2 Root Finding

2.1 Introduction

Roots of an equation

$$f(x) = 0 \tag{2.1}$$

are the zeros, of f, which means, the values of x that makes the value of f zero. Basically equations are categorized into two. If f is a polynomial then the Equation-2.1 is a **polynomial equation** and if f is a non-polynomial then Equation-2.1 is a **transcendental equation**. For the polynomial equations following results hold:

1. Every polynomial equation of degree n has at most n real roots.

2. If n is odd then, the polynomial equation has at least one real root whose sign is opposite to that of the last term.

3. If n is even and the constant term is negative, then the equation has at-least one positive root and at-least one negative root.

4. The imaginary roots occurs in a pair (conjugate-pair). If the coefficients of f are rationals then, the irrational roots occurs in pairs (conjugate-pair).

5. **Descartes' Rule of Signs**

   a). A polynomial equation cannot have more number of positive real roots than the number of changes of signs in the coefficients of $f(x)$.

   b). A polynomial equation cannot have more number of negative real roots than the number of changes of signs in the coefficients of $f(-x)$.

2.2 Bisection Method

Theorem 1 (Bolzano's Theorem)**.** If f(x) is continuous in $[a, b]$, and if $f(a)$ and $f(b)$ are of opposite signs, then $f(c) = 0$ for at least one number $c \in (a, b)$.

The Bisection method is based on Theorem-1. The word "bisection" means "half". Using this method the root c of f is given by $c \approx \frac{a+b}{2}$. Let $x_1 = \frac{a+b}{2}$. If $f(x_1) \neq 0$ then, the root, c lies either in $[a, x_1]$ or in $[x_1, b]$. If $f(a)f(x_1) < 0$ then, c lies in $[a, x_1]$ else, it lies in $[x_1, b]$.

At each step of this method, the given interval is bisected, so the length of the interval is halfed. At nth step the length of the interval is $\frac{|b-a|}{2^n}$. If the tolerance of the given approximation is ϵ then we must have $\frac{|b-a|}{2^n} \leq \epsilon$. And the number of steps required to reach this accuracy is $n \geq log_2(|b-a|)$.

2.2.1 Procedure

1. Choose two real numbers a and b such that $f(a)f(b) < 0$.
2. Set $x_0 = 0$ and $x_1 = \frac{a+b}{2}$.
3. Do
   $\epsilon_r = \left|\frac{x_0 - x_1}{x_0}\right|$
   If $\epsilon_r < tolerance$ then $root = x_1$,
   else
   $x_0 = x_1$ and if $f(a)f(x_1) < 0$ then $x_1 = \frac{a + x_1}{2}$
   if $f(x_1)f(b) < 0$ then $x_1 = \frac{x_1 + b}{2}$.

2.2.2 Exercise

Using Bisection Method:

1. Find a root of $f(x) = x^3 - x - 1 = 0$, correct to 4 decimal places

2.3 Iteration Method

Steps:

1. Re-write the given equation $f(x) = 0$ in the form $x = \phi(x)$. This equation is of **iterative-type**. Meaning we can substitute a value of x in $\phi(x)$ to get another value of x, and continue this process to get the desired value of x if the iteration is of convergent one.
2. Choose an initial root of f, x_0.
3. $x_1 = \phi(x_0)$, $x_2 = \phi(x_1)$ and so on.

The sequence $x_0, x_1, x_2, ...$ may not converge to a definite number. But if the sequence converges to a definite number ζ, then ζ is a root of the given equation.

2.3.1 Exercise

Using Iteration Method:

1. Find a root of $2x - 3 - cosx = 0$, correct to 3 decimal places

2.4 Newton-Rapshon's Method

Steps:

1. Choose an initial guess solution of the given equation $f(x) = 0$, x_0.

2. Let x_1 be a solution, which is more close to the exact solution of $f(x) = 0$. Then Using Taylor's expansion of f about x_0:

$$f(x_1) = f(x_0) + (x_1 - x_0)f'(x_0) + (x_1 + x_0)^2 f''(x_0) + ... = 0$$

Neglecting the second and higher order derivatives, we get

$$f(x_0) + (x_1 - x_0)f'(x_0) = 0 \tag{2.2}$$

The equation-2.8 is a linear equation, so this is an linear approximation. This equation is infact the tangent to the curve of the function $f(x)$ at $(x_0, f(x_0))$. And it is the point x_1 where the tangent meets the $x-axis$. So, the next approximation after x_0 by Newton-Rapshon's method is the point on $x - axis$, where the tangent to the f at x_0 meets the $x - axis$. This point can be solved as follows:

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

3. Successive approximation are given by $x_2, x_3, x_4, ...,$ where $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

2.4.1 Exercise

Using Newton-Rapshon's Method:

1. Find a root of $f(x) = xe^x - 1 = 0$, correct to 4 decimal places

2.5 Secant Method

In Newton-Rapshon's method we use a tangent to the curve to get close to the root of the function. So, Newton-Rapshon's method requires the evaluation of derivatives of the function, which may not always exit. So we replace the tangent, with a secant to approximate the root of the function.

Steps:

1. Choose two initial guess solutions of the given equation $f(x) = 0$, x_{-1} and x_0.

2. The slope of the secant is $\dfrac{f(x_0) - f(x_{-1})}{x_0 - x_{-1}}$.

3. Then equation of the line passing through the points of given by the two initial guesses is $f(x) - f(x_0) = \dfrac{f(x_0) - f(x_{-1})}{x_0 - x_{-1}}(x - x_0)$.

4. x_1 is the point where the secant meets the $x - axis$ so, $f(x_1) = 0$. This gives,

$$\begin{aligned}
0 - f(x_0) &= \frac{f(x_0) - f(x_{-1})}{x_0 - x_{-1}}(x_1 - x_0) \\
x_1 - x_0 &= -\frac{x_0 - x_{-1}}{f(x_0) - f(x_{-1})} f(x_0) \\
x_1 &= x_0 - \frac{x_0 - x_{-1}}{f(x_0) - f(x_{-1})} f(x_0)
\end{aligned}$$

5. This generalizes to

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \tag{2.3}$$

You can get this relation-2.3 just by plugging $f'(x_n) = \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$ as the slope of the tangent in Newton Rapshon's method is just approximated by the slope of the secant in Secant method.

2.6 System of Non-linear equations

For now we consider only a system of two equations. Let a system of two equations be

$$f(x, y) = 0, \qquad g(x, y) = 0 \tag{2.4}$$

2.6.1 Method of Iteration

First we assume that the sytem of equations 2.4 may be written in the form

$$x = F(x, y), \qquad y = G(x, y) \tag{2.5}$$

where the function F and G satisty the following conditions in a **closed** neighborhood of R of the root (α, β):

i) F and G and their firt partial derivatives are continuous in R, and

ii) $\left|\dfrac{\partial F}{\partial x}\right| + \left|\dfrac{\partial F}{\partial y}\right| < 1$ and $\left|\dfrac{\partial G}{\partial x}\right| + \left|\dfrac{\partial G}{\partial y}\right| < 1$, for all (x, y) in R.

If (x_0, y_0) is an initial approximation to the root (α, β), then Equations 2.5 give the sequence

$$\begin{aligned}
x_1 &= F(x_0, y_0), & y_1 &= G(x_0, y_0) \\
x_2 &= F(x_1, y_1), & y_2 &= G(x_1, y_1) \\
&\cdots \\
x_{n+1} &= F(x_n, y_n), & y_{n+1} &= G(x_n, y_n)
\end{aligned} \tag{2.6}$$

For faster convergence, recently computed values of x_i may be used in the evaluation of y_i in Equations. Above conditions are sufficient for convergence and in the limit we obtain,

$$\alpha = F(\alpha, \beta) \quad and \qquad \beta = G(\alpha, \beta) \tag{2.7}$$

Hence (α, β) is the root of the system 2.4.

2.6.2 Newton-Raphson Method

Let $(x_0, y_0$ be an initial approximation to the root of the system of equations in two variables 2.4. If $(x_0 + h, y_0 + k$ is the root of the system, then we must have

$$f(x_0 + h, y_0 + k) = 0 \qquad g(x_0 + h, y_0 + k) = 0$$

Assuming that f and g are sufficiently differentiable, we expand both of these functions by Taylor's series to obtain

$$f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0}... = 0$$
$$g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0}... = 0$$

where, $\dfrac{\partial f}{\partial x_0} = \left[\dfrac{\partial f}{\partial x}\right]_{x=x_0}$, $f_0 = f(x_0, y_0)$, etc

Neglating the second and higher-order derivatives terms, we get,

$$\begin{aligned} h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0}... &= -f_0 \\ h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0}... &= g_0 \end{aligned} \tag{2.8}$$

The system of equations 2.8 possesses a unique solution if

$$D = \begin{vmatrix} \frac{\partial f}{\partial x_0} & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} \end{vmatrix} \neq 0$$

By Cramer's rule

$$h = \frac{1}{D}\begin{vmatrix} -f_0 & \frac{\partial f}{\partial y_0} \\ -g_0 & \frac{\partial g}{\partial y_0} \end{vmatrix} \quad and \quad k = \frac{1}{D}\begin{vmatrix} \frac{\partial f}{\partial y_0} & -f_0 \\ \frac{\partial g}{\partial y_0} & -g_0 \end{vmatrix} \tag{2.9}$$

The new approximations are, therefore

$$x_1 = x_0 + h \quad and \quad y_1 = y_0 + k \tag{2.10}$$

2.6.3 Exercise

1. Find a real root of the system: $y^2 - 5y + 4 = 0$ and $3x^2y - 10x + 7 = 0$ correct to 4 decimal places using initial approximation $(0, 0)$.
2. Solve the system: $x^2 + y = 11$, $x + y^2 = 7$.
3. Solve the system: $x^2 - y^2 = 4$, $x^2 + y^2 = 16$.