Thesis

Sander Beekhuis

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3 Algorithms

Kant and He [2] were the first to design algorithms that determine a regular edge labeling.

Fusy [1] recently developed a different algorithm computing a specific regular edge labeling using a method shrinking a sweepcycle while coloring the outside in accordance with a regular edge labeling.[1]

All algorithms in this section will have the same core (based on [1]). Consisting of shrinking a sweepcycle by so called *valid* paths.[2] But will differ in which valid paths they choose (if there are multiple).

We will start this section with some notation and preliminaries in Subsection 3.1. Then we will state the core algorithm and show that it always computes a regular edge labeling in Subsection 3.2. Afterwards we show in Subsections 3.3, 3.4 and Section 5 how one can adapt the choice of the valid paths to obtain regular edge labellings with certain properties. Namely a the minimal element of the distibutibe lattices of regular edge lablings and redgular edge labelings corresponing to horizontal and vertical rectangualr duals.

3.1 Notation and Preliminaries

Definition (Interior path)**.** We call a path P an internal path of a cycle C if all its edges are in the interior of C and it connects two distinct vertices of C

We will use a script $\mathcal{C}$ to indicate the current sweep cycle. We will repeatedly only consider the path $\mathcal{C} \setminus \{\mathrm{S}\}$. In that case we will always order it from W to E. That these edges are always in $\mathcal{C}$ is a result of Invariant 12 (I1).

We will let $\mathcal{P}$ denote a interior path. Given such a path of k vertices we will index it's nodes by $p_1, \ldots, p_k$ in such a way that p_1 is closer to W then p_k is (and thus that p_k is closer to E then p_1 is).

Then p_1 and p_k indicate the two unique vertices of the walk that are also part of the cycle. We will then let $\mathcal{C}|_{\mathcal{P}}$ denote the part of $\mathcal{C} \setminus \{\mathrm{S}\}$ that is between p_1 and p_k (including). $\mathcal{C}_{\mathcal{P}}$ will denote the cycle we get when we paste $\mathcal{C}|_{\mathcal{P}}$ and $\mathcal{P}$.

3.2 Core

The algorithm will always maintain the following three invariants

Invariants 12

(I1) The cycle $\mathcal{C}$ contains the two edges SW and SE.
(I2) $\mathcal{C} \setminus \{\mathrm{S}\}$ has no chords
(I3) All inner edges of T outside of $\mathcal{C}$ are colored and oriented in such that the innnervertex condition holds.

A cycle satisfying these three invariants will have the same general shape as in figure 3. We note that the cycle has at least 4 vertices because otherwise a seperating triangle is created.

FiXme: We need to add a partial inner vertex condition

[1]The specific regular edge labelling Fusy obtained was the minimal element of the distributive lattice of regular edge labellings.

[2]In Fusy's work cals these *eligible paths*

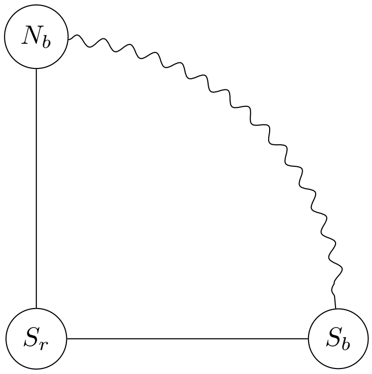


Figure 3: An example of a cycle $\mathcal{C}$ satisfying the invariants

It is also nice to note that the union of the cycle and it's interior form a triangulation of the n-gon since it is a induced subgraph of a triangulation of the 4-gon.

3.2.1 Valid paths

Definition (valid path)**.** We call an internal path $\mathcal{P}$ from w_1 to w_k valid if

(E1) Neither p_1 or p_k is S

(E2) The paths $\mathcal{P}$ and $\mathcal{C}|_{\mathcal{P}}$ both have more than 1 edge [3]

(E3) Every interior edge of $\mathcal{C}_{\mathcal{P}}$ connects a vertex of $\mathcal{P}\setminus\{p_1, p_k\}$ and $\mathcal{C}|_{\mathcal{P}}\setminus\{p_1, p_k\}$. In particular $\mathcal{C}_{\mathcal{P}}$ is a non-separating cycle.

(E4) The path $\mathcal{C}' \setminus \{\mathrm{S}\}$, where $\mathcal{C}'$ is obtained by replacing $\mathcal{C}|_{\mathcal{P}}$ by $\mathcal{P}$ in $\mathcal{C}$, is chordfree.

Remark 13. *"Shrinking" the cycle with an valid path will keep all the invariants true.*

FiXme: We havn't proven this yet
We will show the following proposition.

Theorem 14 (Existence of a eligible path)**.** *When the algorithm's invariant (12 (I1) - 12 (I3)) are satisfied and the cycle $\mathcal{C}$ is separating then there exist a* eligible *internal path.*

FiXme: As outlined in last meeting this proof is not correct as is, it might be best to not read this and skip to the end of this section. We are stuck on the part where we need to find a path satisfying E4. We might proof this from red algo.

Proof. We will first show that there always exists an internal path $\mathcal{P}$. We will then show that a internal path can be found that satisfies conditions $(E1)-(E4)$.

In the proof we will often use that a

[3] i.e. both have an interior vertex

Let us first note that if the cycle C is separating (i.e has a non-empty interior), there is at least one interior vertex v. Since the triangulation of a n-gon is 2-connected there are two ways to go from v to (say) S_r. Hence there is an internal path $\mathcal{P}_0$.

If this path does not satisfy (E1) we can use the following construction. The other vertex where P_0 intersects $\mathcal{C}$ is not S_r. Let us call this vertex x and it's neighbour on the path y. The vertex x might be N_b or S_b but can't be both, hence it has at least one neighbour z on the cycle that is not S_r. Because the triangulation of a n-gon is internally maximally planar we have that yz is an edge. Now xyz is an internal path satisfying (E1). See also figure 4, here we made a choice on which side of y the vertex z lies, but this choice can be made without losing generality.

Hence we have now constructed, or already had, a path that satisfies (E1). Let us for the remainder of the proof denote this path by $\mathcal{P}_1$.

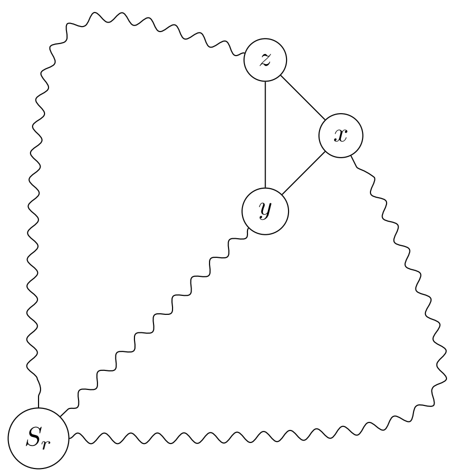


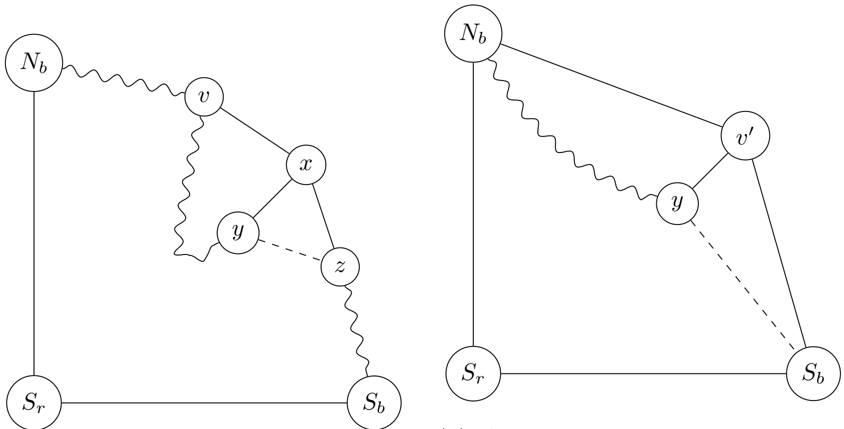
Figure 4: Constructing a path satisfying (E1)

There is a path that also satsifies (E2) If $\mathcal{P}_1$ satisfies (E2) we set $\mathcal{P}_2 = \mathcal{P}_1$ otherwise we will create a path that satisfies (E1) and (E2). If the path $\mathcal{P}_1$ does not satisfy $(E2)$ [4] then there are two possibilities a) $\mathcal{P}_1$ does not have interior vertices and/or b) $[v, v']$ does not have interior vertices. If a) would be true the existence of P_0 would contradict Invariant 12 (I2). Hence the only problem can be that $b)$ occurs.

If $v = N_b$ and $v' = S_b$ we have found a separating triangle given by $S_r N_b S_b$ [5] in original graph. Hence at least one of v or v' is not N_b or S_b. If we call this vertex x its neighbour on the path y and it's neighbour outside $[v, v']$ z. We see that by the interior of $\mathcal{C}$ being maximally planar yz must be an edge. If we now adapt P_1 by replacing yx by yz we have made $[v, v']$ one vertex longer and hence created a path satisfying (E2). In figure 5 we show this procedure in two cases. Executing this procedure does not change that S_r is not one of the

[4] which will be the case if the above construction has been used

[5] this is the cycle $\mathcal{C}$ which is separating



(a) The general case. Note that $x = v'$.

(b) A specific case. Note now that $N_b = v, v' = x$ and $S_b = z$

Figure 5: Creating a path satisfying (E2). The dotted line is the edge we take in the new path $\mathcal{P}_2$

endpoints of the path. Hence we have now created a path $\mathcal{P}_2$ that satisfies (E1) and (E2).

There is a path that also satisfies (E3) If $\mathcal{P}_2$ satisfies $(E3)$, we take $\mathcal{P}_3 = \mathcal{P}_2$. Otherwise we will remedy the defect. We separate five different cases of offending edges. All of the five cases will be easy to remedy giving a path $\mathcal{P}_2'$ still satisfying (E1) and (E2) such that $\mathcal{C}_{\mathcal{P}_2'}$ is strictly contained in $\mathcal{C}_{\mathcal{P}_2}$

a) edges from $[v, v'] \setminus v, v'$to $[v, v'] \setminus v, v'$

b) edges from $\mathcal{P} \setminus v, v'$ to $\mathcal{P} \setminus v, v'$

c) edges incident to v or v and some other vertex on $\mathcal{C}_{\mathcal{P}_2}$

d) edges from $[v, v']$ to some internal vertex

e) edges from $\mathcal{P} \setminus v, v'$ to some internal vertex

The existence of an edge as in a) is forbidden by Invariant 12 (I2). If b) occurs we can simply shortcut our original path $\mathcal{P}_2$ with this edge. If c) occurs this edge can't go to another vertex in $[v, v']$ since that would offend Invariant 12 (I2). Hence they go to a vertex in $\mathcal{P}_2$ and we can shortcut the path as in b).

If d) occurs we simply make a new path and if e) occurs we take a slightly adapted interior path. See figures

Since all of the moves shrink $\mathcal{C}_{\mathcal{P}_2}$ while keeping (E1) and (E2) intact and we can't infinitely shrink this means at a certain point no more moves are available. Since every offending edges allows a move this means that there are no more offending edges. Hence this version of $\mathcal{P}_2'$ satisfies (E3). For the final step of the proof we take $\mathcal{P}_3 = \mathcal{P}_2'$.

There is a path that also satisfies (E4) Suppose that $\mathcal{P}_3$ does not satisfy (E4). Then we can just take the would be interior edge and take this for a nwe path. This is again a finite procedure reducing the sum of $|\mathcal{P}_3| - |[v, v']|$. In the end we have a path satisfying (E1) - (E4).

□

3.3 Minimum distributive lattice element

We get this when we take the "leftmost" eligible path. As is outlined in [1]

FiXme: We might exapand this section

3.4 Horizontal one-sides

Writers note: and what we mean with *cycle border* and *face border*

As an exercise one could try to addapt Fusy's algorithm to generate horizontally one-sided layouts directly, without doing flips in the distributive lattice. It turns out that this is not that diffcult.

Since the horizontal segments correspond to faces in the blue bipolar orientation we want that one of the two borders of the face has a length of at most two. Since every valid path which we update the cycle with splits off one face in the blue bipoolar orientation it is easy to control this property.

Theorem 15. *In the update of the algorithm there is always an eligible path $\mathcal{P}$ available such that either $\mathcal{P}$ or $\mathcal{C}|_{\mathcal{P}}$ is of length* 2.

In order to proof this theorhem we will first show the following lemma.

Lemma 16. *If $\mathcal{P}$ is an eligble path giving raise to a cycle $\mathcal{C}_P$ of which both borders have length of at least* 3*. Then there exist an eligible path $\mathcal{P}'$ such that the pathborder and cycleborder of its cycle $\mathcal{C}_{\mathcal{P}'}$ are both at least* 1 *shorter than those of $\mathcal{C}_{\mathcal{P}}$.*

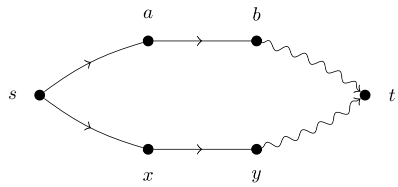
FiXme: Revisit notation after writing section on oriented REL

Proof. In this proof we will frequently use property (E3) of a valid path, we won't mention it every time we use it.

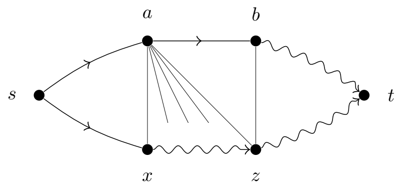
We denote the source by s and the sink by t. We also assign namesa, b and x, y to the first two vertices on both borders, see Figure 6a. Since every interior face of G is a triangle ax is an edge. Now we distinguis two cases, either ay is an edge (case 1) or bx is an edge (case 2). They can't both be an edge at the same time due to planarity, neither can it happen that both of them are not an edge since then the face containing the path $baxy$ is at least of degree 4.

In the first case a may be connected to more vertices on the pathborder, however there is a last one, say z. And this vertex is then also connected to b, otherwise it would not be the last one. Now we can provide an shorther eligible path $\mathcal{P}'$. We start at a go to z and from there we follow the old path $\mathcal{P}$ to t. See figure 6b. It is easy to see that all four properties of an eligible path hold for $\mathcal{P}'$.

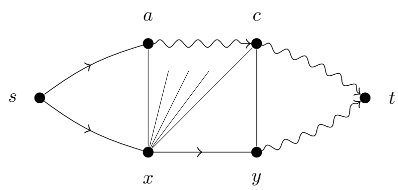
In the second case x may be connected to more vertices along the cycle border, however there is a last one, say c. And this vertex is then also connected to y, otherwise it would not be the last one. Now we can provide an shorther eligible path $\mathcal{P}' = sxz$. See figure 6c. It is straightforward to see that all four properties of an eligible path hold for $\mathcal{P}'$. □



(a) The setting



(b) Case 1



(c) Case 2

Figure 6

Proof of Theorem 15. By Theorem 14 we know there is a eligible path $\mathcal{P}$. If one of the borders of $\mathcal{C}_{\mathcal{P}}$ is of length 2 or less we are done. If this path gives raise to a face $\mathcal{C}_{\mathcal{P}}$ with both borders are both of length at least 3 we can repeatedly apply Lemma 16 until at least one of the borders is of length at most 2. $\square$

If we in every update of the algorithm take the paths from Theorem 15 we end up with the correct faces in the blue bipolar orientation and hence a horizontally one sided rectangular dual.