University of Basel

Fully quantum description of a three level maser, driven by a thermal bath

Master-project

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Contents

1   Introduction

One of the most important questions in thermodynamics is how to convert thermal energy into work. For such tasks there exists many classical engines, for example the steam machines or gasoline engines. This ideas also generalize to quantum systems. In this master-project, a three-level maser, driven by the coupling of a hot and a cold bath are quantified. The three-level maser is a quantum heat engine (QHE). The work extraction from a classical heat engine is often a moving piston. But in this case it is a driving field. In the year 1916 Albert Einstein already discussed three ways of light-matter-interaction (spontaneous emission, absorption, and stimulated emission)[2]. In a paper from 1959. Scovil and Schulz-DuBois investigated whether a laser is a heat engine. In this paper, they take a maser as a device to transform heat into coherent radiation, because heat can make a population inversion [[5]. In their thermodynamic analysis, they use a single-atom laser. They made a groundwork for emerging theory of quantum thermodynamics. In practice and also for the calculations, two different reservoirs are necessary. The high-temperature reservoir can be realized by a fast and accurate estimation of the thermal occupation of propagating microwave modes is highly desirable [4].

2 System and model
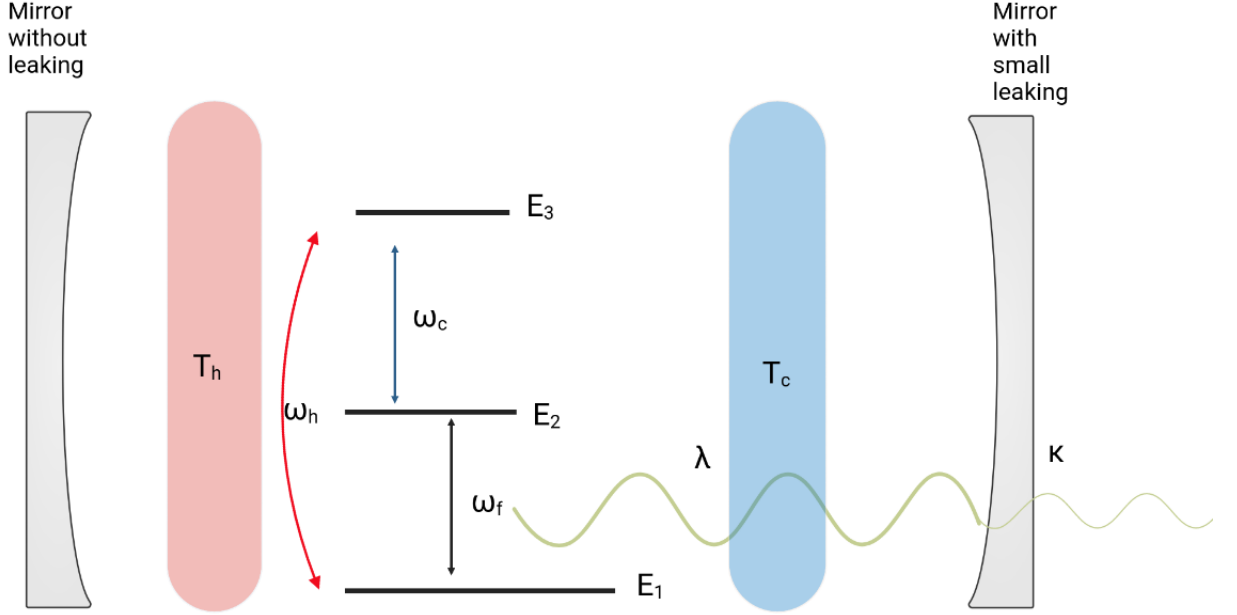
2.1 3-level-Maser Model

Figure 1: Schematic representation of a three-level maser heat engine continuously coupled to two reservoirs of temperatures T_h and T_c. And the three energy levels E_1, E_2, E_3. The system is interacting with a classical single mode field. λ represents the strength of matter-field coupling.

A maser/laser consists of two elements. One of them is a gain medium and the other one is an optical resonator. A gain medium is always a material with an atomic transition between two atomic states. When an atom decays from an energetically higher state to an energetically lower state, a photon is created. In a three level system the three energy levels are E_1, E_2, E_3 shown in Fig. 1. In a first part the system gets pumped from the lowest E_1 level to the highest level E_3. The condition for the third level is that it falls to the middle level E_2 very quickly. On average, the system is almost not in the third state. The resonator should then have a higher decay time, so that a population inversion can build up. This means that a particles is

in the energetically higher state. From this state they come almost exclusively through stimulated emission into the lower state E_1. Our cavity is in resonance, therefore we set $\omega_{cav} = \omega_f$. Emission is a necessary condition for coherent light. Coherent light means all the photons have the same phase and same frequency [2]. Here we consider that higher level will be reached with a interaction of a hot bath. We denote the frequencies of $\omega_h = (E_3 - E_1)/\hbar$, $\omega_c = (E_2 - E_1)/\hbar$ and $\omega_f = (E_2 - E_1)/\hbar$. Efficiency is defined as $\eta = J/P$ In this three-level system each thermal photon from the bath trigger a lasing photon. The efficiency is given by the following formula:

$$\eta_{maser} = \frac{\omega_f}{\omega_h} < 1 - \frac{T_c}{T_h}. \tag{1}$$

2.2   Master-Equation

An arbitrary state on the global state of a cavity and a atom can be described by a total density operator $\rho_{tot}(t)$. The complete description of the total system's state at a given time t is encoded in this density operator ρ [2]. The master equation is:

$$\dot{\rho}(t) = \frac{1}{i\hbar}[H, \rho] + \mathcal{L}_h\rho + \mathcal{L}_c\rho + \mathcal{L}_{cav}\rho. \tag{2}$$

To derive the ρ_{tot} we can solve a specific differential equation. This equation is called Lindblad-master-equation (2). The Basic of this work is a three-level quantum system in a cavity. This three level system is driven by a hot and a cold bath. Those have the temperature T_c and T_h. The cavity is build of two mirrors. One of the cavity have a small leaking, so that a small part of the photons can leave the cavity. This leaking is quantified by a constant κ. In the calculation, the temperature of the thermal bath is constant, therefore it is possible to use the Lindblad-master equation. In this case the master equation is get solved for the steady states. A steady state is a state or condition of a system or process, here the energy states of an atom, that the density matrix of the state does not change in time, or the changes are negligibly. Therefore all observables do not change in time either. It contains the whole description of the three-level system and the cavity. In Fig. 1 is shown a three-level system. H is the the Hamiltonian operator. The interaction with the various environmental heat baths is described by the Liouvillian $\mathcal{L}$. This $\mathcal{L}$ is also called superoperator. The first part of eq (2) is the von Neuman equation, the analogue of the Schrödinger equation for density matrices. This part of the equation is unitary and therefore the process is reversible. The non-unitary part of the equation $\mathcal{L}\rho$ include the superoperator $\mathcal{L}$, which act on the density operator. A superoperator is

a linear operator acting on a vector space of linear operators, as example a density operator. $\mathcal{L}$ consist of three parts. $\mathcal{L}_h$ describe the interaction with the hot bath. $\mathcal{L}_c$ is the contribution from the interaction with the cold bath coupled with the atom. $\mathcal{L}_{cav}$ describe the photons which in the cavity. κ describes the rate of photons which leave the cavity. The Hamiltonian describes the energy. The atomic field system is composed of two crucial parts; the atomic states, the cavity field, and the interaction between the two. The total Hamiltonian,

$$H = H_{free} + H_{int}, \tag{3}$$

consist of two parts; The part of the photons, which describes the photons in the cavity is

$$H_{free} = \sum_{i=1}^{3} \hbar\omega_i \, |i\rangle \langle i| + \hbar\omega_f a^\dagger a, \tag{4}$$

and the interaction Hamiltonian or Jaynes-Cummings Hamiltonian:

$$H_{int} = \hbar g(\sigma_{12} a^\dagger + \sigma_{21} a). \tag{5}$$

The coupling constants g strong for the Hamiltonian and the κ for the Liouvillian part. The coupling constant g is given by $\frac{\Omega}{\hbar}$.

The interaction with the various environmental heat baths is described by the Liouvillian:

$$\mathcal{L}\hat{\rho} = \frac{\gamma_h}{2}(n(\omega_h, T_h) + 1) \cdot \mathcal{D}[\sigma_{13}]\rho + \frac{\gamma_h}{2}n(\omega_h, T_h) \cdot \mathcal{D}[\sigma_{31}]\rho$$
$$+ \frac{\gamma_c}{2}(n(\omega_c, T_c) + 1) \cdot \mathcal{D}[\sigma_{23}]\rho + \frac{\gamma_c}{2}n(\omega_c, T_c) \cdot \mathcal{D}[\sigma_{32}]\rho \qquad (6)$$
$$+ \kappa((\omega_f, T_f) + 1) \cdot \mathcal{D}[a]\rho + \kappa n(\omega_f, T_f) \cdot \mathcal{D}[a^\dagger]\rho.$$

σ_{12} is the transition operator and defined as $|1\rangle\langle 2|$. Similar for $\sigma_{13} = |1\rangle\langle 3|, \sigma_{23} = |2\rangle\langle 3|$. The transition operator describe the transition between an atomic state a to b. $\mathcal{D}$ is defined with following formula

$$\mathcal{D}[A]\rho = (2A\rho A^\dagger - A^\dagger A\rho - \rho A^\dagger A). \qquad (7)$$

Where A represents one of the transition operators σ_{ab} or a ladder operator a. The Bose-Einstein occupation number. It is the mean number of excitations in the reservoir damping the oscillator. It describes the mean occupation number $\langle n(E) \rangle$ of a quantum state of energy E, in thermodynamic equilibrium at absolute temperature T for identical bosons as occupying particles.n depends on the temperature and the frequency. n is defined as: $n(\omega, T) = \frac{1}{\exp[\frac{\hbar\omega_i}{k_b T_i}] - 1}$. The prefactor γ_c, γ_h describes the spontaneous decay rates and are in this calculation generally small. The Liouvillian has different constants.

3 Methods

3.1 Software

For the hole implementation of the tree-level-system in a cavity, I used qutip. Qutip is python library, which allows to solving master equation fast. Further calculations and methods were easily applied in python.

3.2 Implementation of the tree-level-system in qutip

Qutip can be used to solve master equations. To use the qutip functions, it is important to define the constants first. In our case, only the transition between $|1\rangle$ and $|2\rangle$ interact with the light. Defined as constants are the three different Bose-Einstein occupation n_h, n_c and n_f The transition-operators σ_{ab} are made by qutip implementation of the tensor product of $(|a\rangle\langle b|$ and the identity matrices of size of maximum photon number. In the same way, I implemented the other transition operators and $|1\rangle$ and $|2\rangle$are basisstates.For

ower purpose, it will always suffice to truncate the photon's Hilbertspace of $n = 30$. So we get 90 x 90 matrices. The projectors are implemented similarly, with the matrix $|a\rangle \langle a|$ With those its easy to construct the Hamiltons, H_{free} and H_{int}, as in Eq. (4) , Eq .(5). To calculate the the density matrices for steady states we can also use a qutip function. This function needs the total Hamiltonian and a list of the non-unitary operators as arguments. As output of the function steady-state we get the density-matrices for steady-states. [3]

4 Lasing transition

4.1 Wigner function and phase-averaged coherent states (PHAV)

The output of a laser is coherent light. The quantum description of coherent light is a coherent state. The photon number distribution of coherent light is a Poisson distribution. The randomized phase of a coherent state doesn't change the photon-number distribution.

A Wigner function is a representation of a general quantum state of light [1]. The function describe the probability density in phase space.

$$w(x,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi e^{\frac{-ip\xi}{\hbar}} \langle x + \frac{1}{2}\xi| \rho |x - \frac{1}{2}\xi\rangle .$$

(8)

The Wigner function from a coherent state is a Gaussian. The Wigner function of a phase-average-state is non-Gaussian, because p_{nn} of Eq. (10) is a Poisson distribution. The mathematical description of a PHAV is the same as a normal coherent state but with a random phase. So we get a new term of $\exp(i\pi\phi)$ in it. The standard coherent state $|\alpha\rangle$ can be represented by following formula:

$$|\alpha\rangle = e^{\frac{-|\alpha|}{2}} \sum_{n=0}^{\infty} \frac{|\alpha|^n e^{in\phi}}{\sqrt{n!}} |n\rangle .$$

(9)

Get the PHAV state will be obtained by integrating over the phase ϕ.

$$\rho_{PHAV} = \int_{0}^{2\pi} \frac{d\phi}{2\pi} |\alpha(\phi)\rangle \langle\alpha(\phi)| = \sum_{n=0}^{\infty} p_{nn} |n\rangle \langle n| ,$$

(10)

thus is equal to

$$\rho_{PHAV} = \sum_{n=0}^{\infty} \exp(-|\alpha|^2) \frac{|\alpha|^{2n}}{n!} |n\rangle \langle n| .$$

(11)

Eq. (11) is implemented in python to visualize a photon number distribution and Wigner plot Eq (8), shown in Fig 4.1. A Fock plot shows the occupation probability p_{nn}.
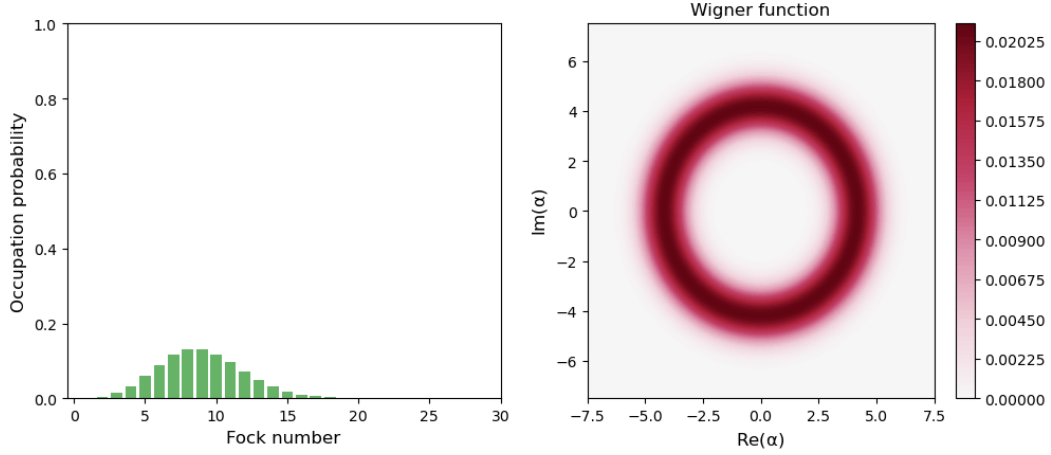

Figure 2: The Wigner-Fock plot of the density matrix of ρ_{PHAV} wit the parameter $\alpha = 3.5$.

For the Wigner function (8) we finally get the following equation

$$w(x,p) = 2 \exp\left[-2\left(|\alpha|^2 + |x + ip|^2\right)\right] I_0(4|\alpha||x + ip|). \qquad (12)$$

I_0 is the first Bessel function. This function is plotted in Fig. 2 The first result are plots from the photon number distribution and the Wigner function. For all calculations, I set the parameters $\hbar$ and kb equal to one. γ_h, γ_c are set to 1. And $\kappa = 0.028\gamma_h$ I tested those with different set of parameters, shown in Fig. 3 and Fig. 5.

Those rings which are shown in Fig. 3 and Fig. 5, are similar to the plot of Fig. 2. If we have a small κ means less photons will not leave and stay in the cavity. we see that in the plot of the photon number distribution. In the first plot I set a high leaking-parameter $\kappa = \frac{1}{\gamma_h}$. Shown in Fig. 3. This means that many photons leave the cavity, and only a few remain in the cavity. We see, that the occupation-number the photon number distribution is most zero and the probability for one photon is just 0.1.
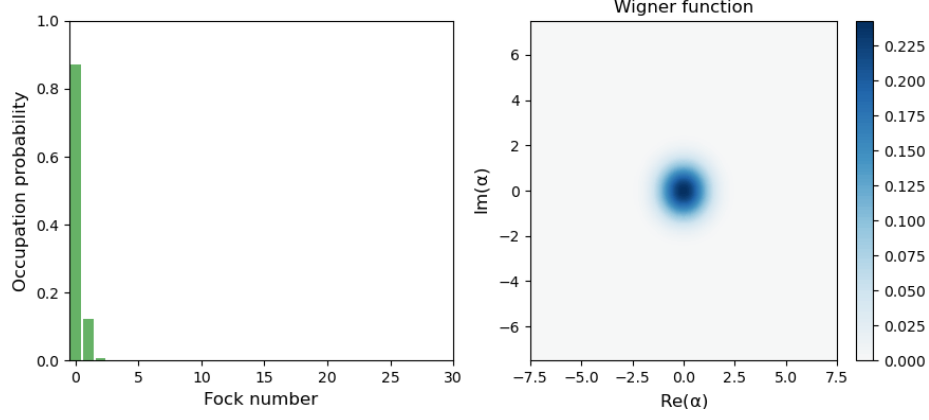
Figure 3: The parameters for the first plot are $n_h = 2.6, n_c = 0.001, n_f = 0.02, \kappa = 1\gamma_h$. With the factor of $\kappa = 1\gamma_h$ the leaking is to high for a lasing state.

In the second plot I took the same parameters again, but with a lower κ. We get more photons in the cavity. We see the distribution in the photon number distribution plot and a PHAV state in the Wigner function.

4.2   Double threshold behaviour

Lasing starts once a threshold for emission is attained, however, increasing the heat flow too much will end up producing thermal light instead of lasing. This configures a double threshold behaviour. The lasing output decreases when n_h is too high and this system shows a double threshold behaviour when the hot bath temperature n_h is too low ($n_h \approx n_c$), the excitation is too weak, and we say that the system is below the lasing threshold; with increasing n_h further, population inversion happens and the lasing light comes out. But when n_h keeps increasing, the lasing gain starts to decrease again and goes below the threshold. This is another critical point, after which the output light becomes thermal radiation again. To avoid this double threshold behaviour, we study a three-level system. This double threshold behaviour can describes by the zeno effect, because n_h is too high, the system is pushed in the highest state. With the plot $Tr[n\rho] = \langle n \rangle$ against different n_h Fig. 4a we see a double threshold behaviour because with too high n_h the expected photon number decrease again. The same thing is seen in the probability for the occupation number ($Tr[H\rho]$) Fig. 4a. With high n_h the probability for the system to stay in a the highest state increases as well. The robability he probability for the system to stay in a the state is described as $P1, P2, P3$ But in this case, the system can not give a lasing output, because lasing comes from the the transition from state $|2\rangle$ to $|1\rangle$ and if $P3$ is too high, the process is not possible anymore. We see in Fig. 4a that the lasing start at the point $P2$ and $P1$ diverging. Thar happens at $n_h = 0.2$. All the simulation are done with steady states.
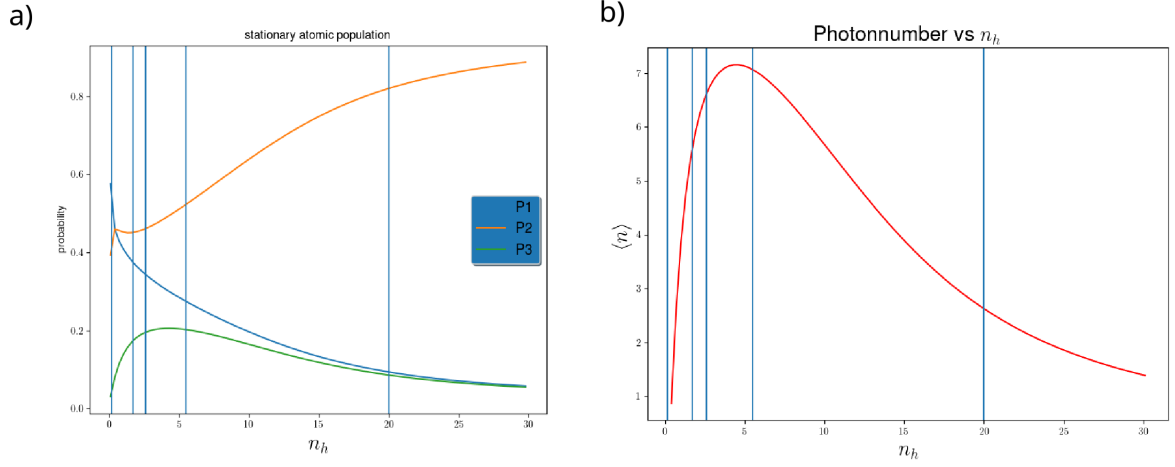
Figure 4: In Fig. 4a the probability for an atom to stay in a state $|0\rangle, |2\rangle$ or $|3\rangle$ vs n_h with the parameters The parameters for the first plot are $n_c = 2.6, n_f = 0.02, \kappa = 0.01$. The blue lines marks the the values for n_h for which the Wigner functions are plotted. In Fig. 4b we see, the the expectation value of the photon number versus different n_h

In a first step, the reduce density matrices ρ_{free} will be used to make Wigner and Fock plots. $\rho_{free} = Tr_{H_{free}}[\rho]$. It is possible to make the partial trace of ρ with qutip, to trace out the reduced density matrices ρ_{free}.
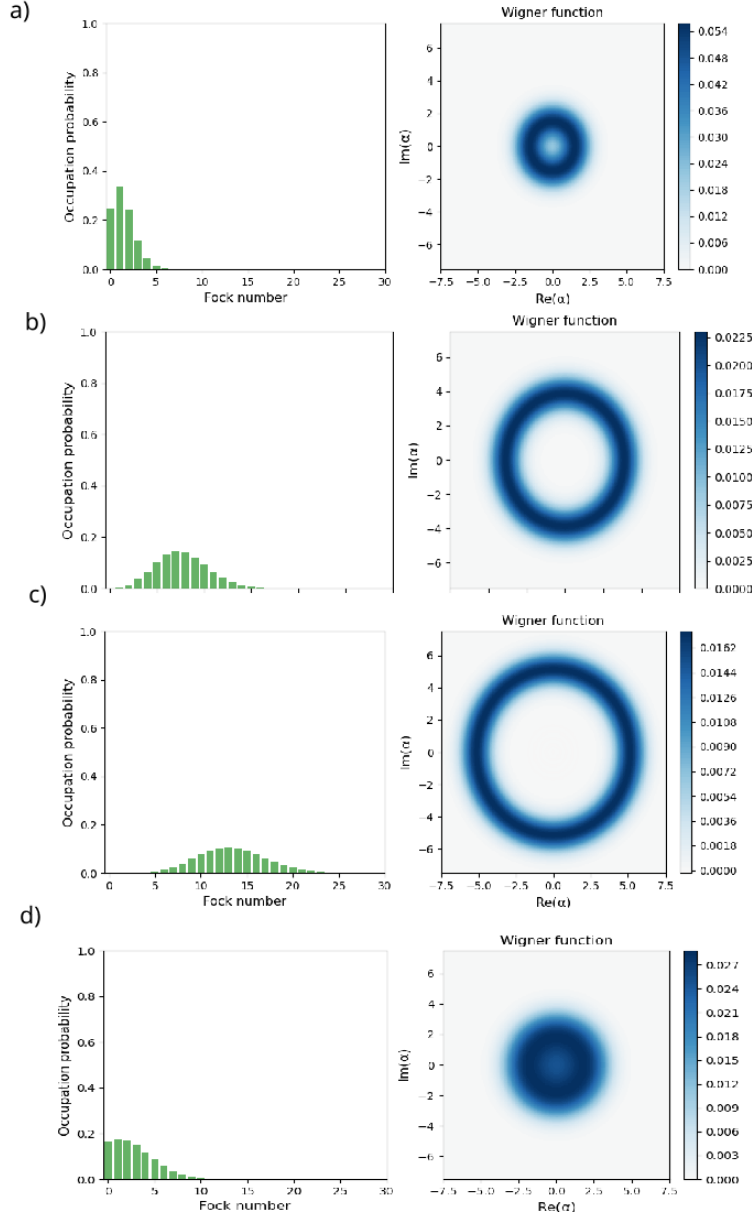
Figure 5: For all of the four the parameters are $n_c = 0.001 n_f = 0.02, \kappa = 0.1\gamma_h$. Cavity with a small leaking. **a)**The parameters for n_h is $n_h = 0.2$. **b)**$n_h = 2.6$ **c)** $n_h = 5.5$ **d)**$n_h = 20$.

We see in Fig 5a that start to get a PHAV If we increase $20 > n_h \gg 0.2$, then the cavity photon number increases again when increasing n_h at the very high temperature regime, as shown in Fig. 5b, Fig 5c. Because, in this

regime, the system has already a population inversion thus the increase of the hot bath temperature T_h can no longer bring in a significant increase to the photons gain. The hot bath no longer has any weakening effect to the lasing, thus more lasing photons can be produced in the cavity, and the lasing power can be increased. But, still, the cavity photon number is limited due to the single atom feature. If we further increase $n_h > 10$, we see the double threshold behavior. In Fig. 5d we get by $n_h = 20$ a thermal state.

5 Thermodynamics

5.1 Heat currents

When we work with density matrices, it is common to work with expectation values with $\langle A \rangle = Tr[A\dot{\rho}]$. A is an operator and describe a measurement. With this, we can calculate the expectation value from an Operator. To calculate the expected heat flow we can take the partial trace from $Tr[H_{frre}, \mathcal{L}[\rho]]$.

$$\langle J_{tot} \rangle = Tr[H_{free} \cdot \mathcal{L}_h[\rho]] + Tr[H_{free} \cdot \mathcal{L}_c[\rho]] + Tr[H_{free} \cdot \mathcal{L}_{cav}[\rho]]. \quad (13)$$

A part of my work is to calculate the occupation number analytically. The calculation is made in two steps. For the warm and the cold bath, we have a transition-operators in the trace. The trick of this calculation is, to get the form $Tr[\sigma_{ab}\rho\sigma_{ab}^{\dagger}]$. The equation gave the following result:

$$\langle J_h \rangle = Tr[H_{free} \cdot \mathcal{L}_h[\rho]] = \hbar\omega_h\gamma_h(2n_h + 1)(P1 - P3), \quad (14)$$

The same calculation can be done for the interaction with the cold bath.

$$\langle J_c \rangle = Tr[H_{free} \cdot \mathcal{L}_c[\rho]] = \hbar\omega_c\gamma_c(2n_c + 1)(P2 - P3), \quad (15)$$

For the calculation the $Tr[H_{free} \cdot \mathcal{L}_{cav}]$, I get the following result

$$\langle J_{cav} \rangle = T[H_{free} \cdot \mathcal{L}_{cav}[\rho]] = 2\hbar\omega_{cav}\kappa(n_{cav} - \langle a^{\dagger}a \rangle). \quad (16)$$

With the density matrices it is possible to calculate the heat flux by taking $Tr[H\mathcal{L}\rho]$. and plot this for different g so the goal of this work is to find a n_h which yield a PHAV state, as in [1] In the second step of the calculation we look for $\langle J_{tot} \rangle$ on different coupling constants g. In other words, in the plot below, the Trace from the density matrices times the Liouvilian against the coupling constant is visualized. The master equation depends on three different Liovillien therms. I calculated the expected heat flow for every different interaction. The cold interaction, the warm and the interaction with the cavity. Fig. 6 shows for the parameters $n_h = n_c = 2.6n_f = 0.02, \kappa = 0.01$ is shown below.
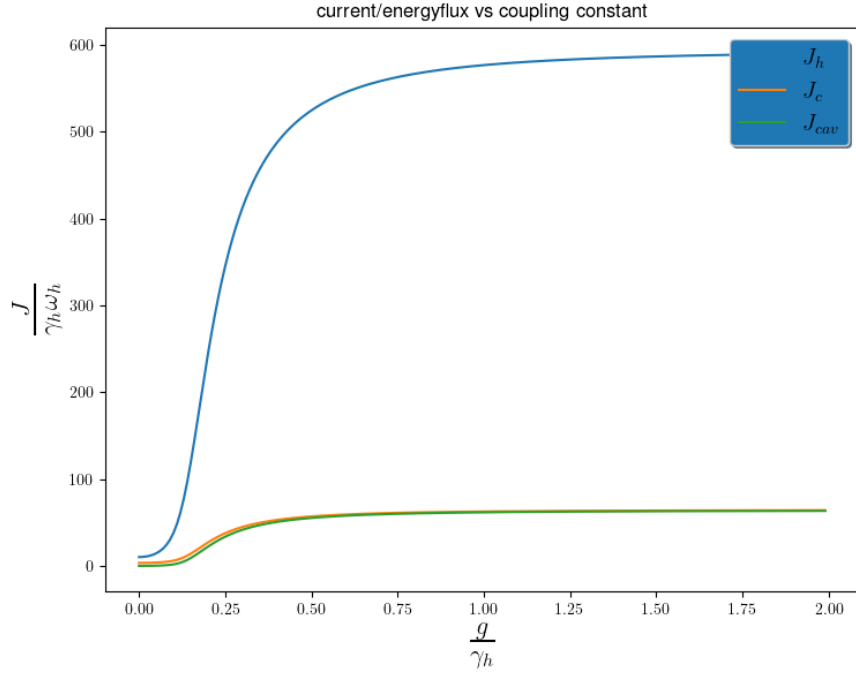
Figure 6: Energy flux vs g with the parameters The parameters for the first plot are $n_h = 2.6, n_c = n_f = 0.02, \kappa = 0.01$

5.2 Entropy production

Another relevant concept is the entropy. It is also a physical physical property. The entropy production is given by the formula
$\dot{\sigma} = \frac{J(n_h)}{T(n_c)} + \frac{J(n_c)}{T(n_h)} + \frac{J(n_{cav})}{T(n_{cav})}$. As already seen in Fig. 4b we have a optimum parameter n_h. in Fig. 4a we see the probability for $P1, P2, P3$ is in the region of $n_h = 5$ well distributed. We see also that this corresponds with the average of the photon number. The higher the entropy, the more the state of the three-level system changes and the more photons are emitted. We see that the entropy production correlate with the average of photon number in Fig 4b.
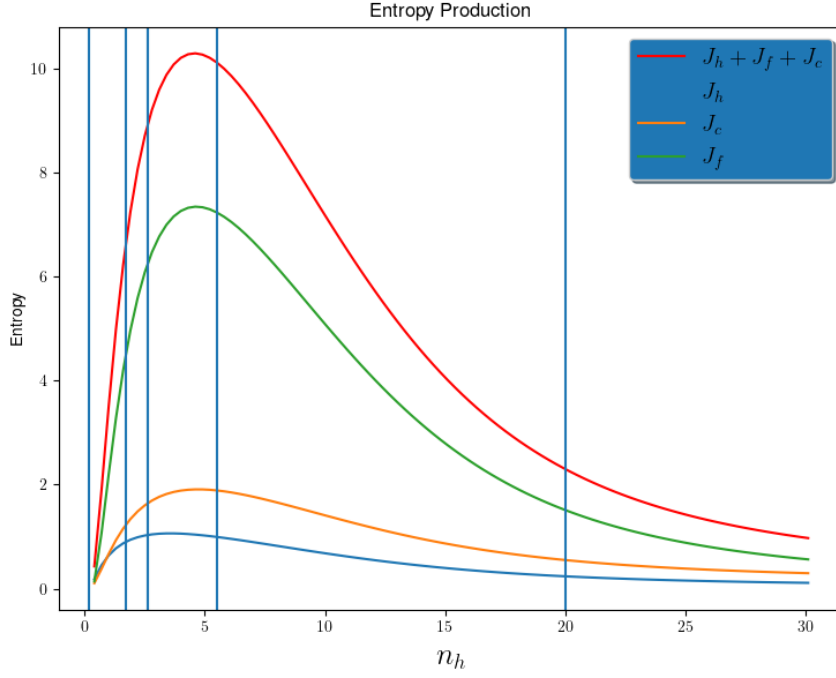

Figure 7: The entropy production for different n_h. The parameters for the first plot are $n_c = 0.001 n_f = 0.02, \kappa = 1$. As in paper [2] is used the values for the frequencies $\omega_f = 30\frac{1}{t}, \omega_h = 150\frac{1}{t}, \omega_c = 120\frac{1}{t}$.

6 Conclusion and outlook

In those numerical calculations are found PHAV states, which in a cavity, driven by the coupling of two different thermal bath. We found the double threshold behaviour from the system, by calculating different energy flux for different n_h With the condition that n_c, n_{cav} is almost zero, the leaking κ is small too, and the hot bath received a value between 1 and 5.5 When we compare the Winger functions of the numerical calculated states with the Wigner plot of a PHAV state Fig. 4a, we see, that pretty similar. As conclusion; it's possible to get a phase average coherent state. In Fig. 6 we see the average of the current, calculated with the formula Eq. (13) For different coupling constants, g we see, that the current increase at most between the $0 < g < 0.25$ In Fig. 4a we see the probability in which state an atom is. If we increase the temperature from the hot bath, we see that the occupation probability of $P2$ increase as well. However, the probability of an atom being in the $P1$ or $P3$ state decreases with increasing n_h. The probability $P1$ and $P2$ is thus inversely proportional to the temperature. The problem is that the cycle of photons can be stopped, because the probability that the photons goes from the ground state into the highest state is too small. Fig. 4a is consistent with the Wigner plots. Fig. 4b helps to get some conclusion about the system starts lasing and when it stops. With a too small temperature, It has not enough energy in the system and to have photons in the upper band. If n_h increase more, a maximum lasing output is reached. If the temperature increasing further, the system is in the highest level, therefore the Rabi oscillation between E_0 and E_2 does not exist anymore. At higher heat of the warm bath, dephasing of the system occurs. Thus, it interacts more with the environment. This has the same effect as if the system is measured often. The Zeno effect then shows that the system no longer oscillates between two different states. Spontaneous emission could be a problem too, that the photons which leave the atoms are not in a coherent state anymore.The consistent numerical and theoretical results we have obtained in the characterization of both PHAVs and their superpositions 2-PHAVs reinforce the possibility of using them for applications to communication protocols [1].

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7 Appendix

The hole Master equation without any substitution:

$$\mathcal{L}\hat{\rho} = \frac{\gamma_h}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_h}{k_b T_h}]-1}+1\right]\cdot\left(2\sigma_{13}\cdot\rho\cdot\sigma_{13}^\dagger - \sigma_{13}^\dagger\sigma_{13}\rho - \rho\sigma_{13}^\dagger\sigma_{13}\right)$$

$$+\frac{\gamma_h}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_h}{k_b T_H}]-1}\right]\cdot\left(2\sigma_{31}\cdot\rho\cdot\sigma_{31}^\dagger - \sigma_{31}^\dagger\sigma_{31}\rho - \rho\sigma_{31}^\dagger\sigma_{31}\right)$$

$$+\frac{\gamma_c}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_c}{k_b T_c}]-1}+1\right]\cdot\left(2\sigma_{23}\cdot\rho\cdot\sigma_{23}^\dagger - \delta_{23}^\dagger\sigma_{23}\rho - \rho\sigma_{23}^\dagger\sigma_{23}\right)$$

$$+\frac{\gamma_c}{2}\left[\frac{1}{\exp[\frac{\hbar\omega_c}{k_b T_c}]-1}\right]\cdot\left(2\sigma_{32}\cdot\rho\cdot\sigma_{32}^\dagger - \sigma_{32}^\dagger\sigma_{32}\rho - \rho\sigma_{32}^\dagger\sigma_{32}\right)$$

$$+\kappa\left[\frac{1}{\exp[\frac{\hbar w_f}{k_b T_f}]-1}+1\right]\cdot\left(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a\right)$$

$$+\kappa\left[\frac{1}{\exp[\frac{\hbar w_f}{k_b T_f}]-1}\right]\cdot\left(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger\right)$$

$$(17)$$

Figure 8: .

$$\text{Tr}\left[H_{free}\cdot\mathcal{L}_h\varrho\right]$$

$$= \text{Tr}\left[\left(\sum_{i=0}^{3}\hbar\omega_i\,|i\rangle\langle i| + \hbar\omega_a\,a^\dagger a\right)\left(\frac{\gamma_h}{2}\left[n(\omega_i;T_h)+1\right]\cdot\left(2\,\delta_{43}\,\varrho\,\delta_{43}^\dagger - \delta_{43}^\dagger\delta_{43}\varrho - \varrho\,\delta_{43}^\dagger\delta_{43}\right)+\right.\right.$$

$$\left.\left.+ \frac{\gamma_h}{2}\left[\bar{n}(\omega_i T_h)\,\left(2\,\delta_{34}\cdot\varrho\,\delta_{34}^\dagger - \delta_{34}^\dagger\delta_{34}\varrho - \varrho\,\delta_{34}^\dagger\delta_{34}\right)\right.\right.$$

$$= \text{Tr}\left[\hbar\omega_h\,\frac{\gamma_h}{2}\left(n(\omega_h,T_h)+1\right)\left(|3\rangle\langle 3|\,2\,\delta_{43}\,\varrho\,\delta_{43}^\dagger - |3\rangle\langle 3|\,\delta_{43}^\dagger\delta_{43}\varrho - |3\rangle\langle 3|\varrho\delta_{43}^\dagger\delta_{43}\right)\right.$$

$$+ \text{Tr}\left[\hbar\omega_h\,\frac{\gamma_h}{2}\,n(\omega_h,T_h)\,\left(|3\rangle\langle 3|\,2\,\delta_{34}\,\varrho\,\delta_{34}^\dagger - |3\rangle\langle 3|\,\delta_{34}^\dagger\delta_{34}\varrho - |3\rangle\langle 3|\varrho\delta_{34}^\dagger\delta_{34}\right)\right.$$

$$+ \text{Tr}\left[\hbar\omega_c\,\frac{\gamma_h}{2}\,(n(\omega_h,T_h)+1)\,\left(|1\rangle\langle 1|\,2\,\delta_{43}\,\varrho\,\delta_{43}^\dagger - |1\rangle\langle 1|\,\delta_{43}^\dagger\delta_{43}\varrho - |1\rangle\langle 1|\varrho\delta_{43}^\dagger\delta_{43}\right)\right.$$

$$+ \text{Tr}\left[\hbar\omega_c\,\frac{\gamma_h}{2}\,n(\omega_h,T_h)\,\left(|1\rangle\langle 1|\,2\,\delta_{34}\,\varrho\,\delta_{34}^\dagger - |1\rangle\langle 1|\,\delta_{34}^\dagger\delta_{34}\varrho - |1\rangle\langle 1|\varrho\delta_{34}^\dagger\delta_{34}\right)\right.$$

$$+ \text{Tr}\left[\hbar\omega_2\,|2\rangle\langle 2|\cdot\mathcal{L}_h\varrho\right] = 0$$

$$+ \text{Tr}\left[\hbar\omega_a\,a\cdot\mathcal{L}_h\varrho\right] = 0 \quad /\!/ \text{ kommutiert}$$

$|3\rangle\langle 3|3\rangle\langle 1|1\rangle\langle 3|\varrho = -P_3 \quad -P_3$

$$= \text{Tr}\left[\hbar\omega_h\,\frac{\gamma_h}{2}\,(n(\omega_h,T_h)+1)\,\left(2\cdot|3\rangle\langle 3|\delta_{43}\varrho\delta_{43}^\dagger - |3\rangle\langle 3|\delta_{43}^\dagger\delta_{43}\varrho - |3\rangle\langle 3|\varrho\delta_{43}^\dagger\delta_{43}\right)\right.$$

$$+ \text{Tr}\left[\hbar\omega_h\,\frac{\gamma_h}{2}\,n(\omega_h,T_h)\cdot\left(2|3\rangle\langle 3|3\rangle\langle 1|\varrho\delta_{43}^\dagger\right)\right] = 1\,(2\cdot\delta_{34}\varrho\,\delta_{13}^\dagger) = +P_1$$

$$+ \text{Tr}\left[\hbar\omega_c\,\frac{\gamma_h}{2}\,n(\omega_h,T_h)+1)\cdot|1\rangle\langle 1|\left(2\cdot\delta_{43}\,\varrho\,\delta_{43}^\dagger\right)\right] = 2|1\rangle\langle 1|1\rangle\langle 3|\varrho|3\rangle\langle 1| = 1\,P_3$$

$$+ \text{Tr}\left[\hbar\omega_c\,\frac{\gamma_h}{2}\,n(\omega_h,T_h)\cdot\left(-|1\rangle\langle 1|\,\delta_{34}^\dagger\delta_{34}\varrho - |1\rangle\langle 1|\varrho\delta_{34}^\dagger\delta_{34}\right)\right] = -2P_1$$

$$-|3\rangle\langle 3|\varrho\,\delta_{43}^\dagger\,\delta_{43} = \varrho|3\rangle\langle 1|1\rangle\langle 3|3\rangle\langle 3| = P_3$$

$$= -\hbar\omega_h\,\gamma_h\cdot n(\omega_h,T_h)+1)\quad P_3$$
$$+ \hbar\omega_h\,\gamma_h\cdot n(\omega_h,T_h)\,P_3 \quad\Bigg\} = -\hbar\omega_h\,\gamma_h\,P_3$$
$$+ \hbar\omega_c\,\gamma_h\cdot(n(\omega_h,T_h)+1)\quad P_1$$
$$- \hbar\omega_c\,\gamma_h\cdot(n(\omega_h,T_h))\,P_2 \quad\Bigg\} = +\hbar\omega_c\,\gamma_h\,P_1$$

$$= \hbar\,\gamma_h\cdot\left(\omega_c\,P_1 - \omega_h\,P_3\right)$$

19

Figure 9: .

$$\text{Tr}\left[H_{free}\cdot\mathcal{L}_{cav}\,\varrho\right]$$

$$=\left(\sum_{i=1}^{3}\hbar\omega_i\,|i\rangle\langle i| + \hbar\omega\,a^\dagger a\right)\cdot\left[k\,(\bar{n}+1)\cdot(2\,a\,\varrho\,a^\dagger - a^\dagger a\,\varrho - \varrho\,a^\dagger a)\right]$$

$$+\left(\sum_{i=1}^{3}\hbar\omega_i\,|i\rangle\langle i| + \hbar\omega\,a^\dagger a\right)\cdot\left[k\,\bar{n}\cdot(2\,a^\dagger\varrho\,a - a\,a^\dagger\varrho - \varrho\,a\,a^\dagger)\right]$$

$$=\text{Tr}\left[\hbar\omega\,k\,(n+1)\left[2\,a^\dagger a\,a\,\varrho\,a^\dagger - a^\dagger a\,a^\dagger a\,\varrho - a^\dagger a\,\varrho\,a^\dagger a\right]\right.$$

$$+\hbar\omega\,kn\left[2\,a^\dagger a\,a^\dagger\varrho\,a - a^\dagger a\,a^\dagger\varrho - a^\dagger a\,\varrho\,a\,a^\dagger\right]$$

$$=\hbar\omega\,k\,(n+1)\left[2\,\text{Tr}\,a^\dagger a^\dagger a\,a\,\varrho\right]\ominus 2\,\text{Tr}\left[a^\dagger a\,a^\dagger a\,\varrho\right]\quad\substack{\to\ominus 2\text{Tr}[a^\dagger a\varrho]-\text{Tr}[a^\dagger a^\dagger a a]\\ Da\;\;a a^\dagger - a^\dagger a = 1}$$

$$+\hbar\omega\,k\,(\bar{n})\cdot\left[2\,\text{Tr}\left[a\,a^\dagger a\,a^\dagger\varrho\right] - \text{Tr}\left[a^\dagger a\,a^\dagger a\,\varrho\right] - \text{Tr}\left[a\,a^\dagger a^\dagger\varrho\right]\right]$$

$$=\ominus\hbar\omega\,k\,(n+1)\cdot\text{Tr}\left[a^\dagger a\,\varrho\right]$$

$$+\hbar\omega\,k\,n\cdot\text{Tr}\left[a\,a^\dagger(a\,a^\dagger - a^\dagger a)\,\varrho\right] + \text{Tr}\left[a\,a^\dagger a\,a^\dagger\varrho\right] - \text{Tr}\left[a^\dagger a\,a\,a^\dagger\varrho\right]$$

$$=\ominus\hbar\omega\,k\,(n+1)\,2\,\text{Tr}\left[a^\dagger a\,\varrho\right] +$$

$$+\hbar\omega\,k\,(n)\,2\,\text{Tr}\left[a\,a^\dagger\varrho\right] \quad + \quad \text{Tr}\left[[a\,a^\dagger - a^\dagger a]\,a\,a^\dagger\varrho\right]$$

$$=-\hbar\omega\,k(n+1)\,2\,\text{Tr}\left[a^\dagger a\,\varrho\right] + \hbar\omega\,kn\,2\,\text{Tr}\left[a\,a^\dagger\varrho\right]$$

$$=2\,\hbar\omega\,k\,\left(\bar{n} - \langle a^\dagger a\rangle\right)$$

20