Thesis

Sander Beekhuis

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1 Preliminaries

All graphs in this thesis are simple. Paths and cycles are always simple while walks are not necessarily simple. A path is a simple walk and a cycle is a closed path.

We will let $\{i, \ldots, j\}$ denote all the integers x such that $i \leq x \leq j$.

The *degree* of a face is the number of vertices it is incident to and a *triangular* face is a face of degree 3.

We will use *k-cycle* to denote a cycle of length k. Moreover a *triangle* is simply a cycle of length 3 (i.e. a 3-cycle).

FiXme: We could cite Jordan's paper here. Should I?

By Jordan's curve theorem a cycle splits the plane into two parts, one bounded and one unbounded. We will call the bounded part the *interior* of this cycle and the unbounded part the *exterior* of this cycle. We will call a cycle *separating* if there are vertices in both it's interior and exterior. An *interior edge* of a cycle is then an edge contained in the interior of the cycle.

We will sometimes give vertices of a path $\mathcal{P}$ explicitly using $p_1 \ldots p_k$. A chord of a walk is an edge that connects two vertices in a walk that is not part of the walk. If a walk has no chords we will call it *chordfree*. Since paths and cycles are special types the same definitions will hold for them.

Once we fix a planar embedding of a graph we can talk about the *rotation* at a vertex v. The rotation at a vertex is clockwise order of the edges incident to v. We will identify these edges with their other endpoints. Two vertices x, y are said to be *consecutive* in the rotation at v when the edges vx and vy are consecutive. In most of this thesis we implicitly fix an embedding since triangulations arr 3-connected and thus only have one planar embedding by a theorem due to Whitney (1933). This statement is proven in for example [1, p. 267]

Given a path $\mathcal{P}$ with vertices $p_1 \ldots p_k$ we will say that a vertex $v \notin \mathcal{P}$ adjacent to $p_i, i \in \{2, \ldots, k-1\}$ lies on the *left* of $\mathcal{P}$ if it lies between p_{i-1} and $pi + 1$ in the rotation at p_i. Otherwise v lies between p_{i+1} and p_{i-1} in the rotation at p_i. In this case we say that v lies to the *right* of $\mathcal{P}$.

We will use the same notion of left and right for edges.

FiXme: We could provide a picture illustrating these concepts

1.1 Plane triangulations

Definition (Plane triangulation)**.** A graph in which all faces are of degree 3.

Definition (Maximal planar graph)**.** A graph such that adding any one edge makes it non-planar.

Theorem 1. *Any graph G is a plane triangulation if and only if it is maximal planar*

Proof. We will prove the equivalence of the negations.

Suppose that G is not maximally planar. Then there is a face F to which we can add an edge while keeping G planar, however this face must then have degree of degree 4 or larger since we can split into two faces with an edges. But a face has at least degree 3. Hence G is not a plane triangulation.

Suppose that G is not a plane triangulation. Then there must be a face F of degree larger then 3. This face will thus admit an extra edge without violating planarity and hence G is not maximally planar. □

Because every face of a plane triangulation is triangular we can make the following remark.

Remark 2. *Every edge is incident to two triangular faces*

Lemma 3. *If two vertices x, y are consecutive in the rotation at v then xy is an edge in G and vxy is a triangle.*

FiXme: Is this more a remark then a lemma?

Proof. in the described situation we have a partial face $\dots xvy \dots$. Since every face of G is a triangle xy must be an edge and vxy must be a triangle. □

1.1.1 Connectedness

Let us first note that any maximally planar graph is 2-connected. Suppose there is a cutvertex, then surly we can add an edge between the components found after removing this cutvertex.

Theorem 4. *Any plane triangulation T is 3-connected.*

Proof. Suppose that T is not 3-connected. Then there must be a 2-cutset S, given by the vertices x and y. Removing this cutset splits the graph into at least two connected components C_i and all components are incident to all cutvertices otherwise we would have found a 1-cutset.

Since S is a cutset, there can't be any edges incident to both C_1 and C_2. But then the edge xy should be separating the 2 components on both sides. This is impossible since we can only draw this edge once. □

FiXme: We could add a figure to make this more clear

Definition (Irreducible triangulation)**.** We call a triangulation irreducible if it has no separating triangles

FiXme: It is called irreducible because there is a reduction that works on separating triangles. We might show this reduction

Theorem 5. *Any irreducible plane triangulation T is 4-connected.*

Proof. Note that any plane triangulation is 3-connected by Theorem 4.

Suppose that T is not 4-connected. Then there must be some 3-cutset (since it is 3-connected) let us denote the vertices of this cutset by x, y and z. Removing this cutset splits the graph into at least two connected components C_i and all components are incident to all cutvertices otherwise we would have found a 2- or 1-cutset.

However, now xy must be an edge in the triangulation T otherwise the graph is not maximal planar (There can't be an edge incident to both C_1 and C_2 because that would negate x, y, z being a cutset.). In the same way yz and xz are edges of T. But then xyz is a separating triangle. This is an contradiction and thus T is 4-connected □

1.2 Triangulations of the k-gon

Definition (Triangulation of the k-gon)**.** We call a graph a triangulation of the k-gon if the outer face has degree k and all interior faces have degree 3.

Vertices bordering the outer face are *outer vertices* while all other vertices are *interior vertices.* Furthermore the cycle formed by all vertices outer vertices is the *outer cycle.*

Sometimes such triangulations of the k-gon are called *(plane) triangulated graphs.*

Definition (Irreducible triangulation of the k-gon)**.** We call a triangulation of the k-gon irreducible if it has no separating triangles.

Note that triangulation of the n-gon $n \geq 4$ is not maximally planar and thus not plane triangulation.

The *completion* of a triangulation of the k-gon $G = (V, E)$. Is the graph $G' = (V', E')$ with vertex set $V' = V \cup \{s\}$ and edge set $E' = E \cup \{sv | v \text{ is a outer vertex}\}$

The completion is plane triangulation. Since the interior of the outer cycle of G always consisted of faces of degree 3. The exterior of the outer cycle consisted of one face of degree k (the outer face) but the completion has turned this into k faces of degree 3.

Theorem 6. *A triangulation of the k-gon G is 2-connected.*

Proof. Suppose that G has a cutvertex v. Then the set $\{s, v\}$ is a 2-cutset of the completion G' of G. This however is in contradiction to Theorem 4 stating that G' is 3-connected. Hence G has no cutvertex and is thus 2-connected. □

Theorem 7. *For every interior vertex v of a triangulation of the k-gon G is connected by at least 3 vertex disjoint paths to different outer vertices.*

Proof. By Theorem 4 the completion G' of G is 3-connected. Hence there are 3 vertex-disjoint paths from v to s. Since v is on the interior and s is on the exterior of the outer cycle $\mathcal{C}$ all these 3 paths cross the outer cycle at least once. These paths cross $\mathcal{C}$ for the first time in different vertices since they are vertex-disjoint. If we shorten the paths to their first crossing with $\mathcal{C}$ we obtain the 3 paths in the theorem. □

FiXme: We can sharpen this to 4 if we have a irreducible triangulation of the k-gon with a chordfree outer cycle

2 Rectangular duals

In this section we will introduce the rectangular dual of a graph. We will prove some simple properties of graphs and their (rectangular) duals.

We define a *rectangular layout* (or simply *layout*) $\mathcal{L}$ to be a partition of a rectangle into finitely many interiorly disjoint rectangles.

We will for simplicity of analysis assume that no four rectangles meet in one point.

We will then define the *dual graph of a layout* $\mathcal{L}$ and denote this graph by $\mathcal{G}(\mathcal{L})$. That is, we represent each rectangle by a vertex and we connect two vertices by an edge exactly when their rectangles are adjacent. Note that this graph is not the same as the *graph dual* of $\mathcal{L}$ when we view it as a graph (namely we don't represent the outer face of $\mathcal{L}$ by a vertex).

FiXme: Probably refer to [5]

So $\mathcal{G}(\mathcal{L})$ is the dual graph of a layout $\mathcal{L}$. In the reverse direction we say a layout $\mathcal{L}$ is a *rectangular dual* of a graph $\mathcal{G}$ if we have that $\mathcal{G} = \mathcal{G}(\mathcal{L})$.

A plane triangulated graph $\mathcal{G}$ does not necessarily have a rectangular dual nor is this dual necessarily unique.

FiXme: in what sense not unique, provide examples

2.1 Extended graphs

When constructing the rectangular dual of a graph G. It will be useful to add four vertices to the graph. We will denote these vertices with $\mathrm{N}, \mathrm{E}, \mathrm{S}, \mathrm{W}$ in corresponding with the four cardinal directions.

We will define define two types of these *extensions*. First we define a *regular extension* $\bar{G}$ which we can apply to any graph G. Then we will define a tight extension $\bar{G}_t$ for which we will require two distinct vertices on the outer cycle of G.

Regular extension A *extension* $\bar{G}$ of G is a augmentation of G with 4 vertices (which we will call it's *poles*). Such that

1. every interior face has degree 3 and the exterior face has degree 4.
2. all poles are incident to the outer face
3. $\bar{\mathcal{G}}$ has no separating triangles (i.e separating 3-cycles).

.ext Such a extended graph does not necessarily exist and is not necessarily unique. However we have the following result due to Kozminski and Kinnen

Theorem 8 (Existence of a rectangular dual). *A plane triangulated graph $\mathcal{G}$ has a rectangular dual if and only if it has an extension $\bar{\mathcal{G}}$*

Proof. This proofed in [5] □

FiXme: Provide location, Kozminski & Kinnen and ungar, See Siam paper

We call any (plane triangulated) graph G that has an extension a *proper* graph.

A proper graph G can have more then one extensions. Each such extension fixes which of the rectangles are in the corners of the rectangular dual $\mathcal{L}$. Hence sometimes such an extension is called a *corner assignment* by other authors.

Note that a graph G with a separating triangle can't be proper, since every possible extension will have a separating triangle.

Tight extension We only define the *tight extension* $\bar{G}_t$ of graph G without separating triangles in two vertices v, v' if the outer cycle is split into two chordfree paths P, P' by v, v'. Otherwise the tight extension is undefined.

We can without loss of generality assume that the order of these vertices and paths is $vPv'P'$ going clockwise along the outer cycle. We will then set $\mathrm{W} = v$, $\mathrm{E} = v'$ and add two vertices N, S. We connect every vertex of P to N and every vertex of P' to S.

Lemma 9. *If it is defined the tight extension of G in v, v' is a extension of $G \setminus \{v, v'\}$.*

Proof. It is clear from the construction that every interior face of $\bar{G}_t$ is of degree 3 and that the outer face is given by NESW an is thus of degree 4. To see that $\bar{G}_t$ has no separating triangles note that G has none and that any separating triangle must thus have one of N, S as a vertex. However a separating triangle containing N or S would imply a chord in P or P'. However in this case the tight extension is not defined. □

A tight extension $\bar{G}_t$ in v, v' is uniquely determined if it exists.

2.2 Different kinds of rectangular duals

Here we talk about area-universal, onesided, vertical/horizontal onesided and pseudo-onesided/ (k,l) -s sided duals. We also state the result by [2] that area-universal duals are the same as onesided duals.

FiXme: This subsection still has to be written

2.3 Regular edge labeling

A regular edge labeling of $\bar{G}$ corresponds to a rectangular dual $\mathcal{L}$ of some fixed extension $\bar{G}$.

A regular edge labeling has certain rules.

We will refer to Kant and He [4] and also note the naming by [3]

FiXme: This subsection still has to be written

2.4 Oriented regular edge labeling

In this subsection we will introduce the unique orientation on a REL. We wills define red faces and blue faces (which we can also do in the unoriented version i guess) with two sides (which is difficult in unoriented settings) and define split and merge vertices as the first and last vertices of such face.

We will also note it is equivalent to the oriented Fusy structure

FiXme: This subsection still has to be written

2.4.1 Being onesided in terms of REL

Lemma 10. *A face F with at least 3 edges on each side contains a Z*

Proof. □

FiXme: TODO

2.4.2 Being pseudo-onesided in terms of REL

3 Fixing a extension

In our explorations to find a lower bound on what kind of *pseudo one-sidedness* is possible we will find it very useful to fix one particular extension $\bar{G}$ of G. Unfortunately if there is no rectangular dual that's (k,l)-sided using some extension $\bar{G}$ of G. This does not imply that G is not (k,l)-sided. There might be another extension of G that has a (k,l)-sided rectangular dual.

Fortunately for us however we can view $\bar{G} = H$ as a graph in it's own right, then G is the interior of a separating 4-cycle of H and we will show this implies that G (as induced subgraph) has to be colored according to the extension $\bar{G}$.

FiXme: right choice of words?

We will thus proof the following theorem in this section.

Theorem 11. *When considering if there are rectangular duals satisfying a certain property. We can consider a fixed extension for a graph.*

Remark 12. *Let $\mathcal{C}$ be a separating 4-cycle of G with interior I. Then in any rectangular dual of G the region enclosed by the rectangles dual to the vertices in $\mathcal{C}$ is a rectangle.*

Remark 13. *Two disjoint rectangles are at most adjacent on one side.*

Lemma 14. *Let $\mathcal{C} = \{a, b, c, d\}$ be a separating 4-cycle of $\bar{G}$ with interior I. Then all interior edges incident to a, b, c and d respectively are red, blue, red and blue or blue, red, blue and red.*

FiXme: We might also want to provide a oriented version of this lemma

Proof. By Remark 12 the union of the rectangles in the interior of $\mathcal{C}$ will be some rectangle in any rectangular dual. We will denote this rectangle by I. Since two disjoint rectangles can only be adjacent to each other at one side all interior edges incident to any vertex of $\mathcal{C}$ are of the same color.

Furthermore a, b, c, d are all adjacent to a different side of I since I has four sides that need to be covered and it is only adjacent to four rectangles. If we then apply the rules of a regular edge labeling we see that if the interior edges of a are one color, those incident to b and d should have the second color. Then of course the interior edges incident to c are again colored with the first color. □

Lemma 14 is useful because it allows us to fix a extension $\bar{G}$ of G by building a *scaffold*. Suppose we want to investigate some extension $\bar{G}$ of G with poles N, E, S and W then we can consider the graph $\bar{G} = H$ as a graph in it's own right. H is a proper graph since it has no irreducible triangles in it's interior (because $\bar{G}$ had none) and it admits a valid extension $\bar{H}$ by connecting the new poles as in Table 1. $\bar{H}$ is shown in Figure 1.

NE	N	E	SE	NW
SE	S	E	NE	SW
SW	S	W	SE	NW
NW	N	W	NE	SW

Table 1: The neighbors of the new poles

FiXme: This lemma implies that any *alternating 4-cycle* is either *left-alternating* or *right-alternating* in the terminology of Fusy

Proof of Theorem 11. The graph H can have more then one extension but they all contain the separating 4-cycle $\mathcal{C} = NESW$ thus by Lemma 14 we see that, without loss of generality, the interior edges of $\mathcal{C}$ incident to N and S are colored red and those incident to E or W are colored blue. This is exactly as if we forced the extension $\bar{G}$ □

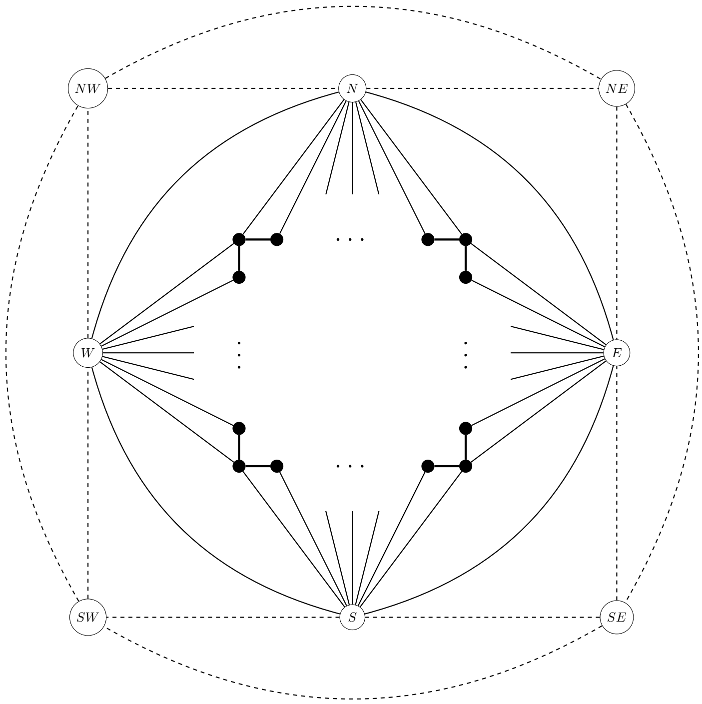


Figure 1: The construction of a scaffold. G is displayed in thick lines and with closed vertices. An arbitrary extension $\bar{G} = H$ is then drawn with thin lines and open vertices. An extension of H is then drawn with dashed edges and open vertices.

3.1 An application: There are graphs that are $(2, \infty)$-sided

We will show this by providing an example graph G with a fixed extension $\bar{G}$ which we can do according to Theorem 11. Consider the graph in Figure 2. Note that most of the interior vertices are of degree 4 and thus the largest part of any regular edge labeling is forced. Those edges that are forced to have a certain color are already colored in Figure 2.

The only edge for which we have freedom to choose a color is the diagonal edge of G. However, if we color this edge blue we get a red $(2, \infty)$ cycle and if we color this edge red we get a blue $(2, \infty)$ cycle. In both cases we will thus obtain a $(2, \infty)$-sided segment in our dual.

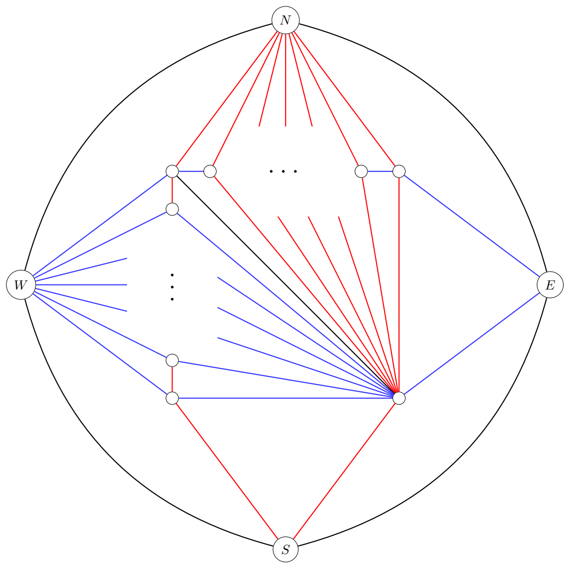


Figure 2: The fixed extension $\bar{G}$

4 Algorithms

FiXme: Harmonize reference invariants and eligible/valid path requirements. and change E to V (for valids)

Kant and He [4] were the first to design algorithms that determine a regular edge labeling.

Fusy [3] recently developed a different algorithm computing a specific regular edge labeling using a method shrinking a sweepcycle while coloring the outside in accordance with a regular edge labeling.[1]

All algorithms in this section will have the same core (based on [3]). Consisting of shrinking a sweepcycle by so called *valid* paths.[2] But will differ in which valid paths they choose (if there are multiple).

We will start this section with some notation and preliminaries in Subsection 4.1. Then we will state the core algorithm and show that it always computes a regular edge labeling in Subsection 4.2. Afterwards we show in Subsections 4.3, 4.4 and Section 5 how one can adapt the choice of the valid paths to obtain regular edge labellings with certain properties. Namely a the minimal element of the distributive lattices of regular edge labellings and regular edge labeling corresponding to horizontal and vertical rectangular duals.

4.1 Notation and Preliminaries

Definition (Interior path)**.** We call a path P an internal path of a cycle C if all vertices except the first and last one are in the interior of C and it connects two distinct vertices of C

We will use a script $\mathcal{C}$ to indicate the current sweep cycle. We will repeatedly only consider the path $\mathcal{C} \setminus \{\mathrm{S}\}$. In that case we will always order it from W to E. That these edges are always in $\mathcal{C}$ is a result of Invariant 15 (I1).

We will let $\mathcal{P}$ denote a interior path. Given such a path of k vertices we will index it's nodes by $p_1, \ldots, p_k$ in such a way that p_1 is closer to W then p_k is (and thus that p_k is closer to E then p_1 is).

Then p_1 and p_k indicate the two unique vertices of the walk that are also part of the cycle. We will then let $\mathcal{C}|_{\mathcal{P}}$ denote the part of $\mathcal{C} \setminus \{\mathrm{S}\}$ that is between p_1 and p_k (including). $\mathcal{C}_{\mathcal{P}}$ will denote the cycle we get when we paste $\mathcal{C}|_{\mathcal{P}}$ and $\mathcal{P}$.

4.2 Core

The algorithm will always maintain the following three invariants

Invariants 15

(I1) The cycle $\mathcal{C}$ contains the two edges SW and SE.
(I2) $\mathcal{C} \setminus \{\mathrm{S}\}$ has no chords
(I3) All inner edges of T outside of $\mathcal{C}$ are colored and oriented in such that the inner vertex condition holds.

FiXme: We need to add a partial inner vertex condition

[1]The specific regular edge labeling Fusy obtained was the minimal element of the distributive lattice of regular edge labellings.

[2]In Fusy's work he calls these *eligible paths*

A cycle satisfying these three invariants will have the same general shape as in figure 3. We note that the cycle has at least 4 vertices because otherwise a separating triangle is created.

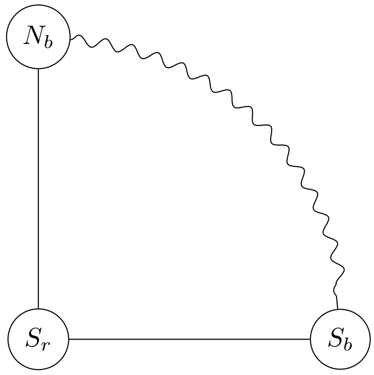


Figure 3: An example of a cycle $\mathcal{C}$ satisfying the invariants

It is also nice to note that the union of the cycle and it's interior form a triangulation of the n-gon since it is a induced subgraph of a triangulation of the 4-gon.

4.2.1 Valid paths

Definition (valid path)**.** We call an internal path $\mathcal{P}$ from w_1 to w_k valid if

(E1) Neither p_1 or p_k is S

(E2) The paths $\mathcal{P}$ and $\mathcal{C}|_{\mathcal{P}}$ both have more than 1 edge [3]

(E3) Every interior edge of $\mathcal{C}_{\mathcal{P}}$ connects a vertex of $\mathcal{P}\setminus\{p_1, p_k\}$ and $\mathcal{C}|_{\mathcal{P}}\setminus\{p_1, p_k\}$. In particular $\mathcal{C}_{\mathcal{P}}$ is a non-separating cycle.

(E4) The path $\mathcal{C}' \setminus \{\mathrm{S}\}$, where $\mathcal{C}'$ is obtained by replacing $\mathcal{C}|_{\mathcal{P}}$ by $\mathcal{P}$ in $\mathcal{C}$, is chordfree.

We note that (E3) and (E4) partially overlap. (E3) already implies that there can't be chords on the left of $\mathcal{C} \setminus \{\mathrm{S}\}$.

Remark 16. *"Shrinking" the cycle with an valid path will keep all the invariants true.*

FiXme: We haven't proven this yet

We will show the following proposition.

Theorem 17 (Existence of a eligible path)**.** *When the algorithm's invariant (15 (I1) - 15 (I3)) are satisfied and the cycle $\mathcal{C}$ is separating then there exist a* eligible *internal path.*

FiXme: As outlined in last meeting this proof is not complete as is, it has been moved to the appendix. We are stuck on the part where we need to find a path satisfying E4. We might proof this from red algo.

[3]i.e. both have an interior vertex

4.3 Minimum distributive lattice element

We get this when we take the "leftmost" eligible path. As is outlined in [3]

FiXme: Expand this subsection

4.4 Horizontal one-sides

FiXme: Define what we mean with *cycle border* and *face border*

As an exercise one could try to adapt Fusy's algorithm to generate horizontally one-sided layouts directly, without doing flips in the distributive lattice. It turns out that this is not that difficult.

Since the horizontal segments correspond to faces in the blue bipolar orientation we want that one of the two borders of the face has a length of at most two. Since every valid path which we update the cycle with splits off one face in the blue bipolar orientation it is easy to control this property.

Theorem 18. *In the update of the algorithm there is always an eligible path $\mathcal{P}$ available such that either $\mathcal{P}$ or $\mathcal{C}|_{\mathcal{P}}$ is of length* 2*.*

In order to proof this theorem we will first show the following lemma.

Lemma 19. *If $\mathcal{P}$ is an eligible path giving raise to a cycle $\mathcal{C}_P$ of which both borders have length of at least* 3*. Then there exist an eligible path $\mathcal{P}'$ such that the path border and cycle border of its cycle $\mathcal{C}_{\mathcal{P}'}$ are both at least* 1 *shorter than those of $\mathcal{C}_{\mathcal{P}}$.*

FiXme: Revisit notation after writing section on oriented REL

Proof. In this proof we will frequently use property (E3) of a valid path, we won't mention it every time we use it.

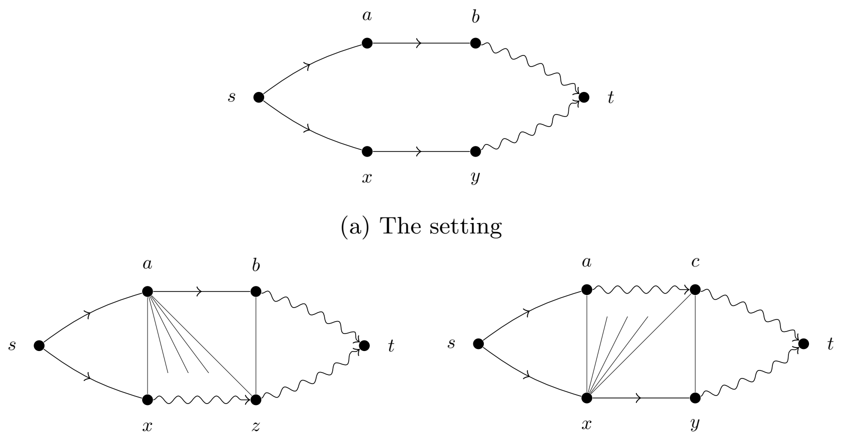
We denote the source by s and the sink by t. We also assign namesa, b and x, y to the first two vertices on both borders, see Figure 4a. Since every interior face of G is a triangle ax is an edge. Now we distinguish two cases, either ay is an edge (case 1) or bx is an edge (case 2). They can't both be an edge at the same time due to planarity, neither can it happen that both of them are not an edge since then the face containing the path $baxy$ is at least of degree 4.

In the first case a may be connected to more vertices on the path border, however there is a last one, say z. And this vertex is then also connected to b, otherwise it would not be the last one. Now we can provide an shorter eligible path $\mathcal{P}'$. We start at a go to z and from there we follow the old path $\mathcal{P}$ to t. See figure 4b. It is easy to see that all four properties of an eligible path hold for $\mathcal{P}'$.

In the second case x may be connected to more vertices along the cycle border, however there is a last one, say c. And this vertex is then also connected to y, otherwise it would not be the last one. Now we can provide an shorter eligible path $\mathcal{P}' = sxz$. See figure 4c. It is straightforward to see that all four properties of an eligible path hold for $\mathcal{P}'$. □

Proof of Theorem 18. By Theorem 17 we know there is a eligible path $\mathcal{P}$. If one of the borders of $\mathcal{C}_{\mathcal{P}}$ is of length 2 or less we are done. If this path gives raise to a face $\mathcal{C}_{\mathcal{P}}$ with both borders are both of length at least 3 we can repeatedly apply Lemma 19 until at least one of the borders is of length at most 2. □

If we in every update of the algorithm take the paths from Theorem 18 we end up with the correct faces in the blue bipolar orientation and hence a horizontally one sided rectangular dual.



(a) The setting

(b) Case 1

(c) Case 2

Figure 4

5 Vertical one-sided dual

We can also adapt Fusy's algorithm to generate a vertically one-sided dual. We then need top generate a regular edge labeling without red faces that have 3 or more edges on both borders.

We will have an additional requirement on top of the requirement that $\bar{G}$ has no separating triangles. We will also require that G has no separating four cycles.

Notational concerns Just as in Section 4 we will use $\mathcal{C}$ to indicate the current sweep line cycle. We will repeatedly only consider the path $\mathcal{C} \setminus \{\mathrm{S}\}$. In that case we will always order it from W to E.

Instead of interior paths we will consider interior walks but we will use similar notation. That is a walk between two distinct vertices of $\mathcal{C}$ of which all vertices except the first and last one are in the interior of $\mathcal{C}$.

We will let $\mathcal{W}$ denote a interior walk. Given such a walk of k vertices we index it's nodes $w_1, \ldots, w_k$ in such a way that w_1 is closer to W then w_k is (and thus that w_k is closer to E then w_1 is).

Then w_1 and w_k indicate the two unique vertices of the walk that are also part of the cycle. We will then let $\mathcal{C}|_{\mathcal{W}}$ denote the part of $\mathcal{C} \setminus \mathrm{S}$ that is between w_1 and w_k (including). $\mathcal{C}_{\mathcal{W}}$ will denote the closed walk formed when we paste $\mathcal{C}|_{W}$ and $\mathcal{W}$.

Since paths are a subclass of walks all of the above notation can also be used for a path $\mathcal{P}$. Note that the closed walk $\mathcal{C}_{\mathcal{P}}$ in this case will actually be a cycle.

5.1 Outline

To describe the algorithm two more definitions are necessary

Definition (Prefence)**.** A prefence $\mathcal{W}$ is a interior walk of $\mathcal{C}$ starting at $v_i \in \mathcal{C}$ and ending at $v_j \in \mathcal{C}$ a both adjacent to S

(P1) For every $v_i \in \mathcal{C} \setminus \{\mathrm{W}, \mathrm{S}, \mathrm{E}\}$ we have that all vertices between v_{i+1} and v_{i-1} in the rotation at v_i are in $\mathcal{W} \setminus \{\mathrm{W}, \mathrm{E}\}$

(P2) For every $w_i \in \mathcal{W} \setminus \{\mathrm{W}, \mathrm{E}\}$ we have that all vertices between w_{i-1} and w_{i+1} in rotation at w_i are in $\mathcal{C} \setminus \{\mathrm{W}, \mathrm{S}, \mathrm{E}\}$

(P3) w_2 and v_{i+1} are consecutive in the rotation at v_i

(P4) v_{j-1} and w_{k-1} are consecutive in the rotation at v_j

We enforce these conditions because they imply (E3) when $\mathcal{W}$ is a path as we will show in Lemma 21.

For a walk however the interior is not clearly defined.

Definition (Fence)**.** A fence is a valid path starting and ending at a vertex adjacent to S

We will show that there is a algorithm if there are no separating 4-cycles in G and no separating 3-cycles in $\bar{G}$.

FiXme: expand on naming/reasons of fence

The algorithm will receive as input a extended graph $\bar{G}$ and will return a regular edge labeling such that all red faces are $(1-\infty)$ using a sweep-cycle approach inspired by Fusy[3].

We will start by creating a prefence W. This may not be a valid path, it doesn't even have to be a path. During the algorithm we will make a number of moves that will turn this prefence into a fence. In each move we shrink C by employing a valid paths and change the prefence.

5.2 Finding a initial prefence

Let v_i denote all the vertices of $\mathcal{C}\backslash\{\text{S}\}$ in the following order W $= v_1\, v_2\, \ldots v_{n-1}\, v_n =$ E. Some intervals of these vertices will be adjacent to S. However, they can't be all adjacent to S since then the sweepcycle will be non-separating since we can't have separating triangles. We denote by v_i the last vertex of fist interval of vertices adjacent to S and by v_j the first vertex of the second interval. As candidate walk we will start with v_i, we will then take the vertices adjacent to v_{i+1} between v_i and v_{i+2} in the rotation at v_{i+1}, followed the vertices between v_{i+1} and v_{i+3} in the rotation at v_{i+2} and so further until we add the vertices between v_{j-2} and v_j in the rotation around v_{j-1} and finally we finish by adding v_j.

We then remove all subsequent duplicate vertices from $\mathcal{W}$.

Lemma 20. *The collection W described above is a prefence.*

Proof. We will first show that W is a walk. We will proof that every vertex is adjacent to the next vertex. Let us suppose that w and w' are two subsequent vertices in W, we will show that ww' is an edge if $\{w, w'\} \cap \{v_i, v_j\} = \emptyset$. Afterwards we will consider this edge case. There are then two cases for w, w'. Either (a) w and w' are vertices adjacent to some v_i subsequent in clockwise order or (b) w was the last vertex adjacent to some v_i and thus w' is the first vertex adjacent to v_{i+1}.

The following two situations can also be seen in Figure 5.

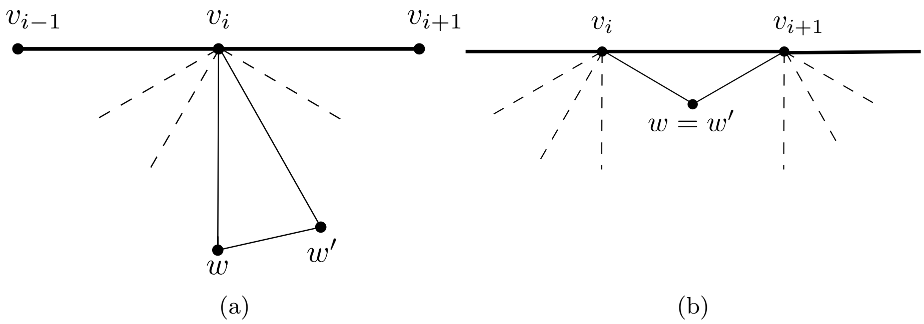


Figure 5: The two main cases of the proof showing that W is a walk

In case (a) we note that v_iw and v_iw' are edges next to each other in clockwise order around v_i. Since every interior face of $\bar{G}$ is a triangle ww' must be an edge. We thus see that w, w' are adjacent and not duplicates.

In case (b) we note that v_iw and v_iv_{i+1} are edges subsequent in clockwise order, hence wv_{i+1} is also an edge. Hence w is the first vertex adjacent to v_{i+1}

after v_i in clockwise order. Thus $w = w'$. They are duplicates and one of them must have been removed.

Now for the edge cases: Let x be the first vertex adjacent to v_{i+1} and let y be the last vertex adjacent to v_{j-1}. v_i and x are vertices adjacent to v_{i+1} subsequent in clockwise order, and hence connected by Lemma 3. In the same way y and v_j are subsequent vertices in the rotation at v_n and hence connected.

Hence $\mathcal{W}$ is a walk. The above also shows that $v_i v_{i+1} x$ and $v_{j-1} v_j y$ are triangles by Lemma 3 and hence $\mathcal{W}$ satisfies properties (P3) and (P4) of being a prefence.

Moreover this walk satisfies (P1) because $\mathcal{W}$ by construction contains all neighbors of any vertex $v_i \in \mathcal{C}|_{\mathcal{W}} \setminus \{v_i, v_j\}$ between v_{i-1} and v_{i+1} in the rotation of v_i.

Finally to see that $\mathcal{W}$ also satisfies (P2). Consider a vertex $w_j \in \mathcal{W} \setminus \{v_i, v_j\}$ then either it is (a) the neighbor of some vertex v_i and only of this vertex or it is (b) the unique vertex neighboring in the interior of the cycle the $\ell + 1$ vertices $v_i, \ldots, v_{i+\ell}$. This is essentially the same case distinction as above. However now (a) $w_{i-1} w_i v_i$ and $v_i w_i w_{i+1}$ or (b) $w_{i-1} w_i v_i$, $v_i w_i v_{i+1}, \ldots, v_{i+\ell-1} w_i v_{i+\ell}$ and $v_{i+\ell} w_i w_{i+1}$ form a set of triangles spanning the area between w_{i-1} and w_{i+1} in the rotation at w_i. Thus any edge not going to $\mathcal{C}|_{\mathcal{W}} \setminus \{v_i, v_j\}$ in this sector will lead to a separating triangle. We however have assumed G has no separating triangles. Hence (P3) holds.[4] $\square$

We then orient $\mathcal{W}$ from v_i (the vertex closest to W)to v_j (the vertex closest to E) and denote it's vertices by $w_1 \ldots w_k$.

5.3 Irregularities

We will distinguish two kinds of *irregularities* in a prefence.

1. The candidate walk is non-simple in a certain vertex. That is, if we traverse the sequence of vertices in $\mathcal{W}$ we see that $w_i = w_j$ for some $i < j$.
2. The candidate walk has a chord on the right. That is, there is an edge $w_i w_j$ on the right of $\mathcal{W}$ with $i < j$ and i and j not subsequent (i.e. $i < j - 1$).

Note that we can't have a chord can on the left of $\mathcal{W}$ ($\mathcal{W}$ being oriented from W to E), since if it would lie on the left of $\mathcal{W}$ the vertices $w_{i+1}, \ldots, w_{j-1}$ would not have been chosen in the construction of the prefence.

Lemma 21. *If a prefence has no irregularities it is a fence.*

Proof. We will show that all the requirements of being a valid path are met.

Path Let us begin by noting that since there are no non-simple points we have a path and not just a walk.

(E1) It is clear that both w_1 and w_k are not S by the construction of the candidate walk.

[4] FiXme: I believe this is still true when separating triangles are allowed to occur. However the prove will have to be different.

(E2) For $\mathcal{W}$ or $\mathcal{C}|_{\mathcal{W}}$ to have only one edge we need to have that v_iv_j is an edge. However, v_iv_j can not be an edge in $\mathcal{C}$ since v_i and v_j are from different intervals of vertices adjacent to S. It can also not be an edge in $\bar{G} \setminus \mathcal{C}$ since that would be a chord of the cycle and these don't exist by Invariant 15 (I2)

(E3) Every interior edge of $\mathcal{C}_{\mathcal{W}}$ with at least one endpoint on the cycle is of the required type by the conditions (P1) - (P4). We note that these edges in particular have both endpoints on the cycle $\mathcal{C}_{\mathcal{W}}$.

Interior edges with both endpoints not on the cycle can a priori exist. However since a triangulation is a connected graph there must then also be an edge with one endpoint on $\mathcal{C}_{\mathcal{W}}$, and one inside $\mathcal{C}_{\mathcal{W}}$ but this can not be if $\mathcal{W}$ is a prefence. However by the argument above both endpoints must then be on $\mathcal{C}_{\mathcal{W}}$, this is a contradiction.

(E4) The cycle $\mathcal{C}'$ only changes between v_i and v_j. There can be no chord with one vertex from cycle $\mathcal{C} \setminus \mathcal{C}|_{\mathcal{W}}$ and one from $\mathcal{W}$ since such a chord would cross $\mathrm{S}v_i$ or $\mathrm{S}v_j$. There is no chord with two vertices in $\mathcal{W}$ since that would be a irregularity and there is no chord with two vertices from $\mathcal{C} \setminus \mathcal{C}|_{\mathcal{W}}$ by Invariant 15 (I2).

Hence, if $\mathcal{W}$ has no irregularities it is a valid path.

Furthermore, $\mathcal{W}$ is a path starting and ending at a vertex adjacent to S because it is prefence. And thus it is a fence. $\square$

Definition (Range of a irregularity)**.** For a non-simple point $w_i = w_j$ with $i < j$ has *range* $\{i, \dots, j\} \subset \mathbb{N}$. A chord w_iw_j with $i < j - 1$ has *range* $\{i, \dots, j\} \subset \mathbb{N}$.

FiXme: Is it better to call this a non-simple point or a non-simple vertex?

Note that a chord can't have the same range as a non-simple point since then w_iw_j will be a loop and we are considering simple graphs. Furthermore two chords have different ranges because we otherwise have a multiedge. Two nonsimple points with the same range are, in fact, the same. This leads us to the following remark.

Remark 22. *Distinct irregularities have distinct ranges.*

Definition (Maximal irregularity)**.** A irregularity is maximal if it's range is not contained[5] in the range of any other irregularity.

Lemma 23. *Maximal irregularities have ranges whose overlap is at most one integer.*

Proof. We let I and J denote two distinct maximal irregularities with ranges $\{i_1, \dots i_2\}$ and $\{j_1, \dots, j2\}$. Let us for the moment suppose that I and J have ranges that overlap more then one integer. Since I and J are both maximal their ranges can not be contained in each other.

Without loss of generality we thus have $i_1 < j_1 < i_2 < j_2$.

Now two chords to the right of $\mathcal{W}$ would cross each other but we have a planar graph so this can't be the case.

[5]Because of Remark 22 being contained is the same as being strictly contained

Now let us without loss of generality suppose that I is a non-simple point. A non-simple point $w_{i_1} = w_{i_2}$ is adjacent to two ranges of vertices in $\mathcal{C} \setminus \{\mathrm{S}\}$. $v_a \dots v_b$ and $v_c \dots v_d$ then $\tilde{C} = w_{i_1} v_b \dots v_c$ is a cycle. And because of the rotation at $w_{i_1} = w_{i_2}$ we have that $w_{i_{1+1}}, \dots, w_{i_{2-1}}$ are inside this cycle while $w_1 \dots w_{i_1 - 1}$ and $w_{i_2+1} \dots w_k$ are outside the cycle. See Figure.

FiXme: We could add figure to clarify.

Now if J is a chord we have $\tilde{C}$, which can't be. If J is also a nonsimple point this would imply that the vertex $w_{j_i} = w_{j_2}$ is at the same time inside and outside $\tilde{C}$ which is clearly impossible. □

5.4 Moves

The algorithm will remove these irregularities by recursing on a subgraph for each maximal irregularity. We shrink the cycle $\mathcal{C}$ with every valid path that is found in the recurrence, in the order they are found. Afterwards we update the prefence by removing $w_{i+1}, \dots, w_{j-1}$. In subsection 5.4.3 we will show that the updated prefence is a prefence for the updated cycle $\mathcal{C}$.

We will first show how to remove these maximal irregularities in Subsections 5.4.1 and 5.4.2. That is, we show which subgraph H we recurse upon for both kinds of irregularity. Furthermore we show that these subgraphs suffice the requirements of the algorithm.

Afterwards, in subsection 5.4.3 we will make sure that the subgraphs we recurse upon are edge-disjoint. That is, they only overlap in border vertices.

It is worth noting that other irregularities contained in such a maximal irregularity are solved in the recurrence.

5.4.1 Chords

If we encounter a chord we will extract a subgraph and recurse on this subgraph. A chord $w_i w_j$ has a triangular face on the left and on the right (like every edge). The third vertex in the face to the left will be called x. x is not necessarily distinct from w_{i+1} and/or w_{j-1} but this is also not necessary for the rest of the argument.

FiXme: We might also work these out in a Figure.

The vertex v_a on the cycle is uniquely determined as the vertices adjacent to both w_i and $wi + 1$. In the same way v_b is the unique neighbor of w_{j-1} and w_j.

We will describe a walk $\mathcal{U}$ running from v_a to v_b. This path consists of all vertices adjacent to w_i in clockwise order from v_a (inclusive) to x(inclusive) and subsequently all vertices adjacent to w_j in clockwise order from x (exclusive) to v_b (inclusive). This path is given in bold in Figure 6.

Lemma 24. *$\mathcal{U}$ is a chordfree path*

Proof. We note that $\mathcal{U}$ is a walk by the same reasoning as is given in Lemma 20.

FiXme: Is it nice to refer to a line of reasoning like this?

$\mathcal{U}$ cant have a non-simple point x' since it would have to be connected to at least two vertices. However a vertex x' that is distinct from x and is connected to both w_i and w_j will induce a separating triangle $w_i x' w_j$. $\mathcal{U}$ also can't be nonsimple at x since x is the the third vertex of the triangular face $w_i w_j x$. Hence $\mathcal{U}$ is a path.

$\mathcal{U}$ can't have chords $u_i u_j$ since they would either induce a separating 3- or 4-cycle either $w_i u_i u_j$ or $w_j u_i u_j$ or $w_i u_i u_j w_j$ depending on the vertex adjacent to u_i and u_j. □

FiXme: We use that we have no 4-cycles here

We then consider the interior of the cycle $\mathcal{C}_{\mathcal{U}}$ and the cycle $\mathcal{C}_{\mathcal{U}}$ itself as the subgraph H. We then take the tight extension at v_a and v_b. We will then recurse on this graph $\bar{H}_t$. See also Figure 6. Since $\mathcal{C}$ is chordfree by invariant 15 (I2) so is $\mathcal{C}|_{\mathcal{U}}$. We have also just shown that $\mathcal{U}$ is chordfree. So $\bar{H}_t$ is indeed defined. Furthermore, since H is a induced subgraph of G, $\bar{H}_t$ contains no separating 4-cycles not involving the poles.

We update the prefence by removing $w_{i+1}, \ldots, w_{j-1}$.

5.4.2 Nonsimple points

Removing a non-simple point is done is a similar manner.

The vertex v_a on $\mathcal{C}$ is uniquely determined as the vertices adjacent to both $w_i = w_j$ and $wi + 1$. In the same way v_b is the unique neighbor of w_{j-1} and $w_j = w_i$. Note that it may be that $w_{i+1} = wj - 1$ this does not matter for the rest of the argument.

FiXme: We may show this in a figure.

We will describe a walk $\mathcal{U}$ running from v_a to v_b. This path consists of all vertices in the rotation at $w_i = w_j$ from v_b (inclusive) to v_a(inclusive). This path is given in bold in Figure 7.

Lemma 25. *$\mathcal{U}$ is a chordfree path.*

Proof. If we orient $\mathcal{U}$ from v_a to v_b we see that $\mathcal{U}$ cant have a non-simple point since such a point would have edges to at least two vertices on the right. However every vertex can only be connected to $w_i = w_j$. Hence $\mathcal{U}$ is a path.

$\mathcal{U}$ can't have chords on the right of the path by the way we construct $\mathcal{U}$. Furthermore $\mathcal{U}$ can't have chords $u_i u_j$ on the left since they would either induce a separating 3-cycle $w_i u_i u_j$. □

FiXme: Here we use no 4-cycles

We then consider the interior of the cycle $\mathcal{C}_{\mathcal{U}}$ and the cycle $\mathcal{C}_{\mathcal{U}}$ itself as the subgraph H.

We then take the tight extension of H at v_a and v_b to recurse on. See also Figure 7. Since $\mathcal{C}$ is chordfree by invariant 15 (I2) so is $\mathcal{C}|_{\mathcal{U}}$. We have also just shown that $\mathcal{U}$ is chordfree. So $\bar{H}_t$ is indeed defined. Furthermore, since H is a induced subgraph of G, $\bar{H}_t$ contains no separating 4-cycles not involving the poles.

We update the prefence by removing $w_{i+1}, \ldots, w_{j-1}$ and we also recognize that $w_i = w_j$ is now a duplicate subsequent occurrence of the same vertex. So we also remove w_j.

5.4.3 Validity

Lemma 26. *After doing a move the updated prefence W is a prefence for the updated cycle C*

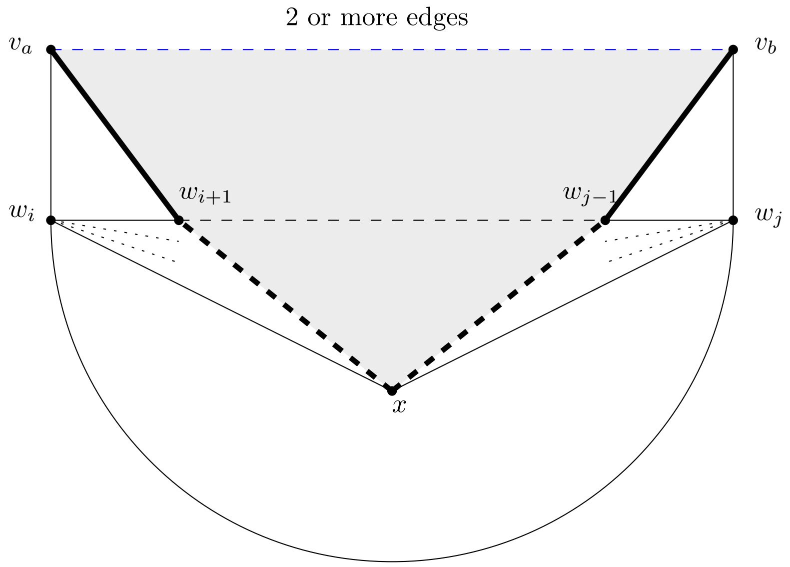
Proof. □

FiXme: TODO

Lemma 27. *Let H_I and H_J be two recursion subgraphs for different maximal irregularities I and J. Then H_I and H_J are edge disjoint.*

Proof. □

FiXme: TODO



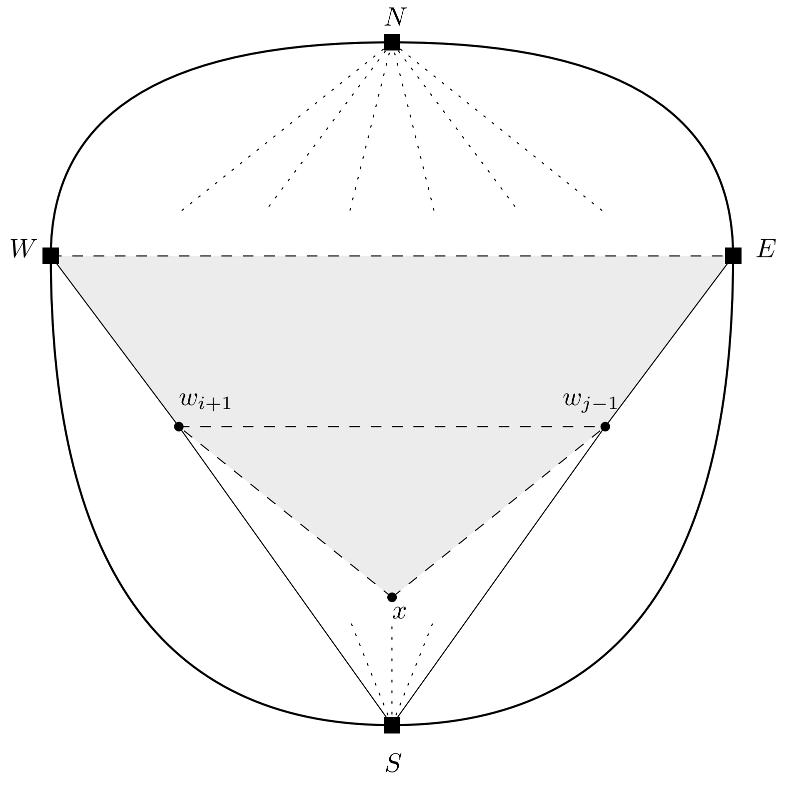


Figure 6: Removing a chord

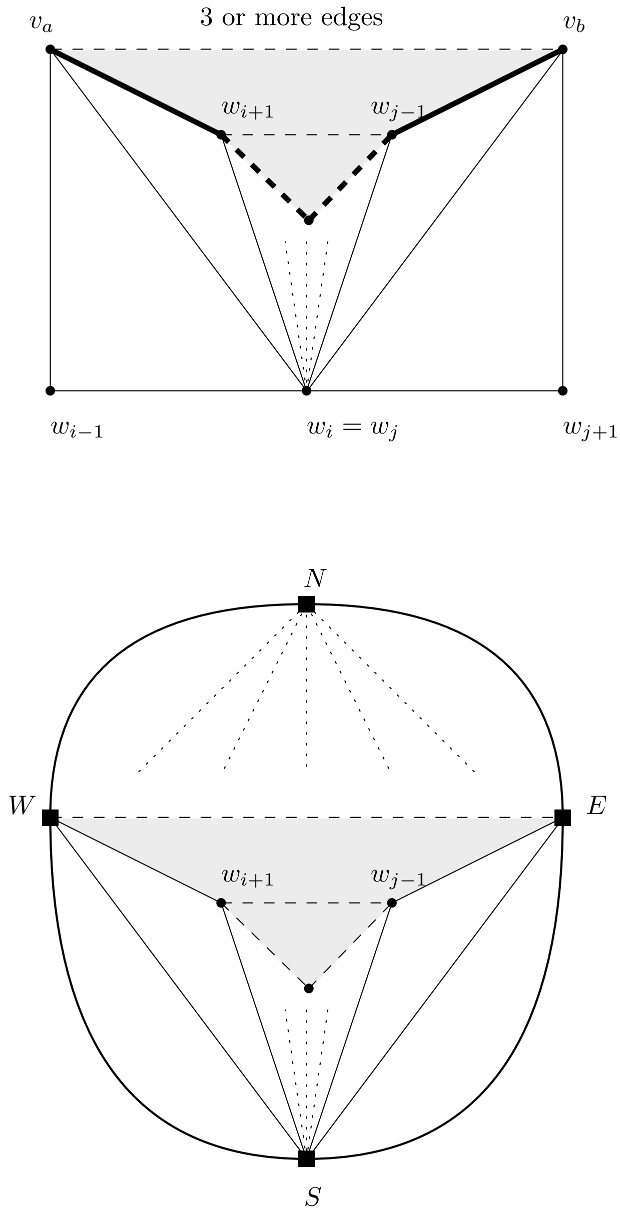


Figure 7: Removing a non-simple point

5.5 Correctness

As long as the interior of $\mathcal{C}$ is nonempty we can find a prefence. And thus we find valid paths. Since we continuously shrink the cycle with valid paths we end up with a regular edge labeling. See core algorithm

The algorithm finishes because it keeps on recursing and shrinking until no graph is left.

5.5.1 The red faces

Let us then argue that the red faces are all $(1-\infty)$ faces, corresponding to one-sided vertical segments. As is shown in Lemma 10 it is sufficient to show no two vertices subsequent on a blue path are first a merge and then a split or vice versa.

We will show the following

Lemma 28. *A split or merge always happens on a vertex that is adjacent to S for some recursion.*

Proof. Every valid path we shrink the cycle by is found as a fence on some recursion level. In this recursion level both w_1 and w_k are adjacent to S. □

Lemma 29. *A path starting at a certain recursion level will stay at that recursion level. It may share vertices with the north boundary of a lower recursion level but never with the south boundary.*

Proof. A valid path can never leave the subgraph H in which its start and endvertex are located. Because it is found as a fence in this subgraph. It can also never run trough a graph H' on a lower recursion level (except for the north boundary path) because in every move the vertices of the prefence in H' are deleted. □

Recall that all our valid paths are oriented from a start vertex to end vertex.

Lemma 30. *A split can't directly be followed by a merge along any valid path during the algorithm.*

Proof. One of paths after the split is no longer on the south boundary of this subgraph H, nor on the south boundary of any other subgraph by Lemma 29. This path hence can't contain a merge.

The other path still potentially follow the south boundary. However merging from the southward side of the path is impossible by Lemma 29 from the northward side is equally impossible since the split and merge have to be neighboring vertices in the rotations of these vertices and thus the path $\mathcal{P}$ that merged must also join again.

But then it is not a valid path. □

However for a blue Z to occur there has to be a valid path that first has a split and then has a merge. Since this can't be all red faces must have only 2 edges on at least one side. Hence the regular edge labeling this algorithm produces corresponds to a vertically one-sided rectangular dual.

FiXme: Show how the algorithm works with some cool examples: For example: The multiple non-simple point $v_i = v_j = v_k$; Example of page $F1$; Example with lots of layered chords