# 4.3 – What Derivatives Tell Us

MATH 2554 – Calculus I

## Theorem (Test for Intervals of Increase and Decrease)

*Suppose $f$ is continuous on an interval $I$ and differentiable at all interior points of $I$. If $f'(x) > 0$ at all interior points of $I$, then $f$ is increasing on $I$. If $f'(x) < 0$ at all interior point of $I$, then $f$ is decreasing on $I$.*

Exercise: Sketch a function on $(-\infty, \infty)$ that has the following properties:

- $f'(x) > 0$ on $(-\infty, -1)$;
- $f'(-1)$ is undefined;
- $f'(x) < 0$ on $(-1, 4)$;
- $f'(x) > 0$ on $(4, \infty)$.

Exercise: Find the intervals on which $f$ is increasing and decreasing where

$$f(x) = 3x^3 - 4x + 12.$$

Graph $f(x)$ and $f'(x)$ on the same graph. What do you notice?

## Identifying Local Maxima and Minima

### Theorem (First Derivative Test)

*Suppose that $f$ is continuous on an interval that contains a critical point $c$ and assume that $f$ is differentiable on an interval containing $c$, except perhaps at $c$ itself.*

1. *If $f'$ changes sign from positive to negative as $x$ increases through $c$, then $f$ has a local maximum at $c$.*
2. *If $f'$ changes sign from negative to positive as $x$ increases through $c$, then $f$ has a local minimum at $c$.*
3. *If $f'$ is positive on both sides near $c$ or negative on both sides near $c$, then $f$ has no local extreme value at $c$.*

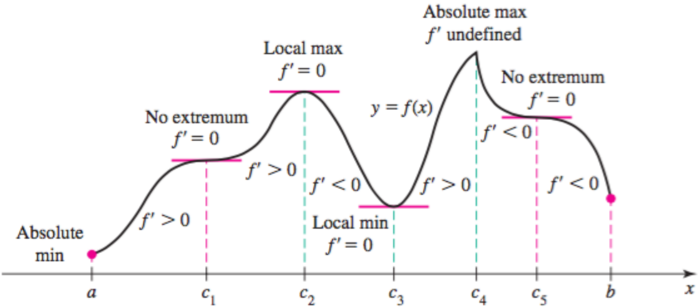


Figure: Critical points versus extrema

Problem: If $f(x) = 2x^3 + 3x^2 - 12x + 1$, identify the critical points on the interval $[-3, 4]$, and use the First Derivative Test to locate the local maximum and minimum values.

What are the absolute max and min?

Uses of the Second Derivative: The second derivative $f''$ determines the intervals on which $f'$ is increasing or decreasing.

Question: What does positivity or negativity of $f''$ determine about $f$?

## Definition (Concavity and Inflection Point)

Let $f$ be differentiable on an open interval $I$. If $f'$ is increasing on $I$, then $f$ is concave up on $I$. If $f'$ is decreasing on $I$, then $f$ is concave down on $I$.

If $f$ is continuous at $c$ and $f$ changes concavity at $c$ (from up to down or vice versa) then $f$ has an inflection point at $c$.

## Theorem (Test for Concavity)

*Suppose that $f''$ exists on an open interval $I$.*

- *If $f'' > 0$ on $I$, then $f$ is concave up on $I$.*
- *If $f'' < 0$ on $I$, then $f$ is concave down on $I$.*
- *If $c$ is a point of $I$ at which $f''$ changes sign at $c$ (from positive to negative or vice versa), then $f$ has an inflection point at $c$.*

Question: What would functions with the following properties look like?

1. $f'(x) > 0$ and $f''(x) > 0$;
2. $f'(x) > 0$ and $f''(x) < 0$;
3. $f'(x) < 0$ and $f''(x) > 0$;
4. $f'(x) < 0$ and $f''(x) < 0$.

## Theorem (Second Derivative Test)

*Suppose that $f''$ is continuous on an open interval containing $c$ with $f'(c) = 0$.*

- *If $f''(c) > 0$, then $f$ has a local minimum at $c$;*
- *If $f''(c) < 0$, then $f$ has a local maximum at $c$;*
- *If $f''(c) = 0$, then the test is inconclusive; $f$ may have a local maximum, local minimum, or neither at $c$.*

Exercise: Let $f(x) = 2x^3 - 6x^2 - 18x$.

1. Determine the intervals on which it is concave up or down, and identify any inflection points.
2. Locate the critical points, and use the Second Derivative Test to determine whether they correspond to local minima or maxima, or whether the test is inconclusive.

## Absolute extreme values on any interval

### Theorem

*Suppose $f$ is continuous on an interval $I$ that contains exactly one local extremum at $x = c$.*

- *If $f(c)$ is a local maximum then it is also the absolute maximum.*
- *If $f(c)$ is a local minimum then it is also the absolute minimum.*

Exercise: Find the absolute extremum of $f(x) = 4x + \frac{1}{\sqrt{x}}$ over $(0, \infty)$.

Homework Problems: Section 4.3 (pp.267-269) #9-15 odd, 27-41 odd, 46-54 even, 59-89 odd, 99