4.9 – Antiderivatives

MATH 2554 – Calculus I

Question: Given a function $f(x)$, can we find a function $F(x)$ so that $F'(x) = f(x)$?

Definition

A function F is an antiderivative of f on an interval I provided $F'(x) = f(x)$, for all $x \in I$.

Example: Suppose $f(x) = 4$.

Question: What is an antiderivative $F(x)$ of $f(x)$?

Answer: $F(x) = 4x$.

Question: Are there any others?

$4x + 1$, $4x + 100$, $4x - 1000$ are all antiderivatives of $f(x) = 4$.

There are always many antiderivatives for a given function! If $F(x)$ is an antiderivative to $f(x)$, then so is $F(x) + C$ where C is any constant.

Theorem (The Family of Antiderivatives)

Let F be any antiderivative of f on an interval I. Then all the antiderivatives of f on I have the form $F + C$, where C is an arbitrary constant.

Exercise: Find ALL antiderivatives of the functions

1. $f(x) = 6x^{-7}$
2. $g(x) = -4\cos 4x$
3. $h(x) = \csc^2 x$.

Notation for Antiderivatives. The notation for taking a derivative is $\frac{d}{dx} f(x)$.

$$\int f(x)\,dx.$$

This notation for taking the antiderivative is called the indefinite integral. The function $f(x)$ is called the integrand, and x is the variable of integration.

Example: $\int 4x^3\,dx = x^4 + C$.

Theorem (Rules for Indefinite Integrals)

1. *The Power Rule:* $\int x^p\,dx = \dfrac{x^{p+1}}{p+1} + C$ *where* $p \neq -1$ *is a real number and* C *is an arbitrary constant.*

2. *Constant Multiple Rule: If* c *is any real number,* $\int cf(x)\,dx = c\int f(x)\,dx.$

3. *Sum Rule:* $\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx.$

Exercise: Compute the following indefinite integrals.

1. $\int 3x^{-2} - 4x^2 + 1 \, dx$
2. $\int 6\sqrt[3]{y} \, dy$
3. $\int 2\cos(2\theta) \, d\theta$
4. $\int x^2 \, dy$. (This is not a typo).

Indefinite Integrals of Trigonometric Functions

	Derivative Equation	Indefinite Integral
1.	$\dfrac{d}{dx}\sin ax = a\cos ax$	$\displaystyle\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
2.	$\dfrac{d}{dx}\cos ax = -a\sin ax$	$\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$
3.	$\dfrac{d}{dx}\tan ax = a\sec^2 ax$	$\displaystyle\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$
4.	$\dfrac{d}{dx}\cot ax = -a\csc^2 ax$	$\displaystyle\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$
5.	$\dfrac{d}{dx}\sec ax = a\sec ax\tan ax$	$\displaystyle\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax + C$
6.	$\dfrac{d}{dx}\csc ax = -a\csc ax\cot ax$	$\displaystyle\int \csc ax\cot ax\,dx = -\frac{1}{a}\csc ax + C$

Example: Find $\displaystyle\int 2\sec^2 2x\,dx$.

Ohter Indefinite Integrals

	Derivative Equation	Indefinite Integral
7.	$\frac{d}{dx} e^{ax} = ae^{ax}$	$\int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$
8.	$\frac{d}{dx} b^x = b^x \ln b$	$\int b^x\, dx = \frac{1}{\ln b} b^x + C$
9.	$\frac{d}{dx} \ln \lvert x \rvert = \frac{1}{x}$	$\int \frac{1}{x}\, dx = \ln \lvert x \rvert + C$
10.	$\frac{d}{dx} \arcsin \frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin \frac{x}{a} + C$
11.	$\frac{d}{dx} \arctan \frac{x}{a} = \frac{a}{a^2 + x^2}$	$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \arctan \frac{x}{a} + C$

Introduction to Differential Equations. An equation involving an unknown function and its derivatives is called a differential equation. Initial Value Problem.

Suppose $g(x)$ is a given function. Consider the problem

$$\begin{cases} f'(x) = g(x) & \text{Differential equation} \\ f(a) = b & \text{Initial condition.} \end{cases}$$

Solve $f'(x) = 7x^6 - 4x^3 + 12$ subject to $f(1) = 24$.

Homework Problems: Section 4.9 (pp.332-333) #11-45 odd, 50-68 even, 73-79 odd