

### 3.1 Introduction

Earlier we discussed the methods of finding an  $n$ th order polynomial passing through  $(n + 1)$  points. In certain cases, it was found that these polynomials give **erroneous** result. Furthermore, it was found that a low-order polynomial approximation in each sub-interval provides a better approximation to the function than fitting a single high-order polynomial to the entire interval. Such piece-wise connecting polynomials are called **spline functions**. The points at which two connecting splines meet are called knots.

The **cubic** spline is the most popular in engineering applications. Before starting cubic splines, we discuss linear and quadratic splines since such a discussion will eventually justify the development of cubic splines.

#### 3.1.1 Linear Splines

Suppose the given data points are  $(x_i, y_i)$ ,  $i = 0, 1, \dots, n$  and  $h_i = x_i - x_{i-1}$ ,  $i = 1, 2, \dots, n$ . Let  $s_i(x)$  be a straight line from  $x_{i-1}$  to  $x_i$ . Then, the slope of  $s_i(x)$  is  $m_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$  and  $s_i(x) = y_{i-1} + m_i(x - x_{i-1})$ .

From the discussion above the  $s_i(x)$  are the **linear splines**.

#### Drawback

The linear splines derived above are continuous in  $[x_0, x_n]$ , but their slopes are discontinuous, i.e their first derivatives are discontinuous.

#### 3.1.2 Quadratic Splines

Let  $s_i(x)$  be a quadratic approximation of the data points in the sub-interval  $[x_{i-1}, x_i]$  satisfying the following conditions:

1.  $s_i(x)$  and  $s'_i(x)$  are continuous on  $[x_0, x_n]$ ,
2.  $s_i(x) = y_i$ ,  $i = 0, 1, 2, \dots, n$

3.  $s'_i(x) = \frac{1}{h_i}[(x_i - x)m_{i-1} + (x - x_{i-1})m_i]$  as  $s'_i(x)$  is linear, where  $m_i = s'_i(x)$ .

**Integrating**  $s'_i(x)$  with respect to  $x$ , we obtain

$$s_i(x) = \frac{1}{h_i} \left[ -\frac{(x_i - x)^2}{2} m_{i-1} + \frac{(x - x_{i-1})}{2} m_i \right] + c_i \quad (3.1)$$

Putting  $x = x_{i-1}$  we get,

$$c_i = y_{i-1} + \frac{h_i}{2} m_{i-1} \quad (3.2)$$

Imposing the continuity condition on the spline functions  $s_i(x)$  and imposing the natural spline condition  $s''_1(x_1) = 0$  we obtain the following relations to calculate  $m_i$ .

$$m_{i-1} + m_i = \frac{2}{h_i}(y_i - y_{i-1}), \quad i = 1, 2, \dots, n \quad \text{and} \quad m_0 = m_1 \quad (3.3)$$

## Drawbacks

The second derivative of the quadratic splines derived above are discontinuous which is an obvious disadvantages. This drawback is removed in the cubic splines.

## 3.2 Cubic Splines

Let  $s_i(x)$  be a cubic approximation of the data points in the sub-interval  $[x_{i-1} - x_i]$  satisfying the following conditions:

1.  $s_i(x)$  is atmost a cubic for  $i = 1, 2, \dots, n$ ,
2.  $s_i(x)$ ,  $s'_i(x)$  and  $s''_i(x)$  are continuous on  $[x_0, x_n]$ ,
3.  $s_i(x) = y_i$ ,  $i = 0, 1, 2, \dots, n$
4.  $s''_i(x) = \frac{1}{h_i}[(x_i - x)M_{i-1} + (x - x_{i-1})M_i]$  as  $s''_i(x)$  is linear, where  $M_i = s''_i(x)$ .

Imposing all these conditions we get,

$$\frac{h_i}{6} M_{i-1} + \frac{1}{3}(h_i + h_{i+1}) M_i + \frac{h_{i+1}}{6} M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}, \quad (i = 1, 2, \dots, n-1) \quad (3.4)$$

For subintervals of equal lengths:

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2} (y_{i+1} - 2y_i + y_{i-1}), \quad (i = 1, 2, \dots, n-1) \quad (3.5)$$

Imposing natural spline condition  $s''_i(x_0) = s''_i(x_n) = 0$  in 3.4, we get,

$$\begin{aligned} 2(h_1 + h_2)M_1 + h_2M_2 &= 6 \left[ \frac{y_2 - y_1}{h_2} - \frac{y_1 - y_0}{h_1} - h_1M_0 \right] \\ h_2M_1 + 2(h_2 + h_3)M_2 + h_3M_3 &= 6 \left[ \frac{y_3 - y_2}{h_3} - \frac{y_2 - y_1}{h_2} \right] \\ h_3M_2 + 2(h_3 + h_4)M_3 + h_4M_4 &= 6 \left[ \frac{y_4 - y_3}{h_4} - \frac{y_3 - y_2}{h_3} \right] \\ &\dots \\ h_{n-1}M_{n-2} + 2(h_{n-1} + h_n)M_{n-1} &= 6 \left[ \frac{y_n - y_{n-1}}{h_n} - \frac{y_{n-1} - y_{n-2}}{h_{n-1}} \right] - h_nM_n \end{aligned}$$

This system is called **tridiagonal system** and there an efficient and an accurate method for solving it.