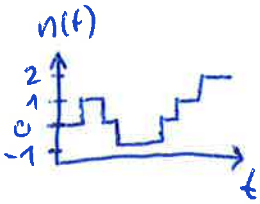
VI. 3 Full counting statistics

Schaller, Open quantum systems far from equilibrium

Goal: Calculate probability distributions, moments, and cumulants for heat and work.

VI. 3.1 Counting particles

n: net amount of particles that entered reservoir



Master equation:

Hamiltonian + other reservoirs

$$\partial_t \hat{\rho}_s = \tilde{\mathcal{L}}_0 \hat{\rho}_s + \kappa n_F \mathcal{D}[\hat{c}^+]\hat{\rho}_s + \kappa(1-n_F)\mathcal{D}[\hat{c}]\hat{\rho}_s \equiv \mathcal{L}\hat{\rho}_s$$

decreases n by 1

$$\mathcal{D}[\hat{c}^+]\hat{\rho}_s = \hat{c}^+\hat{\rho}_s\hat{c} - \frac{1}{2}\{\hat{c}\hat{c}^+, \hat{\rho}_s\}$$

$$\mathcal{D}[\hat{c}]\hat{\rho}_s = \hat{c}\hat{\rho}_s\hat{c}^+ - \frac{1}{2}\{\hat{c}^+\hat{c}, \hat{\rho}_s\}$$

increases n by 1

We may access n by observing the system's jumps

$$\hat{\rho}_s = \sum_{n=-\infty}^{\infty} \hat{\rho}_s(n)$$

n-resolved density matrix

n changes through jumps:

$$\mathcal{J}_+\hat{\rho} = \kappa(1-n_F)\hat{c}\hat{\rho}\hat{c}^+$$

$$\mathcal{J}_-\hat{\rho} = \kappa n_F \hat{c}^+\hat{\rho}\hat{c}$$

no-jump evolution

Write: $\mathcal{L} = \mathcal{L}_0 + \mathcal{J}_+ + \mathcal{J}_-$

jump decreasing n by 1

jump increasing n by 1

$$\partial_t \hat{\rho}_S(n) = \mathcal{L}_0 \hat{\rho}_S(n) + \mathcal{J}_+ \hat{\rho}_S(n-1) + \mathcal{J}_- \hat{\rho}_S(n+1)$$

Summing over n $\Rightarrow$ $\partial_t \hat{\rho}_S = \mathcal{L} \hat{\rho}_S$ ✓

~~Counting field~~

Probability that n particles entered reservoir:

$$p(n) = \mathrm{Tr}\{\hat{\rho}_S(n)\}$$

average: $\sum_{n=-\infty}^{\infty} n\, p(n) = \langle n \rangle$

moments: $\sum_{n=-\infty}^{\infty} n^k p(n) = \langle n^k \rangle$

Counting field

$$\hat{\rho}_S(\chi) = \sum_{n=-\infty}^{\infty} e^{in\chi} \hat{\rho}_S(n)$$

$$\partial_t \hat{\rho}_S(\chi) = \mathcal{L}_0 \hat{\rho}_S(\chi) + \mathcal{J}_+ \sum_n e^{in\chi} \hat{\rho}_S(n-1) + \mathcal{J}_- \sum_n e^{in\chi} \hat{\rho}_S(n+1)$$

$$= \mathcal{L}_0 \hat{\rho}_S(\chi) + e^{i\chi} \mathcal{J}_+ \hat{\rho}_S(\chi) + e^{-i\chi} \mathcal{J}_- \hat{\rho}_S(\chi) \equiv \mathcal{L}(\chi) \hat{\rho}_S(\chi)$$

Formal solution:

$$\hat{\rho}_S(\chi) = e^{\mathcal{L}(\chi) \cdot t} \hat{\rho}_S(\chi, t=0)$$

initial condition: $\hat{\rho}_S(n) = \hat{\rho}_S(t=0)\, \delta_{n,0} \Leftrightarrow \hat{\rho}_S(\chi, t=0) = \hat{\rho}_S(t=0)$

Moment generating function (MGF)

$$\Lambda(\chi) \equiv \sum_n e^{in\chi} P(n) = \mathrm{Tr}\{\hat{\rho}_S(\chi)\}$$

$$\Lambda(\chi=0) = \sum_n P(n) = 1$$

$$\langle n^k \rangle = (-i)^k \partial_\chi^k \Lambda(\chi) \Big|_{\chi=0}$$

Cumulant generating function (CGF)

$$S(\chi) = \ln \Lambda(\chi)$$

$$-i\partial_\chi S(\chi)\Big|_{\chi=0} = \frac{-i\partial_\chi \Lambda(\chi)}{\Lambda(\chi)}\Big|_{\chi=0} = \langle n \rangle \equiv \langle\langle n \rangle\rangle$$

variance

$$\langle\langle n^2 \rangle\rangle = (-i\partial_\chi)^2 S(\chi)\Big|_{\chi=0} = \frac{(-i\partial_\chi)^2 \Lambda(\chi)}{\Lambda(\chi)}\Big|_{\chi=0} - \frac{(-i\partial_\chi \Lambda(\chi))^2}{(\Lambda(\chi))^2}\Big|_{\chi=0} = \langle n^2 \rangle - \langle n \rangle^2$$

$\langle\langle n^3 \rangle\rangle$: skewness

$\vdots$

$$\langle\langle n^k \rangle\rangle = (-i\partial_\chi)^k S(\chi)\Big|_{\chi=0}$$

Long time limit

eigenvalues of $\mathcal{L}(\chi)$

$$e^{\mathcal{L}(\chi)\cdot t} = \sum_j e^{\nu_j t} P_j$$

← projectors depend on eigenvectors

spectral decomposition

$$\Rightarrow \Lambda(\chi) = \sum_j e^{\nu_j t} \mathrm{Tr}\{P_j \hat{\rho}_S(t=0)\} \xrightarrow{t\to\infty} e^{\nu_{max} t} \mathrm{Tr}\{P_{max} \hat{\rho}_S(t=0)\}$$

$$S(\chi) = \nu_{max}\cdot t + \ln(\mathrm{Tr}\{P_{max} \hat{\rho}_S(t=0)\}) \xrightarrow{t\to\infty} \nu_{max}\cdot t$$

- Cumulants become linear in t at large times
- CGF determined by eigenvalue with largest real part!

VI.3.2 Example: transport through a quantum dot

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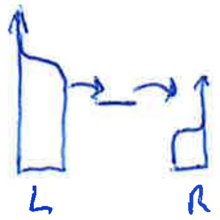

large bias regime: $n_F^L(\varepsilon) = 1$ $\quad n_F^R(\varepsilon) = 0$

Goal: count number of electrons that enter right reservoir

$$\partial_t \hat\rho_s = -i\underbrace{[\varepsilon \hat c^\dagger \hat c, \hat\rho_s]}_{=0} + \kappa_L \mathcal{D}[\hat c^\dagger]\hat\rho_s + \kappa_R \mathcal{D}[\hat c]\hat\rho_s = \mathcal{L}\hat\rho_s$$

Counting field: $\kappa_R \hat c \hat\rho_s \hat c^\dagger \longrightarrow e^{i\chi}\kappa_R \hat c \hat\rho_s \hat c^\dagger$

$$\Rightarrow \partial_t \hat\rho_s(\chi) = \mathcal{L}\hat\rho_s(\chi) + \kappa_R(e^{i\chi}-1)\hat c \hat\rho_s \hat c^\dagger$$

$$\hat\rho_s(\chi) = \begin{pmatrix} \rho_0(\chi) & 0 \\ 0 & \rho_1(\chi) \end{pmatrix} \quad \Rightarrow \partial_t \begin{pmatrix} \rho_0(\chi) \\ \rho_1(\chi) \end{pmatrix} = \underbrace{\begin{pmatrix} -\kappa_L & \kappa_R e^{i\chi} \\ \kappa_L & -\kappa_R \end{pmatrix}}_{L(\chi)} \begin{pmatrix} \rho_0(\chi) \\ \rho_1(\chi) \end{pmatrix}$$

Eigenvalues of $L(\chi)$:

$$\nu_\pm = -\frac{\kappa_L+\kappa_R}{2} \pm \frac{1}{2}\sqrt{(\kappa_L-\kappa_R)^2 + 4e^{i\chi}\kappa_L\kappa_R}$$

$$\Rightarrow \frac{S(\chi)}{t} = -\frac{\kappa_L+\kappa_R}{2} + \sqrt{\left(\frac{\kappa_L-\kappa_R}{2}\right)^2 + e^{i\chi}\kappa_L\kappa_R}$$

see power on page 33

average current

$$\langle\langle n \rangle\rangle = -i\partial_\chi S(\chi)\Big|_{\chi=0} = \frac{\kappa_L\kappa_R}{(\kappa_L+\kappa_R)}\cdot t = \langle I \rangle \cdot t$$

$$\langle\langle n^2 \rangle\rangle = -\partial_\chi^2 S(\chi)\Big|_{\chi=0} = \langle\langle n \rangle\rangle \cdot \frac{\kappa_L^2+\kappa_R^2}{(\kappa_L+\kappa_R)^2}$$

$$\Gamma_L \gg \Gamma_R \quad \Rightarrow \quad S(\chi) = \Gamma_R \cdot t \left(e^{i\chi} - 1\right)$$

$$\Rightarrow \quad \langle\langle n^k \rangle\rangle = \Gamma_R \cdot t$$

$$\Lambda(\chi) = e^{-\Gamma_R t} \exp\left[\Gamma_R t e^{i\chi}\right] = e^{-\Gamma_R t} \sum_{n=0}^{\infty} \frac{(\Gamma_R t)^n}{n!} e^{in\chi}$$

$$\Rightarrow \quad P(n) = \frac{(\Gamma_R t)^n}{n!} e^{-\Gamma_R t} \qquad \text{poissonian statistics}$$

VI.3.3. Counting heat and work

$$\partial_t \hat{\rho}_S(t) = -i\left[\hat{H}_S(t), \hat{\rho}_S(t)\right] + \sum_{\alpha,q} \gamma_q^\alpha(\omega_{\alpha,q}) \mathcal{D}\left[\hat{S}_{\alpha,q}\right] \hat{\rho}_S(t) = \mathcal{L}\hat{\rho}_S(t)$$

$$\left[\hat{S}_{\alpha,q}, \hat{H}_{TD}\right] = \omega_{\alpha,q} \hat{S}_{\alpha,q} \qquad \left[\hat{S}_{\alpha,q}, \hat{N}_S\right] = n_{\alpha,q} \hat{S}_{\alpha,q}$$

$\Rightarrow$ Jump $\hat{S}_{\alpha,q} \hat{\rho}_S \hat{S}_{\alpha,q}^\dagger$ reduces particle number in system by $n_{\alpha,q}$ and internal energy by $\omega_{\alpha,q}$

Joint distribution of heat & work: $P(\{Q_\alpha, P_\alpha\})$

$$\Lambda(\{\chi_\alpha, \lambda_\alpha\}) = \left(\prod_\alpha \int dQ_\alpha e^{iQ_\alpha \chi_\alpha} \int dP_\alpha e^{iP_\alpha \lambda_\alpha}\right) P(\{Q_\alpha, P_\alpha\})$$

$$= \mathrm{Tr}\left\{\hat{\rho}_S(\{\chi_\alpha, \lambda_\alpha\})\right\}$$

$$\partial_t \hat{\rho}_S(\{\chi_\alpha, \lambda_\alpha\}) = \mathcal{L}(\{\chi_\alpha, \lambda_\alpha\}) \hat{\rho}_S(\{\chi_\alpha, \lambda_\alpha\})$$

heat, work of single jump

$$\mathcal{L} \to \mathcal{L}(\{\chi_\alpha, \lambda_\alpha\}) : \quad \hat{S}_{\alpha,q} \hat{\rho}_S \hat{S}_{\alpha,q}^\dagger \to e^{i\chi_\alpha(\omega_{\alpha,q} - \mu_\alpha n_{\alpha,q})} e^{i\lambda_\alpha \mu_\alpha n_{\alpha,q}} \hat{S}_{\alpha,q} \hat{\rho}_S \hat{S}_{\alpha,q}^\dagger$$

Remarks:

i) $\mathcal{L}(\{\chi_\alpha, \lambda_\alpha\})$ can be derived by introducing counting fields on the unitary description with $\hat{H}_{tot}(t)$, and then tracing out the environment using Born-Markov approximations

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ii) The fluctuation theorem can be derived from the master equation. A trajectory γ is then determined by ~~[illegible]~~ outcomes of initial and final measurement on the system, as well as the times and types of all jumps that occur along a trajectory.

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