Electromagnetic field around the pulsar[Pad01]

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Introduction

Pulsars primarily emit radiation due to their magnetic fields and the acceleration of charged particles along these fields. The emission of radiation from pulsars is commonly explained by a model known as the **magnetospheric** model or the **rotating dipole** model. So in an attempt to study the electromagnetic behaviour of the pulsars, we will compute the Electric and Magnetic field in and around a pulsar along with studying the source and strength of the pulsar radiation(s).

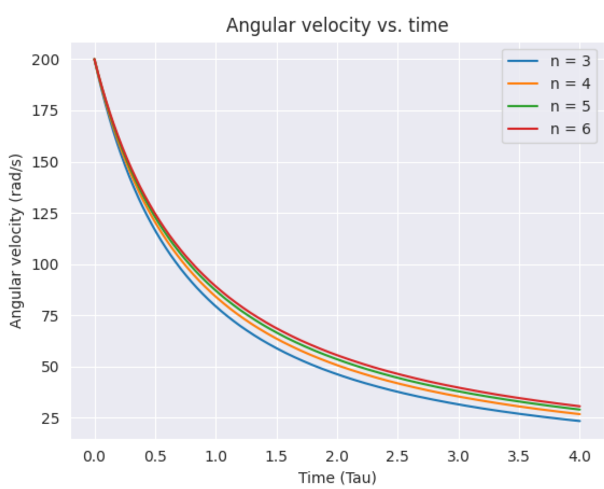
Magnetic dipole radiation

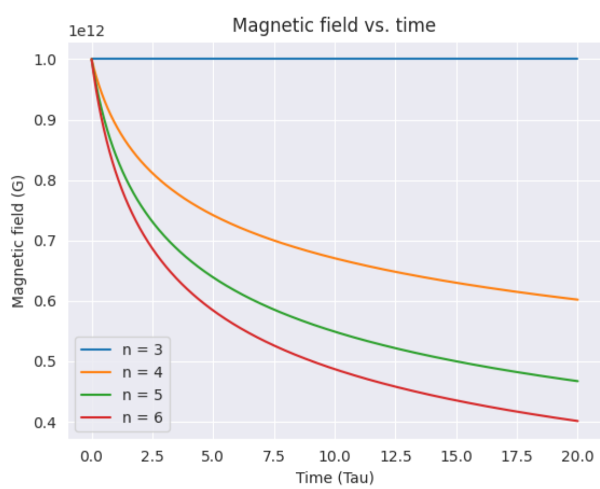
We use Larnon's formula to calculate the Emission rate due to the magnetic dipole and assume that the energy for this is supplied by the rotational kinetic energy of the pulsar.

$$\frac{dE}{dt} = -\frac{2}{3}|\ddot{m}|^2 = -\frac{B^2R^6\omega^4\sin^2\alpha}{6c^3} \quad \text{and} \quad \frac{d(KE)}{dt} = I\omega\dot{\omega} \tag{1}$$

Equating these, we get a differential equation in terms of ω which when generalised for multipole radiation using a breaking index $n = \omega\ddot{\omega}/\dot{\omega}^2$ or $\dot{\omega} \propto \omega^n$, n=3 in case of dipole. This gives us the following relations on

$$\omega(t) = \omega_0(1+\frac{t}{\tau})^{-1/(n-1)} \quad \text{and} \quad B(t) = B_0(1+\frac{t}{\tau})^{-\frac{1}{2}(\frac{n-3}{n-1})} \tag{2}$$





As we can see from the graphs there is a decay in the angular velocity of the pulsar with time, with the decay being fastest for the dipole. On the other hand, there seems to be a constant magnetic field for dipole radiations but with the increase in breaking index, the order of decay is higher.

Estimation of the number of pulsars in the galaxy

Say we measure the radiation energy of the pulsar in the Radio-wave luminosity spectrum, then it is a fixed fraction $f = L_{radio}/L \approx 10^{-5}$ of the total pulsar radiation. Since we also have a limitation imposed by measuring instruments, we consider a certain minimum Flux F_{min} below which we cannot detect radiation. This implies the existence of a maximum radius R_{max} up to where given a fixed luminosity we can detect all pulsars. We then substitute Eq(2.1) in Eq(1.2):

$$L_{radio} = f|\frac{dE}{dt}| = (4\pi R_{max}^2)F_{min} \quad \text{and} \quad \frac{dE}{dt} = I\omega\dot{\omega} \tag{3}$$

$$R_{max}^2 = (\alpha\pi)(\frac{fIP_0^{1/\alpha}}{\tau})(\frac{1}{F_{min}P^{2+(1/\alpha)}}) \quad \text{where} \quad \alpha = \frac{1}{n-1}, P = 2\pi/\omega \tag{4}$$

Now we take a spatial number density $n(P)$ of Pulsars with period between P and $P + dP$ and birth-rate of b. Assuming the galaxy to be of approximately cylindrical in shape with a thickness of h:

$$bdt = n(P)dP \quad => \quad n(P) = b\frac{dT}{dP} = \frac{b\tau}{\alpha P_0}(\frac{P}{P_0})^{(1/\alpha)-1} \tag{5}$$

$$N(P) = n(P)[\pi R_{max}^2 h] \quad => \quad N(P) = \frac{\pi^2 fIhb}{F_{min}}\frac{1}{P^3} \tag{6}$$

We can see that the number of pulsars for a particular period is proportional to P^{-3}. Integrating this expression from P to ∞ would give the total number of Pulsars above a certain period $N(>P) \propto (F_{min}P^2)^{-1}$. Hence, we can say that in a flux-limited survey, we would potentially detect low-period pulsars.

Pulsars with period greater than a value (P_{min}), can be detected anywhere from the galaxy whereas those with periods in range(P_0, P_{min}) can be only detected only if they are at $R < R_{max}(P)$. Hence, the total number of pulsars can be given by:

$$\boxed{N_{tot} = \int_{P_0}^{P_{min}} n(P)\pi R_{gal}^2 hdP + \int_{P_{min}}^{\infty} n(P)\pi R_{max}^2(P)hdP} \tag{7}$$

This was derived under the **assumptions** (1) Pulsars are born with constant parameter values in the galactic disk. (2) Their motion away from the galactic disk is ignorable. (3) They radiate isotropically.

The Aligned Rotator

In all the above calculations we have assumed that the region surrounding the pulsar is a vacuum and that the radiation escapes freely. We also ignored relativistic effects near the neutron star and didn't bother calculating the electromagnetic field in and around the pulsar which is exactly what we are going to do for a simplified case, aligned rotator ($\alpha = 0$). [Magnetic dipole is aligned along the rotational axis]

We first assume that the pulsar to be a conducting, rotating sphere with a Magnetic field B inside. And the scale height given by $H = (k_BT)/(\frac{GMm_H}{R^2}) \approx 0.4(T/10^6)cm$ which is much smaller than the relevant scales. Hence, we can assume that the surface b/w the conductor and the vacuum to have a sharp discontinuity.

The magnetic field configuration inside the neutron star itself can be modeled by (1) **Uniformly magnetized sphere $\vec{B} = B_0\hat{k}$ along the rotating axis** (2) A point dipole at the origin with $\vec{m}$ along $\hat{k}$.

Uniformly Magnetized Sphere

The rotation of the neutron star causes Lorentz Force to act on each charge and it moves until an Electric Field is generated to balance out the magnetic force.

$$\vec{v} = \vec{\omega} \times \vec{r} => |\vec{v}| = \omega r \sin\theta\hat{\phi} \quad \text{and} \quad q(\vec{E}) + \frac{q}{c}(\vec{v} \times \vec{B}) = 0 \tag{8}$$

$$\vec{E} = -\frac{\omega B_0 r \sin\theta}{c}[\sin\theta\hat{r} + \cos\theta\hat{\theta}] \tag{9}$$

We can also verify that $\nabla \times \vec{E} = 0$. Hence, it is a conservative field upon which we can define Electric Potential $\vec{E} = -\nabla V_{in}(r, \theta)$. Integrating this along the line $l \equiv r\sin\theta$, we get

$$V_{in} = \frac{\omega B_0}{2c}r^2\sin^2\theta + \text{constant} = -\frac{\omega B_0 r^2}{3c}[P_2(\cos\theta) - 1] + V_0 \quad \text{where} \quad P_2(x) = (3x^2 - 1)/2 \tag{10}$$

Here, P_2 is the second Legendre Polynomial.
We can now write a similar equation for the potential outside the star, $\vec{E} = -\nabla V_{out}(r, \theta)$ s.t. V_{out} satisfies the Laplace equation and then equate $V_{in} = V_{out}$ at the surface $r = R$

$$\nabla^2 V_{out} = 0 => V_{out}(r, \theta) = \sum_{l=1}^{\infty} \frac{a_l}{r^{l+1}}P_l(\cos\theta) \quad \text{equating to } V_{in} \quad V_0 = -\frac{\omega B_0 R^2}{3} \tag{11}$$

Added to this the Magnetic outside of the pulsar is that of a dipole moment $|m| = B_0 R^3$. This gives us a final expression of the Electric Field and the Magnetic Field inside and outside the pulsar:

$$\boxed{\vec{E}(r,\theta) = \begin{cases} -\frac{\omega B_0 r}{c}[\sin^2\theta\hat{r} + \sin\theta\cos\theta\hat{\theta}] & r < R \\ \frac{\omega B_0 R^5}{cr^4}[\frac{(3\cos^2\theta - 1)}{2}\hat{r} + \sin\theta\cos\theta\hat{\theta}] & r > R \end{cases}} \quad \text{and} \quad \boxed{\vec{B}(r,\theta) = \begin{cases} B_0[\cos\theta\hat{r} - \sin\theta\hat{\theta}] & r < R \\ \frac{B_0 R^3}{2r^3}[2\cos\theta\hat{r} + \sin\theta\hat{\theta}] & r > R \end{cases}} \tag{12}$$

Given the Electric and Magnetic Field distribution, we can now find the charge density and the current distribution using $\rho = (\nabla \cdot \vec{E})/4\pi$ and $J = (c/4\pi)(\nabla \times \vec{B})$ to give the distributions inside and outside the pulsar.

$$\rho = \begin{cases} \frac{\omega B_0}{2\pi c} & r < R \\ 0 & r > R \end{cases} \quad \text{and} \quad \vec{J} = \begin{cases} 0 & r < R \\ 0 & r > R \end{cases} \tag{13}$$

As for the corresponding distributions on the surface,

$$\rho_{sur} = \frac{1}{4\pi}(-V_r^{in} + V_r^{out}) = \frac{\omega B_0 R}{12\pi}[2 - 5P_2(\cos\theta)] \quad \text{and} \quad \vec{J}_{sur} = \frac{c}{4\pi}[\vec{B}^{out} - \vec{B}^{in}] = \frac{cB_0}{8\pi}\sin\theta\hat{\theta} \tag{14}$$

Therefore, the aligned rotator will have (1) a charge density on the surface as well as inside and (2) a current density on its surface. It can be directly verified that the charge density averaged over the star is $\bar{\rho} \approx \omega B_0/c$ and the average current density is $\bar{J} \approx cB_0/R$. To satisfy the condition $\bar{\rho}c << \bar{J}$, we need $\omega R << c$. This condition also ensures that $v << c$ and $E \approx (\omega R/c)B_0 << B_0$, thereby allowing **Magneto-hydrodynamic Approximations (MHD)** to hold.

A typical neutron star has $\omega R/c \approx 10^{-2}$ which holds with our assumption but for larger distances $r \geq R_L = c/\omega$, where R_L is defined as the radius of the **light cylinder**, there are problems with this model.

Magnetosphere

For realistic values of neutron star parameters, its unlikely that the region outside the star is vacuum. In order to conclude this, we calculate the Voltage and the Electric Field near the star's surface.

$$V_{sur} \approx \frac{\omega B_0 R^2}{2c} \approx 3 \times 10^{16} V(\frac{\omega/2\pi}{30s^{-1}})(\frac{B_0}{10^{12}G})(\frac{R}{10km})^2 \quad \text{and} \quad E \approx \frac{\omega R}{c}B_0 \approx 2 \times 10^8 esu(\frac{B_0}{10^{12}G})(\frac{\omega/2\pi}{1s^{-1}}) \tag{15}$$

From this we can see that the electric force due to this field on a charged particle is stronger than the gravitational force (by a factor of $\sim 10^8$ for a proton), and hence the charged particle is pulled out of the neutron star surface to create a magnetosphere around the star.

The charged particles will be pulled out of the surface and will hence spiral around the magnetic field lines while also being accelerated by the presence of an Electric Field component along the Magnetic Field lines given by:

$$E_{||} = \frac{\vec{E} \cdot \vec{B}}{|\vec{B}|} = -\frac{\omega R}{c}B_0\cos^3\theta \tag{16}$$

As for the **field lines** $r(\theta)$ itself, they can be derived as follows:

$$\frac{B_r}{B_\theta} = \frac{dr}{rd\theta} = \frac{2\cos\theta}{\sin\theta} => \boxed{r = K\sin^2\theta} \tag{17}$$

Field lines starting from within an angular position $\theta < \theta_P$ near the polar cap will cross the light cylinder whereas those originating at $\theta > \theta_P$ will loop back to the pulsar. The critical field line, starting at $r = R, \theta = \theta_P$ will touch the light cylinder at $r = R_L, \theta = \pi/2$. And the radius of the polar cap, R_P is the radius of the circle around the pole inside which all field lines escape the light cylinder.

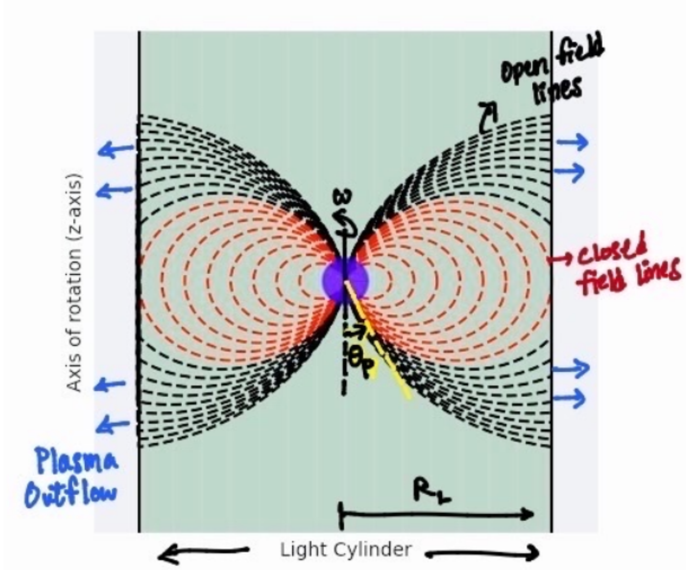
$$\sin^2\theta/r = K => \sin\theta_P = (\frac{R}{R_L})^{1/2} \quad \text{and} \quad R_P = R\sin\theta_P = R(\frac{R}{R_L})^{1/2} = 1.4 \times 10^4 cm(\frac{R}{10km})^{3/2}(\frac{P}{1s})^{-1/2} \tag{18}$$

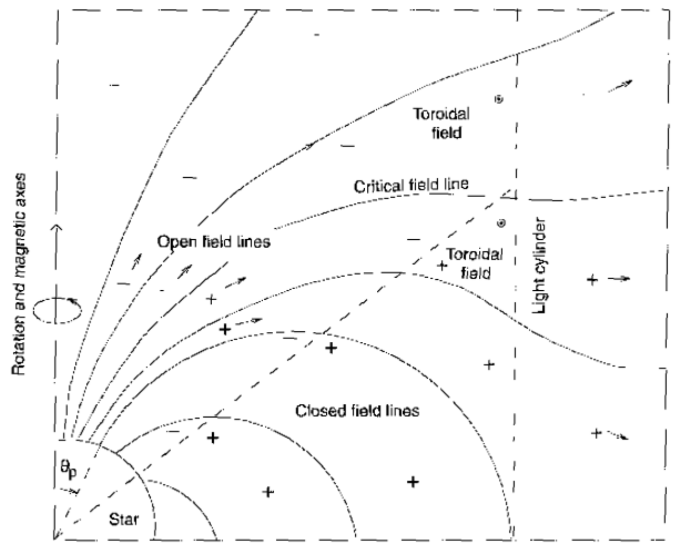
Charges that are pulled out from outside the polar cap region can redistribute themselves around the star, forming a corotating magnetosphere. Neglecting the inertia of the particles, we conclude that the charges will rearrange themselves so that no net electromagnetic force acts on them:

$$\vec{E} + (\frac{\vec{\omega} \times \vec{r}}{c}) \times \vec{B} = 0 \quad \text{and} \quad \nabla \cdot \vec{E} = 4\pi\rho \quad => \quad 4\pi c\rho = \nabla \cdot [\vec{B} \times (\vec{\omega} \times \vec{r})] = -2\vec{\omega} \cdot \vec{B} \tag{19}$$

Therefore, the sign of the charge density is opposite of the sign of B_z for an aligned rotator. Giving the following expression for the number density in a magnetosphere:

$$n_e = 7 \times 10^{-2} B_z(\frac{P}{1s})^{-1} cm^{-3} \tag{20}$$





Charged particles that are pulled out from within the polar cap region will move along the magnetic field lines towards the light cylinder at $r = R_L$, where the relativistic effects will lead to a breakdown of our MHD approximations and prevent the plasma from corotating with the star. The field lines are swept back near the light cylinder and stream off to infinity outside the light cylinder and are called open field lines. Hence, plasma flows away from the pulsar along the open field lines, which arise from a region near the polar cap.

The potential difference ΔV between the centre and the edge of the polar cap:

$$\Delta V = \frac{\omega B_0 r^2}{2c}\sin^2\theta\Big|_{(r=0)}^{(r=R,\theta=\theta_P)} \quad \text{and} \quad \theta_P = (\frac{R}{R_L})^{1/2} << 1 \quad => \quad \Delta V = \frac{\omega B_0 r^2}{2c}\frac{R}{R_L} = 6\times 10^{12}(B_{12})(\frac{P}{1s})^{-2}V \tag{21}$$

The potential difference along the magnetic field lines over a distance of the order of R, given by $\Delta V \approx (\vec{E}\cdot\vec{B}/|\vec{B}|)R$, will also be of the same order. Charged particles flowing along open field lines will be accelerated by this voltage corresponding to energies of $6\times10^{12}eV(B_{12}/P^2)$ making electrons highly relativistic with a gamma factor of $\gamma \approx 10^7(B_{12}/P^2)$.

This outflow of plasma creates an energy-loss mechanism producing a torque on the pulsar that can be estimated as follows. The magnetic field strength at the light cylinder is approximately $(B_0R^3/2R_L^3)$. Because of twisting of the field lines near the light cylinder, we would expect $B_r \approx B_\phi \approx (B_0R^3/2\sqrt{2}R_L^3)$. The moment of the $r\phi$ component of the magnetic stress tensor, $T_{r\phi} = (B_rB_\phi/4\pi)$, will lead to the torque:

$$I\frac{d\omega}{dt} = \frac{B_rB_\phi}{4\pi}(4\pi R_L^2)R_L \approx -\frac{1}{8c^3}(B_0R^3)^2\omega^3 \tag{22}$$

Poloidal and Toroidal Fields

We further analyse the axis-symmetric rotating magnetosphere around a pulsar if we assume that the plasma is force-free, i.e., $\vec{J}\times\vec{B} = 0$ is satisfied, which ensures that the bulk force on the plasma from the magnetic field vanishes. This gives $\vec{J} = \mu(r,\theta,\phi)\vec{B}$ (where $\mu(r,\theta,\phi)$ is a scalar function of spatial coordinates). The configuration is assumed to be stationary in time (although internally, current flows along the magnetic field). Hence, by conservation of the electric charge:

$$0 = \nabla\cdot\vec{J} = \nabla\cdot\mu\vec{B} => (\vec{B}\cdot\nabla)\mu \quad \text{since} \quad \nabla\cdot\vec{B} = 0 \tag{23}$$

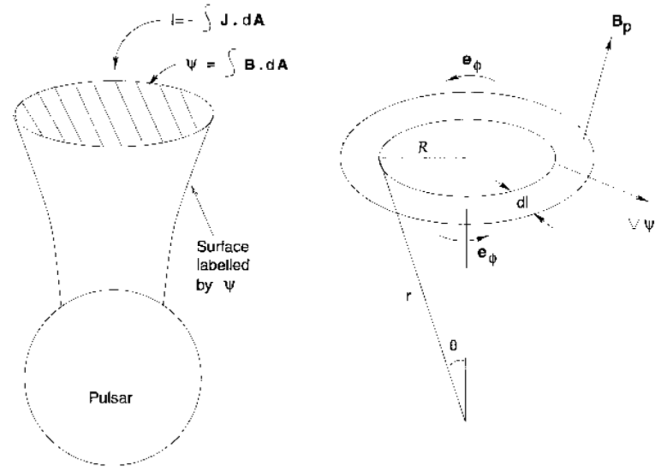
The above result suggests that $\mu = constant$ for a particular field line. Being axis-symmetric, each magnetic field actually sweeps out a non intersecting surface in the 3D. Now we define ψ to be the total magnetic flux contained within this surface:

$$\psi = \int \vec{B}\cdot d\vec{A} = \frac{1}{\mu}\int \vec{J}\cdot d\vec{A} = -\frac{I}{\mu} \tag{24}$$

where I is the current defined by the surface. For further analysis, we separate the magnetic field into a **poloidal part (which has components in the r,θ directions) and a toroidal part (which is in the ϕ direction)**. Formally, $\vec{B}^T = (\vec{B}\cdot\hat{\phi})\hat{\phi}$, $\vec{B}^P = \vec{B} - \vec{B}^T$

Because $\psi(r,\theta,\phi)$ is axis-symmetric and does not change along the poloidal magnetic field, it is clear that $\vec{B}^P$ is perpendicular to both $\nabla\psi$ and $\hat{\phi}$. Let's now consider the magnetic flux through an annulus of width dl and an area orthogonal to $\vec{B}^P$. If $\mathcal{R} \equiv r\sin\theta$ denotes the radial coordinate in the cylindrical $(\mathcal{R},\phi,z)$ coordinate system, then this flux is given by

$$2\pi\mathcal{R}\vec{B}^P dl = \psi(x+dl) - \psi(x) = d\vec{l}\cdot\nabla\psi \tag{25}$$



This can be further solved to give the Poloidal magnetic field strength as well as the Poloidal current density.

$$\boxed{\vec{B}^P = -\frac{1}{2\pi\mathcal{R}}(\hat{\phi}\times\nabla\psi)}, \quad \boxed{\vec{J}^P = +\frac{1}{2\pi\mathcal{R}}(\hat{\phi}\times\nabla I)} \tag{26}$$

The toroidal part of the magnetic field and current density can be determined from Ampere's law applied to a circle of constant $\mathcal{R}, z$, giving:

$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int \vec{J}\cdot d\vec{A} = -\frac{4\pi}{c}I \tag{27}$$

$$\boxed{\vec{B}^T = -\frac{2I}{c\mathcal{R}}\hat{\phi}}, \quad \boxed{\vec{J}^T = -\frac{2\mu I}{c\mathcal{R}}\hat{\phi} = -\frac{2I}{c\mathcal{R}}\frac{dI}{d\psi}\hat{\phi}} \tag{28}$$

The electric field $\vec{E}$ can now be determined using $\vec{E} = -(\vec{v}/c)\times\vec{B}$ since it is a force-free case. Clearly, the electric field is poloidal and is determined entirely by the poloidal part of the magnetic field:

$$\vec{E} = \vec{E}^P = -\frac{\vec{v}}{c}\times B = -(\frac{\omega\mathcal{R}}{c})(\hat{\phi}\times\vec{B}^P) \tag{29}$$

(6.52) We now have all the fields in terms of the functions I and ψ. To determine a relation between these two functions we use the Maxwell's equation $\vec{J} = (c/4\pi)\nabla\times\vec{B}$ as follows:

$$\vec{J}^T = \frac{c}{4\pi}(\nabla\times\vec{B}^P) = -\frac{c}{8\pi^2}\nabla\times(\frac{\hat{\phi}\times\nabla\psi}{\mathcal{R}}) \quad => \quad \nabla\times(\frac{\hat{\phi}\times\nabla\psi}{\mathcal{R}}) = -\frac{16\pi^2}{c^2}(\frac{I}{\mathcal{R}})(\frac{dI}{d\psi})\hat{\phi} \tag{30}$$

$$\mathcal{R}\hat{\phi}\nabla\cdot(\frac{\nabla\psi}{\mathcal{R}^2}) + (\frac{\nabla\psi}{\mathcal{R}^2}\cdot\nabla)\mathcal{R}\hat{\phi} - \mathcal{R}(\hat{\phi}\cdot\nabla)(\frac{\nabla\psi}{\mathcal{R}^2}) = -\frac{16\pi^2}{c^2}(\frac{I}{\mathcal{R}})(\frac{dI}{d\psi})\hat{\phi} \tag{31}$$

Using the facts that $\nabla(\mathcal{R}\hat{\phi}) = 0, \nabla\mathcal{R} = \hat{\mathcal{R}}), \partial\hat{\mathcal{R}}/\partial\phi = \hat{\phi}$ and that $\hat{\phi}, \nabla\psi$ are orthogonal, we get

$$(\frac{\nabla\psi}{\mathcal{R}^2}\cdot\nabla)\mathcal{R}\hat{\phi} = \left(\frac{\nabla\psi}{\mathcal{R}^2}\right)_{\mathcal{R}}\hat{\phi} \quad \text{and} \quad \mathcal{R}(\hat{\phi}\cdot\nabla)(\frac{\nabla\psi}{\mathcal{R}^2}) = \frac{\partial}{\partial\phi}(\frac{\nabla\psi}{\mathcal{R}^2}) = \left(\frac{\nabla\psi}{\mathcal{R}^2}\right)_{\mathcal{R}}\hat{\phi} \tag{32}$$

$$\boxed{\nabla\cdot\left(\frac{\nabla\psi}{\mathcal{R}^2}\right) + \frac{16\pi^2}{c^2}(\frac{I}{\mathcal{R}^2})(\frac{dI}{d\psi}) = 0} \tag{33}$$

In a self-consistent model for the pulsar, the function $I(\psi)$ is determined by the behaviour of the fields and plasma near and outside the light cylinder as a boundary condition. In this region, the MHD approximation breaks down and charged particles are not tied down to move along the field lines. Because the particles can move across the magnetic field, there is a possibility that currents can be closed outside the light cylinder, leading to a consistent picture.

In the absence of such a detailed physical model, we can still try different possible choices for $I(\psi)$ and explore the consequences; the above equation can serve as a good phenomenological tool for such model building. The relation $\vec{J} = \mu\vec{B}$ shows that the magnetic field lines act as current-carrying transmission lines. In such a case, the associated electromagnetic field will lead to a poloidal component of the Poynting flux, given by

$$(T^{0j}\hat{j})^P = \frac{c}{4\pi}(\vec{E}\times\vec{B})^P = \frac{\omega I}{2\pi c}\vec{B}^P \tag{34}$$

In arriving at the last step, we have used the explicit forms obtained for the electric and magnetic fields and have simplified the resulting expression by using elementary vector identities. There is also an accompanying flux of the z component of the angular momentum; the poloidal part of this angular momentum flux is given by

$$\vec{L}^P = -\frac{\mathcal{R}}{4\pi}B_\phi\vec{B}^P = \frac{I}{2\pi c}\vec{B}^P = \frac{1}{\omega}(T^{0j}\hat{j}) \tag{35}$$

The **loss of energy and angular momentum** (and the resulting spin-down of the pulsar) are of the **same order of magnitude** as in the simpler case discussed above. The aligned rotator, however, does not pulse. We can handle this by modifying it as an oblique rotator in which the magnetic field and the rotation axis are misaligned. Many of the qualitative features do not change when the misalignment is introduced.

There are, however, other **fundamental problems with the above model**: (1) To begin with, some of the open-field-line regions have a space charge of one sign out the current is carried by the motion of charges of opposite sign; (2) it is also not clear whether the current closes back properly in this model; (3) finally, we need to demonstrate that charged particles can indeed be pulled out from the surface of the pulsar in the manner assumed.

Pair Production

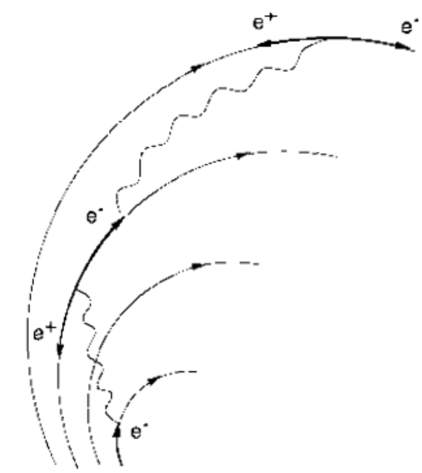
A stray electron in the magnetosphere, spiralling along a magnetic field line, will be accelerated by the electric field component $E_{||}$ parallel to the magnetic field. Near the surface, this field is capable of accelerating electrons to high energies producing highly relativistic electrons with a gamma factor of $\gamma \approx 10^7(B_{12}/P^2)$. The kinetic energy that is due to the spiralling motion, transverse to the local direction of the magnetic field, is quickly lost through **synchrotron emission**.

$$\tau \approx 1.5 \times 10^{-11} s (B_{12})^{-3/2} (\frac{\nu_c}{1GHZ})^{-1/2} \quad \text{where} \quad \nu_c = \gamma^3 \frac{c}{2\pi\mathcal{R}} \tag{36}$$

Here, $\mathcal{R}$ is the radius of curvature of the orbit. We can see that the time scale τ is quite short and hence after this synchrotron emission, the particles start moving along the magnetic field lines. The accelerating electric field along with the curved track of the magnetic field with radius of curvature of the order $\mathcal{R} = (rc/\omega)^{1/2}$ and relativistic energy of the particle $\mathcal{E} = \gamma mc^2$ will once again radiate a synchrotron spectrum.

$$I(\nu) = \frac{e^2}{2\pi c}\frac{c}{\mathcal{R}}(\frac{\nu}{\nu_c})^{1/3}\gamma \quad \text{for } \nu \le \nu_c \quad \text{with} \quad \nu_c = \frac{\gamma^3}{2\pi}(\frac{c\omega}{r})^{1/2} \approx 10^8(eV/h)(\frac{\gamma}{10^7})^3(\frac{\mathcal{R}}{10^8 cm})^{-1} \tag{37}$$

The primary charges around the polar region will radiate gamma rays tangentially, and these rays, if they are energetic enough, will give rise to electron-positron pair production. These secondary charges will be accelerated in opposite directions and will, in turn, give rise to more electron-positron pairs, thereby establishing a cascade.



A photon in vacuum cannot convert itself into an electron-positron pair without violating the conservation of energy and momentum. The photon, however, can convert itself into a **virtual pair of particles** of mass m and charge q for a short period of time $\Delta t \approx (\hbar/mc^2)$, after which the charged particles will re-combine. Whereas if the electric field is strong enough s.t. the work done by the electric field in moving the charged particle over a distance $(\hbar/mc)$ is comparable with the rest-mass energy of the particles, then the virtual pair becomes real and are separated.

$$E > E_c \quad \text{where} \quad qE_c(\frac{h}{mc}) = mc^2 => E_c = \frac{m^2c^3}{q\hbar} \tag{38}$$

The existence of a pure gamma ray photon in the presence of an electric field (from the photon [electromagnetic radiation]) along with a magnetic field which is the sum of the pulsar field and the field due to the photon. Therefore, the Lorentz invariant $E_{tot} - B_{tot}$ can be positive leading to pair production. Photon E = $\hbar\omega$ produces a pair of particles with $\gamma \approx (\hbar\omega/2mc^2)$. The mean free path of these photons $l = \kappa^{-1}$ can be written as:

$$l = (\frac{4.4}{e^2/\hbar c})(\frac{\hbar}{m_e c})(\frac{B_q}{B\sin\theta})\exp(4/3\chi) \quad \text{where} \quad \chi = \frac{h\omega}{m_e c^2}\frac{B\sin\theta}{B_q}, \quad B_q = \frac{m_e^2 c^2}{e\hbar} \approx 4.4 \times 10^{13} G \tag{39}$$

These expressions are valid for $\chi << 1$. The magnetosphere acts as a surface of an opaque solid with attenuation varying from zero to unity over a short distance. We can estimate the location r of this surface by ignoring the spatial variations of B and θ and setting $\kappa r = 1$. Given photon energy, viewing angle, and magnetic-field variation $B(r)$, this equation defines a 3D surface surrounding a pulsar. A photon of energy γ produced inside this surface would have degraded into e^+e^- pairs, leading to a cascade.

In this secondary pair-creation case, the electrons accelerated across the potential drop are able to create energetic photons by curvature radiation, which can produce pairs off the magnetic field.

$$\Delta V \approx \frac{B\omega^2 R^3}{2c^2} \quad \text{and} \quad e\Delta V = \gamma m_e c^2 => \gamma = (\frac{2\pi^2 e}{m_e c^4})\frac{BR^3}{P^2} \approx 1.3\times 10^7(\frac{B}{10^{12}G})(\frac{P}{1s})^{-2} \tag{40}$$

The typical energy of the photons radiated because of curvature radiation will be $\hbar\omega \approx \gamma^3(\hbar c/\mathcal{R})$, where $\mathcal{R}$ is the curvature of the magnetic-field line. To induce pair production, we need to satisfy the condition $(\hbar\omega/2m_e c^2) > 1$. This gives $\boxed{B_{12}P^{-2} \geq 0.025}$

This condition is valid at reasonably strong fields. At lower values of the field, the mean free path of photons produced by curvature radiation must be less than the typical curvature scale length R $l < \mathcal{R}$ which gives:

$$P^6 < \frac{9\pi^6}{2}\frac{m_e^2 R^8}{h^2 c^4}(\frac{B}{B_q})^4 \ln[\frac{R m_e e^2}{4.4\hbar^2}\frac{B}{B_q}] \quad => \quad \boxed{(\frac{B}{10^9 G})(\frac{P}{1s})^{-1.5} > 31} \tag{41}$$

This situation however is different if the primary charges are protons because of their greater mass, they will be accelerated less and their energy might never reach the radiation-reaction limit. The pair cascade will have to be established at a longer distance from the surface of the star where the magnetic field strength is weaker. This has the effect that the synchrotron self-absorption peak will move to lower frequencies. The corresponding upper limit on the period will decrease by a factor $(m_p/m_e)^{2/3}$ and will be $\sim 0.05s$.

Summary

In the above discussion, we have analysed the Electromagnetic field around the Pulsar using various theoretical models. Starting with a Uniformly Conducting Metal Sphere model which behaves like a magnetic dipole, emitting Energy in the form of Dipole radiation and hence slowing down the rotation of the Pulsar. We also used this model to estimate the number of Pulsars in the galaxy.

After deriving the fields, charge distribution and current distribution, we can see that the voltage at the surface of the Pulsar is too high to have a sharp discontinuous boundary with the vacuum as there will be particles from the surface pulled out and aligned to form a Magnetosphere. We then looked at how the presence of the Magnetosphere will influence the Electromagnetic fields and did so by analysing the Fields broken into Poloidal and Toroidal components. And then talked about the distortion in the field lines due to the presence of a light cylinder around the star.

We then looked at the case of e^+e^-pair productions from the high energy gamma radiations in the presence of strong fields and calculated the conditions required in order for pair formation to actually happen in the case of strong magnetic fields ($\sim 10^{12}G$) and comparatively weaker fields.($\sim 10^9 G$)

References

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