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ON FINDING THE RECTANGULAR DUALS OF PLANAR TRIANGULAR GRAPHS*

XIN HE†

Abstract. This paper presents a new linear-time algorithm for finding rectangular duals of planar triangular graphs. The algorithm is conceptually simpler than the previously known algorithm. The coordinates of the rectangular dual constructed by the new algorithm are integers and carry clear combinatorial meaning.

Key words. algorithm, planar graph, rectangular dual

AMS subject classifications. 05C05, 05C38, 68Q25, 68R10

1. Introduction. Let R be a rectangle. A *rectangular subdivision system* of R is a partition of R into a set $\Phi = \{R_1, R_2, \ldots, R_n\}$ of nonintersecting smaller rectangles such that no four rectangles in Φ meet at the same point. A *rectangular dual* of a graph $G = (V, E)$ is a rectangular subdivision system Φ and a one-to-one correspondence $f : V \rightarrow \Phi$ such that two vertices u and v are adjacent in G if and only if their corresponding rectangles $f(u)$ and $f(v)$ share a common boundary. Parts (1) and (2) of Fig. 1 show a graph G and its rectangular dual. If G has a rectangular dual, clearly G must be a planar graph.

The rectangular dual of a graph G finds applications in the floor planning of electronic chips and in architectural design [5], [9]. Each vertex of G represents a circuit module, and the edges represent module adjacencies. A rectangular dual provides a placement of the circuit modules that preserves the required adjacencies.

The problem of finding rectangular duals has been studied in [2], [3], [6], [8], [12]. A linear-time algorithm for this problem was given in [3]. This algorithm is rather complicated, and its correctness proof is incomplete. The algorithm requires real arithmetic for the coordinates of the rectangular dual. We present a new linear-time algorithm for solving this problem. The coordinates of the rectangular dual R constructed by our algorithm are integers and carry clear combinatorial meaning.

The rectangular dual is related to the *tessellation representation* of planar graphs discussed in [14], [15]. The tessellation representation of a plane graph G is a mapping that maps the vertices, edges, and faces of G to the rectangles of the plane such that the incidence relations of G correspond to the geometric adjacencies between the rectangles. A nice linear-time algorithm for constructing the tessellation representation was developed in [15]. Our rectangular-dual algorithm and the algorithm in [15] share certain similarities: both algorithms use two acyclic directed graphs derived from the input graph G. However, the rectangular dual differs from the tessellation representation in at least two aspects. Firstly, in the tessellation representation, the elements (vertices, edges, and faces) of distinct graph-theoretical properties are represented by the same geometric objects (rectangles). The rectangular dual is a more natural representation of planar graphs. Secondly, the rectangular dual is more general than the tessellation representation. Consider a plane graph G. Let G_1 be the plane graph obtained from G as follows: For each face f of G place a new vertex v_f in f and connect v_f to the vertices on the boundary of f. Then the rectangular dual of G_1 is a representation similar to the tessellation representation of G, i.e., the vertices and the faces of G are represented by rectangles and the edges of G are represented by degenerate rectangles (i.e., line segments). On the other hand, it does not seem possible to modify the tessellation representation algorithm so that it constructs the rectangular dual for planar graphs.

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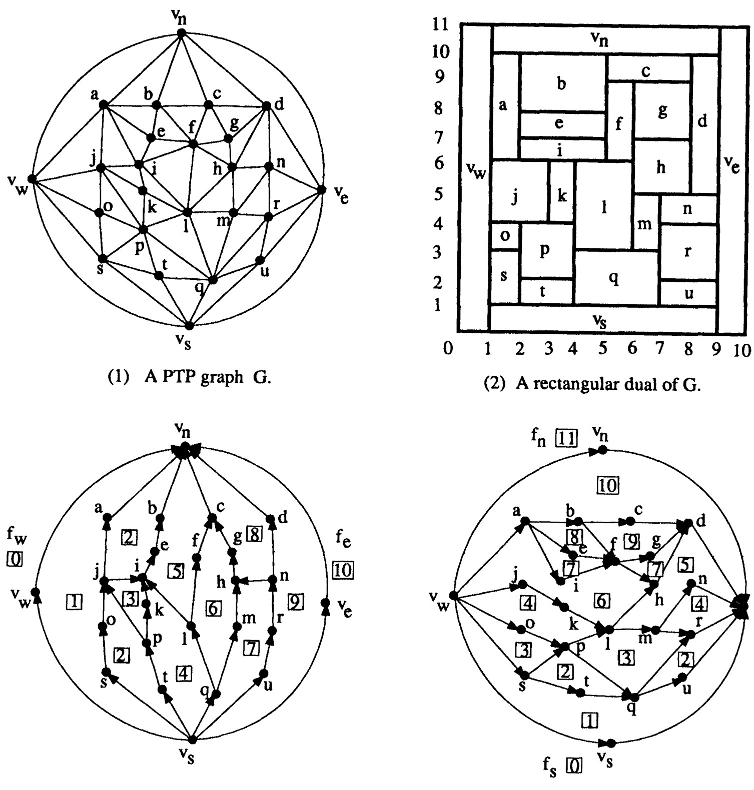
†Department of Computer Science, State University of New York, Buffalo, New York 14260. This research was supported in part by National Science Foundation grant CCR-9011214.



(1) A PTP graph G.

(2) A rectangular dual of G.

(3) S-N net G_1

(4) W-E net G_2

FIG. 1. *A* PTP *graph G and a rectangular dual of G.*

The present paper is organized as follows. Section 2 introduces some definitions and lemmas needed by our algorithm. Section 3 presents the algorithm. Section 4 proves its correctness. Section 5 concludes the paper.

2. Regular edge labeling of planar triangular graphs. Let $G = (V, E)$ be a planar graph. Consider a fixed plane embedding of G. The embedding divides the plane into a number of regions. The unbounded region is called the *exterior face*. Other regions are called *interior faces*. The vertices and the edges on the exterior face are called *exterior vertices* and *exterior edges*, respectively. A cycle C of G divides the plane into its interior region and exterior region. If C contains at least one vertex in its interior region, C is called a *separating cycle* of G. For each vertex v, $N(v)$ denotes the set of neighbors of v and $\mathrm{Star}(v)$ denotes the set of edges incident to v. Whenever these notations are used, it is understood that the members in the set are listed in counterclockwise order around v in the embedding.

Consider a planar graph $H = (V, E)$. Let v_0, v_1, v_2, v_3 be four vertices on the exterior face of H in counterclockwise order. Let P_i ($i = 0, 1, 2, 3$) be the four paths on the exterior face of H consisting of the vertices between v_i and v_{i+1} (where the addition is mod 4). We seek a rectangular dual R_H of H such that the four vertices v_0, v_1, v_2, v_3 correspond to the four corner rectangles of R_H and the vertices on P_0 (P_1, P_2, P_3, respectively) correspond to the rectangles located along the north (west, south, east, respectively) boundary of R_H. Necessary and sufficient conditions for testing whether H has a rectangular dual were discussed in [2], [3], [6]. These conditions, however, can be easily reduced to the following simpler form.

In order to simplify the problem, we modify H as follows: Add four new vertices v_N, v_W, v_S, v_E, and connect v_N (v_W, v_S, v_E, respectively) to every vertex on P_0 (P_1, P_2, P_3, respectively). Then add four new edges (v_N, v_W), (v_W, v_S), (v_S, v_E), (v_E, v_N). Let G be the resulting graph. It is easy to see that H has a rectangular dual R_H, with v_0, v_1, v_2, v_3 corresponding to the four corner rectangles of R_H if and only if G has a rectangular dual R with exactly four rectangles on the boundary of R. Without loss of generality, we will discuss only planar graphs with exactly four vertices on their exterior faces.

If G has a rectangular dual R, then every face of G, except the exterior face, must be a triangle (since no four rectangles of R meet at the same point). Moreover, since at least four rectangles are needed to fully enclose some nonempty area on the plane, any separating cycle of G must have length at least 4. The following theorem states that these two conditions are also sufficient for G to have a rectangular dual.

THEOREM 2.1 [6]. *A planar graph $G = (V, E)$ has a rectangular dual R with four rectangles on the boundary of R if and only if the following two conditions hold*: (1) *Every interior face of G is a triangle, and the exterior face of G is a quadrangle;* (2) *G has no separating triangles.*

A different form of Theorem 2.1 was given in [2], [3]. A graph satisfying the two conditions of Theorem 2.1 is called a *proper triangular planar* (PTP) graph. From now on we discuss only such graphs.

DEFINITION 1. A *regular edge labeling* (REL) of a PTP graph $G = (V, E)$ is a partition of the interior edges of G into two subsets $\{T_1, T_2\}$ of directed edges such that the following hold:

(1) For each interior vertex v the edges in Star(v) appear in counterclockwise order around v as follows: a set of edges in T_1 leaving v; a set of edges in T_2 entering v; a set of edges in T_1 entering v; a set of edges in T_2 leaving v.

(2) All interior edges incident to v_N are in T_1 and entering v_N. All interior edges incident to v_W are in T_2 and leaving v_W. All interior edges incident to v_S are in T_1 and leaving v_S. All interior edges incident to v_E are in T_2 and entering v_E.

From Theorem 2.1 we can easily prove the following.

THEOREM 2.2. *Every* PTP *graph $G = (V, E)$ has a* REL.

Proof. By Theorem 2.1, G has a rectangular dual R. For each $v \in V$ let $R(v)$ denote the rectangle in R corresponding to v. For each interior vertex v label each edge $(v, u) \in$ Star(v) as follows: If $R(u)$ is above $R(v)$, e is in T_1 and directed leaving v. If $R(u)$ is below $R(v)$, e is in T_1 and directed entering v. If $R(u)$ is to the left of $R(v)$, e is in T_2 and directed entering v. If $R(u)$ is to the right of $R(v)$, e is in T_2 and directed leaving v. This labeling satisfies the two conditions of Definition 1. □

Although Theorem 2.2 is proved from Theorem 2.1, our algorithm goes another way around: We find a REL of G first and construct a rectangular dual of G from the REL. We first prove some properties of the REL.

Let $G = (V, E)$ be a PTP graph, and let $\{T_1, T_2\}$ be a REL of G. Let G_1 be the directed subgraph of G induced by the edges in T_1 and the four exterior edges directed as $v_S \to v_W$, $v_W \to v_N$, $v_S \to v_E$, $v_E \to v_N$. Let E_1 denote the edge set of G_1. (E_1 is the union of T_1 and the four exterior edges.) Let G_2 be the directed subgraph of G induced by the edges in T_2 and the four exterior edges directed as $v_W \to v_S$, $v_S \to v_E$, $v_W \to v_N$, $v_N \to v_E$. Let E_2 denote the edge set of G_2. (E_2 is the union of T_2 and the four exterior edges.) We will call G_1 the *S–N net* and G_2 the *W–E net* of G derived from the REL $\{T_1, T_2\}$.

Part (1) of Fig. 1 shows a PTP graph G. An S–N net G_1 and the corresponding W–E net G_2 are shown in parts (3) and (4). (Ignore the integers in the small boxes in parts (3) and (4) for now.)

LEMMA 2.3. (1) G_1 *is acyclic, with* v_S *as the only source and* v_N *as the only sink.*

(2) G_2 *is acyclic, with* v_W *as the only source and* v_E *as the only sink.*

Proof. The lemma is proved by way of contradiction. Suppose either G_1 or G_2 contains a directed cycle. Let $C = \{v_1, \ldots, v_l\}$ be such a cycle such that the total number of vertices that are on C or in the interior of C is minimized. Without loss of generality, suppose C is a cycle in G_1 and is directed in clockwise direction. (The proofs of other cases are similar.)

Case 1: C contains no vertices in its interior. If $l = 3$, then there is no edge in T_2 leaving v_2. This contradicts condition (1) of Definition 1. Suppose $l > 3$. Then there is an edge $e = (v_i, v_j) \in E$ contained in the interior of C. e cannot be in T_1 since otherwise we would have a smaller cycle in G_1, which contradicts the choice of C. So e must be in T_2. But regardless of the direction of e in T_2, condition (1) of Definition 1 is violated either at v_i or at v_j.

Case 2: C contains at least one vertex u_1 in its interior. Start at u_1; we can reach another vertex u_2 by using a T_2 edge. Similarly, from u_2 we can reach another vertex u_3 by using a T_2 edge. Since every vertex u has an incident edge in T_2 leaving u, this process can be repeated again and again. Since C is the smallest cycle in both G_1 and G_2, we cannot have a cycle in G_2 completely contained in the interior of C. Thus we must reach a vertex $v_j \in C$. Then condition (1) of Definition 1 is violated at v_j.

Since we get contradictions in all cases, both G_1 and G_2 are acyclic. Since every vertex v, other than v_S and v_N, has indegree and outdegree at least 1 in G_1, v_S is the only source and v_N is the only sink of G_1. Similarly, v_W is the only source and v_E is the only sink of G_2. □

Both G_1 and G_2 are the so-called *s–t planar graphs.* (An s–t planar graph is a directed acyclic planar graph with exactly one source s and exactly one sink t, and both s and t are on the exterior face of the graph.) The properties of these graphs have been studied in [7], [10], [13]. By using these properties, the structure of G_1 can be summarized as follows:

(a) For each vertex v other than v_S and v_N the edges entering v appear consecutively around v in G_1. The edges leaving v appear consecutively around v in G_1. Let e_1 and e_2 be the leftmost and the rightmost edges in G_1 entering v. Let e_3 and e_4 be the leftmost and the rightmost edges in G_1 leaving v. The face of G_1 with e_1 and e_3 on its boundary is denoted by left(v). The face of G_1 with e_2 and e_4 on its boundary is denoted by right(v). We use f_W to denote left(v_W) and f_E to denote right(v_E). (In other words, the exterior face is divided into two faces f_W and f_E.) For the vertices v_S and v_N define left$(v_S) =$ left$(v_N) = f_W$ and right$(v_S) =$ right$(v_N) = f_E$.

(b) For each interior face f of G_1 the boundary of f consists of two directed paths P_1 and P_2 starting at the same vertex and ending at the same vertex (see part (3) of Fig. 1).

Similarly, the structure of G_2 can be summarized as follows.

(c) For each vertex v other than v_W and v_E the edges entering v appear consecutively around v in G_2. The edges leaving v appear consecutively around v in G_2. Let e_1 and e_2 be the leftmost and the rightmost edges in G_2 entering v. Let e_3 and e_4 be the leftmost and the rightmost edges in G_2 leaving v. The face of G_2 with e_1 and e_3 on its boundary is denoted by above(v). The face of G_2 with e_2 and e_4 on its boundary is denoted by below(v). We use f_N to denote above(v_N) and f_S to denote below(v_S). (In other words, the exterior face is divided into two faces f_N and f_S.) For the vertex v_W and v_E, define above$(v_W) =$ above$(v_E) = f_N$ and below$(v_W) =$ below$(v_E) = f_S$.

(d) For each interior face g of G_2 the boundary of g consists of two directed paths P_1 and P_2 starting at the same vertex and ending at the same vertex (see part (4) of Fig. 1).

3. Algorithm. Let $G = (V, E)$ be a PTP graph, and let $\{T_1, T_2\}$ be a REL of G. Consider the S–N net G_1 derived from $\{T_1, T_2\}$. For each edge $e \in E_1$, let left(e) (right(e), respectively) denote the face of G_1 on the left (right, respectively) of e. Define the *dual graph* of G_1, denoted by G_1^*, as follows. The node set of G_1^* is the set of the interior faces of G_1 plus the two exterior faces f_W and f_E. For each edge $e \in E_1$ there is a corresponding arc e^* in G_1^* directed from the face left(e) to the face right(e). Since G_1 is an s–t graph, G_1^* is also an s–t graph [11]. Namely, G_1^* is a directed acyclic planar graph with f_W as the only source and f_E as the only sink.

Similarly, define the dual graph G_2^* of G_2 as follows. For each edge $e \in E_2$ let above(e) (below(e), respectively) denote the face of G_2 on the left (right, respectively) of e. The nodes of G_2^* are the interior faces of G_2 plus the two exterior faces f_S and f_N. For each edge $e \in E_2$ there is a directed arc e^* in G_2^* from the face below(e) to the face above(e). G_2^* is a directed acyclic planar graph with f_S as the only source and f_N as the only sink.

DEFINITION 2. A *consistent numbering of order* k_1 of G_1^* is a surjective mapping F_1 from the node set of G_1^* to the set of integers $\{0, 1, \ldots, k_1\}$ such that (1) $F_1(f_W) = 0$ and $F_1(f_E) = k_1$, and (2) if there is an arc from the node f to the node g in G_1^*, then $F_1(f) < F_1(g)$.

For an example, a topological ordering [1], [4] of G_1^* is a consistent numbering. As another example, if we define $F_1(f)$ to be the length of the longest path in G_1^* from f_W to f (with $F_1(f_W) = 0$), F_1 is also a consistent numbering. Define the *length* of G_1^* to be the length of the longest path from f_W to f_E in G_1^*. Note that if the length of G_1^* is k, then any consistent numbering of G has order at least k by Definition 2. The consistent numbering of G_2^* can be defined similarly. We now can present our algorithm as follows.

ALGORITHM DUAL:
Input: A PTP graph $G = (V, E)$.
- (1) Find a REL $\{T_1, T_2\}$ of G.
- (2a) Construct the S–N net G_1 derived from $\{T_1, T_2\}$ and its dual graph G_1^*.
- (2b) Compute a consistent numbering F_1 of G_1^*. Let $k_1 = F_1(f_E)$.
- (2c) For each vertex $v \in V$ other than v_S and v_N let $f_1 =$ left(v) and $f_2 =$ right(v) in G_1. Let $x_1(v) = F_1(f_1)$ and $x_2(v) = F_1(f_2)$. Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = k_1 - 1$.
- (3a) Construct the W–E net G_2 derived from $\{T_1, T_2\}$ and its dual graph G_2^*.
- (3b) Compute a consistent numbering F_2 of G_2^*. Let $k_2 = F_2(f_N)$.
- (3c) For each vertex $v \in V$ let $g_1 =$ below(v) and $g_2 =$ above(v) in G_2. Let $y_1(v) = F_2(g_1)$ and $y_2(v) = F_2(g_2)$.
- (4) For each vertex $v \in V$ assign v a rectangle $R(v)$ bounded by two vertical lines with x-coordinates $x_1(v), x_2(v)$ and two horizontal lines with y-coordinates $y_1(v), y_2(v)$.

End.

In §4 we will prove that the algorithm DUAL correctly computes a $k_1 \times k_2$ rectangular dual of G. For an example, the rectangular dual shown in part (2) of Fig. 1 is constructed from the information indicated in parts (3) and (4). In this example $F_1(f)$ is the length of the longest path from f_W to f in G_1^*. $F_2(g)$ is the length of the longest path from f_S to g in G_2^*. In part (3) the integers in the small boxes are the F_1-numbers of the faces of G_1. In part (4) the numbers in the small boxes are the F_2-numbers of the faces of G_2.

To implement the algorithm DUAL, we assume the embedding of G is given. (If not, it can be computed by using the well-known linear-time planarity algorithms.) Step 1 can be carried out by using the $O(n)$ algorithm in [3]. (The algorithm in [3] finds the set T_1, which is called the *path digraph*.) For step (2a) the graph G_1 and the dual graph G_1^* can be constructed from the embedding information of G. The implementation of step (2b) depends on the choice of F_1. The most natural choice, the length of the longest path from f_S to f in G_1^*, can be calculated according to the topological ordering of G_1^* [1], [4]. For step (2c) the left face and the right face of each vertex can be determined from the embedding information. All these steps take $O(n)$ time. Step (3) can be implemented similarly. Step (4) clearly takes $O(n)$ time. Thus the total running time of the algorithm is $O(n)$.

4. Correctness proof. Before we prove the correctness of the algorithm DUAL, we need several definitions. Consider an S–N net G_1 of G. An *S–N path* is a directed path P in G_1 from v_S to v_N. Let P_1 and P_2 be two S–N paths of G_1. (P_1 and P_2 are not necessarily edge disjoint.) We say P_2 is *to the right of* P_1 if every edge $e \in P_2$ is either on P_1 or to the right of P_1.

DEFINITION 3. A *path system* of G_1 is a collection $\{P_0, \ldots, P_{l-1}\}$ of S–N paths of G_1 such that

(1) The union of the paths P_i $(0 \leq i \leq l-1)$ is the edge set E_1 of G_1;

(2) P_i is to the right of P_{i-1} for $1 \leq i \leq l-1$.

DEFINITION 4. Let F_1 be a consistent numbering of G_1^* of order k_1. For each $0 \leq i \leq k_1$

(1) define $\mathrm{FACE}_i = \{f \mid f$ is a face of G_1 with $F_1(f) = i\}$;

(2) define $\mathrm{LB}_i = \{e \in E_1 \mid e$ is on the left boundary of a face $f \in \mathrm{FACE}_i\}$;

(3) define $\mathrm{RB}_i = \{e \in E_1 \mid e$ is on the right boundary of a face $f \in \mathrm{FACE}_i\}$;

(4) define the *standard path system* $\{P_0, \ldots, P_{k_1-1}\}$ of G_1 as $P_0 = RB_0$ and $P_i = P_{i-1} - LB_i \cup RB_i$ for $1 \leq i \leq k_1 - 1$.

Note that $\mathrm{FACE}_0 = \{f_W\}$, $\mathrm{LB}_0 = \emptyset$, $\mathrm{RB}_0 = \{(v_S, v_W), (v_W, v_N)\}$ and that $\mathrm{FACE}_{k_1} = \{f_E\}$, $\mathrm{LB}_{k_1} = \{(v_S, v_E), (v_E, v_N)\}$, $\mathrm{RB}_{k_1} = \emptyset$.

We make the following observations. Consider any edge $e \in E_1$. Let $g_1 = \mathrm{left}(e)$, $g_2 = \mathrm{right}(e)$, $p = F_1(g_1)$, and $q = F_1(g_2)$. Since e is on the right boundary of g_1 and on the left boundary of g_2, $e \in \mathrm{RB}_p$ and $e \in \mathrm{LB}_q$. Since e's corresponding arc e^* is directed from g_1 to g_2 in G_1^*, we have $p < q$. So $\mathrm{LB}_i \cap \mathrm{RB}_i = \emptyset$ for all $0 \leq i \leq k_1$. Since each $e \in E_1$ is in exactly one RB_i $(0 \leq i \leq k_1 - 1)$, E_1 is the disjoint union of the sets RB_i $(0 \leq i \leq k_1 - 1)$. Similarly, E_1 is the disjoint union of the sets LB_i $(1 \leq i \leq k_1)$.

LEMMA 4.1. *Let F_1 be a consistent numbering of G_1^* of order k_1. Then the following hold*:

(a) *The standard path system $\{P_0, P_1, \ldots, P_{k_1-1}\}$ in Definition 4 is a path system of G_1.*

(b) *For each vertex $v \in V$ let $f_1 = \mathrm{left}(v)$ and $f_2 = \mathrm{right}(v)$ in G_1. Define $x_1(v) = F_1(f_1)$ and $x_2(v) = F_1(f_2)$. Then v is on the path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.*

Proof. (a) We prove by induction that the following hold for each i $(0 \leq i \leq k_1 - 1)$: (1) P_i is an S–N path of G_1, and (2) $\mathrm{LB}_{i+1} \subseteq P_i$.

Base step $i = 0$. (1) $P_0 = \{(v_S, v_W), (v_W, v_N)\}$ is an S–N path of G_1.

(2) Let e be an edge in LB_1. Then e is on the left boundary of a face $f \in \mathrm{FACE}_1$. Let e^* be the arc in G_1^* corresponding to e. Since $F_1(f) = 1$, e^* must be directed from f_E to f in G_1^*. This implies $e \in RB_0 = P_0$. Since this is true for all $e \in \mathrm{LB}_1$, we have $\mathrm{LB}_1 \subseteq P_0$.

Induction step. Assume the claims (1) and (2) are true for $i - 1$; we show they are true for i.

(1) By the induction hypothesis P_{i-1} is an S–N path. Suppose $\mathrm{FACE}_i = \{h_1, \ldots, h_l\}$ for some l. Let A_j and B_j be the left and the right boundary of h_j, respectively $(1 \leq j \leq l)$. Since (2) is true for P_{i-1}, the paths A_j $(1 \leq j \leq l)$ are subpaths of P_{i-1}. Since A_j and B_j $(1 \leq j \leq l)$ start at the same vertex and end at the same vertex and P_i is obtained from P_{i-1} by replacing each A_j with B_j, P_i is an S–N path of G_1.

(2) Consider any edge $e \in \mathrm{LB}_{i+1}$. Let $g_1 = \mathrm{left}(e)$ and $g_2 = \mathrm{right}(e)$. Since $e \in \mathrm{LB}_{i+1}$, $F_1(g_2) = i+1$. Suppose $F_1(g_1) = q$ for some q. Then $e \in \mathrm{RB}_q$. Since e^* is directed from g_1 to g_2 in G_1^*, $q < i + 1$. By definition e is added into P_q and deleted when P_{i+1} is constructed. So e is in P_r for all $q \leq r \leq i$. In particular, $e \in P_i$. Thus $\mathrm{LB}_{i+1} \subseteq P_i$. This completes the induction.

Each $e \in E_1$ is in some RB_i $(0 \le i \le k_1 - 1)$ and hence in P_i. Therefore, E_1 is the union of P_i's $(i = 0, \ldots, k_1 - 1)$. From the definition of P_i it is easy to see P_i is to the right of P_{i-1} for all $1 \le i \le k_1 - 1$. Thus $\{P_0, \ldots, P_{k_1-1}\}$ is a path system of G_1.

(b) Since v is on the right boundary of f_1, it is added into the path $P_{x_1(v)}$. Since v is on the left boundary of f_2, it is removed when the path $P_{x_2(v)}$ is constructed. Hence v is on the paths P_i for exactly those indices i with $x_1(v) \le i \le x_2(v) - 1$. □

All of the preceding discussion can be repeated on the W–E net G_2 and its dual graph G_2^*. Let F_2 be a consistent numbering of G_2^* of order k_2. We can construct the standard path system $\{Q_0, \ldots, Q_{k_2-1}\}$ of G_2 from F_2 similar to Definition 4. For each vertex v of G let $g_1 = \mathrm{below}(v)$ and $g_2 = \mathrm{above}(v)$ in G_2. Define $y_1(v) = F_2(g_1)$ and $y_2(v) = F_2(g_2)$. In a manner similar to that of Lemma 4.1, it can be shown that v is on the path Q_j if and only if $y_1(v) \le j \le y_2(v) - 1$.

LEMMA 4.2. *Let G_1 and G_2 be the S–N net and the W–E net derived from a* REL $\{T_1, T_2\}$ *of G. Let F_1 and F_2 be two consistent numberings of G_1^* and G_2^*, respectively. Let u and v be two vertices of G.*

(1) *If $(u, v) \in T_2$ and is directed from u to v in G_2, then $x_2(u) = x_1(v)$.*

(2) *If there is a directed path from u to v in G_2 with length at least* 2, *then $x_2(u) < x_1(v)$.*

(3) *If $(u, v) \in T_1$ and is directed from u to v in G_1, then $y_2(u) = y_1(v)$.*

(4) *If there is a directed path from u to v in G_1 with length at least* 2, *then $y_2(u) < y_1(v)$.*

Proof. We prove only (1) and (2). The proofs of (3) and (4) are similar.

(1) Suppose $(u, v) \in T_2$ and is directed from u to v. Let e_1 (e_2, respectively) be the rightmost outgoing (incoming, respectively) edge of u in G_1. Let e_3 (e_4, respectively) be the leftmost outgoing (incoming, respectively) edge of v in G_1. Let f be the face of G_1 with e_1, e_2, e_3, and e_4 on its boundary. Then $f = \mathrm{right}(u) = \mathrm{left}(v)$ and $x_2(u) = x_1(v) = F_1(f)$.

(2) Let $u = u_0, u_1, \ldots, u_p = v$ $(p \ge 2)$ be a directed path in G_2 from u to v. By (1), $x_2(u_{l-1}) = x_1(u_l)$ for all $1 \le l \le p$. Since $x_1(u_l) < x_2(u_l)$ for all $0 \le l \le p$ and $p \ge 2$, we have $x_2(u) = x_2(u_0) < x_1(u_p) = x_1(v)$. □

From Lemmas 4.1 amd 4.2, we can prove the following.

THEOREM 4.3. *The algorithm* DUAL *correctly constructs a rectangular dual of G in $O(n)$ time.*

Proof. We have shown the algorithm can be implemented in linear time. We next prove the correctness of the algorithm. Let $\{P_0, \ldots, P_{k_1-1}\}$ be the standard path system of G_1 derived from F_1 and let $\{Q_0, \ldots, Q_{k_2-1}\}$ be the standard path system of G_2 derived from F_2. In the rectangular dual R constructed by the algorithm DUAL, each S–N path P_i $(0 \le i \le k_1 - 1)$ corresponds to a vertical strip bounded by the two vertical lines with x-coordinates i and $i+1$. Each W–E path Q_j $(0 \le j \le k_2 - 1)$ corresponds to a horizontal strip bounded by the two horizontal lines with y-coordinates j and $j+1$. Let $R(v)$ be the rectangle with coordinates $x_1(v), x_2(v), y_1(v), y_2(v)$. To show the set $\{R(v) \mid v \in V\}$ forms a rectangular dual of G, we need to prove the following claims.

(1) We show that each unit square R_{ij} $(0 \le i \le k_1 - 1$ and $0 \le j \le k_2 - 1)$ with x-coordinates $i, i+1$ and y-coordinates $j, j+1$ is occupied by a rectangle $R(v)$ for a unique $v \in V$. Consider the S–N path P_i and the W–E path Q_j. Except for the four special cases (a) $i = 0, j = 0$, (b) $i = k_1 - 1, j = 0$, (c) $i = 0, j = k_2 - 1$, (d) $i = k_1 - 1, j = k_2 - 1$, P_i and Q_j intersect at a unique vertex $v \in V$. By Lemma 4.1 (b), v is the unique vertex satisfying all of the following inequalities: $x_1(v) \le i$, $i + 1 \le x_2(v)$, $y_1(v) \le j$, $j + 1 \le y_2(v)$. Hence $R(v)$ is the unique rectangle occupying R_{ij}. For the four special cases this claim is not true. (For example, both v_S and v_W belong to the intersection of P_0 and Q_0.) The four special cases correspond to the four corner unit squares of R. However, the special definition $x_1(v_S) = x_1(v_N) = 1$ and $x_2(v_S) = x_2(v_N) = k_1 - 1$ at step (2b) of the algorithm DUAL

ensures that each of the four unit corner squares of R is occupied by one of $R(v_W)$, $R(v_E)$.

(2) We show if $e = (u, v)$ is an edge in G, then the corresponding rectangles $R(u)$ and $R(v)$ share a common boundary. If e is an exterior edge, this is ensured by the definition of $R(v_N)$, $R(v_W)$, $R(v_S)$, $R(v_E)$. So assume e is an interior edge. Suppose $e \in T_1$ and is directed from u to v. (Other cases are similar.) Let P_i be an S–N path containing e. By Lemma 4.1 (b), $x_1(u) \leq i \leq x_2(u) - 1$ and $x_1(v) \leq i \leq x_2(v) - 1$. By Lemma 4.2 (3), $y_2(u) = y_1(v) = j$ for some j. Thus $R(u)$ and $R(v)$ have the line segment connecting two points (i, j) and $(i + 1, j)$ as their common boundary.

(3) We show that if two rectangles $R(u)$ and $R(v)$ share a common boundary, then (u, v) is an edge in G. Assume the common boundary of $R(u)$ and $R(v)$ contains a horizontal line segment I connecting two points (i, j) and $(i + 1, j)$. (Other cases are similar.) Since $x_1(u) \leq i, i + 1 \leq x_2(u)$ and $x_1(v) \leq i, i + 1 \leq x_2(v)$, both u and v are on the S–N path P_i. We need to show (u, v) is an edge on P_i. If not, there exists a directed path from u to v in G_1 of length at least 2. By Lemma 4.2 (4) we have $y_2(u) < y_1(v)$. This contradicts the assumption that $R(u)$ and $R(v)$ share I as their common boundary.

Thus $e = (u, v)$ is an edge of G if and only if $R(u)$ and $R(v)$ share a common boundary. Hence $\{R(v) \mid v \in V\}$ form a rectangular dual of G. □

5. Conclusion. A new linear-time algorithm for finding a rectangular dual R of a proper triangular planar graph G is presented. The algorithm is based on new understanding of the structure of PTP graphs, which is of independent interest. Our algorithm is conceptually simple. The coordinates of the rectangles of the rectangular dual produced by the algorithm are integers and carry clear combinatorial meaning. This allows us to discuss the heuristics for reducing the width and the height of the rectangular dual R. Several related optimization problems are interesting and deserve further study. Let $w(R)$ and $h(R)$ denote the width and the height of R. How do we find a rectangular dual R of G so that $w(R)$ is minimized, $w(R) + h(R)$ is minimized, or $w(R)h(R)$ is minimized?

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