Secure Discrete-Time Linear-Quadratic Mean-Field Games

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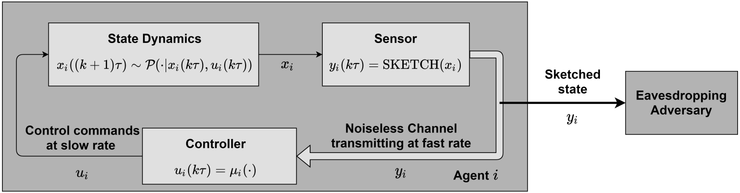

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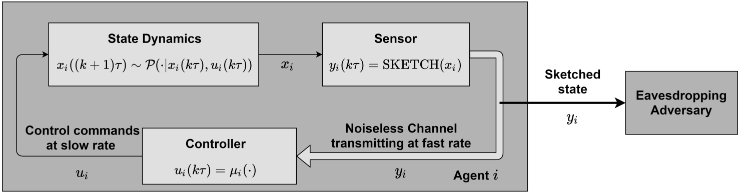

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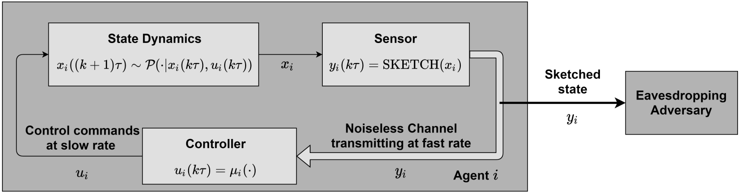

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- Adversary is non-strategic.

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- Furthermore, we show that the MFE of the LQ-MFG is an $(\epsilon + \varepsilon)$-NE of the finite agent game.
- Finally, we empirically investigate the performance of the $(\epsilon + \varepsilon)$-NE.

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- Observations generated at fast rate $1/\Delta$ and control at slow rate $1/\tau$, s.t. $\tau = N\Delta$.
- $x_i(0)$ and $\tilde{w}_i$ are generated i.i.d., second order distribution.
- C_i (private key) chosen uniformly from set of private keys

$$\mathcal{C} = \{C_i | i \in \{1, 2, \ldots, M\}, M < \infty, C_i \in \mathbb{R}^{q \times m}, N > Obs(A, C_i)\}.$$

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- Each agent aims to minimize its cost

$$J_i^n = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big\{ \sum_{k=0}^{T-1} \Big\| x_i(k\tau) - \frac{1}{n-1}\sum_{j\neq i} x_j(k\tau) \Big\|_Q^2 + ||u_i(k\tau)||_R^2 \Big\},$$

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- The solution concept used is that of *Nash Equilibrium*.

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- The cost function of the generic agent given the mean-field trajectory $\bar{x}$ is

$$J(\mu, \bar{x}) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\Big\{ \sum_{k=0}^{T-1} ||x(k\tau) - \bar{x}(k\tau)||_Q^2 + ||u(k\tau)||_R^2 \Big\}.$$

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Definition

The tuple $(\mu^*, \bar{x}^*) \in \mathcal{M} \times \ell^\infty$ is an MFE if $\bar{x}^* = \Lambda(\mu^*)$ and

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- MFE is analog to Nash Equilibrium.

State Reconstruction using Multi-Rate Output Sampling

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- If we denote $y_{[k]} := [y^T((k-1)\tau), \ldots, y^T((k-1)\tau + (N-1)\Delta)]^T$, then

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- Estimation error $w(k\tau)$ is a zero mean random vector with covariance matrix Σ_C. Define set $\mathcal{E}_C := \{\Sigma_C : C \in \mathcal{C}\}$.

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- Define augmented state $z(k\tau) = [x^T(k\tau), \bar{x}^T(k\tau)]^T$. MF trajectory $\bar{x}$ is defined by matrix F.
- Dynamics of augmented state is

$$z((k+1)\tau) = \bar{A}z(k\tau) + \bar{B}u(k\tau) + \bar{w}(k\tau)$$

$$\bar{A} = \begin{bmatrix} A_0 & 0 \\ 0 & F \end{bmatrix}, \bar{B} = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \bar{w}(k\tau) = \begin{bmatrix} w^0(k\tau) \\ 0 \end{bmatrix}$$

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- Cost of generic agent under control law K, $u(k\tau) = -K(\hat{x}^T(k\tau), \bar{x}^T(k\tau))$

$$J(K, F) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\Big\{ \sum_{k=0}^{T-1} \|z(k\tau)\|_Q^2 + \|u(k\tau)\|_R^2 \Big\}.$$

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- The controller $\hat{K}_F$ which minimizes $J(K, F)$ for any stable F is

$$\hat{K}_F = (\bar{B}^T \hat{P} \bar{B} + R)^{-1} \bar{B}^T \hat{P} \bar{A}(I - \hat{\Sigma}_C \Sigma_{\hat{K}_F}^{-1})$$

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- $\Sigma_{\hat{K}_F}$ is the covariance matrix of stationary distribution and is shown to be singular. Hence $\hat{K}_F$ does not exist.

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Assumption

With P given as the unique positive definite solution to the DARE,

$$P = A_0^T P A_0 + Q - A_0^T P B_0 (R + B_0^T P B_0)^{-1} B_0^T P A_0$$

and furthermore that $G_P := -(R + B_0^T P B_0)^{-1} B_0^T$ and $H_P := A_0^T (I + P B_0 G_P)$, we have $\|H_P\|_2 + \frac{\|B_0 G_P\|_2 \|Q\|_2}{(1 - \|H_P\|_2)^2} < 1$

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Theorem

Under above given Assumption, the MFE of LQ-MFG (K^*, F^*) is also the ϵ-MFE of the SLQ-MFG where $\epsilon = \mathcal{O}(\mathrm{tr}(\Sigma_C))$.

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- $\mathrm{tr}(\Sigma_C) \to 0 \implies \epsilon \to 0.$

$(\epsilon + \varepsilon)$-Nash Equilibrium of the secure n-agent LQ game

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Under above given Assumption, let the cost of secure n-agent LQ game under controller μ^* *be* $J_i^n(\mu^{*i}, \mu^{*-i})$*, then*

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where $\epsilon = \mathcal{O}(\sigma_{max})$ *and* $\sigma_{max} := \max_{\Sigma_C \in \mathcal{E}_C} \mathrm{tr}(\Sigma_C)$ *and* $\varepsilon = \mathcal{O}(\sigma_{max}/\sqrt{n-1})$.

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- If $\mathrm{tr}(\sigma_{max}) \to 0$ and $n \to \infty$ then $(\epsilon + \varepsilon) \to 0$.

Performance sensitivity w.r.t. sampling rate

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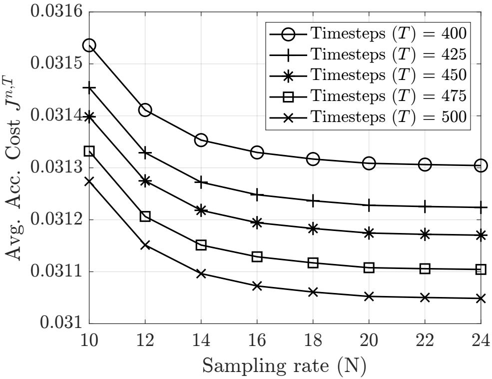

Figure: Average accumulated cost w.r.t. change in sampling rate N.

Performance sensitivity w.r.t. model parameters

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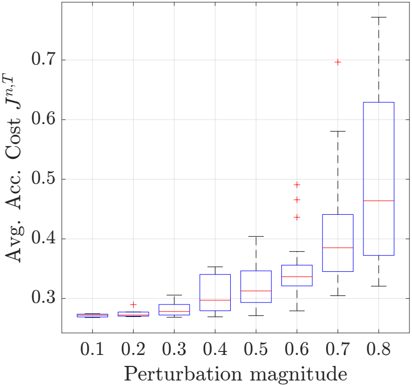

Figure: Average accumulated cost w.r.t. perturbation of the A and B matrices.

Performance sensitivity w.r.t. private keys

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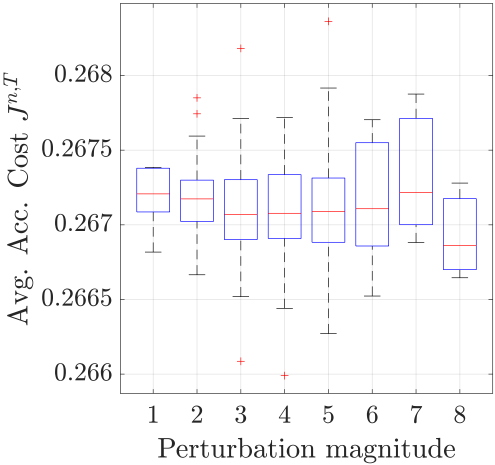

Figure: Average accumulated cost w.r.t. perturbation of the set of keys $\mathcal{C}$.

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- We established that MFE of (standard) LQ-MFG, corresponds to ϵ-MFE of the SLQ-MFG, and an $(\epsilon + \varepsilon)$-Nash equilibrium for the secure $n-$agent dynamic game.
- We empirically demonstrated that performance of the $(\epsilon + \varepsilon)$-Nash equilibrium improves with increasing sampling rate N, deteriorates with variations in model parameters (A, B), and is insensitive to small perturbations in the set of private keys $\mathcal{C}$.