3.6 – Derivatives as Rates of Change

MATH 2554 – Calculus I

Fall 2019

We can compute derivatives. But... so what?

The goal of this section is to discuss several applications of the derivative as a rate of change!

One-Dimensional Motion – Position and Velocity

In Section 2.1, we discussed the definitions of average velocity and instantaneous velocity. We have a position function $s = f(t)$, where the position s of an object was measured at any time t.

In 2.1, we measured the distance away from a given point in terms of a and $a + h$.

$$\Delta s = f(a + \Delta t) - f(a).$$

Here Δt represents how much time has elapsed.

One-Dimensional Motion – Position and Velocity

In our new notation, the average velocity is of the object over the time interval $[a, a + \Delta t]$ is

$$\nu_{\text{av}} = \frac{\Delta s}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

The instantaneous velocity at a is

$$\nu(a) = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = f'(a).$$

One-Dimensional Motion – Speed and Acceleration

Question: What is the difference between speed and velocity?

Answer: Speed is defined by

$$\text{speed} = |\nu|.$$

Acceleration is defined as the rate of change of the velocity.

The velocity at time t	$\nu = \frac{ds}{dt} = f'(t)$
The speed at time t	$\lvert\nu\rvert = \lvert f'(t)\rvert$
The acceleration at time t	$a = \dfrac{d\nu}{dt} = \dfrac{d^2s}{dt^2} = f''(t)$

Questions: Given the height function $s = f(t)$ of an object launched into the air, how would you know:

- Q: The highest point the object reaches?
- A: The velocity the object as it reaches its maximum is 0.
- Q: How long it takes to hit the ground?
- A: Find t_1 so that $f(t_1) = 0$.
- Q: The speed at which the object hits the ground.
- A: $|f'(t_1)|$.

Exercise:

A rock is thrown (downward) off a bridge and its distance s (in feet) from the bridge after t seconds is

$$s(t) = 16t^2 + 4t.$$

1. The velocity of the rock after 2 seconds is
2. The acceleration of the rock after 2 seconds is

Growth Models

Suppose $p = f(t)$ is a function of the growth of some quantity of interest (e.g., population, prices, etc.). The average growth rate of p between times $t = a$ and a later time $t = a + \Delta t$ is the change in p divided by the elapsed time Δt. So:

$$\frac{\Delta p}{\Delta t} = \frac{f(a + \Delta t) - f(a)}{\Delta t}.$$

The instantaneous growth rate of p at a is therefore

$$\frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}.$$

Exercise: The population of the state of Georgia (in thousands) from 1995 ($t = 0$) to 2005 ($t = 10$) is modeled by the polynomial $p(t) = -0.27t^2 + 101t + 7055$.

1. What was the average growth rate from 1995 to 2005?
2. What was the growth rate in 1997?
3. What can you tell about the population growth rate in Georgia between 1995 and 2005?

Economics – Average and Marginal Cost

Suppose a company produces a large amount of a particular quantity.

Associated with manufacturing the quantity is a cost function $C(x)$ that gives the cost of manufacturing x items.

This cost may include a fixed cost to get started as well as a unit cost (or variable cost) in producing one item.

Economics – Average and Marginal Cost If a company produces x items at a cost of $C(x)$, then the

$$\text{Average Cost} = \frac{C(x)}{x}.$$

To determine the average cost of producing Δx *additional* units, we would compute

$$\frac{\Delta C}{\Delta x} = \frac{C(x + \Delta x) - C(x)}{\Delta x}.$$

Economics – Marginal Cost

The marginal cost is defined by

$$\lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = C'(x).$$

Note: Of course, in reality, x is measured by integers, so $\Delta x \to 0$ does not make sense. item.

Exercise:

Suppose the cost of producing x items is

$$C(x) = -0.04x^2 + 100x + 800.$$

1. What is the average cost of producing 500 items?
2. What is the marginal cost when $x = 500$?

Homework Problems: Section 3.6 (pp.186-189) #10-21 all, 23, 24, 29-32, 34