Week 7: Asset Volatility and GARCH

1. Variance and Realized Volatility.

Def: Realized Volatility, Assuming $\{r_t\}$ $t=0,1,2 \cdots N$

$$\sigma_{real}^2 = \frac{1}{N}\sum_{i=1}^{N} \ln\left(S_i/S_{i-1}\right)^2$$

$$= \frac{1}{N}\sum_{i=1}^{N} r_i^2 \quad \cdots\cdots \quad E[r_t^2]$$

Variance of r_t $\quad Var(r_t) = E[(r_t - \mu)^2]$

$\therefore$ The variance is mean-adjusted realized variance.

Realized Volatility: $\quad \sigma_{real} = \sqrt{\sigma_{real}^2} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} r_i^2}$

$$= \sqrt{\frac{1}{N}\sum_{i=1}^{N} \ln\left(S_i/S_{i-1}\right)^2}$$

Quadratiction of $f(x)$:

$$[f,f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=1}^{N} \left[f(x_j) - f(x_{j-1})\right]^2.$$

$f(x_j) = \ln(S_j)$

2. Implied Volatility

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$\Downarrow$$

$$C = BS(r, \sigma, T, S_0, K)$$

Given parameters r, T, S_0, K, We have one to one mapping between C_i and σ_i.

Thus σ_i is ~~called the~~ "implied" by the BS formula.

$\Downarrow$

Volatility Smile / Volatility Term Structure.

3. Why the term "Volatility".

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

$$\gamma_t = \ln S_t \Rightarrow d\gamma_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t.$$

$$\gamma_t = \gamma_0 + \int_0^t (\mu - \frac{\sigma^2}{2})ds + \int_0^t \sigma dW_s.$$

$$\begin{aligned} Var(\gamma_t) &= \cancel{E[\int_0^t} E[\gamma_t - E[\gamma_t]]] \\ &= E[(\int_0^t \sigma dW_t)^2] \\ &= E[\int_0^t \sigma^2 dt] \\ &= \sigma^2 t. \end{aligned}$$

$\therefore$ The variance of γ_t is propotional to σ^2, for a fixed t.

4. Problem with ~~Geometrical Motion~~ B-S.

① No Transaction cost.

② Continuous Hedging $\rightarrow$ week 12

③ Constant Volatility $\rightarrow$ ~~Volatility Models~~.

④ log normal distribution $\rightarrow$ ~~leptokurtic~~

} other volatility Models.

① Lepkurtic:

$$\text{Kurtosis} = E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right] \quad \text{for Normal Kurt} = 3.$$

$$x_t - x_0 = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

QQ Plot. Should be a straight line.

The Geometric Brownian Motion Don't have fat tail distribution.

②. ACF on volatility. $\rightarrow$ $\sigma(t)$ related to $\sigma(t-1)$ $\Rightarrow$ Volatility Clustering.

Volatility is not a constant, but clustered.

③ Volatility Smile.

A Volatility Model need to replicate the smile.

I: GARCH Model

$$\begin{cases} r_t = \mu_t + a_t \\ a_t = \sigma_t \varepsilon_t \\ \sigma_t = \alpha_0 + \alpha_1 \sigma_{t-1} \varepsilon_{t-1} + \beta_1 \sigma_{t-1} \end{cases}$$

Simulating ① Calibrate $\alpha_0, \alpha_1, \beta_1$

② Generate ε_t matrix

③ Generate a_t matrix

④ Generate r_t.

II. Stochastic Volatility Model.

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

$$\Downarrow$$

$$\begin{cases} dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_t^1 \\ dY_t = \phi(Y_t)dt + \psi(Y_t)dW_t^2 \end{cases}$$

$$dW_t^1 \, dW_t^2 = \rho \, dt$$

Simulation: ① Generate dW_t^1 and dW_t^2 correlated.

② Simulate S_t.