3

Fourier Transform

3.1 Fourier Integral

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. The idea of Fourier series extended to the functions that are nonperiodic gives Fourier Integral. The Fourier Integral Representation of a function $f(x)$ is

$$f(x) = \int_0^\infty [A(w)\, cos\, wx + B(w)\, sin\, wx] \tag{3.1}$$

$$\text{where, } A(w) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\, cos\, wv\, dv, \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\, sin\, wv\, dv$$

3.2 Fourier Cosine and Sine Transform

An integral transform is a transformation in the form of an integral that produces from given functions new functions depending on a different variable. The Fourier transforms can be obtained from the Fourier integral.

3.2.1 Fourier Cosine Integral and Fourier Sine Integral

For an even or odd function the Fourier integral becomes simpler. Indeed, if $f(x)$ is an **even** function, then $B(w) = 0$ and if $f(x)$ is an **odd** function, then $A(w) = 0$ and the **Fourier cosine integral and the Fourier sine integral** are:

$$f(x) = \int_0^\infty A(w)\, cos\, wxdw, \qquad f(x) = \int_0^\infty B(w)\, sin\, wxdw \tag{3.2}$$

where

$$A(w) = \frac{2}{\pi}\int_0^\infty f(v)\, cos\, wv\, dv$$

Now, if we set $A(w) = \sqrt{2/pi}\, \hat{f}_c(w)$, where c suggests "cosine."

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, cos\, wx\, dx \tag{3.3}$$

Then,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w)\, cos\, wx\, dw \tag{3.4}$$

The process of obtaining the transform $\hat{f}_c(w)$ from a given f is called **Fourier cosine transform** and the inverse process using the formula 3.4 called **Inverse Fourier cosine transform**. Similarly following gives the **Fourier sine transform** and its inverse transform.

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, sin\, wx\, dx \tag{3.5}$$

Then,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w)\, sin\, wx\, dw \tag{3.6}$$

3.2.2 Properties

1. The Fourier cosine and sine transforms are **linear operations:**
   - $\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$
   - $\mathcal{F}_S(af + bg) = a\mathcal{F}_S(f) + b\mathcal{F}_S(g)$
2. The transform of the derivative of a function f.
   - $\mathcal{F}_c(f'(x)) = w\mathcal{F}_c(f(x)) - \sqrt{\frac{2}{\pi}}f(0)$
   - $\mathcal{F}_S(f'(x)) = -\mathcal{F}_c(f(x))$

3.3 Fourier Transform

The **complex Fourier integral**, is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) e^{iw(x-v)}\, dv\, dw \tag{3.7}$$

This can be written as,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(v) e^{-iwv}\, dv \right] e^{iwx}\, dw$$

Now the function $\hat{f}$ with the following is called the **Fourier transform** of f.

$$\hat{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx \tag{3.8}$$

Then the following is called the **inverse Fourier transform** of $\hat{f}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\hat{w}) e^{iwx}\, dw \tag{3.9}$$

Another notation for the Fourier transform is $\hat{f} = \mathcal{F}(f)$ so $f = \mathcal{F}^{-1}(\hat{f})$.

Theorem 1 (Existence of the Fourier Transform). If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(w)$ of $f(x)$ given by 3.9 exists.

3.3.1 Physical Interpretation

The nature of the representation 3.9 of $f(x)$ becomes clear if we think of it as a superpositon of sinusoidal oscillations of all possible frequences, called a **spectral representation**. Like light is such a superpositon of seven different colors of light. $\hat{f}$ measures the intensity of $f(x)$ in the frequency interval between w and $w + \Delta w$.

3.3.2 Properties

1. **Linearity**

   The Fourier transforms are linear operations:

   - $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$

2. **Shift Formula**

   If $g = f(x + b)$ and $h = f(ax)$ then,

   - $\mathcal{F}(g)(w) = e^{iwb}\,\mathcal{F}(f)(w)$
   - $\mathcal{F}(h)(w) = (1/|a|)\,\mathcal{F}(f)(w/a)$
   - $\mathcal{F}(e^{ibx} f(x))(w) = \mathcal{F}(f)(w - b)$

3. **Derivative Formula**

   The transform of the derivative of a function f.

   - $\mathcal{F}(f')(w) = iw\mathcal{F}_c(f)(w)$
   - $\mathcal{F}(f^{(n)})(w) = (iw)^n\mathcal{F}_c(f)(w)$

3.3.3 Convolution

The convolution $f * g$ of functions f and g is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x - p)\,dp = \int_{-\infty}^{\infty} f(x - p)g(p)\,dp \tag{3.10}$$

Theorem 2. Suppose that the Fourier transform of $f(x)$ and $g(x)$ exist. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g) \tag{3.11}$$

3.3.4 Exercise

Determine the Fourier Transform

a). $f(x) = e^{-a|x|}, \quad a > 0$

b). $f(x) = 4xe^{-x^2}$

Remark. Different books define the Fourier transform in different ways. Why all these variants? These differences are purely technical. The only important point is that e^{-iwx} appears in one and e^{iwx} appears in the other. And the product of the constants before the transform and its inverse must equal to $1/2\pi$.

3.3.5 Fourier Transform Table

S.N	f(x)	$\hat{f} = \{$
1	$\begin{cases} 1 & -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{sinbw}{w}$
2	$\begin{cases} 1 & b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}}$
3	$\frac{1}{x^2 + b^2, \quad (a > 0)}$	$\sqrt{\frac{\pi}{2}}\,\frac{e^{-a\|w\|}}{a}$
4	$\begin{cases} x & 0 < x < b \\ 2x - b & ifb < x < 2b0 \quad \text{otherwise} \end{cases}$	$\frac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi}\,w^2}$
5	$\begin{cases} e^{ax} & x > 0, \quad (a > 0) \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}\,(a + iw)}$
6	$e^{-ax^2}, \quad (a > 0)$	$\frac{1}{\sqrt{2a}}\,e^{-w^2/4a}$
7	$\frac{sin\,ax}{x}, \quad (a > 0)$	$\frac{\pi}{2}$ if $\|w\| < a; 0$ if $\|w\| > a$

Table 3.1: Fourier transform table for elementray functions

3.3.6 Plancherel Identity

If $\int_{\infty}^{\infty} |f(x)|^2\,dx < \infty$, then $\int_{\infty}^{\infty} |\hat{f}(x)|^2\,dx < \infty$ and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\hat{f}(w)|^2$$

This formula is sometimes called the formula for the **conservation of energy**. The left-had side represents the energy of a signal in the time domain, while the right-hand side represents its energy in the frequency.

3.3.7 Exercise

Determine the Fourier transform of:

a). $\dfrac{cos\, ax}{a^2+b^2}$ b). $\dfrac{sin\, ax}{a^2+b^2}$

3.3.8 Applications of Residue

Theorem 3. If f is analytic on all of $\mathbb{C}$ except at the points $z_1, z_2, ..., z_m$, in the **upper half plane** and the points $z_{m+1}, z_{m+2}, ..., z_n$, in the **lower half plane** and that f is absolutely integrable on $\mathbb{R}$, and $\lim\limits_{Rto\infty} |f(z)| = 0$, then the Fourier transform of f is given by,

$$\hat{f}(w) = \begin{cases} -i\ \sum_{j=m+1}^{n}\ Res\{f(z)\, e^{-iwz};\ z_j\}, & w \geq 0 \\ \\ i\ \sum_{j=1}^{m}\ Res\{f(z)\, e^{-iwz};\ z_j\}, & w \leq 0 \end{cases} \tag{3.12}$$

Example 1. Let $f(x) = 1/(x^2+1)$. The analytic extension of f to all of $\mathcal{C}$ is $f(z) = 1/(z^2+1)$. f has simple poles at $-i$, i, and it is easily verified that f satisfies the other conditions of Theorem 3.12. Thus the Fourier transform of f is given by

$$\hat{f}(w) = -i\ Res\{\frac{e^{-iwz}}{(z^2+1)}; -i\} = -i\frac{1}{-2i}e^{-iwi} = \frac{e^{-w}}{2}, \quad w \geq 0$$

$$\hat{f}(w) = i\ Res\{\frac{e^{-iwz}}{(z^2+1)}; i\} = i\frac{1}{2i}e^{-iwi} = \frac{e^{w}}{2}, \quad w \leq 0$$

That is $\hat{f}(w) = e^{-|w|}/2$

3.3.9 Application to Partial Differential Equations

The Heat Equation

The one dimensional homogeneous heat equation on the real line is given by the equation:

$$u_t = ku_{xx}, \qquad -\infty < x < \infty, \quad 0 < t < \infty \tag{3.13}$$

Here u is the temperature function of length and time. We assume the real line is a model of a rod of infinite length that is made of some material of uniform density. And the rod is insulated. k is a positive constant which has a physical meaning that depends upon the material of the rod. This is a simple mathematical model for the distribution of heat, along a rod, over time.

In order to determine a particular solution it requires some conditions. We assume an initial condition of the form:

$$u(x, 0) = f(x), \quad 0 < t < \infty$$

for some f that has Fourier transform.

Procedure

1. Take Fourier transform with respect to the variable x on the both sides of the PDE 3.13. Let, $U(w, t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} u(x, t)\, e^{-iwx}\, dx$. Then,

$$U_t(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t)\, e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k\, u_{xx}(x,t)\, e^{-iwx}\, dx.$$

Now from the derivative of Fourier transform, $\mathcal{F}(f'')(w) = (iw)^2 \mathcal{F}_c(f)(w) = -w^2 \mathcal{F}_c(f)(w)$, we have, so,

$$U_t(w,t) = k\mathcal{F}(u_{xx})(w,t) = -kw^2 U(w,t)$$

$$U_t(w,t) + kw^2 U(w,t) = 0$$

$$U_t + kw^2 U = 0 \tag{3.14}$$

2. The transformed equation 3.14 is an ordinary differential equation of order one and degree one. The general solution whose is

$$U(w,t) = A(w)\, e^{-kw^2 t}$$

, where $A(w)$ is a totally arbitary function of w. The determination of the $A(w)$ gives the partial solution.

3. Using the above initial condition,

$$A(w) = U(w,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx = \hat{f}(w)$$

4. Thus, $U(w,t) = \hat{f}(w)\, e^{-kw^2 t}$ is the solution of the transformed equation. Now we take inverse Fourier transform on both sides to get u which is the solution of the original PDE.

$$u = \mathcal{F}^{-1}\left(\hat{f}(w)\, e^{-kw^2 t}\right)$$