Towards Characterizing Graphs with a Sliceable Rectangular Dual

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Abstract. Let $\mathcal{G}$ be a plane triangulated graph. A rectangular dual of $\mathcal{G}$ is a partition of a rectangle R into a set $\mathcal{R}$ of interior-disjoint rectangles, one for each vertex, such that two regions are adjacent if and only if the corresponding vertices are connected by an edge. A rectangular dual is sliceable if it can be recursively subdivided along horizontal or vertical lines. A graph is *rectangular* if it has a rectangular dual and *sliceable* if it has a sliceable rectangular dual. There is a clear characterization of rectangular graphs. However, a full characterization of sliceable graphs is still lacking. The currently best result (Yeap and Sarrafzadeh, 1995) proves that all rectangular graphs without a separating 4-cycle are sliceable. In this paper we introduce a recursively defined class of graphs and prove that these graphs are precisely the nonsliceable graphs with exactly one separating 4-cycle.

1 Introduction

Let $\mathcal{G}$ be a plane triangulated graph. A *rectangular dual* of $\mathcal{G}$ is a rectangular partition $\mathcal{R}$ such that *(i)* no four rectangles meet in the same point, *(ii)* there is a one-to-one correspondence between the rectangles in $\mathcal{R}$ and the vertices of $\mathcal{G}$, and *(iii)* two rectangles in $\mathcal{R}$ share a common boundary segment if and only if the corresponding vertices of $\mathcal{G}$ are connected. A graph can have exponentially many rectangular duals [6], but might not even have a single one. Rectangular duals have a variety of applications, for example, as rectangular cartograms in cartography or as floorplans in architecture and VLSI design.

There are several types of rectangular duals that are of particular interest. Often it is desirable to assign certain areas to each rectangle. A recent paper by Eppstein et al. [8] studies *area-universal* rectangular duals, which have the property that any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular dual. A rectangular dual is *sliceable* if it can be recursively subdivided along horizontal or vertical lines (such duals are also called guillotine floorplans and can be constructed by glass cuts). While it is generally difficult to determine if an area assignment is feasible and to compute the corresponding layout of the rectangles, it is very easy to do so for sliceable duals. Furthermore, sliceable duals more easily facilitate certain layout steps in VLSI layout. Sliceability does not imply area-universality or vice versa (see Fig. 1).

A graph is *rectangular* if it has a rectangular dual and *sliceable* if it has a sliceable rectangular dual. Ungar [20], Bhasker and Sahni [4], and Koźmiński and Kinnen [12] independently gave equivalent characterizations of the rectangular graphs. Eppstein et al. [8] characterized the area-universal rectangular duals. However, despite an active interest in sliceable rectangular duals, a full characterization of sliceable graphs is still lacking. The currently best result by Yeap and Sarrafzadeh [22] from 1995 proves that all rectangular graphs without a separating 4-cycle are sliceable. Dasgupta and Sur-Kolay [7] modified the approach of Yeap and Sarrafzadeh and claimed two sufficient conditions for sliceability. However, Mumford [15] discovered a critical flaw that invalidates their results.[3]

Related work. Rectangular duals have been studied extensively by the VLSI community. Sliceable layouts more easily facilitate certain steps in the layout process [16]. For instance, the problem of minimizing the perimeter or area of modules in a rectangular layout according to a given measure can be solved in polynomial time for sliceable layouts, but is NP-complete in general [17]. Several papers focus on restricted classes of sliceable and nonsliceable graphs [5,18].

Rectangular duals are also studied in the context of rectangular cartograms, which represent geographic regions by rectangles. The positioning and adjacencies of these rectangles are chosen to suggest

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[3] Confirmed by Dasgupta and Sur-Kolay, personal communication, 2011-2013.

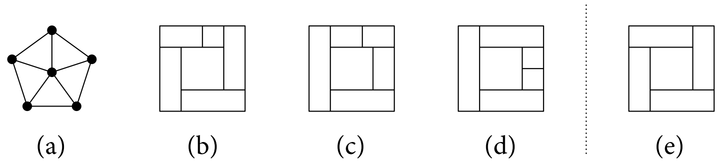


Fig. 1. A graph (a) with rectangular duals (b)-(d) and a rectangular dual of a different graph (e): (b) is not sliceable and not area-universal, (c) is sliceable and not area-universal, (d) is sliceable and area-universal, and (e) is area-universal and not sliceable.

their geographic locations and their areas correspond to the numeric values that the cartogram communicates. Van Kreveld and Speckmann [13] gave the first algorithms to compute rectangular cartograms. Eppstein et al. [8] present a numerical algorithm for area-universal rectangular duals which computes a cartogram with approximately the correct areas. For sliceable rectangular duals one can easily compute a combinatorially equivalent rectangular dual with exactly the specified area assignment, if such a rectangular dual exists. Several papers consider *rectilinear duals*: a generalization of rectangular duals which uses simple (axis-aligned) rectilinear polygons instead of rectangles. Every triangulated graph has a rectilinear dual where every polygon has eight sides, and eight sides are sometimes necessary [10,14,23]. A series of papers studies the question of how many sides are required to respect all adjacencies and area requirements in general. De Berg, Mumford and Speckmann [3] gave the first bound by showing that forty sides per polygon is always sufficient. After several intermediate results, Alam et al. [2] finally closed the gap by proving that eight sides per polygon is always sufficient.

Sliceable rectangular duals are also called *guillotine partitions* or *guillotine layouts*. In this context a different notion of equivalence is used, which is not based on a dual graph. Specifically, two guillotine partitions are equivalent if they have the same *structure tree* [19]. Yao et al. [21] show that the asymptotic number of guillotine partitions is the nth Schröder number. Ackerman et al. [1] derive the asymptotic number of guillotine partitions in higher dimensions.

Results and organization. It is comparatively easy to see that the class of sliceable graphs is not closed under minors. Hence we need to explore different approaches to characterize them. In Section 3 we introduce a recursively defined class of graphs, so-called *rotating pyramids*, which contain exactly one separating 4-cycle. We conjecture that configurations of rotating pyramids determine if a graph is sliceable. We verify our conjecture for the graphs that contain exactly one separating 4-cycle. The non-sliceable graphs in this class are exactly the graphs that reduce to *rotating windmills*: rotating pyramids with a specific corner assignment. In Section 4 we prove that rotating windmills are not sliceable and in Section 5 we argue that all other graphs with exactly one separating 4-cycle are sliceable.

2 Preliminaries

An *extended graph* $E(\mathcal{G})$ of a plane graph $\mathcal{G}$ is an extension of $\mathcal{G}$ with four vertices in such a way that the four vertices form the outer face of $E(\mathcal{G})$. These vertices are labeled $t(\mathcal{G})$, $r(\mathcal{G})$, $b(\mathcal{G})$ and $l(\mathcal{G})$ in clockwise order and are called the *poles* of $E(\mathcal{G})$. The vertices of the original graph $\mathcal{G}$ are called the *interior* vertices. Since choosing the extended graph fixes the vertices that correspond to the four corners (and hence the vertices along the four sides) of the rectangular dual, extended graphs are also called *corner assignments* (Fig. 2).

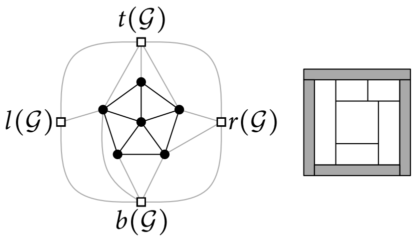


Fig. 2. An extended graph $E(\mathcal{G})$ and the corresponding rectangular dual.

A *separating k-cycle* of an extended graph $E(\mathcal{G})$ is a k-cycle with vertices both inside and outside the cycle. A separating k-cycle is *maximal* if its interior is not contained in any other separating k-cycle of $E(\mathcal{G})$. A *triangle* is a 3-cycle. The *outer cycle* of a plane graph is the cycle formed by the edges incident to the unbounded face. An *irreducible triangulation* is a plane graph without separating triangles and where all interior faces are triangles and the outer face is a quadrangle. A graph $\mathcal{G}$ has a rectangular dual if and only if $\mathcal{G}$ has an extended graph which is an irreducible triangulation [4,12,20].

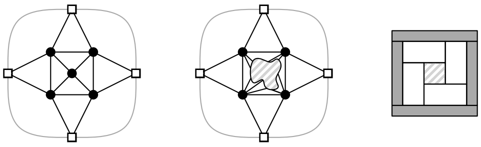


Fig. 3. The windmill, the generalized windmill (the hatched shape is an arbitrary graph), and a rectangular dual of the generalized windmill.

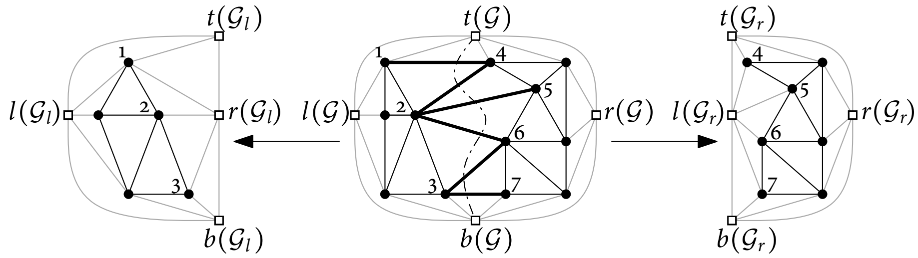
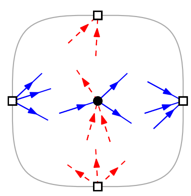


Fig. 4. An extended graph $E(\mathcal{G})$ with a vertical slice indicated by a dash-dotted line and the corresponding $E(\mathcal{G}_\ell)$ and $E(\mathcal{G}_r)$. The edges of the cut-set are bold. The boundary paths are $t(\mathcal{G}), 1, 2, 3, b(\mathcal{G})$ and $t(\mathcal{G}), 4, 5, 6, 7, b(\mathcal{G})$. Both boundary paths are chordless. Figure based on [22].

Sliceable graphs. A rectangular partition is *sliceable* if it can be recursively subdivided along horizontal or vertical lines. An extended graph $E(\mathcal{G})$ is sliceable if and only if it has a sliceable rectangular dual. A graph $\mathcal{G}$ is sliceable if and only if it has a sliceable extended graph. Since a graph has only polynomially many corner assignments, we consider only extended graphs from now on. The smallest nonsliceable extended graph is the *windmill* depicted in Fig. 3. This extended graph can be generalized to a *generalized windmill* by replacing the center vertex with an arbitrary graph. All generalized windmills are nonsliceable.

A *cut* is a partition of the vertices of a graph in two disjoint subsets. The *cut-set* of the cut is the set of edges whose endpoints are in different subsets of the partition. A cut of $\mathcal{G}$ with cut-set S is *vertical* if the edges dual to S form a path from an interior face incident to $t(\mathcal{G})$ to an interior face incident to $b(\mathcal{G})$. Order the edges in the cut-set $e_1, \ldots, e_m$, according to the order in which they are traversed by the dual path. The *left vertex* of e_i is the endpoint of e_i that is in the same component as $l(\mathcal{G})$ in the graph obtained by deleting $t(\mathcal{G})$, $b(\mathcal{G})$, and S from $E(\mathcal{G})$. The *right vertex* is defined analogously. Let the *left boundary walk* $W_\ell = t(\mathcal{G}), u_1, \ldots, u_\ell, b(\mathcal{G})$ be the sequence of left endpoints of $e_1, \ldots, e_m$ (removing consecutive duplicates), and let the *right boundary walk* $W_r = t(\mathcal{G}), v_1, \ldots, v_r, b(\mathcal{G})$ be the sequence of right endpoints of $e_1, \ldots, e_m$ (removing consecutive duplicates). A walk is a path if it visits every vertex at most once. A path $v_1, \ldots, v_k$ is *chordless* if and only if v_i and v_j are not adjacent for each $1 \leq i < j - 1 \leq k$. A vertical cut is a *vertical slice* if its boundary walks are chordless paths (Fig. 4). A vertical slice divides $\mathcal{G}$ into $\mathcal{G}_l$ and $\mathcal{G}_r$. Horizontal cuts, top and bottom boundary walks and horizontal slices are defined analogously.

Regular edge labelings. The equivalence classes of the rectangular duals of an irreducible triangulation $E(\mathcal{G})$ correspond one-to-one to the *regular edge labelings of* $E(\mathcal{G})$. A regular edge labeling of an extended graph $E(\mathcal{G})$ is a partition of the interior edges of $E(\mathcal{G})$ into two subsets of red (dashed) and blue (solid) directed edges such that: *(i)* around each inner vertex in clockwise order we have four contiguous nonempty sets of incoming blue edges, outgoing red edges, outgoing blue edges, and incoming red edges and; *(ii)* $l(\mathcal{G})$ has only outgoing blue edges, $t(\mathcal{G})$ has only incoming red edges, $r(\mathcal{G})$ has only incoming blue edges and $b(\mathcal{G})$ has only outgoing red edges.



A regular edge labeling is *sliceable* if its corresponding rectangular dual is sliceable. One can find a regular edge labeling and construct the corresponding rectangular dual in linear time [11]. A *regular edge coloring* is a regular edge labeling, without the edge directions. A regular edge coloring uniquely determines a regular edge labeling [9, Proposition 2]. A *monochromatic triangle* is a triangle where all edges have the same color. A regular edge labeling (of an irreducible triangulation) induces no monochromatic triangles [9, Lemma 1].

Let $\mathcal{R}$ be a rectangular dual of $E(\mathcal{G})$ and let $\mathcal{L}$ be the regular edge labeling that corresponds to $\mathcal{R}$. Any vertical slice in $\mathcal{R}$ has a blue cut-set and red boundary paths in $\mathcal{L}$. Any horizontal slice in $\mathcal{R}$ has a red cut-set and blue boundary paths (see Fig. 5). A slice is a *first slice* of $E(\mathcal{G})$ if it starts and ends at poles of $E(\mathcal{G})$. Slice a is the only first slice in Fig. 5.

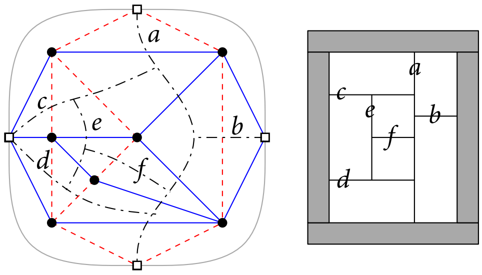


Fig. 5. A regular edge labeling and corresponding rectangular dual. Letters indicate the slices.

k-pyramid extended graphs. A *pyramid* is a 4-cycle with exactly one vertex in its interior. A *k-pyramid extended graph* is an irreducible triangulation $E(\mathcal{G})$ such that $\mathcal{G}$ has no cut-vertices, $\mathcal{G}$ has exactly k separating 4-cycles, and all separating 4-cycles in $E(\mathcal{G})$ are pyramids. We argue that it is sufficient for our investigation of sliceability to consider only k-pyramid extended graphs with $k \geq 1$. Firstly, we may assume $\mathcal{G}$ has no cut-vertex (all omitted proofs are in the full version of the paper):

Lemma 1. *Let $E(\mathcal{G})$ be an extended graph such that $\mathcal{G}$ has a cut-vertex v. Then v is adjacent to two opposite poles, say $t(\mathcal{G})$ and $b(\mathcal{G})$. Slice immediately left and immediately right of v. Then $E(\mathcal{G})$ is sliceable if and only if the three extended graphs that result from the two slices are sliceable.*

Secondly, Mumford [15] showed that it is sufficient to consider extended graphs $E(\mathcal{G})$ such that all separating 4-cycles in $\mathcal{G}$ are pyramids. Her proof directly extends to separating 4-cycles in $E(\mathcal{G})$ instead of $\mathcal{G}$, which immediately proves that generalized windmills (Fig. 3) are nonsliceable. Finally, 0-pyramid extended graphs are always sliceable [22].

Yeap and Sarrafzadeh's algorithm. In Section 5, we explicitly construct slices in a manner which is based on the algorithm by Yeap and Sarrafzadeh [22]. In Theorem 1 below we give a stronger version of their result and also add a missing case which was overlooked in their original analysis. A cycle C in $E(\mathcal{G})$ splits the plane into two parts: a bounded region and an unbounded region. We say that vertices in the bounded region including C are *enclosed* by C.

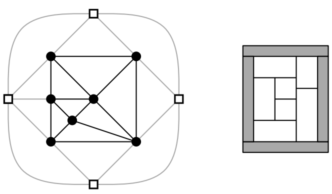
Theorem 1. *Let $E(\mathcal{G})$ be a k-pyramid extended graph ($k \geq 0$). Then there exists a vertical cut S such that (i) the left boundary walk P_ℓ of S is a chordless path that contains only vertices with distance 2 to $r(\mathcal{G})$ in $E(\mathcal{G}) \setminus \{t(\mathcal{G}), l(\mathcal{G}), b(\mathcal{G})\}$ and (ii) if the cycle $C_r := \langle r(\mathcal{G}), P_\ell, r(\mathcal{G})\rangle$ does not enclose a pyramid, then S is a vertical slice. Analogous statements hold for $t(\mathcal{G})$, $l(\mathcal{G})$ and $b(\mathcal{G})$. Consequently, $E(\mathcal{G})$ is sliceable if $k = 0$.*

The following corollary of Lemma 1 gives a final simplification of our problem.

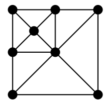
Lemma 2. *Let $E(\mathcal{G})$ be an extended graph with pole p such that p has only one neighbour v in $\mathcal{G}$. Let $E(\mathcal{G}')$ be the extended graph obtained by deleting v from $\mathcal{G}$ and connecting the neighbours of v in $\mathcal{G}$ to p. Then $E(\mathcal{G})$ is sliceable if and only if $E(\mathcal{G}')$ is sliceable.*

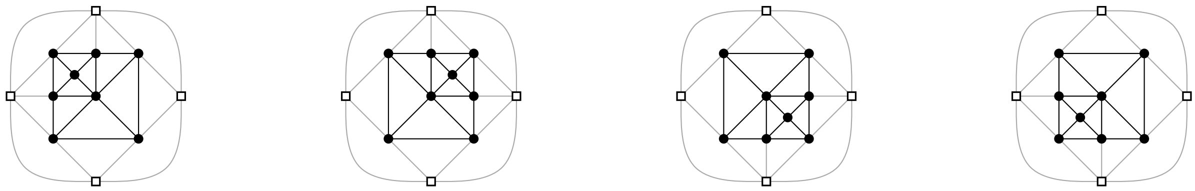
Exhaustively applying Lemma 2 to an extended graph $E(\mathcal{G})$ *reduces* $E(\mathcal{G})$ to an extended graph $E(\mathcal{G}')$. We say that $E(\mathcal{G}')$ is *reduced*. The extended graphs $E(\mathcal{G}_\ell)$ and $E(\mathcal{G}_r)$ resulting from a slice in $E(\mathcal{G})$ might not be reduced even if $E(\mathcal{G})$ is. In this sense, Lemma 2 is different from Lemma 1 and Mumford's observation. In the following we focus on the 1-pyramid extended graphs, among which are both sliceable and nonsliceable extended graphs. The smallest sliceable one is shown on the right. The smallest nonsliceable one is the windmill in Fig. 3.

3 Rotating pyramids and windmills



The graph on the right is the *big pyramid* graph. *Rotating windmills* are recursively defined as follows. The windmill (see Fig. 3) is a rotating windmill. Furthermore, the extended graphs depicted in Fig. 6 are *base rotating windmills*: they are four corner assignments of the big pyramid graph. If $E(\mathcal{G})$ is a rotating windmill other than the windmill, then we can construct another rotating windmill by replacing the pyramid in $E(\mathcal{G})$ with a big pyramid using one of three construction steps, labeled ↑, ↰ and ↱, each depicted in Fig. 7.





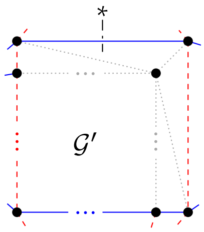
(a) ◰ (b) ◳ (c) ◲ (d) ◱

Fig. 6. The four base rotating windmills.

Intuitively, ↑ extends the rotating windmill in the same direction as the previous extension, ↰ rotates the direction 90° counterclockwise and ↱ rotates the direction 90° clockwise. Note that the construction steps are not allowed to perform a rotation of 180°. We can uniquely identify a rotating windmill by its *construction sequence*. The construction sequence of the windmill is ⊠. The construction sequences of the base rotating windmills are ◰, ◳, ◲ and ◱. If we apply a construction step $s_{k+1} \in \{$↑, ↰, ↱$\}$ to a rotating windmill $bs_1 \cdots s_k$ where $k \geq 0$, $b \in \{$◰, ◳, ◲, ◱$\}$, and $s_1, \ldots, s_k \in \{$↑, ↰, ↱$\}$, then the resulting rotating windmill has construction sequence $bs_1 \cdots s_k s_{k+1}$. Fig. 8 shows three examples. If $E(\mathcal{G})$ is a rotating windmill, then we call $\mathcal{G}$ a *rotating pyramid*. For a given rotating pyramid $\mathcal{G}$, which is not the pyramid, the *inner graph* $\mathcal{G}'$ is defined as the largest strict subgraph of $\mathcal{G}$ such that $\mathcal{G}'$ is a rotating pyramid.

Drawing conventions. We draw the edges of the outer cycle of a rotating pyramid $\mathcal{G}$ as a square. The *top side* of $\mathcal{G}$ is the path from the topleft vertex of $\mathcal{G}$ to the topright vertex (including both). The definitions of *right side*, *bottom side* and *left side* are analogous. Every rotating windmill has two consecutive sides with exactly two vertices, and two consecutive sides with at least two vertices.

Consider the graph $\mathcal{G}$ on the right. The partially drawn edges incident to the vertices on the outer cycle of $\mathcal{G}$ represent connections to vertices not shown in the figure. The inner graph $\mathcal{G}'$ of $\mathcal{G}$ is represented by only its outer cycle; its interior vertices (if any) are not shown. The lines along the top, right, bottom and left sides of $\mathcal{G}'$ contain the $\cdots$-symbol in their center to indicate that there may be zero or more extra vertices on the side. The edges whose color is not uniquely determined are gray (dotted). The start of a slice is denoted with $*$, and the end of a slice is denoted with $\times$ (not shown). Every vertex on the top side of $\mathcal{G}'$ is connected to the topleft vertex in the figure, and every vertex on the right side of $\mathcal{G}'$ is connected to the bottomright vertex in the figure. Since $\mathcal{G}'$ is a rotating pyramid, a maximum of two sides of $\mathcal{G}'$ (and they must be consecutive) can have extra vertices.



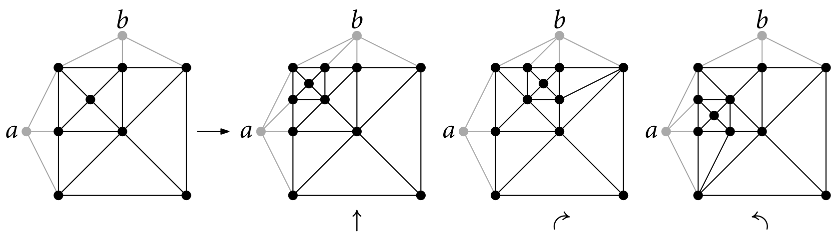


Fig. 7. On the left: the big pyramid in a rotating windmill, along with two of its neighbors in gray. On the right: the results of applying the three construction steps.

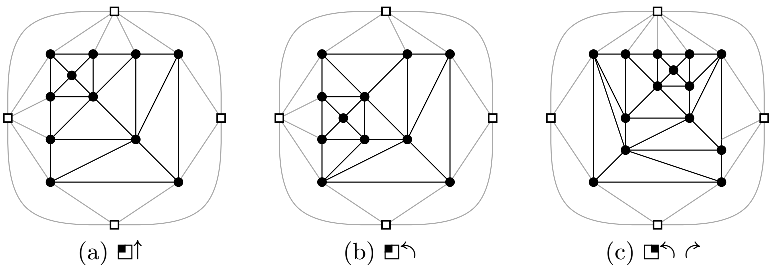


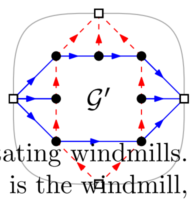
Fig. 8. Three rotating windmills.

4 Rotating windmills are not sliceable

Before we can prove the main result of this section, we need the following lemma:

Lemma 3. *Let $E(\mathcal{G})$ be an extended graph with a sliceable regular edge labeling $\mathcal{L}$. Let $\mathcal{G}'$ be a subgraph of $\mathcal{G}$ such that the outer cycle of $\mathcal{G}'$ under $\mathcal{L}$ has in clockwise order (i) a nonempty path of red edges followed by a nonempty path of blue edges oriented clockwise, and (ii) a nonempty path of red edges followed by a nonempty path of blue edges oriented counterclockwise. Let $E(\mathcal{G}')$ be the extended graph with labeling $\mathcal{L}'$ induced by coloring the edges of $\mathcal{G}'$ according to $\mathcal{L}$. The labeling $\mathcal{L}'$ is a sliceable labeling for $E(\mathcal{G}')$.*

Proof. The figure shows an example of the labeling of the outer cycle of $\mathcal{G}'$, the induced corner assignment $E(\mathcal{G}')$ and the labeling of $E(\mathcal{G}')$. Observe that the slices in $\mathcal{L}'$ are exactly the slices in $\mathcal{L}$ that cut through edges of $\mathcal{G}'$. Since $\mathcal{L}$ is a sliceable labeling of $E(\mathcal{G}')$, the labeling $\mathcal{L}'$ must also be sliceable. □



Theorem 2. *Extended graphs that reduce to rotating windmills are not sliceable.*

Proof. Since the reduction operation preserves sliceability, it is sufficient to consider rotating windmills. We will prove the theorem by structural induction on rotating windmills. Our base case is the windmill, which is not sliceable.

Let $E(\mathcal{G})$ be a rotating windmill and assume that all rotating windmills with fewer vertices are nonsliceable. Assume without loss of generality that the construction sequence of $E(\mathcal{G})$ starts with ◪. For the sake of deriving a contradiction, suppose that $E(\mathcal{G})$ is sliceable and consider a sliceable regular edge labeling. We assume wlog that the first slice in $E(\mathcal{G})$ is a vertical slice from $t(\mathcal{G})$ to $b(\mathcal{G})$. We show that any first slice either *(i)* cannot reach $b(\mathcal{G})$ or *(ii)* cuts $E(\mathcal{G})$ in such a way that a smaller graph is forced into a corner assignment that is a rotating windmill. Both cases result in a contradiction.

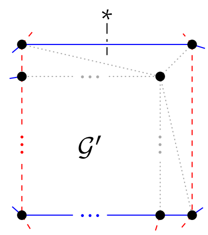


Fig. 9. Graph $\mathcal{G}$.

See Fig. 9. The vertices along the outer cycle of $\mathcal{G}$ are connected to the poles in $E(\mathcal{G})$. Since $t(\mathcal{G})$ has only incoming red edges, the edges along the top side of $\mathcal{G}$ must be blue. A similar reasoning forces the coloring of all edges on the outer cycle of $\mathcal{G}$. Let $\mathcal{G}'$ be the inner graph of $\mathcal{G}$. We distinguish four cases.

Case 1. The first slice does not cut through an edge in the top side of $\mathcal{G}'$, see Fig. 10. As noted previously, the colors of the edges along the outer cycle of $\mathcal{G}$ are forced by the corner assignment. The choice of the slice forces the colors of all dotted edges in Fig. 9. The induced corner assignment of $\mathcal{G}'$ is a rotating windmill $E(\mathcal{G}')$ which is smaller than $E(\mathcal{G})$. By the induction hypothesis, $E(\mathcal{G}')$ is not sliceable. Hence, $E(\mathcal{G})$ is also not sliceable. Contradiction.

Case 2. The top side of $\mathcal{G}'$ has at least two edges and the first slice cuts through the rightmost one, as depicted in Fig. 11(a). The induced corner assignment of $\mathcal{G}'$ is not a rotating windmill, so we cannot immediately conclude that $E(\mathcal{G})$ is not sliceable. Let us consider the structure of $\mathcal{G}'$. Note that the top side of $\mathcal{G}'$ has more than two vertices. This means that the construction sequence of $E(\mathcal{G})$ must start with ◪↷.

The slice that enters $\mathcal{G}'$ in Fig. 11(a) continues at the $*$ in Fig. 11(b). Let $\mathcal{G}''$ be the inner graph of $\mathcal{G}'$. Note that the slice must enter $\mathcal{G}''$: if it did not, we

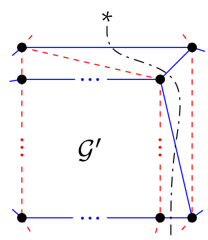
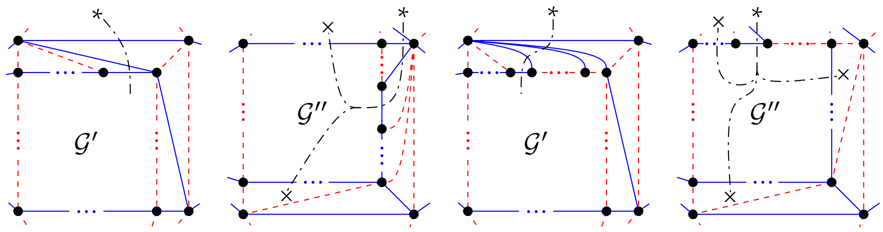


Fig. 10. Case 1.



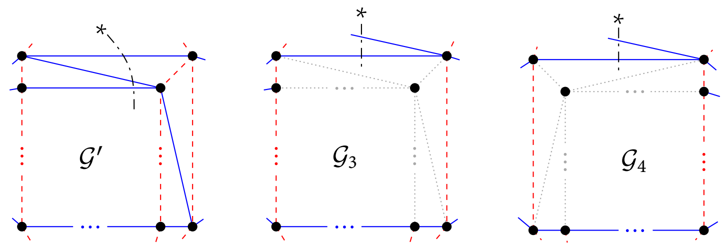
(a) Case 2: $\mathcal{G}$. (b) Case 2: $\mathcal{G}'$. (c) Case 3: $\mathcal{G}$. (d) Case 3: $\mathcal{G}'$.

Fig. 11. (a-b) Graphs $\mathcal{G}$ and $\mathcal{G}'$ in Case 2. (c-d) Graphs $\mathcal{G}$ and $\mathcal{G}'$ in Case 3.

would be in Case 1 again. It follows that the slice must enter $\mathcal{G}''$ through some edge on the right side of $\mathcal{G}''$. This forces the colors of all dotted edges in the figure. The slice cannot leave $\mathcal{G}''$ through an edge on the top or bottom side of $\mathcal{G}''$, since the slice cannot continue to $b(\mathcal{G})$ from there. Since the first slice does not reach $b(\mathcal{G})$, it cannot be the first slice. Contradiction.

Case 3. The top side of $\mathcal{G}'$ has at least two edges and the first slice does not cut through the rightmost one, see Fig. 11(c). Hence, the construction sequence of $E(\mathcal{G})$ must start with ◱↱. The first slice continues at $*$ in Fig. 11(d). Let $\mathcal{G}''$ be the inner graph of $\mathcal{G}'$. All edges in $\mathcal{G}'$ incident to the topright vertex in $\mathcal{G}'$ must be red. This forces the coloring of all remaining edges. So the first slice cannot continue to $b(\mathcal{G})$ after leaving $\mathcal{G}''$: hence it cannot be the first slice. Contradiction.

Case 4. The top side of $\mathcal{G}'$ has exactly one edge e and the first slice cuts through e, see Fig. 12(a). Since $\mathcal{G}'$ has only two vertices on its top side, the construction sequence of $E(\mathcal{G})$ must start with ◲↑ ($\mathcal{G}' = \mathcal{G}_1$) or ◲↰ ($\mathcal{G}' = \mathcal{G}_2$). See Fig. 12(b) for $\mathcal{G}' = \mathcal{G}_1$ and Fig. 12(c) for $\mathcal{G}' = \mathcal{G}_2$. The only difference between $\mathcal{G}_1$ and $\mathcal{G}$ (Fig. 9) is that the topright vertex of $\mathcal{G}_1$ has an extra blue edge. Suppose that $E(\mathcal{G})$ is sliceable for $\mathcal{G}' = \mathcal{G}_1$ (the case $\mathcal{G}' = \mathcal{G}_2$ is similar). Let $\mathcal{L}_\mathcal{G}$ be a sliceable regular edge labeling of $E(\mathcal{G})$ and let $\mathcal{L}_\mathcal{G}[\mathcal{G}_1]$ be the restriction of $\mathcal{L}_\mathcal{G}$ to $\mathcal{G}_1$. All edges along the top side and bottom side of $\mathcal{G}_1$ in $\mathcal{L}_\mathcal{G}[\mathcal{G}_1]$ are blue and all the edges along the left side and right side are red. Let $E(\mathcal{G}_1)$ be the corner assignment of $\mathcal{G}_1$ such that $E(\mathcal{G}_1)$ is a rotating windmill. Coloring the edges of $\mathcal{G}_1$ inside $E(\mathcal{G}_1)$ according to $\mathcal{L}_\mathcal{G}[\mathcal{G}_1]$ yields a sliceable regular edge labeling for $E(\mathcal{G}_1)$ by Lemma 3. But since $E(\mathcal{G}_1)$ is a smaller rotating windmill than $E(\mathcal{G})$, it is not sliceable by the induction hypothesis. Contradiction. □



(a) $\mathcal{G}$. (b) $\mathcal{G}_1$. (c) $\mathcal{G}_2$.

Fig. 12. Case 4: graph $\mathcal{G}$ and two cases for $\mathcal{G}'$: graphs $\mathcal{G}_1$ and $\mathcal{G}_2$.

5 Sliceability of 1-pyramid extended graphs

In this section we prove that all reduced 1-pyramid extended graphs other than rotating windmills are sliceable. Given a 1-pyramid extended graph $E(\mathcal{G})$, let C_p be the cycle defined in Theorem 1 for each pole $p \in \{l(\mathcal{G}), b(\mathcal{G}), r(\mathcal{G}), t(\mathcal{G})\}$.

Lemma 4. *Let $E(\mathcal{G})$ be a reduced 1-pyramid extended graph. Suppose that there exists a slice S that splits $E(\mathcal{G})$ into $E(\mathcal{G}_\ell)$ and $E(\mathcal{G}_r)$, such that $E(\mathcal{G}_\ell)$ (or $E(\mathcal{G}_r)$) can be reduced to a rotating windmill. Then we can construct a reduced 1-pyramid extended graph $E(\mathcal{G}')$ such that $E(\mathcal{G}')$ is not a rotating windmill, $\mathcal{G}'$ is a strict subgraph of $\mathcal{G}$ and $E(\mathcal{G})$ is sliceable if $E(\mathcal{G}')$ is sliceable.*

Proof (sketch). One can argue that that $E(\mathcal{G}_\ell)$ (or $E(\mathcal{G}_r)$) is already be a rotating windmill and then locally change S to a slice that does not induce a rotating windmill in the left or right graph. □

Lemma 5. *Let $E(\mathcal{G})$ be a reduced 1-pyramid extended graph. If C_p encloses the pyramid of $\mathcal{G}$ for all poles p, then $E(\mathcal{G})$ is the windmill.*

Proof (sketch). First, the proof argues that since C_ℓ and C_r both enclose the pyramid, there is a cycle C formed by vertices L from P_ℓ and R from P_r that encloses the pyramid. Since $l(\mathcal{G})$ ($r(\mathcal{G})$) has a path of length two to every vertex on P_ℓ (P_r), one can show that every vertex in $L \setminus R$ must have an edge to a vertex in R. It follows that C is a 4-cycle and since it encloses the pyramid in the 1-pyramid extended graph $E(\mathcal{G})$, the pyramid must be equal to C. Hence, P_ℓ and P_r contain an edge of the outer cycle of the pyramid. By a symmetric argument, P_t and P_b contain an edge of the outer cycle of the pyramid. Next, one can show that every edge of the outer cycle of the pyramid is on a different boundary path. Finally, we can use this property to show that every vertex on the outer cycle of the pyramid is connected to two adjacent poles. It follows that $E(\mathcal{G})$ contains the edges of the windmill. Since $E(\mathcal{G})$ is an irreducible triangulation, no other vertices can be present, which concludes the proof. □

The following algorithm computes a sliceable labeling of a reduced 1-pyramid extended graph that is not a rotating windmill.

1. If $\mathcal{G}$ is a single vertex, we are done.
2. Since $E(\mathcal{G})$ is not a rotating windmill, by Lemma 5, there is a pole p for which C_p does not enclose the pyramid. Use Theorem 1 to compute a slice from p. This slice splits $E(\mathcal{G})$ into $E(\mathcal{G}_\ell)$ and $E(\mathcal{G}_r)$. One of these, say $\mathcal{G}_\ell$, contains the pyramid of $\mathcal{G}$. By Theorem 1, $E(\mathcal{G}_r)$ is sliceable. If $E(\mathcal{G}_\ell)$ can be reduced to a rotating windmill, then proceed to Step 1 with the reduced extended graph $E(\mathcal{G}')$ guaranteed by Lemma 4. Otherwise, reduce $E(\mathcal{G}_\ell)$ using Lemma 2 and go to Step 1 with $E(\mathcal{G}_\ell)$.

The algorithm maintains the invariant that $E(\mathcal{G})$ is a reduced 1-pyramid extended graph that is not a rotating windmill at line 1. Combined with Theorem 2, this concludes the proof of our main result:

Theorem 3. *A 1-pyramid extended graph is sliceable if and only if it cannot be reduced to a rotating windmill.*

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A Rectification of Dasgupta and Sur-Kolay's paper [7]

We first introduce the terminology used by Dasgupta and Sur-Kolay. A *rectangular graph* is a graph that admits a rectangular dual. A cycle is *complex* if it has at least one vertex in its interior. A vertex is a *corner* if it is adjacent to at least two poles. A corner is *nondistinct* if it is adjacent to at least three poles. Their claim is:

Claim ([7]). A rectangular graph $\mathcal{G}$ with n vertices, $n > 4$, is sliceable if it satisfies either of the following two conditions:

1. its outermost cycle is the only complex 4-cycle in $\mathcal{G}$ and at least one of its four vertices is a nondistinct corner;
2. all the complex 4-cycles of $\mathcal{G}$ are maximal.

Note that if $n > 4$, no corner is adjacent to all four poles, so in the context of the claim, nondistinct corners are adjacent to exactly three poles. As originally observed by Mumford [15], the claim is incorrect. Although their proof is similar to the proof given by Yeap and Sarrafzadeh [22] (construct a proper slice, recurse on $\mathcal{G}_l$ and $\mathcal{G}_r$), they fail to show that $\mathcal{G}_l$ and $\mathcal{G}_r$ again satisfy the constraints imposed by the theorem. There exist graphs which are not sliceable for any corner assignment, yet contain only maximal separating 4-cycles [15]. This contradicts the claim. We can prove the existence of such graphs as follows.[4]

Lemma 6. *Let $\mathcal{L}$ be a sliceable regular edge labeling of $E(\mathcal{G})$ and let P be a pyramid in $E(\mathcal{G})$. There must be at least one vertex on the outer cycle of P whose three incident edges in P have the same color.*

Proof. Let P be a pyramid in an extended graph $E(\mathcal{G})$. After removing duplicates with respect to rotation and symmetry, there are only four possible regular edge colorings for P. They are depicted in Fig. 13. Note that (d) is the only nonsliceable dual. In all sliceable duals, the bottom left vertex has three incident edges in P, all of which have the same color. □

Lemma 7. *Let $E(\mathcal{G})$ be a sliceable extended graph. Every pyramid in $E(\mathcal{G})$ must have a vertex with degree 6.*

Proof. It follows from Lemma 6 that every pyramid P must have a vertex v whose three incident edges in P have the same color. Since v is an interior vertex of $E(\mathcal{G})$, it must have four nonempty contiguous sets of incoming blue edges, outgoing red edges, outgoing blue edges and incoming red edges. It follows that the degree of v is at least six. □

One such graph is depicted in Fig. 14. Note that there are five pyramids on the outer cycle of this graph. Suppose for the sake of deriving a contradiction that the graph is sliceable for some corner assignment $E(\mathcal{G})$. It follows from Lemma 7 that every pyramid in $E(\mathcal{G})$ must have a vertex with degree at least six. Every pyramid P in $\mathcal{G}$ has two vertices on the outer cycle with degree 4, and two vertices not on the outer cycle with degree 5. The only way P can have a vertex with degree 6 in $E(\mathcal{G})$ is if a vertex of P on the outer cycle is connected to two poles. But since there are five pyramids and only four poles, this cannot be done. It follows that $\mathcal{G}$ is not sliceable.

[4] The following two lemmas, without proof, can also be found in an unpublished manuscript by Mumford and Speckmann, 2007.

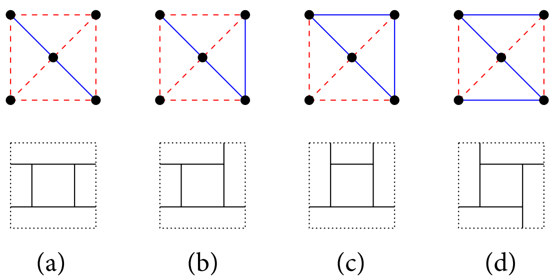

Fig. 13. The four equivalence classes of the regular edge colorings of a pyramid.

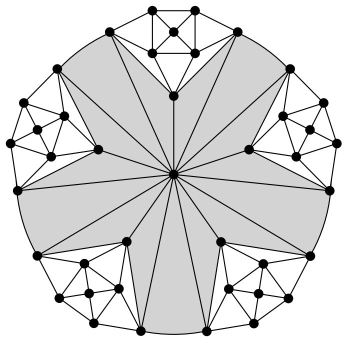

Fig. 14. A graph which is not sliceable for any corner assignment. Original by Mumford [15].

B Omitted proofs

Lemma 1. *Let $E(\mathcal{G})$ be an extended graph such that $\mathcal{G}$ has a cut-vertex v. Then v is adjacent to two opposite poles, say $t(\mathcal{G})$ and $b(\mathcal{G})$. Slice immediately left and immediately right of v. Then $E(\mathcal{G})$ is sliceable if and only if the three extended graphs that result from the two slices are sliceable.*

Proof. In any rectangular dual of $E(\mathcal{G})$, the rectangle R corresponding to v will touch the top side and the bottom side of the dual: R divides the rectangular dual into two smaller rectangular duals. Hence, $E(\mathcal{G})$ is sliceable if and only if the two smaller extended graphs are sliceable (the extended graph with only v is trivially sliceable). □

Theorem 1. *Let $E(\mathcal{G})$ be a k-pyramid extended graph ($k \geq 0$). Then there exists a vertical cut S such that (i) the left boundary walk P_ℓ of S is a chordless path that contains only vertices with distance 2 to $r(\mathcal{G})$ in $E(\mathcal{G}) \setminus \{t(\mathcal{G}), l(\mathcal{G}), b(\mathcal{G})\}$ and (ii) if the cycle $C_r := \langle r(\mathcal{G}), P_\ell, r(\mathcal{G})\rangle$ does not enclose a pyramid, then S is a vertical slice. Analogous statements hold for $t(\mathcal{G})$, $l(\mathcal{G})$ and $b(\mathcal{G})$. Consequently, $E(\mathcal{G})$ is sliceable if $k = 0$.*

Proof. If G contains a vertex v that is adjacent to three poles, say $b(\mathcal{G})$, $r(\mathcal{G})$ and $t(\mathcal{G})$, then the edges incident to v in G form a valid slice. Since v is connected to three poles, the neighbours of v in $\mathcal{G}$ are not adjacent to $r(\mathcal{G})$. Hence, these neighbours have distance exactly two to $r(\mathcal{G})$ and *(ii)* is satisfied. If G contains a cut-vertex v, then we use Lemma 1.

It remains to consider extended graphs $E(\mathcal{G})$ for which $\mathcal{G}$ has no cut-vertex and no vertex adjacent to three poles. Fix any pole, say $r(\mathcal{G})$, and perform a breath-first search on $E(\mathcal{G})$, starting at $r(\mathcal{G})$, labeling every vertex with its distance to $r(\mathcal{G})$. This partitions G into *levels* i of vertices with equal distance i to $r(\mathcal{G})$. Let $E(i, j)$ be the set of edges in $\mathcal{G}$ that connect vertices from level i to vertices from level j. Consider the cut-set $E(1, 2)$. Order the edges in this cut-set $e_1, \ldots, e_m$ according to the order in which they are traversed by the directed path along the faces of $E(\mathcal{G})$ from $t(\mathcal{G})$ to $b(\mathcal{G})$. Let $W_\ell = u_1, \ldots, u_\ell$ be vertices of level 2 in $e_1, \ldots, e_m$ (removing consecutive duplicates) and let $W_r = v_1, \ldots, v_r$ be the vertices of level 1 in $e_1, \ldots, e_m$ (removing consecutive duplicates). Since all vertices in W_r are connected to $r(\mathcal{G})$ and since $E(\mathcal{G})$ is an irreducible triangulation, W_r is a path. However, the left boundary walk W_ℓ is not necessarily a path, as demonstrated in Fig. 15. The original proof by Yeap and Sarrafzadeh assumes the left boundary walk is in fact always a path.

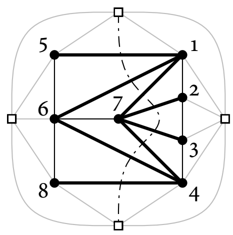

Fig. 15. An extended graph where the left boundary walk is not a path. The right boundary path is $t(\mathcal{G}), 1, 2, 3, 4, b(\mathcal{G})$ and the left boundary walk is $t(\mathcal{G}), 5, 6, 7, 6, 8, b(\mathcal{G})$.

We now show that we can modify the initial cut-set $E(1, 2)$ to obtain a cut-set S_0 that defines two proper boundary paths. If $u_1, \ldots, u_\ell$ are all different, then W_ℓ is a path and we choose $S_0 = E(1, 2)$. Otherwise extend W_ℓ with $u_0 = t(\mathcal{G})$ and $u_{\ell+1} = b(\mathcal{G})$. Now W_ℓ contains a maximal subsequence $u_a, \ldots, u_b$ with $u_a = u_b$ for $0 < a < b < \ell+1$. By definition of W_ℓ, there exist v_c and v_d with $c < d$ and minimal $d - c$ such that the cycle $C = u_a, v_c, v_{c+1}, \ldots, v_{d-1}, v_d, u_b$ contains $u_{a+1}, \ldots, u_{b-1}$. See Fig. 16.

It follows that $u_{a+1}, \ldots, u_{b-1}$ are not connected to any u_i for $i < a$ or $i > b$.

Note that $d - c \geq 3$ since C would be a separating 4-cycle otherwise and that possibly $u_{a+1} = u_{b-1}$. Since $E(\mathcal{G})$ is an irreducible triangulation and by minimality of $d - c$, we know that u_{a+1} is connected to v_c and that u_{b-1} is connected to v_d. Let $P = w_1(= v_c), w_2(= u_{a+1}), \ldots, w_{k-1}(= u_{b-1}), w_k(= v_d)$ be the path formed by the neighbours of u_a from v_c to v_d in clockwise order around u_a. If $k = 3$ then P is chordless since v_c and v_d are not connected. If $k \geq 4$ then suppose that $w_i w_j$ $(i + 1 < j)$ is a chord of P. Note that $i > 1$ or $j < k$ since v_c and v_d are not connected. If $i = 1$ then G contains a separating triangle $u_a v_c w_j u_a$. If $j = k$ then G contains a separating triangle $u_a w_i v_d u_a$. Finally, if $i > 1$ and $j < k$, then $u_a w_i w_j u_a$ is a separating triangle. We conclude that P is always chordless. Now replace the subpath $v_c, \ldots, v_d$ in W_r by P; replace the subwalk $u_a, \ldots, u_b$ in W_ℓ by u_a; and update the cut accordingly. By repeatingly applying this procedure, we eventually obtain a cut S_0 whose two boundary walks are paths. Note that every step removes vertices from the left boundary path, but never adds any. Hence, when this procedure finishes, all vertices on the left boundary path still have distance two to $r(\mathcal{G})$.

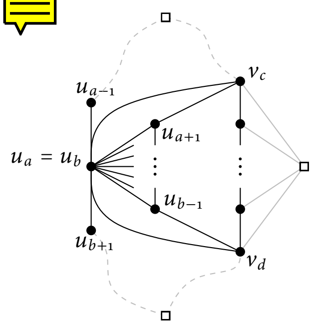

Fig. 16. Iteratively fixing a left boundary walk.

Having fixed the left boundary path, we can now continue with the original proof by Yeap and Sarrafzadeh [22]. Every step in their proof removes vertices from the left boundary path, but never adds any. Hence, when this procedure finishes, all vertices on the left boundary path still have distance two to $r(\mathcal{G})$. Furthermore, in proving that their bypass operation does not introduce chords on the right boundary path, they use only that C_r does not contain a separating 4-cycle. Hence, the algorithm has the properties claimed in Theorem 1. $\square$

Lemma 4. *Let $E(\mathcal{G})$ be a reduced 1-pyramid extended graph. Suppose that there exists a slice S that splits $E(\mathcal{G})$ into $E(\mathcal{G}_\ell)$ and $E(\mathcal{G}_r)$, such that $E(\mathcal{G}_\ell)$ (or $E(\mathcal{G}_r)$) can be reduced to a rotating windmill. Then we can construct a reduced 1-pyramid extended graph $E(\mathcal{G}')$ such that $E(\mathcal{G}')$ is not a rotating windmill, $\mathcal{G}'$ is a strict subgraph of $\mathcal{G}$ and $E(\mathcal{G})$ is sliceable if $E(\mathcal{G}')$ is sliceable.*

Proof. Assume without loss of generality that $\mathcal{G}_\ell$ contains the pyramid and that S is a vertical slice. We begin by showing that $E(\mathcal{G}_\ell)$ is reduced. Suppose that $E(\mathcal{G}_\ell)$ is not reduced and let R be the rotating pyramid of $\mathcal{G}_\ell$. Let V_r be the set of vertices that are removed by reducing $E(\mathcal{G}_\ell)$. If V_r contains a vertex v that is connected by at least two edges to some side of R and by at least two edges to some other side of R, then the reduction step in which v is removed would result in a non-rotating windmill labeling of the outer cycle of R. Furthermore, if a vertex $v \in V_r$ is connected to at least two vertices of a side of R, then that side must be the right side: otherwise v could already have been removed by a reduction step on $E(\mathcal{G})$. It follows that v is not adjacent to $l(\mathcal{G}_\ell)$. Now consider the vertex $v \in V_r$ that is removed by the first reduction step. Since v is adjacent to three poles in $E(\mathcal{G}_\ell)$ but not to $l(\mathcal{G})$, we know that v is adjacent to $r(\mathcal{G}_\ell)$ and $t(\mathcal{G}_\ell)$ and $b(\mathcal{G}_\ell)$. Since S is a vertical slice, we have $t(\mathcal{G}_\ell) = t(\mathcal{G})$ and $b(\mathcal{G}_\ell) = b(\mathcal{G})$. Hence, v is adjacent to $t(\mathcal{G})$ and $b(\mathcal{G})$ in $E(\mathcal{G})$ and since $G_\ell \setminus \{v\}$ and $\mathcal{G}_r$ are nonempty, v is cut-vertex in $E(\mathcal{G})$. This is a contradiction to the assumption that $E(\mathcal{G})$ is a 1-pyramid extended graph; hence $E(\mathcal{G}_\ell)$ is reduced.

Let us consider the graph $E(\mathcal{G}_\ell^+)$ where $\mathcal{G}_\ell^+$ is the union of $\mathcal{G}_\ell$ and the slice (considered as a subgraph) S. Note that any vertical slice in $E(\mathcal{G}_\ell^+)$ is also a vertical slice in $E(\mathcal{G})$. Consider such a vertical slice and suppose that it splits $E(\mathcal{G})$ into a left graph $\mathcal{G}_1$ and a right graph $\mathcal{G}_2$. Suppose that $\mathcal{G}_1$ contains the pyramid (the other case is symmetric). Then by Theorem 1, $E(\mathcal{G}_2)$ is sliceable. Reduce (if necessary) $E(\mathcal{G}_1)$ to obtain a reduced 1-pyramid extended graph $E(\mathcal{G}')$. If $E(\mathcal{G}')$ is sliceable, then so is $E(\mathcal{G})$. Hence, it remains to show that we can always find a slice in $E(\mathcal{G}_\ell^+)$ such that this procedure results in an $E(\mathcal{G}')$ that is not a rotating windmill. We proceed as follows.

Observe that, since $E(\mathcal{G}_\ell)$ is reduced, $\mathcal{G}_\ell$ is a rotating pyramid and the left boundary path of S is the right side of $\mathcal{G}_\ell$. The right boundary path of S must have at least two vertices in $\mathcal{G}_\ell^+$: if it were a single vertex v, then v would be adjacent to two opposite poles in $E(\mathcal{G})$. We perform a case distinction on the construction sequence of the rotating pyramid $\mathcal{G}_\ell$. Fig. 17a shows $E(\mathcal{G}_\ell^+)$ with the slice S. If $\mathcal{G}_\ell$ is a pyramid, then we slice as depicted in Fig. 17b. If the construction sequence of $\mathcal{G}_\ell$ starts with ◰, slice as depicted in Fig. 17c. If the construction sequence starts with ◳, slice as depicted in Fig. 17d. The two other options for the start of the construction sequence are symmetric. In the case of Fig. 17d, some extra care must be taken since there may be two rotating pyramids in $\mathcal{G}_\ell^+$ that contain $\mathcal{X}$. In such a case, Fig. 17e shows how to find a slice if the construction sequence of $\mathcal{G}_\ell$ is ◩↑ and Fig. 17f shows how to find a slice if the construction sequence of $\mathcal{G}_\ell$ starts with ◩↑↑. Note that the case ◩↑↶ is symmetric and for other construction sequences $\mathcal{G}_\ell^+$ contains only one rotating pyramid. Observe that in all cases, the corner assignment $E(\mathcal{G}')$ of the side of the slice that contains the pyramid is either empty (b,e) or reduced and not a rotating windmill (c,d,f). This concludes the proof. ◻

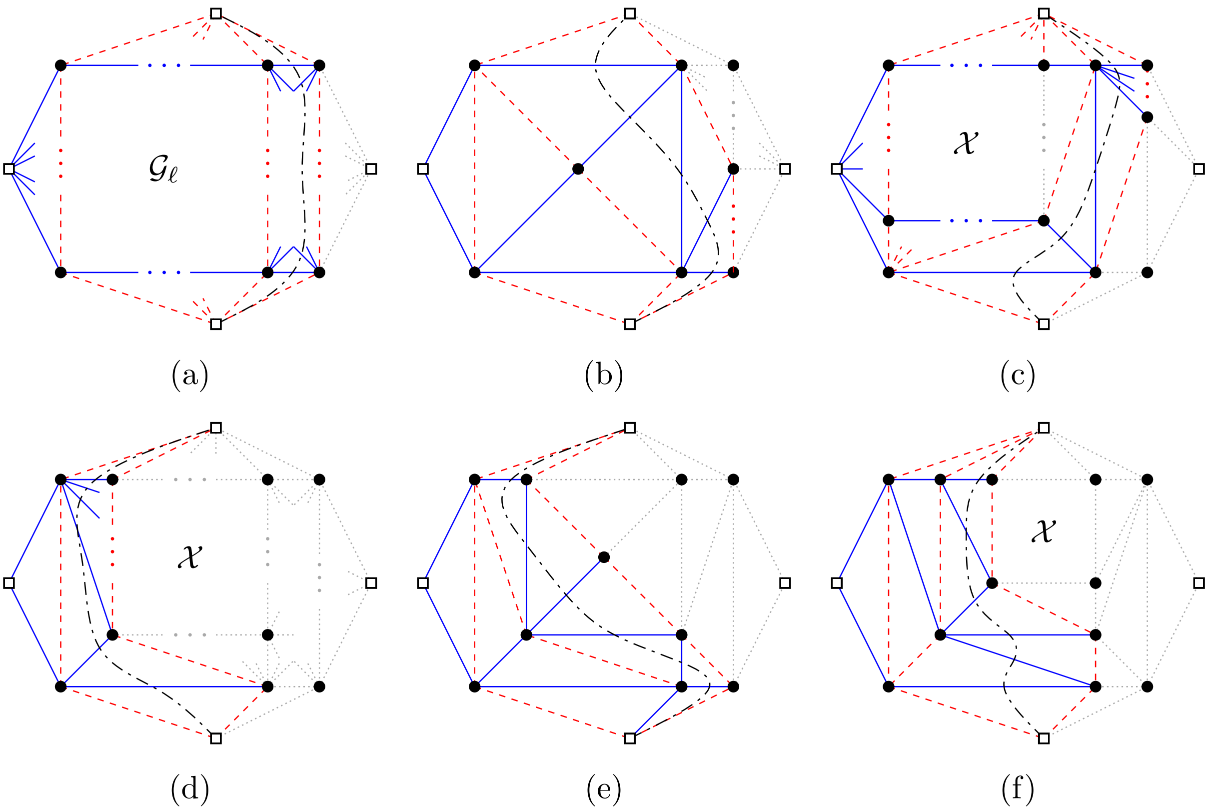

Fig. 17. Finding a vertical slice in $E(\mathcal{G}_\ell^+)$.

Lemma 5. *Let $E(\mathcal{G})$ be a reduced 1-pyramid extended graph. If C_p encloses the pyramid of $\mathcal{G}$ for all poles p, then $E(\mathcal{G})$ is the windmill.*

Proof. We first introduce some terminology. We assume an embedding of $E(\mathcal{G})$ into the plane where the face of size four is the outer face. A cycle C in $E(\mathcal{G})$ thus splits the plane into two parts: a bounded region and an unbounded region. We say that vertices in the bounded region including C are *enclosed* by C. Vertices enclosed by C but not on C are *strictly enclosed* by C. For a pole $p(\mathcal{G})$ of $E(\mathcal{G})$, let $cw(p)$ be the pole adjacent to p in clockwise direction, let $ccw(p)$ be the pole adjacent to p in counterclockwise direction and let $opp(p)$ be the pole opposite p.

Consider the cycles C_ℓ and C_r corresponding to the left and right pole of $E(\mathcal{G})$, respectively. Let P_ℓ and P_r be the restriction of C_ℓ and C_r to $E(\mathcal{G}) \setminus \{l(\mathcal{G}), r(\mathcal{G})\}$. It is easy to verify that $E(\mathcal{G})$ admits an embedding where P_ℓ and P_r are drawn with y-monotone curves. Since P_ℓ and P_r have the same terminal vertices, their union is a set of cycles $C_1, \dots, C_k$ for some $k \geq 1$ ordered in the negative y-direction. Thus, C_1 contains $t(\mathcal{G})$ and C_k contains $b(\mathcal{G})$. A pair of cycles (C_i, C_j) shares zero or more vertices in general and at least one if $|i - j| = 1$.

Each cycle C_i is formed by a subpath L_i of P_ℓ and a subpath R_i of P_r. Recall that L_i and R_i are chordless, all vertices on L_i have distance exactly two to $l(\mathcal{G})$ and all vertices on R_i have distance exactly two to $r(\mathcal{G})$. We call a cycle C_i *important* if C_i is enclosed by C_ℓ and C_r. See Fig. 18.

Proposition 1. *If C_i is important, then for each vertex v on $L_i \setminus R_i$ (i.e., L_i without its terminal vertices), there is a vertex u on R_i such that $\langle \ell(\mathcal{G}), u, v \rangle$ is a path in $E(\mathcal{G})$.*

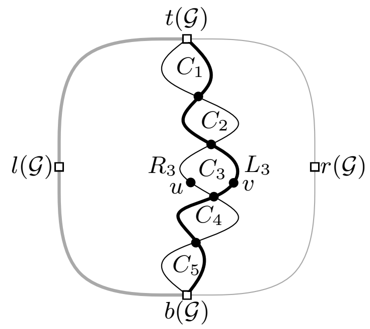

Fig. 18. Cycle C_ℓ is drawn bold. Cycles C_1, C_3 and C_5 are important.

Proof. Since v has distance two to $l(\mathcal{G})$, there must be a path of the form $\langle l(\mathcal{G}), u, v\rangle$ for some u in $E(\mathcal{G})$. Let P^* be the path obtained by replacing L_i by R_i in P_ℓ. Since removing P^* from $E(\mathcal{G})$ disconnects $E(\mathcal{G})$, leaving $l(\mathcal{G})$ and v in different components (since C_i is important), $u \in P^*$. Since all vertices on P_ℓ have distance two to $\ell(\mathcal{G})$, $u \notin P_\ell$. Hence $u \in R_i$, which is what we wanted to prove. □

The analogous statement for $r(\mathcal{G})$ follows by symmetry. Now consider an important cycle C_i. The paths that connect $l(\mathcal{G})$ to L_i and $r(\mathcal{G})$ to R_i induce chords of C_i.

Proposition 2. *The internal faces of the subgraph induced by an important cycle C_i have size at most four and each face of size four has exactly one edge from L_i and exactly one edge from R_i.*

Proof. Consider the subgraph induced by C_i and suppose there is an internal face F with more than four vertices. Then either L_i or R_i must have more than two vertices on F. Without loss of generality, R_i has more than two vertices on F. Let $\langle v_1, v_2, v_3, \ldots\rangle$ be the subpath of R_i on F. But then v_2 is not connected to any vertex of L_i, which is a contradiction to Proposition 1. For the second part of the statement, suppose without loss of generality that F has size four but no edge on R_i. Then F must have three vertices on L_i and we reach a contradiction by the previous argument. Similarly, if F has size four and at least two edges on R_i, then F has at least three vertices on R_i and we reach a contradiction. □

Let P_b and P_t be the restrictions of C_b and C_t to $E(\mathcal{G}) \setminus \{t(\mathcal{G}), b(\mathcal{G})\}$.

Proposition 3. *Every boundary path $P_p \in \{P_\ell, P_r, P_b, P_t\}$ contains exactly one edge e_p that is on the outer cycle of the pyramid. It contains no other vertices of the pyramid. We have $e_\ell \neq e_r$ and $e_t \neq e_b$, but possibly equality for other pairs.*

Proof. Let $C_\boxtimes$ be the cycle C_i (for some i) that contains the pyramid. By the assumption in the lemma, $C_\boxtimes$ is important. Let $F_\boxtimes$ be the face of the subgraph induced by $C_\boxtimes$ that encloses the pyramid in $E(\mathcal{G})$. By Proposition 2, $F_\boxtimes$ has size at most four. Since $E(\mathcal{G})$ has no separating 4-cycle other than the pyramid, $F_\boxtimes$ must be the outer cycle of the pyramid. Hence, by Proposition 2, P_ℓ and P_r each have exactly one edge on the pyramid. The symmetric argumentation for $b(\mathcal{G})$ and $t(\mathcal{G})$ shows that the pyramid must be on P_b and P_t. Since every boundary path P_p is chordless, P_p cannot contain any vertex of the pyramid except the endpoints of e_p (such a vertex would be adjacent to one of the endpoints of e_p). □

Given a path P, let $P[u, v]$ be the path from u to v along P. If u is a terminal vertex of P, then let $N_u(P)$ be the neighbour of u in P. We say that a path is *internal* if it contains exactly one pole. Given poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $q \in \{cw(p), ccw(p)\}$, let $\{s_p(q), t_p(q)\} = e_p$ such that $s_p(q)$ is closer to $q(\mathcal{G})$ along P_p than $t_p(q)$. Note that $s_p(cw(p)) = t_p(ccw(p))$ and vice versa. See Fig. 19(a).

Proposition 4. *Consider a pole $p(\mathcal{G})$ and an internal path $\langle p(\mathcal{G}), u, v\rangle$ for some u and v in $\mathcal{G}$. Let C be a cycle composed of poles and vertices from P_p, $P_{cw(p)}$ and $P_{ccw(p)}$. Let $A = C \cap \{N_{p(\mathcal{G})}(P_{cw(p)}), N_{p(\mathcal{G})}(P_{ccw(p)})\}$. See Fig. 19(b).*

1. *If C strictly encloses v, then C encloses u.*
2. *If C encloses u and $u \notin A$, then C strictly encloses u.*
3. *If $p(\mathcal{G}) \notin C$ and C strictly encloses v, then $u \in A$.*

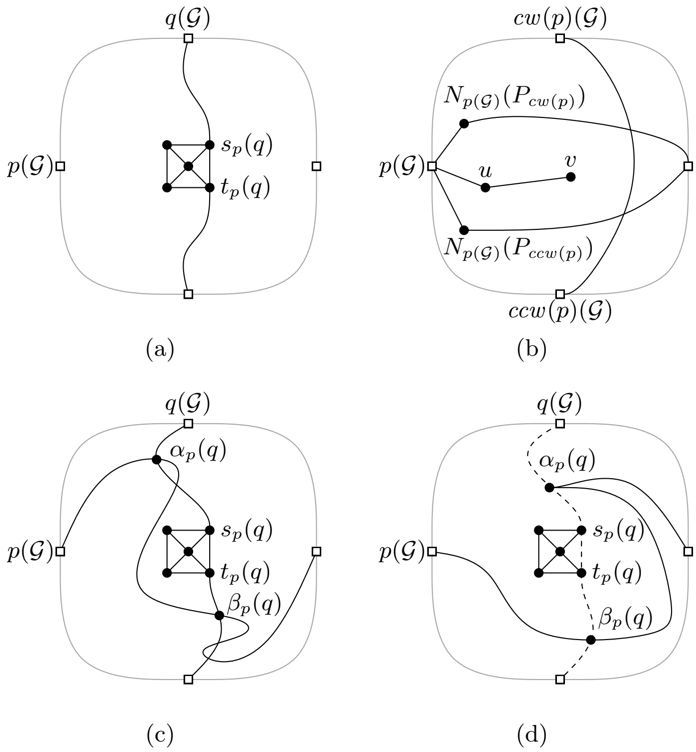

Fig. 19. (a) Definition of $s_p(q)$ and $t_p(q)$. (b) Definitions in Proposition 4. (c) Definition of $\alpha_p(q)$ and $\beta_p(q)$. (d) An extended graph that conforms to Proposition 5.

Proof. The first statement is immediate from the fact that the removal of C disconnects $E(\mathcal{G})$, leaving u and v in different components if u is not enclosed by C. For the second statement, suppose that C encloses u and $u \notin A$. By assumption, u cannot be a pole. Since all vertices on P_p have distance two to $p(\mathcal{G})$, u cannot be on P_p. Since $P_{cw(p)}$ and $P_{ccw(p)}$ are chordless and $u \notin A$, u is not on either $P_{cw(p)}$ or $P_{ccw(p)}$. Hence, C strictly encloses u. For the third statement, suppose that $p(\mathcal{G}) \notin C$ and C strictly encloses v. Then $u \in C$. By the second statement, since C encloses u but not strictly encloses u, we must have $u \in A$. $\square$

Consider poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $q \in \{cw(p), ccw(p)\}$. Let $\alpha_p(q)$ be the first vertex on P_q encountered by traversing P_p from $q(\mathcal{G})$ to $opp(q)(\mathcal{G})$. Let $\beta_p(q)$ be the first vertex on P_q encountered by traversing P_p from $t_p(q)$ to $opp(q)(\mathcal{G})$. Note that $t_p(q) = \beta_p(q)$ if $t_p(q) \in P_q$. See Fig. 19(c).

Proposition 5. *Consider poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $q \in \{cw(p), ccw(p)\}$. Then*

1. $\beta_p(q) \in P_q[p(\mathcal{G}), \alpha_p(q)]$.
2. *The cycle C_q encloses $P_p[q(\mathcal{G}), \beta_p(q)]$.*
3. $P_p[q(\mathcal{G}), \beta_p(q)] \cap P_q \subseteq P_q[\beta_p(q), \alpha_p(q)]$.

Proof. Fig. 19(c) shows an example that cannot occur according to the proposition, whereas Fig. 19(d) shows an example that can occur.

Consider the partition of P_p into maximal subpaths $P_1, \ldots, P_k$ (k even) such that C_q encloses all subpaths with odd index and does not enclose any subpath with even index. Note that P_1 contains $q(\mathcal{G})$ and $\alpha_p(q)$, P_k contains $opp(q)(\mathcal{G})$, each subpath has at least one nonterminal vertex (since P_p is chordless) and every component contains at least one vertex of P_q. Since C_q encloses the pyramid, e_p is completely inside one component, say P_{2x+1} for some $x \geq 0$. This component also includes $\beta_p(q)$. The second statement amounts to proving that $x = 0$. Since the lemma holds trivially when $\alpha_p(q) = \beta_p(q)$, assume for the remainder that $\alpha_p(q) \neq \beta_p(q)$.

To prove the first statement, assume for the sake of obtaining a contradiction that $\alpha_p(q) \in P_q[p(\mathcal{G}), \beta_p(q)]$. Suppose first that some endpoint v_p of e_p is strictly enclosed by C_q. Then v_p is strictly enclosed by the cycle $C = \langle P_p[q(\mathcal{G}), \alpha_p(q)], P_q[\alpha_p(q), opp(p)(\mathcal{G})], q(\mathcal{G})\rangle$[5]. See Fig. 20(a). Note $N_{p(\mathcal{G})}(P_q) \notin C$, since $\alpha_p(q) \in P_p$ and $\alpha_p(q)$ is the vertex closest to $p(\mathcal{G})$ along P_q in $C \cap P_q$. Hence, we get a contradiction from Proposition 4 (3). It follows that e_p is on P_q and hence $\beta_p(q) = t_p(q)$. Then $s_p(q)$ is strictly enclosed by the cycle $C = \langle P_p[q(\mathcal{G}), \alpha_p(q)], P_q[\alpha_p(q), \beta_p(q)], P_p[\beta_p(q), opp(q)(\mathcal{G}), opp(p)(\mathcal{G}), q(\mathcal{G})]\rangle$. By an argument analogous to the one above, we get a contradiction. This proves the first statement.

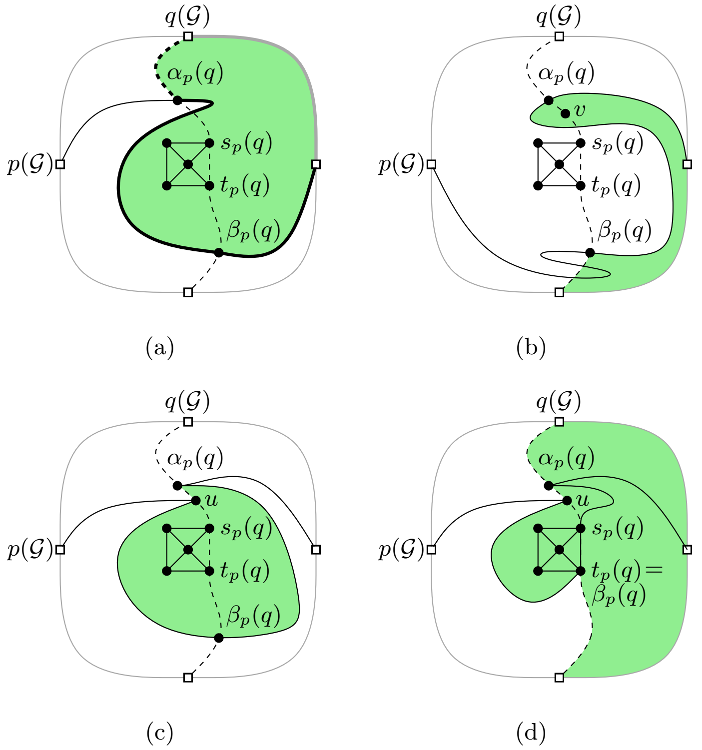

Fig. 20. P_p is dashed and C is shaded in all figures. (a) C strictly encloses a vertex from e_p. (b) C strictly encloses v with P_p dashed. (c) u is on $P_q[p(\mathcal{G}), \beta_p(q)]$ and C strictly encloses some endpoint of e_p. (d) e_p is on P_q.

To prove the second statement, assume for the sake of obtaining a contradiction that $x > 0$ and let v be a nonterminal vertex of P_{2x}. Then the cycle $C = \langle P_p[\beta_p(q), opp(q)(\mathcal{G})], P_q[opp(p)(\mathcal{G}), \beta_p(q)]\rangle$ strictly encloses v. See Fig. 20(b). Note that $p(\mathcal{G}) \notin C$ and $N_{p(\mathcal{G})}(P_q) \notin C$ since all vertices in $P_q[p(\mathcal{G}), \beta_p(q)] \cap C$ are also in P_q. Hence, we get a contradiction from Proposition 4 (3) and thus have $\beta_p(q) \in P_1$. This proves the second statement.

Finally, to prove the third statement, assume for the sake of obtaining a contradiction that there exists a $u \in P_p[q(\mathcal{G}), \beta_p(q)] \cap P_q$ with $u \notin P_q[\beta_p(q), \alpha_p(q)]$. Refer to Fig. 19(d) again. Then either $u \in P_q[p(\mathcal{G}), \beta_p(q)]$ with $u \neq \beta_p(q)$, or $u \in P_q[\alpha_p(q), opp(p)(\mathcal{G})]$ with $u \neq \alpha_p(q)$. The path $P_p[q(\mathcal{G}), \alpha_p(q)]$ divides the cycle C_q into two parts. Since $\alpha_p(q) \neq \beta_p(q)$, $\beta_p(q)$ is not in $P_p[q(\mathcal{G}), \alpha_p(q)]$. Since u must be in the same part as $\beta_p(q)$, by the second statement, $u \in P_q[p(\mathcal{G}), \beta_p(q)]$ with $u \neq \beta_p(q)$. Suppose first that some endpoint v_p of e_p is strictly enclosed by C_q. See Fig. 20(c). Then the cycle $C = \langle P_q[u, \alpha_p(q)], P_p[\alpha_p(q), u]\rangle$ strictly encloses v_p and we get a contradiction from Proposition 4 (3) for the usual reasons. It follows that e_p is on P_q and hence $\beta_p(q) = t_p(q)$. See Fig. 20(d). Then $C = \langle P_p[q(\mathcal{G}), u], P_q[u, \beta_p(q)], P_p[\beta_p(q), opp(q)(\mathcal{G})], opp(p)(\mathcal{G}), q(\mathcal{G})\rangle$ strictly encloses $s_p(q)$ and we get a contradiction from Proposition 4 (3) for the usual reasons. This proves the third statement. □

Proposition 6. *Consider poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $q \in \{cw(p), ccw(p)\}$. Consider an internal path $\langle p(\mathcal{G}), u_p, v_p\rangle$ with $v_p \in P_p[q(\mathcal{G}), \beta_p(q)] \setminus \{\beta_p(q)\}$. Then C_p strictly encloses u_p and C_q encloses u_p.*

[5] This cycle may self-intersect, but this is easily remedied and distracts from the argument.

Proof. Since all neighbours of $p(\mathcal{G})$ must necessarily be in C_p, in particular vertex u_p is strictly enclosed by C_p because u_p is not a pole and does not have distance two to $p(\mathcal{G})$. By Proposition 5 (1), C_q encloses $P_p[q(\mathcal{G}), \beta_p(q)]$ and hence C_q encloses v_p. Suppose for the sake of obtaining a contradiction that C_q does not enclose u_p. See Fig. 21(a). Then $C = \langle P_q[p(\mathcal{G}), \beta_p(q)], P_p[\beta_p(q), opp(q)(\mathcal{G})], p(\mathcal{G})\rangle$ encloses u_p. Since $u_p \notin C_q$ by assumption, by Proposition 4 (2), C strictly encloses u_p. By Proposition 5 (3), $P_q[p(\mathcal{G}), \beta_p(q)]$ has no vertex from $P_p[q(\mathcal{G}), \beta_p(q)] \setminus \{\beta_p(q)\}$. Hence $v_p \notin C$ and since C_q encloses v_p, C does not enclose v_p. But then u_p and v_p cannot be adjacent. Contradiction. Hence, C_q encloses u_p. $\square$

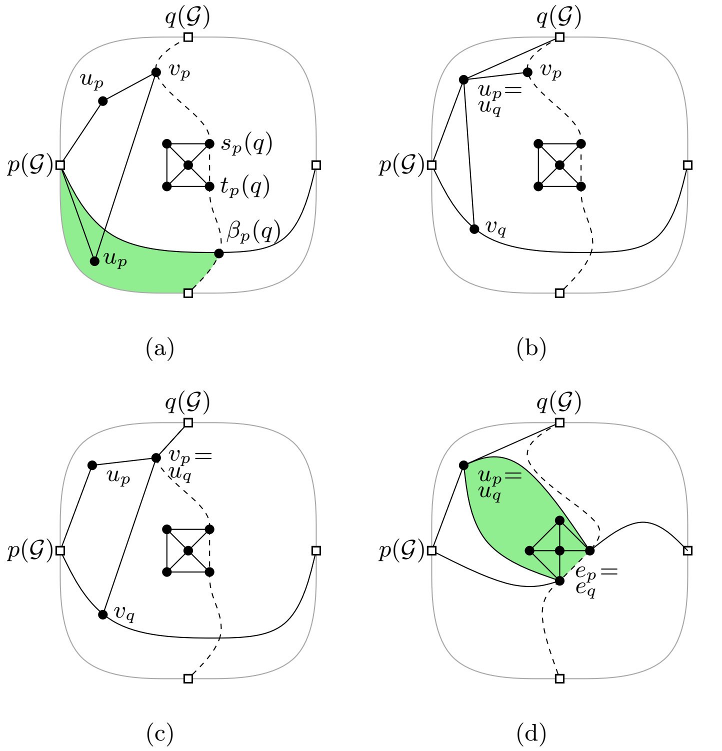

Fig. 21. P_p is dashed and C is shaded in all figures (when applicable). (a) Vertex u_p cannot be in the bottommost position. (b) Case 1 of Proposition 7. (c) Case 2 of Proposition 7. (d) Case 1 of Proposition 8.

Proposition 7. *Consider poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $q \in \{cw(p), ccw(p)\}$. Consider internal paths $\langle p(\mathcal{G}), u_p, v_p\rangle$ and $\langle q(\mathcal{G}), u_q, v_q\rangle$ with $v_p \in P_p[q(\mathcal{G}), \beta_p(q)] \setminus \{\beta_p(q)\}$ and $v_q \in P_q[p(\mathcal{G}), \beta_q(p)] \setminus \{\beta_p(q)\}$. Then*

1. *$u := u_p = u_q$ and u is strictly enclosed by C_p and C_q (see Fig. 21(b)); or*
2. *$u_q = v_p = N_{q(\mathcal{G})}(P_p)$ (see Fig. 21(c)); or*
3. *$u_p = v_q = N_{p(\mathcal{G})}(P_q)$.*

Proof. By Proposition 6, u_p is strictly enclosed by C_p and enclosed by C_q, and u_q is strictly enclosed by C_q and enclosed by C_p. If $u_p \neq N_{p(\mathcal{G})}(P_q)$, the path $\langle p(\mathcal{G}), u_p, v_p\rangle$ divides the cycle C_q into two cycles: one containing $q(\mathcal{G})$ and one containing v_q. Hence, $u_q \in \{p(\mathcal{G}), u_p, v_p\}$ and since $u_q \in \mathcal{G}$ and v_p and $q(\mathcal{G})$ are both on the chordless path P_p, we must have $u_p = u_q$ or $u_q = v_p = N_{q(\mathcal{G})}(P_p)$. If $u_p = N_{p(\mathcal{G})}(P_q)$, then $v_q = u_p$ by a similar argumentation. $\square$

Proposition 8. *Consider poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $p(\mathcal{G}) \neq q(\mathcal{G})$. Then $e_p \neq e_q$.*

Proof. Suppose for the sake of obtaining a contradiction that $e_p = e_q$. Since C_p and C_1 both enclose the pyramid, we must have $s_p(q) = t_q(p)$ and $t_p(q) = s_q(p)$. By Proposition 3, $p \neq opp(q)$. We consider the paths $\langle p(\mathcal{G}), u_p, s_p(q)\rangle$ and $\langle q(\mathcal{G}), u_q, s_q(p)\rangle$ in $E(\mathcal{G})$. By Proposition 7, we are in one of three cases.

Case 1 immediately gives a separating triangle $\langle u, s_q(p) = t_p(q), s_p(q), u\rangle$. See Fig. 21(d). In case 2 we have $u_q = s_p(q) = N_{q(\mathcal{G})}(P_p)$. But since $e_p = e_q$, $s_p(q)$ is on P_q, which is a contradiction to the fact that all vertices on P_q have distance 2 to $q(\mathcal{G})$. Case 3 is symmetric, which concludes the proof. □

Proposition 9. *Consider poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $q \in \{cw(p), ccw(p)\}$. Then e_p and e_q share exactly one vertex.*

Proof. By Proposition 8, e_p and e_q share at most one vertex. Suppose for the sake of obtaining a contradiction that e_p and e_q do not share a vertex. Since C_p and C_q both enclose the pyramid, $t_p(q)$ and $t_q(p)$ must be adjacent. See Fig. 22(a). By Proposition 3 C_q strictly encloses e_p and C_p strictly encloses e_q. Consider the internal paths $\langle p(\mathcal{G}), u_p, t_p(q)\rangle$ and $\langle q(\mathcal{G}), u_q, t_q(p)\rangle$. We distinguish the three cases given by Proposition 7. In the first case, the vertex $u = u_p = u_q$ is strictly enclosed by C_p and C_q. The separating triangle $\langle u = u_p, t_p(q), t_q(p), u_q = u\rangle$ contains the pyramid, both when u is on the outer cycle of the pyramid and when it is not, which is a contradiction to the fact that $E(\mathcal{G})$ is an irreducible triangulation. In the second and third cases, we have $t_p(q) = N_{q(\mathcal{G})}(P_p)$ or $t_q(p) = N_{p(\mathcal{G})}(P_q)$, both of which are impossible. □

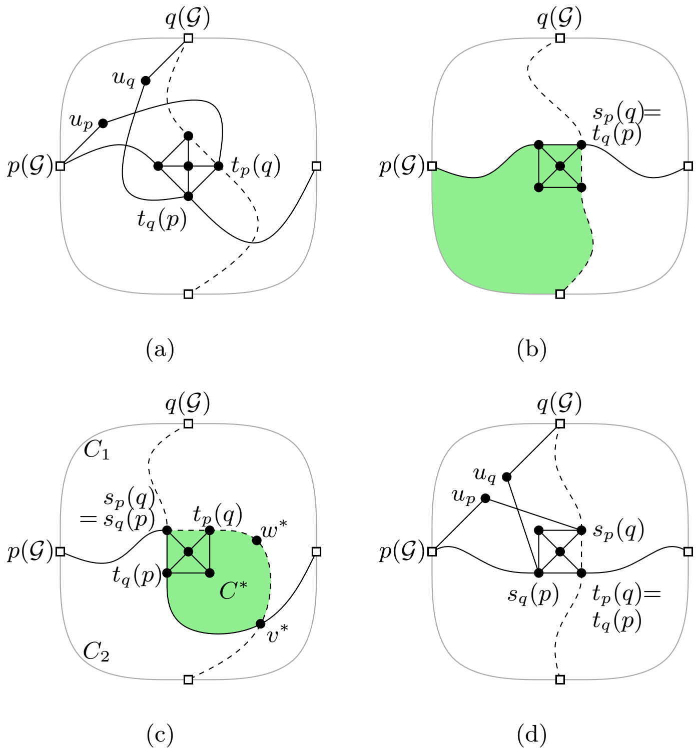

Fig. 22. P_p is dashed and C is shaded in all figures (when applicable). (a) The situation in Proposition 9. (b) The case $s_p(q) = t_q(p)$ in Proposition 10. (c) The case $s_p(q) = s_q(p)$ in Proposition 10. (d) The situation in Proposition 11.

Proposition 10. *Consider poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $q \in \{cw(p), ccw(p)\}$. Then $t_p(q) = t_q(p)$.*

Proof. By Proposition 9, e_p and e_q share exactly one vertex. Suppose for the sake of obtaining a contradiction that $s_p(q) = t_q(p)$ (or vice versa). By Proposition 5 (2), C_p encloses $P_q[p(\mathcal{G}), t_q(p)]$. Let P_p^* be the subpath of P_p from $s_p(q)$ to $opp(q)(\mathcal{G})$. The cycle $C = \langle P_q[t_q(p), p(\mathcal{G})], opp(q)(\mathcal{G}), P_p[opp(q)(\mathcal{G}), s_p(q) = t_q(p)]\rangle$ encloses the pyramid, because C_p encloses the pyramid. See Fig. 22(b). Hence, C_q does not enclose the pyramid. Contradiction. We conclude that $s_p(q) = s_q(p)$ or $t_p(q) = t_q(p)$.

Now suppose for the sake of obtaining a contradiction that $s_p(q) = s_q(p)$. See Fig. 22(c). Consider the Jordan curves that represent the boundary paths P_p and P_q. Though they touch at $s_p(q) = s_q(p)$, they do not properly intersect there: otherwise, C_p and C_q would not enclose the pyramid. But since these curves must properly intersect somewhere (they each connect opposite poles), there must be a vertex $v^* \in P_p[t_p(q), opp(p)(\mathcal{G})] \cap P_q[t_q(p), opp(q)(\mathcal{G})]$ by Proposition 5 (2).

By Proposition 3, v^* is not on the outer cycle of the pyramid. We may assume that $P_p[s_p(q), v^*]$ and $P_q[v^*, s_q(p)]$ do not intersect except at their terminal vertices; otherwise, we can use such an intersection instead of v^*. The cycle $C^* = \langle P_p[s_p(q), v^*], P_q[v^*, s_q(p) = s_p(q)]\rangle$ contains the pyramid. If $P_p[s_p(q), v^*]$ and $P_q[v^*, s_q(p)]$ both contain exactly three vertices, then C^* is a separating 4-cycle. Hence, either $P_p[s_p(q), v^*]$ or $P_q[v^*, s_q(p)]$ must contain another vertex. Without loss of generality, suppose that $P_p[s_p(q), v^*]$ has another vertex w^*. Then C_q strictly encloses w^*, since by minimality of v^*, w^* is not on P_q. Now P_q divides C_p into three cycles, the first two being $C_1 = \langle p(\mathcal{G}), P_p[q(\mathcal{G}), s_p(q) = s_q(p)], P_q[s_q(p), p(\mathcal{G})]\rangle$ and $C_2 = \langle opp(q)(\mathcal{G}), P_q[p(\mathcal{G}), v^*], P_p[v^*, opp(q)(\mathcal{G})]\rangle$. The third and last cycle is C^* and is the only cycle that encloses w^*. Since v^* is on P_p, there must be a $u \in \mathcal{G}$ such that $\langle p(\mathcal{G}), u, v^*\rangle$ is a path in $E(\mathcal{G})$. Note that u is enclosed by either C_1 or C_2. By Proposition 4 (2), it must be strictly enclosed by either C_1 or C_2, unless $u = s_p(q) = s_q(p) = N_{p(\mathcal{G})}(P_q)$. If u is strictly enclosed by C_1 or C_2, we reach a contradiction since w^* is enclosed by neither, and hence u cannot be adjacent to v^*. If $u = s_p(q) = s_q(p) = N_{p(\mathcal{G})}(P_q)$, then $\{u, v^*\}$ still cannot exist since both u and v^* are on P_p but they are not consecutive ($t_p(q)$ separates them on P_p). Contradiction. We conclude that $t_p(q) = t_q(p)$. ◻

Proposition 11. *Consider poles $p(\mathcal{G})$ and $q(\mathcal{G})$ with $q \in \{cw(p), ccw(p)\}$. Let u^* be the vertex on the pyramid not covered by e_p or e_q. Then $E(\mathcal{G})$ contains the edges $\{p(\mathcal{G}), u^*\}$ and $\{q(\mathcal{G}), u^*\}$.*

Proof. By Proposition 10, $t_p(q) = t_q(p)$. See Fig. 22(d). By Proposition 3, the only vertices from the pyramid in P_p and P_q are the ones on e_p and e_q, respectively. Since C_p and C_q both enclose the pyramid, C_p strictly encloses $s_q(p)$ and C_q strictly encloses $s_p(q)$. Consider the internal paths $\langle p(\mathcal{G}), u_p, s_p(q)\rangle$ and $\langle q(\mathcal{G}), u_q, s_q(p)\rangle$. We distinguish the three cases given by Proposition 7. In the first case, C_p and C_q strictly enclose $u = u_p = u_q$. But then $\langle u, s_p(q), t_p(q) = t_q(p), s_q(p), u\rangle$ is a separating 4-cycle (it encloses the pyramid). Hence, u must be u^*, which proves the statement. Alternatively, in the second case, $u_q = s_p(q) = N_{q(\mathcal{G})}(P_p)$. But then $\langle u_q = s_p(q), t_p(q) = t_q(p), s_q(p), u_q\rangle$ is a separating triangle that contains the pyramid. Contradiction. The third case is symmetric. This proves the statement. ◻

By Proposition 8, the edges e_ℓ, e_r, e_b and e_t are all different. By applying Proposition 11 for each pair of adjacent poles, we see that $E(\mathcal{G})$ must contain all the edges of the windmill. And since $E(\mathcal{G})$ is an irreducible triangulation, no other vertices can be present. We conclude that $E(\mathcal{G})$ must be the windmill. ◻