3.5 – Derivatives of Trigonometric Functions

MATH 2554 – Calculus I

Fall 2019

Key Identities: The key identities to computing the derivatives of the trigonometric functions are

- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$

Exercise:

Compute $\lim_{x \to 0} \frac{\sin(7x)}{\sin(2x)}$.

Useful Trig Identities

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \cot^2 x = \csc^2 x$$

$$\tan^2 x + 1 = \sec^2 x \qquad \tan x = \frac{\sin x}{\cos x}$$

$$\sin 2x = 2 \sin x \cos x \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\cos 2x = 1 - 2\sin^2 x \qquad \cot x = \frac{1}{\tan x}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \sec x = \frac{1}{\cos x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \csc x = \frac{1}{\sin x}$$

Theorem (The derivatives of sine and cosine)

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

The difference quotient

$$\begin{aligned}\frac{\cos(x+h)-\cos x}{h} &= \frac{\cos x\cos h - \sin x \sin h - \cos x}{h}\\ &= \cos x\frac{\cos h - 1}{h} - \sin x\frac{\sin h}{h}.\end{aligned}$$

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0} \cos x\frac{\cos h - 1}{h} - \sin x\frac{\sin h}{h} = -\sin x.$$

Exercise: Show that $\dfrac{d}{dx}(\tan x) = \sec^2 x$.

Other Trig Derivatives:

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x.$$

Exercises:

Compute

1. $\dfrac{d}{d\theta}\dfrac{\tan\theta}{1+\tan\theta}$
2. $\dfrac{d}{dt}\sin t\cos t.$

Higher Order Derivatives:

As we have seen before, many functions $f(x)$ can be repeatedly differentiated. Observe that

$$f(x) = \sin x \qquad g(x) = \cos x \quad f'(x) = \cos x$$
$$g'(x) = -\sin x \quad f''(x) = -\sin x \qquad g''(x) = -\cos x$$
$$f'''(x) = -\cos x \qquad g'''(x) = \sin x \quad f^{(4)}(x) = \sin x$$
$$g^{(4)}(x) = \cos x.$$

Homework Problems: Section 3.5 (p. 176-177) #11-51 odd, 58-68 even