Quaternion-based quadcopter model

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Abstract

This document offers a very brief description of a quaternion-based model for the attitude dynamics of the EAGLE quadcopter.

1 Quaternion-based model

1.1 Attitude dynamics

Let $\omega = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3$ be the vector of angular velocities, $q = (q_w, q_x, q_y, q_z) \in \mathbb{H}$ be the quaternion describing the orientation of the quadcopter and $\tau = (\tau_x, \tau_y, \tau_z)$ be the torques exercised along the x, y and z axes respectively. Vectors ω and τ are taken in the local system of coordinates.

The attitude dynamics is described by

$$\dot{q} = \tfrac{1}{2} \cdot q \otimes \begin{bmatrix} 0 \\ \omega \end{bmatrix}, \tag{1a}$$

$$\dot{\omega} = I_{cm}^{-1}(\tau - \omega \times (I_{cm}\omega)), \tag{1b}$$

where $\otimes$ is the Hamilton product of two quaternions, $\times$ is the cross-product of vectors of $\mathbb{R}^3$ and I_{cm} is the inertia matrix

$$I_{cm} = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix} \tag{2}$$

Regarding the moments of inertia I_{xx}, I_{yy} and I_{zz} there are several possibilities to obtain estimates:

i They can be estimated by assuming the the quadcopter has some regular geometry, e.g., to assume it is two perfectly perpendicular and isotropic rods at the ends of which there are attached four motors which have a perfectly cylindrical shape. This way you will be able to use standard textbook equations.

ii To make a detailed CAD design of the quadcopter and calculate the moments of inertia using some software

iii To perform an ad-hop experiment, e.g., to measure the period of natural oscillation or to use the on-board IMU to design and conduct an experiment.

You may start off with some initial estimate using the first option, then design a controller and finally assess how sensitive or robust the resulting closed-loop system is.

1.2 Motor dynamics

Each of the torque references u_x, u_y, u_z produces a corresponding frequency of rotation n_x, n_y, n_z and corresponding torque signals τ_x, τ_y, τ_z. The u-to-τ dynamics is described by the block diagram shown in Figure 1 — notice that this model has input u_j (i.e., u_x or u_y or u_z) and outputs $I_{jj}^{-1}\tau_j$ (i.e., $I_{xx}^{-1}\tau_x$, $I_{yy}^{-1}\tau_y$ and $I_{zz}^{-1}\tau_z$ respectively).

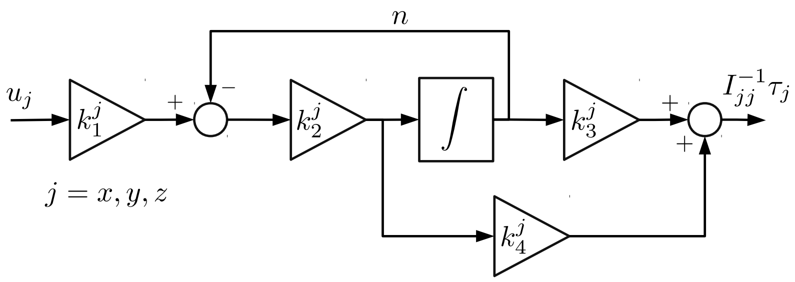

Figure 1: Block diagram of the motor dynamics

The gains k_1, k_2, k_3 and k_4 are given by the following formulas:

$$k_1^x = \frac{V_{\max} - V_{\min}}{60} K_v,$$
$$k_1^y = k_1^z = k_1^x,$$

and

$$k_2 = \frac{1}{\tau_m},$$

where $V_{\max}$ is the max. voltage (around $11.1V$), $V_{\min} = 10\% V_{\max}$, K_v is the *motor speed constant* (in rpm/V), τ_m is the time constant of the motors.

Then, k_3^x and k_3^y are given by

$$k_3^x = \tfrac{\mathrm{d}}{\mathrm{d}n}(C_T \rho n^2 D^4) \tfrac{N_m L}{\sqrt{2} I_{xx}}$$
$$k_3^y = \tfrac{\mathrm{d}}{\mathrm{d}n}(C_T \rho n^2 D^4) \tfrac{N_m L}{\sqrt{2} I_{yy}}$$

where ρ is the density of the air, N_m is the number of motors, C_T is the thrust coefficient of the propellers, D is the diameter of each propeller, L is the arm length of the quadcopter, I_{prop} is the moment of inertia of each propeller, n is the frequency of rotation in rps, I_{xx}, I_{yy} are the moments of inertia about the x and y axes.

Parameter k_3^z is given by

$$k_3^z = \tfrac{\mathrm{d}}{\mathrm{d}n} \left(\frac{C_P \rho n^2 D^5}{2\pi} \right) \frac{N_m}{I_{zz}}$$

where C_P is the power coefficient of the propellers, I_{zz} is the moment of inertia about the z axis.

Parameters $k_4^{x,y}$ and k_4^z are given by

$$k_4^{x,y} = 0,$$
$$k_4^z = 2\pi N_m \frac{I_{\mathrm{prop}} + I_m}{I_{zz}},$$

where I_{prop} is the moment of inertia of each propeller and I_m is each motor's moment of inertia.

A propeller's moment of inertia can be approximated by assuming that it is a rod of total length D which rotates about its center of mass. Then

$$I_{\text{prop}} = m_p \frac{D^2}{12}.$$

1.3 Signals to motors

The signals $v_1, \ldots, v_4$ to the four motors are computed in terms of the torque signals u_x, u_y, u_z and the a *common* signal c as follows

$$\begin{aligned}
v_1 &= c + u_x + u_y - u_z \\
v_2 &= c + u_x - u_y + u_z \\
v_3 &= c - u_x + u_y + u_z \\
v_4 &= c - u_x - u_y - u_z
\end{aligned}$$

where $v_1, \ldots, v_4$ are *relative voltage* signals (they admit values in $[0, 1]$) and will create the torques τ_x, τ_y and τ_z.

1.4 Parameter values

The parameters provided in this section are approximate. You may have to conduct experiments or identify them using input-output data. In any case, these approximate values should be of adequate accuracy to construct a controller that will fly the quadcopter. Be careful with the units of measurement.

You may choose different propellers, but you should find their specifications. You can easily weigh them to get m_p and measure their diameter D. Aerodynamic parameters such as C_T and C_P can be found at `http://m-selig.ae.illinois.edu/props/propDB.html`.

Parameter	Symbol	Value	Units
Number of motors	N_m	4	–
Total mass of the quadcopter	m	1.85	kg
Arm length	L	27	cm
Air density (sea level, 15°)	ρ	1.225	kg/m^3
Gravitational acceleration	g	9.81	m/s^2

Table 1: General properties of the EAGLE quadcopter and various constants.

Parameter	Symbol	Value	Units
Thrust coefficient*	C_T	0.09	–
Power coefficient*	C_P	0.04	–
Propeller mass	m_p	12	g
Propeller diameter	D	11	in

Table 2: Parameters of the propellers (Hobbyking Propeller 11x5 Green).

Parameter	Symbol	Value	Units
Motor speed constant	K_v	700	rpm/V
Motor time constant	τ_m	35	ms
Rotor mass	m_r	42	g
Total motor mass	m_m	102	g

Table 3: Motor parameters (3508-700KV Turnigy Multistar 14 Pole Brushless Multi-Rotor Motor With Extra Long Leads).