Fully description of a tree level maser

Master-Projekt

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Contents

1 Introduction

One of the most important questions in thermodynamics is how to convert thermal energy into work. For such tasks exists many classical engines, as example the steam-machine. In my master-project, I will quantify a three level maser. The three-level maser is a Quantum heat engine (QHE). The work extraction from a classical heat engine is often a moving piston.As example steam machines or a gasoline engine. But in this case it is a driving field. Albert Einstein, already discussed three ways of light-matter-interaction (spontaneous emission, absorption, and stimulated emission) in the year 1916(Quelle Wiki). In paper from 1959 [Quelle altes paper], Scovil and Schulz-DuBois investigated whether a laser is not also a heat engine. In the paper, they take a maser as a device to transform heat into coherent radiation, because heat can make a population inversion. In their thermodynamic analytic, they use a single-atom laser. They made a groundwork for emerging theory of quantum thermodynamics. In practice and also for the calculations, two different reservoirs are necessary. The high-temperature reservoir can be realized by a gas noise lamp and the filter by a wave guide cutting off the lower frequencies. (Quelle main Paper)

2 Theory

2.1 3-level-Maser Model

A Maser/Laser consists of two elements. One of them is a gain medium and the other one is an optical resonator. A gain medium is always a material with an atomic transition between two atomic states. When an atom falls from an energetically higher state to an energetically lower state, a photon is created.

In a three level system the three energy levels are $E1, E2, E3$. Pumping is from the lowest level to the highest level, E3. The condition for the third level is that it falls to the middle level E2 very quickly. On average, the system is almost not in the third state. The resonator should then have a higher decay time, so that a population inversion can build up. This means that several particles are in the energetically higher state. From this state they come almost exclusively through stimulated emission into the lower state E1. Stimulated emission is a necessary condition for coherent light. Coherent light means the light have the same phase and same frequency. (Quelle Wiki)

In the case of this calculation, the higher level will be reached with a interaction of a warm bath.

We denote the frequencies of $\omega_h = (E_2 - E_g)/\hbar$, $\omega_c = (E_1 - E_g)/\hbar$ and $\omega_f = (E_2 - E_1)/\hbar$

The master equation is:

$$\dot{\rho}(t) = \frac{1}{i\hbar}[H, \rho] + \mathcal{L}_h\rho + \mathcal{L}_c\rho + \mathcal{L}_{cav}\rho. \tag{6}$$

The first part of Eq.3 is the von Neuman-equation, the analog of the Schrödinger equation but for density matrices. This part of the equation is unitary and therefore the process is reversible. The non-unitary part of the equation $\mathcal{L}\rho$ include the superoperator $\mathcal{L}$, which act on the density operator. A superoperator is a linear operator acting on a vector space of linear operators, as example a density operator. $\mathcal{L}$ consist of three parts. $\mathcal{L}_h$describe the interaction with the hot bath. $\mathcal{L}_c$ is the contribution from the interaction with the cold bath coupled with the atom. $\mathcal{L}_{cav}$ describe the photons which leave the cavity. So,if we have a small κ means less photons will leave and stay in the cavity. we see that in the Fockplott.

The Hamiltonian describes the energy. The atomic field system is composed of two crucial parts; the atomic states, the cavity field, and the interaction between the two. The interaction Hamiltonian or Jaynes-Cummings Hamiltonian:

$$H_{int} = \hbar g(\sigma_{12}a^\dagger + \sigma_{21}a). \tag{7}$$

And the Hamiltonian of the photons is, which describes the photons in the cavity:

$$H_{free} = \sum_{i=1}^{3} \hbar\omega_i \left|i\right\rangle \left\langle i\right| + \hbar\omega_f a^\dagger a, \tag{8}$$

The total Hamiltonian

$$H = H_{free} + H_{int} \tag{9}$$

The interaction with the various environmental heat baths is described by the Liouvillian:

$$\mathcal{L}\hat{\rho} = \frac{\gamma_h}{2}(n(\omega_h, T_h) + 1) \cdot \mathcal{D}[\sigma_{13}]\rho + \frac{\gamma_h}{2}n(\omega_h, T_h) \cdot \mathcal{D}[\sigma_{31}]\rho$$

$$+\frac{\gamma_c}{2}(n(\omega_c, T_c) + 1) \cdot \mathcal{D}[\sigma_{23}]\rho + \frac{\gamma_c}{2}n(\omega_c, T_c) \cdot \mathcal{D}[\sigma 32]\rho$$

$$\kappa((\omega_f, Tf) + 1) \cdot \mathcal{D}[a]\rho + \kappa n(\omega_f, T_f) \cdot \mathcal{D}[a^{\dagger}]\rho, \tag{10}$$

$\mathcal{D}$ is defined with this formula:

$$\mathcal{D}[A]\rho = (2A\rho A^{\dagger} - A^{\dagger}A\rho - \rho A^{\dagger}A), \tag{11}$$

The Bose-Einstein statistic is a probability distribution in quantum statistics . It describes the mean occupation number $\langle n(E) \rangle$ of a quantum state of energy E, in thermodynamic equilibrium at absolute temperature T for identical bosons as occupying particles. n depends on the temperature and the frequency. n is defined as:

$n(\omega, T) = \frac{1}{\exp[\frac{\hbar\omega_i}{k_b T_i}]-1}$, The prefactor $\gamma_c = \gamma_h$ describes the spontaneous decay rates and are in this calculation relatively low. The Liouvillian have different constants. The coupling constants g strong for the Hamiltonian and the κ for the Liouvillian part. The coupling constant g is given by $\frac{\Omega}{\hbar}$,

2.5 Thermodynamics

When we work with density matrices, its common to work with expectation values with $\langle A \rangle = Tr[A * \rho]$. A is a operator and describe a measurement. With this we can calculate the expectation value from a Operator.

To calculate the expected heat flow we can take the partial trace from

$$\langle J \rangle = Tr[\rho_{free} \cdot \mathcal{L}_h[\rho]] + Tr[\rho_{free} \cdot \mathcal{L}_c[\rho]] + Tr[\rho_{free} \cdot \mathcal{L}_{cav}, \tag{12}$$

A part of my work was to calculate the equation 9 by hand. I made the calculation in two steps. for the warm and the cold path, we have a transition-operators in the trace. The trick of this calculation was, to get the form $Tr[\sigma_{ab}\rho\sigma_{ab}^{\dagger}]$ because this is equal to Pb The equation gave the following result:

$$Tr[\rho_{free} \cdot \mathcal{L}_h[\rho]] = \hbar\omega_h\gamma_h(2n + 1) \cdot (P1 - P3), \tag{13}$$

For the calculation the $T[H_{free} \cdot \mathcal{L}_{cav}]$, I get the following result:

$$T[H_{free} \cdot \mathcal{L}_{cav}[\rho]] = 2\hbar\omega_k(\bar{n} - \langle a^\dagger a \rangle), \tag{14}$$

The efficiency is given by the following formula:

$$\eta_{maser} = \frac{\omega_f}{\omega_h} < 1 - \frac{Tc}{Th}, \tag{15}$$

3 Calculation

3.1 Software

For the hole implementation of the tree-level-system in a cavity, I used qutip. Qutip is library in python, which allows to solve masterequation pretty easy.

3.2 Implementation of the tree-level-system in qutip

In our case only ω_f interact with the light. first i defined the frequencies ω_c, ω_h and ω_f The constants h and the bolzmanfactor k_b are 1. alsou defined as constants are the three different Boseeinstein-distributions n_h, n_c and n_f The transition-operators $Trans_{13}$ are made by following qutip implementation:
"$Trans_{13} = tensor(vg * v1.dag(), qutip.identity(nph))$".
In the same way I implemented also the other transition operators and vg and v1 are basisstates. nph is the maximum of the photonnumber in the cavity. If I set my maximum photon number to 30, I get 90 x 90 matrices. The projectors are implemented similarly. With those its easy to construct the hamiltoniens, H_{free} and H_{int} as in formula 3 and 4. To calculate the the density matrices for steadystates we can alsou use a qutip function, call steadystate(). this function needs the total Hamiltonian and a list of the non-unitary operators as arguments. We can we can construct this list as a multiplication of our transition-operators and the tree different Boseeinstein-distributions times the different γ-factors. as output of the function steady state we get the density-matrices for steady-states.

3.3 Further calculations for thermodynamics

First i made a Fockplot and a wigner function of the reduce density matrices ρ_{free}. Because $\rho = \rho_{atom} \otimes \rho_{free}$, I can make the partial trace of ρ with qutip, to trace out the reduced density matrices $\rho_f ree$. The Fock-plot and the Wigner-plot is also done with a qutip function. with the density matrix times the $L_p\rho$ i calculated on the heat flux by taking the trace of $H\mathcal{L} \cdot \rho$.

and plot this for 200 different g ‘s so the goal of this work is to find Einstein-Bose-distributions which yield a RAHV state. as in the paper (Quelle2)

4 Results

4.1 Lasing transition

The first Result are fockplots and wigner-density-plots. For all calculations, I set the parameters $\hbar$ and k_b equal to one. γ_h, γ_c are set to 35*0.01 I tested those with different set of parameters. sown in fig 2.

In the first plot I set a high leaking-parameter $\kappa = 1$ This means, that many photons leave the cavity and only a few remain in the cavity. We see, that the occupation-number in the fock-plot is most zero and the probability for one photon is just 0.1.

In the second plot I took the same parameters again, but with a lower κ. We get a better distribution in the Fockplot and a RAHV state in the Wignerfunction.

If we further consider $n_h >> 1$, then the cavity photon number approach saturated at the very high temperature regime, as shown in Fig. 4. This is because in this regime, the population has been almost inverted thus the increase of the hot bath temperature Th can no longer bring in a significant increase to the photons gain. The hot bath no longer has any weakening effect to the lasing, thus more lasing photons can be produced in the cavity, and the lasing power can be increased. But still the cavity photon number is limited due to the single atom feature.

4.2 Thermodynamics

In the second step of the calculation of the expectationvalue from energy flow depends on different coupling constants g. In other words I plotted the Trace from the density matrices times the Liouvilian agianst the coupling-constant. The master equation depends on three different Liovillien therms. I calculated the expected heatflow for every different interaction. the cold interaction the warm and the interaction with the caviyt The figure 6 shows for the parameters $n_h = n_c = 2.6n_f = 0.02, \kappa = 0.01$ is shown below.

5 Discussion

6 References

7 Appendix

$$
\begin{aligned}
\mathcal{L}\hat{\rho} = \frac{\gamma_h}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_h}{k_b T_h}]-1}+1\right]\cdot\left(2\sigma_{13}\cdot\rho\cdot\sigma_{13}^{\dagger}-\sigma_{13}^{\dagger}\sigma_{13}\rho-\rho\sigma_{13}^{\dagger}\sigma_{13}\right) \\
+\frac{\gamma_h}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_h}{k_b T_H}]-1}\right]\cdot\left(2\sigma_{31}\cdot\rho\cdot\sigma_{31}^{\dagger}-\sigma_{31}^{\dagger}\sigma_{31}\rho-\rho\sigma_{31}^{\dagger}\sigma_{31}\right) \\
+\frac{\gamma_c}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_c}{k_b T_c}]-1}+1\right]\cdot\left(2\sigma_{23}\cdot\rho\cdot\sigma_{23}^{\dagger}-\delta_{23}^{\dagger}\sigma_{23}\rho-\rho\sigma_{23}^{\dagger}\sigma_{23}\right) \\
+\frac{\gamma_c}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_c}{k_b T_c}]-1}\right]\cdot\left(2\sigma_{32}\cdot\rho\cdot\sigma_{32}^{\dagger}-\sigma_{32}^{\dagger}\sigma_{32}\rho-\rho\sigma_{32}^{\dagger}\sigma_{32}\right) \\
\kappa\left[\frac{1}{\exp[\frac{\hbar w_f}{k_b T_f}]-1}+1\right]\cdot\left(2a\rho a^{\dagger}-a^{\dagger}a\rho-\rho a^{\dagger}a\right) \\
\kappa\left[\frac{1}{\exp[\frac{\hbar w_f}{k_b T_f}]-1}\right]\cdot\left(2a^{\dagger}\rho a-aa^{\dagger}\rho-\rho aa^{\dagger}\right)
\end{aligned}
\tag{16}
$$

2.2 Wigner function

A Wigner function is a representation of a general quantum state of light. The function describe the probability distribution in phase space.

$$w(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\xi e^{\frac{-i}{\hbar}p} \langle x + \frac{1}{2}\xi |\, \rho \,| x - \frac{1}{2}\xi \rangle \tag{1}$$

2.3 Phase-averaged coherent states (PHAV)

The output of a laser is coherent light. The quantum description of coherent light is a coherent state. The photon number distribution of coherent light is a Poisson distribution. The randomized phase of a coherent state doesn't change the photon-number distribution. The Wigner function from a coherent state itself is a Gaussian. But the Wigner function of a phase-average-state has a non-Gaussian Wigner function. The mathematical description of a PHAV is the same as a normal coherent state but with a random phase. So we get a new term of $exp(i\pi\phi)$ in it. The normal coherent state could represented by following formula (Quelle 5):

$$|\alpha\rangle = e^{-1/2|\alpha|} \sum_{n=0}^{\infty} \frac{|\alpha|^n e^{in\phi}}{\sqrt{n!}} |n\rangle\,, \tag{2}$$

To get the phase average state will get reach with the integral around two π.

$$\rho_{PHAV} = \int_0^{2\pi} \frac{d\phi}{2\pi} |\alpha\rangle \langle\alpha| = \sum_{n=0}^{infinity} p_{nn}. \tag{3}$$

this is equal to:

$$\rho_{PHAV} = exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}. \tag{4}$$

For the Wigner function we get finally following equation:

$$W(z) = 2exp\big[-2\big(|\beta|^2 + |z|^2\big)\big) I_0(4|\beta||z|). \tag{5}$$

Useful in the future.(Zitat Paper über PHAV) The consistent experimental and theoretical results we have obtained in the characterization of both PHAVs and their superpositions 2-PHAVs reinforce the possibility of using them for applications to communication protocols.

2.4 Master-Equation

An arbitrary state on this total Hilbert space can be described by a total density operator $\rho_{tot}(t)$.The total density operator is can be written in the Hilbert space $\rho_{tot} = \rho_{atom} + \rho_{photon}$. Encoded in this density operator is a complete description of the total system's state at a given time t. To derive the ρ_{tot} we can solve a specific differential equation. This equation is called Lindblad-master-equation Eq.3. The Basic of our work is a three-level quantum system in a cavity. This three level system is driven by a hot bath

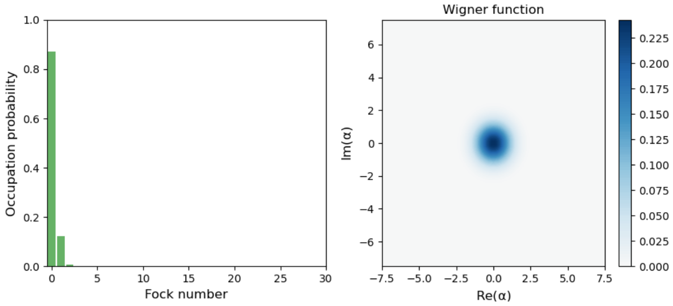

Figure 2: The parameters for the first plot are$n_h = 2.6 n_c = 0.001 n_f = 0.02, \kappa = 1$. The temperature for the warm bath is 460. Cavity with a big leaking

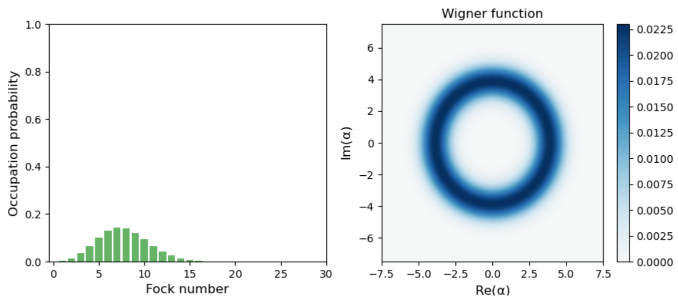

Figure 3: The parameters for the first plot are $n_h = 2.6 n_c = 0.001 n_f = 0.02, \kappa = 0.1$.The temperature for the warm bath is 460. Cavity with a big leaking

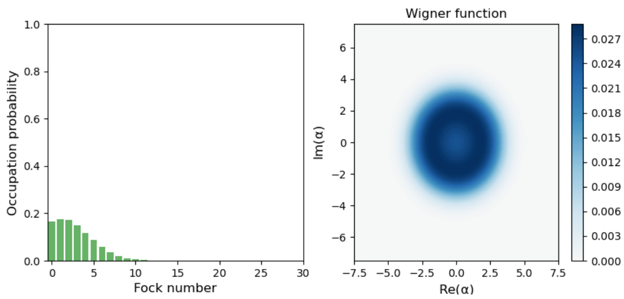

Figure 4: The parameters for the first plot are$n_h = 20 n_c = 0.001 n_f = 0.02, \kappa = 0.001$. The temperature for the warm bath is 3074. Cavity with a small leaking

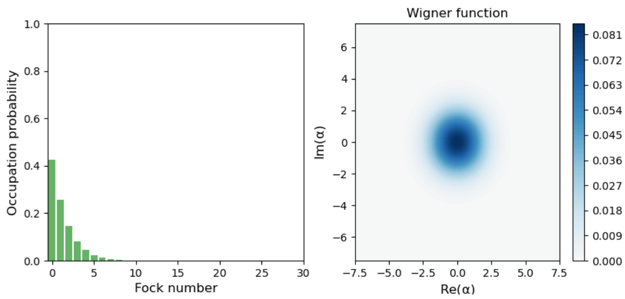

Figure 5: The parameters for the first plot are $n_h = n_c = 2.6n_f = 0.02, \kappa = 0.001$ The temperature for the warm bath is 3074. Cavity with a small leaking

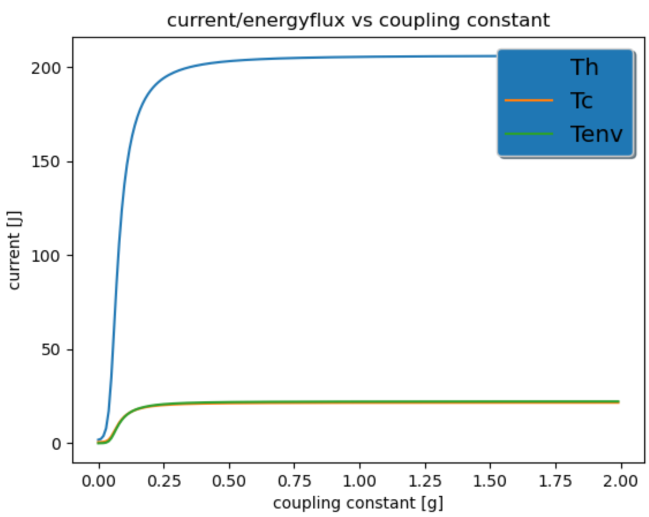

Figure 6: Energy flux vs g with the parameters The parameters for the first plot are $n_h = n_c = 2.6n_f = 0.02, \kappa = 0.01$

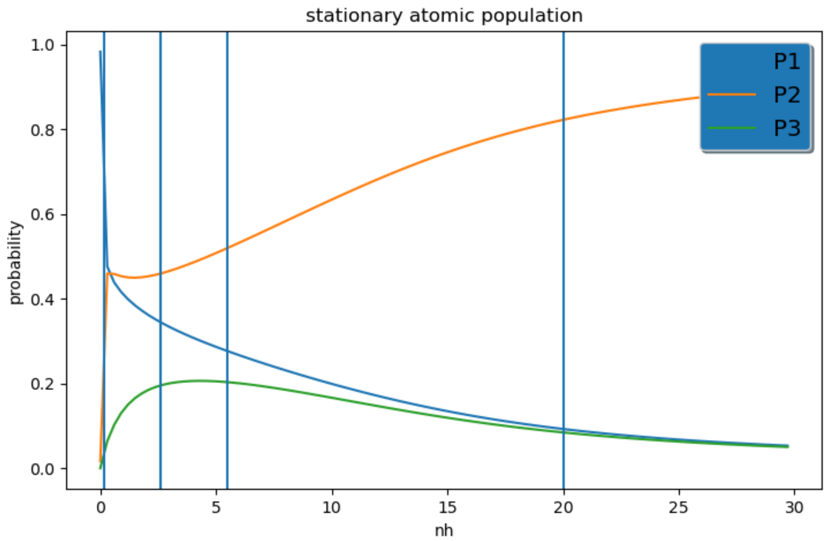

Figure 7: The probability for a atom to sty in a state 0, 1 or 3 vs n_h with the parameters The parameters for the first plot are $n_c = 2.6 n_f = 0.02, \kappa = 0.01$ and n_h is from $0 - 10$