3.7 – The Chain Rule

MATH 2554 – Calculus I

Fall 2019

Goal: The goal is to compute derivative of composition functions. There are functions of the form $h(x) = f(g(x))$ (also written $h(x) = f \circ g(x)$).

Example: Find $h'(x)$ if $h(x) = (2x - 3)^7$.

Intuition:

Suppose that Yvonne (y) can run twice as fast as Uma (u). Therefore $\frac{dy}{du} = 2$. Suppose that Uma can run four times as fast as Xavier (x). So $\frac{du}{dx} = 4$.

Q: How much faster can Yvonne run than Xavier?

A: In this case, we would take both our rates and multiply them together:

$$\frac{dy}{du}\frac{du}{dx} = 2 \cdot 4 = 8.$$

Theorem (The Chain Rule)

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x. Moreover, we can express the derivative as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}\big(f(g(x))\big) = f'\big(g(x)\big)g'(x).$$

Example: Find $f'(x)$ if $f(x) = (2x - 3)^7$.

Exercise:

Compute $\dfrac{dy}{dx}$ where

1. $y = (5x^2 + 11x)^{20}$
2. $y = \left(\dfrac{3x}{4x+2}\right)^{20}$
3. $y = \cos(5x + 1)$
4. $y = \tan(5x^5 - 7x^3 + 2x)$.

Chain Rule for Powers

The function $f(x) = (g(x))^n$ appears with remarkable frequency. In this case, the inner function is $g(x)$ and the outer function is $y = u^n$. Then

$$\frac{d}{dx}\big(g(x)\big)^n = n\big(g(x)\big)^{n-1} g'(x).$$

Exercise: Compute $\dfrac{d}{dt}(1 - e^t)^4$.

The Composition of Three (or More) Functions.

Exercise: Compute $\dfrac{d}{dy}\left[\sqrt{(3y-4)^2+3y}\right]$.

Combining Rules.

Exercise: Compute $\dfrac{d}{d\theta}\left(\theta \sin(e^{\theta})\right)$.

Homework Problems: Section 3.7 (pp.197-198) #11, 15-25, 27-71 odd, 85-87 7-33 odd, 38, 45-67 odd, 71, 73