Cyclically 5-Edge-Connected Cubic Planar Graphs and Shortness Coefficients

P.J. Owens
UNIVERSITY OF SURREY
GUILDFORD, U.K.

ABSTRACT

It is shown that some classes of cyclically 5-edge-connected cubic planar graphs with only one type of face besides pentagons contain non-Hamiltonian members and have shortness coefficients less than unity.

1. INTRODUCTION

For any graph G, let $v(G)$ denote the number of vertices, $h(G)$ the length of a maximum cycle and $h^*(G)$ the length (measured by the number of vertices) of a maximum path. For any infinite class of graphs $\mathscr{G}$, the *shortness coefficient* $\rho(\mathscr{G})$ is defined [4] by

$$\rho(\mathscr{G}) = \lim_{G \in \mathscr{G}} \inf \frac{h(G)}{v(G)}$$

and $\rho^*(\mathscr{G})$ is defined similarly but with h^* in place of h.

Let $P_3(t)$ denote the class of 3-connected cubic planar graphs with at most t types of face and $G_3(p,q)$, where $p < q$, the class of graphs in $P_3(2)$ whose faces are all p-gons or q-gons. It is well known that every cubic planar graph has some faces with less than six edges, so $3 \leq p \leq 5$. Moreover, we assume that $q \geq 6$, since otherwise $G_3(p,q)$ is not infinite and does not contain any non-Hamiltonian graphs. Before we introduce the precise topic of the paper, we give a brief summary of what is known about the existence of non-Hamiltonian graphs in $G_3(p,q)$ and the shortness coefficient of this class, for all p and q.

Journal of Graph Theory, Vol. 6 (1982) 473–479
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When $p = 3$, the 3-connectedness requires that $q \leq 10$. For $q = 8,9$, and 10 the class $G_3(3,q)$ contains some non-Hamiltonian graphs and $\rho(G_3(3,q)) < 1$ [8]. On the other hand, P.R. Goodey [3] has shown that all graphs in G_3 (3,6) are Hamiltonian.

Barnette's conjecture, that every bipartite 3-connected cubic planar graph is Hamiltonian, remains unproved. Its truthwould imply that all graphs in $G_3(4,2k)$, where $k \geq 3$, were Hamiltonian. This has only been proved in the case $k = 3$ (see P.R. Goodey [2]). More is known about G_3 $(4,2k+1)$, namely that it contains some non-Hamiltonian graphs, and has a shortness coefficient less than one, for all $k \geq 3$ (see H. Walther [10] for $k \leq 4$ and [9] for $k = 3$).

The class $G_3(5,q)$ contains some non-Hamiltonian graphs, and has shortness coefficient less than unity, for all $q \geq 7$. Except for the cases $q = 7$ [6] and $q = 10$ [7] these results are due to J. Zaks [11], [12], [13].

A graph is called *cyclically r-edge-connected* if at least r edges must be deleted to disconnect it into two components, each of which contains a cycle. Let $\mathscr{C}$ (r) denote the class of all cyclically r-edge-connected 3-connected cubic planar graphs. Clearly $r \geq 3$ and, since every face of a graph in $\mathscr{C}$ (r) must have at least r edges, $r \leq 5$. Moreover, every graph in $\mathscr{C}$ (5) must have some pentagonal faces. It has been shown [5] that $\rho(\mathscr{C}(4)) \geq 3/4$, which implies that $\rho(\mathscr{C}(5)) \geq 3/4$ also. J. Zaks [14] has shown that $\rho(\mathscr{C}(5)) \leq 85/86$.

In the present paper we consider (for infinitely many values of q) the class $\mathscr{C}(5) \cap G_3(5,q)$ and show not only that it contains non-Hamiltonian members but also that its shortness coefficient is less than unity. Thus $\rho(\mathscr{C}(5) \cap P_3(t)) < 1$ even for $t = 2$, which is a best possible result.

2. THREE LEMMAS

The first lemma is given without proof since it is a well known consequence of Grinberg's theorem (see [1, p.146]).

Lemma 1. Let G be a planar graph with exactly one face whose number of edges is not congruent to 2(mod 3). Then G is non-Hamiltonian. ■

We shall construct graphs which contain induced subgraphs of the type shown in Figure 1. The five "dangling" edges are included to show how a subgraph of type M should be joined to the rest of a graph in which it occurs. The numbers around the labeled circle which we use to represent the subgraph show how many vertices it supplies to each adjoining face.

A cycle which contains every vertex of a graph (or subgraph) is said to *span* it. A path between vertices v_i and v_j is denoted by P_{ij}.

Lemma 2. Let G be any graph with an induced subgraph of type M and let C

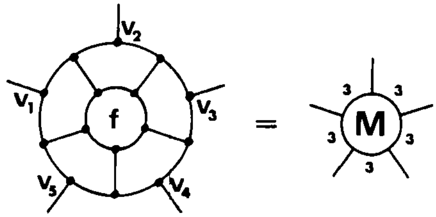

FIGURE 1. Subgraph of type M.

be a cycle in G that is not entirely in M. If C spans M then $C \cap M$ is of the form P_{12} or $P_{12} \cup P_{34}$ (to within a cyclic permutation of suffixes 1,2,3,4,5).

Proof. To within a cyclic permutation of suffixes, the possible forms of the intersection with M of a cycle in G that is not entirely in M are P_{12}, P_{13}, $P_{12} \cup P_{34}$ and $P_{13} \cup P_{45}$. Consider the graph H shown in Figure 2. It contains a copy of M with one dangling edge missing and the other four incident at the vertices of a path $u_3u_4u_5u_1$. Since the exterior face of H is a 9-gon and all other faces are pentagons, Lemma 1 implies that H is non-Hamiltonian. A path P_{13} cannot span M, since otherwise we could add the path $v_3u_3u_4u_5u_1v_1$ and so obtain a spanning cycle in H. Similarly, a pair of paths $P_{13} \cup P_{45}$ cannot span M since otherwise we could add the paths $v_3u_3u_4v_4$, $v_5u_5u_1v_1$ and so obtain a spanning cycle in H. The lemma follows. ■

We now define a subgraph Q_s by taking $5 + 2s$ copies of M, together with $5 + 2s$ pairs of extra vertices, and joining them in a ring as shown (for the case $s = 1$) in Figure 3. Note that Q_0 is the same as the subgraph Q of Zaks [14] and that our third lemma generalizes [14, Lemma 3]. A different proof is required for the general case.

Lemma 3. For any $s \geq 0$, let G be a graph with an induced subgraph of type Q_s and let C be a cycle in G. Then C does not span Q_s.

Proof. We suppose that some cycle C in G does span Q_s and obtain a

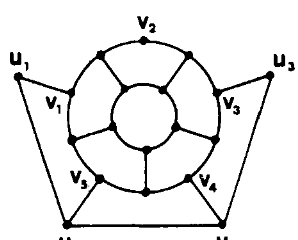

FIGURE 2. The graph H.

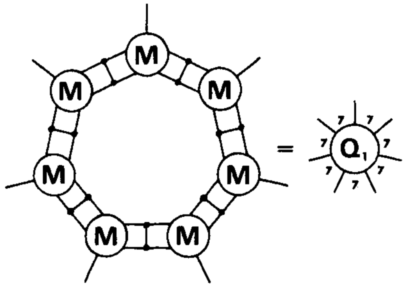

FIGURE 3. The subgraph Q_s (for $s = 1$).

contradiction. Figure 4 shows a part of Q_s, with some of its vertices and edges labeled for reference. The $5 + 2s$ edges which connect Q_s to $G - Q_s$ (such as d_1, d_2) will be called *d-edges* and the $5 + 2s$ vertices situated similarly to z_1 will be called *z-vertices*. Consider the subpath of C that contains z_1 and lies between M_1 and M_2. After allowing for symmetry, there are three different cases to consider.

Case (i). C contains the path $w_2y_1z_1u_1$. Then y_1v_1 is not an edge of C so, by Lemma 2, $C \cap M_1$ connects u_1 to t_1 (that is, either $C \cap M_1$, or one of its components, is a path from u_1 to t_1).

Case (ii). C contains the path $v_1y_1z_1u_1$. Then $C \cap M_1$ must not connect u_1 to v_1, otherwise a cycle would be completed and it would omit (for instance) t_1. Hence, by Lemma 2, $C \cap ,_1$ connects u_1 to t_1.

Case (iii). C contains the path $x_2z_1u_1$. Then $w_2y_1v_1$ is also a subpath of C, since y_1 is in C. It is impossible for $C \cap M_1$ to connect u_1 to v_1 and also $C \cap M_2$ to connect x_2 to w_2, since then a cycle would be completed and it would omit (for instance) t_1. Hence, by Lemma 2, either $C \cap M_1$ connects u_1 to t_1 or $C \cap M_2$ connects x_2 to t_2 (or both).

In all three cases C contains a subpath that starts at z_1, ends with a d-edge (d_1 or d_2) and has no other z-vertices in it. Hence the number of z-vertices in C is no greater than the number of d-edges in C. The latter number must be even, since it is twice the number of components of $C \cap Q_s$, so is at most

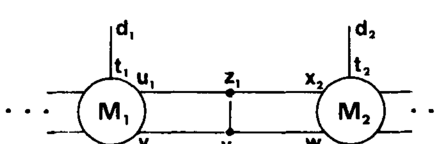

FIGURE 4. Part of Q_s.

$4 + 2s$. Thus C omits at least one z-vertex and this contradicts our initial supposition that C spans Q_s. The lemma follows. ■

3. MAIN RESULTS

Theorem 1. (1) There is a non-Hamiltonian member of $\mathscr{C}(5) \cap P_3(2)$ with only 380 vertices.

(2) For all $s \geq 0$ there is a non-Hamiltonian member of $\mathscr{C}(5) \cap G_3(5,20 + 8s)$ with only $380 + 312s + 64s^2$ vertices.

Proof. Figure 5 shows (for the case $s = 0$) part of a ring subgraph R_s which has $5 + 2s$ dangling edges on each side and supplies $13 + 8s$ vertices to each adjoining face. Let G_0 be the planar graph that consists of two copies of Q_s separated by a ring R_s. All faces within R_s are pentagons and all faces within Q_s, apart from one $(20 + 8s)$-gon, are also pentagons. The $10 + 4s$ faces which lie between R_s and a copy of Q_s have $7 + (13 + 8s) = 20 + 8s$ edges. Since also, by inspection, G_0 is cyclically 5-edge-connected, G_0 is in $\mathscr{C}(5) \cap G_3(5,20 + 8s)$. As G_0 contains two copies of Q_s, Lemma 3 implies that G_0 is non-Hamiltonian and, in fact, that $h(G_0 \leq v(G_0) - 2$. Finally, $v(Q_s) = (5 + 2s)$. 17 and $v(R_s) = (5 + 2s)$ $(42 + 32s)$, so $v(G_0) = 380 + 312s + 64s^2$. This completes the proof of (2) and (1) follows at once from the special case $s = 0$. ■

Theorem 2. (1) $\rho(\mathscr{C}(5) \cap P_3(2)) < 1$.

(2) For all $s \geq 0$, $\rho(\mathscr{C}(5) \cap G_3(5,20 + 8s)) \leq 1 - 1/(360 + 312s + 64s^2) < 1$.

Proof. Denote by X the induced subgraph of G_0 obtained by deleting the vertices of the central pentagon (f in Fig. 1) of any one copy of M in G_0. The faces within X are all 5-gons and $(20 + 8s)$-gons and when X occurs in a graph it supplies three vertices to each of the five adjoining faces, just as M does. Thus the class $\mathscr{C}(5) \cap G_3(5,20 + 8s)$ is closed under replacement of M by X. We consider an infinite sequence $\langle G_n \rangle$ in this class of graphs where (for $n \geq 0$) G_{n+1} is obtained from G_n by replacing one copy of M by a copy of X. Since X contains $4 + 2s$ copies of M in addition to those in its subgraph of type Q_s, we can ensure that after the first step no replacement is made within

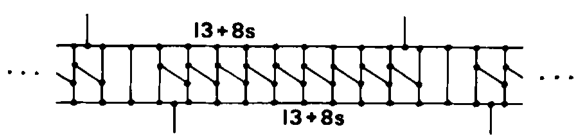

FIGURE 5. Part of a subgraph R_s (for $s = 0$).

an existing copy of Q_s. Thus (for $n > 0$) G_n contains $n + 1$ copies of Q_s and Lemma 3 implies that $h(G_n) \leq v(G_n) - n - 1$. Since

$$v(G_n) - v(G_0) = n(v(X) - v(M)) = n(360 + 312s + 64s^2)$$

it follows that

$$\rho(\mathscr{C}(5) \cap G_3(5,20 + 8s)) \leq 1 - 1/(360 + 312s + 64s^2).$$

This completes the proof of (2), and (1) follows at once from the special case $s = 0$. ■

Corollary. $\rho^*(\mathscr{C}(5) \cap G_3(5,20 + 8s)) \leq 1 - 1/(360 + 312s + 64s^2)$.

Proof. For $n > 0$, at most 2 of the $n + 1$ copies of Q_s in G_n contain ends of a given path in G_n. Hence $h^*(G_n) \leq v(G_n) - n + 1$, and this leads to the same bound for ρ^* as for ρ. ■

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