3.8 – Implicit Differentiation

MATH 2554 – Calculus I

Fact: It is not always the case that functions of x and y can be easily solved for y.

Example: $(x+y)e^{x+y} = x^2$.

In these types of equations, our default assumption is the x is the independent variable and $y = y(x)$ is the dependent variable, that is, y is a function of x.

In these cases, y is not explicitly written in terms of x but instead defined implicitly in terms of x.

We can still compute $\dfrac{dy}{dx}$ in these cases.

Note: It may be the case that we *can* solve for y in terms of x, but the solution may involve two (or more) functions.

Example: Find $\dfrac{dy}{dx}$ if $x^2 + y^2 = 9$.

If we solve for y, we obtain $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$, each of these functions we can differentiate.

The price we pay for implicitly differentiating is that the derivatives often are written in terms of x and y, not just x.

Exercise: Find the tangent line to the circle $x^2 + y^2 = 9$ at the point $(9/5, 12/5)$.

Exercises:

Find:

1. $\dfrac{dy}{dx}$ if $xy + y^3 = 1$.
2. The tangent line to the curve $x^4 - x^2y + y^4 = 1$ at the point $(-1, 1)$

Higher Order Derivatives. We can also find higher order derivatives through repeated differentiation, though it typically involves substituting $\frac{dy}{dx}$ into the formula.

Example: Find $\frac{d^2y}{dx^2}$ if $xy + y^3 = 1$.

Implicit Differentiation also proves a proof of the Power Rule for rational exponents.

Suppose $y = x^{p/q}$.

$$y^q = x^p.$$

Differentiation yields

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

so that

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{(x^{\frac{p}{q}})^{q-1}} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

Homework Problems: Section 3.8 (p.200) #5-25 odd, 31-49 odd