# 2

# Linear Transformation

## 2.1 Introduction

The terms *function*, *mapping*, *map*, and *transformation* are synonymous. Great part of the linear algebra is dedicated to the study of linear transformation.

**Definition 1.** A mapping $T$ from a vector space $V$ to a vector space $W$ is **linear** if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all vectors $x, y \in V$ and for all scalars $\alpha$, $\beta$.
Equivalently, $T$ **linear** if,

i) $T(x + y) = T(x) + T(y)$ for all vectors $x, y \in V$, and

ii) $T(\alpha x) = \alpha T(x)$ for any scalars $\alpha$.

**Example 1.** Show that if $A$ is a $m \times n$ matrix then $T$ defined by $T(x) = Ax$ is a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$.

### 2.1.1 Exercise

1. Is the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x_1, x_2, x_3) = (x_1 + (x_2, 3x_1 - x_2 + x_3, 5x_1 - x_3)$ a linear map? Explain.

2. Show that shift map is not a linear transformation.

3. Is there a linear transformation that maps $(1, 0)$ to $(5, 3, 4)$ and maps $(3, 0)$ to $(1, 3, 2)$?

4. Show that the transformation $T$ defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

5. $T$ is defined by $T(x) = Ax$, find a vector $x$ whose image under $T$ is $b$, determine whether $x$ is unique.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

6. How many rows and columns must a matrix $A$ have in order to define a mapping from $\mathbb{R}^4$ to $\mathbb{R}^5$ by the rule $T(x) = Ax$?

## 2.2 Matrix of a Linear Transformation

**Theorem 1.** Let $A$ be an $m \times n$ matrix. The mapping $x \mapsto Ax$ is linear from $\mathbb{R}^n$ to $\mathbb{R}^m$. Conversely, for a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ there exists an $m \times n$ matrix $A$ such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. In fact, $A$ is the $m \times n$ matrix whose $jth$ column is the vector $T(e_j)$, where $\{e_j : j = 1, 2, ..., n\}$ is the the basis of the domain $\mathbb{R}^n$,i.e $A = [T(e_1) \ ... \ T(e_n)]$.

### 2.2.1 Exercise

Find the standard matrix of the linear transformation $T$.

1. For the dilation transformation $T(x) = 3x$, $x \in \mathbb{R}^2$.

2. Show that the transformation that rotates each point in $\mathbb{R}^2$, about the origin, through an angle $\phi$ counterclockwise, is a linear transformation by find the standard matrix of the transformation.

3. $T(x_1, x_2) = (2x_2 - 3x_1, x_1 - 4x_2, 0, x_2)$

4. $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ first reflects points through the $x_1$-axis then reflects points through the line $x_2 = x_1$

5. $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ that rotates each point through an angle $\phi$, with counterclockwise rotation.

6. Describe the transformation of the following matrices geometrically:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

**Example 2.** Describe the linear mapping that has the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Let the given matrix is denoted by $A$. $A$ is a $2 \times 2$ So, the map is from $\mathbb{R}^2 \mapsto \mathbb{R}^2$. Then for every $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, Now $A\vec{x} = (-x_2, x_1)$ which is a **rotation** transformation that rotates every vector through the angle $90^0$ counterclockwise.

### 2.2.2 Types of Transformation

Geometrically, linear transformation are basically of the following types:

1. Reflection

2. Rotation

3. Contraction and Expansion

4. Shears: Horizontal and Vertical
5. Projections

## 2.3 Kernel and Image of Linear Transformation

Let $T : V \mapsto W$ be a linear transformation.

- **Kernal**

  The kernal of $T$ is the set of all vectors in $V$ that maps to zero vector in $W$. It is denoted by *Ker(T)*. $Ker(T) = \{v \in V\, ; T(v) = 0\}$.In another words, Kernal is the *Null Space* of $T$.

- **Range**

  The range of $T$ is the set of all vectors in $W$ which are the images of the vectors in $V$. It is denoted by $R(t)$. $R(T) = \{w \in W\ :\ w = T(v)\text{for some}v \in V\}$.

### 2.3.1 Exercise

1. Show that $Ker(T)$ is a subspace of $V$ and $R(T)$ is a subspace of $W$.
2. Given the vector space $V$ of all real-valued functions defined on an interval $[a, b]$ such that their first derivative functios are continous on $[a, b]$. Let $W$ be the vector space of all continuous functions of $[a, b]$. Show that $D : V \mapsto V$ that maps $f \in V$ to $f' \in w$ is a linear transformation and find the kernal of $D$.

### 2.3.2 Facts:

1. The linear transformation $T : V \to W$ is one-to-one if, $dim(Ker(T) = 0$,
   i.e $Ker(T) = 0$.
2. The linear transformation $T : V \to W$ is onto if, $dim(R(T) = dim(W)$.

## 2.4 Properties of Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1. Then $T(0) = 0$.
2. $T$ is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution. (*Indirectly it has got to do with the nullity or nullspace of the matrix.*)
3. $T$ is onto if and only if the columns of the corresponding standard matrix spans the $\mathbb{R}^m$. (*So it has got to do with the column space of the matrix.*)

Give an example of one-to-one and not one-to-one linear transformation.