SCFQIP - Homework 3 Toni Berger

1. The Kerr-cat qubit

Def: $|C_\alpha^\pm\rangle = N_\alpha^\pm (|+\alpha\rangle \pm |-\alpha\rangle)$

$N_\alpha^\pm = 1/\sqrt{2(1\pm e^{-2|\alpha|^2})}$

$|\pm\alpha\rangle$: same $\bar{n}$, but opposite phase

$|\pm X\rangle = (|C_\alpha^+\rangle \pm |C_\alpha^-\rangle)/\sqrt{2}$

$|\pm Y\rangle = (|C_\alpha^+\rangle \pm i|C_\alpha^-\rangle)/\sqrt{2}$

$|\pm Z\rangle = |C_\alpha^\pm\rangle$

a) We define $|C_\alpha^+\rangle \hat{=} \begin{pmatrix}1\\0\end{pmatrix}$

$|C_\alpha^-\rangle \hat{=} \begin{pmatrix}0\\1\end{pmatrix}$

$\mathbb{1} = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + \begin{pmatrix}0\\1\end{pmatrix}(0\ 1) \hat{=} |C_\alpha^+\rangle\langle C_\alpha^+| + |C_\alpha^-\rangle\langle C_\alpha^-|$

$X = \begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}(0\ 1) + \begin{pmatrix}0\\1\end{pmatrix}(1\ 0) \hat{=} |C_\alpha^+\rangle\langle C_\alpha^-| + |C_\alpha^-\rangle\langle C_\alpha^+|$

$Y = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \hat{=} -i|C_\alpha^+\rangle\langle C_\alpha^-| + i|C_\alpha^-\rangle\langle C_\alpha^+|$

$Z = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \hat{=} |C_\alpha^+\rangle\langle C_\alpha^+| - |C_\alpha^-\rangle\langle C_\alpha^-|$

$X|\pm X\rangle = X(|C_\alpha^+\rangle \pm |C_\alpha^-\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}}(|C_\alpha^-\rangle \pm |C_\alpha^+\rangle) = \begin{cases} 1/\sqrt{2}\,(|C_\alpha^+\rangle + |C_\alpha^-\rangle) & \to EV = 1 \text{ for } |+X\rangle \\ -1/\sqrt{2}\,(|C_\alpha^+\rangle - |C_\alpha^-\rangle) & \to EV = -1 \text{ for } |-X\rangle \end{cases}$

$Y|\pm Y\rangle = Y(|C_\alpha^+\rangle \pm i|C_\alpha^-\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}}(i|C_\alpha^-\rangle \pm |C_\alpha^+\rangle) = \begin{cases} 1/\sqrt{2}\,(|C_\alpha^+\rangle + i|C_\alpha^-\rangle) & \to \quad '' \quad |+Y\rangle \\ -1/\sqrt{2}\,(|C_\alpha^+\rangle - i|C_\alpha^-\rangle) & \to \quad '' \quad |-Y\rangle \end{cases}$

$Z|\pm Z\rangle = Z|C_\alpha^\pm\rangle = \pm|C_\alpha^\pm\rangle$ ✓

b) Without Normalisation: $|C_\alpha^\pm\rangle \sim (|+\alpha\rangle \pm |-\alpha\rangle)$

We expand $|+\alpha\rangle = e^{-1/2|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$ and $|-\alpha\rangle = e^{-1/2|\alpha|^2}\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{\sqrt{n!}}|n\rangle$

$|C_\alpha^+\rangle = 2e^{-1/2|\alpha|^2}\left(\frac{\alpha^0}{\sqrt{0!}}|0\rangle + 0 + \frac{\alpha^2}{\sqrt{2!}}|2\rangle + 0 + \dots\right)$ → odd terms cancel → has even photon number.

$|C_\alpha^-\rangle = 2e^{-1/2|\alpha|^2}\left(0 + \frac{\alpha^1}{\sqrt{1!}}|1\rangle + 0 + \frac{\alpha^3}{\sqrt{3!}}|3\rangle + 0 + \dots\right)$ → even terms cancel → has odd photon number.

c) $\alpha = 0$, We develop $e^{-2|\alpha|^2} = 1 - 2|\alpha|^2 + \dots$

$e^{-\frac{1}{2}|\alpha|^2} = 1 - \frac{1}{2}|\alpha|^2 + \dots$

$|C_\alpha^+\rangle = 1/\sqrt{2(1+e^{-2|\alpha|^2})}\cdot 2e^{-\frac{1}{2}|\alpha|^2}|0\rangle \simeq 1/\sqrt{2(1+1-2|\alpha|^2)}\cdot 2(1-\frac{1}{2}|\alpha|^2)|0\rangle$

$\simeq 1/\sqrt{1-|2\alpha|^2)}\cdot(1-\frac{1}{2}|\alpha|^2)|0\rangle \simeq |0\rangle$

$|C_\alpha^-\rangle = 1/\sqrt{2(1-e^{-2|\alpha|^2})}\cdot 2e^{-\frac{1}{2}|\alpha|^2}\cdot\alpha|1\rangle \simeq \frac{1}{\sqrt{2(1-1+2|\alpha|^2}}\cdot 2(1-\frac{1}{2}|\alpha|^2)\cdot\alpha|1\rangle \simeq \frac{1}{2|\alpha|}\cdot 2(\alpha - \frac{1}{2}|\alpha|^2\alpha)|1\rangle$

$\simeq |1\rangle$

Note: First order approximation would have been sufficient (probably, I didn't check explicitly)

d) limit $|\alpha|^2 = \bar{n} \to \infty$

I. $P = N_\alpha^+/N_\alpha^- = \dfrac{1/\sqrt{2(1+e^{-2|\alpha|^2})}}{1/\sqrt{2(1-e^{-2|\alpha|^2})}} \simeq \dfrac{1/\sqrt{2(1+0)}}{1/\sqrt{2(1-0)}} = 1 \Rightarrow N_\alpha^+ \sim N_\alpha^- \sim \frac{1}{\sqrt{2}}$

II $|\pm X\rangle = (|C_\alpha^+\rangle \pm |C_\alpha^-\rangle)/\sqrt{2} \approx \frac{1}{2}(|+\alpha\rangle + |-\alpha\rangle \pm (|+\alpha\rangle - |-\alpha\rangle)) = \frac{2}{2}|\pm\alpha\rangle = |\pm\alpha\rangle$

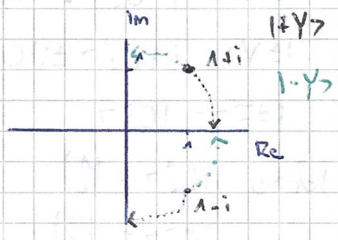
III $|\pm Y\rangle = (|C_\alpha^+\rangle \pm i|C_\alpha^-\rangle)/\sqrt{2} \simeq \frac{1}{2}(|+\alpha\rangle + |-\alpha\rangle \pm i(|+\alpha\rangle - |-\alpha\rangle)) = \frac{1}{2}((1\pm i)|+\alpha\rangle + (1\mp i)|-\alpha\rangle)$

Now we analyze $|\pm Y\rangle$ individually

$|+Y\rangle = \frac{1}{2}((1+i)|+\alpha\rangle + (1-i)|-\alpha\rangle) \Rightarrow$ extract global phase

$\hat{=} \frac{1}{2}(\sqrt{2}|+\alpha\rangle - i\sqrt{2}|-\alpha\rangle) = \frac{1}{\sqrt{2}}(|+\alpha\rangle - i|-\alpha\rangle)$

$|-Y\rangle = \frac{1}{2}((1-i)|+\alpha\rangle + (1+i)|-\alpha\rangle) \hat{=} \frac{1}{\sqrt{2}}(|+\alpha\rangle + i|-\alpha\rangle)$



IV $|\pm Z\rangle = |C_\alpha^\pm\rangle = \frac{1}{\sqrt{2}}(|+\alpha\rangle \pm |-\alpha\rangle)$

V $1-P = 1 - \dfrac{N_\alpha^+}{N_\alpha^-}$ for $\bar{n} = |\alpha|^2 \in \{2,4,6\}$

$$1-p = 1 - \frac{1/\sqrt{2(1+e^{-2|\alpha|^2})}}{1/\sqrt{2(1-e^{-2|\alpha|^2})}} \overset{\text{Wolfram-}\alpha}{\doteq} 1 - \sqrt{\tanh(|\alpha|^2)} = \begin{cases} 0.018 & \text{for } \bar{n}=2 \\ 0.00034 & \bar{n}=4 \\ 0.0000061 & \bar{n}=6 \end{cases}$$

1.2 Photon loss, error suppression and stabilization

Master equation $\dot{\rho} = -\frac{i}{\hbar}(H_{eff}\rho - \rho H_{eff}^\dagger) + \kappa a \rho a^\dagger$ with κ photon loss rate and $H_{eff} = H - \hbar\frac{i}{2}\kappa a^\dagger a$

e) Apply $\sqrt{\kappa} a$ to $\hat{P} = |C_\alpha^+\rangle\langle C_\alpha^+| + |C_\alpha^-\rangle\langle C_\alpha^-|$

$\sqrt{\kappa} a P = \sqrt{\kappa}(a|C_\alpha^+\rangle\langle C_\alpha^+| + \hat{a}|C_\alpha^-\rangle\langle C_\alpha^-|)$

$= \sqrt{\kappa}(a N_\alpha^{+2}(|+\alpha\rangle + |-\alpha\rangle)(\langle+\alpha| + \langle-\alpha|) + a N_\alpha^{-2}(|+\alpha\rangle - |-\alpha\rangle)(\langle+\alpha| - \langle-\alpha|))$

$= \sqrt{\kappa}\alpha(N_\alpha^{+2}(|+\alpha\rangle - |-\alpha\rangle)(\langle+\alpha| + \langle-\alpha|) + N_\alpha^{-2}(|+\alpha\rangle + |-\alpha\rangle)(\langle+\alpha| - \langle-\alpha|))$

$= \sqrt{\kappa}\alpha\left(\frac{N_\alpha^+}{N_\alpha^-}|C_\alpha^-\rangle\langle C_\alpha^+| + \frac{N_\alpha^-}{N_\alpha^+}|C_\alpha^+\rangle\langle C_\alpha^-|\right) \underset{\text{in } C_\alpha^\pm\text{-Basis}}{\doteq} \sqrt{\kappa}\alpha\begin{pmatrix} 0 & N_\alpha^-/N_\alpha^+ \\ N_\alpha^+/N_\alpha^- & 0 \end{pmatrix}$

We can write this in terms of Paulis (here off-diagonal, i.e only X, Y)

$\Rightarrow \sqrt{\kappa} a \hat{P} = \frac{1}{2}\sqrt{\kappa}\alpha\left(\left(\frac{N_\alpha^-}{N_\alpha^+} + \frac{N_\alpha^+}{N_\alpha^-}\right)X + i\left(\frac{N_\alpha^-}{N_\alpha^+} - \frac{N_\alpha^+}{N_\alpha^-}\right)Y\right) = \frac{1}{2}\sqrt{\kappa}\alpha\left(\left(\frac{N_\alpha^{-2} + N_\alpha^{+2}}{N_\alpha^+ N_\alpha^-}\right)X + i\left(\frac{N_\alpha^{-2} - N_\alpha^{+2}}{N_\alpha^+ N_\alpha^-}\right)Y\right)$

$\overset{\text{Wolfram-alpha}}{\doteq} \frac{1}{2}\sqrt{\kappa}\alpha\left(\left(\frac{2}{\sqrt{1-e^{-4|\alpha|^2}}}\right)X + i\left(-\frac{2e^{-2|\alpha|^2}}{\sqrt{1-e^{-4|\alpha|^2}}}\right)Y\right) = \frac{\sqrt{\kappa}\alpha}{\sqrt{1-e^{-4|\alpha|^2}}}(X - ie^{-2|\alpha|^2}Y)$

We can see, that the jump operator corresponds to a superposition of $\underset{X}{\text{bit-flips}}$ and $\underset{Y}{\text{bit-phase-flips}}$ with different complex amplitudes.

f) For large ~~in~~ $\bar{n}$, bit-phase flips (and therefore phase flips) are suppressed exponentially.

Physically, we can argue, that photon loss cannot cause a transition ~~from~~ between $|\pm\alpha\rangle$, since both are Eigenstates of $\hat{a}$.

However, for their superposition, it holds $\hat{a}|C_\alpha^\pm\rangle = \alpha|C_\alpha^\mp\rangle$, which corresponds to a bit-flip.

g) limit $\bar{n} \to \infty$

$p_j = \tau \, Tr[k a \xi a^\dagger]$

Without loss of generality $\hat{\xi} = |\xi\rangle\langle\xi|$ with $|\xi\rangle = a|C_\alpha^+\rangle + b|C_\alpha^-\rangle \Rightarrow \hat{\xi} = \begin{pmatrix} |a|^2 & ab^* \\ a^* b & |b|^2 \end{pmatrix}$

$\Rightarrow X\hat{\xi}X = \begin{pmatrix} |b|^2 & a^* b \\ b^* a & |a|^2 \end{pmatrix}$

$\Rightarrow p_j = \tau \cdot k \cdot |\alpha|^2 \cdot Tr(X\hat{\xi}X) = \tau \cdot k \cdot |\alpha|^2 = \tau \cdot k \cdot \bar{n}$

⇒ Linear dependence on $\bar{n}$ ⇒ The higher the average photon number, the higher the chance for a jump

... but at least it is not exponential ü

h) Parametrically driven nonlinear Hamiltonian $\frac{\hat{H}}{\hbar} = -k a^{\dagger 2} a^2 + \epsilon_2 a^{\dagger 2} + \epsilon_2^* a^2$

We factorize the Hamiltonian $\frac{\hat{H}}{\hbar} = -k\left(a^{\dagger 2} - \frac{\epsilon_2^*}{k}\right)\left(a^2 - \frac{\epsilon_2}{k}\right) + \frac{|\epsilon_2|^2}{k}$

With this form of the Hamiltonian we can guess the Eigenstates $|\pm\alpha\rangle = |\pm\sqrt{\epsilon_2/k}\rangle$

which are Energy degenerate with Energy $\frac{|\epsilon_2|^2}{k}$ (see paper: Engineering the quantum states of light... Puri S. et al. 2017)

Eigenstates will stay the same under evolution with this Hamiltonian.

Since they are Energy-degenerate, the same holds for all of their superpositions.

i) Microwave drive $\hat{H}_{drive}/\hbar = \epsilon_x \hat{a} + \epsilon_x^* \hat{a}^\dagger$

$a \simeq \alpha X \rightarrow a^\dagger \simeq \alpha^* X$

$H_{drive} = \hbar(\epsilon_x \alpha X + \epsilon_x^* \alpha^* X) = \hbar(\epsilon_x \alpha + \epsilon_x^* \alpha^*) X$

We already know the two Eigenstates of this Hamiltonian: $|\pm X\rangle$

Note, that the Rabi frequency is given by $\Omega = \frac{E_+ + E_-}{\hbar}$, where $E_\pm$ are the Eigenenergies of $|\pm X\rangle$ (formula from physics 3)

$E_+ = \hbar(\epsilon_x \alpha + \epsilon_x^* \alpha^*)$ $\quad E_- = -E_+$

$\Rightarrow \Omega = 2(\epsilon_x \alpha + \epsilon_x^* \alpha^*) = 4Re(\epsilon_x \alpha)$

For a transmon, the Rabi frequency would be $\Omega(t) = 2g\alpha(t)$ where g is the vacuum Rabi coupling, which corresponds to the Hamiltonian

$H^d(t) = \xi(t)(e^{-i\omega t} a^\dagger + e^{i\omega t} a)$, which is very similar to the drive, we applied with H_{drive}