Irreducible triangulations of the 4-gon and 4-connectedness

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1 Types of triangulations and their properties

All graphs are presumed simple and have a fixed planar embedding

The *degree* of a face is the number of vertices it is incident to. By a *cycle* we will mean a simple cycle. That is a cycle without repetition of edges or vertices. By Jordan's curve theorhem a cycle splits the plane into two parts, one bounded and one unbounded. We will call the bounded part the *interior* of this cycle and the unbounded part the *exterior* of this cycle.

We will call a cycle *seperating* if there are vertices in both it's interior and exterior. We will use k-*cycle* to denote a cycle of length k. Moreover a *triangle* is simply a cycle of length 3 (i.e. a 3-cycle). graphname

1.1 Plane triangulations

Definition (Plane triangulation)**.** A graph with only faces of degree 3.

Definition (Maximal planar graph)**.** A graph such that adding any one edge leaves it non-planar.

Theorem 1. *Any graph G is a plane triangulation if and only if it is maximal planar*

Proof. We will prove the equivalence of the negations.

Suppose that G is not maximally planar. Then there is a face F to which we can add an edge, however this face must then have degree larger then 4. Hence G is also not a plane triangulation.

Suppose that G is not a plane triangulation. Then there must be a face F of degree larger then 3. This face will thus admit an extra edge without violating planarity and hence G is not maximally planar. □

1.1.1 Connectedness

Theorem 2. *Any plane triangulation T is* 3*-connected.*

Proof. Suppose that T is not 3-connected. Then there must be a 2-cutset S, given by the vertices x and y. Removing this cutset splits the graph into at least two connected components C_i and all components are incident to all cutvertices otherwise we would have found a 1-cutset.

Since S is a cutset, there can't be any edges incident to both C_1 and C_2. But then the edge xy should be separating the 2 components on both sides. This is impossible since we can only draw this edge once. □

Definition (Irreducible triangulation)**.** We call a triangulation irreducible if it has no separating triangles

Theorem 3. *Any irreducible plane triangulation T is 4-connected.*

Proof. Note that any plane triangulation is 3-connected by Theorhem 2.

Suppose that T is not 4-connected. Then there must be some 3-cutset (since it is 3-connected) let us denote the vertices of this cutset by x, y and z. Removing this cutset splits the graph into at least two connected components C_i and all components are incident to all cutvertices otherwise we would have found a 2- or 1-cutset.

However, now xy must be an edge in the triangulation T otherwise the graph is not maximal planar (There can't be an edge incident to both C_1 and C_2 because that would negate x, y, z being a cutset.). In the same way yz and xz are edges of T. But then xyz is a separating triangle. This is an contradiction and thus T is 4-connected □

1.2 Triangulations of the k-gon

Definition (Triangulation of the k-gon)**.** We call a graph a triangulation of the k-gon if the outer face has degree k and all interior faces have degree 3.

Vertices bordering the outer face are *outer vertices* while all other vertices are *interior vertices.* Furthermore the cycle formed by all vertices outer vertices is the *outer cycle.*

Sometimes such triangulations of the k-gon are called *(plane) triangulated graphs.*

Definition (Irreducible triangulation of the k-gon)**.** We call a triangulation of the k-gon irreducible if it has no separating triangles.

Note that triangulation of the n-gon $n \geq 4$ is not maximally planar and thus not plane triangulation.

The *completion* of a triangulation of the k-gon $G = (V, E)$. Is the graph $G' = (V', E')$ with vertex set $V' = V \cup \{s\}$ and edge set $E' = E \cup \{sv | v \text{ is a outer vertex}\}$

The completion is plane triangulation. Since the interior of the outer cycle of G always consisted of faces of degree 3. The exterior of the outer cycle consisted of one face of degree k (the outer face) but the completion has turned this into k faces of degree 3.

Theorem 4. *A triangulation of the k-gon G is 2-connected.*

Proof. Suppose that G has a cutvertex v. Then the set $\{s, v\}$ is a 2-cutset of the completion G' of G. This however is in contradiction to Theorem 2 stating that G' is 3-connected. Hence G has no cutvertex and is thus 2-connected. □

Theorem 5. *A irreducible and chordless triangulation of the k-gon is 3-connected.*

Proof. **Writers note:** Will be provided if this statement turns out to be interesting. Will go via the fact that the completion is a irreducible triangulation. Chordless outer cycle is important, because a chord will form a separating triangle in G'. □

Theorem 6. *Any irreducible triangulation T of the 4-gon with $n \geq 5$ is 3-connected.*

Writers note: This proof could be a corrolary of the above theorhem 5. A chord gives a separating triangle if $n \geq 5$.

Proof. Let us name the four outer vertices a, b, c, d in clockwise order. Let us first note that the diagonals ac and bd can't be an edge since this would create a separating triangle containing the 5th vertex. Let I denote the component of all interior vertices, since every face in the interior is of degree 3 each outer vertices is incident to at least one edge that is also incident to I.

One can now easily check that there is no 2cut set with only exterior vertices. However, a cutset with 1 or 2 interior vertices leads to at least one cycle of degree greater then 3

Hence no 2-cutset of T can't exist and T is 3-connected. □

Theorem 7. *For every interior vertex v of a triangulation of the k-gon G is connected by ate least 3 vertex disjoint paths to different outer vertices.*

Proof. By Theorhem 2 the completion G' of G is 3-connected. Hence there are 3 vertex-disjoint paths from v to s. Since v is on the interior and s is on the exterior of the outer cycle $\mathcal{C}$ all these 3 paths cross the outer cycle at least once. These paths cross $\mathcal{C}$ for the first time in different vertices since they are vertex-disjoint. If we shorten the paths to their first crossing with $\mathcal{C}$ we obtain the 3 paths in the theorem. □

Writers note: We can sharpen this to 4 if we have a irreducible an chordless triangulation of the k-gon

Theorem 8. *Every interior vertex of a triangulation of the n-gon has degree at least 3.*

Proof. Suppose a interior vertex v has degree 1 then clearly the face surrounding v can't have degree 3. Now suppose that an interior vertex v has degree 2. We then let u and w denote it's neighbours and F and F' the face incident to v. See also Figure 1. Then since F and F' are both interior faces they need to be off degree 3 this implies that uw is an edge for both faces. This is impossible and hence every interior vertex has at least degree 3 □

Writers note: If we forbid irreducible triangulations every interior vertex is of degree 4 since the neighbourhood of any internal vertex v looks like a set of triangles.

Writers note: This theorem is currently (17-09) unused.

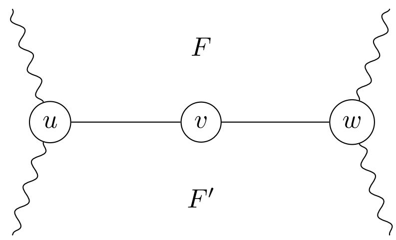

Figure 1: The notation as described in the proof

2 Rectangular duals

In this section we will explain what we mean with the rectangualr dual of a graph. We will prove some simple properties of graphs and their duals.

We define a *rectangular layout* (or simply *layout*) $\mathcal{L}$ to be a partition of a rectangle into finitely many interiorly disjoint rectangles.

We will assume that no four rectangles meet in one point.

We will then look at the *dual graph of a layout* $\mathcal{L}$ and denote this graph by $\mathcal{G}(\mathcal{L})$. That is, we represent each rectangle by a vertex and we connect two vertices by an edge exactly when their rectangles are adjacent. Note that this graph is not the same as the *graph dual* of $\mathcal{L}$ when we view it as a graph (namely we don't represent the outer face of $\mathcal{L}$ by a vertex).

So $\mathcal{G}(\mathcal{L})$ is the dual graph of a layout $\mathcal{L}$. In the reverse direction we say a layout $\mathcal{L}$ is a *rectangular dual* of a graph $\mathcal{G}$ if we have that $\mathcal{G} = \mathcal{G}(\mathcal{L})$.

A plane triangulated graph $\mathcal{G}$ does not necessarily have a rectangular dual nor is this dual necessarily unique.

2.1 Extended graphs

A *extended graph* $\bar{G}$ of G is a augmentation of G with 4 vertices (which we will call it's *poles*). Such that

1. every interior face has degree 3 and the exterior face has degree 4.
2. all poles are incident to the outer face
3. $\bar{\mathcal{G}}$ has no separating triangles (i.e separating 3-cycles).

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We sometimes call an extended graph $\bar{G}$ of G an *extension* of G.

Such a extended graph does not necessarily exist and is not necessarily unique. However we have the following result due to ....

Theorem 9 (Existence of a rectangular dual)**.** *A plane triangulated graph $\mathcal{G}$ has a rectangular dual if and only if it has an extension $\bar{\mathcal{G}}$*

Proof. Kozminski & kinnen and ungar, See siAM paper □

We call any (plane triangulated) graph G that has an extension a *proper* graph.

A proper graph G can have more then one extensions. Each such extension fixes which of the rectangles are in the corners of the rectangular dual $\mathcal{L}$. Hence sometimes such an extension is called a *corner assignment*.

2.2 Regular edge labeling

A regular edge labelling of $\bar{G}$ corresponds to a rectangular dual $\mathcal{L}$ of G with some *corner assignment* fixed.

Or regular edge labelling of a graph.

An *interior edge* of a cycle is an edge on the interior of the cycle (when the cycle is viewed as Jordan curve).

2.2.1 Being onesided in terms of REL

2.2.2 Being psudeo-onesided in terms of REL

3 Fixing a extension

In our explorations to find a lower bound on what kind of *psuedo one-sidedness* is possible we will find it very useful to fix one particular extension $\bar{G}$ of G. Unfortunately if there is no rectangular dual that's (k, l)-sided using the *corner assignment* provided by some extension $\bar{G}$. This does not imply that G is not (k, l)-sided. There might be another extension of G such that under the corner assignment corresponding to this extension G has a (k, l)-sided rectangular dual.

Fortunately for us however we can view $\bar{G} = H$ as a graph in it's own right, then G is the interior of a separating 4-cycle of H and we will show this implies that G (as induced sugraph) has to be coloured according to the extension $\bar{G}$.

Remark 10. *Let $\mathcal{C}$ be a separating 4-cyle of G with interior I. Then in any rectangular dual of G the region enclosed by the rectangles dual to the vertices in $\mathcal{C}$ is a rectangle.*

Remark 11. *Two disjoint rectangles are at most adjacent on one side.*

Lemma 12. *Let $\mathcal{C} = \{a, b, c, d\}$ be a separating 4-cyle of $\bar{G}$ with interior I. Then all interior edges incident to a, b, c and d respectively are red, blue, red and blue or blue, red, blue and red.*

Proof. By Remark 10 the union of the rectangles in the interior of $\mathcal{C}$ will be some rectangle in any rectangular dual. We will denote this rectangle by I. Since two disjoint rectangles can only be adjacent to each other at one side all interior edges incident to any vertex of $\mathcal{C}$ are of the same color.

Furthermore a, b, c, d are all adjacent to a different side of I since I has four sides that need to be covered and it is only adjacent to four rectangles. If we then apply the rules of a regular edge labelling we see that if the interior edges of a are one color, those incident to b and d should have the second color. Then of course the interior edges incident to c are again coloured with the first color. □

This lemma implies that any *alternating 4-cycle* is either *left-alternating* or *right-alternating* in the terminology of Fusy

Furthermore the above Lemma is also very useful in that it allows us to fix a extension $\bar{G}$ of G by building a *scaffold*. Suppose we want to investigate some extension $\bar{G}$ of G with poles N, E, S and W then we can consider the graph $\bar{G} = H$ as a graph in it's own right. H is a proper graph since it has no irreducible triangles in it's interior (because $\bar{G}$ had none) and it admits a valid extension $\bar{H}$ by connecting the new poles NE, SE, SW and NW to N, E, SE, NW, S, E, NE, SW, S, W, SE, NW and N, W, NE, SW respectively. See Figure 2 for this extension.

FiXme: Maybe use a table

Theorem 13. *We can fix an extension, if we want.*

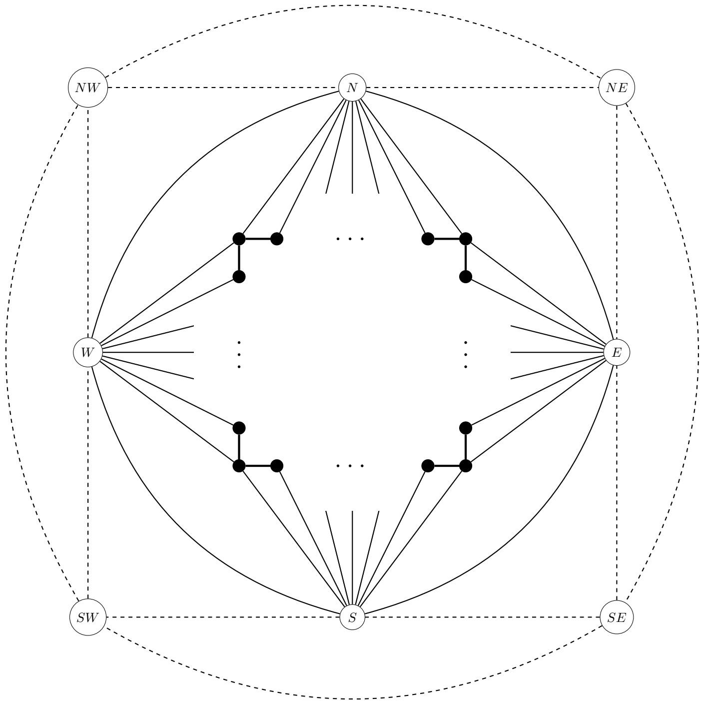

Figure 2: The construction of a scaffold. G is displayed in thick lines and with closed vertices. An arbitrary extension $\bar{G} = H$ is then drawn with thin lines and open vertices. An extension of H is then drawn with dashed edges and open vertices.

The graph H can have more then one extension but they all contain the separating 4-cycle $\mathcal{C} = NESW$ thus by Lemma 12 we see that, without loss of generality, the interior edges of $\mathcal{C}$ incident to N and S are coloured red and those incident to E or W are coloured blue. This is exactly as if we forced the extension $\bar{G}$

3.1 An aplication: There are graphs that are $(2, \infty)$-sided

We will show this by providing an example graph G with a fixed extension $\bar{G}$ which we can do according to Theorem 13. Consider the graph in Figure 3. Note that most of the interior vertices are of degree 4 and thus the largest part of any regular edge labelling is forced. Those edges that are forced to have a certain color are already coloured in Figure 3.

The only edge for which we have freedom to choose a color is the diagonal edge of G. Howeever, if we color this edge blue we get a red $(2, \infty)$ cycle and

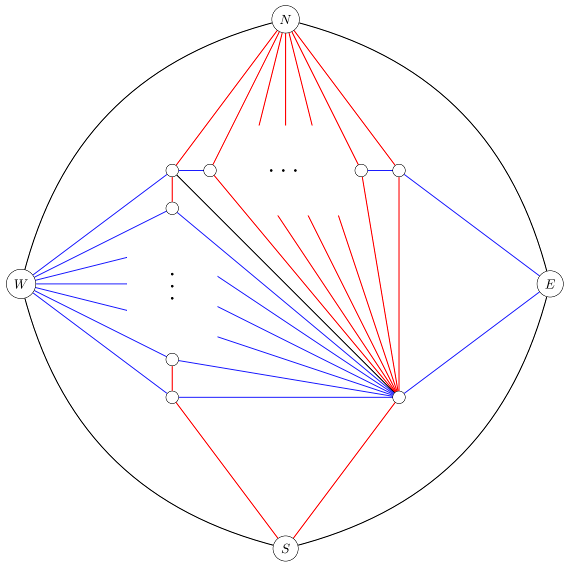

Figure 3: The fixed extension $\bar{G}$

if we color this edge red we get a blue $(2, \infty)$ cycle. In both cases we will thus obtain a $(2, \infty)$-sided segment in our dual.