2

Linear Transformation

2.1 Introduction

The terms *function*, *mapping*, *map*, and *transformation* are synonymous. Great part of the linear algebra is dedicated to the study of linear transformation.

Definition 1. A mapping T from a vector space V to a vector space W is **linear** if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all vectors $x, y \in V$ and for all scalars α, β.
Equivalently, T **linear** if,

i) $T(x + y) = T(x) + T(y)$ for all vectors $x, y \in V$, and

ii) $T(\alpha x) = \alpha T(x)$ for any scalars α.

Example 1. Show that if A is a $m \times n$ matrix then T defined by $T(x) = Ax$ is a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$.

2.1.1 Exercise

1. Is the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x_1, x_2, x_3) = (x_1 + (x_2, 3x_1 - x_2 + x_3, 5x_1 - x_3)$ a linear map? Explain.

2. Show that shift map is not a linear transformation.

3. Is there a linear transformation that maps $(1, 0)$ to $(5, 3, 4)$ and maps $(3, 0)$ to $(1, 3, 2)$?

4. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

5. T is defined by $T(x) = Ax$, find a vector x whose image under T is b, determine whether x is unique.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

6. How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^4$ to $\mathbb{R}^5$ by the rule $T(x) = Ax$?

2.2 Matrix of a Linear Transformation

Theorem 1. Let A be an $m \times n$ matrix. The mapping $x \mapsto Ax$ is linear from $\mathbb{R}^n$ to $\mathbb{R}^m$. Conversely, for a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ there exists an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. In fact, A is the $m \times n$ matrix whose jth column is the vector $T(e_j)$, where $\{e_j : j = 1, 2, ..., n\}$ is the the basis of the domain $\mathbb{R}^n$,i.e $A = [T(e_1) \ ... \ T(e_n)]$.

Example 2. Describe the linear mapping that has the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Let the given matrix is denoted by A. A is a 2×2 So, the map is from $\mathbb{R}^2 \mapsto \mathbb{R}^2$. Then for every $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, Now $A\vec{x} = (-x_2, x_1)$ which is a **rotation** transformation that rotates every vector through the angle 90^0 counterclockwise.

2.2.1 Exercise

Find the standard matrix of the linear transformation T.

1. $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$
2. $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ first reflects points through the x_1-axis then reflects points through the line $x_2 = x_1$
3. $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ that rotates each point through an angle ϕ, with counterclockwise rotation.
4. Describe the transformation of the following matrices geometrically:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

2.3 Kernel and Image of Linear Transformation

Let $T : V \mapsto W$ be a linear transformation.

- **Kernal**

  The kernal of T is the set of all vectors in V that maps to zero vector in W. It is denoted by *Ker(T)*. $Ker(T) = \{v \in V \,;\, T(v) = 0\}$.In another words, Kernal is the *Null Space* of T.

- **Range**

  The range of T is the set of all vectors in W which are the images of the vectors in V. It is denoted by $R(t)$. $R(T) = \{w \in W \,:\, w = T(v) \text{for some} v \in V\}$.

2.3.1 Exercise

1. Show that $Ker(T)$ is a subspace of V and $R(T)$ is a subspace of W.

2. Given the vector space V of all real-valued functions defined on an interval $[a, b]$ such that their first derivative functios are continous on $[a, b]$. Let W be the vector space of all continuous functions of $[a, b]$. Show that $D : V \mapsto V$ that maps $f \in V$ to $f' \in w$ is a linear transformation and find the kernal of D.

2.3.2 Facts:

1. The linear transformation $T : V \to W$ is one-to-one if, $dim(Ker(T) = 0$, i.e $Ker(T) = 0$.

2. The linear transformation $T : V \to W$ is onto if, $dim(R(T) = dim(W)$.