

Solving Ordinary Differential Equations

7.1 Introduction

The analytical methods of solution can be applied to solve only a selected class of differential equations. Those equations which govern physical systems do not possess, in general **closed-form** solutions, and hence recourse must be made to numerical methods for solving such differential equations. To describe various numerical methods for the solution of ordinary differential equations, we consider the general first order initial value problem. The methods so developed can, in general, be applied to the solution of first order equations.

Initial Value Problem: The general first order initial value problem is of the form:

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (7.1)$$

In general there are two classes of the methods for solving 7.1.

1. A series solution.
 - (a) Taylor series method
 - (b) Picard method
2. A set of tabulated values of x and y .
 - (a) Euler's methods
 - (b) Runge-Kutta methods

7.2 Taylor's series method

We consider the IVP 7.1. If $y(x)$ is the exact solution of this IVP, then the Taylor's series for $y(x)$ around $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{(x - x_0)^2}{2!}y''_0 + \dots \quad (7.2)$$

Differentiating 7.1 successively, with respect to x , we obtain, y'', y''', \dots . Since 7.2 is an infinite series, for $y(x_0)$ correct to N decimal places we solve the following equation

n^{th} term of Taylor series \leq Upper limiting error $\implies \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) \leq \frac{1}{2} 10^{-N}$.

If this holds then, the n number of terms is sufficient otherwise, we need to include additional terms.

Example: Estimate the value of $y(0.2)$ using Taylor's series method from the initial value problem: $y'' - xy' - y = 0$ $y(0) = 1$, $y'(0) = 0$

Solution: Given differential equation is

$$y'' = xy' + y \implies y''(0) = 0 \times y'(0) + y(0) = 1$$

Differentiating successively, we get

$$y''' = xy'' + y' + y' = xy'' + 2y' \implies y'''(0) = 0 \times y''(0) + 2y'(0) = 2 \times 0 = 0$$

$$y^{iv} = xy''' + 3y'' \implies y^{iv}(0) = 3y''(0) = 3 \times 1 = 3$$

$$y^v = xy^{iv} + 4y''' \implies y^v(0) = 4y'''(0) = 0$$

$$y^{vi} = xy^v + 5y^{iv} \implies y^{vi} = 5 \times 3 = 15$$

Substituting these values in Taylor's series approximation of $y(x)$, we get

$$y(x) = y_0 + (x-x_0)y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \frac{(x-x_0)^4}{4!}y^{iv}_0 + \frac{(x-x_0)^5}{5!}y^v_0 + \frac{(x-x_0)^6}{6!}y^{vi}_0 \dots$$

$$y(x) = 1 + \frac{x^2}{2!} \times 1 + \frac{x^4}{4!} \times 3 + \frac{x^6}{6!} \times 15 + \dots$$

$$y(x) = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \dots$$

For 4 decimal places accuracy, we solve

$$\frac{x^6}{48} = 1/2 \times 10^{-4} \implies x^6 = 48 \times 0.00005 \implies x = \sqrt[6]{0.0024} = 0.365$$

Since $0.365 > 0.2$, the series up to 6^{th} degree term is sufficient.

$$\therefore y(0.2) = 1 + \frac{0.2^2}{2} + \frac{0.2^4}{8} + \frac{0.2^6}{48} = 1.0050$$

Find $y(0.1)$ correct to 4 decimal places using Taylor's series:

$$y' = x - y^2, y(0) = 1$$



7.3 Picard's Method of Successive Approximation

Integrating the above differential equation 7.1, as follows, we obtain:

$$\begin{aligned} \int_{x_0}^x y' dx &= \int_{x_0}^x f(x, y) dx \\ y &= y_0 + \int_{x_0}^x f(x, y) dx \end{aligned} \quad (7.3)$$

This equation in which the unknown function y appears under the integral sign is called an *integral equation*. Such equation can be solved by the method of successive approximations as follows:

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0) dx \quad (7.4)$$

Proceeding in this way, we obtain:

$$y^{(n)} = y_0 + \int_{x_0}^x f(x, y^{(n-1)}) dx \quad \text{with } y^{(0)} = y_0 \quad (7.5)$$

It has been proved that if $f(x, y)$ is bounded in some region about the point (x_0, y_0) and if $f(x, y)$ satisfies the *Lipschitz condition*, viz.,

$$|f(x, y) - f(x, \bar{y})| \leq K|y - \bar{y}|$$

for some constant K , then the sequence $y^{(n)}$ converges to the solution of the equation 7.1.

7.4.1 Backward Euler's Method

Using the backward difference $\left[\frac{y_{n+1} - y_n}{h} = y'_{n+1} \right]$; we obtain, Backward Euler's Method as follows:

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1}) \quad n = 0, 1, 2, \dots \quad (7.7)$$

We obtain y_{n+1} in $f(x_{n+1}, y_{n+1})$ from the Euler's forward method. Actually,

$$y_{n+1}^1 = y_n + h f(x_n, y_n) \quad n = 0, 1, 2, \dots \text{ and } y_{n+1} = y_n + h f(x_{n+1}, y_{n+1}^1) \quad n = 0, 1, 2, \dots$$

Example: Find $y(1)$ using Euler's method: $y' = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$.

Given, $a = 0, b = 1, y_0 = 1$. Let us choose $n = 5$ so that $h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

Now using Euler's formula on $[x_0, x_1] = [0, 0.2]$, we obtain

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h \frac{y_0^2 - x_0^2}{y_0^2 + x_0^2} = 1 + 0.2 \times \frac{1^2 - 0^2}{1^2 + 0^2} = 1 + 0.2 \times 1 = 1.2$$

Similarly, on $[x_1, x_2] = [0.2, 0.4]$, we obtain

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h \frac{y_1^2 - x_1^2}{y_1^2 + x_1^2} = 1.2 + 0.2 \times \frac{1.2^2 - 0.2^2}{1.2^2 + 0.2^2}$$

$$y_2 = 1.2 + 0.2 \times 0.9459 = 1.3892$$

Similarly, on $[x_2, x_3] = [0.4, 0.6]$, we obtain

$$y_3 = y_2 + hf(x_2, y_2) = y_2 + h \frac{y_2^2 - x_2^2}{y_2^2 + x_2^2} = 1.3892 + 0.2 \times \frac{1.3892^2 - 0.4^2}{1.3892^2 + 0.4^2}$$

$$y_3 = 1.3892 + 0.2 \times 0.8469 = 1.5586$$

Similarly, on $[x_3, x_4] = [0.6, 0.8]$, we obtain

$$y_4 = y_3 + hf(x_3, y_3) = y_3 + h \frac{y_3^2 - x_3^2}{y_3^2 + x_3^2} = 1.5586 + 0.2 \times \frac{1.5586^2 - 0.6^2}{1.5586^2 + 0.6^2}$$

$$y_4 = 1.5586 + 0.2 \times 0.7419 = 1.7069$$

Finally, on $[x_4, x_5] = [0.8, 1]$, we obtain

$$y_5 = y_4 + hf(x_4, y_4) = y_4 + h \frac{y_4^2 - x_4^2}{y_4^2 + x_4^2} = 1.7069 + 0.2 \times \frac{1.7069^2 - 0.8^2}{1.7069^2 + 0.8^2}$$

$$y_5 = 1.7069 + 0.2 \times 0.6398 = 1.8349$$

Exercise: Find $y(2)$ using Euler's method: $y' = y(1 + x^2)$, $y(0) = 1$.

7.4.2 Modified Euler's Method

In the Modified Euler's Method the formula for iteration is

$$y_n^1 = y_n + h f(x_n, y_n) \quad n = 0, 1, 2, \dots$$

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n^1)] \quad (7.8)$$

7.4.3 Exercise

1. Use Euler's method to find $y(0.5)$ using $h = 0.1$:

$$\text{a). } \frac{dy}{dx} = \frac{3}{5}x^3y, \quad y(0) = 1 \qquad \text{b). } \frac{dy}{dx} = 1 + y^2, \quad y(0) = 0$$

2. Use Euler's modified method to find $y(0.2)$ and $y(0.4)$ using $h = 0.2$:

$$\frac{dy}{dx} = x + y, \quad y(0) = 0.$$

Example: Find $y(1)$ using Modified Euler's method: $y' = \frac{y^2 - x^2}{y^2 + x^2}, y(0) = 1$.

Given, $a = 0, b = 1, y_0 = 1$. **Let us choose $n = 5$ so that $h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$**

Now using Modified Euler's formula on $[x_0, x_1] = [0, 0.2]$, we obtain

$$y_1^0 = y_0 + hf(x_0, y_0) = y_0 + h \frac{y_0^2 - x_0^2}{y_0^2 + x_0^2} = 1 + 0.2 \times \frac{1^2 - 0^2}{1^2 + 0^2} = 1 + 0.2 \times 1 = 1.2$$

$$y_1 = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1)] = 1 + \frac{h}{2} \left[\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2} + \frac{(y_1^0)^2 - x_1^2}{(y_1^0)^2 + x_1^2} \right]$$

$$y_1 = 1 + \frac{0.2}{2} \left[\frac{1^2 - 0^2}{1^2 + 0^2} + \frac{1.2^2 - 0.2^2}{1.2^2 + 0.2^2} \right] = 1 + 0.1 \times [1 + 0.9459] = 1.1946$$

Similarly, on $[x_1, x_2] = [0.2, 0.4]$, we obtain

$$y_2^0 = y_1 + hf(x_1, y_1) = y_1 + h \frac{y_1^2 - x_1^2}{y_1^2 + x_1^2} = 1.1946 + 0.2 \times \frac{1.1946^2 - 0.2^2}{1.1946^2 + 0.2^2} = 1.3837$$

$$y_2 = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2)] = 1.1946 + \frac{h}{2} \left[\frac{y_1^2 - x_1^2}{y_1^2 + x_1^2} + \frac{(y_2^0)^2 - x_2^2}{(y_2^0)^2 + x_2^2} \right]$$

$$y_2 = 1.1946 + \frac{0.2}{2} \left[\frac{1.1946^2 - 0.2^2}{1.1946^2 + 0.2^2} + \frac{1.3837^2 - 0.4^2}{1.3837^2 + 0.4^2} \right] = 1.3737$$

Similarly, on $[x_2, x_3] = [0.4, 0.6]$, we obtain $y_3^0 = 1.5425, y_3 = 1.5318$

Similarly, on $[x_3, x_4] = [0.6, 0.8]$, we obtain $y_4^0 = 1.6786, y_4 = 1.6682$

Finally, on $[x_4, x_5] = [0.8, 1]$, we obtain $y_5^0 = 1.7934, y_5 = 1.7833$

Exercise: Find $y(2)$ using Euler's method: $y' = y(1 + x^2), y(0) = 1$.

Example

Find $y(1)$ using Runge-Kutta second order (RK2) method:

$$y' = \frac{y^2 - x^2}{y^2 + x^2}, y(0) = 1.$$

Given, $a = 0, b = 1, y_0 = 1$. **Let us choose $n = 5$ so that $h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$**

Now using Runge-Kutta second order formula on $[x_0, x_1] = [0, 0.2]$, we get

$$k_1 = hf(x_0, y_0) = h \frac{y_0^2 - x_0^2}{y_0^2 + x_0^2} = 0.2 \times \frac{1^2 - 0^2}{1^2 + 0^2} = 0.2$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = 0.2f(0.2, 1.2) = 0.2 \times \frac{1.2^2 - 0.2^2}{1.2^2 + 0.2^2} = 0.1892$$

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2] = 1 + 0.5 \times (0.2 + 0.1892) = 1.1946$$

Similarly, on $[x_1, x_2] = [0.2, 0.4]$, we obtain

$$k_1 = hf(x_1, y_1) = h \frac{y_1^2 - x_1^2}{y_1^2 + x_1^2} = 0.2 \times \frac{1.1946^2 - 0.2^2}{1.1946^2 + 0.2^2} = 0.1891$$

$$k_2 = hf(x_1 + h, y_1 + k_1) = 0.2f(0.4, 1.3837) = 0.2 \times \frac{1.3837^2 - 0.4^2}{1.3837^2 + 0.4^2} = 0.1692$$

$$y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = 1.1946 + 0.5 \times (0.1891 + 0.1692) = 1.3737$$

Similarly, on $[x_2, x_3] = [0.4, 0.6]$, we get $k_1 = 0.1687, k_2 = 0.1474, y_3 = 1.5318$

Similarly, on $[x_3, x_4] = [0.6, 0.8]$, we get $k_1 = 0.1468, k_2 = 0.1260, y_4 = 1.6682$

Finally, on $[x_4, x_5] = [0.8, 1]$, we get $k_1 = 0.1252, k_2 = 0.1051, y_5 = 1.7833$

Exercise: Find $y(2)$ using Runge-Kutta fourth order (RK4) method:

$$y' = y(1 + x^2), y(0) = 1.$$

7.5 Runge-Kutta Methods

Euler's method is less efficient in practical problems. It requires h to be small for obtaining reasonable accuracy. To overcome this issue, Runge-Kutta methods are designed. Let $k_0 = hf(x_0, y_0)$ and $k_1 = hf(x_1, y_0 + hf(x_0, y_0))$. The special case of second order Runge-Kutta

Method gives $y_1 = y_0 + \frac{h}{2}(k_0 + k_1)$, which is equivalent to Modified Euler's Method. So, the iteration formula for the **second-order Runge-Kutta Method** is:

$$\begin{aligned}k_n &= f(x_n, y_n) \\k_{n+1} &= f(x_{n+1}, y_n + hk_n) \\y_{n+1} &= y_n + \frac{h}{2} [k_n + k_{n+1}]\end{aligned}\tag{7.9}$$

7.5.1 Fourth-Order Runge-Kutta Method

The iteration formula for the Fourth-Order Runge-Kutta method is as follows:

$$\begin{aligned}k_n &= f(x_n, y_n) \\k_{n+1} &= f(x_n + (h/2), y_n + (h/2)k_n) \\k_{n+2} &= f(x_n + (h/2), y_n + (h/2)k_{n+1}) \\k_{n+3} &= hf(x_n + h, y_n + hk_{n+2}) \\y_{n+1} &= y_n + \frac{h}{6} [k_n + 2k_{n+1} + 2k_{n+2} + k_{n+3}]\end{aligned}\tag{7.10}$$

7.5.2 Exercise

1. Use Runge-Kutta fourth order formula

(a) to find $y(0.2)$ and $y(0.4)$ given $y' = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$.

(b) to find $y(1.2)$ and $y(1.4)$ given $y' = \frac{3x + y}{x + 2y}$, $y(1) = 1$.

(c) to find $y(0.2)$ given $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

2. Given $\frac{dy}{dx} = y(1 + x^2)$, $y(0) = 1$ find the values for $x = 0.2, 0.4, 0.6$ using the Euler, the modified Euler, and the fourth order Runge-Kutta methods. Compare the results thus obtained.

3. Solve the boundary value problems by finite difference method taking $y(0) = 0$, $y(1) = 1$ for $y = 0.5, 0.25$: a). $y'' - y = 0$ b). $y'' - x^2y' - e^xy = 1/(1 + x)$

7.6 Simultaneous and Higher Order Equations

Consider a pair simultaneous equations:

$$\frac{dy}{dt} = f(t, x, y) \quad \text{and} \quad \frac{dx}{dt} = g(t, x, y)\tag{7.11}$$

with the initial conditions $x = x_0$ and $y = y_0$ when $t = t_0$. Assuming that $\Delta t = h$, $\Delta x = k$, and $\Delta y = l$, the fourth order Runge-Kutta method gives,

$$\begin{aligned}
 k_n &= f(t_n, x_n, y_n) \\
 l_n &= g(t_n, x_n, y_n) \\
 &\dots \\
 y_{n+1} &= y_n + \frac{h}{6} [k_n + 2k_{n+1} + 2k_{n+2} + k_{n+3}] \\
 x_{n+1} &= x_n + \frac{h}{6} [l_n + 2l_{n+1} + 2l_{n+2} + l_{n+3}]
 \end{aligned} \tag{7.12}$$

We now consider the second-order differential equation:

$$y'' = F(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0 \tag{7.13}$$

By setting $z = y'$, the above equation 7.13 becomes a system of first order equations in two equations.

$$y' = z, \quad z' = F(x, y, z), \quad y(x_0) = y_0, \quad z(x_0) = y'_0 \tag{7.14}$$

7.7 Boundary Value Problems

The second order ordinary differential equation consists of two boundary equations as follows:

$$F(x, y, y', y'') = 0, \quad y(x_0) = a, \quad y(x_n) = b$$

One of the popular method of solution to this boundary value problem is the method of finite difference.

7.8 Finite-Difference Method

The finite difference method for the solution of a two-point boundary value problem consists in replacing the derivatives occurring in the differential equation (and in the boundary conditions as well) by means of their finite-difference approximations and then solving the resulting linear system of equations by a standard procedure. We have the following finite difference formula for the derivatives.

1. Forward Difference

$$\begin{aligned}
 \text{a). } y'(x) &\approx \frac{y(x+h) - y(x)}{h} & \text{b). } y''(x) &\approx \frac{y(x+2h) - 2y(x+h) + y(x)}{h^2}
 \end{aligned}$$

Backward Difference

$$\begin{aligned}
 \text{a). } y'(x) &\approx \frac{y(x) - y(x-h)}{h} & \text{b). } y''(x) &\approx \frac{y(x) - 2y(x-h) + y(x-2h)}{h^2}
 \end{aligned}$$

2. Central Difference

$$\text{a). } y'(x) \approx \frac{y(x+h) - y(x-h)}{2h} \quad \text{b). } y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

These formulae can be easily derived by expanding $y(x)$ in the neighborhood of x using Taylor's series. In general for higher order derivatives central difference formula gives better accuracy so, this formula is commonly used.

7.8.1 General Method:

To solve the boundary-value problem 7.15:

$$y'' + f(x)y' + g(x)y = r(x), \quad y(x_0) = a, \quad y(x_n) = b \quad (7.15)$$

1. First we divide the range $[x_0, x_n]$ into n equal subintervals of width h so that $x_i = x_0 + ih, \quad i = 1, 2, \dots, n.$

2. Then $y(x_i) = y_i$ and $y'_i = \frac{y_{i+1} - y_{i-1}}{2h}$ and $y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}.$

3. Substituting these values in 7.15 we get,

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + f_i \frac{y_{i+1} - y_{i-1}}{2h} + g_i y_i = r_i, \quad i = 1, 2, \dots, n-1$$

4. Multiplying through $2h^2$, and gathering the like terms, we get,

$$(2 - hf_i)y_{i-1} + 2(g_i h^2 - 4)y_i + (2 + hf_i)y_{i+1} = 2r_i h^2, \quad i = 1, 2, \dots, n-1$$

with $y_0 = a, \quad y_n = b.$ This comprise of a system of linear equations which can be solved with a suitable method to obtain the solution of the original differential equation 7.15.

Example 1. Solve: $y'' - \cos x y' - \sin x y = e^x, \quad y(0) = 0, \quad y(1) = 1$

Solution Steps:

1. Here, $x_0 = 0$ and $x_n = 1.$ Choosing $n = 4$ we get $h = (1 - 0)/4 = 0.25.$

2. At $x = x_i$ we have $\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \cos x_i \frac{y_{i+1} - y_{i-1}}{2h} - \sin x_i y_i = e^{x_i}$

Substituting $h = 0.25,$ we get, $\frac{y_{i+1} - 2y_i + y_{i-1}}{(0.25)^2} - \cos x_i \frac{y_{i+1} - y_{i-1}}{2(0.25)} - \sin x_i y_i = e^{x_i}$

3. Simplifying: $(16 + 2\cos x_i)y_{i-1} + (\sin x_i - 32)y_i + (16 - 2\cos x_i)y_{i+1} = e^{x_i}$

4. Substituting $x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75$ and $y_0 = 0, \quad y_4 = 1:$

$$(16 + 2\cos(0.25))0 + (\sin(0.25) - 32)y_1 + (16 - 2\cos(0.25))y_2 = e^{0.25}$$

$$(16 + 2\cos(0.5))y_1 + (\sin(0.5) - 32)y_2 + (16 - 2\cos(0.5))y_3 = e^{0.5}$$

$$(16 + 2\cos(0.75))y_2 + (\sin(0.75) - 32)y_3 + (16 - 2\cos(0.75))1 = e^{0.75}$$

5. Simplifying:

$$-32.24 y_1 + 14.06 y_2 = 1.28$$

$$17.75 y_1 - 32.48 y_2 + 14.24 y_3 = 1.64 \quad (7.16)$$

$$17.46 y_2 - 32.68 y_3 + 14.54 = 2.11$$

6. Solving: $y_1 = 0.01, \quad y_2 = 0.38, \quad y_3 = 0.98.$