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ON GENERATING PLANAR GRAPHS*

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Abstract. A 3-valent graph G is cyclically *n-connected* provided one must cut at least n edges in order to separate any two circuits of G. If G is cyclically n-connected but any separation of G by cutting n edges yields a component consisting of a simple circuit, then we say that G is *strongly cyclically* n-connected. We prove that there exists a graph G_0 such that all strongly cyclically 4-connected planar graphs, other than the graph of the cube and the pentagonal prism, can be generated from G_0 by adding edges. We introduce two operations called *adding a pair of edges* and *replacing a face.* We prove that using these two operations together with adding edges we can generate the cyclically 5-connected planar graphs.

1. Introduction

It is a well-known theorem [3] that the planar 3-connected graphs can be generated from the graph S of the simplex by adding edges. That is, if G is a planar 3-connected graph, then there is a sequence $S = G_1$, $G_2, ..., G_n = G$ of planar 3-connected graphs such that G_i is obtained from G_{i-1} by adding an edge e, where either

(i) the vertices of e are vertices of G_{i-1},

(ii) one vertex of e lies on an edge of G_{i-1} and the other is a vertex of G_{i-1},

(iii) each vertex of e lies on an edge of G_{i-1}.

If we restrict ourselves to adding edges of type (iii), then we generate the simple (i.e. 3-valent) planar 3-connected graphs. In studying the 4-color conjecture, a special kind of connectivity for simple graphs is important (see [1, Chapter 17]): a simple planar graph G is said to be *cyclically n-connected* provided no two circuits of G may be separated by removing fewer than n edges. We shall say that G is *strongly cyclically n-connected* provided G is cyclically n-connected and if two circuits of G are separated by the removal of n edges, then one of the component is

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Fig. 1.

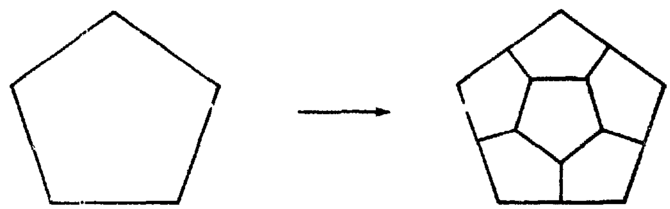

Fig. 2.

a simple circuit. For brevity, we shall write c n-connected and c* n-connected for cyclically n-connected and strongly cyclically n-connected, respectively.

Kotzig [2] has shown that the c 4-connected planar graphs can be generated from the graph of the cube by adding edges. In this paper we show how to generate the c* 4-connected and c 5-connected graphs. For the c 5-connected graphs, we shall need two new operations. The first consists of adding a pair of edges simultaneously across two pentagonal faces as illustrated in Fig. 1.

This operation will be called *adding a pair of edges*. The second operation consists of replacing a pentagonal face by a pentagonal face surrounded by pentagonal faces as in Fig. 2. This operation will be called *replacing a face.*

We shall prove two theorems.

Theorem 1.1. *The* c* *4-connected graphs, with the exception of the graphs of the cube and pentagonal prism, can be generated from the graph* G_0 *(see Fig. 3) by adding edges.*

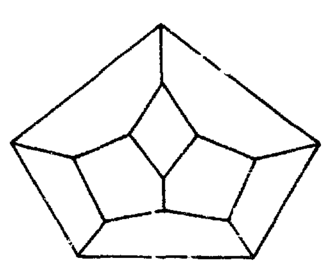

Fig. 3.

Theorem 1.2. *The* c *5-connected graphs can be generated from the graph of the regular dodecahedron by adding edges, adding pairs of edges, and replacing faces.*

2. Preliminaries

We shall need to talk about the inverses of our generating operations. The inverse operations will be called *removing an edge, removing a pair of edges* and *removing a face.* Since we shall find it useful to work in the dual G^* of G, we shall also need the corresponding inverse operation in G^*. If G is a planar 3-valent graph, the following facts about G^* will be useful and are easily verified:

(1) G^* is a planar graph all of whose faces are triangles. (Such a graph will be called *triangular.*)

(2) G is c n-connected if and only if G^* is n-connected (a graph is said to be *n-connected* provided it cannot be disconnected by removing fewer than n vertices).

(3) If G is c* n-connected, then any n-circuit C (i.e., circuit of length n) in G^* without diagonals consists entirely of the neighboring vertices of some vertex v together with the edges joining them. We shall say that C *surrounds* v, and we shall say that G^* is n^**-connected.*

(4) The dual of removing an edge of G is *shrinking an edge* of G^* as illustrated in Fig. 4(a). If shrinking an edge e produces a graph which is n-connected (n^*-connected), we shall say that e is *n-shrinkable* (n^**-shrinkable*).

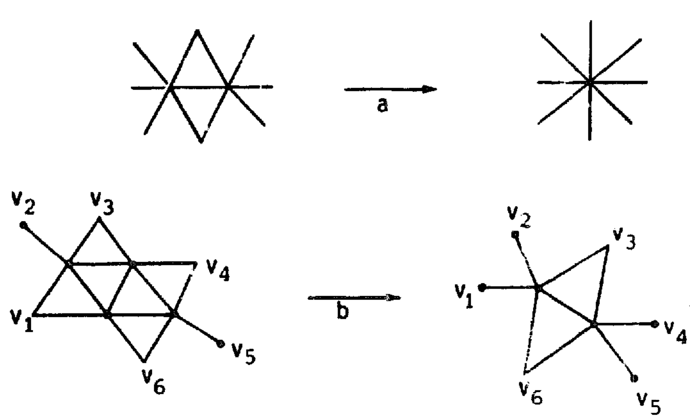

Fig. 4.

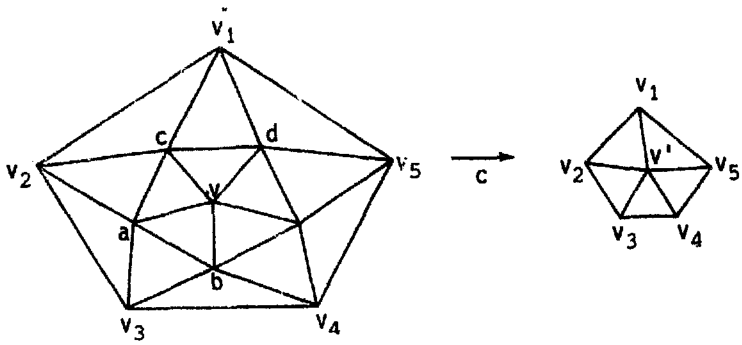

Fig. 5.

(5) The dual of removing a pair of edges of G is *shrinking a pair of edges* (see Fig. 4(b)).

(6) The dual of removing a face is *shrinking a pentagon* as illustrated in Fig. 5.

If G is a triangular graph, any circuit C without diagonals which separates the vertex set of G will be called a *separating circuit.* If C is a separating circuit and does not surround a vertex, then C will be called a proper separating circuit. A circuit of length n will be called an *n-circuit.*

3. The generation of c* 4-connected graphs

Suppose G is the dual of a c* 4-connected graph. We shall show that G has a 4*-shrinkable edge, unless G is the dual of the graph of the cube or the dual of the graph G_0. We consider two cases.

Case 1: G is 5-connected. In this case, we must look for an edge which does not belong to any proper separating 5-circuit. Let v_0 be a 5-valent vertex of G and let its neighbors be $v_1, ..., v_5$ in cyclic order. We shall suppose that $v_1 v_0$ is an edge belonging to a proper separating 5-circuit C_1. Without loss of generality, we assume that $v_0 v_3$ is another edge of C_1. We shall label the other vertices of C_1, so that $C_1 = v_0 v_1 v_6 v_7 v_3$. If we assume that $v_0 v_2$ belongs to a proper separating 5-circuit C_2, then C_2 will have one new vertex v_8 and it must lie in the same region bounded by C_1 in which v_2 lies, for otherwise, C_1 would not be proper.

This gives us two possibilities: either $C_2 = v_0 v_2 v_8 v_7 v_4$ or $C_2 = v_0 v_2 v_8 v_6 v_4$ (by symmetry, we may assume that v_4 belongs to C_2). In the first case, C_2 must surround v_3 and is not proper. In the second case, $v_1 v_6 v_4 v_5$ is a 4-circuit and must have a diagonal (due to the 5-connectedness of G), but adding a diagonal across this circuit creates either a 3 or 4-valent vertex. We conclude that G has a 4*-shrinkable edge.

Case 2: G has a 4-valent vertex v_0. Suppose G has no 4*-shrinkable edges. There are now several possibilities, an edge may fail to be 4*-shrinkable because it belongs to a proper 5-circuit or because it belongs to a 4-circuit which surrounds a vertex. Let the vertex v_0 have neighbors v_1, v_2, v_3 and v_4 in cyclic order.

Case 2(a). The edges v_0v_1 and v_0v_2 both belong to 4-circuits which surround vertices. In this case, G is the graph of the octahedron; its dual is the graph of the cube which is one of the exceptional graphs of our theorem.

Case 2(b). The edge v_0v_1 belongs to a proper separating 5-circuit C_1 while the edge v_0v_2 belongs to a 4-circuit C_2 surrounding a vertex. Let $C_2 = v_0v_2v_5v_4$. By symmetry, we may assume that C_2 surrounds v_1 and thus $C_1 = v_0v_1v_5v_6v_3$. If either of the circuits $v_2v_5v_6v_3$ or $v_4v_5v_6v_3$ has a diagonal, then C_1 would not be proper, thus both of these circuits surround vertices. The graph we now have has a proper 4-circuit $v_3v_4v_5v_7$, where v_7 is the vertex surrounded by $v_2v_5v_6v_3$, which is a contradiction to the 4*-connectedness of G.

Case 2(c). Both v_0v_1 and v_0v_2 belong to proper separating circuits. There are two possible ways that these circuits intersect. Let $C_1 = v_0v_1v_5v_6v_3$ and let C_2 be the other proper separating 5-circuit.

Subcase (i): $C_2 = v_0v_2v_6v_5v_4$. The circuits $v_1v_5v_6v_2$ and $v_4v_5v_6v_3$ both must surround vertices, for otherwise C_1 or C_2 would not be proper. This gives us the graph which is the dual of G_0.

Subcase (ii): $C_2 = v_0v_2v_5v_7v_4$. We consider the circuits $C_3 = v_1v_5v_7v_4$ and $C_4 = v_2v_5v_6v_3$. Both must surround vertices or else C_1 or C_2 would not be proper. Suppose C_3 surrounds v_8 and C_4 surrounds v_9. The edge v_1v_8 must belong to a proper separating 5-circuit if it is not 4*-shrinkable. The only possibility for such a circuit is $v_1v_8v_7v_3v_0$, but if this is a circuit in G, then v_2v_9 is 4*-shrinkable.

We have now shown that if G is not the dual of G_0, or the graph of the cube, then G contains a 4*-shrinkable edge. One need only check that the graph of the pentagonal prism is the only c* 4-connected graph which can be generated from the graph of the cube, and no c* 4-connected graph can be generated from the graph of the pentagonal prism.

As a result of the above, we have Theorem 1.1.

4. The generation of 5-connected graphs

Lemma 4.1. *Let G be a 5-connected triangular graph with no 5-shrinkable edges. If G is not the graph of the icosahedron, then shrinking a pentagon surrounding v preserves the 5-connectivity of the graph.*

Proof. Let G' be the graph obtained by shrinking the pentagon (see Fig. 5) and suppose G' is not 5-connected. Then there is a separating 4-circuit C in G' which includes v'. We may assume without loss of generality that C includes v_1 and v_3. Let v_6 be the fourth vertex of C. Since $v_1 v_2 v_3 v_6$ is a 4-circuit in G, it must have a diagonal which must be $v_2 v_6$. We shall now show that $v_2 v_6$ is 5-shrinkable. Suppose not, then $v_2 v_6$ belongs to a separating 5-circuit in G. The only possible such circuits are $v_6 v_2\, abv_4$ and $v_6 v_2\, cdv_5$, which implies that either $v_6 v_5$ is an edge or $v_6 v_4$ is an edge. Suppose $v_6 v_4$ is an edge, then $v_6 v_1 v_5 v_4$ is a 4-circuit and must have a diagonal which must be $v_6 v_5$, but now G is the graph of the icosahedron. The same argument works if we assume that $v_6 v_5$ is an edge.

Lemma 4.2. *Suppose G is a 5-connected triangular graph (other than the graph of the icosahedron) with four 5-valent vertices as in Fig. 6. Then unless there is a vertex v_{11} such that $v_{11} v_1 v_{10} v_9 v_4$ (or $v_{11} v_2 v_{10} v_7 v_5$) is a separating 5-circuit, we may shrink either the pair $v_{10} v_9, v_7 v_8$ or the pair $v_7 v_{10}, v_8 v_9$; or we may shrink a pentagon.*

Proof. Suppose we cannot shrink $v_{10} v_9, v_7 v_8$, then one (possibly both) of the edges is in a separating 5-circuit other than those surrounding v_{10} and v_8. By symmetry, we may assume that $v_{10} v_9$ is in such a circuit C.

Case 1: the circuit C is $v_3 v_9 v_{10} v_7 v_6$. In this case, $v_3 v_2 v_1 v_6$ is a 4-circuit and must have a diagonal. But no matter which way we add the diagonal, either v_1 or v_2 will have valence less than 5 contradicting the 5-connectedness of G.

Case 2: the circuit C is $v_1 v_{10} v_9 v_3 v_{11}$, for some vertex v_{11} different from $v_1, ..., v_{10}$. Suppose that $v_7 v_{10}, v_9 v_8$ is not shrinkable. As above we may assume that $v_7 v_{10}$ is in a separating 5-circuit C' other than those surrounding v_8 or v_{10}.

Case 2(a): the circuit C' is $v_3 v_9 v_{10} v_7 v_6$. This is the same as Case 1.

Case 2(b): the circuit C' is $v_2 v_{10} v_7 v_5 v_{12}$, for some vertex v_{12} different from $v_1, ..., v_{10}$. By symmetry, this is the type of circuit mentioned in the hypothesis.

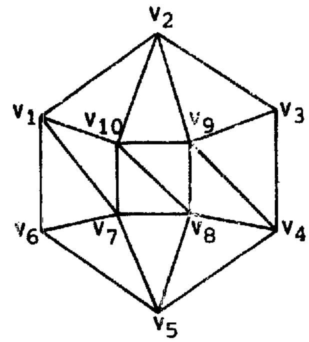

Fig. 6.

Case 2(c): the circuit C' is $v_2v_{11}v_6v_7v_{10}$. In this case, v_{10} is surrounded by 5-valent vertices and we may shrink the pentagon $v_1v_7v_8v_9v_2$.

Case 3: the circuit C is $v_1v_{11}v_4v_9v_{10}$. This is the type of circuit in the hypothesis.

By the symmetry of Fig. 6, we have exhausted all types of circuits.

Lemma 4.3. *If Fig. 6 appears in a 5*-connected triangular graph G, other than the graph of the icosahedron, then one of the pairs of edges* $v_{10}v_9$, v_7v_8 *or* v_7v_{10}, v_8v_9 *is shrinkable.*

Proof. Suppose not, then there is a 5-circuit of the type $v_1v_{10}v_9v_4v_{11}$. This implies that $v_1v_{10}v_8v_4v_{11}$ is a proper separating 5-circuit, which contradicts the 5*-connectedness of G.

Lemma 4.4. *If G is 5*-connected, then it contains Fig. 6 as a subgraph, or contains a 5-shrinkable edge.*

Proof. Let v_0 be any 5-valent vertex of G If there do not exist at least three consecutive 5-valent vertices neighboring v_0, then one of the edges containing v_0 will not belong to any 5-circuit surrounding a vertex and will thus be a 5-shrinkable edge. If, however, these three 5-valent vertices exist, then we have Fig. 6 (with v_0 corresponding to v_8 and the other three vertices corresponding to v_7, v_{10} and v_9).

At this point we have proved Theorem 1.2 for c* 5-connected graphs. Suppose now that G is not 5*-connected, that is, that G contains a proper separating 5-circuit. Suppose further that no edge of G is 5-shrinkable.

We shall assume that G is embedded in the plane and we choose a proper separating 5-circuit C that is minimal in the sense that every other proper separating 5-circuit with an edge inside C has at least one vertex outside C.

Lemma 4.5. *There is no path of length 2 inside C connecting two non-consecutive vertices of C.*

Proof. Let C be the circuit $v_1 v_2 v_3 v_4 v_5$ and assume the path is $v_1 v_0 v_3$. If the circuit $C' = v_1 v_0 v_3 v_4 v_5$ had a diagonal, it would have to be $v_0 v_5$ or $v_0 v_4$. Assume that it is $v_0 v_5$. Then the circuits $v_0 v_3 v_4 v_5$ and $v_1 v_0 v_3 v_2$ must have diagonals and the only possibilities are $v_0 v_4$ and $v_0 v_2$, but this implies that $v_1 v_2 v_3 v_4 v_5$ is not proper. Since C is minimal, C' must surround a vertex. But now, when we add the diagonal to $v_1 v_0 v_3 v_2$, we have that v_0 has valence 4 which is a contradiction.

Corollary 4.6. *Each vertex of C meets at least two edges inside C.*

Lemma 4.7. *Each edge inside C belongs to a 5-circuit that surrounds a vertex.*

Proof. Suppose not. Since no edge is 5-shrinkable, there is an edge inside C belonging to a proper separating 5-circuit C'. By Lemma 4.5, C' must have at least two vertices inside C. We may suppose that C' is $v_6 v_1 v_7 v_8 v_3$ with v_6 outside C and v_7 and v_8 inside C.

The circuit $v_1 v_7 v_8 v_3 v_2$ cannot have diagonals because then there would be only one vertex inside C', namely v_2, and C' would not be proper. By the minimality of C, $v_1 v_7 v_8 v_3 v_2$ must surround a vertex, but now when we add a diagonal to $v_1 v_2 v_3 v_6$, we have that v_2 is only 4-valent.

Lemma 4.8. *Fig. 6 appears in G. With the four 5-valent vertices on or inside C.*

Proof. We form a planar graph H consisting of C, everything inside C and a new vertex w outside that is joined to each vertex of C. We now use the following which follows from Euler's equation for triangular planar graph with vertices all of valence 3 or more (see [1, p. 254]):

$$\sum (6-i)v_i = 12 \,,$$

where v_i is the number of i-valent vertices in G. Since no vertex of H has valence less than 5, it follows that $v_5 \geq 12$ and thus there are at least six 5-valent vertices inside C. Let v_0 be one such vertex inside C. The argument in Lemma 4.4 now gives us the desired subgraph in G (note that H is 5^*-connected).

Lemma 4.9. *Either $v_{10}v_9$ and v_7v_8 or v_7v_{10} and v_8v_9 can be shrunk in G.*

Proof. Suppose not, then without loss of generality we may assume that there is a path $v_1v_{11}v_4$ with v_{11} different from $v_1, ..., v_{10}$ in G. The circuit $v_1v_{10}v_8v_4v_{11}$ is a proper separating circuit in G and thus one of its vertices must lie outside C. If v_1 is the outside vertex, then v_{10} and v_{11} are vertices of C. We consider the possible circuits that could be C. The only possible separating circuits of length 5 containing v_{10} and v_{11} but not v_1 are $v_2v_{10}v_8v_5v_{11}$, $v_2v_{10}v_8v_4v_{11}$, $v_2v_{10}v_7v_6v_{11}$ and $v_2v_{10}v_7v_5v_{11}$. In the first two cases, the circuit separates the set $v_7v_8v_9v_{10}$ which is a contradiction. In the third case, the circuit would have to surround v_1 and would not be proper. In the fourth case, v_2v_{11} being an edge would make $v_2v_9v_4v_{12}$ a separating 4-circuit. We conclude that v_1 cannot be the outside vertex, and similarly v_4 cannot be the outside vertex, thus v_{11} is the outside vertex and v_1 and v_4 are on C.

Suppose v_{10} is on C. The vertex v_8 cannot be on C because C would have to contain v_8v_{10} and thus would separate v_9 from v_7. The vertex v_9 cannot be on C because $v_{10}v_8v_4$ would be a path of length 2 inside C connecting two non-consecutive vertices of C. Neither v_7 nor v_2 can be on C because C would have diagonals. Now we have, however, that no neighbor of v_{10} other than v_1 can be on C, a contradiction. Thus we conclude that v_{10} is not on C. Similarly, v_8 is not on C.

If v_9 is on C, then $v_1v_{10}v_9$ is a path of length 2 inside C connecting non-consecutive vertices of C, thus v_9 is not on C, and similarly v_7 is not on C.

Now we have that none of the vertices v_7, v_8, v_4, v_3, v_5 and v_6 are not on C.

We cannot have that v_2v_4 is an edge because v_3 would be 3-valent; thus if v_2 is on C, we would have that $v_2v_9v_4$ is a path of length two inside connecting non-consecutive vertices of C. We conclude that v_2 is not on C and similarly v_5 is not on C.

We cannot have that $v_1 v_3$ is an edge because v_2 would be only 4-valent. Now if v_3 is on C, then $v_1 v_2 v_3$ would be a path of length two inside C (note that v_2 cannot be outside C when it is joined to v_{10} inside C) joining non-consecutive vertices of C. Thus v_3 and similarly v_6 are not on C.

We now have that the graph in Fig. 6 lies inside C except for the vertices v_1 and v_4 which lie on C. Let v_{12} be the vertex of C that is joined to v_1 and v_4. The circuit $v_{12} v_1 v_6 v_5 v_4$ has no vertices outside C and thus must surround a vertex or have diagonals. If it has diagonals, then either v_6 or v_5 will have valence less than 5. If it surrounds a vertex, then v_6 will be 4-valent. Thus we reach a contradiction, which completes the proof of Theorem 1.2.

Added in proof. Theorem 1.2 has been proved independently by Jean Butler (in: A generation procedure for the simple 3-polytopes with cyclically 5-connected graphs, to appear).

References

[1] B. Grünbaum, Convex Polytopes (Wiley, New York, 1967).
[2] A. Kotzig, Regularly connected trivalent graphs without non-trivial cuts of cardinality 3, Acta. Fac. Rerum Natur. Univ. Comenian. Math. Publ. 21 (1968), 1–14 (1969).
[3] E. Steinitz and H. Rademacher, Vorlesungen über die Theorie der Polyeder (Berlin, 1934).