Statistical ORBIT DETERMINATION, ASEN5070

Lecture 6

Review of Matrix Theory

Fall 2011, 9/2/11

Supplemental Reading:

Appendix B

Colorado Center for Astrodynamics Research
The University of Colorado

Notation

The following matrix notation, definitions, and theorems are used extensively in this class.

- A matrix $\mathbf{A}$ will have elements denoted by a_{ij}, where i refers to the row and j to the column.
- $\mathbf{A}^T$ will denote the transpose of $\mathbf{A}$.
- $\mathbf{A}^{-1}$ will denote the inverse of $\mathbf{A}$.
- $|\mathbf{A}|$ will denote the determinant of $\mathbf{A}$.
- The dimension of a matrix is the number of its rows by the number of its columns.
- An $n \times m$ matrix $\mathbf{A}$ will have n rows and m columns.
- If $m = 1$, the matrix will be called an $n \times 1$ vector.

Matrix Multiplication

- Given

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

- then

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix} \qquad \text{(B.1.2)}$$

provided the elements of $\mathbf{A}$ and $\mathbf{B}$ are conformable.

Fundamentals

- For $\mathbf{A} + \mathbf{B}$ to be defined, $\mathbf{A}$ and $\mathbf{B}$ must have the same dimension.

- The transpose of $\mathbf{A}^T$ equals $\mathbf{A}$; that is, $(\mathbf{A}^T)^T = \mathbf{A}$.

- The inverse of $\mathbf{A}^{-1}$ is $\mathbf{A}$; that is, $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

- The transpose and inverse symbols may be permuted; that is, $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

- $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$.

- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$ if $\mathbf{A}$ and $\mathbf{B}$ are each nonsingular.

Fundamentals

- $(AB)C = A(BC)$, the associative law holds.

- In general $AB \neq BA$, the commutative law does not hold.

- From $AB = 0$ we cannot in general conclude that at least one of A or $B = 0$.

- From $AB = AC$ we cannot in general conclude that $B = C$.

- If $\mathbf{X}$ and $\mathbf{Y}$ are vectors and if $\mathbf{A}$ is a nonsingular matrix and if the equation $\mathbf{Y} = \mathbf{AX}$ holds, then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$.

Matrix Rank

- The *rank* of a matrix is the dimension of its largest square nonsingular submatrix; that is, one whose determinant is nonzero.

- The rank of the product $\mathbf{AB}$ of the two matrices $\mathbf{A}$ and $\mathbf{B}$ is less than or equal to the rank of $\mathbf{A}$ and is less than or equal to the rank of $\mathbf{B}$.

- If $\mathbf{A}$ is an $n \times n$ matrix and if $|\mathbf{A}| = 0$, then the rank of $\mathbf{A}$ is less than n.

- If the rank of $\mathbf{A}$ is $m \leq n$, then the number of linearly independent rows is m; also, the number of linearly independent columns is m ($\mathbf{A}$ is $n \times n$).

- The rank of $\mathbf{AA}^T$ equals the rank of $\mathbf{A}^T\mathbf{A}$, equals the rank of $\mathbf{A}$, equals the rank of $\mathbf{A}^T$.

Quadratic Forms

- The rank of the quadratic form $\mathbf{Y}^T\mathbf{A}\mathbf{Y}$ is defined as the rank of the matrix $\mathbf{A}$ where $\mathbf{Y}$ is a vector and $\mathbf{Y} \neq 0$.

- The quadratic form $\mathbf{Y}^T\mathbf{A}\mathbf{Y}$ is said to be *positive definite* if and only if $\mathbf{Y}^T\mathbf{A}\mathbf{Y} > 0$ for all vectors $\mathbf{Y}$ where $\mathbf{Y} \neq 0$.

- A quadratic form $\mathbf{Y}^T\mathbf{A}\mathbf{Y}$ is said to be *positive semidefinite* if and only if $\mathbf{Y}^T\mathbf{A}\mathbf{Y} \geq 0$ for all $\mathbf{Y}$, and $\mathbf{Y}^T\mathbf{A}\mathbf{Y} = 0$ for some vector $\mathbf{Y} \neq 0$.

Quadratic Forms

- A quadratic form $\mathbf{Y}^T\mathbf{A}\mathbf{Y}$ that may be either positive definite or positive semidefinite is called *nonnegative definite*.

- The matrix $\mathbf{A}$ of a quadratic form $\mathbf{Y}^T\mathbf{A}\mathbf{Y}$ is said to be positive definite (semidefinite) when the quadratic form is positive definite (semidefinite).

- If $\mathbf{A}$ is an $m \times n$ matrix of rank $n < m$, then $\mathbf{A}^T\mathbf{A}$ is positive definite and $\mathbf{A}\mathbf{A}^T$ is positive semidefinite.

- If $\mathbf{A}$ and $\mathbf{B}$ are symmetric conformable matrices, $\mathbf{A}$ is said to be greater than $\mathbf{B}$ if $\mathbf{A} - \mathbf{B}$ is nonnegative definite.

Triangle Matrices

- A triangular matrix has non-zero elements on the diagonal and above (upper triangular) or below (lower triangular).

- Example of a 3×3 upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

- A unitary triangular matrix has diagonal elements equal to 1.

Matrix Square Root

- The square root of an $n \times n$ matrix, P, is defined as $P=AA$, where $A = \sqrt{P}$ is the square root of P.

- A symmetric positive semidefinite matrix has a unique symmetric positive semidefinite square root.

- If $S^T S = P$, where P is symmetric positive semidefinite and S is upper triangular, then an orthogonal matrix Q exists so that

$$S = QA$$

$$\sqrt{AB} \neq \sqrt{A}\sqrt{B}$$

Determinants

- For each square matrix $\mathbf{A}$, there is a uniquely defined scalar called the *determinant* of $\mathbf{A}$ and denoted by $|\mathbf{A}|$.

- The determinant of a diagonal matrix is equal to the product of the diagonal elements.

- If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, then $|\mathbf{AB}| = |\mathbf{BA}| = |\mathbf{A}|\,|\mathbf{B}|$.

- If $\mathbf{A}$ is singular if and only if $|\mathbf{A}| = 0$.

Determinants

- If $\mathbf{C}$ is an $n \times n$ matrix such that $\mathbf{C}^T\mathbf{C} = \mathbf{I}$, then $\mathbf{C}$ is said to be an *orthogonal* matrix, and $\mathbf{C}^T = \mathbf{C}^{-1}$.

- If $\mathbf{C}$ is an orthogonal matrix, then $|\mathbf{C}| = \pm 1$.

- The determinant of a positive definite matrix is positive.

- The determinant of a triangular matrix is equal to the product of the diagonal elements.

- The determinant of a matrix is equal to the product of its eigenvalues.

Matrix Trace

- The *trace* of a matrix $\mathbf{A}$, which will be written $\mathrm{tr}\,(\mathbf{A})$, is equal to the sum of the diagonal elements of $\mathbf{A}$; that is,

$$tr\,(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}. \qquad \text{(B.5.1)}$$

- $\mathrm{tr}\left(\mathbf{AB}\right) = \mathrm{tr}\left(\mathbf{BA}\right)$.

- $\mathrm{tr}\left(\mathbf{ABC}\right) = \mathrm{tr}\left(\mathbf{CAB}\right) = \mathrm{tr}\left(\mathbf{BCA}\right)$; that is, the trace of the product of matrices is invariant under any cyclic permutation of the matrices.

- Note that the trace is defined only for a square matrix.

- If $\mathbf{C}$ is an orthogonal matrix, $\mathrm{tr}\,(\mathbf{C}^T\mathbf{AC}) = \mathrm{tr}\,(\mathbf{A})$.

Eigenvalues and Eigenvectors

- A *characteristic root* (*eigenvalue*) of a $p \times p$ matrix $\mathbf{A}$ is a scalar λ such that $\mathbf{AX} = \lambda\mathbf{X}$ for some vector $\mathbf{X} \neq 0$.

- $\mathbf{X}$ is called the *characteristic vector (eigenvector) of the matrix* $\mathbf{A}$.

- The eigenvalue of a matrix $\mathbf{A}$ can be defined as a scalar λ such that $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

- $|\mathbf{A} - \lambda\mathbf{I}|$ is a pth degree polynomial in λ.

- This polynomial is called the *characteristic polynomial*, and its roots are the eigenvalues of the matrix $\mathbf{A}$.

Eigenvalues and Eigenvectors

- The number of nonzero eigenvalues of a matrix $\mathbf{A}$ is equal to the rank of $\mathbf{A}$.

- The trace of $\mathbf{A}$ is equal to the sum of its eigenvalues.

- The eigenvalues of a symmetric matrix are real.

- The eigenvalues of a positive definite matrix $\mathbf{A}$ are positive; the eigenvalues of a positive semidefinite matrix are nonnegative.

Eigenvalues and Eigenvectors

Example:

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 8 \\ 6 & 16 & 10 \\ 8 & 10 & 25 \end{bmatrix}$$

- The normalized eigenvectors of $\mathbf{A}$ are:

$$\begin{bmatrix} 0.9486 & 0.0684 & 0.3091 \\ -0.2298 & 0.8204 & 0.5235 \\ -0.2178 & -0.5676 & 0.7940 \end{bmatrix}$$

- The eigenvalues of $\mathbf{A}$ are: $D\left(0.7096,\ 9.5818,\ 34.7086\right)$
  1. $\mathbf{A}$ is a positive definite symmetric matrix.
  2. Rank of $\mathbf{A} = 3$: Three nonzero eigenvalues.
  3. The eigenvalues of a symmetric matrix are real.
  4. The sum of the eigenvalues = the trace of $\mathbf{A}$, i.e. 45.

Derivatives

- Let $\mathbf{X}$ be an $n \times 1$ vector and let Z be a scalar that is a function of $\mathbf{X}$. The derivative of Z with respect to the vector $\mathbf{X}$, which will be written $\partial Z / \partial \mathbf{X}$, will mean the $1 \times n$ row vector*

$$\frac{\partial Z}{\partial \mathbf{X}} = \left[\frac{\partial Z}{\partial x_1} \quad \frac{\partial Z}{\partial x_2} \quad \cdots \quad \frac{\partial Z}{\partial x_n} \right]. \qquad (B.7.1)$$

*Generally this partial derivative would be defined as a column vector. However, it is defined as a row vector here because we have defined $\widetilde{H} = \frac{\partial G(\mathbf{X})}{\partial \mathbf{X}}$ as a row vector in the text.

Derivatives

- If $\mathbf{A}$ and $\mathbf{B}$ are $m \times 1$ vectors, which are a function of the $n \times 1$ vector $\mathbf{X}$, and we define

$$\frac{\partial(\mathbf{A}^T\mathbf{B})}{\partial \mathbf{X}}$$

to be a row vector as in Eq. (B.7.1), then

$$\partial(\mathbf{A}^T\mathbf{B}) \,/\, \partial \mathbf{X} = \mathbf{B}^T \frac{\partial \mathbf{A}}{\partial \mathbf{X}} + \mathbf{A}^T \frac{\partial \mathbf{B}}{\partial \mathbf{X}} \qquad \text{(B.7.3)}$$

  - where $\frac{\partial \mathbf{A}}{\partial \mathbf{X}}$ is an $m \times n$ matrix whose ij element is $\frac{\partial A_i}{\partial X_j}$
  - and $\frac{\partial\left(\mathbf{A}^T\mathbf{B}\right)}{\partial \mathbf{X}}$ is a $1 \times n$ row vector.
- The derivative of a matrix product with respect to a scalar is given by

$$\frac{d}{dt}\{\mathbf{A}(t)\mathbf{B}(t)\} = \frac{d\mathbf{A}(t)}{dt}\mathbf{B}(t) + \mathbf{A}(t)\frac{d\mathbf{B}(t)}{dt}. \qquad \text{(B.7.7)}$$

Maxima and Minima

The goal is to find an extrema (maximum or minimum) of a function $f(x_1, x_2 \ldots, x_n)$ that depends on n independent variables $x_1 \cdots x_n$.

- Assuming that first and second order derivatives are continuous, an extrema of $f(x_1 \cdots x_n)$ occurs only at points where

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0$$

Maxima and Minima

- The extrema will be a maximum if the Hessian matrix $\boldsymbol{H}$ is negative definite, where

$$\boldsymbol{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2{}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n{}^2} \end{bmatrix}_{n \times n}$$

- The extrema will be a minimum if $\boldsymbol{H}$ is positive definite.

Maxima and Minima

If there are constraints imposed on the independent variables, i.e., these variables are no longer independent, where the constraints are given by

$$g_l = (x_1, \cdots x_n) = 0, l = 1 \cdots m \text{ where } m < n$$

We may adjoin the constraints to the original function $f(x_1 \cdots x_n)$ using a set of constant unknown Lagrange multipliers, λ_l, $l = 1 \cdots m$, to obtain the Lagrangian function

$$L(x_1 \cdots x_n, \lambda_1 \cdots \lambda_m) = f + \sum_{l=1}^{m} \lambda_l g_l$$

Maxima and Minima

- The extrema of f is now given by

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} \cdots\cdots \frac{\partial L}{\partial x_n} = 0$$

$$g_l\left(x_1 \cdots x_n \right) = 0, \; l = 1 \cdots m$$

- These $n + m$ equations must be solved simultaneously for $x_1 \cdots x_n, \; \lambda_1 \cdots \lambda_m$ in order to obtain the extrema

Maxima and Minima

- The extrema will be a maximum if the Hessian matrix of L is negative definite and a minimum if the Hessian is positive definite, where

$$\boldsymbol{H} = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^{\ 2}} & \cdots & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 x_1} & \cdots & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 L}{\partial x_n^{\ 2}} \end{bmatrix}_{n \times n}$$

Maxima and Minima

Example:

Find the extrema of $f(x_1, x_2) = x_1^2 + x_2^2$

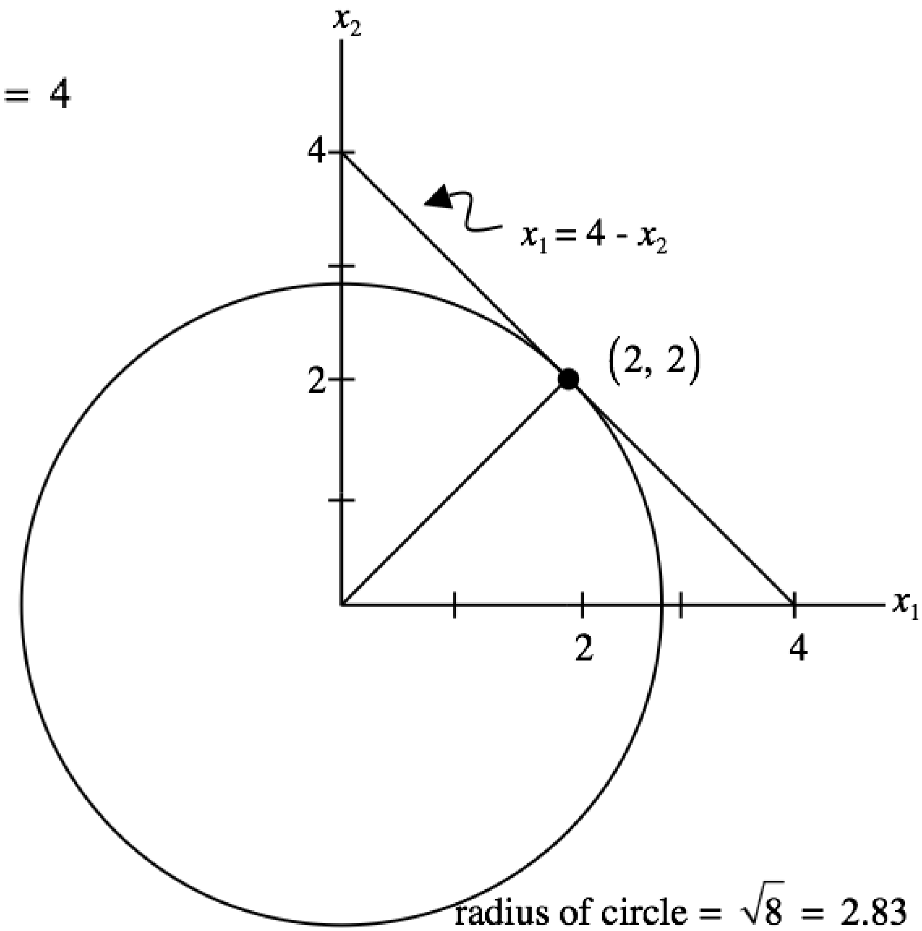
with the constraint $x_1 + x_2 = 4$

$$L = x_1^2 + x_2^2 + \lambda(x_1 + x_2 - 4)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 + \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 + \lambda = 0$$

$$x_1 = x_2 = 2, \lambda = 4$$



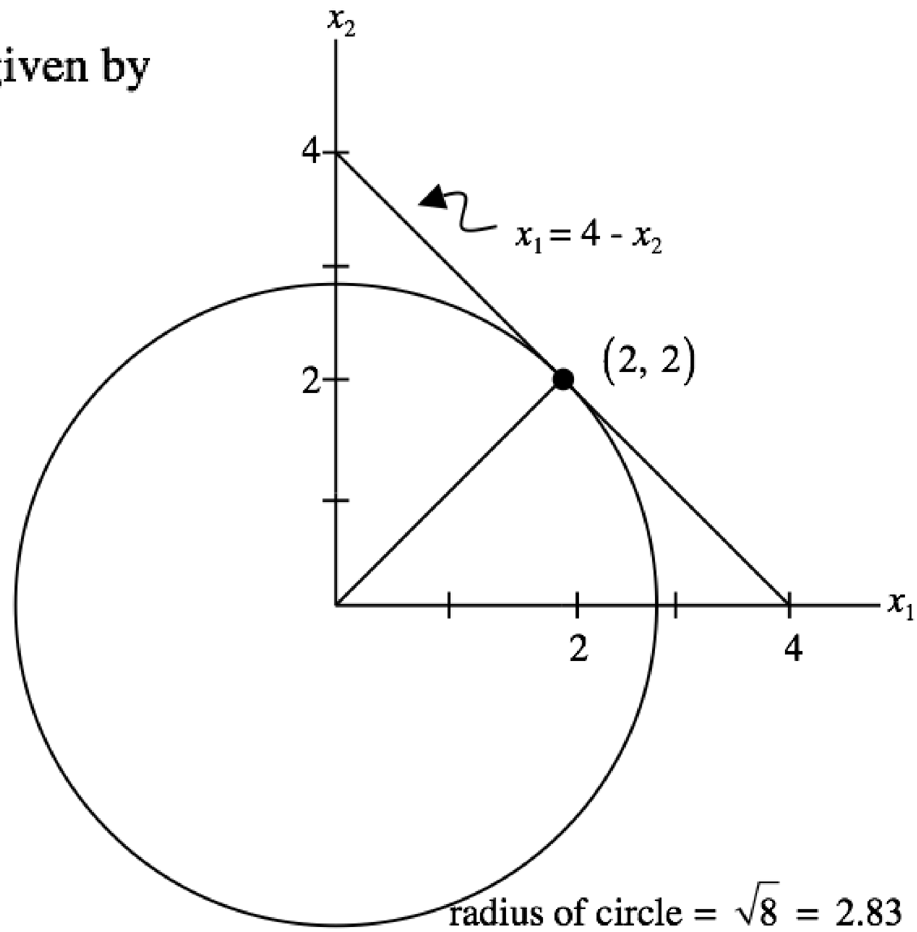
Maxima and Minima

Example:

The Hessian matrix for this example is given by

$$\boldsymbol{H} = \begin{bmatrix} \dfrac{\partial^2 L}{\partial x_1{}^2} & \dfrac{\partial^2 L}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 L}{\partial x_2 \partial x_1} & \dfrac{\partial^2 L}{\partial x_2{}^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

- $\boldsymbol{H}$ is PD so this extrema is a minimum.



Matrix Inversion Theorems

Theorem 1: Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ positive definite (PD) matrices. If $\mathbf{A}^{-1} + \mathbf{B}^{-1}$ is PD, then $\mathbf{A} + \mathbf{B}$ is PD and

$$(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{B}^{-1}(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1}\mathbf{A}^{-1}$$

$$= \mathbf{A}^{-1}(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1}\mathbf{B}^{-1}. \tag{B.9.1}$$

Theorem 2: Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ PD matrices. If $\mathbf{A} + \mathbf{B}$ is PD, then $\mathbf{I} + \mathbf{A}\mathbf{B}^{-1}$ and $\mathbf{I} + \mathbf{B}\mathbf{A}^{-1}$ are PD and

$$(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{B}^{-1} - \mathbf{B}^{-1}(\mathbf{I} + \mathbf{A}\mathbf{B}^{-1})^{-1}\mathbf{A}\mathbf{B}^{-1}$$

$$= \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{I} + \mathbf{B}\mathbf{A}^{-1})^{-1}\mathbf{B}\mathbf{A}^{-1}. \tag{B.9.2}$$

Matrix Inversion Theorems

Theorem 3: If $\mathbf{A}$ and $\mathbf{B}$ are PD matrices of order n and m, respectively, and if $\mathbf{C}$ is of order $n \times m$, then

$$(\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C} + \mathbf{B}^{-1})^{-1}\mathbf{C}^T\mathbf{A}^{-1} = \mathbf{B}\mathbf{C}^T(\mathbf{A} + \mathbf{C}\mathbf{B}\mathbf{C}^T)^{-1} \qquad \text{(B.9.3)}$$

provided the inverse exists.

Theorem 4: The Schur Identity or insideout rule. If $\mathbf{A}$ is a PD $n \times n$ matrix and if $\mathbf{B}$ and $\mathbf{C}$ are any conformable matrices such that $\mathbf{BC}$ is $n \times n$, then

$$(\mathbf{A} + \mathbf{BC})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{I} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}. \qquad \text{(B.9.4)}$$