2.3 – Techniques for Computing Limits

MATH 2554 – Calculus I

Fall 2019

Warm-up Problem: Compute $\lim_{x \to -1} 3x + 5$ graphically and numerically.

Theorem (Limits of Linear Functions)

Let a, b, and m be real numbers. For a linear function $f(x) = mx + b$,

$$\lim_{x \to a} f(x) = f(a) = ma + b.$$

Compute

1. $\lim_{x \to 0} 5x + 10$
2. $\lim_{x \to 5} -\frac{1}{2}x + 11.$

Theorem (Limit Laws)

Assume $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *both exist. The following properties hold, where* c *is a real number* ($c \in \mathbb{R}$)*, and* $m, n > 0$ *are integers* ($m, n \in \mathbb{N}$)*.*

1. *Sum.* $\lim_{x\to a} \big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. *Difference.* $\lim_{x\to a} \big(f(x) - g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. *Constant multiple.* $\lim_{x\to a} \big(cf(x)\big) = c \lim_{x\to a} f(x)$
4. *Product* $\lim_{x\to a} \big(f(x)g(x)\big) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
5. *Quotient* $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, *provided* $g(x) \neq 0$.
6. *Power* $\lim_{x\to a} \big(f(x)\big)^n = \left(\lim_{x\to a} f(x)\right)^n$

Limit Laws Theorem cont'd

7. *Fractional Power* $\lim_{x\to a} (f(x))^{n/m} = \left(\lim_{x\to a} f(x)\right)^{n/m}$, provided $f(x) \geq 0$, for x near a, if m is even and n/m is reduced to lowest terms.

Note: Conclusion 1-6 hold for one-sided limits as well. The following modification of Conclusion 7 holds as well:

7. RH *Fractional Power* $\lim_{x\to a^+} (f(x))^{n/m} = \left(\lim_{x\to a^+} f(x)\right)^{n/m}$, provided $f(x) \geq 0$, for x near a and $x > a$, if m is even and n/m is reduced to lowest terms.

7. LH *Fractional Power* $\lim_{x\to a^-} (f(x))^{n/m} = \left(\lim_{x\to a^-} f(x)\right)^{n/m}$, provided $f(x) \geq 0$, for x near a and $x < a$, if m is even and n/m is reduced to lowest terms.

Theorem (Limits of polynomial and rational functions)

Assume that p and q are polynomials and a is a constant.

1. *Polynomial functions* $\lim_{x \to a} p(x) = p(a)$
2. *Rational Functions* $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$*, provided* $q(a) \neq 0$.

Exercises Use the limit laws to evaluate the following limits

1. $\lim_{x\to 1} \frac{4f(x)g(x)}{h(x)}$, given that $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 1} g(x) = \frac{1}{2}$, and $\lim_{x\to 1} h(x) = -3$.
2. $\lim_{x\to 3} \frac{4x^2 - 6x + 3}{3x - 1}$
3. $\lim_{x\to 2^-} g(x)$ and $\lim_{x\to 2^+} g(x)$, given $g(x) = \begin{cases} x^2 & x \geq 2 \\ 2x - 3 & x < 2. \end{cases}$

Additional Techniques. It may be the case that direct substitution fails, yet we can still evaluate the limit.

Evaluate the following limits:

1. $\lim_{t \to 2} \dfrac{3t^2 - 7t + 2}{t - 2}$
2. $\lim_{h \to 0} \dfrac{\frac{1}{5+h} - \frac{1}{5}}{h}$
3. $\lim_{s \to 3} \dfrac{\sqrt{3s + 16} - 5}{s - 3}$

Theorem (The Squeeze Theorem)

Assume the functions f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all values of x near a, except possibly at a. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

Exercise: Compute $\lim_{x \to 0} x \cos(1/x)$.

Homework Problems: Section 2.3 (pp.76-78): #7-69 odds.