3.9 – Derivatives of Logarithmic and Exponential Functions

MATH 2554 – Calculus I

Armed with Implicit Differentiation, we now have the tools to compute the derivatives of logarithmic and exponential functions. First, however, let's review the relationship between logarithms and exponentials.

Inverse Properties for e^x and $\ln x$.

1. $e^{\ln x} = x$, for $x > 0$, and $\ln(e^x) = x$ for all x.
2. $y = \ln x$ if and only if $x = e^y$.
3. For real numbers and x and $b > 0$, $b^x = e^{\ln b^x} = e^{x \ln b}$.

Computing the derivative of $y = \ln x$.

Step 1. Exponentiate both sides: $e^y = x$.

Step 2. Differentiate both sides. $e^y \dfrac{dy}{dx} = 1$.

Step 3. Solve for $\dfrac{dy}{dx}$. $\dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}$.

Step 4. Conclude $\boxed{\dfrac{d}{dx} \ln x = \dfrac{1}{x}}$.

Computing the derivatives of $y = \ln|x|$.

Observe: $y = \ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0. \end{cases}$

If $x > 0$, then $\ln|x| = \ln x$ so that $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$.

If $x < 0$, then $-x > 0$. Also, we use the Chain Rule to compute

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Conclusion: If $x \neq 0$, $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$.

Exercises

1. Find $\dfrac{d}{dx}\ln(15x)$.
2. Find $\dfrac{d}{dy}y\ln y$.
3. Find $\dfrac{d}{d\theta}\ln(\sin\theta)$.

Key Example: Find the derviative of $y = b^x$.

Exercises

Find the derivative of the following functions:

1. $f(x) = 14^x$
2. $g(y) = 45(3)^{2y}$.

Contextual Example:

The energy (in joules) released by an earthquake of magnitude M is given by the equation

$$E(M) = 25000 \cdot 10^{1.5M}.$$

1. How much energy is released in a magnitude 3.0 earthquake?
2. What size earthquake releases 8 million joules of energy?
3. What is $\dfrac{dE}{dM}$ and what does it tell you?

Derivative of $y = \log_b x$.

Step 1. Exponentiate: $b^y = x$.

Step 2. Differentiate: $b^y \ln b \dfrac{dy}{dx} = 1$.

Step 3. Solve: $\dfrac{dy}{dx} = \dfrac{1}{b^y \ln b} = \dfrac{1}{x \ln b}$.

Logarithmic Differentiation

Combining logarithms and implicit differentiation yields a powerful tool for computing derivatives.

Examples: 1. Compute $\dfrac{d}{dx}x^x$.

2. Compute $\dfrac{d}{dx}x^a$ for any real number a and $x > 0$.

Examples: 1. $\dfrac{d}{dx}\left(\dfrac{x^2(x-1)^3}{(3+5x)^4}\right)$.

2. $\dfrac{d}{dt}(t+1)^t$.

Homework Problems: Section 3.9 (pp.211-212) #9-29 odd, 55-67 odd, 97