4.6 – Linear Approximation and Differentials

MATH 2554 – Calculus I

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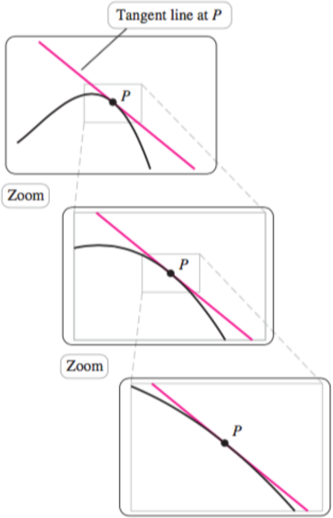

Linear Approximation. The line that best approximates a differentiable function f at the point $P = (a, f(a))$ is the tangent line to f at $(a, f(a))$.

Let $L(x)$ be the linear approximation to $f(x)$ at the point $(a, f(a))$. Then

$$L(x) = f(a) + f'(a)(x - a).$$

Question: From where does this formula come?

- Point-slope form of a line is $y - y_0 = m(x - x_0)$.
- The slope of the tangent line is $f'(a)$.
- Use the point $(x_0, y_0) = (a, f(a))$.

Example: Write the equation of the line that represents the linear approximation to $f(x) = \dfrac{x}{x+1}$ at $a = 1$, and then use the linear approximation to estimate $f(1.1)$.

Exercise: Find the linear approximation to $f(x) = \sqrt{1+x}$ at the point $a = 0$. What is an approximation for $f(0.1)$?

Differentials approximate change analogously to how L approximates f (near a given point).

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x,$$

so

$$f(x + \Delta x) - f(x) \approx \Delta L = f'(x)\Delta x.$$

Notation: We can write $\Delta y = f(x + \Delta x) - f(x)$.

Definition

Let f be differentiable on an interval containing x. A small change in x is denoted by the differential dx. The corresponding change in f is approximated by the differential $dy = f'(x)\,dx$; that is,

$$\Delta y = f(x + dx) - f(x) \approx dy = f'(x)dx.$$

This notation meshes well with Leibniz notation:

$$\frac{dy}{dx} = \frac{f'(x)dx}{dx} = f'(x).$$

Example: Approximating Changes: Use the notation of differentials to approximate the change in $f(x) = x - x^3$ given a small change dx.

Approximate the change in f as x increases from 2 to 2.1.

Homework Problems: Section 4.6 (p.299) #6-8, 19-24, 35-45 odd