5.2 – Definite Integrals

MATH 2554 – Calculus I

In Section 5.1, we saw how to use Riemann sums to approximate the area under a curve. However, the curves we worked with in this section were all non-negative.

Question: What happens when the curve is negative?

Example: Let $f(x) = 8 - 2x^2$ over the interval $[0, 4]$. Use a left, right, and midpoint Riemann sum with $n = 4$ to approximate the area under the curve.

In the previous example, we saw that the areas where f was positive provided positive contributions to the area, while areas where f was negative provided negative contributions.

Definition (Net Area)

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The net area of R is the sum of the areas of the parts of R that lie above the x-axis minus the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

The Definite Integral computes the net area of a region formed by the curve $y = f(x)$ and the x-axis. Definite integrals are computed using (limits of) generalized Riemann sums.

Definition (General partition)

A general partition of an interval $[a, b]$ consists of the n subintervals

$$[x_0, x_1],\ [x_1, x_2], \dots,\ [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$.

$$\Delta x_k = x_k - x_{k-1}, \qquad k = 1, 2, \dots, n.$$

Definition (General Riemann Sum)

Suppose $[x_0, x_1],\ [x_1, x_2], \dots,\ [x_{n-1}, x_n]$ form a general partition of $[a, b]$ with $\Delta_k = x_k - x_{k-1}$. Let x_k^* be any point in $[x_{k-1}, x_k]$ for $k = 1, 2, \dots, n$.

If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a generalized Riemann sum for f on $[a, b]$.

Let

$$\Delta = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}.$$

Question: What happens to the approximation of the net area as $\Delta \to 0$?

Definition (Definite Integral)

A function f defined on $[a, b]$ is integrable on $[a, b]$ if $\lim\limits_{\Delta \to 0} \sum\limits_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a partition. The limit is the definite integral of f from a to b, which we write

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Example and nonexamples of integrable functions

Theorem (Continuous functions are integrable)

If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.

A non-example: The function $f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$ is nonintegrable. Why?

Definite Integrals through Geometry:

Compute

1. $\displaystyle\int_2^6 4x - 3\,dx$
2. $\displaystyle\int_0^3 \sqrt{9 - t^2}\,dt.$

Properties of Definite Integrals

Suppose f and g are integrable functions on $[a, b]$ and c is a constant.

1. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx;$
2. $\displaystyle\int_a^a f(x)\,dx = 0;$
3. $\displaystyle\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$
4. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx;$
5. $\displaystyle\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$ for any $a < p < b$.
6. The function $|f|$ is integrable on $[a, b]$, and $\int_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

Homework Problems: Section 5.2 (pp.364-365) #11-17 odd, 28-30, 35-38, 39-45 odd, 51-53