2.7 – Precise Definitions of Limits

MATH 2554 – Calculus I

Fall 2019

From before:

Definition (Limit of a Function)

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (that is, as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

This "definition" is descriptive, but not rigorous.

Question: How can we turn this definition into a mathematically rigorous statement?

Step 1: Understanding absolute values.

The rigorous definition includes the statements $|x - a| < \delta$ and $|f(x) - L| < \epsilon$.

Example: Rephrase $|x - 4| < \frac{1}{2}$ without using absolute value and demonstrate your answer on the number line.

Now do the same for $0 < |x - a| < \delta$ and $|f(x) - L| < \epsilon$ where $a \in \mathbb{R}$ and $\epsilon > 0$, and $\delta > 0$.

Problem: A carpenter needs to cut a wooden board to a length of 7.6 inches with a tolerance of 0.01 inch, meaning the actual length after the cut can be within 0.01 inch of 7.6 inches and still be usable.

1. What are some possible values of x, where x is the actual length of the board after cutting?
2. Write an inequality to represent the possible usable board lengths.
3. Write an inequality that includes the tolerance and the ideal length to describe all possible values of x.

Recall: The definition of limit says x sufficiently close (but not equal) to a. The number δ is the tolerance, i.e., the number that guarantees that x is "sufficiently close" to a.

Question: How do we use inequalities and absolute values to ensure that $x \neq a$?!?!?!?

Step 2: The translation.

Descriptive Statement: If $f(x)$ is arbitrarily close to L (that is, as close to L as we like) for all x sufficiently close (but not equal) to a.

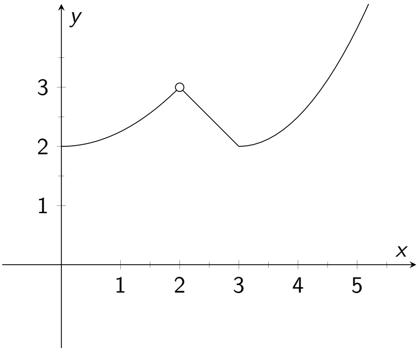
Translation: Given an allowed error tolerance $\epsilon > 0$ around L, I need to find a margin δ around a (namely, $0 < |x - a| < \delta$) so that $f(x)$ is within my allowed tolerance, i.e., $|f(x) - L| < \epsilon$.

Quantitative Statement: Given $\epsilon > 0$, there exists $\delta > 0$ *that depends on* $\epsilon > 0$ so that

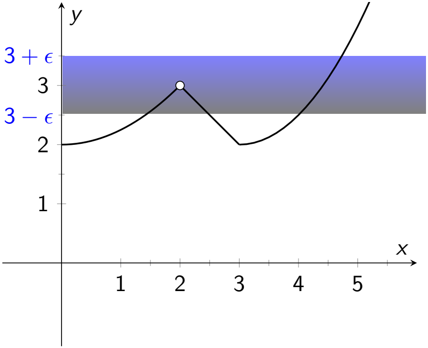
$$|f(x) - L| < \epsilon$$

whenever $0 < |x - a| < \delta$.

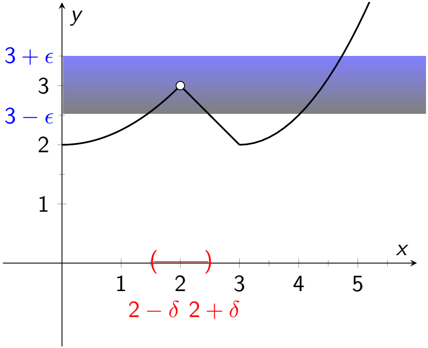
Step 3: Understanding Step 2 graphically. Consider the following function, which is undefined at $x = 2$.



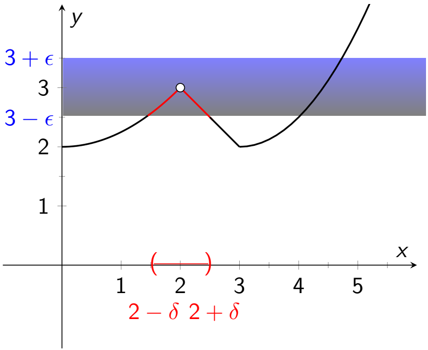
Step 3: The choice of ϵ defines a horizontal strip of y-values from $3-\epsilon$ to $3+\epsilon$. Call this the ϵ-strip.



Step 3: The choice of δ defines an open interval around $x = 2$. Consider the values of the function lying over the interval $(2-\delta, 2+\delta)$.



Step 3: In this example, as the function values are arbitrarily close to 3 provided the inputs are sufficiently close to 2. That is for every ϵ there is a δ (it changes with ϵ) so that *all* the function values lying over the interval $(2-\delta, 2+\delta)$ lie in the ϵ strip.



Notice that the exact value of the function at $x = 2$ does not matter. It this case, the function is not even defined at $x = 2$!

Step 4: Writing the precise definition.

Definition (Limit of a Function)

Assume that $f(x)$ exists for all x in some open interval containing a, except possibly at a. We say that the limit of $f(x)$ as x approaches a is L, written

$$\lim_{x \to a} f(x) = L$$

if for *any* number $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{whenever } 0 < |x - a| < \delta.$$

Homework Problems: Section 2.7 (pp.126-127):#1-7, 9-12, 51,52