16.18) for *any* element, and consequently meets the completeness requirement for variational order $m = 1$. The derivation carries without essential change to three dimensions.[1]

Can you detect a flaw in this conclusion? The fly in the ointment is the last replacement step of (16.18), which assumes that the geometry relations (16.19) *are identically satisfied*. Indeed they are for all the example elements presented in the previous sections. But if the new shape functions are constructed directly by the methods of Chapter 18, *a posteriori* checks of those identities are necessary.

§16.6.2. Completeness Checks

The first check in (16.19) is easy: *the sum of shape functions must be unity*. This is also called the *unit sum condition*. It can be easily verified by hand for simple elements. Here are two examples.

Example 16.1. Check for the linear triangle: directly from the definition of triangular coordinates,

$$N_1^e + N_2^e + N_3^e = \zeta_1 + \zeta_2 + \zeta_3 = 1. \tag{16.20}$$

[1] This derivation is due to B. M. Irons. See for example [397, p. 75]. The property was known since the mid 1960s and contributed substantially to the rapid acceptance of iso-P elements.

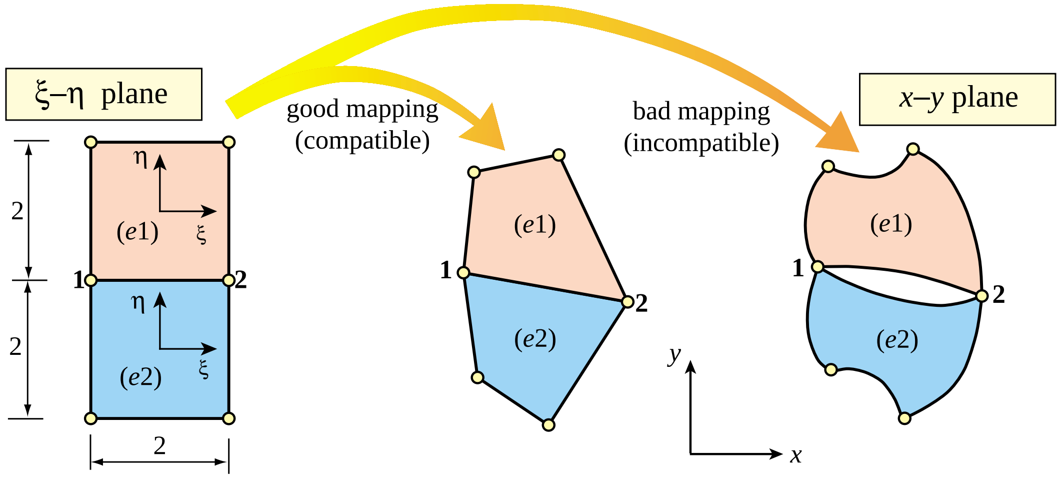

FIGURE 16.9. Good and bad isoparametric mappings of 4-node quadrilateral from the $\{\xi,\eta\}$ reference plane onto the $\{x,y\}$ physical plane.

Example 16.2. Check for the 4-node bilinear quadrilateral:

$$
\begin{aligned}
N_1^e + N_2^e + N_3^e + N_4^e &= \tfrac{1}{4}(1-\xi-\eta+\xi\eta) + \tfrac{1}{4}(1+\xi-\eta-\xi\eta) \\
&\quad + \tfrac{1}{4}(1+\xi+\eta+\xi\eta) + \tfrac{1}{4}(1-\xi+\eta-\xi\eta) = 1
\end{aligned}
\tag{16.21}
$$

For more complicated elements see Exercises 16.2 and 16.3.

The other two checks are less obvious. For specificity consider the 4-node bilinear quadrilateral. The geometry definition equations are

$$
x = \sum_{i=1}^{4} x_i N_i^e(\xi,\eta), \qquad y = \sum_{i=1}^{4} y_i N_i^e(\xi,\eta). \tag{16.22}
$$

Given the corner coordinates, $\{x_i, y_i\}$ and a point $P(x,y)$ one can try to solve for $\{\xi,\eta\}$. This solution requires nontrivial work because it involves two coupled quadratics, but can be done. Reinserting into (16.22) simply gives back x and y, and nothing is gained.[2]

The correct question to pose is: is the correct geometry of the quadrilateral preserved by the mapping from $\{\xi,\eta\}$ to $\{x,y\}$? In particular, are the sides straight lines? Figure 16.9 illustrate these questions. Two side-two squares: $(e1)$ and $(e2)$, contiguous in the $\{\xi,\eta\}$ reference plane, are mapped to quadrilaterals $(e1)$ and $(e2)$ in the $\{x,y\}$ physical plane through (16.22). The common side 1-2 must remain a straight line to preclude interelement gaps or interpenetration.

We are therefore lead to consider *geometric compatibility* upon mapping. But this is equivalent to the question of *interelement displacement compatibility*, which is stipulated as item (C) in §18.1. The statement "the displacement along a side must be uniquely determined by nodal displacements on that side" translates to "the coordinates of a side must be uniquely determined by nodal coordinates on that side." Summarizing:

[2] This tautology is actually a blessing, since finding explicit expressions for the natural coordinates in terms of x and y rapidly becomes impossible for higher order elements. See, for example, the complications that already arise for the bilinear quadrilateral in §23.3.

$$\boxed{\text{Unit-sum condition + interelement compatibility} \rightarrow \text{completeness.}} \tag{16.23}$$

This subdivision of work significantly reduces the labor involved in element testing.

§16.6.3. *Completeness for Higher Variational Index

The completeness conditions for variational index 2 are far more demanding because they involve quadratic motions. No simple isoparametric configurations satisfy those conditions. Consequently isoparametric formulations have limited importance in the finite element analysis of plate and shell bending.

§16.7. Iso-P Elements in One and Three Dimensions

The reader should not think that the concept of isoparametric representation is confined to two-dimensional elements. It applies without conceptual changes to one and three dimensions *as long as the variational index remains one*.[3] Three-dimensional solid elements are covered in an advanced course. The use of the isoparametric formulation to construct a 3-node bar element is the topic of Exercises 16.4 through 16.7.

Notes and Bibliography

A detailed presentation of the isoparametric concept, with annotated references to the original 1960 papers may be found in the textbook [397].

This matrix representation for isoparametric elements used here was introduced in [204].

The term *Lagrangian element* in the mathematical FEM literature identifies quadrilateral and hexahedra (brick) elements that include all polynomial terms $\xi^i \eta^j$ (in 2D) or $\xi^i \eta^j \mu^k$ (in 3D) with $i \leq n$, $j \leq n$ and $k \leq n$, as part of the shape function interpolation. Such elements have $(n+1)^2$ nodes in 2D and $(n+1)^3$ nodes in 3D, and the interpolation is said to be n-bicomplete. For example, if $n = 2$, the biquadratic quadrilateral with $(2+1)^2 = 9$ nodes is Lagrangian and 2-bicomplete. (The qualifier "Lagrangian" in this context refers to Lagrange's interpolation formula, not to Lagrange multipliers.)

References

Referenced items have been moved to Appendix R

[3] A limitation explained in §16.6.3.

Homework Exercises for Chapter 16

The Isoparametric Representation

EXERCISE 16.1 [D:10] What is the physical interpretation of the shape-function unit-sum condition discussed in §16.6? Hint: the element must respond exactly in terms of displacements to rigid-body translations in the x and y directions.

EXERCISE 16.2 [A:15] Check by algebra that the sum of the shape functions for the six-node quadratic triangle (16.11) is exactly one regardless of natural coordinates values. Hint: show that the sum is expressable as $2S_1^2 - S_1$, where $S_1 = \zeta_1 + \zeta_2 + \zeta_3$.

EXERCISE 16.3 [A/C:15] Complete the table of shape functions (16.23) of the nine-node biquadratic quadrilateral. Verify that their sum is exactly one.

EXERCISE 16.4 [A:20] Consider a three-node bar element referred to the natural coordinate ξ. The two end nodes and the midnode are identified as 1, 2 and 3, respectively. The natural coordinates of nodes 1, 2 and 3 are $\xi = -1$, $\xi = 1$ and $\xi = 0$, respectively. The variation of the shape functions $N_1(\xi)$, $N_2(\xi)$ and $N_3(\xi)$ is sketched in Figure E16.1. These functions must be quadratic polynomials in ξ:

$$N_1^e(\xi) = a_0 + a_1\xi + a_2\xi^2, \quad N_2^e(\xi) = b_0 + b_1\xi + b_2\xi^2, \quad N_3^e(\xi) = c_0 + c_1\xi + c_2\xi^2. \tag{E16.1}$$

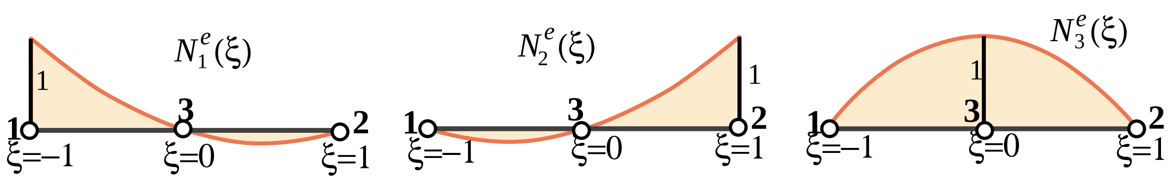

FIGURE E16.1. Isoparametric shape functions for 3-node bar element (sketch). Node 3 has been drawn at the 1–2 midpoint but it may be moved away from it, as in Exercises E16.5 and E16.6.

Determine the coefficients a_0, through c_2 using the node value conditions depicted in Figure E16.1; for example $N_1^e = 1$, 0 and 0 for $\xi = -1$, 0 and 1 at nodes 1, 3 and 2, respectively. Proceeding this way show that

$$N_1^e(\xi) = -\tfrac{1}{2}\xi(1-\xi), \qquad N_2^e(\xi) = \tfrac{1}{2}\xi(1+\xi), \qquad N_3^e(\xi) = 1 - \xi^2. \tag{E16.2}$$

Verify that their sum is identically one.

EXERCISE 16.5

[A/C:15+10+15+5] A 3-node straight bar element is defined by 3 nodes: 1, 2 and 3, with axial coordinates x_1, x_2 and x_3, respectively, as illustrated in Figure E16.2. The element has axial rigidity EA and length $\ell = x_2 - x_1$. The axial displacement is $u(x)$. The 3 degrees of freedom are the axial node displacements u_1, u_2 and u_3. The isoparametric definition of the element is

$$\begin{bmatrix} 1 \\ x \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}, \tag{E16.3}$$

in which $N_i^e(\xi)$ are the shape functions (E16.2) of the previous Exercise. Node 3 lies between 1 and 2 but is not necessarily at the midpoint $x = \frac{1}{2}\ell$. For convenience define

$$x_1 = 0, \qquad x_2 = \ell, \qquad x_3 = (\tfrac{1}{2} + \alpha)\ell, \tag{E16.4}$$

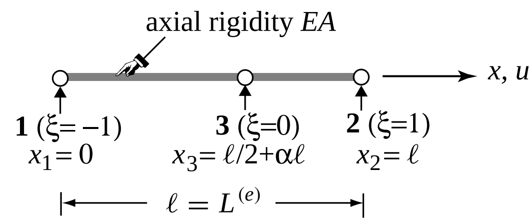

FIGURE E16.2. The 3-node bar element in its local system.

where $-\frac{1}{2} < \alpha < \frac{1}{2}$ characterizes the location of node 3 with respect to the element center. If $\alpha = 0$ node 3 is located at the midpoint between 1 and 2. See Figure E16.2.

(a) From (E16.4) and the second equation of (E16.3) get the Jacobian $J = dx/d\xi$ in terms of ℓ, α and ξ. Show that: (i) if $-\frac{1}{4} < \alpha < \frac{1}{4}$ then $J > 0$ over the whole element $-1 \le \xi \le 1$; (ii) if $\alpha = 0$, $J = \ell/2$ is constant over the element.

(b) Obtain the 1×3 strain-displacement matrix $\mathbf{B}$ relating $e = du/dx = \mathbf{B}\,\mathbf{u}^e$, where $\mathbf{u}^e$ is the column 3-vector of node displacements u_1, u_2 and u_3. The entries of $\mathbf{B}$ are functions of ℓ, α and ξ. Hint: $\mathbf{B} = d\mathbf{N}/dx = J^{-1} d\mathbf{N}/d\xi$, where $\mathbf{N} = [\, N_1 \;\; N_2 \;\; N_3 \,]$ and J comes from item (a).

(c) Show that the element stiffness matrix is given by

$$\mathbf{K}^e = \int_0^\ell EA\,\mathbf{B}^T\mathbf{B}\,dx = \int_{-1}^1 EA\,\mathbf{B}^T\mathbf{B}\,J\,d\xi. \tag{E16.5}$$

Evaluate the rightmost integral for arbitrary α but constant EA using the 2-point Gauss quadrature rule (E13.7). Specialize the result to $\alpha = 0$, for which you should get $K_{11} = K_{22} = 7EA/(3\ell)$, $K_{33} = 16EA/(3\ell)$, $K_{12} = EA/(3\ell)$ and $K_{13} = K_{23} = -8EA/(3\ell)$, with eigenvalues $\{8EA/\ell, 2EA/\ell, 0\}$. Note: use of a CAS is recommended for this item to save time.

(d) What is the minimum number of Gauss points needed to integrate $\mathbf{K}^e$ exactly if $\alpha = 0$?

EXERCISE 16.6 [A/C:20] This Exercise is a continuation of the foregoing one, and addresses the question of why $\mathbf{K}^e$ was computed by numerical integration in item (c). Why not use exact integration? The answer is that the exact stiffness for arbitrary α is numerically useless. To see why, try the following script in *Mathematica*:

```
ClearAll[EA,L,alpha,xi]; (* Define J and B={{B1,B2,B3}} here *)
Ke=Simplify[Integrate[EA*Transpose[B].B*J,{xi,-1,1},
   Assumptions->alpha>0&&alpha<1/4&&EA>0&&L>0]];
Print["exact Ke=",Ke//MatrixForm];
Print["exact Ke for alpha=0",Simplify[Ke/.alpha->0]//MatrixForm];
Keseries=Normal[Series[Ke,{alpha,0,2}]];
Print["Ke series about alpha=0:",Keseries//MatrixForm];
Print["Ke for alpha=0",Simplify[Keseries/.alpha->0]//MatrixForm];
```

At the start of this script define `J` and `B` with the results of items (a) and (b), respectively. Then run the script. The line `Print["exact Ke for alpha=0",Simplify[Ke/.alpha->0]//MatrixForm]` will trigger error messages. Comment on why the exact stiffness cannot be evaluated directly at $\alpha = 0$ (look at the printed expression before this one). A Taylor series expansion about $\alpha = 0$ circumvents these difficulties but the 2-point Gauss integration rule gives the correct answer without the gyrations.

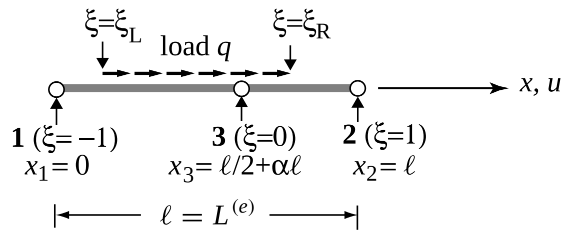

FIGURE E16.3. The 3-node bar element under a "box" axial load q.

EXERCISE 16.7 [A/C:20] Construct the consistent force vector for the 3-node bar element of the foregoing exercise, if the bar is loaded by a uniform axial force q (given per unit of x length) that extends from $\xi = \xi_L$ through $\xi = \xi_R$, and is zero otherwise. Here $-1 \leq \xi_L < \xi_R \leq 1$. See Figure E16.3. Use

$$\mathbf{f}^e = \int_{-\xi_L}^{\xi_R} q\,\mathbf{N}^T\,J\,d\xi, \tag{E16.6}$$

with the $J = dx/d\xi$ found in Exercise 16.5(a) and analytical integration. The answer is quite complicated and nearly hopeless by hand. Specialize the result to $\alpha = 0$, $\xi_L = -1$ and $\xi_R = 1$.