**Rectangular Layout**

**This talk will be about rectangular layout’s satisfying certain properties. So of course we must first define what a rectangular layout is. A rectangular layout is a rectangle subdivided into finitely many interior-disjoint rectangles.**

**Applications**

**These rectangular layouts are in use by map makers. For example, the actual size of a state does not see much about it’s production or population. Mapmakers then use a rectangular layout to depict the states allowing them to deform these regions easily. While maintaining the correct adjacencies.**

**Applications**

**This is a beautiful example of a handcrafted rectangular layout of the states of the USA deformed to match population sizes. Note that the author managed to fit in large rivers and mountain ranges as straight line segments**

**Applications**

**Of course there are more applications of these layouts. For example in architecture, where the rectangles describe different rooms or in chip design where the rectangles give the placement of different modules.**

**Area-universal layouts**

**Not all rectangular layouts are equally usefull for these applications. An usefull class of layouts is given by the area-universal layouts. These layouts have the same adjacencies no matter what areas we assign to each rectangle.**

**This is usefull in for example building design if you allways want the kitchen and dining room to share a border. Among other constraints.**

**Other uses are in animations of statistics (like population) over time while maintaining adjacencies.**

**It turns out that a layout is area-universal exactly when it is one-sided. As is shown by Eppstein et al. in 2012.**

**One-sided layouts**

**To understand the previous statement we need to know what such an one-sided layout is. Before we can say this we need the concept of a maximal segment. A maximal segment is a line segment in this layout that can’t be extended any further. The blue and green line in this slide are both maximal segments.**

**A layout is one-sided if every maximal segment is the side of a single rectangle. This layout is not onesided. As can be seen by the red line segment.**

**One-sided layout**

**Is this layout one sided? \*pause\* No there is again a maximal segment that is not the side of a single rectangle.**

**Equivalent layouts**

**We will consider layouts as being equivalent when their rectangles have the same adjacencies with the same orientation (left of, right of, above and below).**

**This one is not equivalent because we have a different adjacency. We now have a adjacency between yellow and purple.**

**This one is not equivalent because the yellow-purple adjacency flipped orientation.**

**Separating cycles**

**We will soon bring graphs in to play in order to describe adjacencies in such a rectangular layout. Hence I introduce the concept of a separating cycle already. A separating cycle is a cycle whose removal would split the graph into two components. The vertices of such a cycles are a cutset.**

**Another view of a separating cycle is a cycle with a non-empty interior and exterior.**

**Rectangular duals**

**Kozminski and Kinnen manged showed for which adjacency requirements a rectangular layout can be created. They did this by describing the required adjacencies as a graph and introducing the rectangular dual.**

**A rectangular dual of certain graph G is a rectangular layout in which each rectangle of the layout corresponds to a node of G and two rectangles are adjacent exactly when their nodes are connected by an edge.**

**Kozminski and Kinnen showed that the graphs admitting such a rectangular dual with 4 rectangles on the outer boundary satisfy the following two requirements:**

1. **Every interior face is a triangle and the exterior faces is a quadrangle**
2. **The graph has no separating triangle**

**It is easy to see that the second requirement is a necessary requirement. Suppose that we have a rectangular dual of a graph with a separating triangle. Then this dual would have a an area bounded by only three rectangles. This is of course impossible.**

**Rectangular duals**

**This rectangular dual is not unique. The left and right figure a both rectangular duals of the central graph. But they are nonequivalent layouts.**

**Rectangular duals**

**This theorem leads us to only consider graphs that fulfill all the requirements. We will call the four outer vertices poles (North, east, south and west).**

**If a graph is plane triangulated (all interior vertices are triangles ) and without separating triangles but the outer face is not of the right degree. We can add 4 vertices and consider the graph now obtained, if it fulfils the non-separating triangle requirement this modified graph admits a rectangular dual. We won’t go into detail here but in this case you have to check all polynomially many ways of adding these vertices.**

**If we exclude the trivial case of a graph with only 4 vertices the area bounded by these 4 poles. Will be an area bounded by 4 rectangles and thus must be a rectangle itself. This is because every convex corner must be created by bordering to two different rectangles. And the rectangle is the only rectilinear shape with just 4 convex corners.**

**By convention the outer rectangles are called the north, east, south and west rectangle.**

**This since the north rectangle is above all other rectangles (ignoring east and south) and similar things hold for the other rectangles.**

**We often don’t care how the rectangles representing the poles are adjacent since this won’t significantly change the rectangular dual but only this outside frame.**

**Regular edge labeling**

**We had already seen that some graphs admit several rectangular duals. That is, non-equivalent layouts. We now want to determine exactly one such layout. Because layouts are equivalent if they have the same adjacencies and orientation it is logical to color and orient the graph to determine the orientation.**

**We will first show how given such a rectangular dual we can construct such a colloring and orientation. We refer to this as an regular edge labelling of the graph.**

**We color an edge blue if two rectangles are horizontally adjacent and orient this edge from left to right.**

**We do this for all adjacencies.**

**We color an edge red if two rectangles are vertically adjacent. We orient this edge from top to bottom.**

**We do this for all adjacencies.**

**We don’t orient or color the edges on the outer face. This because we don’t find the different orientations of the adjacencies between the four outer rectangles interesting. These four outer rectangles are also not displayed in this figure.**

**Since we will use these REL’s a lot we will take some time to treat 5 of their properties. The first two properties together form a different way to define a REL**

**Exterior vertex condition**

**If we create a regular edge labelling in this way there are two important properties of this labeling. One of these properties is the exterior vertex condition.**

**Recall that the outer four rectangles form a rectangular frame. All other rectangles (except for the other exterior ones) are going to be below the north rectangles. And thus all interior edges are going to be incoming red.**

**[Give one more example]**

**The same thing holds for the other 2 vertices.**

**All edges between two exterior vertices are uncolored and unoriented. Since we don’t care how these are related to each other.**

**Interior vertex condition**

**The second property is on interior vertices.**

**Since an interior vertex is surrounded on all four sides by other rectangles every interior vertex has the following pattern of edges in clockwise order: outgoing red, outgoing blue, incoming red and incoming blue.**

**It is clear that every regular edge labeling derived from a rectangular dual has these properties. But these two properties actually define a labeling that corresponds to a rectangular dual. Every labeling and orientation satisfying these two properties corresponds to a valid rectangular dual. Which can be constructed by an algorithm by Kant and He.**

**Acyclic flows**

**But first let us consider three more implied properties of the REL.**

**Firstly, both the blue and red subgraph are acyclic flows.**

**In the dual this is very clear. As otherwise one would have a group of rectangles none of which is for example leftmost.**

**But we can also show this by contradiction using the interior vertex condition. Suppose we have a blue cycle. Then all red flows passing the cycle must be directed inward in the same direction (let’s assume inwards but otherwise the same argument works). We follow the inward flow and arrive at an vertex, this vertex can’t be on the cycle since then it wouldn’t receive incoming red.**

**However this cycle has also at least one outgoing red edge. Following this we reach another vertex not on the cycle.**

**Going from one vertex to the next we must eventually hit a vertex for the second time. Creating a cycle.**

**However then in this cycle is another cycle and so on. But this is incompatible with the finite number of vertices inside the cycle. (We are using normal finite graphs).**

**No monocolored triangles**

**This second derived property of a REL can also be observed in both the rectangular dual as the rel. We can’t have any triangular face whose three edges are all of the same color**

**A monocolored triangle would correspond to three rectangles all vertically adjacent to each other. This is off course impossible.**

**In the REL it is also easy to see that this is impossible. Suppose we have a blue monocolored triangle then there is one vertex where there is both a incoming and outgoing blue edge. But then we skip one of the red parts of the interior vertex condition. Thus such a triangle can’t exist**

**Separating 4-cycle**

**There is a more derived property that also holds for these regular edge labeling.**

**That is any separating 4-cycle has one vertex where all the interior edges are incoming red, one vertex where all the interior edges are incoming blue, one vertex where all the interior edges are outgoing red and one where all interior edges are outgoing blue. This is because the 4 vertices in the separating 4-cycle form a rectangular frame.**

**This is all very similar to the exterior vertex condition. The difference being that we don’t know which of the vertices is going to have incoming blue interior edges until we see at least one interior edge whose color is determined.**

**Separating 4-cycle**

**This is just an example of a larger separating 4-cycle so you can see this principle indeed holds for any separating 4-cycle. In this case our 4-cycle contains a number of exterior vertices so there is no freedom in the labeling of the interior edges adjacent to the cycle. We will use this technique in an upcoming counterexample.**

**One-sided duals**

**Since we are interested in area universal-duals a natural question is to look if every graph that permits a rectangular dual permits a one-sided or area-universal dual (recall that these two things are the same). Unfortunately this is not the case. The following counterexample has been found by Rindsma. She proved this enumaring all the possible rel’s of this graph.**

**k-sided duals**

**So if we can’t hope that all graphs are one-sided, what is the next best thing? If one of the sides of each maximal line segment would only contain a retilvyl low number of rectangles then the chances of adjacencies breaking when we resize these rectangles are not that large.**

**Hence we introduce a k-sided segment as a maximal segment that has at most k rectangles on one of it’s sides. A k-sided dual is then one for which all maximal segments are k-sided.**

**This is a 2-sided layout**

**And this is a 3-sided layout**

**And our main research question becomes: Do graphs without a 1-sided dual admit a k-sided dual for some, hopefully small constant, k?**

**k-sided regular edge labeling**

**How would such a k-sided maximal segment look in the regular edge labeling. Since every rectangle corresponds to a vertex such a k-sided segment corresponds to a blue or red face with one of it’s directed paths starting at the splitting vertex on the left and ending at the merging vertex at the right having at most k+2 vertices.**

**∞-sided duals**

Unfortunately k-sided duals will not be possible for all types of graphs. Specifically, graphs with separating 4-cycles containing vertices that are poles can be not k-sided for any k.

Consider the following graph. The white components are just filler components that can arbitrarily large. Larger components lead to larger values of k.

We will color this graph using the properties of REL’s we just described and see that it always has a large blue or large red face.

[Walk trough steps]

**First we color the exterior vertices**

**Then we use the following separating 4-cycles**

**And color them**

**And we color some edges that are forced by the property that we can’t have monocolored triangles.**

**The thick black edge is a problem because if we color it red we get a blue face with a lot of rectangles on both side and If we color it blue we get such a red face.**

**Conjecture**

**We conjecture the following for graphs without separating 4-cycles.**

**Firstly that none are onesided. This hinges on a remark by Eppstein et all that for graphs without a separating 4-cycle being onesided is the same as a graph having a unique regular edge labeling. This however is unlikely since such a graph without separating 4-cycles has a lot of degree 5 and higher vertices giving us a lot of freedom on the inner vertex condition.**

**Moreover we conjecture that all these graphs are 2-sided. This is mostly due to the fact that we havn’t found a counterexample so far. We are adapting a sweep cycle algorithm by Fusy in order to get an algorithm that provably finds these REL’s (first for any constant k and not just 2). The general idea of this algorithm is starting with an unlabeled graph and shrinking the cycle containing the unlabeled part in a structured manner.**

**Thank you all for listening**