3.10 – Derivatives of Inverse Trigonometric Functions

MATH 2554 – Calculus I

Recall: If a function f is one-to-one, then its inverse exists and denoted f^{-1}.

If f is invertible and $y = f(x)$, then $x = f^{-1}(y)$.

Example: If $y = x^3 + 1$, then $x = (y-1)^{1/3}$. So $f(x) = x^3 + 1$ has inverse $f^{-1}(y) = (y-1)^{1/3}$.

Note: f^{-1} is not the same as $\dfrac{1}{f(x)}$!! Instead, $\big(f(x)\big)^{-1} = \dfrac{1}{f(x)}$.

Notation: On an approriate domain, the trig functions are invertible. We will use the notation

$$\arcsin x = \sin^{-1} x, \quad \arccos x = \cos^{-1} x, \text{ etc.}$$

interchangably.

Question: What are the domains and ranges of $\arcsin x$, $\arccos x$, and $\arctan x$?

Implicit Differentiation allows us to compute the derivatives of the inverse trig functions.

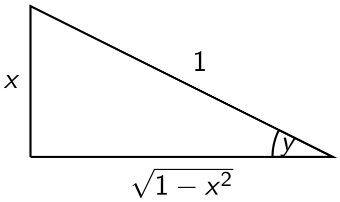
Let $y = \arcsin x$.

$$1 = (\cos y)\frac{dy}{dx}.$$

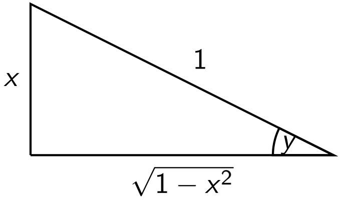
This means $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$.

Question: How can we find $\cos y$?

Answer: Draw a triangle!!



Implicit Differentiation allows us to compute the derivatives of the inverse trig functions.



We see that $\cos y = \sqrt{1-x^2}$ so that $\dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}}$.

Question: How do we know that $\cos y = \sqrt{1-x^2}$ and not $-\sqrt{1-x^2}$?

Examples: Compute

1. $\dfrac{d}{dx}\left[\sin^{-1}(4x^2 - 3)\right]$
2. $\dfrac{d}{ds}\cos(\sin^{-1} s)$.

Example:

Compute $\dfrac{d}{dx}\arctan x$.

Repeating this calculation for the remainder of the trig functions shows

$$\begin{aligned}
\frac{d}{dx}\sin^{-1}x &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\cos^{-1}x &= -\frac{1}{\sqrt{1-x^2}} & -1 < x < 1 \\
\frac{d}{dx}\tan^{-1}x &= \frac{1}{1+x^2} & \frac{d}{dx}\cot^{-1}x &= -\frac{1}{1+x^2} & -\infty < x < \infty \\
\frac{d}{dx}\sec^{-1}x &= \frac{1}{|x|\sqrt{x^2-1}} & \frac{d}{dx}\csc^{-1}x &= -\frac{1}{|x|\sqrt{x^2-1}} & |x| > 1.
\end{aligned}$$

Compute $\dfrac{d}{dt}\arctan(1/t)$.

Derivatives of Invertible Functions.

Example: Let $y = mx + b$. Then $\dfrac{dy}{dx} = m$.

Solving for x yields $x = f^{-1}(y) = \dfrac{y}{m} - \dfrac{b}{m}$.

Consequently, $\dfrac{df^{-1}}{dy} = \dfrac{1}{m}$.

This is not a coincidence!

Theorem

Let f be differentiable and invertible on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}, \quad \textit{where } y_0 = f(x_0).$$

Exercise: Determine the derivative to the inverse of $f(x) = x^2 + 1$ at the point $(5, 2)$.

Homework Problems: Section 3.10 (pp.225-227) #15-43 odd, 50-52