2.6 – Continuity

MATH 2554 – Calculus I

Fall 2019

Warm-up Problem: Let $f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 8 & x = 2 \end{cases}$.

What is

1. $f(2)$
2. $\lim_{x \to 2} f(x)$?

Definition

A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

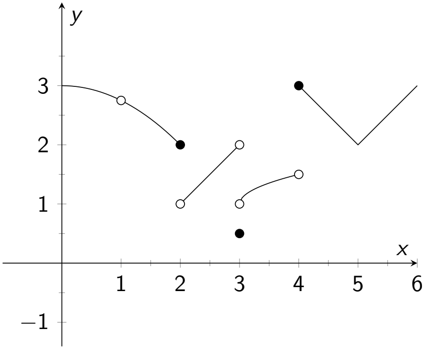
Intuitively, a function f is continuous at $x = a$ if the graph of f contains no holes or breaks at $x = a$. In other words, the graph near $x = a$ can be drawn without lifting a pencil.

It is helpful to think of a definition is as guide for establishing a property. In the case of continuity, the definition tells us we need to check that the following three properties hold:

Continuity Checklist

1. $f(a)$ is defined (that is, a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Example: In the following example, where is the function f continuous and where are the points of discontinuity? At the points of discontinuity, which aspects of the checklist fail?



Theorem

If f and g are continuous at a, then the following functions are continuous at a. Assume that $c \in \mathbb{R}$ is a constant and $n \in \mathbb{N}$.

a. $f + g$ b. $f - g$ c. cf

d. fg e. f/g, *provided* $g(a) \neq 0$ f. $\big(f(x)\big)^n$.

Polynomials are continuous for all x.

Example Where does $r(x) = x^3 + \dfrac{2x^2 - x + 1}{(x-5)(x-1000)}$ have points of discontinuity?

Theorem

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.

Example Where is $\left(1 + \dfrac{1}{x}\right)^{35}$ continuous?

Continuity on an interval. Let $[a, b]$ be an interval. We have established the criteria for f to be continuous on (a, b).

What about the endpoints?

Definition

A function f is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$ and continuous from the left at b if $\lim_{x \to b^-} f(x) = f(b)$. We say f is continuous on $[a, b]$ if f is continuous at every point in (a, b), continuous from the right at a and continuous from the left at b.

Other examples of continuous functions

- Trig functions: $\sin x$ and $\cos x$ are continuous everywhere. Therefore, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are continuous on their domains.
- Exponentials: Exponential functions a^x are continuous for all x when $a > 0$ (e.g., 2^x, e^x).
- Inverse Functions: If f is continuous on an interval I *and* the inverse function f^{-1} exists on the interval $f(I)$, then f^{-1} is continuous on $f(I)$.

Theorem (Continuity with Roots)

Suppose that m and n are positive integers that share no common factor.

1. *If m is an odd integer, then $\left(f(x)\right)^{n/m}$ is continuous wherever $f(x)$ is continuous.*
2. *If m is even, then $\left(f(x)\right)^{n/m}$ is continuous wherever $f(x)$ is positive and continuous.*

Example: Where is $g(x) = \sqrt[4]{1 - x^2}$ continuous?

Theorem (The Intermediate Value Theorem)

Suppose that f is continuous on $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.

Example: Prove that $f(x) = -x^5 - 4x^2 + 2\sqrt{x} + 5$ has a zero in the interval $(0, 3)$.

Homework Problems: Section 2.6 (pp.112-114): #5-13 odds, 17-37 odds, 67, 85, 87