Chapter 4 Cont.. Spline Functions

3.1 Introduction

Earlier we discussed the methods of finding an nth order polynomial passing through $(n+1)$ points. In certain cases, it was found that these polynomials give **erroneous** result. Furthermore, it was found that a low-order polynomial approximation in each sub-interval provides a better approximation to the function than fitting a single high-order polynomial to the entire interval. Such piece-wise connecting polynomials are called **spline functions**. The points at which two connecting splines meet are called knots.
The **cubic** spline is the most popular in engineering applications. Before starting cubic splines, we discuss linear and quadratic splines since such a discussion will eventually justify the development of cubic splines.

3.1.1 Linear Splines

Suppose the given data points are $(x_i, y_i),\ i = 0, 1, ..., n$ and $h_i = x_i - x_{i-1},\ i = 1, 2, ..., n$. Let $s_i(x)$ be a straight line from x_{i-1} to x_i. Then, the slope of $s_i(x)$ is
$m_i = \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$ and $s_i(x) = y_i + m_i(x - x_i)$.
From the discussion above the $s_i(x)$ are the **linear splines**.

Drawback

The linear splines derived above are continuous in $[x_0, x_n]$, but their slpes are discontinuous, i.e their first derivatives are discontinuous.

3.1.2 Quadratic Splines

Let $s_i(x)$ be a quadratic approximation of the data points in the sub-interval $[x_{i-1} - x_i]$ satisfying the following conditions:

1. $s_i(x)$ and $s_i'(x)$ are continuous on $[x_0, x_n]$,

2. $s_i(x) = y_i, \qquad i = 0, 1, 2, ..., n$

3. $s_i'(x) = \frac{1}{h_i}[(x_i - x)m_{i-1} + (x - x_{i-1})m_i$ as $s_i'(x)$ is linear, where $m_i = s_i'(x)$.

Integrating $s_i'(x)$ with respect to x, we obtain

$$s_i(x) = \frac{1}{h_i}\left[-\frac{(x_i - x)^2}{2}\,m_{i-1} + \frac{(x - x_{i-1})}{2}\,m_i\right] + c_i \tag{3.1}$$

Putting $x = x_{i-1}$ we get,

$$c_i = y_{i-1} + \frac{h_i}{2}m_{i-1} \tag{3.2}$$

Imposing the continuity condition on the spline functions $s_i(x)$ and imposing the natural spline condition $s_1''(x_1) = 0$ we obtain the following realtions to calculate m_i.

$$m_{i-1} + m_i = \frac{2}{h_i}(y_i - y_{i-1}), \qquad i = 1,\ 2,\ ...,\ n \quad \text{and} \quad m_0 = m_1 \tag{3.3}$$

Drawbacks

The second derivative of the quadratic splines derived above are discontinuous which is an obvious disadvantages. This drawback is removed in the cubic splines.

3.2 Cubic Splines

Let $s_i(x)$ be a cubic approximation of the data points in the sub-interval $[x_{i-1} - x_i]$ satisfying the following conditions:

1. $s_i(x)$ is atmost a cubic for $i = 1,\ 2,\ ...,\ n$,
2. $s_i(x)$, $s_i'(x)$ and $s_i''(x)$ are continuous on $[x_0, x_n]$,
3. $s_i(x) = y_i, \qquad i = 0,\ 1,\ 2,\ ...,\ n$
4. $s_i''(x) = \frac{1}{h_i}[(x_i - x)M_{i-1} + (x - x_{i-1})M_i$ as $s_i''(x)$ is linear, where $M_i = s_i''(x)$.

Imposing all these conditions we get,

$$\frac{h_i}{6}\,M_{i-1} + \frac{1}{3}(h_i + h_{i+1})\,M_i + \frac{h_{i+1}}{6}\,M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}, \qquad (i = 1, 2, ..., n-1) \tag{3.4}$$

For subintervals of equal lengths:

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\,(y_{i+1} - 2y_i + y_{i-1}), \qquad (i = 1, 2, ..., n-1) \tag{3.5}$$

Imposing natural spline condition $s_i''(x_0) = s_i''(x_n) = 0$ in 3.4, we get,

$$2(h_1 + h_2)M_1 + h_2M_2 = 6\left[\frac{y_2 - y_1}{h_2} - \frac{y_1 - y_0}{h_1} - h_1M_0\right]$$

$$h_2M_1 + 2(h_2 + h_3)M_2 + h_3M_3 = 6\left[\frac{y_3 - y_2}{h_3} - \frac{y_2 - y_1}{h_2}\right]$$

$$h_3M_2 + 2(h_3 + h_4)M_3 + h_4M_4 = 6\left[\frac{y_4 - y_3}{h_4} - \frac{y_3 - y_2}{h_3}\right]$$

$$...$$

$$h_{n-1}M_{n-2} + 2(h_{n-1} + h_n)M_{n-1} = 6\left[\frac{y_n - y_{n-1}}{h_n} - \frac{y_{n-1} - y_{n-2}}{h_{n-1}}\right] - h_nM_n$$

This system is called **tridiagonal system** and there an efficient and an accurate method for solving it.