1 1D compressible Nonviscous Linear Equations

$$
\begin{aligned}
\frac{\partial m}{\partial t} &= -\rho - \frac{\partial p}{\partial x} \\
\frac{\partial \rho}{\partial t} &= -\frac{\partial m}{\partial x} + S_\rho \\
\frac{\partial p}{\partial t} &= -N^2 m - \frac{\partial m}{\partial x}, \\
N^2 &= -\frac{1}{\rho_0}\frac{d\rho_0}{dx} - 1, \beta = -\frac{1}{\rho_0}\frac{d\rho_0}{dx}
\end{aligned} \tag{1}
$$

1.1 Constant N^2

Separation of Variables

$$
\begin{aligned}
2\pi\sigma &= \sqrt{4\pi^2 k^2 + \frac{1}{4}N^4 + \frac{1}{2}N^2 + \frac{1}{4}} \\
m &= e^{-\frac{1}{2}\beta x}\sin(2\pi k x)\sin(2\pi\sigma t) \\
\rho &= e^{-\frac{1}{2}\beta x}\left[-\frac{N^2+1}{4\pi\sigma}\sin(2\pi k x) + \frac{k}{\sigma}\cos(2\pi k x)\right]\cos(2\pi\sigma t) \\
p &= e^{-\frac{1}{2}\beta x}\left[\frac{N^2-1}{4\sigma\pi}\sin(2\pi k x) + \frac{k}{\sigma}\cos(2\pi k x)\right]\cos(2\pi\sigma t)
\end{aligned} \tag{2}
$$

1.2 Linear N^2

Method of Manufactured Solutions

$$
\begin{aligned}
N^2(x) &= ax \\
m &= e^{-\frac{ax+1}{2}x}\sin(2\pi k x)\sin(2\pi\sigma t) \\
\rho &= e^{-\frac{ax+1}{2}x}\left[\frac{1}{\pi\sigma}\left(2k^2 - 2\sigma^2)\pi^2 - \frac{ax}{4} - \frac{1}{8}\right)\sin(2\pi k x) + \frac{k(ax+1)}{\sigma}\cos(2\pi k x)\right]\cos(2\pi\sigma t) \\
p &= e^{-\frac{ax+1}{2}x}\left[\frac{-1}{4\sigma\pi}\sin(2\pi k x) + \frac{k}{\sigma}\cos(2\pi k x)\right]\cos(2\pi\sigma t) \\
S_\rho &= -2\left[\left((2k^2 - s\sigma^2)\pi^2 + \frac{ax}{4} + \frac{1}{8}\right)\sin(2\pi k x) + ax\pi k\cos(2\pi k x)\right]e^{-\frac{ax+1}{2}x}\sin(2\pi\sigma t)
\end{aligned} \tag{3}
$$

$a > 0$, $k \in \mathbb{Z}$ and σ can be freely chosen.

2 1D compressible Viscous Linear Equations

$$
\begin{aligned}
\frac{\partial m}{\partial t} &= -\rho - \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 m}{\partial x^2} + 2\beta \frac{\partial m}{\partial x} - 2\beta^2 m - \frac{1}{\rho_0} \frac{d^2 \rho_0}{dx^2} m \right] + S_m \\
\frac{\partial \rho}{\partial t} &= -\frac{\partial m}{\partial x} + S_\rho \\
\frac{\partial p}{\partial t} &= -N^2 m - \frac{\partial m}{\partial x} + S_p, \\
N^2 &= -\frac{1}{\rho_0} \frac{d\rho_0}{dx} - 1, \beta = -\frac{1}{\rho_0} \frac{d\rho_0}{dx}
\end{aligned}
\tag{4}
$$

2.1 $\beta = 0, N^2 = -1$

Separation of Variables

$$
\begin{aligned}
\sigma &= k\sqrt{1 - \pi^2 k^2 \nu^2} \\
m &= e^{-2\pi^2 k^2 \nu t} \sin(2\pi k x) \sin(2\pi \sigma t) \\
\rho &= \cos(2\pi k x) e^{-2\pi^2 k^2 \nu t} \left[\frac{\sigma}{k} \cos(2\pi \sigma t) + 2\pi k \nu \sin(2\pi \sigma t) \right] \\
p &= \left[\cos(2\pi k x) - \frac{1}{2\pi k} \sin(2\pi k x) \right] e^{-2\pi^2 k^2 \nu t} \left[\frac{\sigma}{k} \cos(2\pi \sigma t) + \pi k \nu \sin(2\pi \sigma t) \right]
\end{aligned}
\tag{5}
$$

$\nu^2 < \frac{1}{k^2 \pi^2}$

2.2 Constant N^2

Method of Manufactured Solutions

$$
\begin{aligned}
m &= e^{-\frac{1}{2}\beta x} \sin(2\pi k x) e^{-2\pi^2 k^2 \nu t} \sin(2\pi \sigma t) \\
\rho &= \frac{-2e^{-\frac{1}{2}\beta x} e^{-2\pi^2 k^2 \nu t}}{\pi \left(\pi^2 k^4 \nu^2 + \sigma^2\right)} \Bigg(\Bigg[\left(k^6 \pi^4 \nu^2 + \left(\left(\frac{11\, \nu^2 \beta^2}{8} - 1 \right) k^4 + k^2 \sigma^2 \right) \pi^2 - \frac{\left((\beta - 2)\, k^2 - 22\, \beta\, \sigma^2 \right) \beta}{16} \right) \nu \pi \sin(2\pi \sigma t) \\
&\quad + \sigma \left(k^4 \pi^4 \nu^2 + \left(-k^2 + \sigma^2 \right) \pi^2 - \frac{\beta^2}{16} + \frac{\beta}{8} \right) \cos(2\pi \sigma t) \Bigg] \sin(2\pi k x) \\
&\quad - k\pi \left[\pi\, \nu \left(\pi^2 \beta\, k^4 \nu^2 + \beta\, \sigma^2 + \frac{k^2}{2} \right) \sin(2\pi \sigma t) + \frac{\sigma}{2} \cos(2\pi \sigma t) \right] \cos(2\pi k x) \\
p &= e^{-\frac{1}{2}\beta x} e^{-2\pi^2 k^2 \nu t} \frac{\left(4\, k\pi\, \cos\left(2\, k\pi\, x \right) + \beta\, \sin\left(2\, k\pi\, x \right) - 2\, \sin\left(2\, k\pi\, x \right) \right) \left(k^2 \sin\left(2\, \pi\, \sigma\, t \right) \pi\, \nu + \cos\left(2\, \pi\, \sigma\, t \right) \sigma \right)}{4\, \pi\, \left(\pi^2 k^4 \nu^2 + \sigma^2 \right)} \\
S_\rho &= 4e^{-\frac{1}{2}\beta x} e^{-2\pi^2 k^2 \nu t} \Bigg(\nu\, \beta\, \pi\, \sigma \left[k\pi\, \cos\left(2\, k\pi\, x \right) - \frac{11\, \beta\, \sin\left(2\, k\pi\, x \right)}{8} \right] \cos(2\pi \sigma t) \\
&\quad + \left[\left(k^4 \pi^4 \nu^2 + \left(\frac{11\, \beta^2 k^2 \nu^2}{8} - k^2 + \sigma^2 \right) \pi^2 - \frac{\beta^2}{16} \right) \sin\left(2\, k\pi\, x \right) - k^3 \pi^3 \nu^2 \beta\, \cos\left(2\, k\pi\, x \right) \right] \sin(2\pi \sigma t) \Bigg)
\end{aligned}
\tag{6}
$$

$k \in \mathbb{Z}$ and σ can be freely chosen.

2.3 Linear N^2

Method of Manufactured Solutions