# 5

# Constructing MoM Members

## TABLE OF CONTENTS

## §5.1. Introduction

The truss member used as example in Chapters 2–4 is an instance of a *structural element*. Such elements may be formulated directly using concepts and modeling techniques developed in Mechanics of Materials (MoM).[1] The construction does not involve the more advanced tools that are required for the continuum finite elements that appear in Part II.

This Chapter presents an overview of the technique to construct the element stiffness equations of "MoM members" using simple matrix operations. These simplified equations come in handy for a surprisingly large number of applications, particularly in skeletal structures. Focus is on *simplex elements*, which may be formed directly as a sequence of matrix operations. Non-simplex elements are presented as a recipe because their proper formulation requires work theorems not yet studied.

The physical interpretation of the FEM is still emphasized. Consequently we continue to speak of structures built up of *members* (elements) connected at *joints* (nodes).

## §5.2. Formulation of MoM Members

### §5.2.1. What They Look Like

MoM-based formulations are largely restricted to *intrinsically one-dimensional members*. These are structural components one of whose dimensions, called *longitudinal*, is significantly larger than the other two, called the *transverse dimensions*. Such members are amenable to the simplified structural theories developed in MoM textbooks. This Chapter covers only *straight* members with geometry defined by the two end joints.[2] The member *cross sections* are defined by the intersection of planes normal to the longitudinal dimension with the member. See Figure 5.1. Note that although the individual member will be idealized as being one-dimensional in its intrinsic or local coordinate system, it often functions as component of a two- or three-dimensional structure.

This class of structural components embodies bars, beams, beam-columns, shafts and spars. Although geometrically similar, the names distinguish the main kind of internal forces the member resists and transmits: axial forces for bars, bending and shear forces for beams, axial compression and bending for beam-columns, torsion forces for shafts, and shear forces for spars.

Members are connected at their end joints by displacement degrees of freedom. For truss (bar) and spar members those freedoms are translational components of the joint displacements. For other types, notably beams and shafts, nodal rotations are chosen as additional degrees of freedom.

Structures fabricated with MoM members are generally three-dimensional. Their geometry is defined with respect to a global Cartesian coordinate system $\{x, y, z\}$. Two-dimensional idealizations are useful simplifications should the nature of the geometry and loading allow the reduction of the structural model to one plane of symmetry, which is chosen to be the $\{x, y\}$ plane. Plane trusses and plane frameworks are examples of such simplifications.

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[1] Mechanics of Materials was called Strength of Materials in older texts. It covers bars, beams, shafts, arches, thin plates and shells, but only one-dimensional models are considered in introductory undergraduate courses. MoM involves *ab initio* phenomenological assumptions such as "plane sections remain plane" or "shear effects can be neglected in thin beams." These came about as the byproduct of two centuries of structural engineering practice, justified by success. A similar acronym (MOM) is used in Electrical Engineering for something completely different: the Method of Moments.

[2] Advanced Mechanics of Materials includes curved members. Plane arch elements are studied in Chapter 13.

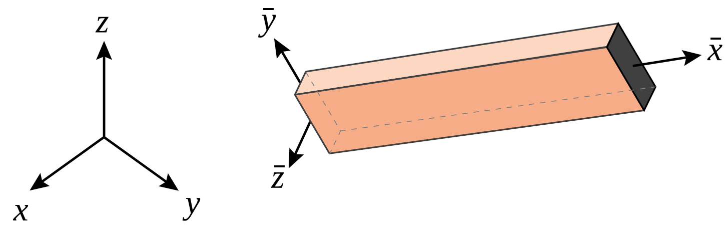


FIGURE 5.1. A Mechanics of Materials (MoM) member is a structural element one of whose dimensions (the longitudinal dimension) is significantly larger than the other two. Local axes $\{\bar{x}, \bar{y}, \bar{z}\}$ are chosen as indicated. Although the depicted member is prismatic, some applications utilize tapered or stepped members, the cross section of which varies as a function of $\bar{x}$.

In this Chapter we study generic structural members that fit the preceding class. An individual member is identified by $e$ but this superscript will be usually suppressed in the equations below to reduce clutter. The local axes are denoted by $\{\bar{x}, \bar{y}, \bar{z}\}$, with $\bar{x}$ along the longitudinal direction. See Figure 5.1.

The mathematical model of a MoM member is obtained by an idealization process. The model represents the member as a line segment that connects the two end joints, as depicted in Figure 5.2.

### §5.2.2. End Quantities, Degrees of Freedom, Joint Forces

The set of mathematical variables used to link members are called *end quantities* or *connectors*. In the Direct Stiffness Method (DSM) these are joint displacements (the degrees of freedom) and the joint forces. These quantities are related by the member stiffness equations.

The degrees of freedom at the end joints $i$ and $j$ are collected in the joint displacement vector $\bar{\mathbf{u}}$. This may include translations only, rotations only, or a combination of translations and rotations.

The vector of joint forces $\bar{\mathbf{f}}$ groups components in one to one correspondence with $\bar{\mathbf{u}}$. Component pairs must be *conjugate* in the sense of the Principle of Virtual Work. For example if the $\bar{x}$-translation at joint $i$: $\bar{u}_{xi}$ appears in $\bar{\mathbf{u}}$, the corresponding entry in $\bar{\mathbf{f}}$ is the $\bar{x}$-force $\bar{f}_{xi}$ at $i$. If the rotation about $\bar{z}$ at joint $j$: $\bar{\theta}_{zj}$ appears in $\bar{\mathbf{u}}$, the corresponding entry in $\bar{\mathbf{f}}$ is the $z$-moment $\bar{m}_{zj}$.

### §5.2.3. Internal Quantities

Internal quantities are mechanical actions that take place within the member. Those actions involve stresses and deformations. Accordingly two types of internal quantities appear:

*Internal member forces* form a finite set of stress-resultant quantities collected in an array $\mathbf{p}$. They are obtained by integrating stresses over each cross section, and thus are also called generalized stresses in structural mechanics. This set characterizes the forces resisted by the material. Stresses at any point in a section may be recovered if $\mathbf{p}$ is known.

*Member deformations* form a finite set of quantities, chosen in one-to one correspondence with internal member forces, and collected in an array $\mathbf{v}$. This set characterizes the deformations experienced by the material. They are also called generalized strains in structural theory. Strains at any point in the member can be recovered if $\mathbf{v}$ is known.

As in the case of end quantities, internal forces and deformations are paired in one to one correspondence. For example, the axial force in a bar member must be paired either with an average

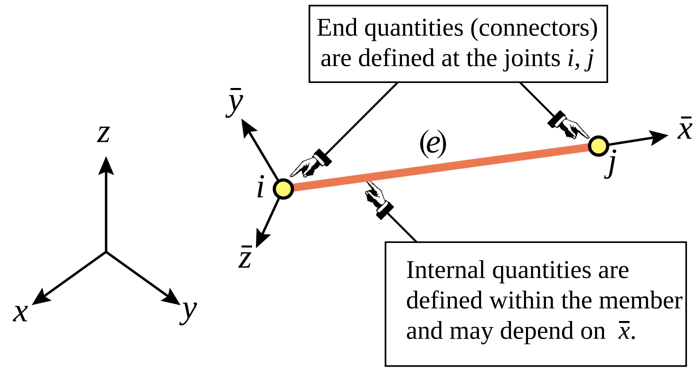


FIGURE 5.2. The FE mathematical idealization of a MoM member. The model is one-dimensional in $\bar{x}$. The two end joints are the site of end quantities: joint forces and displacements, that interconnect members. Internal quantities characterize internal forces, stresses and deformations in the member.

axial deformation, or with the total elongation. Pairs that mutually correspond in the sense of the Principle of Virtual Work are called *conjugate*. Unlike the case of end quantities, conjugacy of internal quantities is not a mandatory requirement although it simplifies some expressions.

### §5.2.4. Discrete Field Equations, Tonti Diagram

The matrix equations that connect $\bar{\mathbf{u}}$, $\mathbf{v}$, $\mathbf{p}$ and $\bar{\mathbf{f}}$ are called the *discrete field equations*. There are three of them.

The member deformations $\mathbf{v}$ are linked to the joint displacements $\bar{\mathbf{u}}$ by the kinematic compatibility conditions, also called the deformation-displacement or strain-displacement equations:

$$\boxed{\mathbf{v} = \mathbf{B}\,\bar{\mathbf{u}}.} \tag{5.1}$$

The internal member forces are linked to the member deformations by the constitutive equations. In the absence of initial strain effects those equations are homogeneous:

$$\boxed{\mathbf{p} = \mathbf{S}\,\mathbf{v}.} \tag{5.2}$$

Finally, the internal member forces are linked to the joint forces by the equilibrium equations. If the internal forces $\mathbf{p}$ are *constant over the member*, the relation is simply

$$\boxed{\bar{\mathbf{f}} = \mathbf{A}^T\,\mathbf{p}.} \tag{5.3}$$

In (5.3) the transpose of $\mathbf{A}$ is used for convenience.[3]

The foregoing equations can be presented graphically as shown in Figure 5.3. This is a discrete version of the so-called *Tonti diagrams*, which represent governing equations as arrows linking boxes containing kinematic and static quantities. Tonti diagrams for field (continuum) equations are introduced in Chapter 11.

[3] If $\mathbf{p}$ is a function of $\bar{x}$ the relation is of differential type. This form is studied in §5.4.

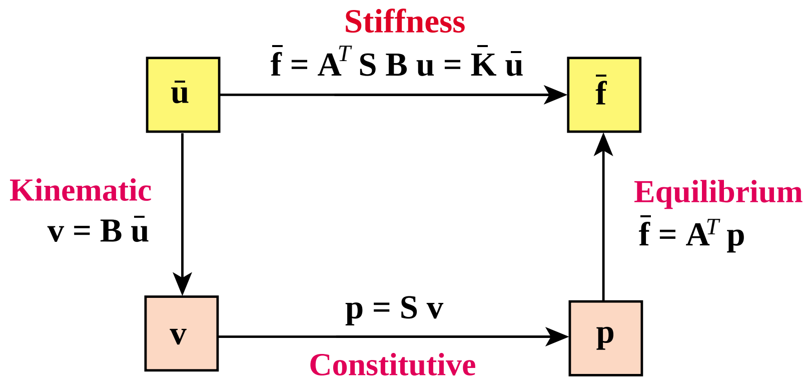


FIGURE 5.3. Tonti diagram of the three discrete field equations (5.1)–(5.3) and the stiffness equation (5.4) fora *simplex* MoM member. Internal and end quantities appear inside the orange and yellow boxes, respectively.

Matrices **B**, **S** and **A** receive the following names in the literature:

- **A** Equilibrium, leverage
- **S** Rigidity, material, constitutive[4]
- **B** Compatibility, deformation-displacement, strain-displacement

If the element is sufficiently simple, the determination of these three matrices can be carried out through MoM techniques. If the construction requires more advanced tools, however, recourse to the general methodology of finite elements and variational principles is necessary.

## §5.3. Simplex MoM Members

Throughout this section we assume that the *internal quantities are constant over the member length*. Such members are called *simplex elements*. If so the matrices **A**, **B** and **S** are *independent* of member cross section. For simplex elements the derivation of the element stiffness equations reduces to a straightforward sequence of matrix multiplications.

Under the constancy-along-$\bar{x}$ assumption, elimination of the interior quantities **p** and **v** from (5.1) through (5.3) yields the element stiffness relation

$$\bar{\mathbf{f}} = \mathbf{A}^T \mathbf{S}\,\mathbf{B}\,\bar{\mathbf{u}} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}, \tag{5.4}$$

whence the element stiffness matrix is

$$\boxed{\bar{\mathbf{K}} = \mathbf{A}^T \mathbf{S}\,\mathbf{B}.} \tag{5.5}$$

If both pairs: $\{\mathbf{p}, \mathbf{v}\}$ and $\{\bar{\mathbf{f}}, \bar{\mathbf{u}}\}$, are conjugate in the sense of the Principle of Virtual Work, it can be shown that $\mathbf{A} = \mathbf{B}$ and that **S** is symmetric. In that case

$$\boxed{\bar{\mathbf{K}} = \mathbf{B}^T \mathbf{S}\mathbf{B}.} \tag{5.6}$$

is a symmetric matrix. Symmetry is computationally desirable for reasons outlined in Part III.

[4] The name *rigidity matrix* for **S** is preferable. It is a member integrated version of the cross section constitutive equations. The latter are usually denoted by symbol **R**, as in §5.4.

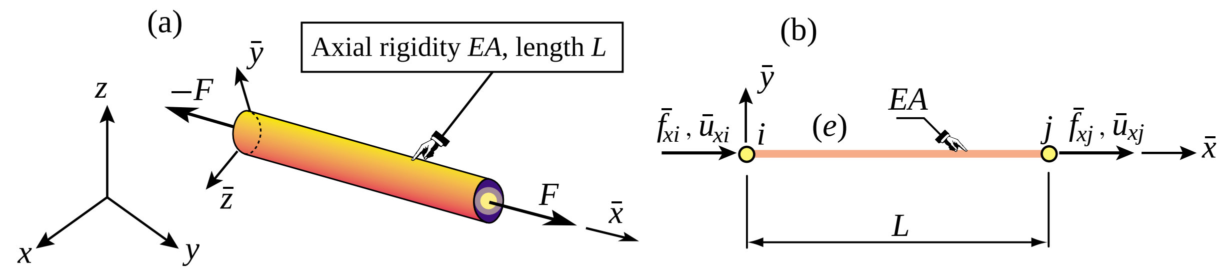


FIGURE 5.4. The prismatic bar (also called truss) member: (a) individual member shown in 3D space, (b) idealization as generic member.

**Remark 5.1**. If $\bar{\mathbf{f}}$ and $\bar{\mathbf{u}}$ are conjugate (as required in §5..2.2) but $\mathbf{p}$ and $\mathbf{v}$ are not, $\bar{\mathbf{K}}$ must come out to be symmetric even if $\mathbf{S}$ is unsymmetric and $\mathbf{A} \neq \mathbf{B}$. However there are more opportunities to go wrong.

### §5.3.1. The Bar Element Revisited

The simplest MoM element is the prismatic bar or truss member already derived in Chapter 2. See Figure 5.4. This qualifies as simplex because all internal quantities are constant. One minor difference in the derivation below is that the joint displacements and forces in the $\bar{y}$ direction are omitted in the generic element because they contribute nothing to the stiffness equations. In the FEM terminology, freedoms associated with zero stiffness are called *inactive*. Three choices for internal deformation and force variables are considered next. The results confirm that the element stiffness equations coalesce, as expected, since the end quantities $\bar{\mathbf{f}}$ and $\bar{\mathbf{u}}$ stay the same.

*Derivation Using Axial Elongation and Axial Force.* The member axial elongation $d$ is taken as deformation measure, and the member axial force $F$ as internal force measure. Hence $\mathbf{v}$ and $\mathbf{p}$ reduce to the scalars $\equiv d$ and $p \equiv F$, respectively. The chain of discrete field equations is easily constructed:

$$d = [\,-1 \quad 1\,] \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{xj} \end{bmatrix} = \mathbf{B}\bar{\mathbf{u}}, \qquad F = \frac{EA}{L} d = S\,d, \qquad \bar{\mathbf{f}} = \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{xj} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} F = \mathbf{A}^T\,F. \tag{5.7}$$

Consequently

$$\bar{\mathbf{K}} = \mathbf{A}^T S \mathbf{B} = S \mathbf{B}^T \mathbf{B} = S\,[\,-1 \quad 1\,] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{5.8}$$

Note that $\mathbf{A} = \mathbf{B}$ because $F$ and $d$ are conjugate: $F\,d$ is work. The foregoing equations can be represented graphyically with the discrete Tonti diagram of Figure 5.5.

*Derivation Using Mean Axial Strain and Axial Force.* Instead of $d$ we may use the mean axial strain $\bar{e} = d/L$ as deformation measure whereas $F$ is kept as internal force measure. The only change is that $\mathbf{B}$ becomes $[\,-1 \quad 1\,]/L$ whereas $S$ becomes $EA$. Matrix $\mathbf{A}$ does not change. The product $\mathbf{A}^T S\mathbf{B}$ gives the same $\bar{\mathbf{K}}$ as in (5.8), as can be expected. Now $\mathbf{A}^T$ is not equal to $\mathbf{B}$ because $F$ and $\bar{e}$ are not conjugate, but they differ only by a factor $1/L$.

*Derivation Using Mean Axial Strain and Axial Stress.* We keep the mean axial strain $\bar{e} = d/L$ as deformation measure but take the mean axial stress $\bar{\sigma} = F/A$ (which is *not* conjugate to $\bar{e}$) as

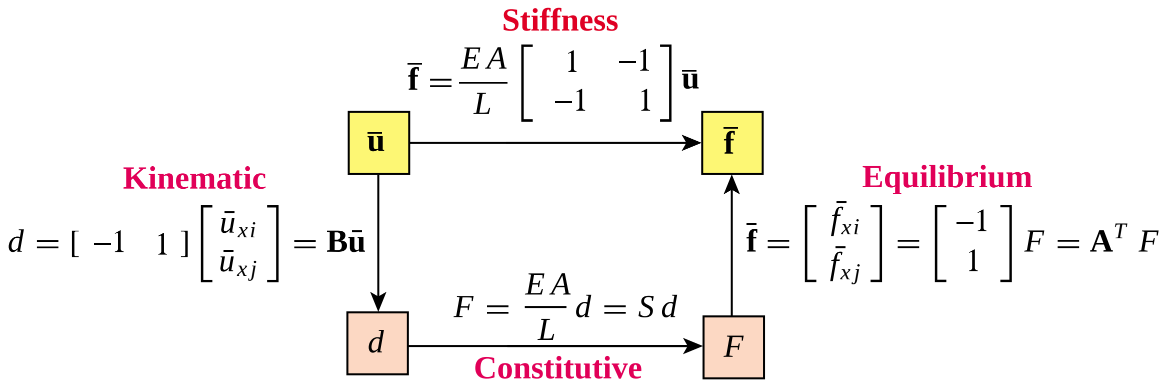


FIGURE 5.5. Tonti diagram for the bar element discrete equations (5.7)–(5.8).

internal force measure. Now $\mathbf{B} = [\,-1 \quad 1\,]/L$, $S = E$ and $\mathbf{A}^T = A\,[\,-1 \quad 1\,]$. The product $\mathbf{A}^T S\mathbf{B}$ gives again the same $\bar{\mathbf{K}}$. as shown in (5.8).

*Transformation to Global Coordinates*. Since $\bar{u}_{yi}$ and $\bar{u}_{yj}$ are not part of (5.7) and (5.8) the displacement transformation matrix from local to global $\{x, y\}$ coordinates is $2 \times 4$, instead of $4 \times 4$ as in §2.8.1. On restoring the element identifier $e$ the appropriate local-to-global transformation is

$$\bar{\mathbf{u}}^e = \begin{bmatrix} \bar{u}^e_{xi} \\ \bar{u}^e_{xj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} u^e_{xi} \\ u^e_{yi} \\ u^e_{xj} \\ u^e_{yj} \end{bmatrix} = \mathbf{T}^e\,\mathbf{u}^e, \tag{5.9}$$

in which $c = \cos\varphi^e$, $s = \sin\varphi^e$, and $\varphi^e$ is the angle from $x$ to $\bar{x}$, cf. Figure 2.10 of Chapter 2. The $4 \times 4$ globalized element stiffness matrix $\mathbf{K}^e = (\mathbf{T}^e)^T\,\bar{\mathbf{K}}^e\,\mathbf{T}^e$ agrees with (2.18). The extension of (5.9) to 3D is handled as an Exercise.

### §5.3.2. The Spar Element

The *spar* or *shear-web* member has two joints (end nodes): $i$ and $j$. This member can only resist and transmit a *constant shear force* $V$ in the plane of the web, which is chosen to be the $\{\bar{x}, \bar{y}\}$ plane. See Figure 5.6. It is often used for modeling high-aspect aircraft wing structures, as illustrated in Figure 5.7. We consider here only *prismatic* spar members of uniform material and constant cross section, which thus qualify as simplex.

The active degrees of freedom for a generic spar member of length $L$, as depicted in Figure 5.6(b), are $\bar{u}_{yi}$ and $\bar{u}_{yj}$. Let $G$ be the shear modulus and $A_s$ the effective shear area. The latter is a concept developed in Mechanics of Materials; for a narrow rectangular cross section, $A_s = 5A/6$.

The shear rigidity is $GA_s$. As deformation measure the mean shear strain $\gamma = V/(GA_s)$ is chosen. The kinematic, constitutive, and equilibrium equations are

$$\gamma = \frac{1}{L}[\,-1 \quad 1\,]\begin{bmatrix} \bar{u}_{yi} \\ \bar{u}_{yj} \end{bmatrix} = \mathbf{B}\bar{\mathbf{u}}, \qquad V = GA_s\gamma = S\gamma, \qquad \bar{\mathbf{f}} = \begin{bmatrix} \bar{f}_{yi} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} V = \mathbf{A}^T V, \tag{5.10}$$

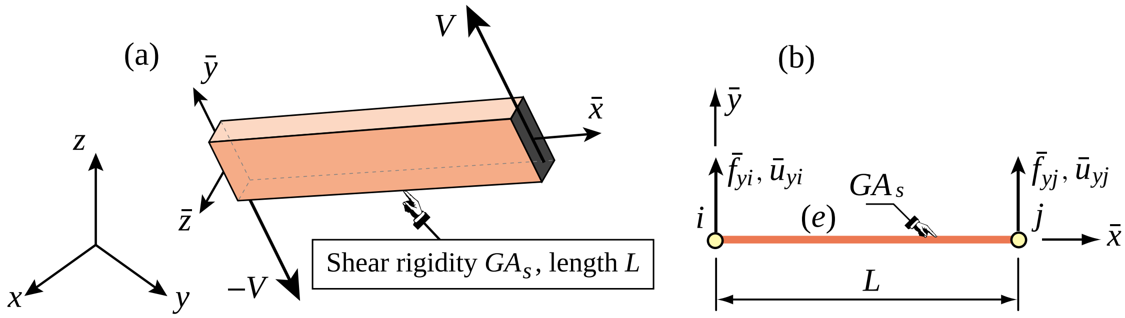


FIGURE 5.6. The prismatic spar (also called shear-web) member: (a) individual member shown in 3D space, (b) idealization as generic member in local system.

Note that $\mathbf{A} \neq \mathbf{B}$ because $V$ and $\gamma$ are not work-conjugate. (This difference is easily adjusted for, however; see Exercise 5.1.) The local stiffness equations follow as

$$\bar{\mathbf{f}} = \begin{bmatrix} \bar{f}_{yi} \\ \bar{f}_{yj} \end{bmatrix} = \mathbf{A}^T\, S\, \mathbf{B}\bar{\mathbf{u}} = \frac{GA_s}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_{yi} \\ \bar{u}_{yj} \end{bmatrix} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}. \tag{5.11}$$

If the spar member is used in a *two dimensional* context, the displacement transformation from local to global coordinates $\{x, y\}$ is

$$\bar{\mathbf{u}}^e = \begin{bmatrix} \bar{u}^e_{yi} \\ \bar{u}^e_{yj} \end{bmatrix} = \begin{bmatrix} -s & c & 0 & 0 \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u^e_{xi} \\ u^e_{yi} \\ u^e_{xj} \\ u^e_{yj} \end{bmatrix} = \mathbf{T}^e\,\mathbf{u}^e, \tag{5.12}$$

in which $c = \cos\varphi^e$, $s = \sin\varphi^e$, and $\varphi^e$ is the angle from $x$ to $\bar{x}$, cf. Figure 2.10 of Chapter 2. The $4 \times 4$ globalized spar stiffness matrix is then $\mathbf{K}^e = (\mathbf{T}^e)^T\,\bar{\mathbf{K}}^e\,\mathbf{T}^e$. The full expression is worked out in Exercise 5.2.

More often, however, the spar member will be a component in a *three-dimensional structural model*, e.g. the aircraft wing shown in Figure 5.7. If so the determination of the $2 \times 6$ transformation matrix is more involved, as an "orientation node" is required. This is the topic of Exercise 5.5.

### §5.3.3. The Shaft Element

The *shaft*, also called *torque member*, has two joints (end nodes): $i$ and $j$. A shaft can only resist and transmit a *constant torque* or *twisting moment* $T$ along its longitudinal axis $\bar{x}$, as pictured in Figure 5.8(a).

We consider here only *prismatic* shaft members with uniform material and constant cross section, which thus qualify as simplex. The active degrees of freedom of a generic shaft member of length $L$, depicted in Figure 5.8(b), are $\bar{\theta}_{xi}$ and $\bar{\theta}_{xj}$. These are the infinitesimal end rotations about $\bar{x}$, positive according to the right-hand rule. The associated joint (node) forces are end moments denoted as $\bar{m}_{xi}$ and $\bar{m}_{xj}$.

The only internal force is the torque $T$, positive if acting as pictured in Figure 5.8(a). Let $G$ be the

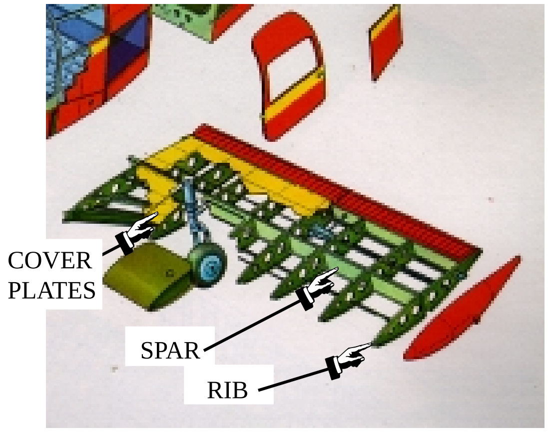


FIGURE 5.7. Spar members in aircraft wing (Piper Cherokee). For more impressive aircraft structures see CAETE slides.

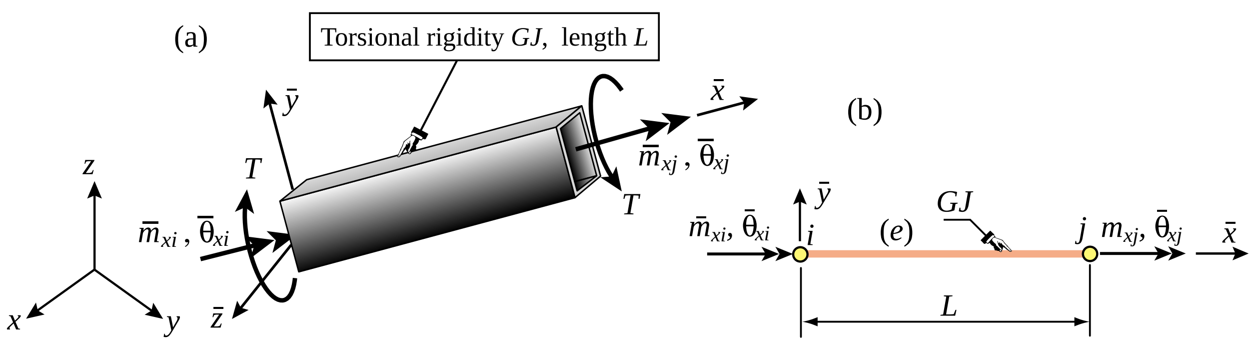


FIGURE 5.8. The prismatic shaft (also called torque member): (a) individual member shown in 3D space, (b) idealization as generic member in the local system.

shear modulus and $GJ$ the effective torsional rigidity.[5] As deformation measure pick the relative twist angle $\phi = \bar{\theta}_{xj} - \bar{\theta}_{xi}$. The kinematic, constitutive, and equilibrium equations provided by Mechanics of Materials are

$$\phi = [\,-1 \quad 1\,] \begin{bmatrix} \bar{\theta}_{xi} \\ \bar{\theta}_{xj} \end{bmatrix} = \mathbf{B}\bar{\mathbf{u}}, \qquad T = \frac{GJ}{L}\phi = S\gamma, \qquad \bar{\mathbf{f}} = \begin{bmatrix} \bar{m}_{xi} \\ \bar{m}_{xj} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} T = \mathbf{B}^T T. \tag{5.13}$$

From these the local stiffness equations follow as

$$\bar{\mathbf{f}} = \begin{bmatrix} \bar{m}_{xi} \\ \bar{m}_{xj} \end{bmatrix} = \mathbf{B}^T S \mathbf{B}\bar{\mathbf{u}} = \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \bar{\theta}_{xi} \\ \bar{\theta}_{xj} \end{bmatrix} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}. \tag{5.14}$$

If the shaft is used in a two-dimensional context, the displacement transformation to global coordi-

[5] $J$ has dimension of (length)$^4$. For a circular or annular cross section it reduces to the polar moment of inertia about $\bar{x}$. The determination of $J$ for noncircular cross sections is covered in Mechanics of Materials textbooks.

nates $\{x, y\}$ within the framework of infinitesimal rotations, is

$$\bar{\mathbf{u}}^e = \begin{bmatrix} \bar{\theta}^e_{xi} \\ \bar{\theta}^e_{xj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} \theta^e_{xi} \\ \theta^e_{yi} \\ \theta^e_{xj} \\ \theta^e_{yj} \end{bmatrix} = \mathbf{T}^e\,\boldsymbol{\theta}^e, \tag{5.15}$$

where as usual $c = \cos\varphi^e$, $s = \sin\varphi^e$, and $\varphi^e$ is the angle from $x$ to $\bar{x}$. Note that $\boldsymbol{\theta}^e$ collects only global node rotations components. This operation is elaborated further in Exercise 5.3.

## §5.4. *Non-Simplex MoM Members

The straightforward formulation of simplex MoM elements does not immediately carry over to the case in which internal quantities $\mathbf{p}$ and $\mathbf{v}$ vary over the member; that is, depend on $\bar{x}$. The dependence may be due to element type, varying cross section, or both. As a result, one or more of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{S}$ depend on $\bar{x}$. Such members are called *non-simplex*.

The matrix multiplication rule (5.4) cannot be used to construct the element stiffness matrix $\bar{\mathbf{K}}^e$ of non-simplex members. This can be grasped by observing that $\mathbf{A}(\bar{x})^T\mathbf{S}(\bar{x})\mathbf{B}(\bar{x})$ would depend on $\bar{x}$. On the other hand, $\bar{\mathbf{K}}^e$ must be independent of $\bar{x}$ because it relates the end quantities $\bar{\mathbf{u}}$ and $\bar{\mathbf{f}}$.

### §5.4.1. *Formulation Rules

The derivation of non-simplex MoM elements requires use of the work principles of mechanics, for example the Principle of Virtual Work or PVW. Thus, more care must be exercised in the choice of conjugate internal quantities. The following rules can be justified through the arguments presented in Part II. They are stated here as recipe, and apply only to displacement-assumed elements.

*Rule 1.* Select internal deformations $\mathbf{v}(\bar{x})$ and internal forces $\mathbf{p}(\bar{x})$ that are conjugate in the PVW sense. Link deformations to node displacements by $\mathbf{v}(\bar{x}) = \mathbf{B}(\bar{x})\mathbf{u}$.

*Rule 2.* From the PVW it may be shown (see **Remark 5.2** below) that the force equilibrium equation exists only in a differential sense:

$$\mathbf{B}^T d\mathbf{p} = d\bar{\mathbf{f}}. \tag{5.16}$$

Here $d$ in $d\mathbf{p}$ denotes differentiation with respect to $\bar{x}$. The meaning of $d\mathbf{p}$ is simply $\mathbf{p}(\bar{x})\,d\bar{x}$. That is, the differential of internal forces as one passes from cross-section $\bar{x}$ to a neighboring one $\bar{x}+d\bar{x}$. The interpretation of $d\bar{\mathbf{f}}$ is less immediate because $\bar{\mathbf{f}}$ is not a function of $\bar{x}$. It actually means the contribution of that member slice to the building of the node force vector $\bar{\mathbf{f}}$. See (5.18) and (5.19) below.

*Rule 3.* The constitutive relation is

$$\mathbf{p} = \mathbf{R}\mathbf{v}, \tag{5.17}$$

in which $\mathbf{R}$, which may depend on $\bar{x}$, must be symmetric. Note that symbol $\mathbf{R}$ in (5.17) replaces the $\mathbf{S}$ of (5.2). Matrix $\mathbf{R}$ pertains to a specific cross section whereas $\mathbf{S}$ applies to the entire member. This distinction is further elaborated in Exercise 5.9.

The discrete relations supplied by the foregoing rules are displayed in the discrete Tonti diagram of Figure 5.9.

Internal quantities are now eliminated starting from the differential equilibrium relation (5.16):

$$d\bar{\mathbf{f}} = \mathbf{B}^T d\mathbf{p} = \mathbf{B}^T\mathbf{p}\,d\bar{x} = \mathbf{B}^T\,\mathbf{R}\,\mathbf{v}\,d\bar{x} = \mathbf{B}^T\,\mathbf{R}\,\mathbf{B}\,\bar{\mathbf{u}}\,d\bar{x} = \mathbf{B}^T\,\mathbf{R}\,\mathbf{B}\,d\bar{x}\;\bar{\mathbf{u}}. \tag{5.18}$$

Integrating both sides over the member length $L$ yields

$$\bar{\mathbf{f}} = \int_0^L d\bar{\mathbf{f}} = \int_0^L \mathbf{B}^T\,\mathbf{R}\,\mathbf{B}\,d\bar{x}\;\bar{\mathbf{u}} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}, \tag{5.19}$$

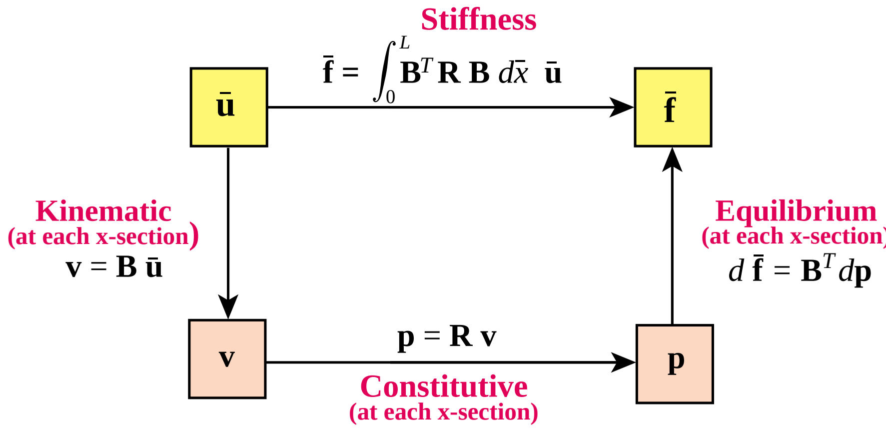


FIGURE 5.9. Discrete Tonti diagram of the equations for a non-simplex MoM member.

because $\bar{\mathbf{u}}$ does not depend on $\bar{x}$. Consequently the local element stiffness matrix is

$$\boxed{\bar{\mathbf{K}} = \int_0^L \mathbf{B}^T\,\mathbf{R}\,\mathbf{B}\;d\bar{x}} \tag{5.20}$$

The recipe (5.20) will be justified in Part II through energy methods. It will be seen that it generalizes to arbitrary displacement-assumed finite elements in any number of space dimensions. It is used in the derivation of the stiffness equations of the plane beam element in Chapter 12. The reduction of (5.20) to (5.6) when the dependence on $\bar{x}$ disappears is the subject of Exercise 5.8.

**Remark 5.2**. The proof of (5.16) follows by equating expressions of the virtual work of a slice of length $d\bar{x}$ undergoing virtual node displacements $\delta\bar{\mathbf{u}}$ and associated deformations $\delta\mathbf{v}$: $d\bar{\mathbf{f}}^T\;\delta\bar{\mathbf{u}} = d\mathbf{p}^T\;\delta\mathbf{v} = d\mathbf{p}^T\;(\mathbf{B}\,\bar{\mathbf{u}}) = (\mathbf{B}^T\;d\mathbf{p})^T\;\delta\bar{\mathbf{u}}$. Since $\delta\bar{\mathbf{u}}$ is arbitrary, $\mathbf{B}^T\;d\mathbf{p} = d\bar{\mathbf{f}}$.

## §5.4.2. *Examples

**Example 5.1**. A two-node bar element has constant elastic modulus $E$ but a continuously varying area: $A_i$, $A_j$ and $A_m$ at $i$, $j$ and $m$, respectively, where $m$ is the midpoint between end joints $i$ and $j$. This variation can be fitted by

$$A(\bar{x}) = A_i N_i(\bar{x}) + A_j N_j(\bar{x}) + A_m N_m(\bar{x}). \tag{5.21}$$

Here $N_i(\bar{x}) = -\frac{1}{2}\xi(1-\xi)$, $N_j(\bar{x}) = \frac{1}{2}\xi(1+\xi)$ and $N_m(\bar{x}) = 1-\xi^2$, with $\xi = 2x/L - 1$, are interpolating polynomials further studied in Part II as “element shape functions.”

As internal quantities take the strain $e$ and the axial force $p = EAe$, which are conjugate quantities. Assuming the strain $e$ to be uniform over the element (this is characteristic of a displacement assumed element and is justified through the method of shape functions explained in Part II.) the MoM equations are

$$e = \mathbf{B}\bar{\mathbf{u}}, \quad p = EA(\bar{x})\,e = R(\bar{x})\,e, \quad d\bar{\mathbf{f}} = \mathbf{B}^T d\mathbf{p}, \quad \mathbf{B} = \frac{1}{L}\,[\,-1 \quad 1\,]. \tag{5.22}$$

Inserting into (5.20) and carrying out the integration yields

$$\bar{\mathbf{K}} = \frac{E\bar{A}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \text{with} \quad \bar{A} = \tfrac{1}{6}(A_i + A_j) + \tfrac{2}{3}A_m. \tag{5.23}$$

**Example 5.2**. Same as in the previous case but now the strain $e$ is taken to be $e = p/(EA)$, whereas the axial force $p$ is constant and defined by $p = -\bar{f}_{xi} = \bar{f}_{xj}$. The integrals become rational functions of $\bar{x}$ and are best evaluated through *Mathematica*. The completion of this Example is the matter of an Exercise.

## Notes and Bibliography

The derivation of MoM elements using straightforward matrix algebra is typical of pre-1962 Matrix Structural Analysis (MSA). The excellent book of Pestel and Leckie [572], unfortunately out of print, epitomizes that approach. Historically this idea interweaved with Generation 1 of FEM, as outlined in Appendix O. By 1970 simplified derivations had fallen out of favor as yokelish. But these elements do not need improvement. They still work fine: a bar or beam stiffness today is the same as 40 years ago.[6]

The Mechanics of Materials books by Beer-Johnston [67] and Popov [588] may be cited as being widely used in US undergraduate courses. But they are not the only ones. A September 2003 in-print book search through `www3.addall.com` on "Mechanics of Materials" returns 99 hits whereas one on "Strength of Materials" (the older name) compiles 112. Folding multiple editions and hardback/paperback variants one gets about 60 books; by all accounts an impressive number.

Spar members are discussed only in MoM books that focus on aircraft structures, since they are primarily used in modeling shear web action. On the other hand, bars, shafts and beams are standard fare.

The framework presented here is a tiny part of MSA. A panoramic view, including linkage to continuum formulations from the MSA viewpoint, is presented in [230].

The source of Tonti diagrams is discussed in Chapter 11.

## References

Referenced items have been moved to Appendix R.

[6] The chief technical difference is the heavier use of differential equations prior to 1962, as opposed to the energy methods in vogue today. The end result for simple one-dimensional models is the same.

## Homework Exercises for Chapter 5
## Constructing MoM Members

**EXERCISE 5.1** [A:10] Explain how to select the deformation variable $v$ (paired to $V$) of the spar member formulated in §5.3.2, so that $\mathbf{A} = \mathbf{B}$. Draw the Tonti diagram with the discrete equations for that choice of $v$ and $p$, using Figure 5.5 as guide (that is, with the actual matrix equations along the arrows).

**EXERCISE 5.2** [A:15] Obtain the $4 \times 4$ global element stiffness matrix of a prismatic spar member in a two dimensional Cartesian system $\{x, y\}$. Start from (5.11). Indicate where the transformation (5.12) comes from (Hint: read §2.8). Evaluate $\mathbf{K}^e = (\mathbf{T}^e)^T\, \bar{\mathbf{K}}^e\, \mathbf{T}^e$ in closed form.

**EXERCISE 5.3** [A:15] Obtain the $4 \times 4$ global element stiffness matrix of a prismatic shaft element in a two dimensional Cartesian system $\{x, y\}$. Include only node rotation freedoms in the global displacement vector. Start from (**?**). Justify the transformation (5.15) (Hint: infinitesimal rotations transform as vectors). Evaluate $\mathbf{K}^e = (\mathbf{T}^e)^T\, \bar{\mathbf{K}}^e\, \mathbf{T}^e$ in closed form.

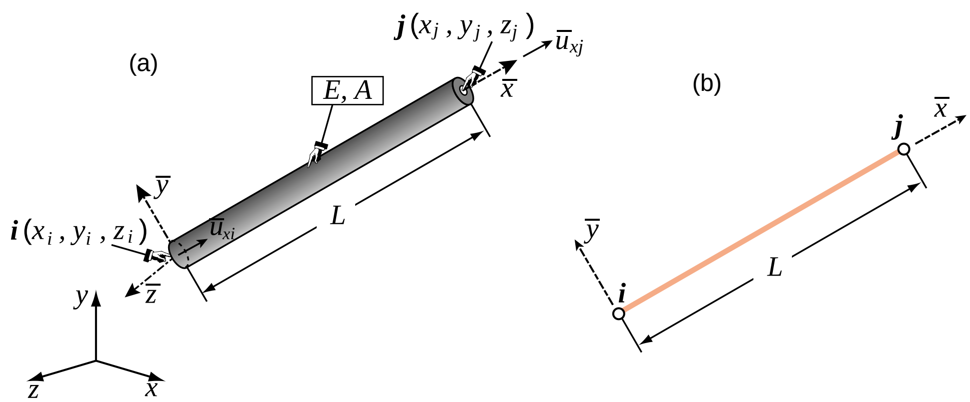


FIGURE E5.1. Bar element in 3D for Exercise 5.4.

**EXERCISE 5.4** [A+N:15(10+5)] A bar element moving in three dimensional space is completely defined by the global coordinates $\{x_i, y_i, z_i\}$, $\{x_j, y_j, z_j\}$ of its end nodes $i$ and $j$, as illustrated in Figure E5.1. The $2 \times 6$ displacement transformation matrix $\mathbf{T}$, with superscript $e$ dropped for brevity, links $\bar{\mathbf{u}}^e = \mathbf{T}\mathbf{u}^e$. Here $\bar{\mathbf{u}}^e$ contains the two local displacements $\bar{u}_{xi}$ and $\bar{u}_{xj}$ whereas $\mathbf{u}^e$ contains the six global displacements $u_{xi}, u_{yi}, u_{zi}, u_{xj}, u_{yj}, u_{zj}$.

(a) From vector mechanics show that

$$\mathbf{T} = \frac{1}{L}\begin{bmatrix} x_{ji} & y_{ji} & z_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{ji} & y_{ji} & z_{ji} \end{bmatrix} = \begin{bmatrix} c_{xji} & c_{yji} & c_{zji} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{xji} & c_{yji} & c_{zji} \end{bmatrix} \tag{E5.1}$$

in which $L$ is the element length, $x_{ji} = x_j - x_i$, etc., and $c_{xji} = x_{ji}/L$, etc., are the direction cosines of the vector going from $i$ to $j$.

(b) Evaluate $\mathbf{T}$ for a bar going from node $i$ at $\{1, 2, 3\}$ to node $j$ at $\{3, 8, 6\}$.

**EXERCISE 5.5** [A+N:30(10+15+5)] A spar element in three dimensional space is only partially defined by the global coordinates $\{x_i, y_i, z_i\}$, $\{x_j, y_j, z_j\}$ of its end nodes $i$ and $j$, as illustrated in Figure E5.2. The problem is that axis $\bar{y}$, which defines the direction of shear force transmission, is not uniquely defined by $i$ and $j$.[7] Most FEM programs use the *orientation node* method to complete the definition. A third node $k$, not

[7] The same ambiguity arises in beam elements in 3D space. These elements are covered in Part III.

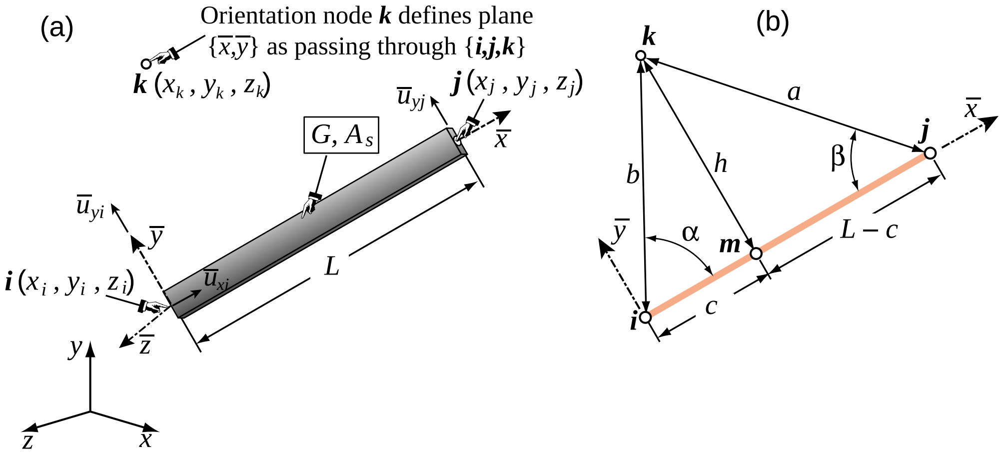


FIGURE E5.2. Spar element in 3D for Exercise 5.5.

colinear with $i$ and $j$, is provided by the user. Nodes $\{i,\ j,\ k\}$ define the $\{\bar{x},\ \bar{y}\}$ plane and thus $\bar{z}$. The projection of $k$ on line $ij$ is point $m$. The distance $h > 0$ from $m$ to $k$ is called $h$ as shown in Figure E5.2(b). The $2 \times 6$ displacement transformation matrix $\mathbf{T}$, with superscript $e$ omitted to reduce clutter, relates $\bar{\mathbf{u}}^e = \mathbf{T}\mathbf{u}^e$. Here $\bar{\mathbf{u}}^e$ contains the local transverse displacements $\bar{u}_{yi}$ and $\bar{u}_{yj}$ whereas $\mathbf{u}^e$ contains the global displacements $u_{xi}, u_{yi}, u_{zi}, u_{xj}, u_{yj}, u_{zj}$.

(a) Show that

$$\mathbf{T} = \frac{1}{h}\begin{bmatrix} x_{km} & y_{km} & z_{km} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{km} & y_{km} & z_{km} \end{bmatrix} = \begin{bmatrix} c_{xkm} & c_{ykm} & c_{zkm} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{xkm} & c_{ykm} & c_{zkm} \end{bmatrix} \tag{E5.2}$$

in which $x_{km} = x_k - x_m$, etc., and $c_{xkm} = x_{km}/h$, etc., are the direction cosines of the vector going from $m$ to $k$. (Assume that the position of $m$ is known. That computation is carried out in the next item.)

(b) Work out the formulas to compute the coordinates of point $m$ in terms of the coordinates of $\{i,\ j,\ k\}$. Assume $a$, $b$ and $L$ are computed immediately from the input data. Using the notation of Figure E5.2(b) and elementary trigonometry, show that $h = 2A/L$, where $A = \sqrt{p(p-a)(p-b)(p-L)}$ with $p = \frac{1}{2}(L+a+b)$ (Heron's formula), $\cos\alpha = (L^2+b^2-a^2)/(2bL)$, $\cos\beta = (L^2+a^2-b^2)/(2aL)$, $c = b\cos\alpha$, $L - c = a\cos\beta$, $x_m = x_i(L-c)/L + x_j c/L$, etc.[8]

(c) Evaluate $\mathbf{T}$ for a spar member going from node $i$ at $\{1, 2, 3\}$ to node $j$ at $\{3, 8, 6\}$. with $k$ at $\{4, 5, 6\}$.

**EXERCISE 5.6** [A:20] Explain how thermal effects can be generally incorporated in the constitutive equation (5.2) to produce an initial force vector for a simplex element.

**EXERCISE 5.7** [A:15] Draw the discrete Tonti diagram for the prismatic shaft element. Use 5.5 as a guide (that is, with the actual matrix equations along the arrows).

**EXERCISE 5.8** [A:15] If the matrices $\mathbf{B}$ and $\mathbf{R}$ are constant over the element length $L$, show that expression (5.20) of the element stiffness matrix for a non-simplex member reduces to (5.6), in which $\mathbf{S} = L\mathbf{R}$.

**EXERCISE 5.9** [A:20] Explain in detail the quickie derivation of footnote 6. (Knowledge of the Principle of Virtual Work is required to do this exercise.)

[8] An alternative and more elegant procedure, found by a student in 1999, can be sketched as follows. From Figure E5.2(b) obviously the two subtriangles $imk$ and $jkm$ are right-angled at $m$ and share side $km$ of length $h$. Apply Pythagoras' theorem twice, and subtract so as to cancel out $h^2$ and $c^2$, getting a linear equation for $c$ that can be solved directly.

**EXERCISE 5.10** [A:25(10+5+10)] Consider a non-simplex element in which $\mathbf{R}$ varies with $\bar{x}$ but $\mathbf{B} = \mathbf{A}$ is constant.

(a) From (5.20) prove that

$$\bar{\mathbf{K}} = L\mathbf{B}^T\bar{\mathbf{R}}\,\mathbf{B}, \quad \text{with} \quad \bar{\mathbf{R}} = \frac{1}{L}\int_0^L \mathbf{R}(\bar{x})\,d\bar{x} \tag{E5.3}$$

(b) Apply (E5.3) to obtain $\bar{\mathbf{K}}$ for a tapered bar with area defined by the linear law $A = A_i(1-\bar{x}/L) + A_j\bar{x}/L$, where $A_i$ and $A_j$ are the end areas at $i$ and $j$, respectively. Take $\mathbf{B} = [\,-1 \quad 1\,]/L$.

(c) Apply (E5.3) to verify the result (5.23) for a bar with parabolically varying cross section.

**EXERCISE 5.11** [A/C+N:30(25+5)] A prismatic bar element in 3D space is referred to a global coordinate system $\{x, y, z\}$, as in Figure E5.1. The end nodes are located at $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$.[9] The elastic modulus $E$ and the cross section area $A$ are constant along the length. Denote $x_{21} = x_2 - x_1$, $y_{21} = y_2 - y_1$, $z_{21} = z_2 - z_1$ and $L = \sqrt{x_{21}^2 + y_{21}^2 + z_{21}^2}$.

(a) Show that the element stiffness matrix in *global* coordinates can be compactly written[10] A plodding way is to start from the local stiffness (5.8) and transform to global using (**?**)

$$\mathbf{K}^e = \frac{EA}{L^3}\,\mathbf{B}^T\mathbf{B}, \quad \text{in which} \quad \mathbf{B} = [\,-x_{21} \quad -y_{21} \quad -z_{21} \quad x_{21} \quad y_{21} \quad z_{21}\,]. \tag{E5.4}$$

(b) Compute $\mathbf{K}^e$ if the nodes are at $\{1, 2, 3\}$ and $\{3, 8, 6\}$, with elastic modulus $E = 343$ and cross section area $A = 1$. Note: the computation can be either done by hand or with the help of a program such as the following *Mathematica* module, which is used in Part III:

```
Stiffness3DBar[ncoor_,mprop_,fprop_,opt_]:= Module[
 {x1,x2,y1,y2,z1,z2,x21,y21,z21,Em,Gm,rho,alpha,A,
  num,L,LL,LLL,B,Ke},  {{x1,y1,z1},{x2,y2,z2}}=ncoor;
  {x21,y21,z21}={x2-x1,y2-y1,z2-z1};
  {Em,Gm,rho,alpha}=mprop; {A}=fprop; {num}=opt;
  If [num,{x21,y21,z21,Em,A}=N[{x21,y21,z21,Em,A}]];
  LL=x21^2+y21^2+z21^2; L=PowerExpand[Sqrt[LL]];
  LLL=Simplify[LL*L]; B={{-x21,-y21,-z21,x21,y21,z21}};
  Ke=(Em*A/LLL)*Transpose[B].B;
Return[Ke]];

ClearAll[Em,A]; Em=343; A=1;
```

[9] End nodes are labeled 1 and 2 instead of $i$ and $j$ to agree with the code listed below.

[10] There are several ways of arriving at this result. Some are faster and more elegant than others. Here is a sketch of one of the ways. Denote by $L_0$ and $L$ the lengths of the bar in the undeformed and deformed configurations, respectively. Then

$$\tfrac{1}{2}(L^2 - L_0^2) = x_{21}(u_{x2} - u_{x1}) + y_{21}(u_{y2} - u_{y1}) + z_{21}(u_{z2} - u_{z1}) + Q \approx \mathbf{B}\mathbf{u},$$

in which $Q$ is a quadratic function of node displacements which is therefore dropped in the small-displacement linear theory. Also on account of small displacements

$$\tfrac{1}{2}(L^2 - L_0^2) = \tfrac{1}{2}(L + L_0)(L - L_0) \approx L\,\Delta L.$$

Hence the small axial strain is $e = \Delta L/L = (1/L^2)\mathbf{B}\mathbf{u}^e$, which begins the Tonti diagram. Next is $F = EA\,e$. Finally you must show that force equilibrium at nodes requires $\mathbf{f}^e = (1/L)\mathbf{B}^T F$. Multiplying through gives (E5.4).

```
ncoor={{0,0,0},{2,6,3}}; mprop={Em,0,0,0}; fprop={A}; opt={False};
Ke=Stiffness3DBar[ncoor,mprop,fprop,opt];
Print["Stiffness of 3D Bar Element:"];
Print[Ke//MatrixForm];
Print["eigs of Ke: ",Eigenvalues[Ke]];
```

As a check, the six eigenvalues of this particular $\mathbf{K}^e$ should be 98 and five zeros.

**EXERCISE 5.12** [A/C:25] Complete Example 5.2.