[DRAFT] All solutions to the hardest logic puzzle ever

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[NOTE: obviously I will first state the formulation of the puzzle.]

We will present here a systematic way of solving the hardest logic puzzle ever, as it was (presumably) intended in its original formulation by Boolos. That means that we take it that Random truly answers randomly, and we require that all questions be truly yes\no-questions. We take the latter to mean that in any given circumstance, for any God, there is always a correct answer to the question. The fact that we will solve the puzzle systematically, implies that in some sense we will describe all possible solutions to it. In some sense, because there will still be a large degree of freedom as to exactly how one formulates the questions. More precisely, what we shall do for each of the three questions, is to characterize to which statement all possible solutions have to be equivalent.

Name the three Gods X, Y and Z; their descriptions T, F and R; and the three questions $Q1$, $Q2$ and $Q3$. We will pose $Q1$ to X. The six possibilities will be noted as:

$$a = (R, F, T) \tag{1}$$
$$b = (R, T, F) \tag{2}$$
$$c = (T, F, R) \tag{3}$$
$$d = (F, T, R) \tag{4}$$
$$e = (T, R, F) \tag{5}$$
$$f = (F, R, T) \tag{6}$$

We define S as the set of all remaining possible solutions, so initially $S = \{a, b, c, d, e, f\}$.

The best any binary question can do is reduce the possibilities by half. Since the puzzle requires that there is only one possibility left after three questions, we need to have at most 4 remaining possibilities after $Q1$. On the other hand, if $X = R$, then regardless of $Q1$, the answer gives us no information whatsoever. Therefore it is impossible to exclude this possibility after $Q1$, leaving us at best with a $4-4$ split of the six possibilities between an answer of *Da* and *Ja*. Combining these two observations we can conclude that after $Q1$ we have 4 possibilities left, amongst which a and b.

We need the answer to $Q2$ to split these 4 options into 2 and 2. This implies that we should pose $Q2$ to a God who is known not to be Random, for otherwise there will be at least one option which figures in both sides of the split after $Q2$. Since we cannot rule out that $X = R$ after $Q1$, $Q1$ should split the possibilities into $Y \neq R$ on the one side and $Z \neq R$ on the other. Therefore $Q1$ should be such that

$$Da! \Rightarrow \{a,b,c,d\} \tag{7}$$
$$Ja! \Rightarrow \{a,b,e,f\} \tag{8}$$

Where we take $Da!$ (resp. $Ja!$) to mean that Da (resp. Ja) was answered, and where obviously the roles of Da and Ja may be reversed.

Whatever $Q1$, if $X = R$ then the above will hold. So for the moment we may simply assume that $X \neq R$, giving $S = \{c,d,e,f\}$ and

$$Da! \Rightarrow \{c,d\} \tag{9}$$
$$Ja! \Rightarrow \{e,f\} \tag{10}$$

Any question can be seen as a proposition followed by a question mark, so we can write $Q1$ as $A?$. There are four mutually exclusive and exhaustive possibilities regarding the answer given ($Da!$ or $Ja!$) and the fact whether $Da = Yes$ or not, and what we require is that for each of these possibilities A leads us to the set of solutions that go along with it. A should therefore simply be equivalent to a description of the set of solutions for each of those possibilities.

$$A \iff ((Da! \wedge Da = Yes \Rightarrow S_1) \wedge (Da! \wedge Ja = Yes \Rightarrow S_2) \wedge (Ja! \wedge Ja = Yes \Rightarrow S_3) \wedge (Ja! \wedge Da = Yes \Rightarrow S_4))$$

The truth of A implies for each of the four possibilities to which set the solution belongs, and the falsehood of A implies for each possibility that the solution belongs to the complement of this set (with respect to S). The challenge lies in figuring out which choices of the sets S_i are appropriate.

As a first step, we can figure out the implications of there not being paradoxical questions. Disallowing such questions means that regardless of whether $Da = Yes$ or not, if $X = T$ he should be able to answer consistenly, and likewise for $X = F$.

Assume that $Da = Yes$. This means that

$$A \iff (Da! \Rightarrow S_1 \wedge Ja! \Rightarrow S_4)$$

Now if $X = T$, and he replies $Da!$, he is stating that the solution is an element of S_1. Therefore, if he wishes to state that the solution is not an element of S_1 (and thus an element of $S \setminus S_1$), he has no option but to reply $Ja!$. But that implies stating $\neg A$, and thus that the solution is an element of $S \setminus S_4$. From this it follows that we are forced to choose sets S_1 and S_4 such that $S_1 = S_4$.[1] Starting now from the assumption that $Ja = Yes$, a similar argument also forces us to choose sets S_2 and S_3 such that $S_2 = S_3$.

As a second step, we start by noting that is impossible to find out whether $Da = Yes$ or $Ja = Yes$. Indeed, this would require discriminating between 12 possibilities, which is cleary impossible given that using three binary questions can result in at most 8 different possible outcomes. Since we do not know the meaning of the word Da, we have to be able to interpret a response of $Da!$ as indicating a specific set of solutions independently of its meaning. Assume that the response to $Q1$ was $Da!$. We are then left with

$$A \iff (Da = Yes \Rightarrow S_1 \wedge Ja = Yes \Rightarrow S_2)$$

[1] The same conlusion can be reached assuming $X = F$.

Again we start by assuming that $X = T$. If it were the case that $Da = Yes$, then A is true and the solution is in S_1. If on the other hand it were the case that $Ja = Yes$, then A is false and thus the solution is in $S \setminus S_2$. Since we just established that both cases need to indicate the same set of solutions, we are forced to choose sets S_1 and S_2 such that $S_1 = S \setminus S_2$.[2]

Taking into account both steps, we end up with

$$A \Longleftrightarrow ((Da! \wedge Da = Yes \Rightarrow S_1) \wedge (Da! \wedge Ja = Yes \Rightarrow S \setminus S_1) \wedge (Ja! \wedge Ja = Yes \Rightarrow S \setminus S_1) \wedge (Ja! \wedge Da = Yes \Rightarrow S_1))$$

which can be expressed more simply as

$$A \Longleftrightarrow (Da = Yes \Leftrightarrow S_1)$$

Recall that we want $Da! \Leftrightarrow \{c, d\}$ to hold. We need to find out which choices of S_1 satisfy this criterion.

Assume c is the correct solution. We thus want X to answer $Da!$. Since c implies $X = T$, he will speak truly. In a world where $Da = Yes$, X will thus be affirming A. Taking together the fact that the equivalence between $Da = Yes$ and S_1 is true, and that $Da = Yes$ is indeed the case, it must be that $c \in S_1$. Likewise, in a world where $Ja = Yes$, X will be affirming $\neg A$. In this case there is an equivalence between $Ja = Yes$ and S_1, implying again that $c \in S_1$.

Now assume d is the correct solution. X should again answer $Da!$. Solution d implies $X = F$. If $Da = Yes$, we can infer that $\neg A$ holds and therefore $Da = Yes \Leftrightarrow S \setminus S_1$. Together with the fact that indeed $Da = Yes$, we get that $d \in S \setminus S_1$.

One can go over the other two solutions in exactly the same way, ultimately arriving at the unique choice for S_1 being $S_1 = \{c, f\}$. Remember however that we arrived at this conclusion under the assumption that $X \neq R$. Therefore in general all possible questions $Q1$ will be of the form

$$A \Longleftrightarrow ((Da = Yes \Rightarrow S_1) \wedge (Ja = Yes \Rightarrow S_2))$$

where $\{c, f\} \subseteq S_1$ and $\{d, e\} \subseteq S_2$. Each of the solutions a and b may be placed in either S_1, S_2, in both, or may be left out altogether. Clearly in this setting the simplest choice is $S_1 = \{c, f\}$ and $S_2 = \{a, b, d, e\}$, so that we end up with

$$A \Longleftrightarrow (Da = Yes \Leftrightarrow \{c, f\})$$

Boolos makes the same choice in his article, be it that the roles of Da and Ja are reversed. Rabern & Rabern choose $S_1 = \{a, b, d, e\}$ and $S_2 = \{a, b, c, f\}$. [NOTE: I do intend to explain this in a longer version.]

On to the second question. We have found out after $Q1$ that either $Y \neq R$, or that $Z \neq R$, but for the sake of simplicity let's rename the Gods so that we have found out $X \neq R$, leaving us with the same set of possible solutions as we encountered before: $S = \{c, d, e, f\}$. We need to split up these possiblities into two sets of two.

[2]Like with the previous argument, the same conlusion can be reached assuming $X = F$.

One way of doing so is simply by making the same choice as we did for the first question, namely

$$Da! \Rightarrow \{c,d\} \tag{11}$$
$$Ja! \Rightarrow \{e,f\} \tag{12}$$

We can hence repeat the previous argumentation, and writing $Q2$ as B? we arrive back at

$$B \Longleftrightarrow (Da = Yes \Leftrightarrow \{c,f\})$$

The difference here is that we no longer need to worry about solutions a and b, and we have a unique solution.

A second way to split up the possibities would be

$$Da! \Rightarrow \{c,f\} \tag{13}$$
$$Ja! \Rightarrow \{e,d\} \tag{14}$$

[NOTE: I will show that this leads to a contradiction, and thus this option is impossible.]

The third and last way of splitting up the possibilities would be

$$Da! \Rightarrow \{c,e\} \tag{15}$$
$$Ja! \Rightarrow \{d,f\} \tag{16}$$

[NOTE: I will show that $S_1 = S$, and thus $B \Longleftrightarrow Da = Yes$]

Finally we arrive at the third question, $Q3$, which we will write as C?. Depending on which choice was made for $Q2$, we can end up here with either $S = \{c,d\}$ (which is completely symmetric to $S = \{e,f\}$) or with $S = \{c,e\}$ (which is completely symmetric to $S = \{d,f\}$).

In the first case we want that

$$Da! \Rightarrow \{c\} \tag{17}$$
$$Ja! \Rightarrow \{d\} \tag{18}$$

which is a simpler version of the third possibility we considered regarding $Q2$. In a similar fashion the result is that $S_1 = S$, and thus

$$C \Longleftrightarrow Da = Yes$$

Likewise, in the second case we want that

$$Da! \Rightarrow \{c\} \tag{19}$$
$$Ja! \Rightarrow \{e\} \tag{20}$$

and we end up with a simpler version of the first possibility we considered regarding $Q2$, resulting in

$$C \Longleftrightarrow (Da = Yes \Leftrightarrow \{c\})$$