4.2 – The Mean Value Theorem

MATH 2554 – Calculus I

The Mean Value Theorem is nicknamed "The Midwife of Calculus" because of its instrumental role in proving the Fundamental Theorem of Calculus, Taylor's Theorem, and many of the Big Theorems in calculus.

The main tool in proving the Mean Value Theorem is Rolle's Theorem (a special case).

As a warmup for Rolle's Theorem, if $f(x)$ is a differentiable function on $[a, b]$ and $f(a) = f(b)$, what might the graph of f look like?

Theorem (Rolle's Theorem)

Let f be a continuous function on a closed interval $[a, b]$ and be differentiable on (a, b) with $f(a) = f(b)$. There is at least one point c in (a, b) with $f'(c) = 0$.

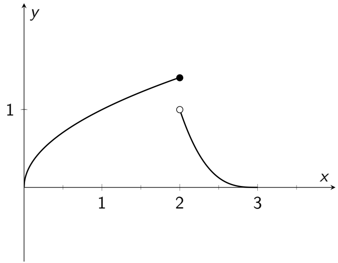
Question: Why is Rolle's Theorem true?

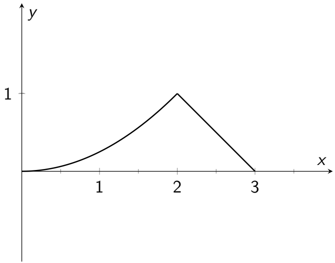
Case 1: f is a constant function, so $f'(x) = 0$ for all x in (a, b), and there is nothing to prove.

Case 2: f is not identically constant. Either the absolute max or the absolute min of f on $[a, b]$ is not $f(a)$. Consequently, that absolute extrema has to be a local extrema and hence have zero derivative.

Question: What happens if the hypotheses of Rolle's Theorem are relaxed?

Question: What happens if the hypotheses of Rolle's Theorem are relaxed?





Example: Show that the derivative of the function

$$f(x) = (x^2 - 4)\sin\left(\ln\left(x^2 + e^{e^{e^x + x^2}}\right)\right)$$

vanishes at some point on the interval $[-2, 2]$.

Theorem (The Mean Value Theorem)

If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note: The Mean Value Theorem says that there exists a point c in (a, b) where the instantaneous rate of change of f at c is the **same** as the average rate of change of f over $[a, b]$.

Rolle's Theorem and the Mean Value Theorem are existence theorems. Their most useful applications are in cases where you cannot find c explicitly (like in your homework). Quite frankly, those problems are silly.

Consequences of the Mean Value Theorem

Theorem (Zero Derivatives Implies Constant Function)

If f is differentiable and $f'(x) = 0$ at all points on an interval I, then f is a constant function on I.

Theorem (Functions with Equal Derivatives Differ by a Constant)

If two functions have the property that $f'(x) = g'(x)$ for all x on an interval I, then $f(x) - g(x) = C$ on I, where C is a constant. That is, f and g differ by a constant.

Increasing and Decreasing Functions

Definition (Increasing and Decreasing Functions)

Suppose a function f is defined on an interval I. We say that f is increasing on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is decreasing on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

Exercise: Sketch the graph of an increasing function and a decreasing function.

More Consequences

Theorem (Intervals of Increase and Decrease)

Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

Exercise: A plane begins its takeoff at 2:00pm on a 2500-mile flight. The plane arrives at its destination at 7:30pm. Explain why there are at least two times during the flight when the speed of the plane is 400 miles per hour.

Homework Problems: Section 4.2 (pp.254-256) #5,7,8,13,14,21-31 odd