Quadcopter model

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1 Introduction

A quadcopter is an aerial vehicle that is able to hover. It has four identical rotors arranged at the corners of a square body, and its propellers or blades have a fixed angle of attack. Figure 1 shows the diagram of the quadcopter that you will have to control. The rotors are paired, and each pair rotates in a different direction. Motors 1 and 3 rotate clockwise when looked from above, whereas motors 2 and 4 have a counter-clockwise rotation. When all the motors rotate at the same angular velocity, the torques τ_1, τ_2, τ_3 and τ_4 (these are the counter torques applied to the aircraft as a consequence of the motors rotation) will cancel each other out and the quadcopter will not splin about its z^b axis ($\dot{\psi} = 0$). The quadcopter will hover when the angular velocities are such taht the total thrust $(f_1 + f_2 + f_3 + f_4)$ generated by the rotors is equal to the force of gravity. In order to describe the movement of the quadrotor and its attitude, two frames of reference are used, namely the inertial frame and the body frame (see Figure 1). The inertial frame is defined by the ground, with gravity pointing in the negative z direction. The body frame is defined by the orientation of the quadcopter, with the rotor axes pointing in the positive z^b direction and the arms pointing in the x^b and y^b directions. The attitude of the quadcopter is determined by three angles, namely roll-ϕ, pitch-θ and yaw-ψ. The way of changing these angles by playing with the angular velocities of the rotors is illustrated in Figure 2. Changes in the roll and pitch are accompanied by translational motion. It should be clear that a quadcopter is an underactuated vehicle, since it only has four actuators (rotors) for controlling six degrees of freedom (six translational, x, y, z, and three rotational, ϕ, θ, ψ). We will denote the angular velocities of the motors by $\omega_1, \ldots, \omega_4$, and the angular velocity at which the quadcopter hovers by ω_{hover}.

2 Quadcopter model

The translational motion of the quadcopter in the inertial frame is described by the following set of equations:

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + RT^b + F_D \tag{1}$$

where x, y, z are the coordinates of the position of the quadcopter in the inertial frame, m is the mass of the system, g is the acceleration due to gravity, F_D is the drag force due to air friction, T^b is the thrust vector in the body frame, and R is the rotation matrix that relates the body frame with the inertial frame and is defined as follows:

$$R = \begin{pmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta \\ \cos\theta\cos\psi & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \tag{2}$$

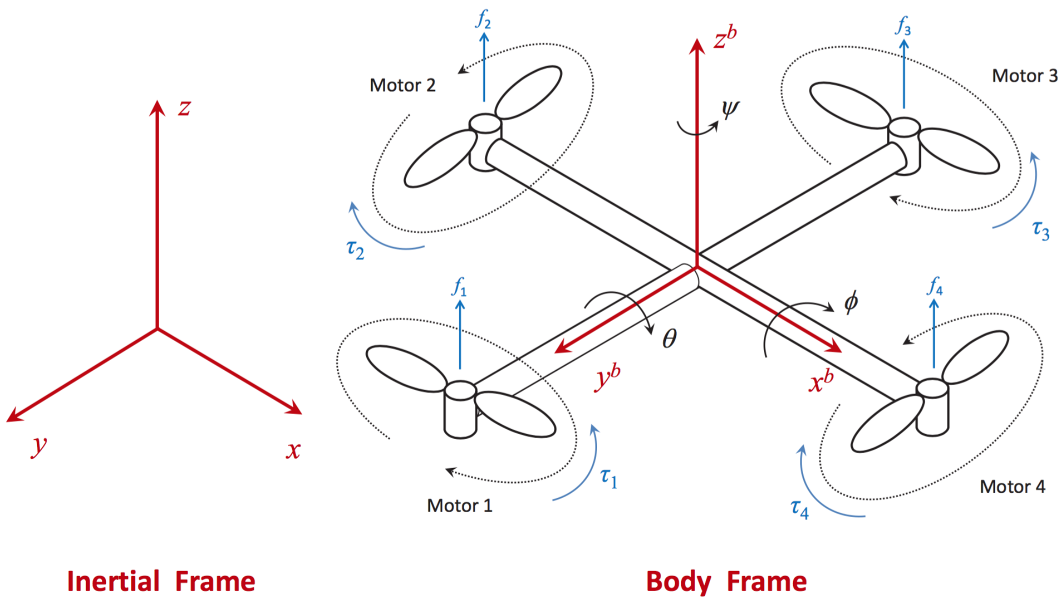

Figure 1: Quadcopter configuration. The roll, pitch, yaw angles are denoted by ϕ, θ, ψ respectively. Motors 1 and 3 rotate clockwise, while motors 2 and 4 rotate counter-clockwise. The thrust generated by the rotors are indicated by f_1, f_2, f_3, f_4, while $\tau_1, \tau_2, \tau_3, \tau_4$ are the torques applied to the aircraft (counter torques) as a consequence of the spinning of the rotors.

The drag force F_D is modelled as a force proportional to the linear velocity in each direction

$$F_D = -k_d \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}.$$

Here k_d is the air friction coefficient. The thrust f_i generated by the i-th rotor is given by the following expression

$$f_i = k\omega_i^2$$

where k is the propeller/rotor lift coefficient and ω_i is the angular velocity of the i-th motor. The total thrust vector T^b generated by the four rotors (in the body frame) is therefore given by

$$T^b = k \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 \omega_i^2 \end{pmatrix}.$$

We can assume that the dynamics of the motors is much faster than the one of the quadcopter, and therefore is not taken into account. Since the angular velocity of each rotor is tipically proportional to the applied voltage, we have that

$$\omega_i^2 = c_m V_i^2$$

where c_m is a constant and V_i is the voltage applied to the i-th motor.

While it is convenient to have linear equations of motion in the inertial frame, the rotational equations of motion are useful to us in the body frame, so that we can express

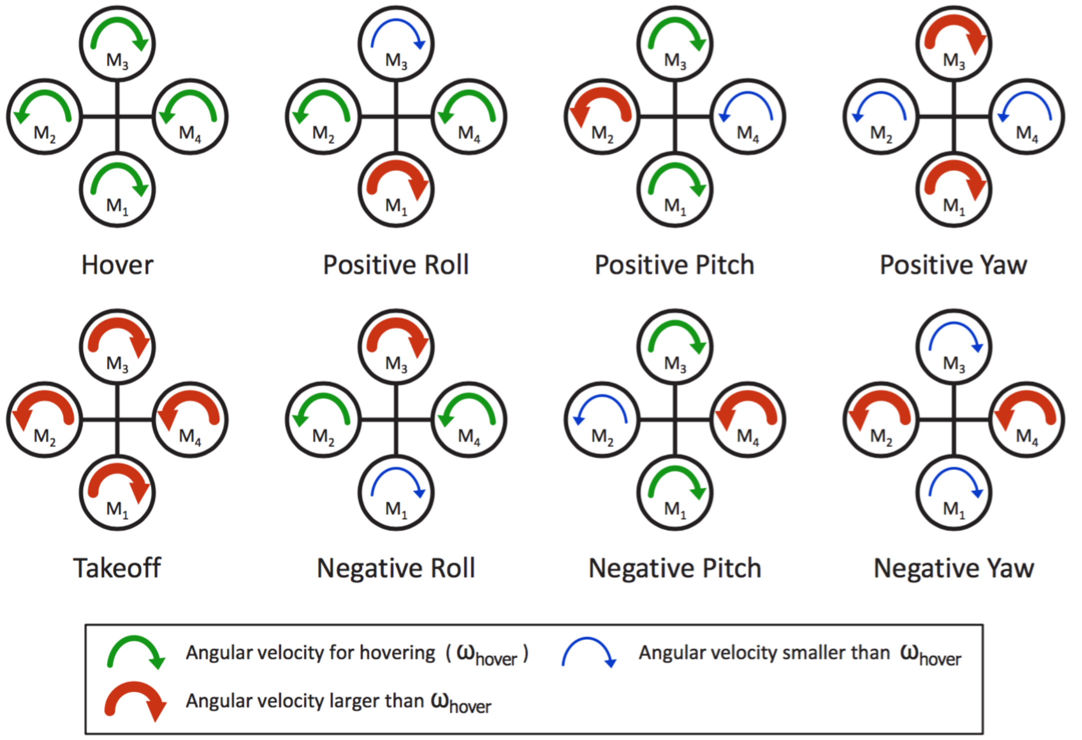

Figure 2: Illustration of the quadcopter motion obtained by varying the angular velocities of its rotors.

rotations about the center of the quadcopter instead of about the inertial center. To this end we can use Euler's equations for rigid body dynamics, which are defined as follows:

$$I\dot{\omega} + \omega \times (I\omega) = \tau \tag{3}$$

where $\times$ denotes cross product, I is the inertial matrix, $\omega = (\omega_x \omega_y \omega_z)$ is the angular velocity vector, and $\tau = (\tau_\phi \tau_\theta \tau_\psi)$ is the vector of external torques.

We can model the quadcopter as two thin uniform rods crossed at the origin with a point mass (motor) at each end. This results in a diagonal inertia matrix of the following form:

$$I = \begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix}$$

where I_{xx}, I_{yy}, I_{zz} are the moments of inertia of the quadcopter about the x^b, y^b, z^b axes respectively. After computing the cross product, equation (3) reduces to:

$$\begin{pmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{pmatrix} \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} = \begin{pmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{pmatrix} - \begin{pmatrix} (I_{yy} - I_{zz})\omega_y\omega_z \\ (I_{zz} - I_{xx})\omega_x\omega_z \\ (I_{xx} - I_{yy})\omega_x\omega_y \end{pmatrix}. \tag{4}$$

The roll and pitch torques are derived from standard mechanics as follows:

$$\tau_\phi = L(f_1 - f_3) = Lk(\omega_1^2 - \omega_3^2) = Lkc_m(V_1^2 - V_3^2),$$
$$\tau_\phi = L(f_2 - f_4) = Lk(\omega_2^2 - \omega_4^2) = Lkc_m(V_2^2 - V_4^2),$$

where L is the distance between the rotor and the quadcopter center (radius). As it was discussed earlier, all the rotors apply torques to the aircraft about its z^b axis, while they rotate. In order to have an angular acceleration about the z^b axis, the total torque generated by the rotors has to overcome the drag forces. The total torque about the z^b axis, that is the yaw torque τ_ψ, is given by the following equation

$$\tau_\psi = b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) = bc_m(V_1^2 - V_2^2 + V_3^2 - V_4^2)$$

where b is the propeller drag coefficient. The roll, pitch and way rates are related to the components of the angular velocity vector by means of the following expression:

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = Q\omega = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{5}$$

where Q is a projection matrix.

Finally, from equations (1), (2), (4) and (5), the equations of the dynamics are as follows:

$$\begin{aligned}
\dot{x} &= v_x && (6) \\
\dot{y} &= v_y && (7) \\
\dot{z} &= v_z && (8) \\
\dot{v}_x &= -\tfrac{k_d}{m} v_x + \tfrac{kc_m}{m}(\sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta)(V_1^2 + V_2^2 + V_3^2 + V_4^2) && (9) \\
\dot{v}_y &= -\tfrac{k_d}{m} v_y + \tfrac{kc_m}{m}(\cos\phi\sin\psi\sin\theta + \cos\psi\sin\phi)(V_1^2 + V_2^2 + V_3^2 + V_4^2) && (10) \\
\dot{v}_z &= -\tfrac{k_d}{m} v_z - g + \tfrac{kc_m}{m}(\cos\theta\cos\phi)(V_1^2 + V_2^2 + V_3^2 + V_4^2) && (11) \\
\dot{\phi} &= \omega_x + \omega_y(\sin\phi\tan\theta) + \omega_z(\cos\phi\tan\theta) && (12) \\
\dot{\theta} &= \omega_y\cos\phi - \omega_z\sin\phi && (13) \\
\dot{\psi} &= \tfrac{\sin\phi}{\cos\theta}\omega_y + \tfrac{\cos\phi}{\cos\theta}\omega_z && (14) \\
\dot{\omega}_x &= Lkc_m I_{xx}^{-1}(V_1^2 - V_3^2) - (I_{yy} - I_{zz})I_{xx}^{-1}\omega_y\omega_z && (15) \\
\dot{\omega}_y &= Lkc_m I_{yy}^{-1}(V_2^2 - V_4^2) - (I_{zz} - I_{xx})I_{yy}^{-1}\omega_x\omega_z && (16) \\
\dot{\omega}_z &= bc_m I_{zz}^{-1}(V_1^2 - V_2^2 + V_3^2 - V_4^2) - (I_{xx} - I_{yy})I_{zz}^{-1}\omega_y\omega_z && (17)
\end{aligned}$$