4.8 – Newton's Method

MATH 2554 – Calculus I

Newton's Method is a powerful and applicable tool for approximating zeros of differentiable functions.

Most functions are too complicated to find their zeros analytically, so approximation methods have tremendous importance in applications. approximation techniques are still used because they are often computationally much faster and many also come with estimates of the error.

Newton's Method is the granddaddy of all modern approximation methods and has a geometrically compelling argument.

A Geometric Argument.

Step 1. Pick a point x_0 near the zero r and draw the tangent line to $y = f(x)$ at the point $(x_0, f(x_0))$.

Step 2. The point $(x_1, 0)$ at which the tangent line intersects the x-axis becomes the new approximation to r.

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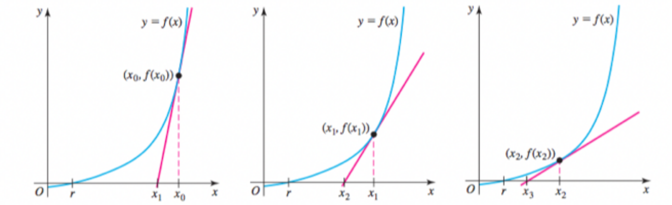


Step 3. Iterate this process so that the new approximation to r at the $(n+1)$st step, denoted x_{n+1}, is the intersection of the tangent line to the curve $y = f(x)$ at the point $(x_n, f(x_n))$ with the x-axis.

Question: How do I find x_{n+1}?

Answer: The tangent line to $y = f(x)$ at $(x_n, f(x_n))$ is

$$y - f(x_n) = f'(x_n)(x - x_n).$$

$$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n).$$

Solving for x_{n+1} shows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Procedure: Newton's Method for Approximating Roots of $f(x) = 0$

1. Choose an initial approximation x_0 as close to the a root as possible.
2. For $n = 1, 2, \ldots,$ define

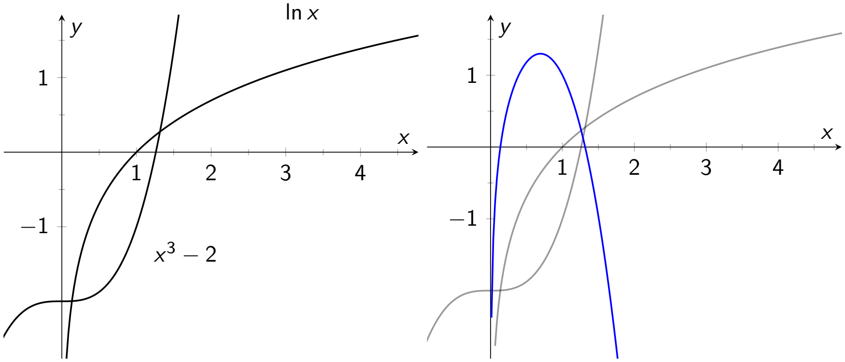
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

   provided $f'(x_n) \neq 0$.
3. End the calculations when a termination condition is met.

Example: Use Newton's Method and compute x_1, x_2, and x_3 for $f(x) = x^3 + x^2 + 1$ when $x_0 = -2$.

Example: We can also use Newton's Method to approximate intersection points. Do so for $y = \ln x$ and $y = x^3 - 2$.

To choose an initial point, it's very helpful to have a graph of the two functions. Consider the graph of $y = \ln x$ and $y = x^3 - 2$.



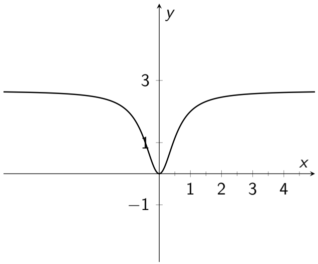
Intersections are the zeros of $y = \ln x - x^3 + 2$ graphed in blue on the right.

Newton's Method is not perfect. When Newton's Method fails, it can be dramatic!

It can happen that

1. The sequence $\{x_n\}$ can wander;
2. The sequence $\{x_n\}$ can converge very slowly;
3. The sequence $\{x_n\}$ can diverge!

Use Newton's Method to approximate the zero to $f(x) = \dfrac{8x^2}{3x^2+1}$ where $x_0 = 1$.



Here's the "approximation" (blue good, red bad, green awful)

n	x_n	x_n	x_n
0	1	0.15	1.1
1	-1	0.0699375	-1.4465
2	1	0.034456	3.81665
3	-1	0.0171665	-81.4865
4	1	0.00857564	$8.11572 \cdot 10^5$
5	-1	0.00428687	$-8.01692 \cdot 10^{17}$

Homework Problems: Section 4.8 (p.318-319) #5-15 odd, 21-29 odd, 37,53