4

Interpolation

4.1 Introduction

The central problem of numerical analysis is: given a set of tabular values $(x_0, y_0), (x_1, y_1), (x_2, y_2), ..., (x_n$ satisfying the realtion $y = f(x)$, where the explict nature of $f(x)$ is not known, is to find a simpler function $\phi(x)$ such that $\phi(x)$ and $f(x)$ agree at the set of the tabulated points. Such process is called **interpolation**. If $\phi(x)$ is a polynomial, then the process is called *polynomial interpolation*, and $\phi(x)$ a interpolating polynomial. In this chapter we are concerned with only polynomial interpolation.

Theorem 1 (Weiestrass Theorem)**.** If $f(x)$ is continuous in $x_0 \geq x \geq x_n$, then given $\epsilon > 0$, there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon$, for all $x \in (x_0, x_n)$.

So, this gives the justification for approximating a function with a polynomial function.

4.2 Finite Differences

Assume that we have a table of values $(x_i, y_i),\ i = 0, 1, ..., n$ of any function $y = f(x)$. Suppose the values of x are equally spaced, i.e., $x_i = x_0 + ih,\ i = 0, 1, ..., n$ for some constant h.
Then, the values $y_1 - y_0$, $y_2 - y_1$, ..., $y_n - y_{n-1}$, are called the differences of y. If $y_{i+1} - y_i$ is taken as the i^{th} difference, then we are considering it as a forward difference, if it is taken as the $(i + 1)^{th}$ differnce then, we are considering it as a backward differnce, and if it is taken as $((i + 1)/2)^{th}$ then, we are considering it as a central differnce. More on these topics in coming subsections.

4.2.1 Forward Differences

The forward differnce Δy_i is defined as $\Delta y_i = y_{i+1} - y_i$, where Δ is called the *forward difference operator*. We have,

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \quad ..., \quad \Delta y_{n-1} = y_n - y_{n-1} \tag{4.1}$$

These differnces $\Delta y_i,\ i = 0, 1, ..., (n - 1)$ are called *first forward differences.*

- For $(n+1)$ points, there are n forward differnces.
- These n-forward differnces begin at Δy_0, and end at Δy_{n-1}. There is no Δy_n.

Second Forward Differnces

The forward differences of the first forward differences are called second forward differences. They are denoted by $\Delta^2 y_i,\ i = 0, 1, ..., (n-2)$. That is,

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0, \quad \Delta^2 y_1 = \Delta y_2 - \Delta y_1, \quad ..., \quad \Delta^2 y_{n-2} = \Delta y_n - \Delta y_{n-1} \tag{4.2}$$

Here, $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 - y_0$.

Higer Order Forward Differnces

Here, $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = \Delta y_2 - \Delta y_1 - (\Delta y_1 - \Delta y_0) = ... = y_3 - 3y_2 + 3y_1 - y_0$.
The coefficients occuring are the **binomial coefficients**.

Programming

For practical computations, the following notation is useful: $y_j = DEL(0, j)$, and

$$\Delta y_j = DEL(0, j+1) - DEL(0, j)$$
$$\Delta^i y_j = DEL(i-1, j+1) - DEL(i, j)$$

4.2.2 Backward Differences

The backward differnce ∇y_{i+1} is defined as $\nabla y_{i+1} = y_{i+1} - y_i$, where ∇ is called the *backward difference operator*. We have,

$$\nabla y_1 = y_1 - y_0, \quad \nabla y_2 = y_2 - y_1, \quad ..., \quad \nabla y_n = y_n - y_{n-1} \tag{4.3}$$

These differnces $\nabla y_{i+1},\ i = 0, 1, ..., (n-1)$ are called *first backward differences.*

- For $(n+1)$ points, there are n backward differnces.
- These n-backward differnces begin at ∇y_1, and end at ∇y_n. There is no ∇y_0.

Second Backward Differnces

The backward differences of the first backward differences are called second backward differences. They are denoted by $\nabla^2 y_{i+2},\ i = 0, 1, ..., (n-2)$. That is,

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \quad \nabla^2 y_3 = \nabla y_3 - \nabla y_2, \quad ..., \quad \nabla^2 y_n = \nabla y_n - \nabla y_{n-1} \tag{4.4}$$

Here, $\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 - y_0$.

Higer Order Backward Differnces

Here, $\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = \nabla y_3 - \nabla y_2 - (\nabla y_2 - \nabla y_1) = ... = y_3 - 3y_2 + 3y_1 - y_0$.
The coefficients occuring are the **binomial coefficients**.

4.2.3 Central Differences

The central differnce $\delta y_{i+1/2}$ is defined as $\delta y_{i+1/2} = y_{i+1} - y_i$, where δ is called the *central difference operator*. We have,

$$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1, \quad ..., \quad \delta y_{(2n-1)/2} = y_n - y_{n-1} \tag{4.5}$$

These differnces $\delta y_{i+1/2}$, $i = 0, 1, ..., (n-1)$ are called *first central differences.*

- For $(n+1)$ points, there are n first central differnces.
- These n-central first differnces begin at $\delta y_{1/2}$, and end at $\delta y_{(2n-1)/2}$.

Second Central Differnces

The central differences of the first central differences are called second central differences. They are denoted by $\delta^2 y_i$, $i = 1, ..., (n-1)$ $\delta^2 y_i = \delta y_{i+1/2} - \delta y_{(i-1)+1/2}$. That is,

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}, \quad \delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}, \quad ..., \quad \delta^2 y_{n-1} = \delta y_{(2n-1)/2} - \delta y_{(2n-3)/2} \tag{4.6}$$

Here, $\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2} = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 - y_0$.

Higer Order Central Differnces

Here, $\delta^3 y_{3/2} = \delta^2 y_2 - \delta^2 y_1 = \delta y_{5/2} - \delta y_{3/2} - (\delta y_{3/2} - \delta y_{1/2}) = ... = y_3 - 3y_2 + 3y_1 - y_0$.

4.2.4 Equivallency of the differences

From the above subsections we can see that, the same numbers occurs at the same positions whether we use forward backward or central differences. That is,

$$\Delta y_0 = \nabla y_1 = \delta y_{1/2}, \qquad \Delta^2 y_0 = \nabla^2 y_1 = \delta^2 y_1, \qquad \Delta^3 y_0 = \nabla^3 y_3 = \delta^3 y_{3/2}$$

4.3 Newton's formula for Interpolation

Given $(n+1)$ points, (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) of x and $y(x)$, with x values equally spaced, we find a polynomial $y_n(x)$ of nth degree such that, $y(x)$ and $y_n(x)$ agree at the given points. If we consider $y_n(x)$ to be the polynomial:

$$\begin{aligned} y_n x(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ + ... + a_n(x - x_0)(x - x_1)...(x - x_{n-1}). \end{aligned} \tag{4.7}$$

Then, $a_0 = y_0, \quad a_1 = \dfrac{y - y_1}{x_1 - x_0} = \dfrac{\Delta y_0}{h}, \quad a_2 = \dfrac{\Delta^2 y_0}{2!\, h^2}, \quad a_3 = \dfrac{\Delta^3 y_0}{3!\, h^3}, \; ..., \; a_n = \dfrac{\Delta^n y_0}{n!\, h^n}$

where, h is the $x_i - x_j, \quad i \neq j$.
Setting $x = x_0 + ph$, we get,

$$\begin{aligned} y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... \\ + \frac{p(p-1)(p-2)...(p-(n-1))}{n!}\Delta^n y_0 \end{aligned} \tag{4.8}$$

This formula 4.8 is the **Newton's forward difference interpolation** formula.

4.3.1 Newton's backward difference Interpolation

If we consider $y_n(x)$ to be the polynomial:

$$\begin{aligned} y_n x(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) \\ +... + a_n(x - x_n)(x - x_{n-1})(x - x_{n-2})...(x - x_1). \end{aligned} \tag{4.9}$$

Then we get the following formula for $y_n(x)$.

$$\begin{aligned} y_n(x) = y_n + p\nabla y_n + \frac{p(p-1)}{2!}\nabla^2 y_n + \frac{p(p-1)(p-2)}{3!}\nabla^3 y_n + ... \\ + \frac{p(p-1)(p-2)...(p-(n-1))}{n!}\nabla^n y_n \end{aligned} \tag{4.10}$$

This formula 4.10 is the **Newton's backward difference interpolation** formula.

Remark. The Newton's forward difference interpolation formula is useful for interpolating near the begining of the tabular values and the Newton's backward difference interpolation formula is useful for interpolating near the end of the tabular values.

4.4 Interpolation with Unevenly spaced points

This section discusses the inerpolation method for unequally spaced values of the argument.

4.4.1 Lagrange's Interpolation Formula

Let $y(x)$ be continuous and differentiable $(n+1)$ times in the interval (a, b). Given $(n+1)$ points, (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) of x and $y(x)$, where x values not necessarily equally spaced, we find a polynomial $L_n(x)$ of nth degree such that, $y(x)$ and $L_n(x)$ agree at the given points.
The formula $\sum_{i=0}^{n} l_i(x)\, y_i$, where $l_i(x) = \dfrac{(x-x_0)(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$ is called the Lagrange's Interpolating formula.

4.4.2 Disadvantage of Lagrange's Interpolation

The Lagrange's Interpolation formula has the disadvantage that if another interpolation point were added, then interpolation coefficients $l_i(x)$ will have to be recomputed. We therefore see an interpolation polynomial which has the property that a polynomial of higer degree may be derived from it by simply adding new terms. Newton's general interpolation formula is one such formula.

4.5 Divided Differences

Given $(n+1)$ points, (x_0, y_0), (x_1, y_1), ..., (x_n, y_n) of x and $y(x)$, with x values equally spaced, the divided differences of order $1, 2, ..., n$ are defined by the relations:

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$
$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$
$$...$$
$$[x_0, x_1, x_2, ..., x_n] = \frac{[x_1, x_2, ..., x_n] - [x_0, x_1, ..., x_{n-1}]}{x_n - x_0}$$

It is easy to see that, $[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} = [x_1, x_0]$. Again,

$$\begin{aligned}
[x_0, x_1, x_2] &= \frac{[x_2, x_0] - [x_0, x_1]}{x_2 - x_0} \\
&= \frac{1}{x_2 - x_0}\left[\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right] \\
&= \frac{1}{x_2 - x0}\left[\frac{y_2}{x_2 - x_1} - y_1\left(\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_0}\right) + \frac{y_0}{x_1 - x_0}\right] \\
&= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}
\end{aligned}$$

4.5.1 Newton's General Interpolation Formula

We have, $[x, x_0] = \frac{y - y_0}{x - x_0} \implies y = y_0 + (x - x_0)[x, x_0]$

And $[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1} \implies = [x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$

From these two relations we get,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \tag{4.11}$$

Similarly we can get $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$ and so on... Proceeding this way we get,

$$\begin{aligned}
y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \\
(x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + ... \\
+(x - x_0)(x - x_1)(x - x_2)...(x - x_n)[x, x_0, x_1, ..., x_n]
\end{aligned} \tag{4.12}$$

This Equation 4.12 is called the Newton's general interpolation formula with divided differnces.

4.6 Inverse Interpolation

Given a set of values of x and y, the process of finding the value of x for a certain value of y is called inverse inerpolation.

- When the values of x are at unequal intervals, the most obvious way of performing this process is by interchanging x and y in Lagrange's method.
- When the values of x are equally spaced, the method of successive approximations on Newton's method should be used.