3

Eigenvectors, Eigenvalues and Diagonalization

3.1 Introduction

Let A be any square matrix, real or complex. A number λ is an **eigenvalue** of A if the equation

$$Ax = \lambda x$$

is true for some nonzero vector x. The vector x is and **eigenvector** associated with the eigenvalue λ. Both the eigenvalue and the eigenvector may be complex.

3.1.1 Exercise

1. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & -8 \end{bmatrix}$?

2. Is $\lambda = 2$ a eigenvalue of a. $\begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$?

Theorem 1. A scalar λ is an eigenvalue of a matrix A if and only if $Det(A - \lambda I) = 0$.

The equation $Det(A - \lambda I) = 0$ is called the **characteristic equation** of A. It is the equation from which we can compute the eigenvalues of A. The function $p : p(\lambda) = Det(A - \lambda I)$ is the **characteristic polynomial** of A.

3.1.2 Eigenspace

For an eigenvalue λ of a matrix A, the set $\{x \; : \; Ax = \lambda x\}$ forms a vector space. This forms a vector space because the vector x is a nonzero vector for it be an eigenvector. If x is a nonzero solution of $Ax = \lambda x \implies (A - \lambda I)x = 0$, which is a homogeneous system, then this homogeneous system has infinitely many solution. And this vector space is called eigenspace.

3.1.3 Exercise

1. What are the characteristic equation and the eigenvalues of the following matrices? For each eigenvalue, find an eigenvector.

   a. $\begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 4 & 1 & 1 \\ 2 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ c. $\begin{bmatrix} 2 & -i & 0 \\ i & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. Find a basis for the eigenspace corresponding to each eigenvalue.

   a. $\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ c. $\begin{bmatrix} 4 & 0 & 0 \\ -2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Find the eigenvalue-eigenvector pairs. Explore the geometric effect of letting $x^{(k)} = Ax^{(k-1)}$ and $k = 0, 1, 2, ...$

4. Prove that A and A^t have the same eigenvalues.

3.1.4 Important Results

Theorem 2. The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 3. If $v_1, ..., v_r$ are eigenvectors that corresponds to distinct eigenvalues $\lambda_1, ..., \lambda_r$of an $n \times n$ matrix A, then the set $\{v_1, ..., v_r\}$ is linearly independent.

Theorem 4. Zero is an eigenvalue of A if and only if A is not invertible.

3.2 Diagonalization

A square matrix A is said to be diagonalizable if, there exists an invertible matrix P and a diagonal matrix D, such that

$$A = PDP^{-1}.$$

If $A = PDP^{-1}$ then A is said to be similar to D.
This is standardize as if, A is similar to a diagonal matrix.

If $A = PDP^{-1}$ then prove that $A^k = PD^kP^{-1}$.

3.2.1 The Diagonalization Procedure

Suppose A is an $n \times n$ matrix with n different eigenvalues: $\lambda_1, \lambda_2, ..., \lambda_n$ so that the set of n corresponding eigenvectors $v_1, v_2, ..., v_n$ are linearly independent. Now, $Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2, ..., Av_n = \lambda_n v_n$. Let P be the matrix whose columns are the eigenvectors, i.e $P = [v_1,\ v_2,\ ...\ v_n]$. Then

$$AP = [\lambda_1 v_1,\ \lambda_2 v_2,\ ...\ \lambda_n v_n] = [v_1,\ v_2,\ ,...,\ v_n] \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \lambda_1 \end{bmatrix} = PD \tag{3.1}$$

where $D = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \lambda_1 \end{bmatrix}$, so that $AP = PD$ gives, $A = PDP^{-1}$. This is summarized as the following theorem.

Theorem 3 (**The Diagonalization Theorem**)
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors of P.

Theorem 4. An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

3.2.2 Exercise

1. Diagonalize the matrices, if possible:

   i). $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$

   ii). $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

   iii). $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

   iv). $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

2. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

3.3 Matrices Whose Eigenvalues are not Distinct

If $n \times n$ A has n distinct eigenvalues, with corresponding eigenvectors, then A is automatically diagonalizable. Now, we will look at a case where A has fewer than n distinct eigenvalues, and it is still possible to diagonalize A.

Theorem 5. Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, ... \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n. And this happens if and only if the dimension of the eigenspace for each λ_k equals to the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

3.3.1 Multiplicities of an eigenvalue

- The algebraic multiplicity of λ is the number of times $(\lambda - t)$ occurs as a factor of in its characteristic polynomial.
- The geometric multiplicity of an eigenvalue λ is the dimension of the corresponding eigenspace E_λ. i.e $dim\{E_\lambda\}$.

3.3.2 Exercise

Diagonalizable the following matrix, if possible.

1. $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$
2. $\begin{bmatrix} 4 & -9 \\ 4 & 8 \end{bmatrix}$

3.3.3 Similarity

If A an B are $n \times n$ matrices, then A **similar to** B if there is an invertible matrix P such that

$$A = PBP^{-1}$$

or equivalently $B = P^{-1}AP$. Writing $Q = P^{-1}$, we can say that B is similar to A, and we simply say A and B are similar.

Theorem 6. If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof. Given, $B = P^{-1}AP$ we have, $(B - \lambda I) = P^{-1}AP - \lambda P^{-1}P = P^{-1}(A - \lambda I)P$. Then

$$\begin{aligned} det(B - \lambda I) &= \det[P^{-1}(A - \lambda I)P] \\ &= \det(P^{-1}).\ \det(A - \lambda I).\ \det(P) \\ &= \det(P^{-1}).\ \det(P).\ \det(A - \lambda I) \\ &= \det(A - \lambda I) \end{aligned}$$

As, $\det(P^{-1}).\ \det(P) = \det(P^{-1}.P) = det(I) = 1$. Hence A and B has same characteristic polynomial. And the proof follows. □

3.4 Linear Transformation and Diagonalization

We will see that the transformation $x \to Ax$ is essentially same as the very simple mapping $x \to Dx$ when viewed from the proper perspective.

Theorem 7. Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$. If B is the basis for $\mathbb{R}$, formed from the columns of P, then D is the $\mathcal{B}$-matix for the transformation $x \mapsto Ax$.

Proof. Let the order of matrix A be n. Since A is diagonalizable, A has n linearly independent eigenvectors: $\mathcal{B} = \{v_1, ..., v_n\}$, that forms a basis of $\mathbb{R}^n$. Suppose for $x = x_1e_1 + ... + x_ne_n$ we have $x = x'_1v_1 + ... + x'_nv_n$.

$$\begin{aligned} Ax = A(x'_1v_1 + ... + x'_nv_n) &= x'_1Av_1 + ... + x'_nAv_n \\ &= [Av_1 \;...\; Av_n](x'_1 \;...\; x'_n)^t \\ &= [\lambda_1v_1 \;...\; \lambda_nv_n](x'_1 \;...\; x'_n)^t \\ &= \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \lambda_1 \end{bmatrix} x \qquad (3.2) \\ &= Dx \end{aligned}$$

Here, $[\lambda_1v_1 \;...\; \lambda_nv_n] = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \lambda_1 \end{bmatrix}$. This is because for $(x'_1 \;...\; x'_n)^t$, the vectors $(v_1, ..., v_n)$ are the unit vectors, just like $e_1, ..., e_n$ are for $x_1, ..., x_n$. □

Remark. This proof is not need for the students.

3.4.1 Exercise

Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that the transformation matrix of the transformation given by the following matrix is Diagonal.

1. $\begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$
2. $\begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$