# Thesis

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October 13, 2016

# 1 Types of triangulations and their properties

## 1.1 Preliminaries

All graphs are presumed simple and have a fixed planar embedding. In this thesis paths and cycles are always simple while walks are not necessarily simple.

The *degree* of a face is the number of vertices it is incident to. By a *cycle* we will mean a simple cycle. That is a cycle without repetition of edges or vertices. By Jordan's curve theorem a cycle splits the plane into two parts, one bounded and one unbounded. We will call the bounded part the *interior* of this cycle and the unbounded part the *exterior* of this cycle.

We will call a cycle *separating* if there are vertices in both it's interior and exterior. We will use *k-cycle* to denote a cycle of length $k$. Moreover a *triangle* is simply a cycle of length 3 (i.e. a 3-cycle).

## 1.2 Plane triangulations

**Definition** (Plane triangulation)**.** A graph with only faces of degree 3.

**Definition** (Maximal planar graph)**.** A graph such that adding any one edge leaves it non-planar.

**Theorem 1.** *Any graph $G$ is a plane triangulation if and only if it is maximal planar*

*Proof.* We will prove the equivalence of the negations.

Suppose that $G$ is not maximally planar. Then there is a face $F$ to which we can add an edge, however this face must then have degree larger then 4. Hence $G$ is also not a plane triangulation.

Suppose that $G$ is not a plane triangulation. Then there must be a face $F$ of degree larger then 3. This face will thus admit an extra edge without violating planarity and hence $G$ is not maximally planar. □

### 1.2.1 Connectedness

**Theorem 2.** *Any plane triangulation $T$ is 3-connected.*

*Proof.* Suppose that $T$ is not 3-connected. Then there must be a 2-cutset $S$, given by the vertices $x$ and $y$. Removing this cutset splits the graph into at least two connected components $C_i$ and all components are incident to all cutvertices otherwise we would have found a 1-cutset.

Since $S$ is a cutset, there can't be any edges incident to both $C_1$ and $C_2$. But then the edge $xy$ should be separating the 2 components on both sides. This is impossible since we can only draw this edge once. □

**Definition** (Irreducible triangulation)**.** We call a triangulation irreducible if it has no separating triangles

**Theorem 3.** *Any irreducible plane triangulation $T$ is 4-connected.*

*Proof.* Note that any plane triangulation is 3-connected by Theorhem 2.

Suppose that $T$ is not 4-connected. Then there must be some 3-cutset (since it is 3-connected) let us denote the vertices of this cutset by $x, y$ and $z$. Removing this cutset splits the graph into at least two connected components $C_i$ and all components are incident to all cutvertices otherwise we would have found a 2- or 1-cutset.

However, now $xy$ must be an edge in the triangulation $T$ otherwise the graph is not maximal planar (There can't be an edge incident to both $C_1$ and $C_2$ because that would negate $x, y, z$ being a cutset.). In the same way $yz$ and $xz$ are edges of $T$. But then $xyz$ is a separating triangle. This is an contradiction and thus $T$ is 4-connected □

## 1.3 Triangulations of the $k$-gon

**Definition** (Triangulation of the $k$-gon)**.** We call a graph a triangulation of the $k$-gon if the outer face has degree $k$ and all interior faces have degree 3.

Vertices bordering the outer face are *outer vertices* while all other vertices are *interior vertices*. Furthermore the cycle formed by all vertices outer vertices is the *outer cycle*.

Sometimes such triangulations of the $k$-gon are called *(plane) triangulated graphs*.

**Definition** (Irreducible triangulation of the $k$-gon)**.** We call a triangulation of the $k$-gon irreducible if it has no separating triangles.

Note that triangulation of the $n$-gon $n \geq 4$ is not maximally planar and thus not plane triangulation.

The *completion* of a triangulation of the $k$-gon $G = (V, E)$. Is the graph $G' = (V', E')$ with vertex set $V' = V \cup \{s\}$ and edge set $E' = E \cup \{sv | v \text{ is a outer vertex}\}$

The completion is plane triangulation. Since the interior of the outer cycle of $G$ always consisted of faces of degree 3. The exterior of the outer cycle consisted of one face of degree $k$ (the outer face) but the completion has turned this into $k$ faces of degree 3.

**Theorem 4.** *A triangulation of the $k$-gon $G$ is 2-connected.*

*Proof.* Suppose that $G$ has a cutvertex $v$. Then the set $\{s, v\}$ is a 2-cutset of the completion $G'$ of $G$. This however is in contradiction to Theorem 2 stating that $G'$ is 3-connected. Hence $G$ has no cutvertex and is thus 2-connected. □

**Theorem 5.** *A irreducible and chordless triangulation of the $k$-gon is 3-connected.*

*Proof.* **Writers note:** Will be provided if this statement turns out to be interesting. Will go via the fact that the completion is a irreducible triangulation. Chordless outer cycle is important, because a chord will form a separating triangle in $G'$. □

**Theorem 6.** *Any irreducible triangulation $T$ of the 4-gon with $n \geq 5$ is 3-connected.*

**Writers note:** This proof could be a corrolary of the above theorhem 5. A chord gives a separating triangle if $n \geq 5$.

*Proof.* Let us name the four outer vertices $a, b, c, d$ in clockwise order. Let us first note that the diagonals $ac$ and $bd$ can't be an edge since this would create a separating triangle containing the 5th vertex. Let $I$ denote the component of all interior vertices, since every face in the interior is of degree 3 each outer vertices is incident to at least one edge that is also incident to $I$.

One can now easily check that there is no 2cut set with only exterior vertices. However, a cutset with 1 or 2 interior vertices leads to at least one cycle of degree greater then 3

Hence no 2-cutset of $T$ can't exist and $T$ is 3-connected. □

**Theorem 7.** *For every interior vertex $v$ of a triangulation of the $k$-gon $G$ is connected by ate least 3 vertex disjoint paths to different outer vertices.*

*Proof.* By Theorhem 2 the completion $G'$ of $G$ is 3-connected. Hence there are 3 vertex-disjoint paths from $v$ to $s$. Since $v$ is on the interior and $s$ is on the exterior of the outer cycle $\mathcal{C}$ all these 3 paths cross the outer cycle at least once. These paths cross $\mathcal{C}$ for the first time in different vertices since they are vertex-disjoint. If we shorten the paths to their first crossing with $\mathcal{C}$ we obtain the 3 paths in the theorem. □

**Writers note:** We can sharpen this to 4 if we have a irreducible an chordless triangulation of the $k$-gon

**Theorem 8.** *Every interior vertex of a triangulation of the $n$-gon has degree at least 3.*

*Proof.* Suppose a interior vertex $v$ has degree 1 then clearly the face surrounding $v$ can't have degree 3. Now suppose that an interior vertex $v$ has degree 2. We then let $u$ and $w$ denote it's neighbours and $F$ and $F'$ the face incident to $v$. See also Figure 1. Then since $F$ and $F'$ are both interior faces they need to be off degree 3 this implies that $uw$ is an edge for both faces. This is impossible and hence every interior vertex has at least degree 3 □

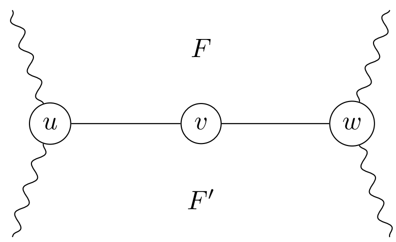


Figure 1: The notation as described in the proof

**Writers note:** If we forbid irreducible triangulations every interior vertex is of degree 4 since the neighbourhood of any internal vertex $v$ looks like a set of triangles.

**Writers note:** This theorem is currently (17-09) unused.

# 2 Rectangular duals

In this section we will explain what we mean with the rectangualr dual of a graph. We will prove some simple properties of graphs and their duals.

We define a *rectangular layout* (or simply *layout*) $\mathcal{L}$ to be a partition of a rectangle into finitely many interiorly disjoint rectangles.

We will assume that no four rectangles meet in one point.

We will then look at the *dual graph of a layout* $\mathcal{L}$ and denote this graph by $\mathcal{G}(\mathcal{L})$. That is, we represent each rectangle by a vertex and we connect two vertices by an edge exactly when their rectangles are adjacent. Note that this graph is not the same as the *graph dual* of $\mathcal{L}$ when we view it as a graph (namely we don't represent the outer face of $\mathcal{L}$ by a vertex).

So $\mathcal{G}(\mathcal{L})$ is the dual graph of a layout $\mathcal{L}$. In the reverse direction we say a layout $\mathcal{L}$ is a *rectangular dual* of a graph $\mathcal{G}$ if we have that $\mathcal{G} = \mathcal{G}(\mathcal{L})$.

A plane triangulated graph $\mathcal{G}$ does not necessarily have a rectangular dual nor is this dual necessarily unique.

## 2.1 Extended graphs

A *extended graph* $\bar{G}$ of $G$ is a augmentation of $G$ with 4 vertices (which we will call it's *poles*). Such that

1. every interior face has degree 3 and the exterior face has degree 4.
2. all poles are incident to the outer face
3. $\bar{\mathcal{G}}$ has no separating triangles (i.e separating 3-cycles).

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We sometimes call an extended graph $\bar{G}$ of $G$ an *extension* of $G$.

Such a extended graph does not necessarily exist and is not necessarily unique. However we have the following result due to ....

**Theorem 9** (Existence of a rectangular dual)**.** *A plane triangulated graph $\mathcal{G}$ has a rectangular dual if and only if it has an extension $\bar{\mathcal{G}}$*

*Proof.* Kozminski & kinnen and ungar, See siAM paper □

We call any (plane triangulated) graph $G$ that has an extension a *proper* graph.

A proper graph $G$ can have more then one extensions. Each such extension fixes which of the rectangles are in the corners of the rectangular dual $\mathcal{L}$. Hence sometimes such an extension is called a *corner assignment*.

## 2.2 Regular edge labeling

A regular edge labelling of $\bar{G}$ corresponds to a rectangular dual $\mathcal{L}$ of $G$ with some *corner assignment* fixed.

Or regular edge labelling of a graph.

An *interior edge* of a cycle is an edge on the interior of the cycle (when the cycle is viewed as Jordan curve).

### 2.2.1 Being onesided in terms of REL

### 2.2.2 Being psudeo-onesided in terms of REL

# 3 Fixing a extension

In our explorations to find a lower bound on what kind of *psuedo one-sidedness* is possible we will find it very useful to fix one particular extension $\bar{G}$ of $G$. Unfortunately if there is no rectangular dual that's $(k, l)$-sided using the *corner assignment* provided by some extension $\bar{G}$. This does not imply that $G$ is not $(k, l)$-sided. There might be another extension of $G$ such that under the corner assignment corresponding to this extension $G$ has a $(k, l)$-sided rectangular dual.

Fortunately for us however we can view $\bar{G} = H$ as a graph in it's own right, then $G$ is the interior of a separating 4-cycle of $H$ and we will show this implies that $G$ (as induced sugraph) has to be coloured according to the extension $\bar{G}$.

**Remark 10.** *Let $\mathcal{C}$ be a separating 4-cyle of $G$ with interior $I$. Then in any rectangular dual of $G$ the region enclosed by the rectangles dual to the vertices in $\mathcal{C}$ is a rectangle.*

**Remark 11.** *Two disjoint rectangles are at most adjacent on one side.*

**Lemma 12.** *Let $\mathcal{C} = \{a, b, c, d\}$ be a separating 4-cyle of $\bar{G}$ with interior $I$. Then all interior edges incident to $a, b, c$ and $d$ respectively are red, blue, red and blue or blue, red, blue and red.*

*Proof.* By Remark 10 the union of the rectangles in the interior of $\mathcal{C}$ will be some rectangle in any rectangular dual. We will denote this rectangle by $I$. Since two disjoint rectangles can only be adjacent to each other at one side all interior edges incident to any vertex of $\mathcal{C}$ are of the same color.

Furthermore $a, b, c, d$ are all adjacent to a different side of $I$ since $I$ has four sides that need to be covered and it is only adjacent to four rectangles. If we then apply the rules of a regular edge labelling we see that if the interior edges of $a$ are one color, those incident to $b$ and $d$ should have the second color. Then of course the interior edges incident to $c$ are again coloured with the first color. □

This lemma implies that any *alternating 4-cycle* is either *left-alternating* or *right-alternating* in the terminology of Fusy

Furthermore the above Lemma is also very useful in that it allows us to fix a extension $\bar{G}$ of $G$ by building a *scaffold*. Suppose we want to investigate some extension $\bar{G}$ of $G$ with poles $N$, $E$, $S$ and $W$ then we can consider the graph $\bar{G} = H$ as a graph in it's own right. $H$ is a proper graph since it has no irreducible triangles in it's interior (because $\bar{G}$ had none) and it admits a valid extension $\bar{H}$ by connecting the new poles $NE, SE, SW$ and $NW$ to $N, E, SE, NW$, $S, E, NE, SW$, $S, W, SE, NW$ and $N, W, NE, SW$ respectively. See Figure 2 for this extension.

FiXme: Maybe use a table

**Theorem 13.** *We can fix an extension, if we want.*

The graph $H$ can have more then one extension but they all contain the separating 4-cycle $\mathcal{C} = NESW$ thus by Lemma 12 we see that, without loss of generality, the interior edges of $\mathcal{C}$ incident to $N$ and $S$ are coloured red and those incident to $E$ or $W$ are coloured blue. This is exactly as if we forced the extension $\bar{G}$

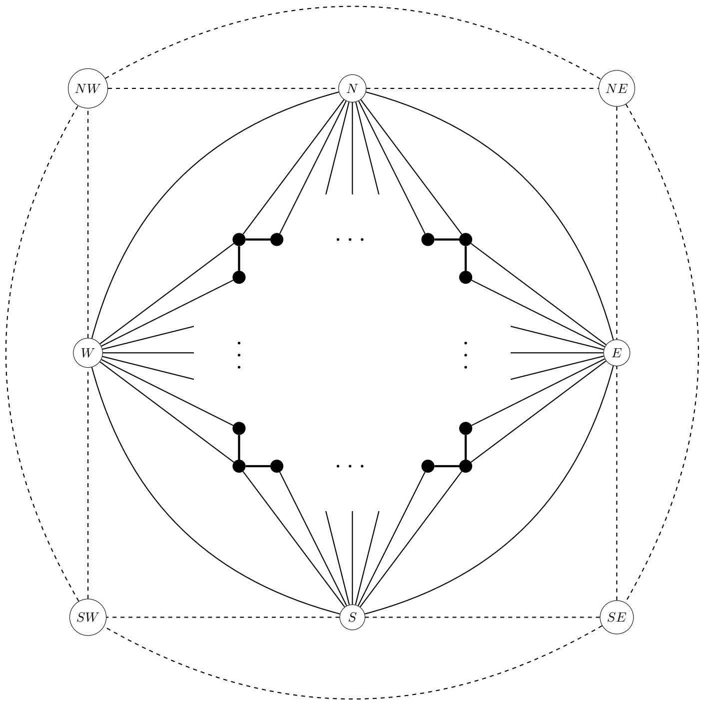



Figure 2: The construction of a scaffold. $G$ is displayed in thick lines and with closed vertices. An arbitrary extension $\bar{G} = H$ is then drawn with thin lines and open vertices. An extension of $H$ is then drawn with dashed edges and open vertices.

### 3.1 An aplication: There are graphs that are $(2, \infty)$-sided

We will show this by providing an example graph $G$ with a fixed extension $\bar{G}$ which we can do according to Theorem 13. Consider the graph in Figure 3. Note that most of the interior vertices are of degree 4 and thus the largest part of any regular edge labelling is forced. Those edges that are forced to have a certain color are already coloured in Figure 3.

The only edge for which we have freedom to choose a color is the diagonal edge of $G$. Howeever, if we color this edge blue we get a red $(2, \infty)$ cycle and if we color this edge red we get a blue $(2, \infty)$ cycle. In both cases we will thus obtain a $(2, \infty)$-sided segment in our dual.

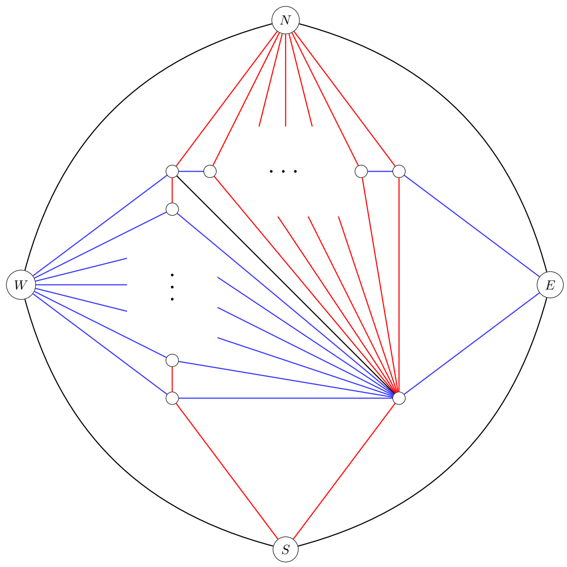



Figure 3: The fixed extension $\bar{G}$

# 4 Algorithms

Kant and He [**KH**] were the first to desing algorithms that determine a regular edge labeling.

Fusy [2, 1] recently developed a different algorithm computing a specific regular edge labeling using a method shrinkg a sweepcycle while coloring the outside in accordance with a regular edge labeling.[1]

All algorithms in this section will have the same core (based on [2]). Consisitng of shrinking a sweepcycle by socalled *valid* paths[2]. But will differ in which valid paths they choose (if there are multiple).

We will start this section with some notation and preliminaries in Subsection 4.1. Then we will state the core algorithm and show that it always computes a regular edge labeling in Subsection 4.2. Afterwards we show in Subsections 4.3, 4.4, 4.5. How one can adapt the choice of the valid paths to obtain regular edge labellings with certain properties.

FiXme: what properties

## 4.1 Notation and Preliminaries

**Definition** (Interior path)**.** We call a path $P$ an internal path of a cycle $C$ if all its edges are in the interior of $C$ and it connects two distinct vertices of $C$

FiXme: is interior defined somewhere?

We will use a script $\mathcal{C}$ to indicate the current sweep cycle. We will repeatedly only consider the path $\mathcal{C} \setminus \{\text{S}\}$. In that case we will always order it from W to E.

We will let $\mathcal{P}$ denote a interior path. Given such a path of $k$ vertices we will index it's nodes by $p_1, \dots, p_k$ in such a way that $p_1$ is closer to W then $p_k$ is (and thus that $p_k$ is closer to E then $p_1$ is).

Then $p_1$ and $p_k$ indicate the two unique vertices of the walk that are also part of the cycle. We will then let $\mathcal{C}|_{\mathcal{P}}$ denote the part of $\mathcal{C} \setminus \{\text{S}\}$ that is between $p_1$ and $p_k$ (including). $\mathcal{C}_{\mathcal{P}}$ will denote the cycle we get when we paste $\mathcal{C}|_{\mathcal{P}}$ and $\mathcal{P}$.

## 4.2 Core

The algorithm will always maintain the following three invariants

### Invariants 14

(I1) The cycle $\mathcal{C}$ contains the two edges SW and SE.

(I2) $\mathcal{C} \setminus \{\text{S}\}$ has no chords in it's interior

(I3) All inner edges of $T$ outside of $\mathcal{C}$ are colored and oriented in such that the innnervertex condition holds.

FiXme: def chord and chord in interior (or just simplify to has no chords

A cycle satisfying these three invariants will have the same general shape as in figure 4. We note that the cycle has at least 4 vertices because otherwise a seperating triangle is created.

It is also nice to note that the union of the cycle and it's interior form a triangulation of the $n$-gon since it is a induced subgraph of a triangulation of the 4-gon.

[1] The specific regular edge labelling Fusy obtained was the minimal element of the lattice of regular edge labellings.

[2] In Fusy's work these are called *eligible paths*

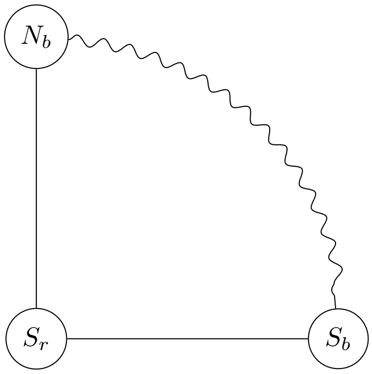



Figure 4: An example of a cycle $\mathcal{C}$ satisfying the invariants

#### 4.2.1 Eligible paths

**Definition** (valid path)**.** We call an internal path $\mathcal{P}$ from $w_1$ to $w_k$ valid if

(E1) Neither $w_1$ or $w_k$ is S

(E2) The paths $\mathcal{P}$ and $\mathcal{C}|_{\mathcal{P}}$ both are more then 1 edge [3]

(E3) Each edge in the interior to $\mathcal{C}_{\mathcal{P}}$ connects a vertex of $\mathcal{P} \setminus \{w_1, w_k\}$ and $\mathcal{C}|_{\mathcal{W}} \setminus \{w_1, w_k\}$. In particular $\mathcal{C}_{\mathcal{P}}$ is a non-separating cycle.

(E4) The cycle $\mathcal{C}'$ obtained by replacing $\mathcal{C}|_{\mathcal{P}}$ by $\mathcal{P}$ in $\mathcal{C}$ has no chord that doesn't involve S.

**Remark 15.** *"Shrinking" the cycle with an valid path will keep all the invariants true.*

We will show the following proposition.

FiXme: proof this

**Theorem 16** (Existence of a eligible path)**.** *When the algorithm's invariant (14 (I1) - 14 (I3)) are satisfied and the cycle $\mathcal{C}$ is separating then there exist a* eligible *internal path.*

FiXme: This is very hard to prove, mustly the part that we can always find a path satisfying E4. See page 11. Prrof this from red algo.

*Proof.* We will first show that there always exists an internal path $\mathcal{P}$. We will then show that a internal path can be found that satisfies conditions $(E1)-(E4)$.

In the proof we will often use that a

Let us first note that if the cycle $C$ is separating (i.e has a non-empty interior), there is at least one interior vertex $v$. Since the triangulation of a $n$-gon is 2-connected there are two ways to go from $v$ to (say) $S_r$. Hence there is an internal path $\mathcal{P}_0$.

If this path does not satisfy (E1) we can use the following construction. The other vertex where $P_0$ intersects $\mathcal{C}$ is not $S_r$. Let us call this vertex $x$ and it's neighbour on the path $y$. The vertex $x$ might be $N_b$ or $S_b$ but can't be both, hence it has at least one neighbour $z$ on the cycle that is not $S_r$. Because the triangulation of a $n$-gon is internally maximally planar we have that $yz$ is an edge. Now $xyz$ is an internal path satisfying (E1). See also figure 5, here we

[3] i.e. both have an interior vertex

made a choice on which side of $y$ the vertex $z$ lies, but this choice can be made without losing generality.

Hence we have now constructed, or already had, a path that satisfies (E1). Let us for the remainder of the proof denote this path by $\mathcal{P}_1$.

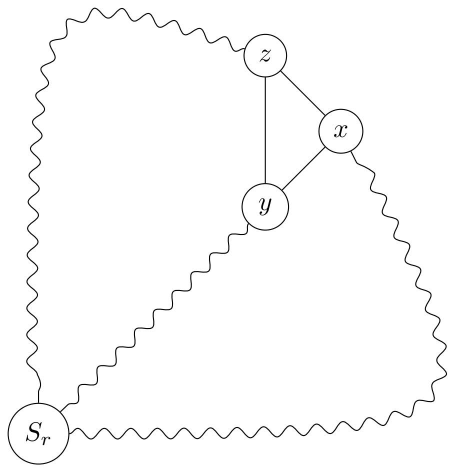



Figure 5: Constructing a path satisfying (E1)

**There is a path that also satsifies (E2)** If $\mathcal{P}_1$ satisfies (E2) we set $\mathcal{P}_2 = \mathcal{P}_1$ otherwise we will create a path that satisfies (E1) and (E2). If the path $\mathcal{P}_1$ does not satisfy $(E2)$ [4] then there are two possibilities a) $\mathcal{P}_1$ does not have interior vertices and/or b) $[v, v']$ does not have interior vertices. If a) would be true the existence of $P_0$ would contradict Invariant 14 (I2). Hence the only problem can be that $b$) occurs.

If $v = N_b$ and $v' = S_b$ we have found a separating triangle given by $S_r N_b S_b$ [5] in original graph. Hence at least one of $v$ or $v'$ is not $N_b$ or $S_b$. If we call this vertex $x$ its neighbour on the path $y$ and it's neighbour outside $[v, v']$ $z$. We see that by the interior of $\mathcal{C}$ being maximally planar $yz$ must be an edge. If we now adapt $P_1$ by replacing $yx$ by $yz$ we have made $[v, v']$ one vertex longer and hence created a path satisfying (E2). In figure 6 we show this procedure in two cases. Executing this procedure does not change that $S_r$ is not one of the endpoints of the path. Hence we have now created a path $\mathcal{P}_2$ that satisfies (E1) and (E2).

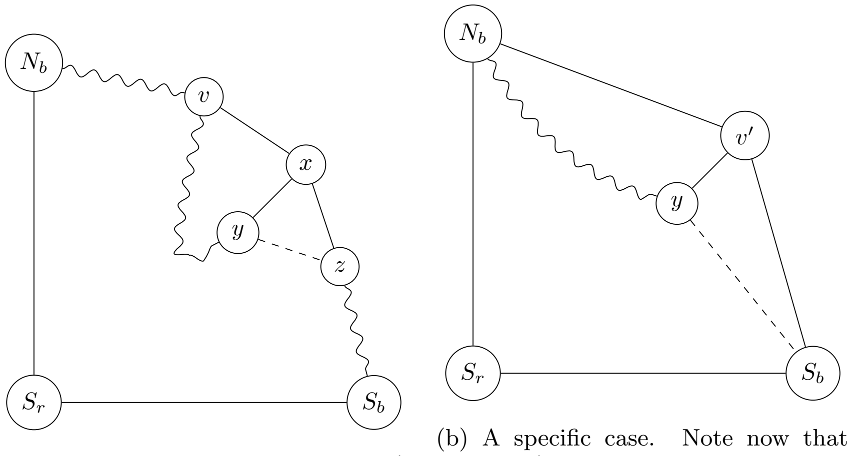
**There is a path that also satisfies (E3)** If $\mathcal{P}_2$ satisfies $(E3)$, we take $\mathcal{P}_3 = \mathcal{P}_2$. Otherwise we will remedy the defect. We separate five different cases of offending edges. All of the five cases will be easy to remedy giving a path $\mathcal{P}_2'$ still satisfying (E1) and (E2) such that $\mathcal{C}_{\mathcal{P}_2'}$ is strictly contained in $\mathcal{C}_{\mathcal{P}_2}$

a) edges from $[v, v'] \setminus v, v'$to $[v, v'] \setminus v, v'$

b) edges from $\mathcal{P} \setminus v, v'$ to $\mathcal{P} \setminus v, v'$

[4]which will be the case if the above construction has been used

[5]this is the cycle $\mathcal{C}$ which is separating




(a) The general case. Note that $x = v'$.

(b) A specific case. Note now that $N_b = v, v' = x$ and $S_b = z$

Figure 6: Creating a path satisfying (E2). The dotted line is the edge we take in the new path $\mathcal{P}_2$

c) edges incident to $v$ or $v$ and some other vertex on $\mathcal{C}_{\mathcal{P}_2}$

d) edges from $[v, v']$ to some internal vertex

e) edges from $\mathcal{P} \setminus v, v'$ to some internal vertex

The existence of an edge as in a) is forbidden by Invariant 14 (I2). If b) occurs we can simply shortcut our original path $\mathcal{P}_2$ with this edge. If c) occurs this edge can't go to another vertex in $[v, v']$ since that would offend Invariant 14 (I2). Hence they go to a vertex in $\mathcal{P}_2$ and we can shortcut the path as in b).

If d) occurs we simply make a new path and if e) occurs we take a slightly adapted interior path. See figures

Since all of the moves shrink $\mathcal{C}_{\mathcal{P}_2}$ while keeping (E1) and (E2) intact and we can't infinitely shrink this means at a certain point no more moves are available. Since every offending edges allows a move this means that there are no more offending edges. Hence this version of $\mathcal{P}'_2$ satisfies (E3). For the final step of the proof we take $\mathcal{P}_3 = \mathcal{P}'_2$.

**There is a path that also satisfies (E4)** Suppose that $\mathcal{P}_3$ does not satisfy (E4). Then we can just take the would be interior edge and take this for a nwe path. This is again a finite procedure reducing the sum of $|\mathcal{P}_3| - |[v, v']|$. In the end we have a path satisfying (E1) - (E4).

□

### 4.3 Minimum distributive lattice element

We get this when we take the "leftmost" eligible path.

FiXme: write section, potentially just refer to Fusy2006

### 4.4 Horizontal one-sides

**Writers note:** we should define the border of a face of a bipolar orientation somewhere. **Writers note:** and what we mean with *cycle border* and *face* *border* **Writers note:** as well as the notation of $\mathcal{F}_\mathcal{P}$ the face of a path

As an exercise one could try to addapt Fusy's algorithm to generate horizontally one-sided layouts directly, without doing flips in the distributive lattice. It turns out that this is not that diffcult.

Since the horizontal segments correspond to face in the blue bipolar orientation we want that one of the two borders of the face has a length of at most two. Since every eligible path we take splits off one face in the blue bipoolar orientation it is easy to control this property.

**Theorem 17.** *In the update of the algorithm there is always an eligible path $\mathcal{P}$ available such that either $\mathcal{P}$ or $[v, v']$ is of length 2.*

In order to proof this theorhem we will first show the following lemma.

**Lemma 18.** *If $\mathcal{P}$ is an eligble path giving raise to a face $\mathcal{F}_P$ of which both border have length at least 3. Then there exist an eligible path $\mathcal{P}'$ such that the pathborder and cycleborder of its face $\mathcal{F}_{\mathcal{P}'}$ are both at least 1 shorter than those of $\mathcal{F}_\mathcal{P}$.*

*Proof.* In this proof we will frequently use property (E3) of a Eligible path, we won't mention it every time we use it.

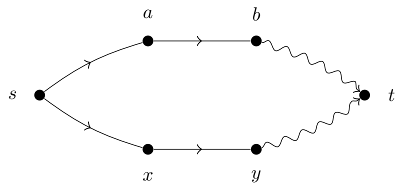
We denote the source by $s$ and the sink by $t$. We also assign names$a, b$ and $x, y$ to the first two vertices on both borders, see Figure 7a. Since every interior face of $G$ is a triangle $ax$ is an edge. Now we distinguis two cases, either $ay$ is an edge (case 1) or $bx$ is an edge (case 2). Tey can't both be an edge at the same time due to planarity, neither can it happen that both of them are not an edge since then the face containing the path $baxy$ is at least of degree 4.

In the first case $a$ may be connected to more vertices on the pathborder, however there is a last one, say $z$. And this vertex is then also connected to $b$, otherwise it would not be the last one. Now we can provide an shorther eligible path $\mathcal{P}'$ we start at $a$ go to $z$ and from there we follow the old path $\mathcal{P}$ to $t$. See figure 7b. It is easy to see that all four properties of an eligible path hold for $\mathcal{P}'$.

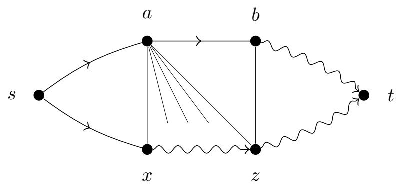
In the second case $x$ may be connected to more vertices along the cycle border, however there is a last one, say $c$. And this vertex is then also connected to $y$, otherwise it would not be the last one. Now we can provide an shorther eligible path $\mathcal{P}' = sxz$. See figure 7c. It is straightforward to see that all four properties of an eligible path hold for $\mathcal{P}'$. ☐

*Proof of Theorem 17.* By Theorem 16 we know there is a eligible path $\mathcal{P}$. If one of the borders of $\mathcal{F}_\mathcal{P}$ is of length 2 or less we are done. If this path gives raise to a face $\mathcal{F}_\mathcal{P}$ with both borders are both of length at least 3 we can repeatedly apply Lemma 18 until at least one of the borders is of length at most 2. ☐

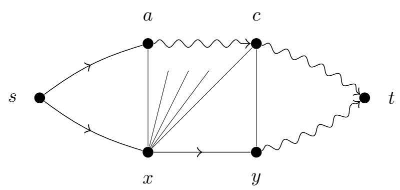
If we in every update of the algorithm take the paths from Theorem 17 we end up with the correct faces in the blue bipolar orientation and hence a horizontally one sided rectangular dual.




(a) The setting




(b) Case 1




(c) Case 2

Figure 7

## 4.5 Vertical one-sided

We can also adapt Fusy's algorithm to generate a vertically one-sided dual. We do this by picking the eligible path with the leftmost starting point and letting it run for as long as possible. In the final oriented regular edge labelling we want to prevent red faces that have 3 or more edges on both borders.

If a red face has

**Lemma 19.** *A face $F$ with at least 3 edges on each side contains a $Z$*

But a $z$ can only be the result of a seqeuncce of eligble paths that does not satisfy the requirements.

# 5 The red algo

**Notational concerns** We will use $\mathcal{C}$ to indicate the current sweep line cycle. We will repeatedly only consider the path $\mathcal{C} \setminus \{\mathrm{S}\}$. In that case we will always order it from W to E.

We will let $\mathcal{W}$ denote a interior walk . Given such a walk of $k$ vertices we index it's nodes $w_1, \ldots, w_k$ in such a way that $w_1$ is closer to $W$ then $w_k$ is (and thus that $w_k$ is closer to $E$ then $w_1$ is).

FiXme: have i defined this already

Then $w_1$ and $w_k$ indicate the two unique vertices of the walk that are also part of the cycle. We will then let $\mathcal{C}|_{\mathcal{W}}$ denote the part of $\mathcal{C} \setminus \mathrm{S}$ that is between $w_1$ and $w_k$ (including). $\mathcal{C}_{\mathcal{W}}$ will denote the closed walk formed when we paste $\mathcal{C}|_{W}$ and $\mathcal{W}$.

Since paths are a subclass of walks all of the above notation can also be used for a path $\mathcal{P}$. Note that the closed walk $\mathcal{C}_{\mathcal{P}}$ in this case will actually be a cycle.

**prelim** *nondistinct corner.*

*chordfree path*

Chords to the left/right of a path

**Lemma 20.** *If a boundray path is without chords adding a pole to it will not create a separating triangle (cf. Yeap)*

## 5.1 Outline

To describe the algorithm two more definitions are necessary

**Definition** (prefence)**.** A prefence $\mathcal{W}$ is a interior walk of $\mathcal{C}$ starting at W and ending at E such that

(P1) all neighbors of $v_i \in \mathcal{C} \setminus \{\mathrm{W}, \mathrm{S}, \mathrm{E}\}$ that are between $v_{i-1}$ and $v_{i+1}$ in clockwise order are in $\mathcal{W} \setminus \{\mathrm{W}, \mathrm{E}\}$

(P2) all the neighbours of $w_i \in \mathcal{W} \setminus \{\mathrm{W}, \mathrm{E}\}$ that are between $w_{i-1}$ and $w_{i+1}$ in clockwise order (if any) are in $\mathcal{C} \setminus \{\mathrm{W}, \mathrm{S}, \mathrm{E}\}$

(P3) $w_1$ and $v_1$ are next to each other in the clockwise order around W

(P4) $w_k$ and $v_n$ are next to each other in the clockwise order around E

We enforce these conditions because they imply (E3) when $\mathcal{W}$ is a path. Every edge with one endpoint on the cycle is of the required type by the conditions. Interior edges with both endpoints not on the cycle can a priori exist. However since we have a connected graph there must then also be an edge with one endpoint in $\mathcal{C}_{\mathcal{W}}$, but this can not be if $\mathcal{W}$ is a prefence.

For a walk however the interior is not clearly defined.

**Definition** (fence)**.** A fence is a valid path starting at W and ending at E

We will show that there is a algorithm if there are no separating 4-cycles in $G$ and no separating 3-cycles in $\bar{G}$.

FiXme: expand on name-ing/reasons of fence

If graph $G$ has non-distinct corners or cutvertices or it is empty we treat them separately and recurse on a smaller graph.

FiXme: TODO

The main algorithm will receive as input a extended graph $\bar{G}$ without non-distinct corners and no separating 4 cycles and will return a regular edge labeling such that all red faces are $(1-\infty)$ using a sweep-cycle approach inspired by Fusy [2].

FiXme: spelling Fusy and cite

We will start by creating a walk $W$. This walk may not be a valid path, it doesn't even have to be a path. During the algorithm we will make a number of moves that will turn this candidate walk into a valid path. In each move we shrink $C$ by employing a valid path and change the candidate walk.

One invariant we will always maintain is that the area bounded by $\mathcal{C}_{\mathcal{W}}$ will never have interior vertices. .

FiXme: What is exactly the area bounded by a closed walk

## 5.2 Treating nondistinct corners and cutvertices of $G$

-Still to be written -

## 5.3 Finding a initial prefence

FiXme: change this to handle nondistinct corners

Let $v_i$ denote all the vertices of $\mathcal{C} \setminus \{\mathrm{W}, \mathrm{S}, \mathrm{E}\}$ in the order that they occur on $\mathcal{C} \setminus \{\mathrm{S}\}$. That is $\mathcal{C} \setminus \{\mathrm{S}\}$ is given by $\mathrm{W}v_1 \dots v_n\mathrm{E}$. As candidate walk we will start with W, we will then take the vertices adjacent to $v_1$ between E and $v_2$ in clockwise order (exclusive), followed the vertices adjacent to $v_2$ between $v_1$ and $v_3$ in clockwise order and so further until we finally add the vertices adjacent to $v_n$ between $v_{n-1}$ and E in clockwise order and finally we finish by adding $E$.

We then remove all subsequent duplicate vertices from $\mathcal{W}$.

**Lemma 21.** *The collection $W$ described above is a prefence.*

FiXme: introduce a term for "edges subsequent to each other in clockwise order around $v$"

*Proof.* We will first show that $W$ is a walk. We will proof that every vertex is adjacent to the next vertex. Let us suppose that $w$ and $w'$ are two subsequent vertices in $W$, we will show that $ww'$ is an edge if $\{w, w'\} \cap \{\mathrm{W}, \mathrm{E}\} = \emptyset$ afterwards we will consider this edge case. There are then two main case for $w, w'$. Either $(a)$ $w$ and $w'$ are vertices adjecent to some $v_i$ subsequent in clockwise order or $(b)$ $w$ was the last vertex adjacent to some $v_i$ and thus $w'$ is the first vertex adjacent to $v_{i+1}$.

The following two situations can also be seen in Figure 8.

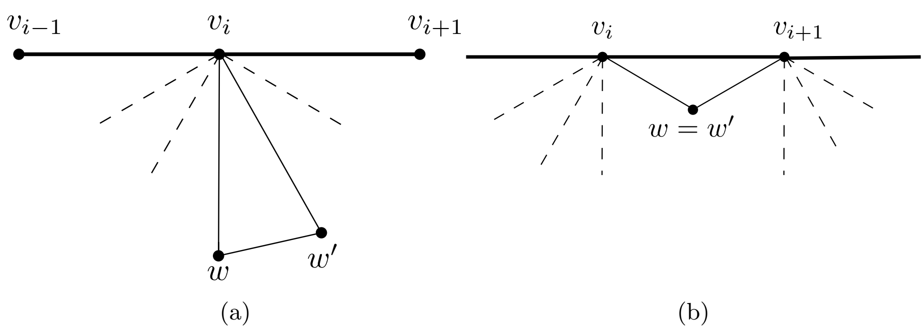



Figure 8: The two main cases of the proof showing that $W$ is a walk

In case $(a)$ we note that $v_i w$ and $v_i w'$ are edges next to each other in clockwise order around $v_i$. Since every interior face of $\bar{G}$ is a triangle $ww'$ must be an edge. We thus see that $w, w'$ are adjacent and not duplicates.

In case $(b)$ we note that $v_i w$ and $v_i v_{i+1}$ are edges subsequent in clockwise order, hence $wv_{i+1}$ is also an edge. Hence $w$ is the first vertex adjacent to $v_{i+1}$ after $v_i$ in clockwise order. Thus $w = w'$. They are duplicates and one of them must have been removed.

Now for the edge cases: Let $w_1$ be the first vertex adjacent to $v_1$ and let $w_m$ be the last vertex adjacent to $v_n$. W and $w_1$ are vertices adjacent to $v_1$ subsequent in clockwise order, and hence connected. $w_m$ and E are vertices adjacent to $v_n$ subsequent in clockwise order and hence connected.

Hence $\mathcal{W}$ is a walk. The above also shows that $\mathrm{W}w_1v_1$ and $\mathrm{E}v_nw_m$ are triangles and hence $\mathcal{W}$ satisfies properties (P3) and (P4) of being a prefence.

Moreover this walk satisfies (P1) because $\mathcal{W}$ by construction contains all neighbors of any vertex $v_i \in \mathcal{C} \setminus \{\mathrm{W}, \mathrm{S}, \mathrm{E}\}$ between $v_{i-1}$ and $v_{i+1}$ in clockwise order.

Finally to see that $\mathcal{W}$ also satisfies (P2). Consider a vertex $w_j \in \mathcal{W} \setminus \{\mathrm{W}, \mathrm{E}\}$ then either it is $(a)$ the neighbor of some vertex $v_i$ and only of this vertex or it is $(b)$ the unique vertex neighboring of $\ell+1$ number vertices $v_i, \dots, v_{i+\ell}$ in the interior of the cycle. This is essentially the same case distinction as above. However now $(a)$ $w_{i-1}w_iv_i$ and $v_iw_iw_{i+1}$ or $(b)$ $w_{i-1}w_iv_i$, $v_iw_iv_{i+1}, \dots, v_{i+\ell-1}w_iv_{i+\ell}$ and $v_{i+\ell}w_iw_{i+1}$ form a set of triangles spanning the area between $w_{i-1}$ and $w_{i+1}$ in clockwise order. Thus any edge not going to $\mathcal{C} \setminus \{\mathrm{W}, \mathrm{S}, \mathrm{E}\}$ in this sector will lead to a separating triangle. We however have assumed $G$ has no separating triangles. Hence (P3) holds.[6] □

## 5.4 Irregularities

We will distinguish two kinds of *irregularities* in a prefence.

1. The candidate walk is non-simple in a certain vertex. That is, if we traverse the sequence of vertices in $\mathcal{W}$ we see that $w_i = w_j$ for some $i < j$.

2. The candidate walk has a chord. That is, there is an edge $w_iw_j$ in $G$ with $i < j$ and $i$ and $j$ not subsequent (i.e. $i < j - 1$).

   Note that such a chord can only lie on the right of $\mathcal{W}$ ($\mathcal{W}$ being oriented from W to E), since if it would lie on the left of $\mathcal{W}$ the vertices $w_{i+1}, \dots, w_{j-1}$ would not have been chosen by the construction. can we orient a walk like this, maybe make more explicit) FiXme: (

**Lemma 22.** *If a prefence has no irregularities it is fence.*

*Proof.* We will show that all the requirements of being a valid path are met. FiXme: refer by labels instead of text

Path Let us begin by noting that since there are no non-simple points we actually have a path and not just a walk.

(E1) It is clear that both $w_1$ and $w_k$ are not S by the construction of the candidate walk.

[6] I believe this is still true when separating triangles are allowed to occur. However the prove will have to be different.

(E2) For $\mathcal{W}$ or $\mathcal{C}|_{\mathcal{W}}$ to have only one edge we need to have that WE is an edge (since $\mathcal{W}$ is constructed as walk from W to E). However, then one of the 3-cycles WEN or WES is separating since the graph $G$ is non-empty. Hence both $\mathcal{W}$ and $\mathcal{C}|_{\mathcal{W}}$ have more than one edge.

(E3) We note that $\mathcal{W}$ is both a path and a prefence. Hence (E3) is satisfied. Since every edge with one endpoint on the cycle is of the required type by the conditions of a prefence. Interior edges with both endpoints not on the cycle can a priori exist. However since we have a connected graph there must then also be an edge with one endpoint in $\mathcal{C}_{\mathcal{W}}$, but this can not be if $\mathcal{W}$ is a prefence.

(E4) The cycle $\mathcal{C}'$ is simply S$\mathcal{W}$ (since $\mathcal{W}$ is walk from W to E by construction) and since $\mathcal{W}$ has no chords $\mathcal{C}'$ has none not involving S.

Hence, if $\mathcal{W}$ has no irregularities it is a valid path.

Furthermore, $\mathcal{W}$ is a path from W to E because it is prefence. And thus □

**Definition** (Range of a irregularity)**.** For a non-simple point $w_i = w_j$ with $i < j$ has *range* $\{i, \ldots, j\} \subset \mathbb{N}$. A chord $w_i w_j$ with $i < j - 1$ has *range* $\{i, \ldots, j\} \subset \mathbb{N}$.

FiXme: Is point the right word? it is def not a vertex

Note that a chord can't have the same range as a non-simple point since then $w_i w_j$ will be a loop and we are considering simple graphs. Furthermore two chords have different ranges because we otherwise have a multiedge. Two nonsimple points with the same range are, in fact, the same. This leads us to the following remark.

**Remark 23.** *Different irregularities have different ranges.*

**Definition** (Maximal irregularity)**.** A irregularity is maximal if it's range is not strictly contained[7] in the range of any other irregularity.

**Lemma 24.** *Maximal irregularities have ranges whose overlap is at most one integer.*

*Proof.* We let $I$ and $J$ denote two distinct maximal irregularities with ranges $\{i_1, \ldots i_2\}$ and $\{j_1, \ldots, j2\}$. Let us for the moment suppose that $I$ and $J$ have ranges that overlap more then one. Since $I$ and $J$ are both maximal their ranges can not be strictly contained in each other and by Remark 23 they can't be equal. Hence the ranges must partially overlap.

FiXme: we might redefine range to make this nicer. However it may make the rest of the algo more ugly. Revisit

Without loss of generality we then have $i_1 < j_1 < i_2 < j_2$. Any additional equality in this chain would offend the ranges not being contained in each other or the overlap being larger then one integer.

Now two chords, both laying to the right of $\mathcal{W}$, would cross each other in this case (but we have a planar graph).

FiXme: I need to note somewhere that every vertex in the candidate walk is adjacent to a subpath of $\mathcal{C} \setminus \{\mathrm{S}\}$

A non-simple point $w_{i_1} = w_{i_2}$ is adjacent to two ranges of vertices in $\mathcal{C} \setminus \{\mathrm{S}\}$. $v_a \ldots v_b$ and $v_c \ldots v_d$ we need that $b$ and $c$ are not subsequent otherwise we have a separating 3 cycle $w_{i_1} v_b v_c$, now however $\tilde{(C)} = w_{i_1} v_b \ldots v_c$ is a cycle. And because of the clockwise order of adjacencies around the vertices of $\mathcal{C} \setminus \{\mathrm{S}\}$ we have that $w_{i_1+1}, \ldots, w_{i_2-1}$ are inside this cycle while $w_1 \ldots w_{i_1-1}$ and $w_{i_2+1} \ldots w_k$ are outside the cycle. See Figure.

FiXme: add figure

[7] Because of Remark 23 being contained is the same as being strictly contained

Now $J$ being a chord will imply a edge crossing $\tilde{C}$, which can't be. The same argumentation holds symmetrically for $J$ being a non-simple point and $I$ being a chord. Two nonsimple points would imply that the vertex $w_{j_i} = w_{j_2}$ is at the same time inside and outside $\tilde{C}$. □

## 5.5 Moves

The algorithm will remove these irregularities by recursing on a subgraph for each maximal irregularity. We shrink the cycle $\mathcal{C}$ with every valid path that is found in the recurrence, in the order they are found. Afterwards we update the candidate walk by removing $w_{i+1}, \ldots, w_{j-1}$ in the case of a chord or $w_{i+1}, \ldots, w_j$ in the case of a nonsimple point. In subsection 5.5.3 we will show that the updated prefence $\mathcal{W}$ walk is an prefence for the upadated cycle $\mathcal{C}$.

We will first show how to remove these maximal irregularities in Subsections 5.5.1 and 5.5.2. That is, we show which subgraph $H$ we recurse upon for both kinds of irregularity. Furthermore we show that this subgraph suffices the requirements of the algorithm.

Afterwards, in subsection 5.5.3 we will make sure that the subgraphs we recurse upon are edge-disjoint. That is, they only overlap in border vertices.

It is worth noting that other irregularities contained in such a maximal irregularity are solved in the recurrence.

### 5.5.1 Chords

If we encounter a chord we will extract a subgraph and recurse on this subgraph. A chord $w_iw_j$ has a triangular face on the left and on the right (like every edge). The third vertex in the face to the left will be called $x$. $x$ is not necessarily distinct from $w_{i+1}$ and/or $w_{j-1}$ but that is also not necessary for the rest of the argument.

The vertex $v_a$ on the cycle is uniquely determined as the vertices adjacent to both $w_i$ and $wi+1$. In the same way $v_b$ is the unique neighbor of $w_{j-1}$ and $w_j$. FiXme: work out these in a example

We will describe a path $\mathcal{U}$ running from $v_a$ to $v_b$. This path consists of all vertices adjacent to $w_i$ in clockwise order from $v_a$ (inclusive) to $x$(inclusive) and subsequently all vertices adjacent to $w_j$ in clockwise order from $x$ (exclusive) to $v_b$ (inclusive). This path is given in bold in Figure 9.

FiXme: note this is a walk by a similar argument as $\mathcal{W}$

FiXme: restructure

**Lemma 25.** *$\mathcal{U}$ is a chordfree path*

*Proof.* $\mathcal{U}$ cant have a non-simple point $x'$ since it would have to be connected to at least two vertices. However a vertex $x'$ that is distinct from $x$ and is connected to both $w_i$ and $w_j$ will induce a separating triangle $w_ix'w_j$.

FiXme: make "connected" more precise: connected where?

$\mathcal{U}$ can't have chords $u_iu_j$ since they would either induce a separating 3- or 4-cycle either $w_iu_iu_j$ or $w_ju_iu_j$ or $w_iu_iu_jw_j$ depending on the vertex adjacent to $u_i$ and $u_j$. □

FiXme: Here we use no 4-cycles

We then consider the interior of the cycle $\mathcal{C}_\mathcal{U}$ and the cycle $\mathcal{C}_\mathcal{U}$ itself as the subgraph $H$. We then set $v_a$ = W and $v_b$ = E and we connect all vertices in $\mathcal{C}|_\mathcal{U}$ to a new north pole N and all vertices in $\mathcal{U}$ to a new south pole S. We then arrive at the graph $H'$ upon which we will recurse. See also Figure 9. Since $\mathcal{C}$ is chordfree by invariant 14 (I2) so is $\mathcal{C}|_\mathcal{U}$. We have also just shown that $\mathcal{U}$ is chordfree. Hence adding the north and south pole doesn't create separating triangles. Furthermore since $H$ is a induced subgraph of $G$ it contains no separating 4-cycles not involving the poles.

FiXme: adapt candidate walk

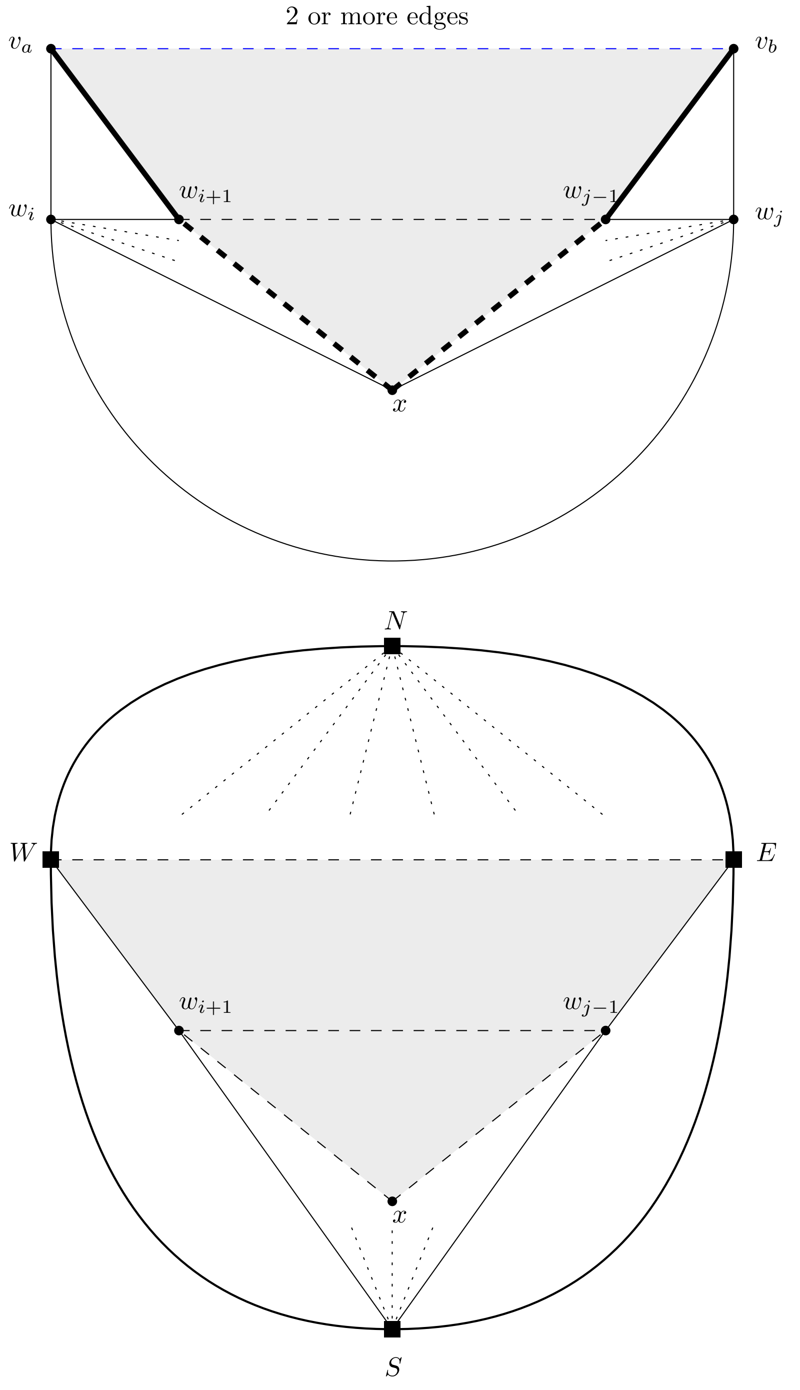



Figure 9: Removing a chord

### 5.5.2 Nonsimple points

Removing a non-simple point is done is a similar manner.

The vertex $v_a$ on $\mathcal{C}$ is uniquely determined as the vertices adjacent to both $w_i = w_j$ and $wi + 1$. In the same way $v_b$ is the unique neighbor of $w_{j-1}$ and $w_j = w_i$. Note that it may be that $w_{i+1} = wj - 1$ this does not matter for the rest of the argument.

We will describe a path $\mathcal{U}$ running from $v_a$ to $v_b$. This path consists of all vertices adjacent to $w_i = w_j$ in clockwise order from $v_a$ (inclusive) to $v_b$(inclusive). This path is given in bold in Figure 10.

FiXme: show example

**Lemma 26.** *$\mathcal{U}$ is in fact a path, moreover it has no chords.*

*Proof.* $\mathcal{U}$ cant have a non-simple point $x$ since such a point would have to be connected to at least two vertices. However every vertex is only connected to $w_i = w_j$.

FiXme: restructure

$\mathcal{U}$ can't have chords on the right of the path by construction. Furthermore $\mathcal{U}$ can't have chords $u_iu_j$ on the left since they would either induce a separating 3-cycle $w_iu_iu_j$. □

FiXme: make "connected" more precise: connected where? On the right of the path

We then consider the interior of the cycle $\mathcal{C}_\mathcal{U}$ and the cycle $\mathcal{C}_\mathcal{U}$ itself as the subgraph $H$. We then set $v_a$ = W and $v_b$ = E and we connect all vertices in $\mathcal{C}|_\mathcal{U}$ to a new north pole N and all vertices in $\mathcal{U}$ to a new south pole S. We then arrive at the graph $H'$ upon which we will recurse. See also Figure 9. Since $\mathcal{C}$ is chordfree by invariant 14 (I2) so is $\mathcal{C}|_\mathcal{U}$. We have also just shown that $\mathcal{U}$ is chordfree. Hence adding the north and south pole doesn't create separating triangles. Furthermore since $H$ is a induced subgraph of $G$ it contains no separating 4-cycles not involving the poles.

FiXme: Here we use no 4-cycles

FiXme: adapt candidate walk

### 5.5.3 Validity

**Lemma 27.** *After doing a move the updated prefence $W$ is a prefence for the updated cycle $C$*

*Proof.* □

FiXme: TODO

**Lemma 28.** *Let $H_I$ and $H_J$ be two recursion subgraphs for different maximal irregularities $I$ and $J$. Then $H_I$ and $H_J$ are edge disjoint.*

*Proof.* □

FiXme: TODO

## 5.6 Correctness

As long as the interior of $\mathcal{C}$ is nonempty we can find

The algorithm finishes because it keeps on recursing and shrinking until no graph is left.

### 5.6.1 The red faces

Let us then argue that the red faces are all $(1-\infty)$ faces, corresponding to one-sided vertical segments. Let us note that every time we do a move for a maximal irregularity with range $\{i, \ldots, j\}$. We will shrink the cycle by the valid paths found in the recursion, in the order they are found in the recursion. Insofar

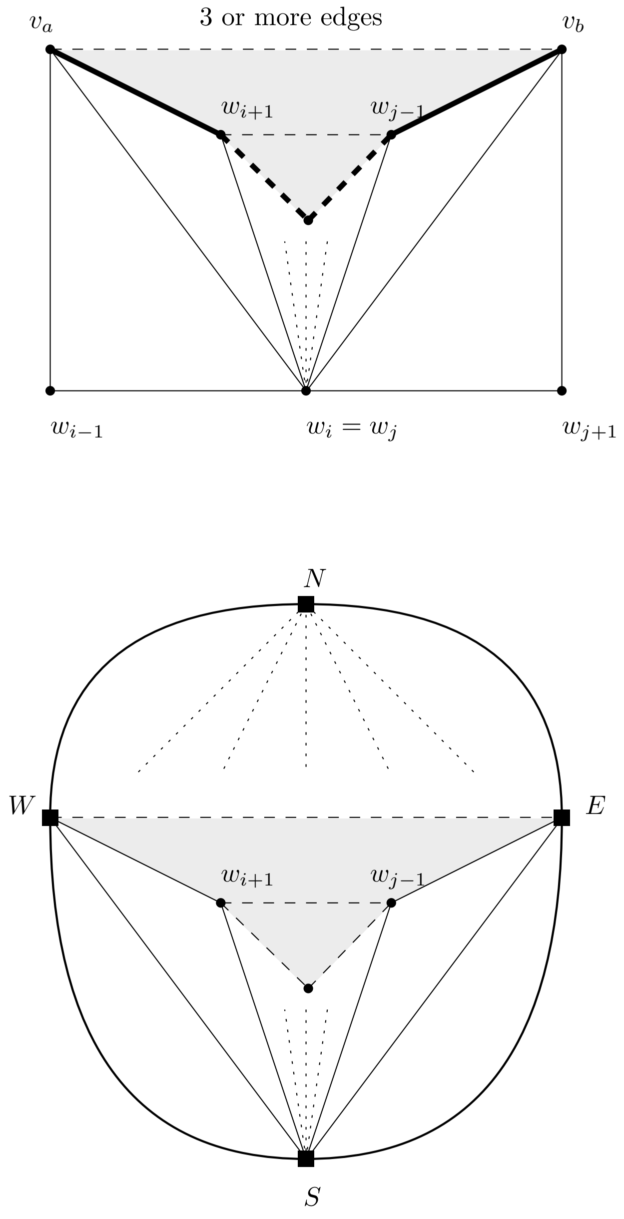



Figure 10: Removing a non-simple point

these paths are connected to $\mathcal{C}$ they start on or after the unique neighbor of $w_i$ and $w_{i+1}$ on the cycle and the end on or before the unique neighbor of $w_{j-1}$ and $w_j$. Since the ranges of maximal irregularties at most overlap by 1 integer (Lemma 24) their paths do not overlap

# 6 TODO

Cool examples: The multiple non-simple point $v_i = v_j = v_k$ Example of page $F1$

example with lots of chords