9

Eigenvalues and Eigenvectors

9.1 Introduction

Let A be any square matrix, real or complex. A number λ is an **eigenvalue** of A if the equation

$$Ax = \lambda x$$

is true for some nonzero vector x. The vector x is and **eigenvector** associated with the eigenvalue λ. Both the eigenvalue and the eigenvector may be complex.

Theorem 1. A scalar λ is an eigenvalue of a matrix A if and only if $Det(A - \lambda I) = 0$.

The equation $Det(A - \lambda I) = 0$ is called the **characteristic equation** of A. It is the equation from which we can compute the eigenvalues of A. The function
$p : p(\lambda) = Det(A - \lambda I)$ is the **characteristic polynomial** of A.

9.1.1 Eigenspace

For an eigenvalue λ of a matrix A, the set $\{x \ : \ Ax = \lambda x\}$ forms a vector space. This forms a vector space because the vector x is a nonzero vector for it be an eigenvector. If x is a nonzero solution of $Ax = \lambda x \implies (A - \lambda I)x = 0$, which is a homogeneous system, then this homogeneous system has infinitely many solution. And this vector space is called eigenspace.

9.1.2 Exercise

1. What are the characteristic equation and the eigenvalues of the following matrices? For each eigenvalue, find an eigenvector.

   a. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 4 & 1 & 1 \\ 2 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ c. $\begin{bmatrix} 2 & -i & 0 \\ i & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Find the eigenvalue-eigenvector pairs. Explore the geometric effect of letting $x^{(k)} = Ax^{(k-1)}$ and $k = 0, 1, 2, ...$