# The correctness of Fusy's algorithm

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September 15, 2016

The algorithm will always maintain the following three invariants

**Invariants 1**

(I1) The cycle $\mathcal{C}$ contains the two edges $S_rS_b$ and $S_rN_b$.
(I2) No edge in the interior of $\mathcal{C}$ connects two vertices in $\mathcal{C} \setminus S_r$
(I3) All inner edges of $T$ outside of $\mathcal{C}$ are colored and oriented in such that the innnervertex condition holds.

A cycle satisfying these three invariants will have the same general shape as in figure **??**. We note that the cycle has at least 4 vertices because otherwise a seperating triangle is created.

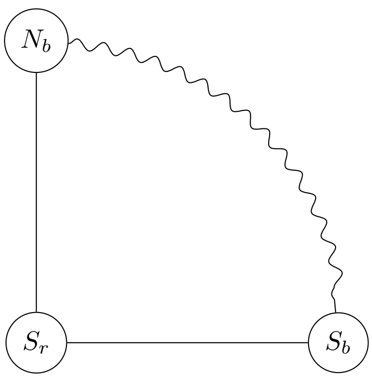



Figure 1: An example of a cycle $\mathcal{C}$ satisfying the invariants

It is also nice to note that the union of the cycle and it's interior form a triangulation of the $n$-gon since it is a induced subgraph of a triangulation of the 4-gon.

If we remove $S_r$ from $\mathcal{C}$ we are left with a path from $N_b$ to $S_b$. We can then order nodes of the path by their distance (over the path) to $S_b$. Thus $N_b$ is maximal while $S_R$ is minimal. For any two vertices $v > v'$ in this path we will denote by $[v, v']$ the subpath from $v$ to $v'$.

**Definition** (internal path)**.** We call a path $\mathcal{P}$ an internal path of $\mathcal{C}$ if all its edges are in the interior of $C$ and it connects two distinct vertices of $\mathcal{C}$

**Definition** (eligible path)**.** We call an internal path $\mathcal{P}$ from $v$ to $v'$ eligible if

(E1) Neither $v$ or $v'$ is $S_r$

(E2) The paths $\mathcal{P}$ and $[v, v']$ both have at least 3 vertices [1]

(E3) Each edge in the interior to $\mathcal{C}_\mathcal{P}$ connects a vertex of $\mathcal{P}\backslash v, v'$ and $[v, v']\backslash v, v'$. In particular $\mathcal{C}_\mathcal{P}$ is a non-separating cycle.

(E4) The cycle $\mathcal{C}'$ obtained by replacing $[v, v']$ by $\mathcal{P}$ in $\mathcal{C}$ has no interior edge connecting the two vertices of $C \setminus S_r$.

# 1 If the invariants are satisfied there always is an *eligible* internal path

We will show the following proposition.

**Proposition 2.** *When the algorithm's invariant (1 (I1) - 1 (I3)) are satisfied and the cycle $\mathcal{C}$ is separating then there exist a* eligible *internal path.*

*Proof.* We will first show that there always exists an internal path $\mathcal{P}$. We will then show that a internal path can be found that satisfies conditions $(E1)-(E4)$.

In the proof we will often use that a

Let us first note that if the cycle $C$ is separating (i.e has a non-empty interior), there is at least one interior vertex $v$. Since the triangulation of a $n$-gon is 2-connected there are two ways to go from $v$ to (say) $S_r$. Hence there is an internal path $\mathcal{P}_0$.

If this path does not satisfy (E1) we can use the following construction. The other vertex where $P_0$ intersects $\mathcal{C}$ is not $S_r$. Let us call this vertex $x$ and it's neighbour on the path $y$. The vertex $x$ might be $N_b$ or $S_b$ but can't be both, hence it has at least one neighbour $z$ on the cycle that is not $S_r$. Because the triangulation of a $n$-gon is internally maximally planar we have that $yz$ is an edge. Now $xyz$ is an internal path satisfying (E1). See also figure 3, here we made a choice on which side of $y$ the vertex $z$ lies, but this choice can be made without losing generality.

Hence we have now constructed, or already had, a path that satisfies (E1). Let us for the remainder of the proof denote this path by $\mathcal{P}_1$.

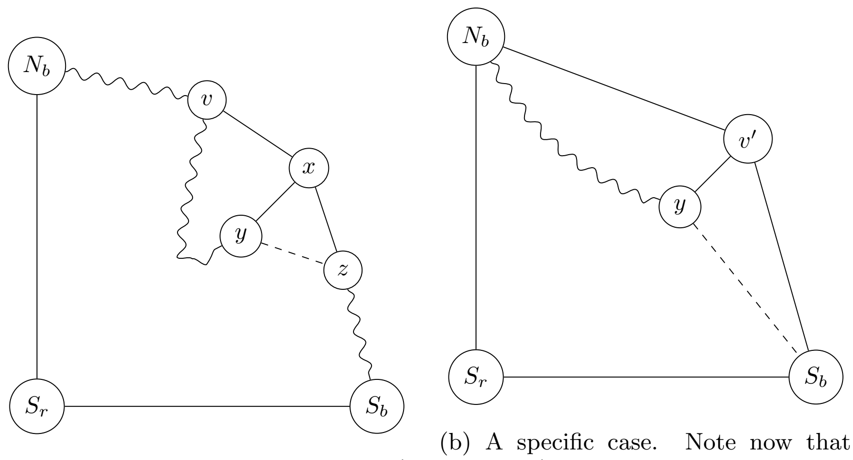
If $\mathcal{P}_1$ satisfies (E2) we set $\mathcal{P}_2 = \mathcal{P}_1$ otherwise we will create a path that satisfies (E1) and (E2). If the path $\mathcal{P}_1$ does not satisfy $(E2)$ [2] then there are two possibilities a) $\mathcal{P}_1$ does not have interior vertices and/or b) $[v, v']$ does not have interior vertices. If a) would be true the existence of $P_0$ would contradict Invariant 1 (I2). Hence the only problem can be that $b)$ occurs.

If $v = N_b$ and $v' = S_b$ we have found a separating triangle given by $S_rN_bS_b$ [3] in original graph. Hence at least one of $v$ or $v'$ is not $N_b$ or $S_b$. If we call this vertex $x$ its neighbour on the path $y$ and it's neighbour outside $[v, v']$ $z$. We see that by the interior of $\mathcal{C}$ being maximally planar $yz$ must be an edge. If we now adapt $P_1$ by replacing $yx$ by $yz$ we have made $[v, v']$ one vertex longer and hence created a path satisfying (E2). In figure 2 we show this procedure in two cases. Executing this procedure does not change that $S_r$ is not one of the

[1] that is, they have an interior vertex

[2] which will be the case if the above construction has been used

[3] this is the cycle $\mathcal{C}$ which is separating




(a) The general case. Note that $x = v'$.

(b) A specific case. Note now that $N_b = v, v' = x$ and $S_b = z$

Figure 2: Creating a path satisfying (E2). The dotted line is the edge we take in the new path $\mathcal{P}_2$

endpoints of the path. Hence we have now created a path $\mathcal{P}_2$ that satisfies (E1) and (E2).

If $\mathcal{P}_2$ satisfies $(E3)$, we take $\mathcal{P}_3 = \mathcal{P}_2$. Otherwise we will remedy the defect. We separate five different cases of offending edges. All of the five cases will be easy to remedy giving a path $\mathcal{P}_2'$ still satisfying (E1) and (E2) such that $\mathcal{C}_{\mathcal{P}_2'}$ is strictly contained in $\mathcal{C}_{\mathcal{P}_2}$

a) edges from $[v, v'] \setminus v, v'$to $[v, v'] \setminus v, v'$

b) edges from $\mathcal{P} \setminus v, v'$ to $\mathcal{P} \setminus v, v'$

c) edges incident to $v$ or $v$ and some other vertex on $\mathcal{C}_{\mathcal{P}_2}$

d) edges from $[v, v']$ to some internal vertex

e) edges from $\mathcal{P} \setminus v, v'$ to some internal vertex

The existence of an edge as in a) is forbidden by Invariant 1 (I2). If b) occurs we can simply shortcut our original path $\mathcal{P}_2$ with this edge. If c) occurs this edge can't go to another vertex in $[v, v']$ since that would offend Invariant 1 (I2). Hence they go to a vertex in $\mathcal{P}_2$ and we can shortcut the path as in b).

If d) occurs we simply make a new path and if e) occurs we take a slightly adapted interior path. See figures

Since all of the moves shrink $\mathcal{C}_{\mathcal{P}_2}$ while keeping (E1) and (E2) intact and we can't idenfinitly shrink this means at a certain point no more moves are availble. Since every offending edges allows a move this means that there are no more offending edges. Hence this version of $\mathcal{P}_2'$ satisfies (E3). For the final step of the proof we take $\mathcal{P}_3 = \mathcal{P}_2'$.

Suppose that $\mathcal{P}_3$ does not satisfy (E4). Then we can just take the would be interior edge and take this for a nwe path. This is again a finite procedure reducing the sum of $|\mathcal{P}_3| - |[v, v']|$. In the end we have a path satisfying (E1) - (E4).

□

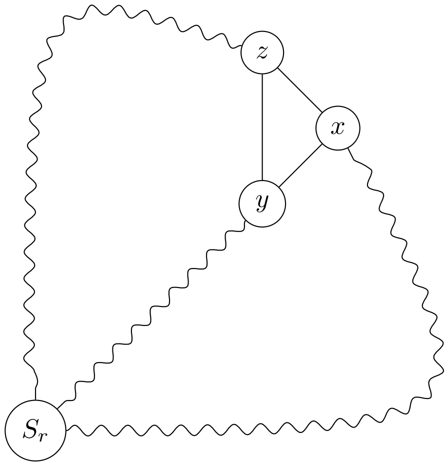



Figure 3: Constructing a path satisfying (E1)