4.7 – L'Hôpital's Rule

MATH 2554 – Calculus I

L'Hôpital's Rule is an extremely useful technique for evaluating a class of limits for which simple analytic techniques do not work.

Theorem (L'Hôpital's Rule, Indeterminate form: 0/0)

Suppose f and g are differentiable on an open interval I containing a value a such that $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right-hand side exists or is $\pm\infty$. The rule also applies when $x \to a$ is replaced with $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Exercise: Compute

1. $\lim_{x \to 0} \dfrac{e^x - 1}{x^2 + 3x}$.
2. $\lim_{y \to 0} \dfrac{\tan 3y}{\tan 6y}$.

Theorem (L'Hôpital's Rule, Indeterminate form: ∞/∞)

Suppose f and g are differentiable on an open interval I containing a value a such that $g'(x) \neq 0$ on I when $x \neq a$. If $\lim\limits_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right-hand side exists or is $\pm\infty$. The rule also applies when $x \to a$ is replaced with $x \to \pm\infty$, $x \to a^+$, or $x \to a^-$.

Exercise: Compute

1. $\lim_{x\to\infty} \dfrac{e^{3x}}{4e^{3x}+5}$.
2. $\lim_{y\to\infty} \dfrac{\ln(3x+5e^{x})}{\ln(7x+3e^{2x})}$.
3. $\lim_{s\to\infty} \dfrac{s}{\sqrt{s^2+4}}$

Related Indeterminate Forms: $0 \cdot \infty$ and $\infty - \infty$. L'Hôpital's Rule can often be used to evaluate limits in these forms, but the expressions must first be manipulated to an indeterminate form of the type $0/0$ or ∞/∞.

Example: Find

1. $\lim_{x \to \infty} x^2 \left(1 - \cos \frac{1}{x^2}\right)$
2. $\lim_{x \to \infty} x - \sqrt{x^2 - 1}$. (Hint: factor out an x)

Related Indeterminate Forms: 1^∞, 0^0, and ∞^0. Using logarithms and exponentials, we can handle forms of this type. Observe:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln(f(x)^{g(x)})} = \lim_{x\to a} e^{g(x)\ln f(x)}.$$

Thus, if $\lim_{x\to a} f(x)^{g(x)}$ has the indeterminate form $1^\infty, 0^0$, or ∞^0, then use the following procedure:

1. Analyze $L = \lim_{x\to a} g(x)\ln f(x)$. This limit can be put into the indeterminate form $0/0$ or ∞/∞.
2. When L is finite, $\lim_{x\to a} f(x)^{g(x)} = e^L$. If $L = \infty$, then $\lim_{x\to a} f(x)^{g(x)} = \infty$. If $L = -\infty$, then $\lim_{x\to a} f(x)^{g(x)} = 0$.

Find

1. $\lim_{x \to 0^+} x^x$
2. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

Growth Rates of Functions

Definition

Suppose f and g are functions with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. We say f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, equivalently, if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty.$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

where $0 < M < \infty$ (i.e. M is nonzero and finite).

Notation: If f grows faster than g as $x \to \infty$, we write $g \ll f$.

Homework Problems: Section 4.7 (pp.310-311) #13-81 odd