ASEN 5070:
Statistical Orbit Determination I

Exam #3
Due Monday, Dec 17 at 11:59 pm

Open Book, Computer, and Notes

Complete the following problems and turn in an electronic submission to D2L by the due date, just like we've done with our homeworks during the semester. There is a D2L Dropbox called "Exam 3". Scanned sheets are completely acceptable. We'll grade on completeness and correctness, and it must be clear enough to read and grade, but it's fine if it's hand-written (except plots of course!). The due date will be enforced unless you get prior written permission to turn it in late. Any unexcused late submissions will be docked points.

Please do not give, request, or accept help from others on this exam, except via email to the instructors.

If you have any questions, please send us an email! Email all three of us to get the quickest response. We hope you enjoy this (haha), and perhaps even learn a little more about Stat OD doing this exam.

1. 10% A random variable has the cumulative distribution function given by:

$$F(x) = \begin{cases} 0, & x < 0 \\ kx^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

where k is some constant and $F(x)$ is continuous. Answer the following:

a) 2% Find k.

b) 3% What is the probability that a realization of x falls between $-\infty$ and 0.5?

c) 5% What is the corresponding probability density function over all x? Is it a continuous function over all x?

2. 13%

   a. 2% True/False: If the matrix A is a matrix of constants, you can use a Laplace Transform to solve for an analytical expression for $\Phi(t, t_0)$.

   b. 2% True/False: If the equations of motion for a particle are linear and the observation state equations are not linear, you can use a Laplace Transform to solve for an analytical expression for $\Phi(t, t_0)$.

   c. 2% True/False: The state transition matrix is used to map the state **X**(t) from one time to another; it is only used to map the state deviation vector ***x***(t) if the system is linear.

   d. 2% True/False: The state transition matrix that is used to map a state deviation vector from time t_1 to time t_{10} may be constructed via the following:

$$\Phi(t_{10}, t_1) = \Phi(t_{10}, t_9) + \Phi(t_9, t_8) + \cdots + \Phi(t_2, t_1)$$

   e. 1% for each right answer. Which of the following matrices could be a viable State Transition Matrix?

I. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

II. $\begin{bmatrix} t - t_0 & 0 & 0 \\ 0 & (t-t_0)^2 & 0 \\ 0 & 0 & -(t-t_0) \end{bmatrix}$

III. $\begin{bmatrix} 1 & (t-t_0) & 2(t-t_0)^2 \\ (t-t_0) & 1 & (t-t_0) \\ 2(t-t_0)^2 & (t-t_0) & 1 \end{bmatrix}$

IV. $\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1 & 1/2 \\ 1/3 & 1/2 & 1 \end{bmatrix}$

V. $\begin{bmatrix} (t-t_0)^2 + (t-t_0) + 1 & 0 & (t-t_0)/12 \\ -(t-t_0) & (t-t_0)^3 + 1 & (t-t_0) \\ 100(t-t_0) & 10(t-t_0) & 1 \end{bmatrix}$

3. 17%
   a. 2% each: Which of the following could be a viable covariance matrix for two parameters?

I. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

II. $\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$

III. $\begin{bmatrix} 4 & -5 \\ -5 & 9 \end{bmatrix}$

IV. $\begin{bmatrix} 4 & 8 \\ 8 & 9 \end{bmatrix}$

V. $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

VI. $\begin{bmatrix} 101 & 0.001 \\ 0.001 & 0.03 \end{bmatrix}$

   b. 5% Given the following covariance matrix

$$\begin{bmatrix} 9 & 7.5 \\ 7.5 & 25 \end{bmatrix}$$

What is the correlation coefficient?

4. 30% There is a file on the course website in the Exam 3 area called "Exam3_Problem4_data.txt" that contains eight independent and uncorrelated solutions to the same problem, each including a corresponding error covariance matrix. Perhaps these came from eight different navigators using different data. The solutions all correspond to the same epoch. We'll refer to the i^{th} solution as $\hat{x}_i$ and its associated error covariance matrix as P_i. Use Sections 4.16 and 4.17 as a guide for the following:

a. 6% Use this file to determine the *best* estimate of the state deviation vector, $\hat{x}$, using all of the information provided. Also determine the error covariance P for that estimate.

b. 5% Produce a figure that includes the following:
   - The best estimate of the state deviation vector as a point on the plot.
   - The best estimate's corresponding 1σ, 2σ, and 3σ error ellipses. That is, plot concentric ellipses radiating away from the best estimate, and label each ellipse. The ellipses should be oriented along the covariance matrix's principal axes (see Section 4.16).
   - All of the other estimates as points.
   - Using only this figure, do any of the solutions appear suspicious? That is, do any of them seem like they could be the result of faulty assumptions/ math/data?

c. 3% Consider Solution #6 ($\hat{x}_6$). Compute the number of standard deviations that this solution is away from the *best* estimate generated in (a) using the best estimate covariance ellipsoid, P.

d. 3% Now compute the number of standard deviations that separates the best estimate solution from Solution 6 using Solution 6's error covariance, P_6.

e. 5% Produce another figure, identical to that generated in part (b), but this time also plot each solution's 1σ error ellipse. Be sure to rotate each ellipse along the corresponding matrix's principal axes. I suggest plotting the solution's estimate and its error ellipse in the same color and using several colors to make them aesthetically differentiable. I'd also suggest plotting the *best* estimate's ellipses with a thicker linewidth to help them stand out among each solution. Points will be awarded for clarity. Feel free to zoom in and show other views of the figure.

f. 3% If you are the navigation team lead and the team presented the plot from part (e) to you, would you feel satisfied that every solution could describe the actual state of the system? Do any solutions appear suspicious to you? Would you be satisfied to fly the *best* estimate in this case?

g. 3% Sample the *best* estimate's covariance matrix, P, 1000 times using the Monte Carlo technique and plot those samples in a new figure. Include the *best* estimate and at least the 1σ error ellipse on the same figure.

h. 2% Use the data generated in (g) to compute a numerical covariance for the Monte Carlo's results. Compare this numerical covariance with the best estimate's error covariance from part (a). Use the following equations:

$$\bar{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i \qquad P_{\text{MonteCarlo}} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T$$

5. 30% Problem 4-42. Use the same data file that we used for HW 11 (this file is also online in the Exam 3 area, called “Exam3_Problem5_data.txt”). Use $\sigma = 2.49$ and $\beta = 0.045$ for this problem. Note that these are not the optimal values determined from Exercise 4-41, but these will illustrate the smoothed estimates more clearly. We will grade on the following:

a) 20% Clean and complete plot that resembles Figure 4.19.2 (but with different data of course).

b) 5% Compute the RMS of the Filtered values and of the Smoothed values. Compare them: is the Smoothed RMS smaller than the Filtered RMS?

c) 5% Produce a histogram of the Filtered values and of the Smoothed values. The Matlab function “hist(data,bins)” computes a histogram given the data and the number of bins you’re interested in. The function “histfit” fits a Normal curve to the bins. Does the Smoothed histogram conform to a normal curve better than the Filtered histogram?