University of Basel

Fully quantum description of a tree level maser, driven by a thermal bath

Master-Projekt

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1 Introduction

One of the most important questions in thermodynamics is how to convert thermal energy into work. For such tasks exists many classical engines, as example the steam-machine. In my master-project, I will quantify a three level maser. The three-level maser is a Quantum heat engine (QHE). The work extraction from a classical heat engine is often a moving piston.As example steam machines or a gasoline engine. But in this case it is a driving field. Albert Einstein, already discussed three ways of light-matter-interaction (spontaneous emission, absorption, and stimulated emission) in the year 1916(Quelle Wiki). In paper from 1959 [Quelle altes paper], Scovil and Schulz-DuBois investigated whether a laser is not also a heat engine. In the paper, they take a maser as a device to transform heat into coherent radiation, because heat can make a population inversion. In their thermodynamic analytic, they use a single-atom laser. They made a groundwork for emerging theory of quantum thermodynamics. In practice and also for the calculations, two different reservoirs are necessary. The high-temperature reservoir can be realized by a gas noise lamp and the filter by a wave guide cutting off the lower frequencies. (Quelle main Paper)

2 Theory

2.1 3-level-Maser Model

A Maser/Laser consists of two elements. One of them is a gain medium and the other one is an optical resonator. A gain medium is always a material with an atomic transition between two atomic states. When an atom falls from an energetically higher state to an energetically lower state, a photon is created.

In a three level system the three energy levels are $E1, E2, E3$. Pumping is from the lowest level to the highest level, E3. The condition for the third level is that it falls to the middle level E2 very quickly. On average, the system is almost not in the third state. The resonator should then have a higher decay time, so that a population inversion can build up. This means that several particles are in the energetically higher state. From this state they come almost exclusively through stimulated emission into the lower state E1. Stimulated emission is a necessary condition for coherent light. Coherent light means the light have the same phase and same frequency. (Quelle Wiki)

In the case of this calculation, the higher level will be reached with a interaction of a warm bath.

We denote the frequencies of $\omega_h = (E_2 - E_g)/\hbar$, $\omega_c = (E_1 - E_g)/\hbar$ and $\omega_f = (E_2 - E_1)/\hbar$

2.2 Wigner function

A Wigner function is a representation of a general quantum state of light. The function describe the probability distribution in phase space.

$$w(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\xi e^{\frac{-i}{\hbar}p} \langle x + \frac{1}{2}\xi|\, \rho \,|x - \frac{1}{2}\xi\rangle \tag{1}$$

2.3 Phase-averaged coherent states (PHAV)

The output of a laser is coherent light. The quantum description of coherent light is a coherent state. The photon number distribution of coherent light is a Poisson distribution. The randomized phase of a coherent state doesn't change the photon-number distribution. The Wigner function from a coherent state itself is a Gaussian. But the Wigner function of a phase-average-state has a non-Gaussian Wigner function. The mathematical description of a PHAV is the same as a normal coherent state but with a random phase. So we get a new term of $exp(i\pi\phi)$ in it. The normal coherent state could represented by following formula (Quelle 5):

$$|\alpha\rangle = e^{-1/2|\alpha|}\sum_{n=0}^{\infty}\frac{|\alpha|^n e^{in\phi}}{\sqrt{n!}}|n\rangle\,, \tag{2}$$

To get the phase average state will get reach with the integral around two π.

$$\rho_{PHAV} = \int_0^{2\pi}\frac{d\phi}{2\pi}|\alpha\rangle\langle\alpha| = \sum_{n=0}^{infinity} p_{nn}. \tag{3}$$

this is equal to:

$$\rho_{PHAV} = exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}. \tag{4}$$

For the Wigner function we get finally following equation:

$$W(z) = 2exp\big[-2\big(|\beta|^2 + |z|^2\big)\big)I_0(4|\beta||z|). \tag{5}$$

Useful in the future.(Zitat Paper über PHAV) The consistent experimental and theoretical results we have obtained in the characterization of both PHAVs and their superpositions 2-PHAVs reinforce the possibility of using them for applications to communication protocols.

2.4 Master-Equation

An arbitrary state on this total Hilbert space can be described by a total density operator $\rho_{tot}(t)$.The total density operator is can be written in the Hilbert space $\rho_{tot} = \rho_{atom} + \rho_{photon}$. Encoded in this density operator is a complete description of the total system's state at a given time t. To derive the ρ_{tot} we can solve a specific differential equation. This equation is called Lindblad-master-equation Eq.3. The Basic of our work is a three-level quantum system in a cavity. This three level system is driven by a hot bath and a cold bath. Those have the temperature T_c and T_h. The cavity is build of two mirrors. One of the cavity have a small leaking, so that a small part of the photons can leave the cavity. This leaking is quantified by a constant κ. In the calculation, the thermal bath is constant, therefore we can use the Lindblad-masterequation.

In this case w solve the master-equation for steady states A steady state is a state or condition of a system or process, here the energy states of an atom, that does not change in time, or the changes are negligibly. Therefore, it contains the hole description of the internal state of both the hot and cold heat baths, and of the work environment. In the following figure is shown a three-level system:

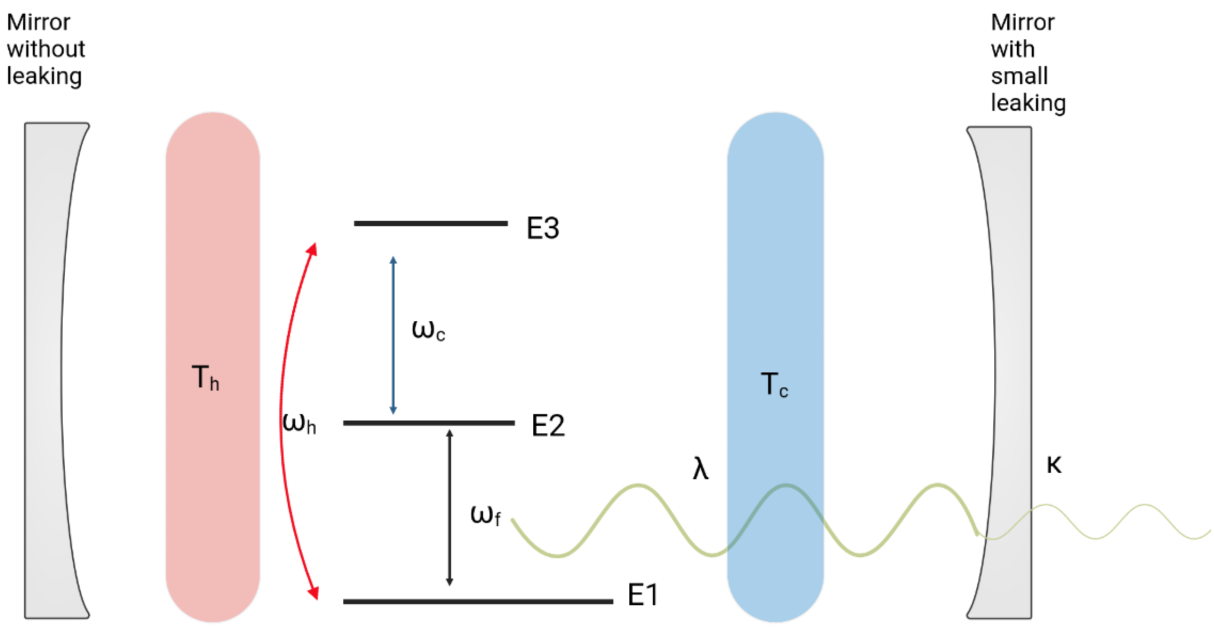

Figure 1: Schematic representation of three-level laser heat engine continuously coupled to two reservoirs of temperatures T_h and T_c having coupling constants Γ_h and Γ_c, respectively. The system is interacting with a classical single mode field. λ represents the strength of matter-field coupling. Source[6]

The master equation is:

$$\dot{\rho}(t) = \frac{1}{i\hbar}[H, \rho] + \mathcal{L}_h\rho + \mathcal{L}_c\rho + \mathcal{L}_{cav}\rho. \tag{6}$$

The first part of Eq.3 is the von Neuman-equation, the analog of the Schrödinger equation but for density matrices. This part of the equation is unitary and therefore the process is reversible. The non-unitary part of the equation $\mathcal{L}\rho$ include the superoperator $\mathcal{L}$, which act on the density operator. A superoperator is a linear operator acting on a vector space of linear operators, as example a density operator. $\mathcal{L}$ consist of three parts. $\mathcal{L}_h$describe the interaction with the hot bath. $\mathcal{L}_c$ is the contribution from the interaction with the cold bath coupled with the atom. $\mathcal{L}_{cav}$ describe the photons which leave the cavity. So,if we have a small κ means less photons will leave and stay in the cavity. we see that in the Fockplott.

The Hamiltonian describes the energy. The atomic field system is composed of two crucial parts; the atomic states, the cavity field, and the interaction between the two. The interaction Hamiltonian or Jaynes-Cummings Hamiltonian:

$$H_{int} = \hbar g(\sigma_{12} a^\dagger + \sigma_{21} a). \tag{7}$$

And the Hamiltonian of the photons is, which describes the photons in the cavity:

$$H_{free} = \sum_{i=1}^{3} \hbar \omega_i \left| i \right\rangle \left\langle i \right| + \hbar \omega_f a^\dagger a, \tag{8}$$

The total Hamiltonian

$$H = H_{free} + H_{int} \tag{9}$$

The interaction with the various environmental heat baths is described by the Liouvillian:

$$\begin{aligned}\mathcal{L}\hat{\rho} = \frac{\gamma_h}{2}(n(\omega_h, T_h) + 1) \cdot \mathcal{D}[\sigma_{13}]\rho + \frac{\gamma_h}{2} n(\omega_h, T_h) \cdot \mathcal{D}[\sigma_{31}]\rho \\ + \frac{\gamma_c}{2}(n(\omega_c, T_c) + 1) \cdot \mathcal{D}[\sigma_{23}]\rho + \frac{\gamma_c}{2} n(\omega_c, T_c) \cdot \mathcal{D}[\sigma_{32}]\rho \\ \kappa((\omega_f, T_f) + 1) \cdot \mathcal{D}[a]\rho + \kappa n(\omega_f, T_f) \cdot \mathcal{D}[a^\dagger]\rho.\end{aligned} \tag{10}$$

The σ_{ab} is the transition operator and defined as $|a\rangle\langle b|$. It describe the transition between a atomic state a to b $\mathcal{D}$ is defined with this formula:

$$\mathcal{D}[A]\rho = (2A\rho A^\dagger - A^\dagger A\rho - \rho A^\dagger A), \tag{11}$$

The Bose-Einstein ocupation number. It is the mean number of excitations in the reservoir damping the oscillator . It describes the mean occupation number $\langle n(E)\rangle$ of a quantum state of energy E, in thermodynamic equilibrium at absolute temperature T for identical bosons as occupying particles. n depends on the temperature and the frequency. n is defined as:

$n(\omega, T) = \frac{1}{\exp[\frac{\hbar\omega_i}{k_b T_i}]-1}$, The prefactor $\gamma_c = \gamma_h$ describes the spontaneous decay rates and are in this calculation generally small. The Liouvillian has different constants. The coupling constants g strong for the Hamiltonian and the κ for the Liouvillian part. The coupling constant g is given by $\frac{\Omega}{\hbar}$,

2.5 Thermodynamics

When we work with density matrices, its common to work with expectation values with $\langle A\rangle = Tr[A\dot{\rho}]$. A is a operator and describe a measurement. With this we can calculate the expectation value from an Operator.

To calculate the expected heat flow we can take the partial trace from

$$\langle J\rangle = Tr[\rho_{free} \cdot \mathcal{L}_h[\rho]] + Tr[\rho_{free} \cdot \mathcal{L}_c[\rho]] + Tr[\rho_{free} \cdot \mathcal{L}_{cav}]. \tag{12}$$

A part of my work is to calculate the occupation number analytically. I made the calculation in two steps. for the warm and the cold path, we have a transition-operators in the trace. The trick of this calculation is, to get the form $Tr[\sigma_{ab}\rho\sigma_{ab}^\dagger]$ because this is equal to Pb The equation gave the following result:

$$Tr[\rho_{free} \cdot \mathcal{L}_h[\rho]] = \hbar\omega_h\gamma_h(2n+1) \cdot (P1 - P3), \tag{13}$$

For the calculation the $Tr[H_{free} \cdot \mathcal{L}_{cav}]$, I get the following result:

$$T[H_{free} \cdot \mathcal{L}_{cav}[\rho]] = 2\hbar\omega_k(\bar{n} - \langle a^\dagger a\rangle). \tag{14}$$

The efficiency is given by the following formula:

$$\eta_{maser} = \frac{\omega_f}{\omega_h} < 1 - \frac{Tc}{Th}. \tag{15}$$

3 Methods

3.1 Software

For the hole implementation of the tree-level-system in a cavity, I used qutip. Qutip is library in python, which allows to solve masterequation pretty easy. Further calculation and methods was easily applied in python.

3.2 Implementation of the tree-level-system in qutip

Qutip can be used to solve master equations. For that we have to define constants. In our case only ω_f interact with the light.

The constants $\hbar$ and the bolzmanfactor k_B are 1. alsou defined as constants are the three different Bose Einstein occupation n_h, n_c and n_f The transition-operators σ_{ab} are made by following qutip implementation:
$\sigma ab = tensor(|a\rangle \langle b| \cdot I_{(nph)}".$
In the same way I implemented also the other transition operators and vg and v1 are basisstates. nph is the maximum of the photonnumber in the cavity. If I set my maximum photon number to 30, I get 90 x 90 matrices. The projectors are implemented similarly, but wit the matrix $|a\rangle \langle a|$ With those its easy to construct the Hamiltoniens, H_{free} and H_{int}, as in Eq.3 , Eq.4 To calculate the the density matrices for steady states we can also use a qutip function, call steadystate(). this function needs the total Hamiltonian and a list of the non-unitary operators as arguments. We can we can construct this list as a multiplication of our transition-operators and the tree different Bose Einstein occupation numbers times the different γ-factors. As output of the function steady state we get the density-matrices for steady-states.

3.3 Further calculations for thermodynamics

First i made a Fockplot and a wigner function of the reduce density matrices ρ_{free}. Because $\rho = \rho_{atom} \otimes \rho_{free}$, I can make the partial trace of ρ with qutip, to trace out the reduced density matrices $\rho_f ree$. The Fock-plot and the Wigner-plot is also done with a qutip function. with the density matrix times the $L_p\rho$ i calculated on the heat flux by taking the trace of $H\mathcal{L} \cdot \rho$. and plot this for 200 different g 's so the goal of this work is to find Einstein-Bose-distributions which yield a RAHV state. as in the paper (Quelle2) A other useful scientific concept is the entropy. its also a physical physical property. The entropy production is given by the formula $\dot{\sigma} = \sum_i^3 \frac{Ji(n_h,n_c,n_cav)}{T_i((n_h,n_c,n_{cav}))}$

4 Results

4.1 Lasing transition

The first Result are Fockplots and Wigner-density-plots. For all calculations, I set the parameters $\hbar$ and k_b equal to one. γ_h, γ_c are set to 35*0.01 I tested those with different set of parameters. shown in Fig 2.

In the first plot I set a high leaking-parameter $\kappa = 1$. This means that many photons leave the cavity, and only a few remain in the cavity. We see, that the occupation-number in the fock-plot is most zero and the probability for one photon is just 0.1.

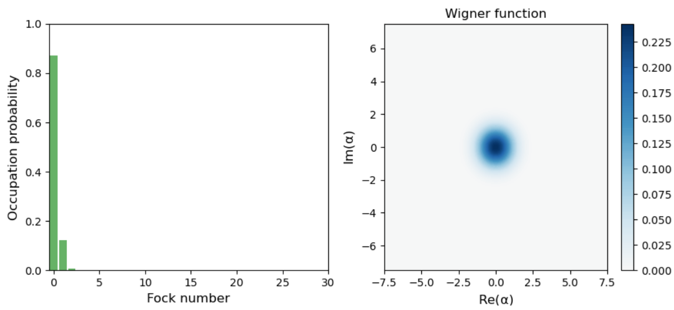

Figure 2: The parameters for the first plot are$n_h = 2.6 n_c = 0.001 n_f = 0.02, \kappa = 1$. The temperature for the warm bath is 460. Cavity with a big leaking

In the second plot I took the same parameters again, but with a lower κ. We get a better distribution in the Fockplot and a RAHV state in the Wigner function.

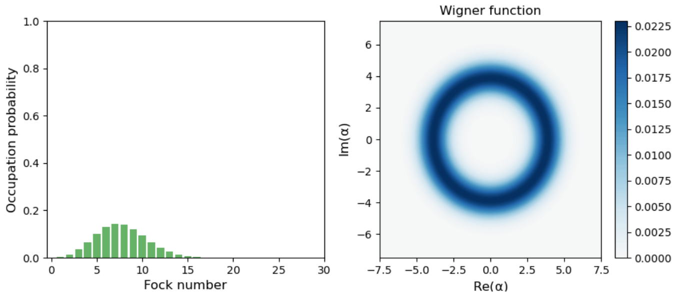

Figure 3: The parameters for the first plot are $n_h = 2.6 n_c = 0.001 n_f = 0.02, \kappa = 0.1$.The temperature for the warm bath is 460. Cavity with a big leaking

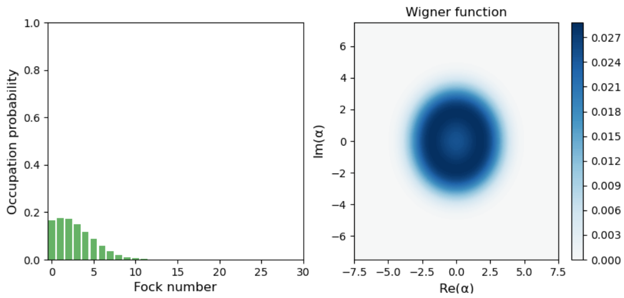

Figure 4: The parameters for the first plot are$n_h = 20 n_c = 0.001 n_f = 0.02, \kappa = 0.001$. The temperature for the warm bath is 3074. Cavity with a small leaking

If we further consider $n_h >> 1$, then the cavity photon number approach saturated at the very high temperature regime, as shown in Fig. 4. This is because, in this regime, the population has been almost inverted thus the increase of the hot bath temperature T_h can no longer bring in a significant increase to the photons gain. The hot bath no longer has any weakening effect to the lasing, thus more lasing photons can be produced in the cavity, and the lasing power can be increased. But, still, the cavity photon number is limited due to the single atom feature.

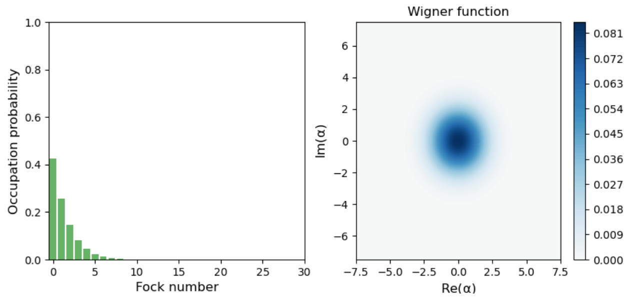

Figure 5: The parameters for the first plot are $n_h = n_c = 2.6 n_f = 0.02, \kappa = 0.001$ The temperature for the warm bath is 3074. Cavity with a small leaking

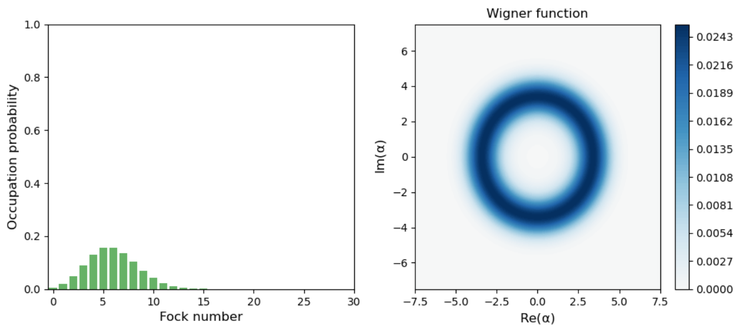

Figure 6: The parameters for the first plot are $n_h = 1.7 n_c = 0.01 n_f = 0.02, \kappa = 0.001$ The temperature for the warm bath is 3074. Cavity with a small leaking

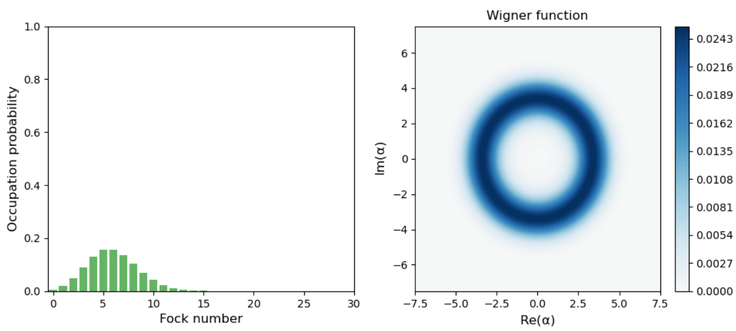

Figure 7: The parameters for the first plot are $n_h = 5.5 n_c = 0.01 n_f = 0.02, \kappa = 0.001$ The temperature for the warm bath is 3074. Cavity with a small leaking

4.2 Thermodynamics

In the second step of the calculation of the expectation value from energy flow depends on different coupling constants g. In other words I plotted the Trace from the density matrices times the Liouvilian against the coupling constant. The master equation depends on three different Liovillien therms. I calculated the expected heat flow for every different interaction. the cold interaction the warm and the interaction with the cavity The figure 6 shows for the parameters $n_h = n_c = 2.6 n_f = 0.02, \kappa = 0.01$ is shown below.

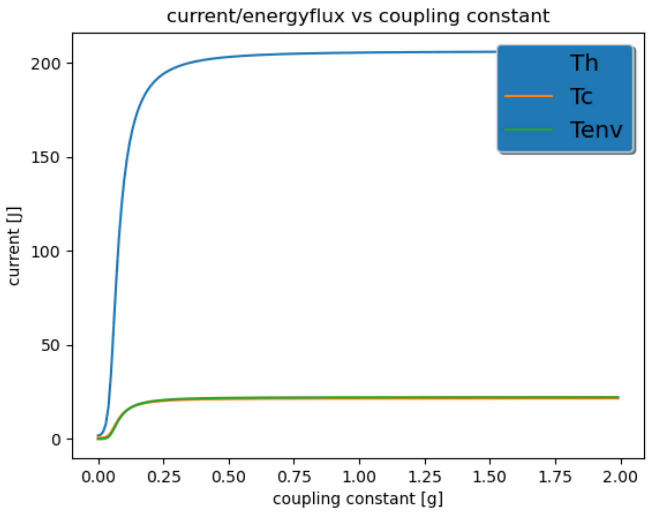

Figure 8: Energy flux vs g with the parameters The parameters for the first plot are $n_h = n_c = 2.6 n_f = 0.02, \kappa = 0.01$

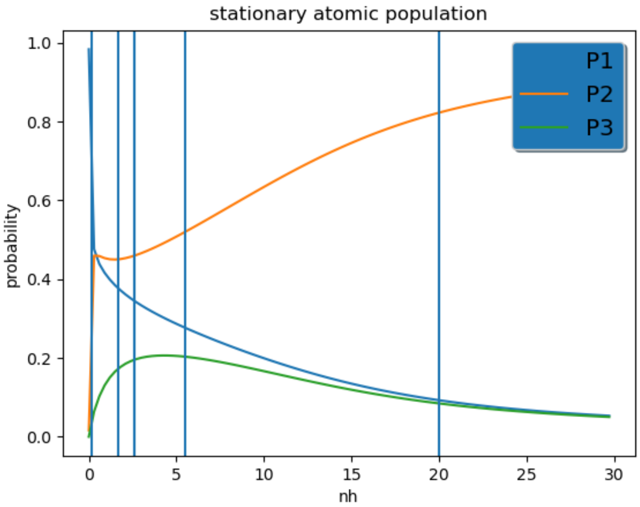

Figure 9: The probability for a atom to sty in a state 0, 1 or 3 vs n_h with the parameters The parameters for the first plot are $n_c = 2.6 n_f = 0.02, \kappa = 0.01$ and n_h is from $0 - 10$ the blue lines marks the the values for n_h for which the Wigner functions are plotted

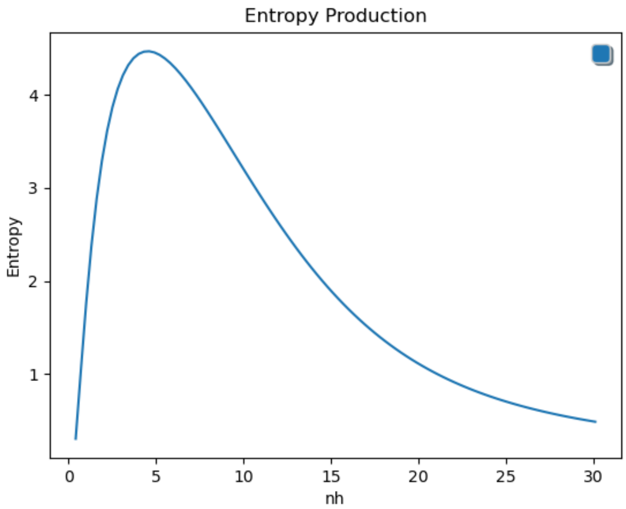

Figure 10: The entropy production for n_h from 0-30

5 Discussion

Wth the condition that n_c, n_cav is almost zero, the leaking κ is small too and the hot bath have a value between 1 and 5.5 When we compare the winger functions of the numerical calculated states with the Wigner plot of a PHAV state, we see, that is pretty similar. As conclusion; its possible to get a phase average coherent state. In Figure 8 we see the average of the current, calculated with the formula Eq.13. for different coupling constants g we see, that the current increase at most between the $0 < g < 0.25$

In figure 9 we we see the probability in which state a atom is. If we increase the temperature from the hot bath, we see that the occupation probability of P 2 increase as well. However, the probability of an atom being in the P1 or P3 state decreases with increasing nh. The probability of P1 and P2 is thus inversely proportional to the temperature. the problem is that the cycle of photen can be stopped, becausethe probability that the photons goes from the ground state in the highest state is to small.

A other problem which has not been discussed in this report is, that the temperatures can reach the ionization temperature.

6 References

7 Appendix

$$
\begin{aligned}
\mathcal{L}\hat{\rho} = & \frac{\gamma_h}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_h}{k_b T_h}]-1}+1\right]\cdot\left(2\sigma_{13}\cdot\rho\cdot\sigma_{13}^{\dagger}-\sigma_{13}^{\dagger}\sigma_{13}\rho-\rho\sigma_{13}^{\dagger}\sigma_{13}\right)\\
& +\frac{\gamma_h}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_h}{k_b T_H}]-1}\right]\cdot\left(2\sigma_{31}\cdot\rho\cdot\sigma_{31}^{\dagger}-\sigma_{31}^{\dagger}\sigma_{31}\rho-\rho\sigma_{31}^{\dagger}\sigma_{31}\right)\\
& +\frac{\gamma_c}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_c}{k_b T_c}]-1}+1\right]\cdot\left(2\sigma_{23}\cdot\rho\cdot\sigma_{23}^{\dagger}-\delta_{23}^{\dagger}\sigma_{23}\rho-\rho\sigma_{23}^{\dagger}\sigma_{23}\right)\\
& +\frac{\gamma_c}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_c}{k_b T_c}]-1}\right]\cdot\left(2\sigma_{32}\cdot\rho\cdot\sigma_{32}^{\dagger}-\sigma_{32}^{\dagger}\sigma_{32}\rho-\rho\sigma_{32}^{\dagger}\sigma_{32}\right)\\
& \kappa\left[\frac{1}{\exp[\frac{\hbar w_f}{k_b T_f}]-1}+1\right]\cdot\left(2a\rho a^{\dagger}-a^{\dagger}a\rho-\rho a^{\dagger}a\right)\\
& \kappa\left[\frac{1}{\exp[\frac{\hbar w_f}{k_b T_f}]-1}\right]\cdot\left(2a^{\dagger}\rho a-aa^{\dagger}\rho-\rho aa^{\dagger}\right)
\end{aligned}
\tag{16}
$$