Adversarial LQ Mean-Field Games over Multigraphs

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UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Opinion Dynamics on a multigraph

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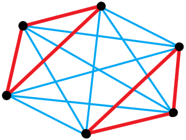

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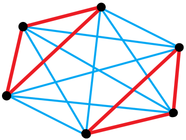

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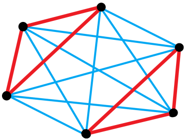

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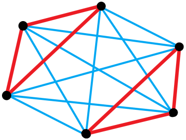

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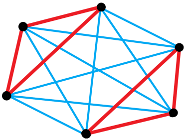

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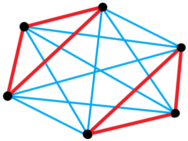

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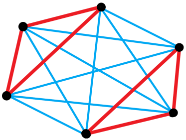

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- Adversary manipulates mass opinion (Media Houses).
- Assumption: Local graphs are non-overlapping.

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- First to investigate the local(physical) vs global(virtual) graph dichotomy in multigraphs.
- Risk-sensitive and robust MFGs[3] considers adversarial agents but does not consider sparse graph structure.

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- Numerical Results.

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- Numerical Results.
- Future directions.

Formulation (finite population)

- Agent $i \in [N]$ has linear dynamics (uncoupled),

$$Z^i_{t+1} = AZ^i_t + BU^i_t + CV_t + W^i_t, t \in (0, \dots, T-1),$$

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$$J_N^i(\pi^i, \pi^{-i}, V) =$$

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Definition (Nash Equilibrium)

The set $\pi^* = (\pi^{1*}, \ldots, \pi^{N*})$ and adversary policy $V^* = (V_0^*, \ldots, V_{T-1}^*)$ constitute Nash equilibrium if,

$$J_N^i(\pi^{i*}, \pi^{-i*}, V^*) \leq J_N^i(\pi^i, \pi^{-i*}, V^*), \quad \pi^i \in \Pi^i, i \in [N]$$

$$J_N^0(\pi^*, V^*) \leq J_N^0(\pi^*, V), \quad V \in \mathcal{V}$$

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- $\bar{Z} = (\bar{Z}_0, \ldots, \bar{Z}_T)$ represents global mean-field, $Y = (Y_0, \ldots, Y_T)$ represents local mean-field.

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[4] P. Turnes Jr and L. Monteiro, "An epidemic model to evaluate the homogeneous mixing assumption," Communications in Nonlinear Science and Numerical Simulation,vol. 19, no. 11, pp. 4042–4047,2014.

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Definition (Mean-Field Equilibrium)

The tuple $(\mu^*, V^*, \bar{Z}^*, Y^*)$ is a mean-field equilibrium if,
Optimality: $\mu^* = \text{argmin}_\mu J(\mu, \bar{Z}^*, V^*)$, $V^* = \text{argmin}_V J^0(V, \mu^*, \bar{Z}^*)$,
Consistency: $\bar{Z}^*$ and Y^* are aggregate behaviors *consistent* with μ^* and V^*.

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Form of generic agent's equilibrium control

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Theorem

The generic agent's equilibrium control adapted to filtration $\mathcal{F}_t^Z \vee \mathcal{F}_t^Y$ is given by

$$U_t^* = -R_t^{-1} B^T \zeta_{t+1}$$

where ζ_t can be constructed as

$$\begin{aligned}
\zeta_t &= A^T \zeta_{t+1} + (Q_t + \bar{Q}_t + \tilde{Q}_t) Z_t - \bar{Q}_t \bar{Z}_t - \tilde{Q}_t Y_t - M_t^\zeta, \\
\zeta_T &= (Q_T + \bar{Q}_T + \tilde{Q}_T) Z_T - \bar{Q}_T \bar{Z}_T - \tilde{Q}_T Y_T, \\
M_t^\zeta &= A^T \zeta_{t+1} - A^T \mathbb{E}\left[\zeta_{t+1} \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^Y\right].
\end{aligned}$$

where M_t^ζ is a martingale difference sequence adapted to filtration $\mathcal{F}_t^Z \vee \mathcal{F}_t^Y$.

Form of adversary's equilibrium control

Form of adversary's equilibrium control

Theorem

If the following condition is satisfied

$$S_t - C^T \hat{P}_{t+1} C > 0, \tag{1}$$

where the matrix $\hat{P}_t$ is defined recursively by

$$\hat{P}_t = Q_t + A^T \hat{P}_{t+1} A + A^T \hat{P}_{t+1} C (S_t - C^T \hat{P}_{t+1} C)^{-1} C^T \hat{P}_{t+1} A, \quad \hat{P}_T = Q_T,$$

then the equilibrium control policy of the adversary is,

$$V_t^* = -S_t^{-1} C^T \bar{\zeta}_{t+1}^0, \bar{\zeta}_t^0 = A^T \bar{\zeta}_{t+1}^0 - Q_t \bar{Z}_t, \bar{\zeta}_T^0 = -Q_T \bar{Z}_T, \tag{2}$$

where $\bar{\zeta}_t^0$ is the adversary's co-state and $\bar{Z}$ is the global MF of the agents.

Equilibrium Global MF dynamics

Equilibrium Global MF dynamics

Theorem

If the preceding assumptions are satisfied, then equilibrium global MF follows linear dynamics,

$$\bar{Z}^*_{t+1} = E_t^{-1} A \bar{Z}^*_t = \bar{F}_t \bar{Z}^*_t, \tag{3}$$

where $E_t = (I + BR_t^{-1} B^T \bar{P}_{t+1} - CS_t^{-1} C^T \bar{P}_{t+1})$ *and* $\bar{P}_t$ *is given by the Riccati equation,*

$$\bar{P}_t = A^T \bar{P}_{t+1} E_t^{-1} A + Q_t, P_T = Q_T$$

and the equilibrium adversarial policy,

$$V^*_t = S_t^{-1} C^T \bar{P}_t \bar{F}_t \bar{Z}^*_t. \tag{4}$$

is linear in the equilibrium global MF.

Equilibrium Local MF dynamics

Equilibrium Local MF dynamics

Theorem

If the preceding assumptions are satisfied, then local equilibrium MF has linear Gaussian dynamics driven by the equilibrium global MF:

$$Y^*_{t+1} = \tilde{F}^1_t Y^*_t + \tilde{F}^2_t \bar{Z}^*_t + \tilde{E}^{-1}_t \tilde{W}_t \tag{5}$$

where $\tilde{W}_t = \sum_{j \in \mathcal{N}} W^j_t / |\mathcal{N}|$,

$$\begin{aligned}
\tilde{E}_t =& (I + BR^{-1}B^T\tilde{P}_{t+1} - CS^{-1}_t C^T \tilde{P}_{t+1}), \tilde{F}^1_t = \tilde{E}^{-1}_t A, \\
\tilde{F}^2_t =& \tilde{E}^{-1}_t (BR^{-1}_t B^T - CS^{-1}_t C^T)) \sum_{i=0}^{T-t} \prod_{j=1}^{i} \tilde{H}_{t+j} \bar{Q}_{t+i} \bar{F}_{t+j} \\
\tilde{P}_t =& A^T \tilde{P}_{t+1} \tilde{E}^{-1}_t A + Q_t + \bar{Q}_t, \quad \tilde{P}_T = Q_T + \bar{Q}_T,
\end{aligned}$$

and $\tilde{H}_t = A^T(I - \tilde{P}_{t+1}\tilde{E}^{-1}_t BR^{-1}B^T)$.

MFE Characterization

- Generic agent's problem can be cast as time varying linear quadratic regulator (TV-LQR) problem.

$$U_t^* = \mu_t^*(Z_t, Y_t^*, \bar{Z}_t^*) = \bar{K}_t^* \begin{pmatrix} Z_t \\ Y_t^* \\ \bar{Z}_t^* \end{pmatrix}$$

MFE Characterization

Conclusion:

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- Then global mean-field (at equilibrium) has deterministic dynamics, so can be computed offline.

MFE Characterization

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- Then global mean-field (at equilibrium) has deterministic dynamics, so can be computed offline.
- The adversary actions (at equilibrium) depend *only* on the global mean-field.

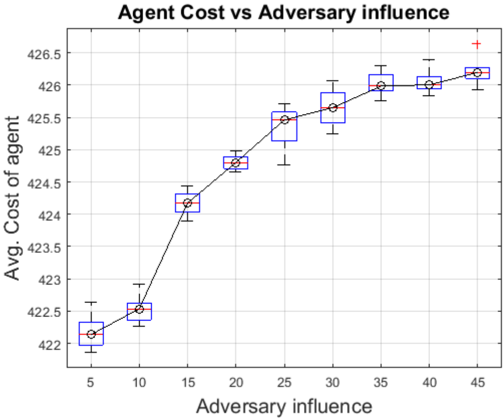
MFE Characterization

<u>**Conclusion:**</u>

- Then global mean-field (at equilibrium) has deterministic dynamics, so can be computed offline.
- The adversary actions (at equilibrium) depend *only* on the global mean-field.
- The agent's actions (at equilibrium) have linear dependence on,
  - the agent's state,
  - the local mean-field,
  - the global mean-field.

Numerical Results (Affect of Adversary)

- Number of agents $N = 10,000$
- Neighborhood size $|\mathcal{N}(i)| = 1000$.



Future Directions

- ϵ-Nash analysis of the MFE.
- More general local network structure given *homogenous mixing hypothesis*,
  - Overlapping local graphs,
  - Agents can have types specifying local connectivity.
- Multiple adversaries.