1 Vector Calculus

1.1 Basics

scalars, vectors, their notations, length of a vector, norm of a vector, unit vector, angle between two vectors, inner product, orthogonal vectors, orthonormal vectors,

Physically, a vector is a quantity having both magnitude and direction. Mathematically, a vector is an element of a vector space. A vector is a compound mathematical object constructed from simpler objects called scalars.

- **Equality of Vectors**

  Two vectors a and b are said to be equal if they have same length and same direction. Hence a vector can be arbitrarily translated; i.e, its initial point can be chosen arbitrarily.

- **Plane Vectors and Space Vectors**

  The vectors of the vector space $\mathbb{R}^2$ are called plane vectors and the vectors of the vector space $\mathbb{R}^3$ are called space vectors. Plane vectors are $2-D$ vectors and space vectors are $3-D$ vectors.

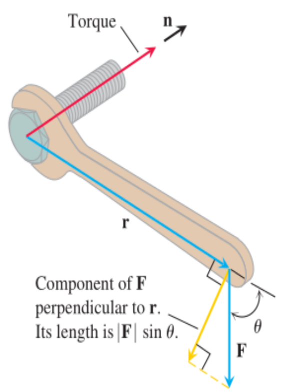


FIGURE 12.32 The torque vector describes the tendency of the force **F** to drive the bolt forward.

Torque

When we turn a bolt by applying a force **F** to a wrench (Figure 12.32), we produce a torque that causes the bolt to rotate. The **torque vector** points in the direction of the axis of the bolt according to the right-hand rule (so the rotation is counterclockwise when viewed from the *tip* of the vector). The magnitude of the torque depends on how far out on the wrench the force is applied and on how much of the force is perpendicular to the wrench at the point of application. The number we use to measure the torque's magnitude is the product of the length of the lever arm **r** and the scalar component of **F** perpendicular to **r**. In the notation of Figure 12.32,

$$\text{Magnitude of torque vector} = |\mathbf{r}||\mathbf{F}|\sin\theta,$$

or $|\mathbf{r}\times\mathbf{F}|$. If we let $\mathbf{n}$ be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is $\mathbf{r}\times\mathbf{F}$, or

$$\text{Torque vector} = (|\mathbf{r}||\mathbf{F}|\sin\theta)\,\mathbf{n}.$$

Recall that we defined $\mathbf{u}\times\mathbf{v}$ to be $\mathbf{0}$ when $\mathbf{u}$ and $\mathbf{v}$ are parallel. This is consistent with the torque interpretation as well. If the force **F** in Figure 12.32 is parallel to the wrench, meaning that we are trying to turn the bolt by pushing or pulling along the line of the wrench's handle, the torque produced is zero.

- **Vector Product**

  Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be two space vectors. Then, their vector product is denoted by $a \times b$ and defined as a new vector such that
  $a \times b = (a_2b_3 - b_2a_3, a_3b_1 - b_3a_1, a_1b_2 - b_1a_2)$

- **Right Handed Coordinate System**

  Here the coordinate system is right-handed, meaning the unit vectors $\hat{i}, \quad \hat{j}, \quad \hat{k}$ forms a right-handed triple. *What is a right-handed triple?*

1.2 Vector Calculus

1.2.1 Scalar Functions and Vector Functions

- A function whose co-domain is a scalar set is called a scalar function. For example, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x$ is a **scalar function**. We say that a scalar function defines a scalar field in that domain. Examples of scalar fields are the temperature field in a body or the pressure field of the air in the earth's atmosphere.

- A function whose co-domain is a vector space is called a **vector function**. For example, $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $v(x, y) = (y, x)$ is a vector function. Moreover, a vector function on D is a rule that assigns a vector to each element in D. The common vector functions are the plane vector functions and the space vector functions.

- So, the mathematics of a vector function is a mathematics of a multivariable function and the vector calculus is **multivariable calculus**.

- A **vector field** is a field that is defined by a vector function. For example magnetic field of a bar magnet is a vector field, velocity field of a rotating disc is a vector field, they are characterized by the vector functions, force and velocity. Another vector field is the earth's gravitational field. This one is a three dimensional vector field. *Try drawing these fields in your copy.*

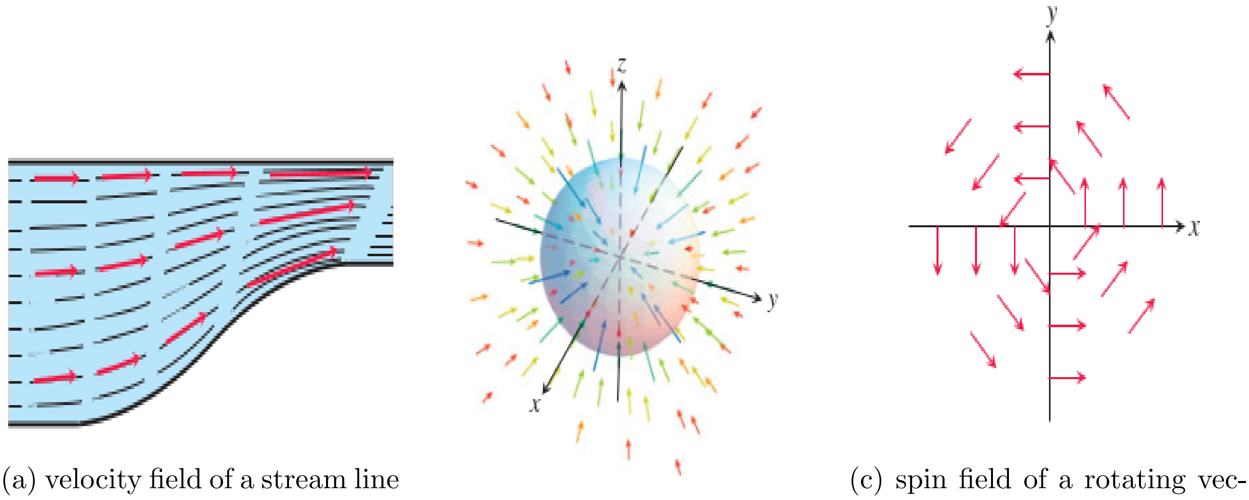
1.2.2 Vector Field

Mathematically, a vector field is a field that is defined by a vector function.

- Suppose a region in the plane or in space is occupied by a moving fluid, such as water. The fluid is made up of a large number of particles, and at any instant of time, a particle has a velocity $\vec{v}$. At different points of the region at a given time, these velocities can vary. We can think of a velocity vector being attached to each point of the fluid representing the velocity of a particle at that point. Such a fluid flow is an example of a vector field.

- The tangent T for a curve and normal vectors N for a surface in a space both form vector fields along the curve.

- The gradient of a scalar function in a space we obtain a three-dimensional field on the surface.



(a) velocity field of a stream line flow

(b) gravitational field

(c) spin field of a rotating vectors

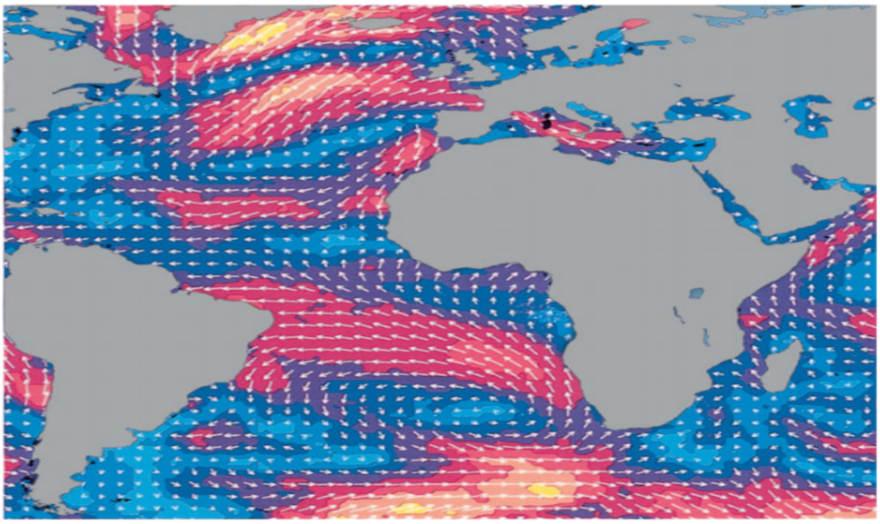


Figure 1.2: Wind measurement over world's oceans.

1.2.3 Curves in Space

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I \tag{1.1}$$

The points $(x, y, z) = (f(t), g(t), h(t)), \quad t \in I$ make up the curve in space that we call the particle's path. The equation 1.1 parametrize the curve. A curve in space can also be represented in vector form. The vector

$$r(t) = \vec{OP} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \tag{1.2}$$

The equation 1.2 defines a **vector valued function** or a vector function. The circle $x^2 + y^2 = 1$ is a scalar function whereas, the same circle $(x, y) = (cost, sint)$ is a vector function. A major application of vector calculus concerns curves and surfaces and their use in physics and geometry. This field is called **differential geometry**. It plays a role in mechanics, computer-aided and traditional engineering design, geodesy and geography, space travel, and relativity theory.
Curves in space may occur as paths of moving bodies. This and other applications motivate *parametric representations* with parameter t, which may be time or something else: $r(t) = (x(t),\ y(t),\ z(t)) = x(t)\hat{i} + y(t)\hat{k} + z(t)\hat{k}$. To each value of t there corresponds a point of C with position vector $r(t)$. Moreover, the sense of increasing t, called the positive sense on the curve, induces an **orientation** of the curve, a direction of travel along the curve. The sense of decreasing t is then called the negative sense on the curve.

Example 1. The ellipse in the xy-plane with center at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$

. The corresponding vector function with the parameter t is

$$r(t) = (acost,\ bsint, 0) = acost\ \hat{i} + bsint\ \hat{j}$$

- Similarly the parametric equation of a **straight line** is $r(t) = \vec{a} + t\vec{b}$.

- The parametric equation of **circular helix** that lies on the cylinder $x^2 + y^2 = a^2$ is $r(t) = (acost,\ asint,\ ct)$. If $c > 0$ the helix is shaped like a right-handed screw, else it looks like a left-handed screw. What happens if $c = 0$?

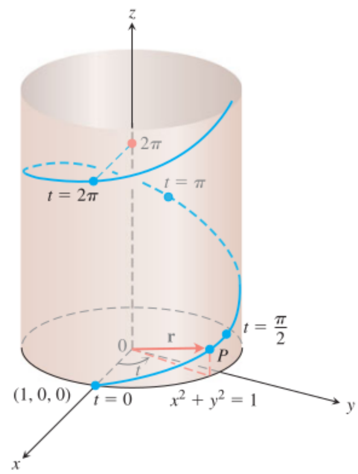


FIGURE 13.3 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ (Example 1).

1.2.4 Exercise

1. Write a parametric equation of a general plane.

2. Find a parametric equation of the following:

   a). Circle of radius 3, center $(4, 6)$. b). Parabola $y = 4x^2$

3. What curves are represented by the following?

   a). $(2+cos3t, -2+sin3t, 5)$ b). $(t, 1/t, 0)$ c). $(cosht,\ sinht,\ 0)$

4. Show that setting $t = -t$ reverses the orientation of $(acost,\ asint,\ 0)$.

1.2.5 Derivative of a Vector function

When a vector-valued function changes, the change can occur in both magnitude and direction, so the derivative is itself a vector. The integral of a vector valued function is also a vector. We use the calculus of these functions to describe the paths and motions of objects moving in plane or in a space.

The vector function $r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ has a derivative at t if f, g, and h have derivatives at t. The derivative is the vector function

$$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}$$

The vector $r'(t)$, when different from 0, is defined to be the vector **tangent** to the curve at t. The **tangent line** to the curve at a point defined by t_0 is defined to be the line through the point parallel to $r'(t_0)$. We require $r'(t) \neq 0$ for a **smooth** curve to make sure the curve has a continuously turning tangent at each point. On a smooth curve, there are no sharp corners or cusps.

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

1.2.6 Exercise

1. Explain geometrically, how $r'(t)$ is a vector along the tangent to the curve r at t.
2. Find the velocity, speed and acceleration of a particle that is moving along the curve $r(t) = 2cost\hat{i} + 2sint\hat{j} + cos^2t\hat{k}$ in time t.
3. Explain the derivative of a vector function of constant length. *(Take the help of dot product.)*
4. What kind of surfaces are the level surfaces $f(x, y, z) = const$?

a). $f = x^2 + y^2 + 4z^2$ b). $f = z - \sqrt{x^2 - y^2}$ c). $f = 4x + 3y - 5z$

5. Sketch the vector fields of:

   a). $v = i - j$ b). $v = yi + xj$ c). $v = (x - y)i + (x + y)j$

6. Find the first and second derivative of: $[4cost,\ 4sint,\ 2t]$

7. Find the first partial derivatives of: $[e^x cosy,\ e^x siny]$

1.3 Multivariable Calculus Review

Many functions depend on more than one independent variable. For instance, the volume of a right circular cylinder is a function $V = \pi r^2 h$ of its radius and height, so it a function $V(r, h)$ of two variables r and h. Real-valued functions of several independent real variables are defined analogously to functions in the single-variable case.

Definition 1. Suppose D is a set of $n - tuples$ of real numbers $(x_1, x_2, ..., x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, ..., x_n)$$

to each element of D. The set D is the function's **domain**, and f is said to be a function of n **independent variables** x_1 to x_2.

If f is a function of two independent variables, we usually call the independent variables x and y and the dependent variable z, and we picture domain of f as a region in the xy-plane. If f is a function of three independent variables, we usually call the independent variables x,y and z and the dependent variable w, and we picture the domain as a region in space.

1.3.1 Functions of two Variables

Let S be a subset of $\mathbb{R} \times \mathbb{R}$. Then a function $f : S \to \mathbb{R}$ is called function of two variables.

Limit for functions of two variables

The concept of limit for functions of two variables is analogous to the concept of limit for functions of one variable. If L is the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0), then the situation is denoted by $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$.

Continuity

The concept of limit for functions of two variables is analogous to the concept of limit for functions of one variable.

A function $f(x,y)$ is **continuous** at the point (x_0,y_0) if

1. f is defined at (x_0,y_0),
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists,
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

A function is continuous if it is continuous at every point of its domain.

1.3.2 Partial Derivatives of a Function of Two Variables

The calculus of several variables is similar to single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get "partial derivative".

- The **partial derivative** of $f(x,y)$ with respect to x at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h},$$

provided the limit exits.

- The partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{k\to 0}\frac{f(x_0,y_0+k)-f(x_0,y_0)}{k},$$

provided the limit exits.

The partial derivative $\dfrac{\partial f}{\partial x}$ is also denoted by f_x, and $\dfrac{\partial f}{\partial y}$ by f_y.

Second-Order Partial Derivatives

The second-order partial derivatives of a function of two variables are as follows:

a). $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$ b). $f_{yy} = \dfrac{\partial^2 f}{\partial y^2}$ c). $f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y}$ d). $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$

Theorem 1 (The Mixed Derivative Theorem)**.** If $f(x,y)$ and its partial derivatives $f_x,\ f_y,\ f_{xy},\ f_{yx}$ are defined throughout an open region containing (a,b) and are all **continuous** at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

1.3.3 The Chain Rule

Theorem 2 (One independent variable and Two intermediate variables)**.** If $w = f(x,y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t)$ is differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Theorem 3 (Two independent variable and Two intermediate variables)**.** Suppose $w = f(x, y)$ and $x = x(s, t)$, $y = y(s, t)$. If all there functions are differentiable, then w has partial derivatives with respect to s and t, given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

1.3.4 Implicit Differentiation

Theorem 4. Suppose $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$. Then at any point where $F_y \neq 0$, $\quad \dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

1.4 Gradient Vectors and Directional Derivatives

We shall see that some of the vector fields in applications-not all of them!- can be obtained from scalar fields. This is a considerable advantage because scalar fields can be handled more easily. The relation between these two kinds of fields cab be obtained by **gradient**, which is thus of great practical importance.

Definition 2. The gradient vector (**gradient**) of a scalar function $f(x, y, z)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

.

Gradients are useful in several ways, notably in giving the rate of change of $f(x, y, z)$ in any direction in space, in obtaining surface normal vectors, and in deriving vector fields, from scalar fields, as we are going to show in this section.

Definition 3. If f is differentiable in a neighborhood of P_0, then the **directional derivative** of f at P_0, in the direction of the unit vector $\hat{u}$, is the **dot product** of the (**gradient**) of f at a point P_0 and the vector $\hat{u}$

$$\nabla f.\hat{u}$$

But $\nabla f.\hat{u} = |\nabla f||\hat{u}|cos\theta$ and this value becomes maximum if $cos\theta = 1 \implies \theta = 0$. That is when $\hat{u}$ has the same direction as ∇f. So, the rate of change of f at P_0 is maximum along the direction of ∇f.

Gradient as Surface Normal Vectors

Let f be a differentiable scalar function in space. Then $f(x, y, z) = c$ is a surface S, called the **level surface** of f. Then if the gradient of f at a point P of S is not the zero vector, it is a *normal vector* of S at P.

Proof. Let C be a curve on S through the point P of S. As a curve in space, C has a representation $r(t) = [x(t),\ y(t),\ z(t)]$. For C to lie on the surface S, the components of $r(t)$ must satisfy $f(x, y, z) = c$, that is

$$f(x(t), y(t), z(t)) = c \tag{1.3}$$

Now a tangent vector of C is $r'(t) = [x'(t),\ y'(t),\ z'(t)]$. And the tangent vector of all curves on S passing through P will generally form a plane, called the **tangent plane** of S P. The normal of this plane is called the **surface normal** to S at P. A vector in the direction of the surface normal is called a **surface normal vector'** of S at P. By chain rule in 1.3 we get,

$$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = \nabla f.r' = 0$$

Hence ∇f is orthogonal to all the vectors r' in the tangent plane, so that it is normal vector S at P. □

1.4.1 Vector field that are Gradient of Scalar field: (Potentials)

A vector field that can obtained from a scalar field is generally obtained by operating gradient on that scalar field. So, for such a vector field $v(P)$, there exists a scalar field $f(P)$ such that $v(P) = grad(P)$. The function $f(P)$ is called a **potential function**. Such vector field are characterized by a property called **conservative field**, meaning energy is conserved; meaning no energy is lost or gained in displacing a body in a loop. Conservative vector field play a central role in physics and engineering. An example of a conservative field is gravitational field.

1.4.2 Exercise

1. Find the gradient of the function at the given point.

   a). $f(x, y) = y - x, \quad (2, 1)$ b). $f(x, y) = ln(x^2 + y^2) \quad (1, 1)$

2. Given the velocity potential f of a flow, find the velocity $v = \nabla.f$ of the flow and its value at P.

   a). $f = x^2 + y^2 + z^2, \quad P = (3, 2, 2)$ b). $f = e^x siny, \quad P = (1, \pi)$

3. Heat flows in the direction of maximum decrease of temperature T. Find this direction in general and at P.

   a).$T = x^2 - y^2, \quad P = (2, 1)$ b). $T = tan^{-1}(y/x), \quad P = (2, 2)$

4. Find a normal vector of the surface at the given point P.

a). $z = x^2 + y^2$, $P = (3, 4, 25)$ b). $ax + by + cz = d$, $P = (x, y, z)$

5. Find the derivative of the function at P in the direction of $\vec{u}$.

   a.) $f = 2xy - 3y^2$ $P = (5, 5)$, $\vec{u} = 4\hat{i} + 3\hat{j}$

   b.) $f = xyz$ $P = (-1, 1, 3)$, $\vec{u} = 1\hat{i} - 2\hat{j} + 2\hat{k}$

6. Interpret the gradient of a function at a point, both geometrically and analytically.

1.4.3 Exercise

1. Find the divergence of the following vector functions.

   a). $[x^3 + y^3,\ 3xy^2,\ 3zy^2]$ b).$[e^{2x}cos2y,\ e^{2x}sin2y,\ 5e^{2z}]$

   c). $[x^2 + y^2,\ 2xyz,\ z^2 + x^2]$ d).$x^2y^2z^2[x,\ y,\ z]$

2. Show that the flow with velocity vector $v = y\hat{i}$ is incompressible. ($div\, v = 0$).

1.5 Divergence of a vector field

From a scalar field we can obtain a vector field by the gradient. Conversely, from a vector field we can obtain a scalar field by the divergence.

Let $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field in $\mathbb{R}^3$ such that the partial derivatives $\frac{\partial F_1}{\partial x}$, $\frac{\partial F_2}{\partial y}$, $\frac{\partial F_3}{\partial z}$ exist. Then the divergence of F is denoted by $div\, F$, and defined by

$$div\, F = \nabla.F = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Thus divergence of F is a scalar product of ∇ and F.

1. Physically, the divergence of a vector field F gives its flux, i.e., rate of flow per unit time per unit volume or per unit area.
2. A vector point function F is said to be **solenoidal** if $div\, F = 0$.
3. If F is constant then $div\, F = 0$.

1.6 Curl of a Vector Field

Let $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector field in $\mathbb{R}^3$ such that the partial derivatives $\frac{\partial F_1}{\partial x}$, $\frac{\partial F_2}{\partial y}$, $\frac{\partial F_3}{\partial z}$ exist. Then the curl of F is denoted by $curl\, F$, and defined by

$$curl\, F = \nabla \times F = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

Thus the curl of F is a vector product of ∇ and F.

Theorem 5. The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

Proof. A rotation of a rigid body B about a fixed axis in space can be described by a vector w of magnitude ω in the direction of the axis of rotation, where $\omega > 0$ is the angular speed of the rotation. If r is the position vector of a particle on the rigid body then, its velocity is given by $v = w \times r$.

With respect to Cartesian coordinate system having z axis as the axis of rotation. Then $w = [0, 0, w] = \omega\hat{k}, \quad v = w \times r = [-\omega y, \omega x, 0] = -\omega y\hat{\imath} + \omega x\hat{\jmath}$

Now,

$$curl\, v = \nabla \times v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\,\hat{k} = 2w$$

□

1.6.1 Exercise

1. Show that the divergence of a gradient is the Laplacian.

2. Prove that gradient fields are irrotational, i.e., $curl(gradf) = 0$.

3. Prove that curl fields are solenoidal, i.e., $div(curlf) = 0$.

1.6.2 Exercise

1. Find *curl* of the following vector fields.

   a). $[y,\ 2x^2,\ 0]$

   b). $[siny,\ cosz,\ -tanx]$

   c). $[y^n,\ z^n,\ x^n], \ \ (n > 0)$

   d). $[e^x cosy,\ e^x siny, 0]$

2. Let v be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? $v = [y, -x, 0]$