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Orthogonality and Least Squares

4.1 Inner Product

Let $u = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix}$ be any two vectors in $\mathbb{R}^n$. Then the number $u^T v$ is called the **inner product** of u and v. This inner product is also commonly known as **dot product** and denoted by **u.v**.

4.1.1 Properties of Inner Product

1. $u.v = v.u$
2. $(u + v).w = u.w + v.w$
3. $(\alpha u).v = \alpha(u.v) = u.(cv)$
4. $u.u \geq 0$ and $u.u = 0 \iff u = 0$

4.1.2 The Length of a Vector

The length of a vector v is called the **norm** of v.
It is denoted by $\|v\|$ and defined by $\|v\| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$ so that, $\|v\|^2 = v.v$ There are several kinds of norms actually, this particular norm is called **Euclidean norm**. For any scalar α, $\|\alpha v\| = |\alpha|\|v\|$. A vector whose length is unity is called a **unit vector**. If we divide a nonzero vector v by its length, we obtain a unit vector u. This process is called **normalizing** of the vector v.

Distance between vectors

For u and v in a vector space V, the distance between them is written as dist(u,v) and is defined as $dist(u, v) = \|u - v\|$.

4.2 Orthogonal Vectors

The two vectors u and v are orthogonal vectors if their dot product is zero,i.e $u.v = 0$. Observe that the zero vector is orthogonal to every vector as $0^T v = 0$ for all v.

Theorem 1 (The Pythagorean Theorem)
Two vectors u and v are orthogonal if and only if $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

4.2.1 Orthogonal Complement

- If a vector z is orthogonal to every vectors in a subspace W then, z is said to be orthogonal to W.
- The set of all vectors that are orthogonal to W is called the **orthogonal complement** of W. It is denoted by $W^\perp$. $W^\perp = \{z \ : \forall v \in W \ z.v = 0\}$

Theorem 2

1. A vector x is in $W^\perp$ if and only if x is orthogonal to every vector in a set that is spans W.
2. $W^\perp$ is also a subspace.
3. Row space is orthogonal complement of the Null space for a matrix.

4.2.2 Orthogonal Sets

A set of vectors $\{u_1, ...u_p\}$ in a vector space V is said to be **orthogonal set** if each pair of distinct vectors from the set is orthogonal,i.e for all $u_i, u_j \in V$ we have $u_i.u_j = 0$ whenever $i \neq j$.