5.1 – Approximating Areas Under Curves

MATH 2554 – Calculus I

In the previous 2 chapters, we learned that the derivative of a function tells us the rate of change of a function as well as the slope of the tangent line to the curve.

The motivating problem that leads to the development of the integral is the desire to compute the area under a curve.

Question: If we know the velocity function of a particular object, what does that tell us about its position function?

Toy Example: Suppose you ride your bike at a constant velocity of 8 miles per hour for 1.5 hours.

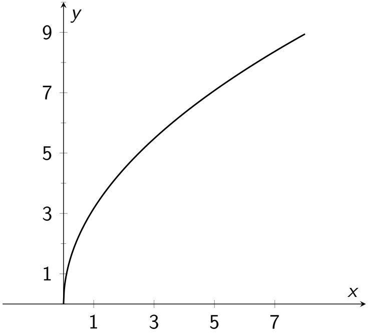
1. What is the velocity function that models this scenario?
2. What does the graph of the velocity function look like?
3. What is the position function for this scenario?
4. What is the antiderivative of the velocity function?
5. Where is the displacement (e.g, the distance you have traveled) represented when looking at the graph of the velocity function?

In the previous example, the velocity was constant. In most cases, this is not accurate (or possible). How can we find displacement when the velocity is changing over an interval?

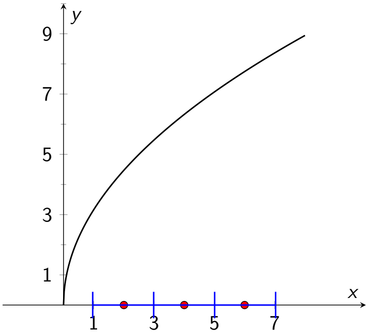
Our strategy is to divide the time interval into subintervals and approximate the velocity on each subinterval with a constant velocity. Then for each subinterval, the displacement can be approximated and summed.

Note: This provides us with only an approximation, but with a larger number of subintervals, the approximation becomes increasingly accurate.

Example: Consider the velocity function $\nu(t) = \sqrt{10t}$ of an object moving along the line for $1 \leq t \leq 7$.



Example: Consider the velocity function $v(t) = \sqrt{10t}$ of an object moving along the line for $1 \leq t \leq 7$.



1. Divide the time interval into $n = 3$ subintervals and approximate the velocity via a midpoint approximation. Estimate the displacement of the object on $[1, 7]$.
2. Divide the time interval into $n = 6$ subintervals and approximate the velocity via a midpoint approximation. Estimate the displacement of the object on $[1, 7]$.

Working towards a Riemann Sum.

Idea: The more subintervals into which the original interval is divided, the better the approximation.

Definition

Let $[a, b]$ be an interval and n a positive integer.

$$[x_0, x_1],\ [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

where $a = x_0$, $b = x_n$, and $\Delta x = \dfrac{b-a}{n}$.

$$x_k = a + k\Delta x, \quad k = 0, 1, 2, \ldots, n$$

Definition (Riemann Sum)

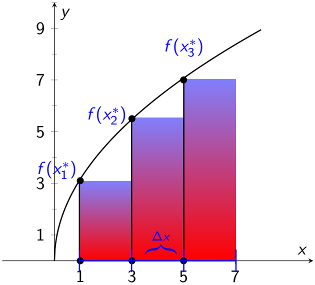
Suppose f is defined on a closed interval $[a, b]$, which is divided into n equal subintervals, $[x_0, x_1]$, $[x_1, x_2], \ldots, [x_{n-1}, x_n]$ where $a = x_0$ and $b = x_n$. In each subinterval $[x_{k-1}, x_k]$, choose any point x_k^* (so that $x_{k-1} \leq x_k^* \leq x_k$). Then the sum

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

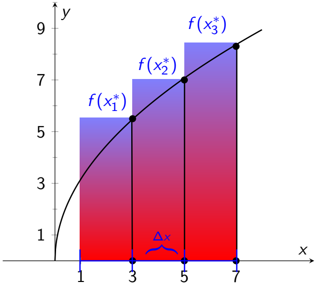
is called a Riemann sum.

- a left Riemann sum if x_k^* is the left endpoint of $[x_{k-1}, x_k]$ (that is, $x_k^* = x_{k-1}$);
- a right Riemann sum if x_k^* is the right endpoint of $[x_{k-1}, x_k]$ (that is, $x_k^* = x_k$);
- a midpoint Riemann sum if x_k^* is the midpoint of $[x_{k-1}, x_k]$ (that is, $x_k^* = \dfrac{x_{k-1} + x_k}{2}$);

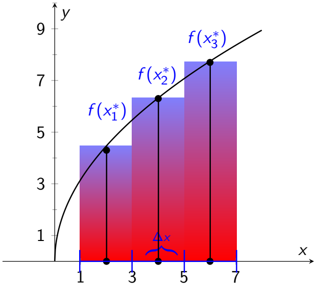
Visualizing Riemann Sums:



$x_k^* = x_{k-1}$



$x_k^* = x_k$



$x_k^* = \frac{x_{k-1}+x_k}{2}$

Example: Compute the left, right, and midpoint Riemann sums for the function $f(x) = 2x^3$ on the interval $[0, 8]$ with $n = 4$.

Example: True/False: If $f(x)$ is positive and decreasing on (a, b), then a given left Riemann sum will be less than the corresponding right Riemann sum for $f(x)$ on $[a, b]$.

$\sum$ notation

We use $\sum$ notation to compactly express sums. Suppose that $\{a_1, \ldots, a_N\}$ is a set of numbers. Then

$$a_1 + a_2 + \cdots + a_N = \sum_{k=1}^{N} a_k .$$

Exercise: Convert the following into $\sum$ notation or out of $\sum$ notation

1. $\displaystyle\sum_{j=0}^{6} 2^j$
2. $\displaystyle\sum_{\ell=-2}^{10} 2\ell$
3. $-1+4+7+10+13$
4. $\sin 1+\sin 2+\sin 3+\sin 4+\sin 5+\sin 6$
5. $f(x_1^*)\Delta x+f(x_2^*)\Delta x+\cdots+f(x_n^*)\Delta x$

Homework Problems: Section 5.1 (pp.348-349) #9, 11-14, 15a, 17-33odd, 40, 41, 47-49