2.5 – Limits at Infinity

MATH 2554 – Calculus I

Fall 2019

Warm-up Problem: Consider the function $f(x) = -3 + \dfrac{x^2}{x^3 + 25}$.

Evaluate $f(x)$ at $x = 100, 1000, 10000$.

What is your conjecture about $\lim_{x \to \infty} f(x)$?

Definition (Limits at Infinity and Horizontal Asymptotes)

If $f(x)$ becomes arbitrarily close to a finite number L for all sufficiently large and positive x, then we write

$$\lim_{x \to \infty} f(x) = L$$

and say the limit of $f(x)$ as x approaches infinity is L.

In this case, the line $y = L$ is a horizontal asymptote of f.

The limit at negative infinity, $\lim_{x \to -\infty} f(x) = M$ is defined analogously. When this limit exists, the line $y = M$ is a horizontal asymptote.

A Limit at Infinity that's also an Infinite Limit

Let $n > 0$ be a positive integer.

Compute $\lim_{x \to \infty} x^n$ and $\lim_{x \to -\infty} x^n$.

Definition (End behavior)

The behavior of a function as $x \to \pm\infty$ is called the end behavior of a function.

Theorem (The end behavior of powers and polynomials)

Let $n \in \mathbb{N}$ and p be the polynomial

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \quad \textit{where } a_n \neq 0.$$

Then

1. $\lim_{x\to\pm\infty} x^n = \infty$ *when n is even.*
2. $\lim_{x\to\infty} x^n = \infty$ *and* $\lim_{x\to-\infty} x^n = -\infty$ *when n is odd.*
3. $\lim_{x\to\pm\infty} \frac{1}{x^n} = \lim_{x\to\pm\infty} x^{-n} = 0.$
4. $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n$ *where the limit (which is infinite) depends on $\deg p = n$ and the sign of a_n.*

Example What are the end behaviors of $f(x) = -3x^3 + 4x^2 - 1$ as $x \to \pm\infty$.

End behaviors of rational functions. The goal is to understand the pattern of the end behaviors as $x \to \pm\infty$ of the following functions:

1. $f(x) = \dfrac{x+1}{2x^2+3}$
2. $g(x) = \dfrac{4x^3-3x}{2x^3+5x^2+x+2}$
3. $h(x) = \dfrac{6x^4-1}{4x^3+3x^2+2x+6}$

Theorem (End behavior and asymptotics of rational functions)

Suppose that $f(x) = \frac{p(x)}{q(x)}$ *is a rational function where*

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0$$
$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + a_0$$

with $a_m \neq 0$ *and* $b_n \neq 0$.

1. *If* $m < n$, *then* $\lim\limits_{x \to \pm\infty} f(x) = 0$ *and* $y = 0$ *is a horizontal asymptote.*
2. *If* $m = n$, *then* $\lim\limits_{x \to \pm\infty} f(x) = \dfrac{a_m}{b_n}$ *and* $y = \dfrac{a_m}{b_n}$ *is a horizontal asymptote.*
3. *If* $m > n$, *then* $\lim_{x \to \pm\infty} f(x) = \infty$ *or* $-\infty$ *and* f *has no horizontal asymptote.*

Special cases of the end behavior of rational functions. As before, let $f(x) = \frac{p(x)}{q(x)}$ is a rational function where

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + a_0$$

- Vertical asymptotes. If f is in reduced form (this means p and q share no common factors), then the vertical asymptotes of f occur at the zeros of q.
- Slant asymptotes. If $m = n + 1$, then $\lim_{x\to\infty} f(x) = \infty$ or $-\infty$. While f has no horizontal asymptote, f has a slant asymptote.

What is the slant asymptote of $h(x) = \dfrac{6x^4 - 1}{4x^3 + 3x^2 + 2x + 6}$?

Algebraic and Transcendental Functions. Find the end behaviors of the following functions:

- algebraic $f(x) = \dfrac{4x^3}{2x^3 + \sqrt{9x^6 + 15x^4}}$
- transcendental $g(x) = e^x$
- trigonometric $h(x) = \cos x$

Homework Problems: Section 2.5 (pp.100-101): #3-10, 17-36 odds, 37-53 odds, 71-79 odds,