Statistical ORBIT DETERMINATION, ASEN5070

Lecture 7

Fundamentals of Orbit Determination

Fall 2011, 9/7/2011

Supplemental Reading:

Chapter 4

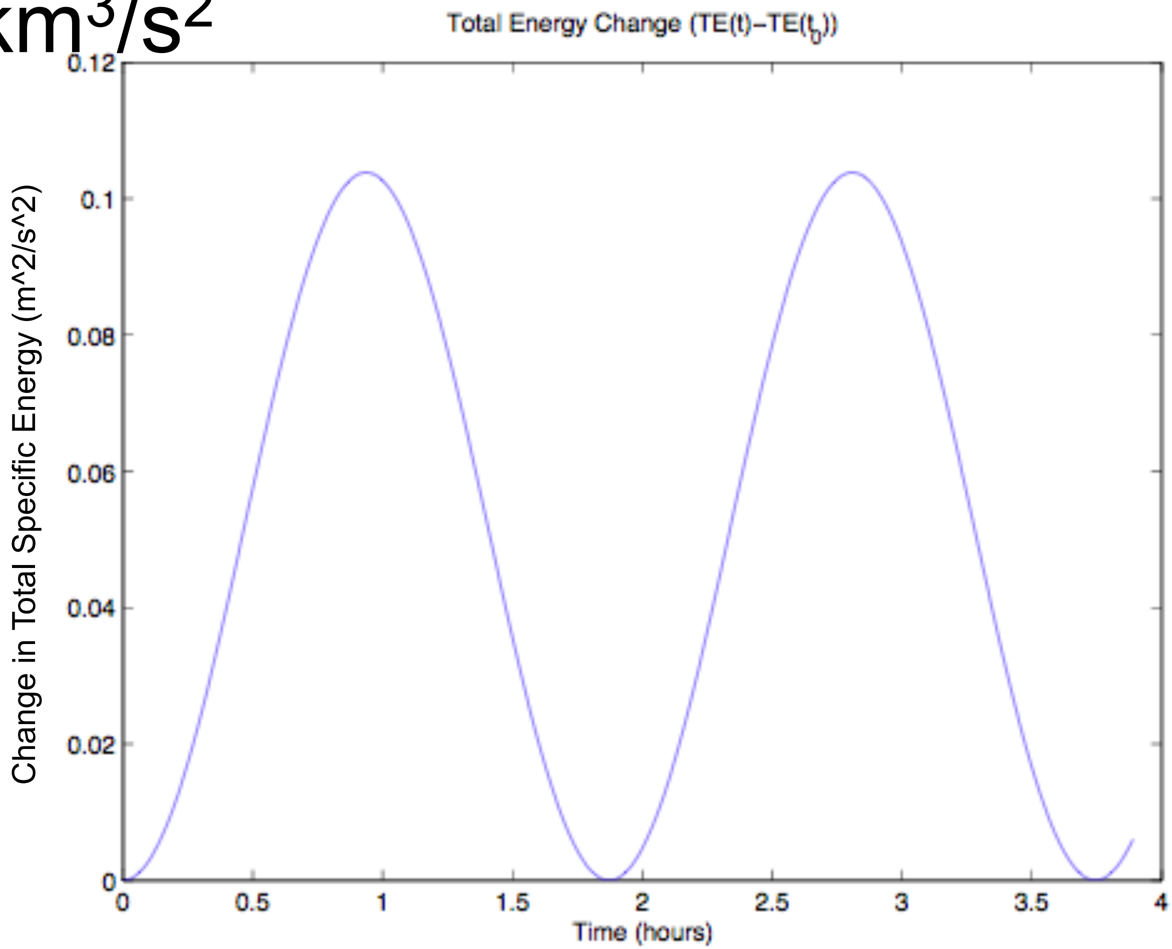
Colorado Center for Astrodynamics Research
The University of Colorado

Accuracy/Consistency of MU

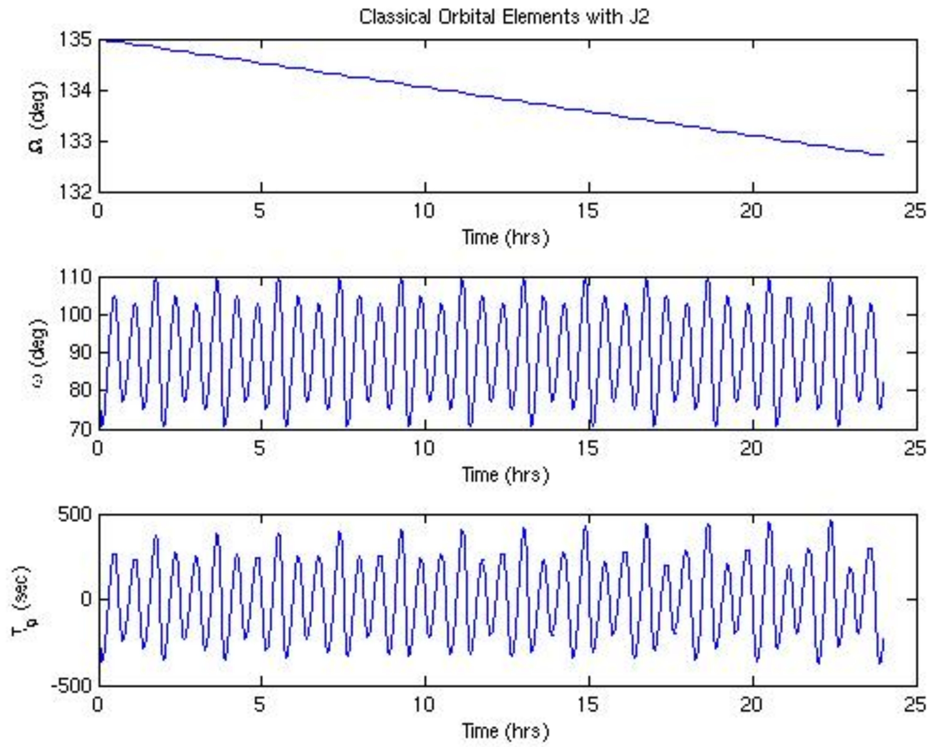
Error in μ of 0.5 km^3/s^2

Integrator μ = 398600 km^3/s^2

Energy Calculation μ = 398600.5 km^3/s^2



Orbit Elements HW #2



T_p Calculation

T_p is determined from the following equations:

$$M = E - e\sin E \qquad n = \sqrt{\frac{\mu}{a^3}} \qquad t - T_p = M/n$$

However, as time t increases, T_p is not constrained to an orbital period and thus increases as a step function. To resolve this, MOD T_p with the orbital period.

$$T_p = \mathrm{MOD}(T_p, P)$$

A situation may arise in which the calculation for the mean Anomaly, M, and true anomaly, v, do not agree resulting in the mean anomaly to be past perigee while the true anomaly is behind perigee (this is an artifact of numerical integration).

T_p Calculation

To correct this, we will introduce the angle of periapse θ_p:

$$\theta_p = nT_p$$

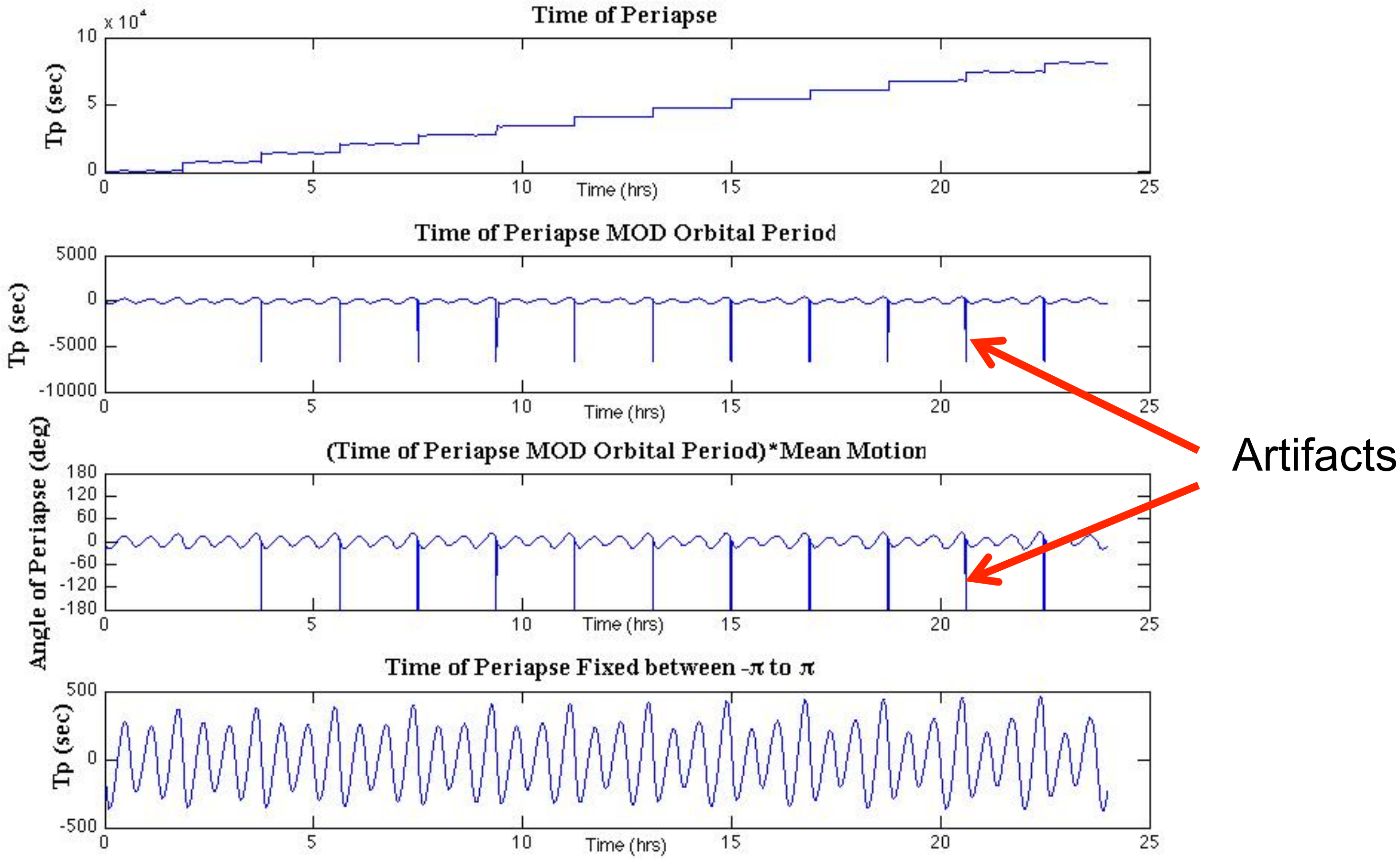
From this, one will notice that the artifacts occur when

$$-\pi < \theta_p \leq \pi$$

Thus, constraining θ_p to be between –π to π will remove the artifacts. The angle of periapse θ_p can then be converted back to time of periapse T_p by

$$T_p = \frac{\theta_p}{n}$$

T_p Results



Time of Periapse
x 10^4
Tp (sec)
Time (hrs)
Time of Periapse MOD Orbital Period
Tp (sec)
Time (hrs)
(Time of Periapse MOD Orbital Period)*Mean Motion
Angle of Periapse (deg)
Time (hrs)
Time of Periapse Fixed between -π to π
Tp (sec)
Time (hrs)
Artifacts

T_p Code

t(i) = epoch time

```
M(i) = E - ecc(i)*sin(E);

n(i) = sqrt(mu/a(i)^3);

%   Determine Tp from T-Tp = M/n
Tp(i) = t(i) - M(i)/n(i);

%   Mod the Epoch Time with the Period
Tp3(i) = mod(t(i),2*pi/n(i)) - M(i)/n(i);

%   Turn that time into an Angle of Periapse
Tp4(i) = Tp3(i)*n(i);
Tp5(i) = Tp4(i);

%   Brandon's Code
Tp2(i) = mod(t(i),(2*pi)/n(i))*n(i) - (E - ecc(i)*sin(E));

%   Make sure the angle is between negative pi to pi
if Tp2(i) < -pi
    Tp2(i) = Tp2(i) + 2*pi;
    Tp5(i) = Tp5(i) + 2*pi;
else if Tp2(i) > pi
    Tp2(i) = Tp2(i) - 2*pi;
    Tp5(i) = Tp5(i) - 2*pi;
end

%   Divide angle of Periapse by Mean Motion to get Time of Periapse
Tp2(i) = Tp2(i)/n(i);
Tp5(i) = Tp5(i)/n(i);
```

Sample T_p MATLAB Code

```
mean_mot = sqrt(MU./(semi.*semi.*semi));
time_peri_angle =  mod(Tout,(2*pi)./mean_mot).*mean_mot ...
                 - (eccentric - ecc.*sin(eccentric));

I = find( time_peri_angle < -pi );
time_peri_angle(I) = time_peri_angle(I) + 2*pi;

I = find( time_peri_angle > pi );
time_peri_angle(I) = time_peri_angle(I) - 2*pi;

time_peri = time_peri_angle./mean_mot;
```

Linearization of the OD Process

General case, the governing relations involve the non-linear expressions:

$$\dot{\mathbf{X}} = F(\mathbf{X}, t), \qquad \mathbf{X}(t_k) \equiv \mathbf{X}_k \qquad (4.2.1)$$

$$\mathbf{Y}_i = G(\mathbf{X}_i, t_i) + \epsilon_i; \qquad i = 1, \ldots, \ell \qquad (4.2.2)$$

$\boldsymbol{X}_k$ = the unknown *n*-dimensional state vector at time t_k

$\boldsymbol{Y}_i$ = for $i = 1, \ldots, l$ is a *p*-dimensional set of **observations**

$\widehat{\boldsymbol{X}}_k$ = best estimate of the unknown value of $\boldsymbol{X}_k$

In general $p < n$ and $m = p \times l \gg n$

Linearization of the OD Process

Formulation of:

$$\dot{\mathbf{X}} = F(\mathbf{X}, t), \qquad \mathbf{X}(t_k) \equiv \mathbf{X}_k \tag{4.2.1}$$

$$\mathbf{Y}_i = G(\mathbf{X}_i, t_i) + \boldsymbol{\epsilon}_i; \qquad i = 1, \ldots, \ell \tag{4.2.2}$$

(1) The inability to observe the state directly

(2) Non-linear relations between the observations and state

(3) Fewer observations at any time epoch that there are state vector components $p < n$

(4) Errors in the observations represented by $\boldsymbol{\epsilon}_i$

Linearization of the OD Process

Replace the **nonlinear** orbit determination problem to estimate the **state vector** with a **linear** orbit determination problem to determine the **deviation** from some reference solution

$$\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t), \qquad \mathbf{y}(t) = \mathbf{Y}(t) - \mathbf{Y}^*(t) \quad (4.2.3)$$

And thus

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{X}}(t) - \dot{\mathbf{X}}^*(t). \quad (4.2.4)$$

Linearization of the OD Process

Expanding Eqs. (4.2.1) and (4.2.2) in a Taylor series about the reference trajectory yields:

$$
\begin{aligned}
\dot{\mathbf{X}}(t) &= F(\mathbf{X}, t) = F(\mathbf{X}^*, t) + \left[\frac{\partial F(t)}{\partial \mathbf{X}(t)}\right]^* [\mathbf{X}(t) - \mathbf{X}^*(t)] \\
&\quad + O_F\,[\mathbf{X}(t) - \mathbf{X}^*(t)] \\
\mathbf{Y}_i &= G(\mathbf{X}_i, t_i) + \boldsymbol{\epsilon}_i = G(\mathbf{X}_i^*, t_i) + \left[\frac{\partial G}{\partial \mathbf{X}}\right]_i^* [\mathbf{X}(t_i) - \mathbf{X}^*(t_i)]_i \\
&\quad + O_G\,[\mathbf{X}(t_i) - \mathbf{X}^*(t_i)] + \boldsymbol{\epsilon}_i
\end{aligned}
\qquad (4.2.5)
$$

Linearization of the OD Process

Neglect higher order terms in Eq. (4.2.5) and apply the following conditions:

$$\dot{X}^* = F(X^*, t) \qquad Y_i^* = G(X_i^*, t)$$

Rewriting Eq. (4.2.5):

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) \\ \mathbf{y}_i &= \widetilde{H}_i \mathbf{x}_i + \boldsymbol{\epsilon}_i \qquad (i = 1, \ldots, \ell) \end{aligned} \qquad (4.2.6)$$

Where:

$$A(t) = \left[\frac{\partial F(t)}{\partial \mathbf{X}(t)}\right]^* \qquad \widetilde{H}_i = \left[\frac{\partial G}{\partial \mathbf{X}}\right]_i^*$$

Linearization of the OD Process

Hence, the original **non-linear** estimation problem is replaced by the **linear** estimation problem described by Eq. (4.2.6)

$$
\begin{aligned}
\mathbf{x}(t) &= \mathbf{X}(t) - \mathbf{X}^*(t), \\
\mathbf{x}_i &= \mathbf{X}(t_i) - \mathbf{X}^*(t_i) \\
\mathbf{y}_i &= \mathbf{Y}_i - G(\mathbf{X}_i^*, t_i).
\end{aligned}
$$

Generally,

- uppercase **X** and **Y** will represent the state and observation vectors

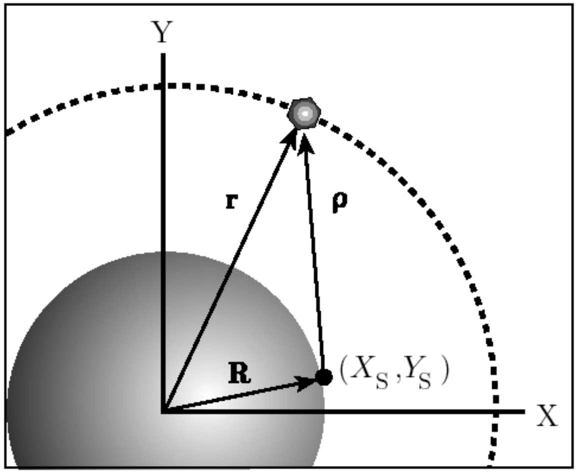
- lowercase **x** and **y** will represent the state and observation <u>deviation</u> vectors

Example 4.2.1

Example 4.2.1

Compute the A matrix and the $\widetilde{H}$ matrix for a satellite in a plane under the influence of only a *central force*. Assume that the satellite is being tracked with range observations, ρ, from a single ground station. Assume that the station coordinates, (X_S, Y_S), and the gravitational parameter are unknown. Then, the state vector, $\mathbf{X}$, is given by

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ U \\ V \\ \mu \\ X_S \\ Y_S \end{bmatrix}$$



where U and V are velocity components and X_S and Y_S are coordinates of the tracking station. From Newton's Second Law and the law of gravitation,

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3}$$

Example 4.2.1

Or in component form:

$$\ddot{X} = -\frac{\mu X}{r^3} \qquad \ddot{Y} = -\frac{\mu Y}{r^3}$$

Expressed in first order form:

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{U} \\ \dot{V} \\ \dot{\mu} \\ \dot{X}_S \\ \dot{Y}_S \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \end{bmatrix} = \begin{bmatrix} U \\ V \\ -\frac{\mu X}{r^3} \\ -\frac{\mu Y}{r^3} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 4.2.1

$$A(t) = \frac{\partial F(\mathbf{X}^*, t)}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} & \frac{\partial F_1}{\partial U} & \frac{\partial F_1}{\partial V} & \frac{\partial F_1}{\partial \mu} & \frac{\partial F_1}{\partial X_S} & \frac{\partial F_1}{\partial Y_S} \\ \frac{\partial F_2}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_2}{\partial Y_S} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_7}{\partial X} & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{\partial F_7}{\partial Y_S} \end{bmatrix}^*$$

Example 4.2.1

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\mu}{r^3} + \frac{3\mu X^2}{r^5} & \frac{3\mu XY}{r^5} & 0 & 0 & -\frac{X}{r^3} & 0 & 0 \\ \frac{3\mu XY}{r^5} & -\frac{\mu}{r^3} + \frac{3\mu Y^2}{r^5} & 0 & 0 & -\frac{Y}{r^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^*$$

Example 4.2.1

The $\widetilde{H}$ matrix is given by

$$\widetilde{H} = \frac{\partial \rho}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial \rho}{\partial X} & \frac{\partial \rho}{\partial Y} & \frac{\partial \rho}{\partial U} & \frac{\partial \rho}{\partial V} & \frac{\partial \rho}{\partial \mu} & \frac{\partial \rho}{\partial X_{\mathrm{S}}} & \frac{\partial \rho}{\partial Y_{\mathrm{S}}} \end{bmatrix}^{*}$$

where

$$\rho = \left[(X - X_{\mathrm{S}})^2 + (Y - Y_{\mathrm{S}})^2 \right]^{1/2}.$$

It follows then that

$$\widetilde{H} = \begin{bmatrix} \frac{X - X_{\mathrm{S}}}{\rho} & \frac{Y - Y_{\mathrm{S}}}{\rho} & 0 & 0 & 0 & -\frac{(X - X_{\mathrm{S}})}{\rho} & -\frac{(Y - Y_{\mathrm{S}})}{\rho} \end{bmatrix}^{*}$$

State Transition Matrix

State Transition Matrix

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$$

$$\mathbf{y}_i = \widetilde{H}_i \mathbf{x}_i + \boldsymbol{\epsilon}_i \qquad (i = 1, \ldots, \ell) \tag{4.2.6}$$

$$A(t) = \left[\frac{\partial F(t)}{\partial \mathbf{X}(t)}\right]^* \qquad \widetilde{H}_i = \left[\frac{\partial G}{\partial \mathbf{X}}\right]_i^*$$

The first of Eq. (4.2.6) represents a system of linear differential equations with time-dependent coefficients. The symbol $[\,]^*$ indicates that the values of $\mathbf{X}$ are derived from a particular solution to the equations $\dot{\mathbf{X}} = F(\mathbf{X}, t)$ which is generated with the initial conditions $\mathbf{X}(t_0) = \mathbf{X}_0^*$. The general solution for this system, $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$, can be expressed as

$$\mathbf{x}(t) = \Phi(t, t_k)\mathbf{x}_k \tag{4.2.7}$$

where $\mathbf{x}_k$ is the value of $\mathbf{x}$ at t_k; that is, $\mathbf{x}_k = \mathbf{x}(t_k)$. The matrix $\Phi(t_i, t_k)$ is called the state transition matrix and was introduced in Chapter 1, Section 1.2.5.

State Transition Matrix

$$
\begin{aligned}
1. \quad \Phi(t_k, t_k) &= I \\
2. \quad \Phi(t_i, t_k) &= \Phi(t_i, t_j)\Phi(t_j, t_k) \\
3. \quad \Phi(t_i, t_k) &= \Phi^{-1}(t_k, t_i).
\end{aligned}
\tag{4.2.8}
$$

$$\dot{\Phi}(t, t_k) = A(t)\Phi(t, t_k) \tag{4.2.10}$$

with initial conditions

$$\Phi(t_k, t_k) = I.$$

STM for a Linear System

Given the system:

$$\dot{x} = ax + by$$
$$\dot{y} = ky$$

Find the STM for:

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Equations are linear in the dependent variables (*x*,*y*) and their derivatives (in space state form)

STM for a Linear System

Equations are linear in the dependent variables (*x*,*y*) and their derivatives (in space state form)

$$\dot{X} = AX = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Whose solutions is

$$X(t) = \Phi(t, t_0) X_0$$

Where

$$\dot{\Phi}(t, t_0) = A\Phi(t, t_0)$$

STM for a Linear System

Since the differential equations for $\Phi(t, t_0)$ are linear with constant coefficients we can solve them using Laplace Transforms:

$$\Phi(t, t_0) = \mathcal{L}^{-1}(SI - A)$$

$$SI - A = \begin{bmatrix} s-a & -b \\ 0 & s-b \end{bmatrix} \qquad (SI - A)^{-1} = \begin{bmatrix} \dfrac{1}{s-a} & \dfrac{b}{(s-a)(s-b)} \\ 0 & \dfrac{1}{s-k} \end{bmatrix}$$

$$\therefore \Phi(t, t_0) = \begin{bmatrix} e^{at} & \dfrac{b}{a-k}\left(e^{at} - e^{kt}\right) \\ 0 & e^{kt} \end{bmatrix}$$

STM for a Linear System

If **A** is a constant matrix, there are a number of ways (including Laplace Transforms) to solve the equation

$$\dot{\Phi}(t,t_0) = A\Phi(t,t_0)$$

For example, we may integrate the equations directly:

$$\dot{\Phi}(t,t_0) = \begin{bmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} \\ \dot{\phi}_{21} & \dot{\phi}_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}\phi_{11} + A_{12}\phi_{21} & A_{11}\phi_{12} + A_{12}\phi_{22} \\ A_{21}\phi_{11} + A_{22}\phi_{21} & A_{21}\phi_{12} + A_{22}\phi_{22} \end{bmatrix}$$

STM for a Linear System

Note that the columns of $\dot{\Phi}$ are independent.

Hence, if $A = A(t)$ and we must use numerical integration we could integrate the rows of $\dot{\Phi}$ independently as *n* systems of $n \times 1$ equations as opposed to an $n \times n$ system of simultaneous equations.

STM for a Linear System

Evaluating the equations for $\dot{\Phi}$ yields:

$$\dot{\phi}_{11} = a\phi_{11} + b\phi_{21} \qquad (1)$$

$$\dot{\phi}_{21} = k\phi_{21} \qquad (2)$$

$$\dot{\phi}_{12} = a\phi_{12} + b\phi_{22} \qquad (3)$$

$$\dot{\phi}_{22} = k\phi_{22} \qquad (4)$$

with I.C. $\Phi\left(t, t_0\right) = I$

Note that the columns of $\dot{\Phi}$ are independent i.e., Eqns. (1) and (2) are independent of (3) and (4)

STM for a Linear System

Solutions

From Eq. (2) $$\frac{d\phi_{21}}{\phi_{21}} = k dt \longrightarrow \ln\phi_{21} = kt + \ln C$$

$$\frac{\phi_{21}}{C} = e^{kt} \longrightarrow \phi_{21} = Ce^{kt}$$

$$@\, t = t_0 = 0 \quad \phi_{21} = 0$$

$$\therefore C = 0 \text{ and } \Phi_{21} = 0$$

STM for a Linear System

Likewise, from Eq. (4)

$$\phi_{22} = Ce^{kt}$$

$$@\ t = t_0 = 0 \qquad \phi_{22} = 1$$

$$\therefore C = 1 \text{ and } \Phi_{22} = e^{kt}$$

Thus Eq. (1) becomes

$$\dot{\phi}_{11} = a\phi_{11} \text{ and } \phi_{11} = e^{at}$$

STM for a Linear System

Finally, from Eq. (3),

$$\dot{\phi}_{12} = a\phi_{12} + be^{kt} \quad (5)$$

The homogeneous equation

$$\dot{\phi}_{12} = a\phi_{12}$$

has the solution

$$\phi_{12} = ce^{at}$$

To get a particular solution note that e^{kt} has the derivative ke^{kt}

STM for a Linear System

So try

$$\phi_{12p} = C_1 e^{kt}$$

substitute the particular equation into Eq. (5)

$$C_1 k e^{kt} = a C_1 e^{kt} + b e^{kt}$$

Then

$$C_1 k - a C_1 - b = 0 \longrightarrow C_1 = \frac{b}{k-a}$$

STM for a Linear System

The general solution is the sum of homogeneous and particular solutions

$$\phi_{12} = Ce^{at} + \frac{b}{k-a}e^{kt}$$

$$@\ t = t_0 = 0\ ,\quad \phi_{12} = 0$$

$$\therefore \qquad C = -\frac{b}{k-a}$$

Hence

$$\phi_{21} = -\frac{b}{k-a}e^{at} + \frac{b}{k-a}e^{kt} \quad = \frac{b}{a-k}\left(e^{at} - e^{kt}\right)$$

and

$$\Phi(t,t_0) = \begin{bmatrix} e^{at} & \frac{b}{a-k}\left(e^{at} - e^{kt}\right) \\ 0 & e^{kt} \end{bmatrix}$$

Note that @ t=0

$$\Phi(t,t_0) = I$$

Relating the Observations to an Epoch State

$$\mathbf{x}(t) = \Phi(t, t_k)\mathbf{x}_k \tag{4.2.7}$$

$$\mathbf{y}_i = \widetilde{H}_i \mathbf{x}_i + \boldsymbol{\epsilon}_i \qquad (i = 1, \ldots, \ell)$$

$$\begin{aligned}
\mathbf{y}_1 &= \widetilde{H}_1 \Phi(t_1, t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_1 \\
\mathbf{y}_2 &= \widetilde{H}_2 \Phi(t_2, t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_2 \\
&\vdots \\
\mathbf{y}_\ell &= \widetilde{H}_\ell \Phi(t_\ell, t_k)\mathbf{x}_k + \boldsymbol{\epsilon}_\ell .
\end{aligned} \tag{4.2.37}$$

Relating the Observations to an Epoch State

Using the following definition:

$$\mathbf{y} \equiv \begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix}; \quad H \equiv \begin{bmatrix} \widetilde{H}_1 \Phi(t_1, t_k) \\ \vdots \\ \widetilde{H}_\ell \Phi(t_\ell, t_k) \end{bmatrix}; \quad \boldsymbol{\epsilon} \equiv \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_\ell \end{bmatrix} \tag{4.2.38}$$

and if the subscript on $\mathbf{x}_k$ is dropped for convenience, then Eq. (4.2.37) can be expressed as follows:

$$\mathbf{y} = H\mathbf{x} + \boldsymbol{\epsilon} \tag{4.2.39}$$

Relating the Observations to an Epoch State

$$y = Hx + \epsilon \qquad (4.2.39)$$

$\boldsymbol{y}$ is an $m \times 1$ vector $\qquad$ $\boldsymbol{x}$ is an $n \times 1$ vector

$\boldsymbol{\epsilon}$ is an $m \times 1$ vector

H is an $m \times n$ mapping matrix

$m = p \times l$ is the total number of observations

$m > n$ is an essential condition

Have m unknown observation errrors

Relating the Observations to an Epoch State

$$\mathbf{y} = H\mathbf{x} + \epsilon \tag{4.2.39}$$

Results in:

m unknown observation errors

$m + n$ total unknowns

m equations

The least squares criterion provides us with conditions on the m observation errors that allow a solution for the n state variables, $\boldsymbol{X}_k$, at the epoch time t_k

Review of Variables