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Generation of entanglement between two three-level atoms interacting with a time-dependent damping field

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ABSTRACT

In this article, the interaction between two Λ-type three-level atoms and a single-mode cavity field is discussed. The damping parameter is taken into account. Also, the field and the atoms are assumed to be coupled with modulated coupling parameter which depends explicitly on time. Even though the system seems complicated, the analytical form of the wave function and the probability amplitudes are exactly obtained. To comprehend the degree of entanglement between subsystems (field-atom and atom-atom), the dynamics of entanglement through various measures, namely, concurrence and negativity is evaluated. Also, the Mandel parameter and normal squeezing of the resultant state have been calculated. In each case, the influences of damping parameter on the above criteria are numerically analyzed, in detail. It is illustrated that the degree of entanglement can be generated by taking the evolved parameters, appropriately.

1. Introduction

The idea of entanglement is perhaps one of the most amazing characteristics of quantum mechanics. Entanglement is one of the most remarkable nonclassical properties in quantum theory and also a fundamental resource for quantum information processing [1,2]. Quantum entanglement is seen not only as a riddle, but also as a resource to be handled for quantum computing and communication [3,4], for instance in the examination of dense coding, quantum teleportation, decoherence in quantum computers and quantum cryptographic [5–8]. Recently, there becomes been great attention to describe the interaction between atom-atom, field-field and atom-field. In fact, these interactions have been widely studied in many papers [9–13]. In the recent years, a great attention is paid for investigating the interaction between a three-level atom and electromagnetic cavity field. It was proved that key distributions based on three-level quantum systems are more secure against eavesdropping than those based on two-level systems [14,15]. So it is important to point out that the increased insight into the dynamics of the three-level systems may be helpful in developing quantum information theory [16–20]. Studying the purity of a three-level atomic system in the presence of Stark shift contributions and the effects of gravity field are presented in Ref. [21]. Investigating the entanglement of a multi-photon three-level atom near the edge of a photonic band gap is presented in Ref. [22]. The dynamics of entanglement of a three-level atom in motion interacting one and two coupled modes has been studied [23,24]. The entanglement between the three-level atom for different configurations and a cavity field (correlated and non-correlated) in the presence of nonlinearities (intensity-dependent coupling and cross Kerr medium and non-linear Kerr medium) is well studied in many cases for the initial states of the system [25–27] as well as when the atom and the field are assumed to be coupled with modulated coupling parameter which depends explicitly on time [28]. The autocorrelation function of microcavity-emitting field in the non-linear regime is investigated [29]. Information dynamics of a three-level atom interacting with a damped cavity field is recently investigated taking into consideration that the optical cavity is coupled to the environment [30].

The noise spectra of an emitted field from a quantum well embedded in semiconductor microcavity in the strong coupling regime have been studied [31]. On the other hand the physical properties of the excited coherent states through the Mandel's parameter and the Wehrl entropy have been studied [32]. In this regard and after wonderful work that was presented by Bateman [33]. Caldirola and Kanai have introduced the Hamiltonian of a harmonic oscillator with a time-dependent mass [34,35]. Which is known as Caldirola-Kanai (CK) Hamiltonian. This Hamiltonian was investigated in several quantum systems to study the physical properties such as plasma environments and the mesoscopic (RLC) circuits [36,37]. The coherent states and

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squeezed states of the CK Hamiltonian were also studied [38,39], and shown that the eigenfunctions of the CK Hamiltonian satisfy the minimum uncertainty relation in a generalized form [40].

On the other hand the damping in the interaction of a two-photon field with a two-level atom through quantized (CK) Hamiltonian has been studied [41], the damping in the interaction of two Λ-type three-level atoms and a single-mode quantized field with the (CK) Hamiltonian has been studied [42] In this article, we study the interaction between two Λ-type three-level atoms and a single-mode quantized field with the (CK) Hamiltonian. Furthermore, the field and the atoms are assumed to be coupled with modulated coupling parameter which depends explicitly on time. Under an approximation where fast oscillations are ignored the probability amplitudes and the associated atoms-field state vectors are obtained. Moreover, we study the influence of the damping parameters on the time behavior of the physical properties such as concurrence, negativity, Mandel parameter, and normal squeezing. The rest of this article is organized as follows. In the next section, we derive the form of the probability amplitudes for the considered system. In Section 3, by considering different values of the coupling variation parameter, the effect of damping on the physical properties are numerically investigated. Finally, in Section 4, a summary is provided.

2. Description of the model

We devote this section to propose the Hamiltonian model which describes the interaction between two Λ-type three-level atoms and a single-mode quantized field with the (CK) Hamiltonian. To do so we consider the (CK) Hamiltonian which describes a damped harmonic oscillator with a time-dependent mass [34,35]. This Hamiltonian is given by

$$H_{ck} = \frac{\hat{p}^2}{2m_0}\exp(-2\gamma t) + \frac{1}{2}m_0\omega^2\hat{q}^2\exp(2\gamma t), \tag{1}$$

where ω, m_0, and γ are frequency, initial mass, and damping parameter, respectively. In this regard, it is noted that the canonical transformations it is possible to build generalized creation and annihilation operators which are described in terms of the standard $\hat{a}^\dagger$ and $\hat{a}$ as [43]

$$\begin{aligned}\hat{Q} &= \frac{1}{2\sqrt{\Omega\omega}}(\zeta_+\hat{a} + \zeta_-\hat{a}^\dagger),\\ \hat{Q}^\dagger &= \frac{1}{2\sqrt{\Omega\omega}}(\zeta_+^*\hat{a}^\dagger + \zeta_-^*\hat{a}),\end{aligned} \tag{2}$$

where, $\Omega = \omega\sqrt{1-\eta^2}$, $\eta = \frac{\gamma}{\omega}$ and $\zeta_\pm = \Omega + i\gamma \pm \omega$. The relation $[\hat{Q}, \hat{Q}^\dagger] = 1$ is easily verified. Accordingly, the transformed Hamiltonian in terms of $\hat{Q}$, $\hat{Q}^\dagger$ reads [44].

$$H_{ck} = \hbar\Omega(\hat{Q}^\dagger\hat{Q} + \frac{1}{2}). \tag{3}$$

The obtained Hamiltonian is reduced from a variable mass Hamiltonian to a quantized Hamiltonian that does not explicitly depend on time.

Here, the atomic levels are indicated by $|1\rangle$, $|2\rangle$ and $|3\rangle$ with energies $\omega_1 > \omega_2 > \omega_3$, transition $|2\rangle \leftrightarrow |3\rangle$ is forbidden in the electric-dipole approximation; the allowed transitions are $|1\rangle \leftrightarrow |2\rangle$ and $|1\rangle \leftrightarrow |3\rangle$ Fig. 1. The free atomic and the field Hamiltonians are given by

$$H_0 = \sum_{j=A,B}(\omega_1\hat{\sigma}_{11}^j + \omega_2\hat{\sigma}_{22}^j + \omega_3\hat{\sigma}_{33}^j) + \Omega\hat{Q}^\dagger\hat{Q}, \tag{4}$$

Accordingly, the atoms-field Hamiltonian in the dipole approximation can be written as

$$H_{AB-f} = \sum_{j=A,B} g_1^j(t)(\hat{\sigma}_{12}^j + \hat{\sigma}_{21}^j)(\hat{Q} + \hat{Q}^\dagger) + g_2^j(t)(\hat{\sigma}_{13}^j + \hat{\sigma}_{31}^j)(\hat{Q} + \hat{Q}^\dagger), \tag{5}$$

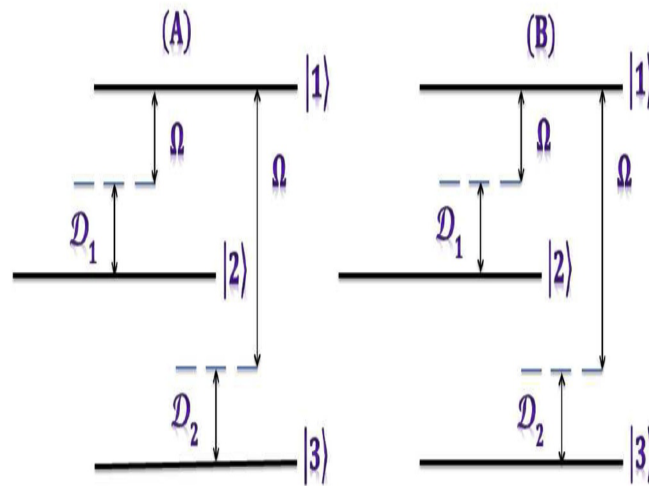

Fig. 1. Energy level diagram for two Λ-type three-level atoms coupled to a single-mode cavity field of frequency Ω with detunings $\mathcal{D}_1$, $\mathcal{D}_2$.

where $g_l^j(t)$, $(l = 1, 2)$ are the time coupling parameters between the field and the atoms and it may take the form $g_l^j(t) = \lambda_l^j\cos(\mu t)$ where μ is the coupling variation parameter and λ_l^j is an arbitrary constant.

We can represent H_{AB-f} in the interaction picture as

$$\begin{aligned}\mathcal{H}_I(t) &= e^{iH_0t}H_{AB-f}e^{-iH_0t}\\ &= \textstyle\sum_{j=A,B} g_1^j(t)(e^{it(\omega_1-\omega_2)}\hat{\sigma}_{12}^j + e^{-it(\omega_1-\omega_2)}\hat{\sigma}_{21}^j)(e^{-i\Omega t}\hat{Q} + e^{i\Omega t}\hat{Q}^\dagger)\\ &\quad + g_2^j(t)(e^{it(\omega_1-\omega_3)}\hat{\sigma}_{13}^j + e^{-it(\omega_1-\omega_3)}\hat{\sigma}_{31}^j)(e^{-i\Omega t}\hat{Q} + e^{i\Omega t}\hat{Q}^\dagger),\end{aligned} \tag{6}$$

If we replacing the terms $\hat{Q}$ and $\hat{Q}^\dagger$ from Equation (2) into Equation (6), the Hamiltonian in the interaction picture and in the rotating wave approximation is obtained as

$$\begin{aligned}\mathcal{H}_I(t) &= \textstyle\sum_{j=A,B}\frac{g_1^j(t)}{2\sqrt{\omega\Omega}}(e^{it\mathcal{D}_1}\zeta_+\hat{a}\hat{\sigma}_{12}^j + e^{-it\mathcal{D}_1}\zeta_+^*\hat{a}^\dagger\hat{\sigma}_{21}^j)\\ &\quad + \frac{g_2^j(t)}{2\sqrt{\omega\Omega}}(e^{it\mathcal{D}_2}\zeta_+\hat{a}\hat{\sigma}_{13}^j + e^{-it\mathcal{D}_2}\zeta_+^*\hat{a}^\dagger\hat{\sigma}_{31}^j).\end{aligned} \tag{7}$$

where the detuning parameters $\mathcal{D}_1$ and $\mathcal{D}_2$ are given by

$$\mathcal{D}_1 = (\omega_1 - \omega_2) - \Omega, \qquad \mathcal{D}_2 = (\omega_1 - \omega_3) - \Omega. \tag{8}$$

Now, in order to reach the solution of the atoms-field system described by the above Hamiltonian, there are two different but equivalent methods, namely Heisenberg operators and probability amplitudes approaches [45]. However, the offered formalism for the considered system is based on the method of probability amplitudes. But, before applying this way to reach our aim, it is necessary to write the trigonometric function in an exponential form $g_l^j(t) = \frac{\lambda_l^j}{2}(e^{i\mu t} + e^{-i\mu t})$, so the Hamiltonian take the following form

$$\begin{aligned}\mathcal{H}_I(t) &= \textstyle\sum_{j=A,B}\frac{\lambda_1^j}{4\sqrt{\omega\Omega}}((e^{i\mu t} + e^{-i\mu t})e^{it\mathcal{D}_1}\zeta_+\hat{a}\hat{\sigma}_{12}^j + (e^{i\mu t} + e^{-i\mu t})e^{-it\mathcal{D}_1}\zeta_+^*\hat{a}^\dagger\hat{\sigma}_{21}^j)\\ &\quad + \frac{\lambda_2^j}{4\sqrt{\omega\Omega}}((e^{i\mu t} + e^{-i\mu t})e^{it\mathcal{D}_2}\zeta_+\hat{a}\hat{\sigma}_{13}^j + (e^{i\mu t} + e^{-i\mu t})e^{-it\mathcal{D}_2}\zeta_+^*\hat{a}^\dagger\hat{\sigma}_{31}^j).\end{aligned} \tag{9}$$

We note that there are two exponential terms in the above Hamiltonian $e^{\pm i(\mathcal{D}_{1,2}+\mu)t}$ and $e^{\pm i(\mathcal{D}_{1,2}-\mu)t}$, as an approximation we neglect the rapidly oscillating (counter rotating) terms $e^{\pm i(\mathcal{D}_{1,2}+\mu)t}$, this approximation is completely similar to the RWA and has been used widely to solve the physical models [46]. So, the Hamiltonian of the system in the interaction picture can be written as

$$\mathcal{H}_I(t) = \sum_{j=A,B} \frac{\lambda_1^j}{4\sqrt{\omega\Omega}} (e^{it\Delta_1}\zeta_+ \hat{a}\hat{\sigma}_{12}^j + e^{-it\Delta_1}\zeta_+^* \hat{a}^\dagger \hat{\sigma}_{21}^j) + \frac{\lambda_2^j}{4\sqrt{\omega\Omega}} (e^{it\Delta_2}\zeta_+ \hat{a}\hat{\sigma}_{13}^j + e^{-it\Delta_2}\zeta_+^* \hat{a}^\dagger \hat{\sigma}_{31}^j). \quad (10)$$

where Δ_1 and Δ_2 are given by

$$\Delta_1 = \mathcal{D}_1 - \mu, \qquad \Delta_2 = \mathcal{D}_2 - \mu. \quad (11)$$

Now, for simplicity, we consider the resonance case $\Delta_1 = \Delta_2$. Also, without loss of generality, we suppose $\lambda_1^j = \lambda_2^j = \lambda$, we can recast the Hamiltonian (10) as follows

$$\mathcal{H}_I(t) = \sum_{j=A,B} \frac{\lambda}{4\sqrt{\omega\Omega}} (e^{it\Delta_1}\zeta_+ \hat{a}\hat{\sigma}_{12}^j + e^{-it\Delta_1}\zeta_+^* \hat{a}^\dagger \hat{\sigma}_{21}^j) + \frac{\lambda}{4\sqrt{\omega\Omega}} (e^{it\Delta_1}\zeta_+ \hat{a}\hat{\sigma}_{13}^j + e^{-it\Delta_1}\zeta_+^* \hat{a}^\dagger \hat{\sigma}_{31}^j). \quad (12)$$

The wave function at any time **t** may be written in the following form

$$\begin{aligned}|\psi(t)\rangle = \sum_{n=0}^{+\infty} [&\Theta_1(n, t)|1, 1, n\rangle + \Theta_2(n+1, t)(|1, 2, n+1\rangle + |2, 1, n+1\rangle) \\ &+ \Theta_3(n+1, t)(|1, 3, n+1\rangle + |3, 1, n+1\rangle) \\ &+ \Theta_4(n+2, t)(|2, 3, n+2\rangle + |3, 2, n+2\rangle) \\ &+ \Theta_5(n+2, t)|2, 2, n+2\rangle + \Theta_6(n+2, t)|3, 3, n+2\rangle]. \end{aligned} \quad (13)$$

where, $\Theta_k (k = 1, 2..6)$ are the time-dependent probability amplitudes, the time-dependent Schrödinger equation of the wave function (13) i.e. $i\frac{\partial}{\partial}|\psi(t)\rangle = \mathcal{H}_I|\psi(t)\rangle$ the coupling differential equations for the probability amplitudes can be written as

$$\begin{aligned}
i\dot{\Theta}_1 &= 2\psi_{n+1}\Theta_2 e^{i\Delta_1 t} + 2\psi_{n+1}\Theta_3 e^{i\Delta_1 t}, \\
i\dot{\Theta}_2 &= \psi_{n+2}\Theta_5 e^{i\Delta_1 t} + \psi_{n+1}^*\Theta_1 e^{-i\Delta_1 t} + \psi_{n+2}\Theta_4 e^{i\Delta_1 t} \\
i\dot{\Theta}_3 &= \psi_{n+2}\Theta_4 e^{i\Delta_1 t} + \psi_{n+2}\Theta_6 e^{i\Delta_1 t} + \psi_{n+1}^*\Theta_1 e^{-i\Delta_1 t}, \\
i\dot{\Theta}_4 &= \psi_{n+2}^*\Theta_3 e^{-i\Delta_1 t} + \psi_{n+2}^*\Theta_2 e^{-i\Delta_1 t}, \\
i\dot{\Theta}_5 &= 2\psi_{n+2}^*\Theta_2 e^{-i\Delta_1 t}, \\
i\dot{\Theta}_6 &= 2\psi_{n+2}^*\Theta_3 e^{-i\Delta_1 t},
\end{aligned} \quad (14)$$

where

$$\psi_{n+1} = \lambda\sqrt{n+1}\zeta_+/4\sqrt{\omega\Omega}, \qquad \psi_{n+2} = \lambda\sqrt{n+2}\zeta_+/4\sqrt{\omega\Omega}. \quad (15)$$

In this case, we assumed that $\omega^2 \gg \gamma^2$. So that we have $\Omega \simeq \omega - \gamma^2/2\omega$. By considering the field to be initially prepared in the coherent states $|\alpha\rangle$ and the atoms to be initially prepared in the excited states $|1, 1\rangle$, the wave function at the initial time is given by

$$|\psi(0)\rangle = \sum_{n=0}^{+\infty} c_n|1, 1, n\rangle \quad (16)$$

where $c_n = e^{-\frac{\bar{n}}{2}}\frac{\bar{n}^{\frac{n}{2}}}{\sqrt{n!}}$. The system in Equation (14) can be solved with the help of Laplace transform after taking the following transformation

$$\begin{aligned}
\Theta_1(n, t) &= \bar{\Theta}_1(n, t)e^{i\Delta_1 t}, \quad \Theta_2(n+1, t) = \bar{\Theta}_2(n+1, t), \\
\Theta_3(n+1, t) &= \bar{\Theta}_3(n+1, t), \quad \Theta_4(n+2, t) = \bar{\Theta}_4(n+2, t)e^{-i\Delta_1 t}, \\
\Theta_5(n+2, t) &= \bar{\Theta}_5(n+2, t)e^{-i\Delta_1 t}, \quad \Theta_6(n+2, t) = \bar{\Theta}_6(n+2, t)e^{-i\Delta_1 t},
\end{aligned} \quad (17)$$

where $\bar{\Theta}_k$ is an arbitrary function. After straightforward calculations, the probability amplitudes can be calculated as

$$\begin{aligned}
\Theta_1(n, t) &= \sum_{e=1}^6 \frac{\Phi_1(s_e)}{\Omega_e} e^{(s_e + i\Delta_1)t}, \\
\Theta_2(n+1, t) &= \sum_{e=1}^6 \frac{\Phi_2(s_e)}{\Omega_e} e^{s_e t}, \\
\Theta_3(n+1, t) &= \Theta_2(n+1, t) \\
\Theta_4(n+2, t) &= \sum_{e=1}^6 \frac{\Phi_4(s_e)}{\Omega_e} e^{(s_e - i\Delta_1)t}, \\
\Theta_5(n+2, t) &= \Theta_6(n+2, t) = \Theta_4(n+2, t)
\end{aligned} \quad (18)$$

where

$$\begin{aligned}
\Omega_e &= \prod_{d=1\cdot\cdot 6}^{e\neq d} (s_e - s_d), \\
\Phi_1(s) &= (s - i\Delta_1)(8|\psi_{n+2}|^4 + s(s - i\Delta_1)(6|\psi_{n+2}|^2 + s(s - i\Delta_1))), \\
\Phi_2(s) &= -i\psi_{n+1}^*(s - i\Delta_1)^2(2|\psi_{n+2}|^2 + s(s - i\Delta_1)), \\
\Phi_3(s) &= \Phi_2(s), \\
\Phi_4(s) &= \Phi_5(s) = \Phi_6(s) = -2\psi_{n+1}^*\psi_{n+2}^*(s - i\Delta_1)(2|\psi_{n+2}|^2 + s(s - i\Delta_1)).
\end{aligned} \quad (19)$$

and

$$\begin{aligned}
s_1 &= i\Delta_1, \\
s_2 &= \frac{1}{2}(i\Delta_1 + \sqrt{-8|\psi_{n+2}|^2 - \Delta_1^2}), \\
s_3 &= \frac{1}{2}(i\Delta_1 - \sqrt{-8|\psi_{n+2}|^2 - \Delta_1^2}), \\
s_4 &= \frac{\sqrt[3]{2B^2} - 2\sqrt[3]{3}\mu_1}{\sqrt[3]{36B}} \\
s_5 &= \frac{(-\sqrt[6]{3} + i\sqrt[3]{9})\sqrt[3]{2B^2} + (2\sqrt{3} + 6i)\mu_1}{\sqrt{3}\sqrt[3]{96B}} \\
s_6 &= \frac{(-\sqrt[6]{3} - i\sqrt[3]{9})\sqrt[3]{2B^2} + (2\sqrt{3} - 6i)\mu_1}{\sqrt{3}\sqrt[3]{96B}},
\end{aligned} \quad (20)$$

where

$$\begin{aligned}
A &= \sqrt{81\mu_0^2 + 12\mu_1^3} - 9\mu_0, \\
\mu_0 &= -4i\Delta_1(|\psi_{n+1}|^2 - |\psi_{n+2}|^2) \\
\mu_1 &= 4|\psi_{n+1}|^2 + 4|\psi_{n+2}|^2 + \Delta_1^2
\end{aligned} \quad (21)$$

In what follows, we turn our attention to study some physical properties of the considered system.

3. Some physical properties

Now, the above calculations can be used to discuss some physical properties of this system such as concurrence as a measure of entanglement between atoms and field and negativity as a measure of entanglement between two atoms. Note that in all situations, we set $\delta_1 = \delta_2 = \lambda$ such that $\delta_1 = (\omega_1 - \omega_2) - \omega$, $\delta_2 = (\omega_1 - \omega_3) - \omega$ and $\lambda = 0.01\omega$.

3.1. The entanglement between atoms and field: concurrence

The concurrence given by Wootters and Hill [47,48] is a proper measure of the entanglement of any state of two atoms, pure or mixed. For a pure state $|\psi(t)\rangle$ on $(K \times L)$-dimensional Hilbert space $M = M_K \otimes M_L$. The concurrence can be defined as follows [49,50]

$$C(t) = \sqrt{2(|\langle\psi(t)|\psi(t)\rangle|^2 - Tr(\varrho_L^2(t)))}, \quad (22)$$

where $\varrho_L(t) = Tr_K(|\psi(t)\rangle\langle\psi(t)|)$ is the reduced density operator of the subsystem with dimension L and Tr_K is the partial trace over M_K. It is remarkable to mention that, the concurrence fluctuates between $\sqrt{2(L-1)/L}$ for maximally entangled state and 0 for separable state. Herein we calculate the concurrence to get the degree of entanglement (DEM) between the atoms and the field. To reach this goal, we need to calculate the atomic reduced density matrix ϱ_{AB} which are calculated by taking trace over the field as

$$\varrho_{AB} = \mathrm{Tr_f}(|\psi(t)\rangle\langle\psi(t)|) \quad (23)$$

with the atomic basis $\{|1, 1\rangle, |1, 2\rangle, |1, 3\rangle, |2, 1\rangle, |2, 2\rangle, |2, 3\rangle, |3, 1\rangle, |3, 2\rangle, |3, 3\rangle\}$, the diagonal matrix elements are given by,

$$\begin{aligned}
\varrho_{11} &= \sum_{n=0}^{\infty} |c_n|^2|\Theta_1(n, t)|^2, & \varrho_{22} &= \sum_{n=0}^{\infty} |c_n|^2|\Theta_2(n+1, t)|^2, \\
\varrho_{33} &= \varrho_{22}, & \varrho_{44} &= \sum_{n=0}^{\infty} |c_n|^2|\Theta_4(n+2, t)|^2, \\
\varrho_{55} &= \varrho_{44}, & \varrho_{66} &= \varrho_{44},
\end{aligned} \quad (24)$$

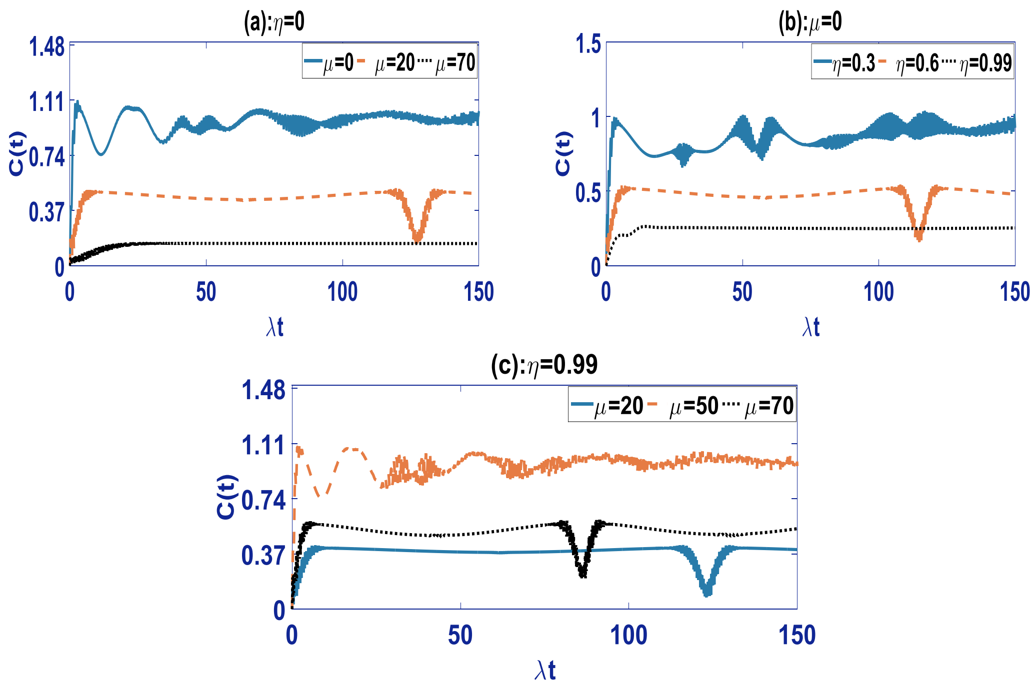

Fig. 2. Concurrence versus scaled time λt, with $\bar{n}=25$, $\delta_1=\delta_2=\lambda$ and $\lambda/\omega=0.01$.

and the off-diagonal matrix elements are given by

$$\begin{aligned} \varrho_{12} &= \textstyle\sum_{n=0}^{\infty} c_{n+1}c_n^{*}\Theta_1(n+1,t)\Theta_2(n+1,t)^{*}, & \varrho_{45} &= \varrho_{44}, \\ \varrho_{14} &= \textstyle\sum_{n=0}^{\infty} c_{n+2}c_n^{*}\Theta_1(n+2,t)\Theta_4(n+2,t)^{*} & \varrho_{46} &= \varrho_{44}, \\ \varrho_{24} &= \textstyle\sum_{n=0}^{\infty} c_{n+1}c_n^{*}\Theta_2(n+2,t)\Theta_4(n+2,t)^{*}, & \varrho_{56} &= \varrho_{44}, \\ \varrho_{23} &= \varrho_{22}, & \varrho_{21} &= \varrho_{12}^{*}, \\ \varrho_{34} &= \varrho_{24}, & \varrho_{41} &= \varrho_{14}^{*}. \end{aligned} \tag{25}$$

So we can rewrite concurrence in the following form

$$C(t)=\sqrt{2\sum_{l,m=1,2..9}^{l\neq m}(\varrho_{ll}\varrho_{mm}-\varrho_{lm}\varrho_{ml})}. \tag{26}$$

In Fig. 2, we have plotted the concurrence versus scaled time λt when $\hat{n}=25$. In Fig. 2a, we set three dissimilar values of the coupling variation parameter μ in the absence of damping parameter where $\eta=0$. We observe that when $\mu=0$, solid line, the concurrence reaches the maximum value of entanglement after the beginning of the interaction. Also, the lower points of the concurrence go up as time goes on. On the other hand, when we taking into account the μ parameter, we note that the (DEM) is pulled down as μ increases. In Fig. 2b, we set three dissimilar values of the damping parameter η in the absence of the coupling variation parameter ($\mu=0$), we note that when $\eta=0.3$, the maximum value of entanglement decreasing compared by the case $\eta=0$ (solid line in Fig. 2a). Moreover, we note that by increasing μ parameter (Fig. 2a) on the general behavior of concurrence is similar to the effect of increasing damping parameter η see (Fig. 2b).

In Fig. 2c, for the three dissimilar values of the coupling variation parameter μ in the presence of damping parameter $\eta=0.99$. It is clear that, the coupling variation parameter plays a dramatic role in the degree of entanglement, as gradual increase in the degree of entanglement by increasing the coupling variation parameter from 0 to 50, then it goes back to decreasing in the period $\mu>50$, it is more clearly Fig. 3 the concurrence plotted for versus μ with the different values of the scaled timeλt. It is found that in the absence of the damping parameter the entanglement decreases dramatically by increasing the values of coupling variation parameter see Fig. 3a. On the other hand when damping parameter takes a suitable value $\eta=0.99$ the entanglement increases gradually, by increasing the coupling variation parameter from 0 to 50, we see that the entanglement decreased in the period $\mu>50$ see Fig. 3b.

3.2. The entanglement between the two atoms: negativity

In this subsection, we are chiefly interested in analyzing the effect of the coupling variation and damping parameters on the entanglement between the two atoms. From equation (23), we can study the evolution of entanglement for the atoms A and B. We will use the definition of the negativity [51,52]

$$N(\varrho_{AB})=\frac{\sum_k |u_k^{T_A}|-1}{2}, \tag{27}$$

where $u_k^{T_A}$ are the eigenvalues of the partial transpose of ϱ_{AB} with respect to subsystem A.To display the effect of damping on the degree of entanglement between two atoms. In Fig. 4 we have plotted the time evolution of negativity when the field is initially in a coherent state with the mean photon number $\bar{n}=25$. In Fig. 4a for undamped instances, for the three different values of the coupling variation parameter μ, we see that when $\mu=0$, (solid-line), the negativity rapidly oscillates around its upper value (nearly 0.24). On the other hand, when we taking into account the μ parameter we note that the negativity is pulled down as μ increases. In Fig. 4b, we set three dissimilar values of the damping parameter η in the absence of the coupling variation parameter where $\mu=0$, we note that the effect of increasing μ parameter, the general behavior of negativity has a destructive effect on the entanglement, the dotted curve where the damping is 0.99, the rebirth of entanglement happened in short time and the death of entanglement happened in long time. In Fig. 4c, for three dissimilar values of the coupling variation parameter μ, in the presence of damping parameter $\eta=0.99$, in this case, the coupling variation parameter plays a different role in the degree of entanglement, we note that effect of increasing μ parameter from 0 to 50 on the behavior of the negativity has a constructive effect on the entanglement. While a devastating effect on the entanglement occur of the higher values μ. In Fig. 5, we have plotted the negativity for versus μ with the same data in Fig. 3. In Fig. 5a, we can see the sudden death phenomenon of entanglement if there is not taken

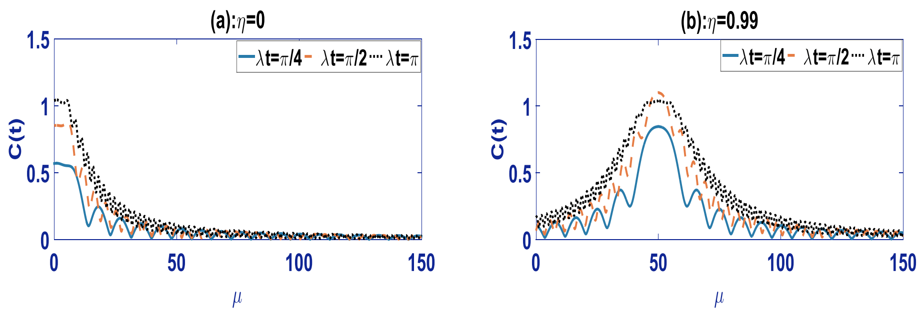

Fig. 3. Concurrence versus μ, with $\bar{n} = 25$, $\delta_1 = \delta_2 = \lambda$ and $\lambda/\omega = 0.01$, and for different values λt.

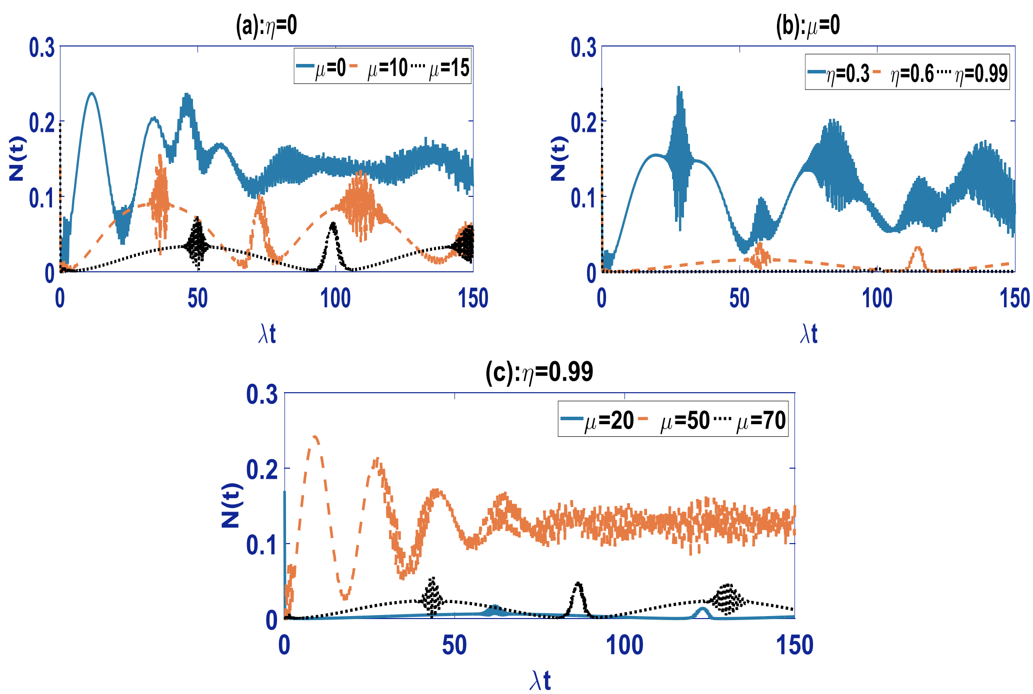

Fig. 4. Negativity versus scaled time λt, with $\bar{n} = 25$, $\delta_1 = \delta_2 = \lambda$ and $\lambda/\omega = 0.01$.

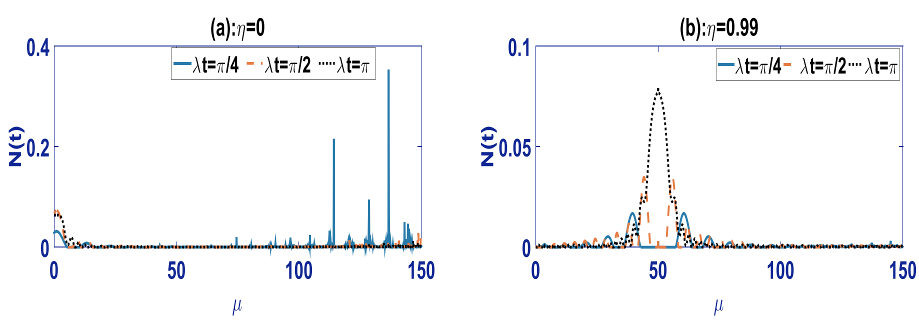

Fig. 5. Negativity versus μ, with $\bar{n} = 25$, $\delta_1 = \delta_2 = \lambda$ and $\lambda/\omega = 0.01$, and for different values λt.

the damping parameter into account, while Fig. 5b illustrates the case when the damping taken into account, we can see occur the sudden death and the sudden birth of entanglement.

3.3. Mandel Q_m parameter

The sub-Poissonian statistics can be studied via the Mandel Q_m-parameter. In specific, if $Q_m > 0$, the system exhibits super-Poissonian while, if $Q_m < 0$, the statistics is sub-Poissonian statistics. The Mandel Q_m-parameter is defined as [53]

$$Q_m = \frac{\langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2 - \langle a^\dagger a \rangle}{\langle a^\dagger a \rangle}. \tag{28}$$

We study the influences of the coupling variation parameter on the temporal evolution of the Mandel Q_m parameter for two cases, first case when we do not take the damping parameter into account as in Fig. 6a,b,c, the other case when we take the damping parameter into account as in Fig. 6d,e,f. For the first case, we observe that when $\mu = 0$, Mandel parameter changes between negative and positive values which means that the photons illustrate sub or super-Poissonian statistics for various intervals of times see Fig. 6a. On the other hand, when μ parameter takes a suitable value where $\mu = 15$, we observe that a

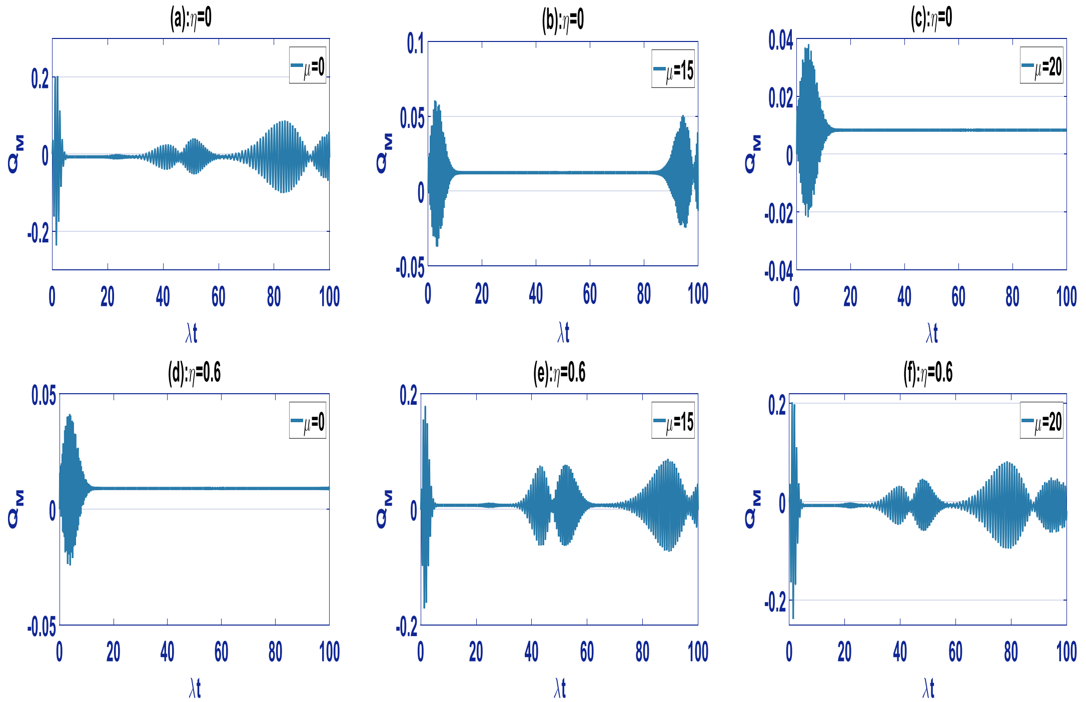

Fig. 6. Mandel Q_m parameter versus scaled time λt, with $\bar{n} = 25$, $\delta_1 = \delta_2 = \lambda$ and $\lambda/\omega = 0.01$.

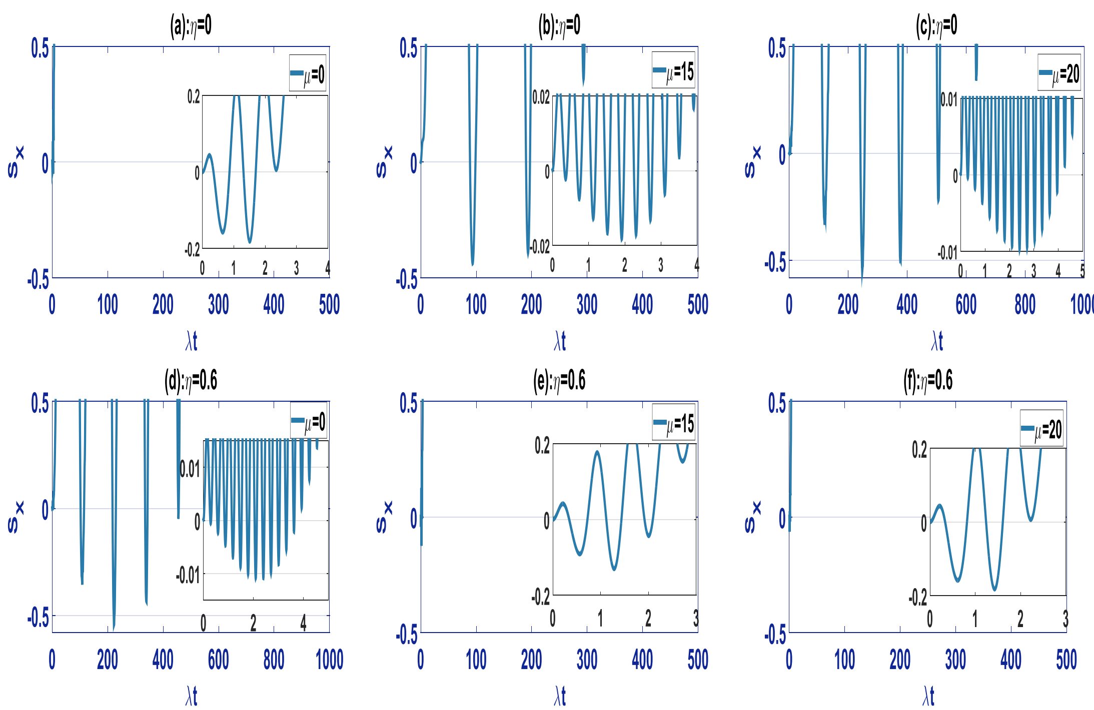

Fig. 7. Normal squeezing S_x parameter plotted as a function of scaled time λt, with $\bar{n} = 25$, $\delta_1 = \delta_2 = \lambda$ and $\lambda/\omega = 0.01$, and for different values of the damping parameter η. All initial atomic states are as stated in Fig. 2.

collapses period is clearly evident in the positive region see Fig. 6b. Further, we note that increasing the value of μ, not only leads to a great increase in a collapses periods but also leads to a great decreasing in the revival periods, so that they appear only at the beginning of the interaction and then disappear as time goes on see Fig. 6c. As soon as we take the influence of the damping parameter into consideration a reverse change occurs in the function behavior. Where increasing the value of μ, the causes of the function fluctuate between positive and negative values see Fig. 6d,e,f.

3.4. Normal squeezing

To discuss the normal squeezing properties of the obtained state $|\psi(t)\rangle$, we introduce two quadrature field operators $\hat{x} = \frac{a + a^{\dagger}}{\sqrt{2}}$ and

$\hat{p} = \frac{a + a^{\dagger}}{\sqrt{2}i}$, the variances of position $(\Delta x)^2$ and momentum $(\Delta p)^2$, the squeezing of x and p is expressed by $S_l = 2(\Delta l)^2 - 1$ with $l = x,\ p$. Squeezing occurs in l component $-1 < S_l < 0$. These parameters can be rewritten as follows:

$$S_x = 2\langle \hat{a}^{\dagger}\hat{a}\rangle + 2\mathrm{Re}\langle \hat{a}^{\dagger 2}\rangle - 4(\mathrm{Im}\langle \hat{a}^{\dagger}\rangle)^2$$
$$S_p = 2\langle \hat{a}^{\dagger}\hat{a}\rangle - 2\mathrm{Re}\langle \hat{a}^{\dagger 2}\rangle - 4(\mathrm{Im}\langle \hat{a}^{\dagger}\rangle)^2 \tag{29}$$

In Fig. 7, we have plotted the quantity S_x versus scaled time λt with the same data in Fig. 6. For the first case when $\mu = 0$, we noted that the squeezing occurs for a short time $(0.3 < \lambda t < 2)$ see Fig. 7a. Increasing the value of μ leads to increases in the amount and maximum value of squeezing see Fig. 7b,c. As soon as we take the influence of the damping parameter into consideration. Increasing the value of μ leads to decrease in the amount and maximum value of squeezing see Fig. 7d,e,f.

4. Summary

In this work, we have studied a Hamiltonian model which describes the interaction between the field and two Λ-type three-level atoms. The system has been taken to include the effect of damping parameter via quantized the field by using (CK) Hamiltonian. The coupling parameter between the atoms and the field is modulated to be time dependent. Under the initial conditions, an exact solution for the wave function in Schrödinger picture is obtained. We have managed to discuss the degree of entanglement between atoms-field and atom-atom by using the concurrence and negativity respectively as well as both the Mandel Q_m-parameter and normal squeezing. The results for the effect of damping and the coupling variation parameter on these features showed that the coupling variation parameter plays an important role in the time evolution of entanglement. We conclude the main results as follows:

i) The effects of the damping parameter in the absence of the coupling variation parameter are similar to the coupling variation parameter effects in the absence of the damping parameter. We observe that there is decreased of the DEM, we not decrease of the revival time by increase of both the damping parameter and the coupling variation parameter.

ii) In the presence of the damping parameter, when we studied the effect of increasing the coupling variation parameter on the DEM of the system, we noticed that increase in the DEM in the period from $0 \le \mu \le 50$, while significantly decreased in $\mu > 50$.

iii) In the absence of damping, the Mandel parameter becomes more positive by increasing the coupling variation parameter, while oscillating between positive and negative values in the absence of μ parameter. On the other hand, when the damping parameter takes into account the Mandel parameter becomes more positive in the absence of μ parameter while oscillating between positive and negative values by increasing the coupling variation parameter.

iv) In the absence of damping, the squeezing becomes stronger by increasing the coupling variation parameter while it is decreasing in the absence of μ parameter. On the other hand, when the damping parameter takes into account the squeezing becomes weaker by increasing the coupling variation parameter while it is increasing in the absence of μ parameter.

References

[1] L. Chen, E. Chitambar, R. Duan, Z. Ji, A. Winter, Phys. Rev. Lett. 105 (2010) 200501.
[2] S. Ahadpour, F. Mirmasoudi, Theor. Math. Phys. 195 (2018) 628.
[3] H.A. Hessian, A.-B.A. Mohamed, A.H. Homid, Int. J. Quantum Inf. 13 (2015) 1550056.
[4] N. Jing, B. Yu, Quant. Inf. Process. 17 (2017) 11128.
[5] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W.K. Wootters, Phys. Rev. Lett. 70 (1993) 1895.
[6] C.H. Bennett, S.J. Wiesner, Phys. Rev. Lett. 69 (1992) 2881.
[7] Z.L. Xiang, S. Ashhab, J.Q. You, F. Nori, Rev. Mod. Phys. 85 (2013) 623.
[8] P.D. Nation, J.R. Johansson, M.P. Blencowe, F. Nori, Rev. Mod. Phys. 84 (2012) 1.
[9] E.T. Jaynes, F.W. Cummings, Proc. IEEE 51 (1963) 89.
[10] G. Cattapan, P. Lottib, Physica E 57 (2014) 118.
[11] M. Ghorbani, M. Javad Faghihi, H. Safari, JOSA B 34 (2017) 1884.
[12] J. Hide, V. Vedral, Physica E 45 (2010) 359.
[13] H.R. Baghshahi, M.K. Tavassoly, M.J. Faghihi, Laser Phys. 26 (2014) 125203.
[14] N.J. Cerf, M. Bourennane, A. Karlsson, N. Gisin, Phys. Rev. Lett. 88 (2002) 127902.
[15] P.K. Jha, H. Eleuch, Y.V. Rostovtsev, Phys. Rev. A 82 (2010) 045805.
[16] H.-I. Yoo, J.H. Eberly, Phys. Rep. 118 (1985) 239.
[17] A.M. Abdel-Hafez, A.S.F. Obada, M.M.A. Ahmad, Physica A 144 (1987) 530.
[18] A.-S.F. Obada, A.M. Abdel-Hafez, J. Mod. Optic. 34 (1987) 665.
[19] M. Alexanian, S.K. Bose, Phys. Rev. A 52 (1995) 2218.
[20] M.H. Mahran, A.-R.A. El-Samman, A.-S.F. Obada, J. Mod. Optic. 36 (1989) 53.
[21] A.-S.F. Obada, M.M.A. Ahmed, A. Salah, A.M. Farouk, J. Mod. Optic. 63 (2016) 2315.
[22] M. Abdel-Aty, Laser Phys. 16 (2006) 1381.
[23] M.J. Faghihi, M.K. Tavassoly, M. Hatami, Physica A 407 (2014) 100.
[24] M.J. Faghihi, M.K. Tavassoly, M.R. Hooshmandasl, JOSA B 30 (2013) 1109.
[25] M.J. Faghihi, M.K. Tavassoly, JOSA B 30 (2013) 2810.
[26] M. Ghorbani, H. Safari, M.J. Faghihi, JOSA B 33 (2016) 1022.
[27] A.-S.F. Obada, S.A. Hanoura, A.A. Eied, Laser Phys. 23 (2013) 055201.
[28] N.H. Abd El-Wahab, A. Salah, A.S.A. Rady, A.-N.A. Osman, Differ. Equ. Dyn. Syst. (2016), https://doi.org/10.1007/s12591-016-0291-0.
[29] H. Eleuch1, N. Rachid, Eur. Phys. J. D 57 (2010) 259.
[30] A.-B.A. Mohamed, H. Eleuch, Eur. Phys. J. Plus 132 (2017) 75.
[31] H. Eleucha, Eur. Phys. J. D 49 (2008) 391.
[32] K. Berrada, S. Abdel-Khalek, H. Eleuch, Y. Hassouni, Quant. Inf. Process. 18 (2013) 69.
[33] H. Bateman, Phy. Rev. 38 (1931) 815.
[34] E. Kanai, Prog. Theor. Phys. 3 (1948) 440.
[35] P. Caldirola, Il Nuovo Cimento 18 (1941) 393.
[36] I.A. Pedrosa, A.P. Pinheiro, Prog. Theor. Phys. 125 (2011) 1133.
[37] J.R. Choi, S. Lakehal, M. Maamache, S. Prog, Electromagn. Res. Lett. 44 (2014) 71.
[38] J.R. Choi, Int. J. Theor. Phys. 45 (2006) 176.
[39] J.R. Choi, Pramana 62 (2004) 13.
[40] S.P.J. Kim, Phys. A. Math. Gen. 36 (2003) 12089.
[41] A. Dehghani1, B. Mojaveri, R. Jafarzadeh Bahrbeig, Int. J. Theor. Phys. 65 (2018) 3982.
[42] T.M. El-Shahat, M. Kh. Ismail, A.F. Al Naim, J. Russ. Laser Res. 39 (2018) 231.
[43] M.S. Abdalla, Phys. Rev. A 170 (1991) 393.
[44] R. Daneshmand, M.K. Tavassoly. Ann. Phys. 529 (5) 1600246.
[45] M.O. Scully, M.S. Zubairy, Quantum Optics, Cambridge University Press, Cambridge, 2001.
[46] W.H. Louisell, A. Yariv, A.E. Siegman, Phys. Rev. 124 (1961) 1646.
[47] S. Hill, W.K. Wootters, Phys. Rev. Lett. 78 (1997) 5022.
[48] W.K. Wootters, Phys. Rev. Lett. 80 (1998) 2245.
[49] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, Rev. Mod. Phys. 81 (2009) 865.
[50] A.-S.F. Obada, M.M.A. Ahmed, A.M. Farouk, A. Salah, Eur. Phys. J. D 71 (2017) 338.
[51] A. Peres, Phys. Rev. Lett. 77 (1996) 1413.
[52] P. cki. Horode, Phys. Lett. A 232 (1997) 333.
[53] M. L, J. Russ. Laser Res. 4 (1979) 205.