S

Spatial Applications of Matrices

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In this Appendix we summarize some geometric applications of matrices in 3D space. Indexed homogeneous Cartesian coordinates[1] $\{x_0, x_1, x_2, x_3\}$ are used. To pass to physical coordinates, divide x_1, x_2 and x_3 by x_0: $\{x_1/x_0, x_2/x_0, x_3/x_0$. If $x_0 = 0$ the physical coordinates are at infinity.

§S.1. Points, Planes

Points in 3D space will de identified by X, Y, P, Q, etc. Their coordinates are put in the 4-vectors

$$\mathbf{x} = [\, x_0 \;\; x_1 \;\; x_2 \;\; x_3 \,]^T, \quad \mathbf{y} = [\, y_0 \;\; y_1 \;\; y_2 \;\; y_3 \,]^T, \quad \mathbf{p} = [\, p_0 \;\; p_1 \;\; p_2 \;\; p_3 \,]^T, \text{etc} \tag{S.1}$$

Planes in 3D space will be identified by A, B, C, etc. The equation of plane A is written $a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$ or

$$\mathbf{a}^T \mathbf{x} = 0, \quad \text{or} \quad \mathbf{x}^T \mathbf{a} = 0 \tag{S.2}$$

The plane A is thus defined by the 4-vector $\mathbf{a}$.

Example S.1. Find the coordinates of the point where line joining points P and Q intersects plane A.

Solution. Any point of PQ is R where

$$\mathbf{r} = \lambda \mathbf{p} + \mu \mathbf{q} \tag{S.3}$$

If R is on plane A, then $\mathbf{a}^T \mathbf{x} = 0$ so that $\lambda \mathbf{a}^T \mathbf{p} + \mu \mathbf{a}^T \mathbf{q} = 0$. Absorbing a suitable multiplier into the coordinates of R we obtain its vector in the form

$$\mathbf{r} = (\mathbf{a}^T \mathbf{q})\mathbf{p} - (\mathbf{a}^T \mathbf{p})\mathbf{q}. \tag{S.4}$$

§S.2. Lines

Let $\mathbf{x}$ and $\mathbf{y}$ be the coordinates of points X and Y on a given line L. The 4×4 antisymmetric matrix

$$\mathbf{L} = \mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T \tag{S.5}$$

is called the *coordinate matrix* of the line. It can be shown that $\mathbf{L}$ determines the line L to within a scale factor.

Let A and B be two planes through L. The 4×4 antisymmetric matrix

$$\mathbf{L}^* = \mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T \tag{S.6}$$

is called the *dual coordinate matrix* of line L. It can be shown that[2]

$$\mathbf{L}^* \mathbf{L} = \mathbf{0}. \tag{S.7}$$

[1] These coordinates were independently invented in 1827 by Møbius and Feuerbach, and further developed in 1946 by E. A. Maxwell at Cambridge. See E. A. Maxwell, *General Homogeneous Coordinates in Space of Three Dimensions*, Cambridge University Press, 1951.

[2] See E. A. Maxwell, loc. cit., page 150.

Example S.2. Find where line L defined by two points X and Y meets a plane A.

Solution. Consider the vector

$$\mathbf{p} = \mathbf{L}\mathbf{a} = (\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T)\mathbf{a} = (\mathbf{y}^T\mathbf{a})\mathbf{x} - (\mathbf{x}^T\mathbf{a})\mathbf{y} \tag{S.8}$$

From the last form the point P of coordinates $\mathbf{p}$ must lie on the line that joins X and Y. Moreover since $\mathbf{L}$ is antisymmetrical, $\mathbf{a}^T\mathbf{L}\mathbf{a} = 0$ so that $\mathbf{a}^T\mathbf{p} = 0$. Thus P is the intersection of line L and plane A.

Example S.3. Find the plane A that joins line L to a point X.

Solution.

$$\mathbf{a} = \mathbf{L}^*\,\mathbf{x} \tag{S.9}$$

where $\mathbf{L}^*$ is the dual of $\mathbf{L}$. The demonstration is trivial.

Two immediate corollaries: (i) Line L lies in the plane A if $\mathbf{L}\mathbf{a} = \mathbf{0}$; (ii) Line L passes through the point X if $\mathbf{L}^*\mathbf{x} = \mathbf{0}$.

Example S.4.

Consider two lines L_1 and L_2 with coordinate matrices $\mathbf{L}_1$, $\mathbf{L}_2$ and dual matrices $\tilde{\mathbf{L}}_1$ and $\tilde{\mathbf{L}}_2$, respectively. Find the conditions for the lines to intersect.

Solution. Any of the four equivalent conditions

$$\mathbf{L}_1\mathbf{L}_2^*\mathbf{L}_1 = \mathbf{0}, \quad \mathbf{L}_1^*\mathbf{L}_2\mathbf{L}_1^* = \mathbf{0}, \quad \mathbf{L}_2\mathbf{L}_1^*\mathbf{L}_2 = \mathbf{0}, \quad \mathbf{L}_2^*\mathbf{L}_1\mathbf{L}_2^* = \mathbf{0}. \tag{S.10}$$

For the proof see E. A. Maxwell, loc. cit, page 154.