3.3 – Rules of Differentiation

MATH 2554 – Calculus I

Fall 2019

Problem: Computing derivatives using the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

is challenging!!

Solution: Prove theorems that allow us to differentiate large classes of functions!

Theorem (Constant Functions)

If c is a constant, then $\frac{d}{dx}(c) = 0$.

This theorem makes sense since the function $y = c$ has slope 0.

Also, if $f(x) = c$, then

$$f(x) - f(a) = c - c = 0.$$

This means if $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = \frac{0}{x - a} = 0$$

and consequently $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0.$

Theorem (Power Rule)

If $n \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

If $n \geq 2$ is an integer, then the proof relies on the factoring formula

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

This means

$$\frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}$$

$$\lim_{x \to a} \frac{x^n - a^n}{x-a} = a^{n-1} + a^{n-2}a + \cdots + a \cdot a^{n-2} + a^{n-1} = na^{n-1}.$$

Other rules include:

- Constant Multiple Rule $\frac{d}{dx}\big(cf(x)\big) = c\frac{df}{dx}$.
- Sum Rule $(f+g)'(x) = f'(x) + g'(x)$.
- The Exponential Function $\frac{d}{dx}(e^x) = e^x$.
- Higher Order Derivatives $\frac{d^2f}{dx^2} = f''(x) = \frac{d}{dx}\big(f'(x)\big)$. In general,

$$\frac{d^n}{dx^n}\big(f(x)\big) = f^{(n)}(x) = \frac{d}{dx}\big(f^{(n-1)}(x)\big).$$

Notation Alert: $f^n(x)$ is very different than $f^{(n)}(x)$!!!

Example: $f^3(x) = \big(f(x)\big)^3$ while $f^{(3)}(x) = \frac{d}{dx}\bigg(\frac{d}{dx}\bigg(\frac{d}{dx}\big(f(x)\big)\bigg)\bigg)$

An argument for the existence of e

Let's examine $y = 2^x$ and $y = 3^x$. If $f(x) = b^x$, then

$$\frac{f(x+h) - f(x)}{h} = \frac{b^{x+h} - b^x}{h} = \frac{b^x(b^h - 1)}{h}.$$

A suitably smart argument shows the limit exists for all $b > 0$ and

$$\lim_{h \to 0} \frac{2^h - 1}{h} < 1 \quad \text{and} \quad \lim_{h \to 0} \frac{3^h - 1}{h} > 1.$$

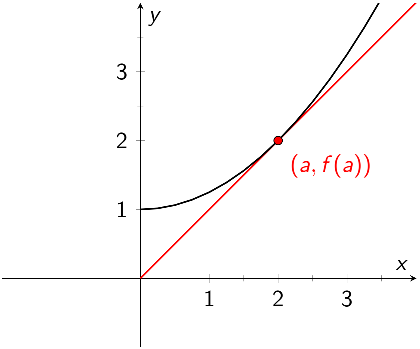
$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Examples: Compute $f'(x)$ when

1. $f(x) = 3x^2 - 2x + e^x + 1$
2. $f(x) = 3x^{75} - 72x^{-1} + 200.$

Tangent Lines:

Let $f(x)$ be a function that's differentiable at $x = a$. Find an equation for the line tangent to $f(x)$ at $x = a$.



The slope $m = f'(a)$. Therefore, the tangent line at the point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a).$$

Exercises:

Find equations for the lines tangent to $f(x)$ for $x = a$.

1. $f(x) = 3x^3 - 7/x + 1$, $a = 1$.
2. $f(x) = 2x(x^2 - 3x) + 4$, $a = -5$.

Homework Problems: Section 3.3 (pp. 160-162): #15-41 odd, 68-72, 76