Superconducting Circuits for Quantum Information Processing
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Homework 3

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Help session date:	Thu 26.05.2022
Due date:	Thu 02.06.2022
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Total points: 30

In this homework we will investigate a bosonic Schrödinger-cat qubit, called the "Kerr-cat" qubit, both analytically and with numerical simulations. This qubit is an example of a noise-biased qubit and, in contrast to other bosonic qubits, it is implemented in an anharmonic oscillator.

1 The Kerr-cat qubit

1.1 Code states

A so-called Schrödinger-cat qubit is encoded into superpositions of macroscopically distinct states which are robust against noise. Here, these states are coherent states of microwave light in a superconducting cavity. Their Schrödinger-cat-like superpositions are defined as:

$$|\mathcal{C}_\alpha^\pm\rangle = \mathcal{N}_\alpha^\pm \left(|+\alpha\rangle \pm |-\alpha\rangle\right), \tag{1}$$

with the normalization factors: $\mathcal{N}_\pm = 1/\sqrt{2(1 \pm e^{-2|\alpha|^2})}$. Note that the coherent states $|\pm\alpha\rangle$ have the same average photon number $\bar{n} = |\alpha|^2$, but opposite phase and are therefore maximally far from each other in oscillator phase space. The states spanning the Bloch sphere of the Schrödinger-cat qubit are:

$$\begin{aligned}
|\pm X\rangle &= \left(|\mathcal{C}_\alpha^+\rangle \pm |\mathcal{C}_\alpha^-\rangle\right)/\sqrt{2}, \\
|\pm Y\rangle &= \left(|\mathcal{C}_\alpha^+\rangle \pm i\,|\mathcal{C}_\alpha^-\rangle\right)/\sqrt{2}, \\
|\pm Z\rangle &= |\mathcal{C}_\alpha^\pm\rangle\,.
\end{aligned}$$

2 pts. (a) Express the identity and Pauli operators acting on the Bloch sphere of this qubit in terms of $|\mathcal{C}_\alpha^\pm\rangle$, with $|\mathcal{C}_\alpha^+\rangle$ as the basis vector $\begin{pmatrix}1\\0\end{pmatrix}$. Double-check your results by computing the eigenvalues of the Pauli operators with the states $|\pm X\rangle, |\pm Y\rangle, |\pm Z\rangle$.

2 pts. (b) Show that the states $|\mathcal{C}_\alpha^\pm\rangle$ contain only even and odd photon numbers respectively. This property is referred to as even and odd photon number parity. Note that this is true irrespective of the global normalization factor meaning you do not need to include it in the calculation.

2 pts. (c) Derive the expressions of the states $|\mathcal{C}_\alpha^\pm\rangle$ in the limit $\alpha \to 0$.

3 pts. (d) Prove that in the limit $|\alpha|^2 = \bar{n} \to \infty$ we get $p = \mathcal{N}_\alpha^+/\mathcal{N}_\alpha^- \to 1$ and that the following approximations become exact:

$$\begin{aligned}
|\pm X\rangle &\approx |\pm\alpha\rangle, \\
|\pm Y\rangle &\approx (|+\alpha\rangle \mp i\,|-\alpha\rangle)/\sqrt{2}, \\
|\pm Z\rangle &\approx (|+\alpha\rangle \pm |-\alpha\rangle)/\sqrt{2}.
\end{aligned}$$

Because of the exponential dependence of $\mathcal{N}_\alpha^\pm$ on $\bar{n}$, these approximations are very good even for moderately large photon numbers. Compute $1-p$ for $\bar{n}=2$, $\bar{n}=4$, and $\bar{n}=6$ (the answer can be an upper bound).

1.2 Photon loss, error suppression, and stabilization

Under photon loss with rate κ, the density matrix of the system evolves according to the master equation

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}(\hat{H}_{\text{eff}}\hat{\rho} - \rho\hat{H}_{\text{eff}}^\dagger) + \kappa\hat{a}\hat{\rho}\hat{a}^\dagger. \tag{2}$$

Here, $\hat{H}_{\text{eff}} = \hat{H} - \hbar\frac{i}{2}\kappa\hat{a}^\dagger\hat{a}$ is an effective non-hermitian Hamiltonian that incorporates the deterministic energy decay. The last term on the right side of eqn (2) represents stochastic photon jumps. We will at first focus only on these jumps.

3 pts. (e) By applying the jump operator $\sqrt{\kappa}\hat{a}$ to the projector onto the code space of the cat qubit $\hat{P} = |\mathcal{C}_\alpha^+\rangle\langle\mathcal{C}_\alpha^+| + |\mathcal{C}_\alpha^-\rangle\langle\mathcal{C}_\alpha^-|$, derive the expression of this operator in terms of the Pauli matrices you found earlier. Give an interpretation in terms of bit flips, phase flips, and bit-phase flips.

2 pts. (f) In the limit of large $\bar{n}$, what happens to the errors caused by photon loss on the cat-qubit Bloch sphere? Give a physical interpretation based on what you now know about coherent states as well as the states $|\mathcal{C}_\alpha^\pm\rangle$.

1 pt. (g) In the same limit as in the previous point, calculate the probability $p_\mathrm{j} = \tau \mathrm{Tr}[\kappa \hat{a}\hat{\rho}\hat{a}^\dagger]$ that such a jump happens during a time τ. Give an interpretation of how this probability depends on the average photon number $\bar{n}$.

Now that we have better understood the quantum-jump part of the density matrix evolution, we will focus on the part describing the deterministic evolution. As you have seen in problem set 10, under this evolution the coherent states would decay to the vacuum state over time. In order to compensate for this decay, we introduce a parametrically driven nonlinear Hamiltonian:

$$\hat{H}/\hbar = -K\hat{a}^{\dagger 2}\hat{a}^2 + \epsilon_2 \hat{a}^{\dagger 2} + \epsilon_2^* \hat{a}^2 \ . \tag{3}$$

Here, K is the Kerr nonlinearity which we have already seen when discussing the transmon qubit (we have simplified the notation with $K = E_c/2$). We have also added a parametric drive of strength ϵ_2. This drive is similar to a coherent drive, but it adds two photons to the system instead of one at a time. The Hamiltonian written above is already in the rotating frame of the anharmonic oscillator and the parametric drive is resonant.

3 pts. (h) Show that this Hamiltonian has two degenerate eigenstates which are coherent states with amplitudes $\pm\alpha = \pm\sqrt{\epsilon_2/K}$. What does this mean for the evolution of these states and all of their superpositions under this Hamiltonian?
Hint: Try to factorize the Hamiltonian.

1.3 Single-qubit gate

We can perform a gate operation on this qubit by applying an additional microwave drive given by $\hat{H}_\mathrm{drive}/\hbar = \epsilon_x \hat{a} + \epsilon_x^* \hat{a}^\dagger$. Here, we have again written $\hat{H}_\mathrm{drive}$ in the rotating frame of the oscillator and chosen the drive to be resonant. This eliminates the fast time-dependence of the drive terms, but the drive strength ϵ_x is still a complex number (it has an amplitude and a phase).

2 pts. (i) Use the expression of $\hat{a}$ you found earlier to show that this leads to a Rabi oscillation on the cat-qubit Bloch sphere with a rate $\Omega = 4\mathrm{Re}(\epsilon_x \alpha)$. You can calculate the expression using the limit $p \to 1$. How is this different from what you would expect to happen when applying the same microwave drive to a transmon qubit?
Hint: Remember that the Rabi rate is defined as $\Omega/2\hat{\sigma}_i$, where i is the axis around which the qubit rotates.

2 Playing with the cat

Now, open the attached notebook, run each part and find a physical explanation for the following simulations.

1 pt. (a) We modify the Hamiltonian of eqn. (3) such that the parametric drive is ramped up very slowly compared to $1/K$ ($t_{\mathrm{pd}} \gg 1/K$). Use as an initial state first $|\psi\rangle_0 = |0\rangle$, and then $|\psi\rangle_0 = |1\rangle$. What are the steady states of the system after the drive has fully ramped on? Which states of the cat-qubit do the states correspond to? Find a physical explanation for why the system goes from these initial states to these final states in terms of the photon number parity and the Hamiltonian they are subject to.

1 pt. (b) Now, repeat the previous point but change the initial states to $|\psi\rangle_0 = 1/\sqrt{2}(|0\rangle + |1\rangle)$ and $|\psi\rangle_0 = 1/\sqrt{2}(|0\rangle + i\,|1\rangle)$. Based on the results of point 1(c) and 2(a), what do you expect the final states to be? Where do they lie on the cat-qubit Bloch sphere?

2 pts. (c) Add the single photon drive Hamiltonian of problem 1(i) after ramping on from $|\psi\rangle_0 = |0\rangle$ and explain its action on the state. How would you implement a π pulse for this qubit? Now apply the same drive phase-shifted by $\pi/2$, can you explain what you see based on exercise 1(i)?

1 pt. (d) Increase the intensity of the drive (ϵ_x) above $4\bar{n}K$. How do the Pauli operators evolve? Can you find an explanation for their behaviour?

2 pts. (e) Now, run the script in which the initial state $|\psi\rangle_0 = |\alpha\rangle$ is a coherent state and the Hamiltonian does not include either of the two drives ($\epsilon_2 = \epsilon_x = 0$). Can you explain what you see for different evolution times $T_{\mathrm{ev}} = \{\frac{\pi}{4K}, \frac{\pi}{3K}, \frac{\pi}{2K}, \frac{\pi}{K}\}$? How could this be used to implement a gate on the Kerr-cat qubit?

3 pts. (f) Lastly, add losses to the system for the three states $|+Z\rangle$, $|+Y\rangle$ and $|+X\rangle$. Explain the behaviour for the three states, also by visualizing the expectation value of the Pauli operators over time. Use this simulation to estimate T_{bf} the characteristic decay time associated with bit flips on the cat-qubit Bloch sphere.