Thesis

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Notational concerns We will use $\mathcal{C}$ to indicate the current sweep line cycle. We will repeatedly only consider the path $\mathcal{C} \setminus \{\mathrm{S}\}$. In that case we will always order it from W to E.

We will let $\mathcal{W}$ denote a interior walk . Given such a walk of k vertices we index it's nodes $w_1, \ldots, w_k$ in such a way that w_1 is closer to W then w_k is (and thus that w_k is closer to E then w_1 is).

FiXme: have i defined this already

Then w_1 and w_k indicate the two unique vertices of the walk that are also part of the cycle. We will then let $\mathcal{C}|_{\mathcal{W}}$ denote the part of $\mathcal{C} \setminus \mathrm{S}$ that is between w_1 and w_k (including). $\mathcal{C}_{\mathcal{W}}$ will denote the closed walk formed when we paste $\mathcal{C}|_W$ and $\mathcal{W}$.

Since paths are a subclass of walks all of the above notation can also be used for a path $\mathcal{P}$. Note that the closed walk $\mathcal{C}_{\mathcal{P}}$ in this case will actually be a cycle.

prelim *nondistinct corner.*

chordfree path

Chords to the left/right of a path

Lemma 1. *If a boundray path is without chords adding a pole to it will not create a sep triangle (cf. Yeap)*

FiXme: to prove

0.1 Outline

We will show that there is a algorithm if there are no separating 4-cycles in G and no separating 3-cycles in $\bar{G}$.

If graph G has non-distinct corners or cutvertices or it is empty we treat them separately and recurse on a smaller graph.

FiXme: TODO

The main algorithm will receive as input a extended graph $\bar{G}$ without non-distinct corners and no separating 4 cycles and will return a regular edge labeling such that all red faces are $(1-\infty)$ using a sweep-cycle approach inspired by Fusy [**Fusy2006**].

FiXme: spelling Fusy and cite

We will start by creating a walk W. This walk may not be a valid path, it doesn't even have to be a path. During the algorithm we will make a number of moves that will turn this candidate walk into a valid path. In each move we shrink C by employing a valid path and change the candidate walk.

One invariant we will always maintain is that the area bounded by $\mathcal{C}_{\mathcal{W}}$ will never have interior vertices. .

FiXme: What is exactly the area bounded by a closed walk

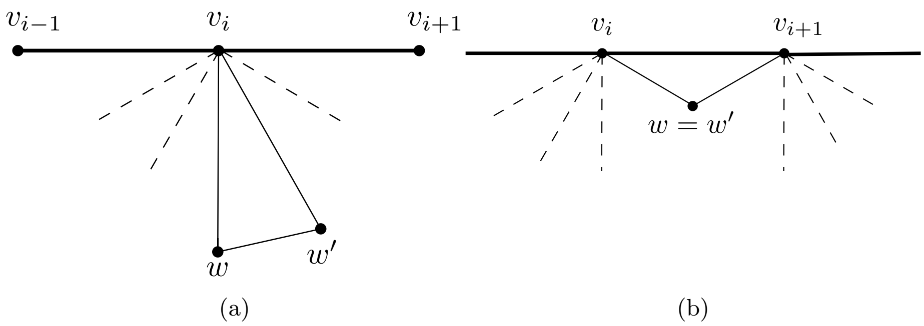


Figure 1: The two main cases of the proof showing that W is a walk after removing duplicates.

0.2 Treating nondistinct corners and cutvertices of G

-Still to be written -

0.3 The initial candidate walk

Let v_i denote all the vertices of $\mathcal{C} \setminus \{\mathrm{W}, \mathrm{S}, \mathrm{E}\}$ in the order that they occur on $\mathcal{C} \setminus \{\mathrm{S}\}$. That is $\mathcal{C} \setminus \{\mathrm{S}\}$ is given by $\mathrm{W}v_1 \ldots v_n\mathrm{E}$. As candidate walk we will start with W, we will then take the vertices adjacent to v_1 between E and v_2 in clockwise order (exclusive), followed the vertices adjacent to v_2 between v_1 and v_3 in clockwise order and so further until we finally add the vertices adjacent to v_n between v_{n-1} and E in clockwise order and finally we finish by adding E.

FiXme: change this to handle nondistinct corners

Lemma 2. *After removing subsequent duplicates the collection W described above is indeed a walk.*

FiXme: introduce a term for "edges subsequent to each other in clockwise order around v"

Proof. To show that W is a walk it's sufficient to show that every vertex is adjacent to the next vertex. Let us suppose that w and w' are two subsequent vertices in W, we will show that they are connected if $\{w, w'\} \cap \{\mathrm{W}, \mathrm{E}\} = \emptyset$ after that we will consider this edge case. There are then two main case for w, w'. Either (a) w and w' are vertices adjecent to some v_i subsequent in clockwise order or (b) w was the last vertex adjacent to some v_i and thus w' is the first vertex adjacent to v_{i+1}.

The following two situations can also be seen in Figure 1.

In case (a) we note that $v_i w$ and $v_i w'$ are edges next to each other in clockwise order around v_i. Since every interior face of $\bar{G}$ is a triangle ww' must be an edge. We thus see that w, w' are adjacent and not duplicates.

In case (b) we note that $v_i w$ and $v_i v_{i+1}$ are edges subsequent in clockwise order, hence wv_{i+1} is also an edge. Hence w is the first vertex adjacent to v_{i+1} after v_i in clockwise order. Thus $w = w'$, they are duplicates and we will remove w.

Now for the edge cases: Let w_1 be the first vertex adjacent to v_1 and let w_m be the last vertex adjacent to v_n. W and w_1 are vertices adjacent to v_1 subsequent in clockwise order, and hence connected. w_m and E are vertices adjacent to v_n subsequent in clockwise order and hence connected. $\square$

0.4 Irregularities

We will distinguish two kinds of *irregularties* on the candidate walk.

1. The candidate walk is non-simple in a certain vertex. That is, if we traverse the sequence of vertices in $\mathcal{W}$ we see that $w_i = w_j$ for some $i < j$.

2. The candidate walk has a chord. That is, there is an edge $w_i w_j$ in G with $i < j$ and i and j not subsequent (i.e. $i < j - 1$).

   Note that such a chord can only lie on the right of $\mathcal{W}$ ($\mathcal{W}$ being oriented from W to E), since if it would lie on the left of $\mathcal{W}$ the vertices $w_{i+1}, \dots, w_{j-1}$ would not have been chosen by the construction.

Lemma 3. *If a candiate walk has no irregularities it is a valid path.*

Proof. We will show that all the requirements of being a valid path are met.

FiXme: refer by labels instead of text

Path Let us begin by noting that since there are no non-simple points we actually have a path and not just a walk.

(E1) It is clear that both w_1 and w_k are not S by the construction of the candidate walk.

(E2) For $\mathcal{W}$ or $\mathcal{C}|_{\mathcal{W}}$ to have only one edge we need to have that WE is an edge (since $\mathcal{W}$ is constructed as walk from W to E). However, then one of the 3-cycles WEN or WES is separating since the graph G is non-empty. Hence both $\mathcal{W}$ and $\mathcal{C}|_{\mathcal{W}}$ have more than one edge.

(E3) There are no interior edges in $\mathcal{C}_{\mathcal{W}}$ that are adjacent to $w_1 = W$ or $w_k = E$ since $w_2 v_1$ and $w_{k-1} v_n$ are edges in $\bar{G}$. This can be seen in the construction of the path. But it is also enough to realize that $w_1 = \mathrm{W}$ and w_2 are subsequent neighbors in the clockwise order around v_1. The same holds for w_{k-1} and $w_k = \mathrm{E}$ around v_n.

Furthermore there are no interior edges with both vertices adjacent to $\mathcal{C}|_{\mathcal{W}}$ because these edges would be chords offending Invariant **??** .

FiXme: fix ref

Finally there are no interior edges with both adjacencies to vertices in $\mathcal{W}$ because $\mathcal{W}$ has no chords since it has no irregularities.

(E4) The cycle $\mathcal{C}'$ is simply S$\mathcal{W}$ (since $\mathcal{W}$ is walk from W to E by construction) and since $\mathcal{W}$ has no chords $\mathcal{C}'$ has none not involving S.

Hence, if $\mathcal{W}$ has no irregularities it is a valid path. ☐

Definition (Range of a irregularity)**.** For a non-simple point $w_i = w_j$ with $i < j$ has *range* $\{i, \dots, j\} \subset \mathbb{N}$. A chord $w_i w_j$ with $i < j - 1$ has *range* $\{i, \dots, j\} \subset \mathbb{N}$.

FiXme: Is point the right word? it is def not a vertex

Note that a chord can't have the same range as a non-simple point since then $w_i w_j$ will be a loop and we are considering simple graphs. Furthermore two chords have different ranges because we otherwise have a multiedge. Two nonsimple points with the same range are, in fact, the same. This leads us to the following remark.

Remark 4. *Different irregularities have different ranges.*

Definition (Maximal irregularity)**.** A irregularity is maximal if it's range is not strictly contained[1] in the range of any other irregularity.

Lemma 5. *Maximal irregularities have ranges whose overlap is at most one integer.*

FiXme: we might redifine range to make this nicer. However it may make the rest of the algo more ugly. Revisit

Proof. We let I and J denote two distinct maximal irregularities with ranges $\{i_1, \dots i_2\}$ and $\{j_1, \dots, j2\}$. Let us for the moment suppose that I and J have ranges that overlap more then one. Since I and J are both maximal their ranges can not be strictly contained in each other and by Remark 4 they can't be equal. Hence the ranges must partially overlap.

Without loss of generality we then have $i_1 < j_1 < i_2 < j_2$. Any additional equality in this chain would offend the ranges not being contained in each other or the overlap being larger then one integer.

FiXme: I need to note somewhere that every vertex in the candidate walk is adjacent to a subpath of $\mathcal{C} \setminus \{\mathrm{S}\}$

Now two chords, both laying to the right of $\mathcal{W}$, would cross each other in this case (but we have a planar graph).

A non-simple point $w_{i_1} = w_{i_2}$ is adjacent to two ranges of vertices in $\mathcal{C} \setminus \{\mathrm{S}\}$. $v_a \dots v_b$ and $v_c \dots v_d$ we need that b and c are not subsequent otherwise we have a separating 3 cycle $w_{i_1} v_b v_c$, now however $\tilde(C) = w_{i_1} v_b \dots v_c$ is a cycle. And because of the clockwise order of adjacencies around the vertices of $\mathcal{C} \setminus \{\mathrm{S}\}$ we have that $w_{i_1+1}, \dots, w_{i_2-1}$ are inside this cycle while $w_1 \dots w_{i_1-1}$ and $w_{i_2+1} \dots w_k$ are outside the cycle. See Figure.

FiXme: add figure

Now J being a chord will imply a edge crossing $\tilde{C}$, which can't be. The same argumentation holds symmetrically for J being a non-simple point and I being a chord. Two nonsimple points would imply that the vertex $w_{j_i} = w_{j_2}$ is at the same time inside and outside $\tilde{C}$. □

0.5 Moves

We will now show how to remove these maximal irregularities. These maximal irregularities don't influence each other because their ranges only overlap at most one. Other irregularities contained in such a maximal irregularity are solved in the recurrence.

0.5.1 Chords

If we encounter a chord we will extract a subgraph and recurse on this subgraph. A chord $w_i w_j$ has a triangular face on the left and on the right (like every edge). The third vertex in the face to the left will be called x. x is not necessarily distinct from w_{i+1} and/or w_{j-1} but that is also not necessary for the rest of the argument.

FiXme: work out these in a example

The vertex v_a on the cycle is uniquely determined as the vertices adjacent to both w_i and $wi + 1$. In the same way v_b is the unique neighbor of w_{j-1} and w_j.

We will describe a path $\mathcal{U}$ running from v_a to v_b. This path consists of all vertices adjacent to w_i in clockwise order from v_a (inclusive) to x(inclusive) and subsequently all vertices adjacent to w_j in clockwise order from x (exclusive) to v_b (inclusive). This path is given in bold in Figure 2.

Lemma 6. *$\mathcal{U}$ is in fact a path, moreover it has no chords.*

Proof. $\mathcal{U}$ cant have a non-simple point x' since it would have to be connected to at least two vertices. However a vertex x' that is distinct from x and is connected to both w_i and w_j will induce a separating triangle $w_i x' w_j$.

FiXme: restructure

FiXme: make "connected" more precise: connected where?

$\mathcal{U}$ can't have chords $u_i u_j$ since they would either induce a separating 3- or 4-cycle either $w_i u_i u_j$ or $w_j u_i u_j$ or $w_i u_i u_j w_j$ depending on the vertex adjacent to u_i and u_j. □

FiXme: Here we use no 4-cycles

We then consider the interior of the cycle $\mathcal{C}_\mathcal{U}$ and the cycle $\mathcal{C}_\mathcal{U}$ itself as the subgraph H. We then set v_a = W and v_b = E and we connect all vertices in $\mathcal{C}|_\mathcal{U}$ to a new north pole N and all vertices in $\mathcal{U}$ to a new south pole S. We then arrive at the graph H' upon which we will recurse. See also Figure 2. Since $\mathcal{C}$ is chordfree by invariant **??** so is $\mathcal{C}|_\mathcal{U}$. We have also just shown that $\mathcal{U}$ is chordfree. Hence adding the north and south pole doesn't create seperating triangles. Furthermore since H is a induced subgraph of G it contains no seperating 4-cycles not involving the poles.

FiXme: ref

0.5.2 Nonsimple points

Removing a non-simple point is done is a similar manner.

The vertex v_a on $\mathcal{C}$ is uniquely determined as the vertices adjacent to both $w_i = w_j$ and $wi + 1$. In the same way v_b is the unique neighbor of w_{j-1} and $w_j = w_i$. Note that it may be that $w_{i+1} = wj - 1$ this does not matter for the rest of the argument.

We will describe a path $\mathcal{U}$ running from v_a to v_b. This path consists of all vertices adjacent to $w_i = w_j$ in clockwise order from v_a (inclusive) to v_b(inclusive). This path is given in bold in Figure 3.

FiXme: show example

Lemma 7. *$\mathcal{U}$ is in fact a path, moreover it has no chords.*

Proof. $\mathcal{U}$ cant have a non-simple point x since such a point would have to be connected to at least two vertices. However every vertex is only connected to $w_i = w_j$.

FiXme: restructure

FiXme: make "connected" more precise: connected where? On the right of the path

$\mathcal{U}$ can't have chords on the right of the path by construction. Furthermore $\mathcal{U}$ can't have chords $u_i u_j$ on the left since they would either induce a separating 3-cycle $w_i u_i u_j$. □

FiXme: Here we use no 4-cycles

We then consider the interior of the cycle $\mathcal{C}_\mathcal{U}$ and the cycle $\mathcal{C}_\mathcal{U}$ itself as the subgraph H. We then set v_a = W and v_b = E and we connect all vertices in $\mathcal{C}|_\mathcal{U}$ to a new north pole N and all vertices in $\mathcal{U}$ to a new south pole S. We then arrive at the graph H' upon which we will recurse. See also Figure 2. Since $\mathcal{C}$ is chordfree by invariant **??** so is $\mathcal{C}|_\mathcal{U}$. We have also just shown that $\mathcal{U}$ is chordfree. Hence adding the north and south pole doesn't create separating triangles. Furthermore since H is a induced subgraph of G it contains no separating 4-cycles not involving the poles.

FiXme: ref

1 TODO

Cool examples: The multiple non-simple point $v_i = v_j = v_k$ Example of page $F1$

[1] Because of Remark 4 being contained is the same as being strictly contained

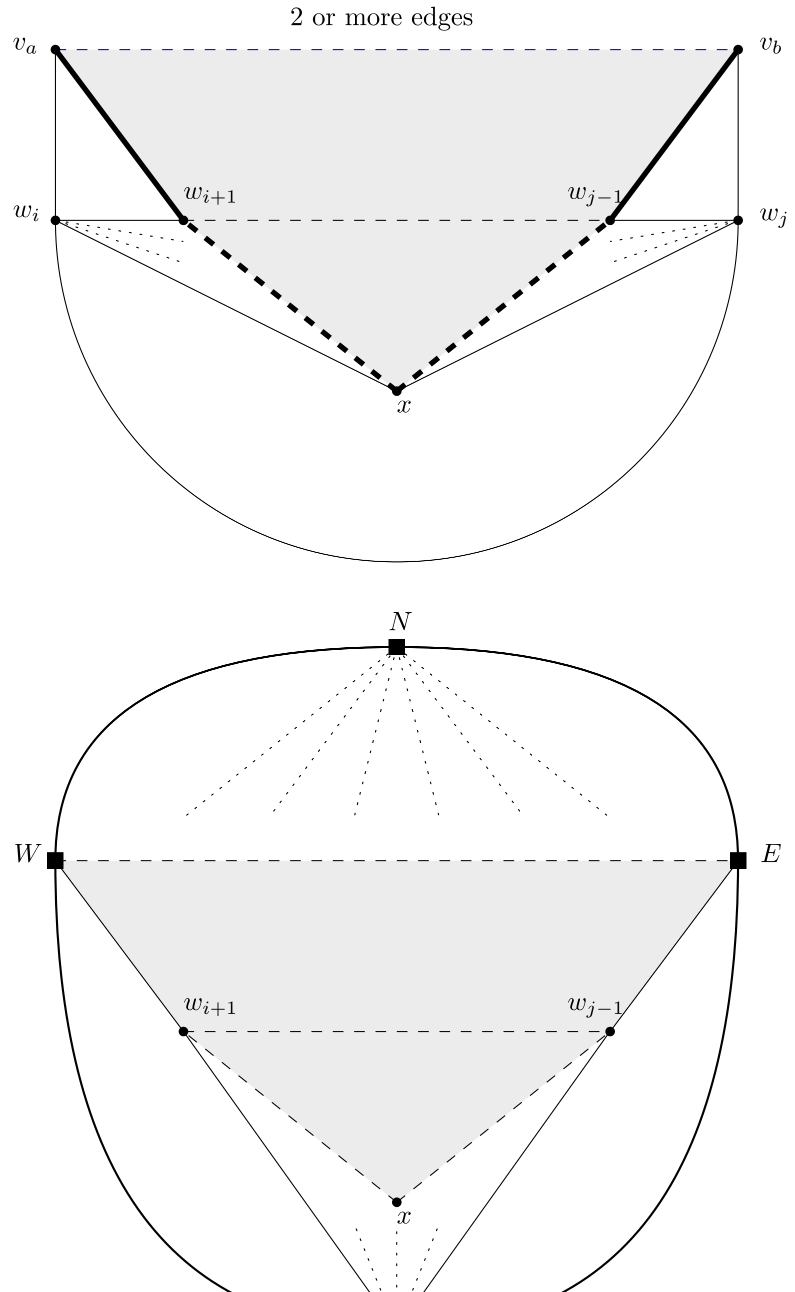
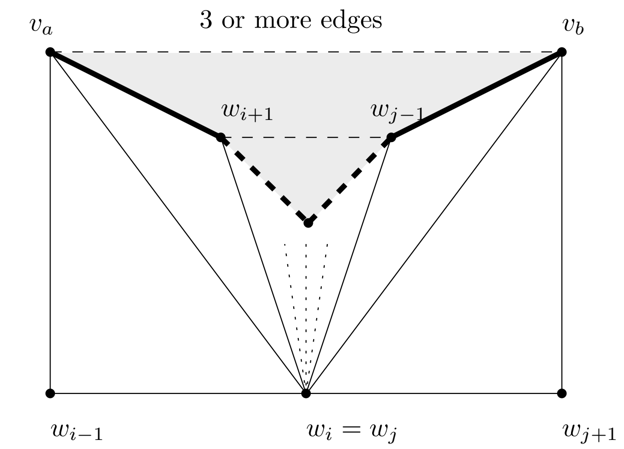


Figure 2: Removing a chord

example with lots of chords



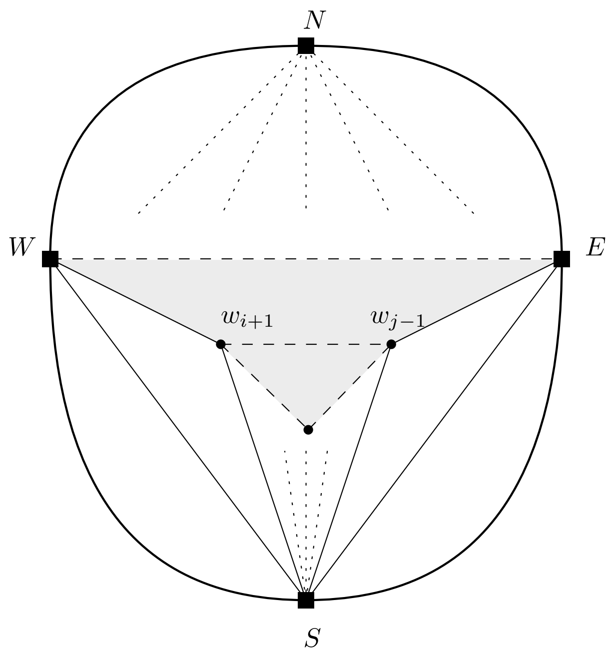


Figure 3: Removing a non-simple point