3.4 – The Product and Quotient Rules

MATH 2554 – Calculus I

Fall 2019

Recall: If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}\Big(f(x) + g(x)\Big) = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}\Big(f(x) - g(x)\Big) = f'(x) - g'(x) \qquad \text{Difference Rule}$$

Question: What about products and quotients?

Example: $f(x) = x^2$, $g(x) = x^3$.

$f'(x) =$

$g'(x) =$

$\big(f(x)g(x)\big)' =$

Theorem (Product Rule)

If f and g and differentiable at x, then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).$$

The difference quotient

$$\begin{aligned}
&\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \frac{f(x+h) - f(x)}{h} g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}
\end{aligned}$$

Product Rule Proof, cont'd

This means $\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$ exists and, moreover,

$$\begin{aligned} \frac{d}{dx}\big(f(x)g(x)\big) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right] \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

Examples: Compute

$$\frac{d}{dx}\left(x^2 \cdot x^3\right) =$$

$$\frac{d}{dw}\left((2w^2 + 3w + 1)(w^4 + e^w)\right) =$$

$$\frac{d}{ds}\left(s^2 4(s+2)\right) =$$

Question: What about quotients?

Let $f(x) = x^3$ and $g(x) = x^2$. Then $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{d}{dx}x = 1$.

However, $\dfrac{f'(x)}{g'(x)} = \dfrac{3x^2}{2x} = \dfrac{3}{2}x$.

Theorem (Quotient Rule)

If f and g are differentiable at x, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

The proof of the Quotient Rule is similar to, but slightly more complicated than, the proof of the Product Rule...

Here's an almost proof of the Quotient Rule.

Let $q(x) = \dfrac{f(x)}{g(x)}$.

$$f'(x) = q'(x)g(x) + q(x)g'(x).$$

$$\begin{aligned} q'(x) &= \frac{f'(x) - q(x)g'(x)}{g(x)} \\ &= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}. \end{aligned}$$

Very Subtle Question: Why is this only a fake proof??

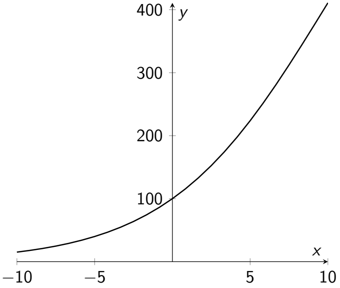
Exercise: Compute $\dfrac{d}{dx}\left\{\dfrac{4x^3+2x-3}{x+1}\right\}$.

Other examples: Compute $f'(x)$ if $f(x)$ is

1. e^{kx} where k is a positive integer.
2. x^n where n is a negative integer.
3. Combinations of the rules: $f(x) = \frac{(3-x)e^x}{x^2}$

Application: Rates of Change. The derivative provides information about the instantaneous rate of change of the function being differentiated.

For example, suppose that the population of a culture can be modeled by the function $p(t) = \frac{800}{1+7e^{-0.2t}}$.



We can find the instantaneous growth rate of the population at any time $t \geq 0$ as well as the steady-state population.

Homework Problems: Section 3.4 (pp. 168-169): #7-59 multiples of 3, 69, 71, 73, 74, 77-81 odd