Therefore,

$$\Phi(t_i, t_j) = \Phi(t_i, t_k)\Phi(t_k, t_j).$$

c. $\Phi^{-1}(t_i, t_k) = \Phi(t_k, t_i)$.
Use $x(t_i) = \Phi(t_i, t_k)x(t_k)$ or

$$\begin{aligned} x(t_k) &= \Phi^{-1}(t_i, t_k)x(t_i) \\ &= \Phi(t_k, t_i)x(t_i). \end{aligned}$$

Hence,

$$\Phi^{-1}(t_i, t_k) = \Phi(t_k, t_i)$$

d. $\dot{\Phi}^{-1}(t_i, t_k) = -\Phi^{-1}(t_i, t_k)A(t_i)$.
Differentiating the identity

$$\begin{aligned} \Phi\Phi^{-1} &= I \\ \dot{\Phi}\Phi^{-1} + \Phi\dot{\Phi}^{-1} &= 0 \end{aligned}$$

Substituting $\dot{\Phi} = A\Phi$ yields,

$$A\Phi\Phi^{-1} + \Phi\dot{\Phi}^{-1} = 0$$

and

$$\dot{\Phi}^{-1}(t_i, t_k) = -\Phi^{-1}(t_i, t_k)A(t_i)$$

3. We wish to minimize the performance index

$$J(x) = 1/2(\mathbf{y} - H\mathbf{x})^T W(\mathbf{y} - H\mathbf{x}) + 1/2(\bar{\mathbf{x}} - \mathbf{x})^T \overline{W}(\bar{\mathbf{x}} - \mathbf{x})$$

Using Eq. (B.7.3)

$$\begin{aligned} \frac{\partial J(x)}{\partial(x)} = 0 &= -H^T W(\mathbf{y} - H\hat{\mathbf{x}}) - \overline{W}(\bar{\mathbf{x}} - \hat{\mathbf{x}}) \\ &= (H^T W H + \overline{W})\hat{\mathbf{x}} - H^T W\mathbf{y} - \overline{W}\bar{\mathbf{x}} = 0 \end{aligned}$$

or

$$\hat{\mathbf{x}} = (H^T W H + \overline{W})^{-1}(H^T W\mathbf{y} + \overline{W}\bar{\mathbf{x}})$$

4.

$$\begin{bmatrix} \dot{\Phi}_{11} & \dot{\Phi}_{12} \\ \dot{\Phi}_{21} & \dot{\Phi}_{22} \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & g \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$$

We may integrate the four differential equations. However, using Laplace Transforms is simpler

$$\begin{aligned} \Phi = \mathcal{L}^{-1}[SI - A]^{-1} &= \mathcal{L}^{-1} \begin{bmatrix} s-a & 0 \\ -b & s-g \end{bmatrix}^{-1} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{(s-a)(s-g)} \begin{bmatrix} s-g & 0 \\ b & s-a \end{bmatrix} \right\} \end{aligned}$$