Chapter 4 Cont.. Spline Functions

3.1 Introduction

Earlier we discussed the methods of finding an nth order polynomial passing through $(n+1)$ points. In certain cases, it was found that these polynomials give **erroneous** result. Furthermore, it was found that a low-order polynomial approximation in each sub-interval provides a better approximation to the function than fitting a single high-order polynomial to the entire interval. Such piece-wise connecting polynomials are called **spline functions**. The points at which two connecting splines meet are called knots.
The **cubic** spline is the most popular in engineering applications. Before starting cubic splines, we discuss linear and quadratic splines since such a discussion will eventually justify the development of cubic splines.

3.1.1 Linear Splines

Suppose the given data points are $(x_i, y_i),\ i = 0, 1, ..., n$ and $h_i = x_i - x_{i-1},\ i = 1,\ 2,\ ...,\ n$. Let $s_i(x)$ be a straight line from x_{i-1} to x_i. Then, the slope of $s_i(x)$ is
$m_i = \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}$ and $s_i(x) = y_{i-1} + m_i(x - x_{i-1})$.
From the discussion above the $s_i(x)$ are the **linear splines**.

Drawback

The linear splines derived above are continuous in $[x_0, x_n]$, but their slopes are discontinuous, i.e their first derivatives are discontinuous.

3.1.2 Quadratic Splines

Let $s_i(x)$ be a quadratic approximation of the data points in the sub-interval $[x_{i-1} - x_i]$ satisfying the following conditions:

1. $s_i(x)$ and $s_i'(x)$ are continuous on $[x_0, x_n]$,

2. $s_i(x_i) = y_i, \qquad i = 0,\ 1,\ 2,\ ...,\ n$

3. $s_i'(x) = \frac{1}{h_i}[(x_i - x)m_{i-1} + (x - x_{i-1})m_i]$ as $s_i'(x)$ is linear, where $m_i = s_i'(x)$.

Integrating $s_i'(x)$ with respect to x, we obtain

$$s_i(x) = \frac{1}{h_i}\left[-\frac{(x_i - x)^2}{2}m_{i-1} + \frac{(x - x_{i-1})^2}{2}m_i\right] + c_i \tag{3.1}$$

Putting $x = x_{i-1}$ we get,

$$c_i = y_{i-1} + \frac{h_i}{2}m_{i-1} \tag{3.2}$$

Imposing the continuity condition on the spline functions $s_i(x)$ we get,

$$m_{i-1} + m_i = \frac{2}{h_i}(y_i - y_{i-1}), \qquad i = 1,\, 2,\, ...,\, n \tag{3.3}$$

Imposing the natural spline condition $s_1''(x_1) = 0$ we obtain, $m_0 = m_1$

Drawbacks

The second derivative of the quadratic splines derived above are discontinuous which is an obvious disadvantages. This drawback is removed in the cubic splines.

3.2 Cubic Splines

When computers were not available, the draftsman used a device to draw a smooth curve through a given set of points such that the slope and the curvature are also continuous along the curve, that is $f(x),\ f'(x),\ f''(x)$ are continuous on the curve. Such a device was called a **spline** and plotting of the curve was called **spline fitting**.

Let $s_i(x)$ be a cubic approximation of the data points in the sub-interval $[x_{i-1} - x_i]$ satisfying the following conditions:

1. $s_i(x)$ is atmost a cubic for $i = 1,\, 2,\, ...,\, n$,
2. $s_i(x)$, $s_i'(x)$ and $s_i''(x)$ are continuous on $[x_0, x_n]$,
3. $s_i(x) = y_i, \qquad i = 0,\, 1,\, 2,\, ...,\, n$
4. $\mathbf{s_i''(x) = \frac{1}{h_i}[(x_i - x)M_{i-1} + (x - x_{i-1})M_i]}$ as $s_i''(x)$ is linear, where $M_i = s_i''(x)$.

Integrating the condition-4, twice with respect to x: we get,

$$s_i(x) = \frac{1}{h_i}\left[-\frac{(x_i - x)^3}{6}M_{i-1} + \frac{(x - x_{i-1})^3}{6}M_i\right] + c_i(x_i - x) + d_i(x - x_{i-1}) \tag{3.4}$$

Using the condition: $s_i(x_{i-1}) = y_{i-1}$ and $s_i(x_i) = y_i$

$$c_i = \frac{1}{h_i}\left[y_{i-1} - \frac{h_i^2}{6}M_{i-1}\right], \qquad d_i = \frac{1}{h_i}\left[y_i - \frac{h_i^2}{6}M_i\right] \tag{3.5}$$

Imposing all these conditions we get,

$$\boxed{\frac{h_i}{6}\,M_{i-1}+\frac{1}{3}(h_i+h_{i+1})\,M_i+\frac{h_{i+1}}{6}\,M_{i+1}=\frac{y_{i+1}-y_i}{h_{i+1}}-\frac{y_i-y_{i-1}}{h_i},\qquad (i=1,2,...,n-1)} \tag{3.6}$$

For subintervals of equal lengths:

$$M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}\,(y_{i+1}-2y_i+y_{i-1}),\qquad (i=1,2,...,n-1) \tag{3.7}$$

Imposing **natural spline** condition $s_i''(x_0)=s_i''(x_n)=0$ in 3.6, we get,

$$\begin{aligned}
2(h_1+h_2)M_1+h_2M_2 &= 6\left[\frac{y_2-y_1}{h_2}-\frac{y_1-y_0}{h_1}-h_1M_0\right]\\
h_2M_1+2(h_2+h_3)M_2+h_3M_3 &= 6\left[\frac{y_3-y_2}{h_3}-\frac{y_2-y_1}{h_2}\right]\\
h_3M_2+2(h_3+h_4)M_3+h_4M_4 &= 6\left[\frac{y_4-y_3}{h_4}-\frac{y_3-y_2}{h_3}\right]\\
&\dots\\
h_{n-1}M_{n-2}+2(h_{n-1}+h_n)M_{n-1} &= 6\left[\frac{y_n-y_{n-1}}{h_n}-\frac{y_{n-1}-y_{n-2}}{h_{n-1}}\right]-h_nM_n
\end{aligned}$$

This system is called **tridiagonal system** and there an efficient and an accurate method for solving it.

Example 1. Obtain the natural cubic spline approximation for the function defined by the data: $(0,1),\ (1,2),\ (2,33),(3,244)$. Hence find an estimate of $y(2.5)$.

Solution

For the equally x-spaced data we obtain:

$$M_0+4M_1+M_2=6(y_2-2y_1+y_0) \tag{3.8}$$

$$M_1+4M_2+M_3=6(y_3-2y_2+y_1) \tag{3.9}$$

Using $M_0=0=M_3$, we get, $4M_1+M_2=180,\quad M_1+4M_2=1080$. Then,

$$s_3(x)=\frac{1}{h_3}\left[-\frac{(x_3-x)^3}{6}\,M_2+\frac{(x-x_2)^3}{6}\,M_3\right]+$$
$$\frac{1}{h_3}\left[y_2-\frac{h_3^2}{6}M_2\right](x_3-x)+\frac{1}{h_3}\left[y_3-\frac{h_3^2}{6}M_3\right](x-x_2)$$
$$=-46x^3+414x^2-985x+725$$

$$s(2.5)=-46(2.5)^3=414(2.5)^2-982(2.5)+715=121.25$$

3.2.1 Exercise

1. For the data points: $(0,0),\ (\pi/2,1),\ (\pi,0)$, determine the following:

   (a) natural quadratic splines

   (b) natural cubic splines

(c) $y(\pi/6)$ using natural cubic spline

2. Determine $y(\pi/6)$ using the natural cubic splines from the data points:
$(0,0),\ (\pi/4, 1/\sqrt{2}),\ (\pi/2, 1),\ (3\pi/4, 1/\sqrt{2}),\ (\pi, 0)$.

3. What do you understand by natural spline? Explain why the natural cubic spline condition have $M_0 = 0$ and $M_n = 0$.